AUTOMORPHISMS OF INTEGRAL GROUP RINGS

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ABSTRACT

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By

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For the most part, our attention is focused on a slightly restricted subgroup of the group $\mathcal{A}$ of all ring automorphisms of an integral group ring $Z(G)$ of a finite group G over the rational integers. This subgroup is denoted by $\mathcal{N}\mathcal{A}$ and consists of those elements of $\mathcal{A}$ which are "normalized" in a natural way. Information about $\mathcal{N}\mathcal{A}$ is easily converted to information about $\mathcal{A}$. When the commutator subgroup G' of G is abelian, it is shown that $Aut(G)$, the group of all automorphisms of G, has a normal complement M in $\mathcal{N}\mathcal{A}$. Using this decomposition of $\mathcal{N}\mathcal{A}$, we give, in Chapter 1, several conditions which guarantee that all elements f of $\mathcal{N}\mathcal{A}$ can be written, for all $g \in G$, as $f(g) = u\sigma(g)u^{-1}$ for some $\sigma \in Aut(G)$ and some unit u of $Q(G)$, the group ring of G over the field of rational numbers. Such an f is said to have an <u>elementary representation</u>. Among the sufficient conditions for all elements of $\mathcal{N}\mathcal{A}$ to have an elementary representation are: (1) G' has 1,2, or 3 elements, (2) G has at most one non-linear irreducible representation over the field of complex numbers, (3) G is nilpotent with nilpotence class ≤ 2, (4) G has a cyclic normal subgroup of prime index. A key lemma for the fourth result is that if $f \in \mathcal{N}\mathcal{A}$ and

$g, g_1 \in G$ are such that $f(\bar{C}_g) = \bar{C}_{g_1}$ then the order of g and the order of g_1 are equal; $\bar{C}_g$, $\bar{C}_{g_1}$ denote the class sums corresponding to g and g_1, respectively, and this result is valid even if G' is not abelian.

Chapter 2 contains several preliminary results about $\mathcal{NA}$. If G' is abelian and M is as described above, it is of interest to know when every element of M fixes all class sums corresponding to elements of G. This question has an affirmative answer if G satisfies either of (1), (2) or (3) above. Although the question is not answered, in general, it is shown that if M is non-trivial then M contains non-trivial elements which fix all class sums corresponding to elements of G. Without the assumption that G' is abelian, we discuss when $\mathcal{NA} = \mathrm{Aut}(G)$. Some properties which the elements of $\mathrm{Aut}(G)$ and $\mathcal{NA}$ share are also described. $f(\Delta(K))$ is analyzed if $f \in \mathcal{NA}$ has an elementary representation and $K \trianglelefteq G$; here $\Delta(K)$ denotes the kernel of the canonical ring homomorphism of $Z(G)$ onto $Z(G/K)$. Finally, the center of $\mathcal{NA}$ is examined and shown to be trivial if the center of G is trivial.

Examples are given in Chapter 3 to show that all elements of $\mathcal{NA}$ can have elementary representations even when G' is non-abelian and simple.

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NOTATION AND TERMINOLOGY

G = a finite group

$K \leq G$ = K is a subgroup of G

$K \trianglelefteq G$ = K is a normal subgroup of G

K' = the derived subgroup of K

$\mathfrak{Z}(G)$ = the center of the group G

$R(G)$ = the group ring of G over the ring R

$\mathfrak{Z}(R(G))$ = the center of $R(G)$

Z, Q, C = rational integers, rational field, complex field, respectively

ℓ = the augmentation map from $Z(G)$ onto Z given by $\ell(\sum_g a_g g) = \sum_g a_g$

$\Delta(S) = \{ \sum_{s \in S} (s-1)t(s) \mid t(s) \in Z(G)$ is arbitrary$\}$, where S is a finite subset of $Z(G)$

Λ = Z - submodule of $Z(G)$ spanned by all differences $ab-ba$ with $a,b \in Z(G)$

$\Lambda_p = \Lambda + pZ(G)$, where p is a prime integer

$Aut(G)$ = group of all automorphisms of the group G

$\mathcal{A}$ = the group of all ring automorphisms of $Z(G)$

$\mathcal{NA} = \{f \in \mathcal{A} \mid \ell(f(g)) = 1$ for all $g \in G\}$ = the set of all <u>normalized automorphisms</u> of $Z(G)$

I = set of elements of $\mathcal{A}$ which fix $\mathfrak{Z}(Z(G))$ elementwise

T_u = endomorphism of $Z(G)$ given by $x \to uxu^{-1}$ for all $x \in Z(G)$; u is a unit in $Q(G)$ such that $uxu^{-1} \in Z(G)$ for all $x \in Z(G)$

H = <u>group basis</u> for $Z(G)$. That is, H is a subset of $Z(G)$ such that

(1) H is a multiplicative subgroup of the group of units of $Z(G)$

(2) H is a free Z-basis for $Z(G)$

(3) $\ell(h) = 1$ for all $h \in H$.

Let A be a finite group and $a \in A$.

$|A|$, $|a|$ = the order of the group A and the element a respectively

C_a = the set of all conjugates of a in A

$|C_a|$ = the number of elements in C_a

$\overline{C}_a$ = the element of $Z(A)$ which is the sum of all elements in C_a

Suppose $x = \sum_g a_g g$ is an element of $Z(G)$ and $g_1 \in G$. The phrase "x sums to n on C_{g_1}" means that $\sum_{g \in C_{g_1}} a_g = n$.

The notation 2.3.1 means the 1st result in the 3rd Section of Chapter 2. The results are numbered consecutively in each section without regard to the words lemma, proposition, and theorem.

INTRODUCTION

The study of integral group rings $Z(G)$[1] has received considerable research attention for many years with much interest being focused on the extent to which $Z(G)$ determines G; this has been called the group ring problem. In this connection, we mention a recent paper of E.C. Dade, in [4], in which he exhibited 2 non-isomorphic groups G_1, G_2 such that $F(G_1) \cong F(G_2)$ for all fields F. This is in contrast to a result of Whitcomb in [14] which shows that $Z(G_1) \not\cong Z(G_2)$. One method of studying an object is to see how it can be moved around; in particular, we are interested in the group of all ring automorphisms of $Z(G)$. Although the group ring problem is not discussed in this thesis, it is possible that some of the results will aid in solving a part of it; see Obayashi [8].

For the most part, our work is concerned with a slightly specialized subgroup of $\mathcal{A}$, namely the subgroup $\mathcal{NA}$ of all normalized automorphisms of $Z(G)$. First, we show that, when G' is abelian, $\mathrm{Aut}(G)$ has a normal complement $M = \{f \in \mathcal{NA} \mid f(g) - g \in \Delta(G')\Delta(G)$ for all $g \in G\}$ in $\mathcal{NA}$. In Chapter 1, several conditions are given which guarantee that each element f of $\mathcal{NA}$ is such that, for all $g \in G$, $f(g) = u\sigma(g)u^{-1}$ for some $\sigma \in \mathrm{Aut}(G)$ and some unit u in $Q(G)$; both σ and u depend on f. If f can be described in this way, we say it has an <u>elementary representation</u>. In all of the results

[1] See the previous pages for notation and terminology.

concerning elementary representations in Chapter 1, G' is abelian and an elementary representation is obtained for all elements of $\mathcal{NA}$ by showing that all elements of M have an elementary representation. If G is nilpotent with nilpotence class ≤ 2 or if G has at most one non-linear irreducible representation over C, we show that $M \subseteq I$. This implies that the elements of $\mathcal{NA}$ have elementary representations. When G has nilpotence class ≤ 2, this is a result of Sehgal in [12]. If $G = D_4$ is the dihedral group of order 8, then G is nilpotent of class 2 so that every $f \in \mathcal{NA}$ is of the form $f = \Phi_1 \circ \sigma$ where $\sigma \in \mathrm{Aut}(D_4)$ and $\Phi_1 \in I$ satisfies $\Phi_1(g) - g \in \Delta(D_4')\Delta(D_4)$ for all $g \in D_4$. This is a recent result of Obayashi in [8]. Perhaps the most satisfying result we have obtained is that if G has a cyclic normal subgroup of prime index, then all elements of $\mathcal{NA}$ have elementary representations. A key lemma for this result is that if $f \in \mathcal{NA}$ and $f(\bar{C}_g) = \bar{C}_{g_1}$ for $g, g_1 \in G$ then $|g| = |g_1|$; here G' is not necessarily abelian. It is also demonstrated that all elements of $\mathcal{NA}$ have elementary representations if G' has 1, 2 or 3 elements.

Chapter 2 contains preliminary results about $\mathcal{A}$ and $\mathcal{NA}$. If G' is abelian and M is as above, it is of interest when $M \subseteq I$. This question is not answered, in general, but it is shown that if M is not trivial then $M \cap I$ is not trivial. Then the assumption that G' is abelian is dropped and we discuss when $\mathcal{NA} = \mathrm{Aut}(G)$. Two properties of $\mathrm{Aut}(G)$ enjoyed by $\mathcal{NA}$, in addition to those of Chapter 1, are also disclosed. $f(\Delta(K))$ is analyzed if $f \in \mathcal{NA}$ has an elementary representation and $K \trianglelefteq G$. Finally, the center of $\mathcal{NA}$ is studied and shown to be trivial if $Z(G)$ is trivial.

In Chapter 3, we describe a method for studying the question of an elementary representation for elements of $\mathcal{NA}$ for arbitrary G. Using this method, several examples are given to show that all elements of $\mathcal{NA}$ can have elementary representations even when G' is non-abelian and simple. At present, we know of no group G for which some element of $\mathcal{NA}$ fails to have an elementary representation and it is hoped that more research can be done on this topic.

CHAPTER 1

ELEMENTARY REPRESENTATIONS FOR NORMALIZED AUTOMORPHISMS

Section 1. Results from the Literature. We record here some results which will be needed from the literature. The first 3 propositions are listed in [9] and the proofs are included in the appendix.

Proposition 1.1.1 $\lambda \in \Lambda$ iff λ sums to 0 on all conjugacy classes of G.

Proposition 1.1.2 $\lambda \in \Lambda_p$ iff λ sums to an integral multiple of p on all conjugacy classes of G.

Proposition 1.1.3 If $x, y \in Z(G)$ are congruent modulo Λ_p, then x^p and y^p are congruent modulo Λ_p.

The next 4 results are taken from [14]. We emphasize that H always denotes a group basis for $Z(G)$.

Theorem 1.1.4 (Glauberman) If $h \in H$, then $\overline{C}_h = \overline{C}_g$ for some $g \in G$. ($\overline{C}_h$ denotes the element of $Z(G)$ obtained by adding all conjugates of h in H.) This theorem shows that the center of H and the center of G coincide. Also, it enables one to set up an isomorphism Φ from the lattice of normal subgroups of H onto the lattice of normal subgroups of G. If $M \trianglelefteq H$ then $\Phi(M) = K$ where $K = \{g \in G | \overline{C}_g = \overline{C}_h$ for some $h \in M\}$.

Theorem 1.1.5 If $\Phi(M) = K$ then $|M| = |K|$, $\Phi(M') = K'$, and $\Delta(M) = \Delta(K)$.

Theorem 1.1.6 If B is an abelian normal subgroup of H and $\Phi(B) = A$, then there is an isomorphism θ of A onto B such that if $a \in A$ and $\theta(a) = b$ then a and b are congruent modulo $\Delta(A)\Delta(G)$.

Theorem 1.1.7 If G' is abelian then each $h \in H$ is congruent to a unique $g \in G$ modulo $\Delta(G')\Delta(G)$ and the mapping $h \to g$ is an isomorphism of H onto G.

The next theorem is proved exactly like Theorem 1 of [12].

Theorem 1.1.8 Suppose $f_1, f_2 \in \mathcal{A}$ are such that $f_1(\overline{C}_g) = f_2(\overline{C}_g)$ for all $g \in G$. Then there is a unit u in $Q(G)$ such that $f_2(g) = u\, f_1(g)u^{-1}$ for all $g \in G$.

Section 2. A Normal Complement for Aut(G) in $\mathcal{NA}$. In this section we record the fact that $\mathcal{NA}$ is a subgroup of finite index in $\mathcal{A}$ and exhibit, when G' is abelian, a normal complement for Aut(G) in $\mathcal{NA}$. Here we are identifying $\sigma \in \text{Aut}(G)$ with the element of $\mathcal{NA}$ obtained by extending σ linearly to all of $Z(G)$.

Let $f \in \mathcal{A}$. For each $g \in G$, set $\hat{f}(g) = \ell(f(g))\cdot f(g)$. One easily checks that $\hat{f}$ extends linearly to an element of $\mathcal{A}$. Since the only units of Z are ± 1, it is clear that $\hat{f} \in \mathcal{NA}$. Also, $\hat{f}(G)$ is a group basis for $Z(G)$; this is true for any element of $\mathcal{NA}$. If $g \in G$, we have, by Theorem 1.1.4, $\hat{f}(\overline{C}_g) = \overline{C}_{\hat{f}(g)} = \overline{C}_{g_1}$ for some $g_1 \in G$. But if $g, x_2gx_2^{-1}, \ldots, x_ngx_n^{-1}$ is the set of all conjugates of g in G then $\hat{f}(\overline{C}_g) = \hat{f}(g + x_2gx_2^{-1} + \ldots + x_ngx_n^{-1})$ $= \ell(f(g))\cdot f(g) + \ell(f(x_2gx_2^{-1}))\cdot f(x_2gx_2^{-1}) + \ldots + \ell(f(x_ngx_n^{-1}))f(x_ngx_n^{-1})$ $= \ell(f(g))f(\overline{C}_g)$. Thus, if $f \in \mathcal{A}$ and $g \in G$ then $f(\overline{C}_g) = \pm C_{g_1}$ for some $g_1 \in G$. By Theorem 1.1.8, we see that the subgroup I of all elements of $\mathcal{A}$ which fix the center of $Z(G)$ pointwise is a normal subgroup of $\mathcal{A}$ of finite index.

Lemma 1.2.1 $\mathcal{NA}$ is a subgroup of $\mathcal{A}$ of finite index.

Proof: It can be verified that an element $f \in \mathcal{A}$ is in $\mathcal{NA}$ iff for each $g \in G$, $f(\overline{C}_g) = \overline{C}_{g_1}$ for some $g_1 \in G$ depending, of course, on g. Thus $I \subseteq \mathcal{NA}$. Since $[\mathcal{A} : I]$ is finite and equal to $[\mathcal{A} : \mathcal{NA}][\mathcal{NA} : I]$, $\mathcal{NA}$ is of finite index in $\mathcal{A}$.

We will show that $\mathrm{Aut}(G)$ has a normal complement in $\mathcal{NA}$ by exhibiting a surjective group homomorphism $\beta: \mathcal{NA} \to \mathrm{Aut}(G)$ and noting that the exact sequence $1 \to \ker\beta \overset{\text{inj.}}{\to} \mathcal{NA} \overset{\beta}{\to} \mathrm{Aut}(G) \to 1$ is split by the natural injection of $\mathrm{Aut}(G)$ into $\mathcal{NA}$.

Let $f \in \mathcal{NA}$ so that $f(G)$ is a group basis for $Z(G)$. If G' is abelian, we can use Theorem 1.1.7 to exhibit an isomorphism i_f of $f(G)$ onto G by setting, for each $g \in G$, $i_f(f(g)) = g_1$ if $f(g)$ is congruent to g_1 modulo $\Delta(G')\Delta(G)$. Denote by σ_f the element of $\mathrm{Aut}(G)$ given by $i_f \circ f|_G$ and set $\beta(f) = \sigma_f$ for each $f \in \mathcal{NA}$. In order to show that β is a group homomorphism, we need part of the next lemma.

Lemma 1.2.2 Suppose that G' is abelian and that K is a characteristic subgroup of G such that $K \leq G'$ or $G' \leq K$. Then $f(\Delta(K)) = \Delta(K)$ for every $f \in \mathcal{NA}$.

Proof: We note that $f(\Delta(K)) \subseteq \Delta(K)$ for every $f \in \mathcal{NA}$ implies that $f(\Delta(K)) = \Delta(K)$ for every $f \in \mathcal{NA}$, since $\mathcal{NA}$ is a subgroup of $\mathcal{A}$. Since $\Delta(K)$ is an ideal of $Z(G)$, $f(\Delta(K)) \subseteq \Delta(K)$ iff $f(k-1) \in \Delta(K)$ for each $k \in K$.

Suppose that $K \leq G'$ and that $\Phi(f(K)) = A$. Since Φ preserves intersections and $\Phi(f(G')) = \Phi(f(G)') = G'$, we have

$A = \Phi(f(K)) = \Phi(f(K) \cap f(G')) = \Phi(f(K)) \cap \Phi(f(G')) = A \cap G'$. Hence $A \leq G'$. Now let $a \in A$. By Theorem 1.1.6, since $f(K)$ is abelian, we know that $f(k) \equiv a \mod \Delta(A)\Delta(G)$ for some $k \in K$. Thus $f(k) \equiv a \mod \Delta(G')\Delta(G)$ since $\Delta(A) \subseteq \Delta(G')$. This shows that $a = \sigma_f(k) \in K$, since K is characteristic in G. Whence $A \subseteq K$ and, since Φ preserves order, $A = K$. Thus, if $k \in K$, there is a $k_1 \in K$ such that $f(k) - k_1 \in \Delta(K)\Delta(G) \subseteq \Delta(K)$. $f(k) - k_1 = f(k-1) - (k_1-1) \in \Delta(K)$ implies that $f(k-1) \in \Delta(K)$ since $(k_1-1) \in \Delta(K)$. This completes one part of the proof.

Suppose $G' \leq K$. If $k \in K$ then $f(k) - \sigma_f(k) \in \Delta(G')\Delta(G) \subseteq \Delta(G') \subseteq \Delta(K)$. Since K is characteristic in G, $\sigma_f(k) \in K$ and $f(k-1) \in \Delta(K)$ as before. This completes the proof.

Lemma 1.2.3 β is a surjective homomorphism.

Proof: Let $\tau \in \mathrm{Aut}(G)$. Then for each $g \in G$, $\tau(g) \equiv \tau(g) \mod \Delta(G')\Delta(G)$ and $\beta(\tau)(g) = \tau(g)$. This shows that β is surjective. To show that β is a homomorphism, let $f_1, f_2 \in \mathcal{NA}$ and $g \in G$. Then $f_1 \circ f_2(g) = f_1(f_2(g)) = f_1(\sigma_{f_2}(g) + \gamma)$ for some $\gamma \in \Delta(G')\Delta(G)$. Whence $f_1 \circ f_2(g) = f_1(\sigma_{f_2}(g)) + f_1(\gamma) \equiv \sigma_{f_1}(\sigma_{f_2}(g)) \mod \Delta(G')\Delta(G)$, by Lemma 1.2.2. Thus $\beta(f_1 \circ f_2)(g) = \beta(f_1) \circ \beta(f_2)(g)$ and β is a homomorphism.

Theorem 1.2.4 If G' is abelian, then $\mathrm{Aut}(G)$ has a normal complement M in $\mathcal{NA}$. In fact, $M = \{f \in \mathcal{NA} \mid f(g) \equiv g \mod \Delta(G')\Delta(G)$ for each $g \in G\}$.

Proof: We need only note that the natural injection of $\mathrm{Aut}(G)$ into $\mathcal{NA}$ splits the sequence $1 \to \ker \beta \xrightarrow{\text{inj.}} \mathcal{NA} \xrightarrow{\beta} \mathrm{Aut}(G) \to 1$ and that $M = \ker \beta$.

Section 3. Some Sufficient Conditions for Elementary Representations.

The following definition is motivated by Theorem 2 of [12].

Definition. We say that $f \in \mathcal{A}$ has an elementary representation ($\mathcal{E}.\mathcal{R}.$) if $f = T_u \circ \sigma$ for some $\sigma \in \mathrm{Aut}(G)$ and some unit u in $Q(G)$. (Recall that $T_u(x) = uxu^{-1}$ for all $x \in Z(G)$.)

The remainder of this chapter is devoted to giving some sufficient conditions for all elements of $\mathcal{NA}$ to have an elementary representation. We remark that information about $\mathcal{NA}$ quickly converts to information about $\mathcal{A}$; see Section 2 for the construction of an $\hat{f} \in \mathcal{NA}$ from an $f \in \mathcal{A}$.

All of the sufficient conditions for the elements of $\mathcal{NA}$ to have an $\mathcal{E}.\mathcal{R}.$ include the assumption that G' is abelian. However, this is not a necessary condition; in Chapter 3 we give examples to show that all of the elements of $\mathcal{NA}$ may have an $\mathcal{E}.\mathcal{R}.$ when G' is non-abelian and simple. When G' is abelian, the decomposition of $\mathcal{NA}$ in Theorem 1.2.4 allows the statement that all elements of $\mathcal{NA}$ have an $\mathcal{E}.\mathcal{R}.$ iff all elements of $\ker \beta$ have an elementary representation.

Our method for showing that $f \in \mathcal{NA}$ has an $\mathcal{E}.\mathcal{R}.$ is to exhibit a $\sigma \in \mathrm{Aut}(G)$ which does the same thing to class sums as f; that is, $f(\overline{C}_g) = \sigma(\overline{C}_g)$ for all $g \in G$; an application of Theorem 1.1.8 says that $f = T_u \circ \sigma$ for some unit u in $Q(G)$. In case G is abelian, $f|_G \in \mathrm{Aut}(G)$ for each $f \in \mathcal{NA}$; this follows from G. Higman's result, in [5], that the only units u of finite order in $Z(G)$ with $\ell(u) = 1$ are the elements of G. Later we will see that $\mathcal{NA} = \mathrm{Aut}(G)$ implies that all subgroups of G are normal.

The next 2 lemmas will be needed in subsequent proofs; G' is not necessarily abelian.

<u>Lemma 1.3.1</u> If $h \in H$, $g \in G$ are such that $\overline{C}_h = \overline{C}_g$ then h sums to 1 on C_g and to 0 on all other conjugacy classes of G. (Since for each $h \in H$ there is always a $g \in G$ with $\overline{C}_h = \overline{C}_g$, by Theorem 1.1.4, this shows that each element of a group basis sums to 1 on some conjugacy class of G and to 0 on all other conjugacy classes of G.)

<u>Proof</u>: The result will follow from Proposition 1.1.1 if we show that $h-g \in \Lambda$. Applying ℓ to $\overline{C}_h = \overline{C}_g$, we see that $|C_h| = |C_g| = n$, say. If u is a unit in $Z(G)$ and $x \in Z(G)$ then $u(xu^{-1}) - x \in \Lambda$. Combining this comment with $\overline{C}_h = \overline{C}_g$, we see that $n(h-g) \in \Lambda$. By Proposition 1.1.1, $h-g \in \Lambda$.

<u>Lemma 1.3.2</u> If $h \in H$, then $h \equiv g_1 \bmod \Delta(G')\Delta(G)$ for some $g_1 \in G$. In fact, if $\overline{C}_h = \overline{C}_g$ with $g \in G$, then g_1 may be chosen from gG'.

<u>Proof</u>: As in the proof of Lemma 1.3.1, $\overline{C}_h = \overline{C}_g$ implies $h-g \in \Lambda$. Clearly $\Lambda \subseteq \Delta(G')$ so that $h-g = \sum_{a \in G'} (a-1)t(a)$ for some elements $t(a) \in Z(G)$. Computing as Whitcomb did in proving Theorem 1.1.7, we have that $h-g \equiv \sum_{a \in G'} (a-1)\ell(t(a)) \equiv \sum_{a \in G'} (a^{\ell(t(a))} - 1) \equiv \prod_{a \in G'} a^{\ell(t(a))} - 1$ mod $\Delta(G')\Delta(G)$. Thus $h \equiv (\prod_{a \in G'} a^{\ell(t(a))} \cdot g) \bmod \Delta(G')\Delta(G)$. Whence one value of g_1 is $\prod_{a \in G} a^{\ell(t(a))} \cdot g \in gG'$.

The next lemma yields another proof for Theorem 2 of [12]. The manner in which it is used to prove this theorem (here, Theorem 1.1.4) makes the following question of interest. For what groups G does the conclusion of Lemma 1.3.3 hold?

Lemma 1.3.3 Suppose G is nilpotent of nilpotence class ≤ 2. If $h \in H$, $g \in G$ are such that $h-g \in \Delta(G')\Delta(G)$ then h sums to 1 on C_g and to 0 on all other conjugacy classes of G.

Proof: The proof is by induction on $|G|$. If G is abelian, then $G' = 1$, $h = g$ and the result is trivial. Since G has class ≤ 2, the set $K = \{[g,x] = g^{-1}x^{-1}gx \mid x \in G\}$ is a normal subgroup of G. Let $\Pi: Z(G) \to Z(G/K)$ be the ring epimorphism induced by the canonical homomorphism of G onto G/K. Since $\Pi(H)$ is a group basis for $Z(G/K)$, by Lemma 3.1 of [2], and $\Pi(h) - gK \in \Delta((G/K)')\Delta(G/K)$, we have, by induction, that $\Pi(h)$ sums to 1 on the conjugacy class of gK in G/K and to 0 on all other conjugacy classes of G/K. By Lemma 1.3.1, we know that h sums to 1 on some class, say C_{g_1}, of G and to 0 on all others. Therefore, g_1K is a conjugate of gK so that $g_1K = xgx^{-1}K$ for some $x \in G$. That is, for some $k = [g,x_2] \in K$, $g_1 = xgx^{-1}k = xgx^{-1}g^{-1}x_2^{-1}gx_2 = g[g,x^{-1}][g,x_2] = g[g,x^{-1}x_2]$ = a conjugate of g. Thus $\overline{C}_g = \overline{C}_{g_1}$ and the proof is complete.

Theorem 1.3.4 (Sehgal [12]) If G is nilpotent of nilpotence class ≤ 2, then each $f \in \mathfrak{N}\mathfrak{a}$ has an elementary representation.

Proof: Since $G' \le Z(G)$, we have the decomposition of Theorem 1.2.4, so it suffices to show that each $f \in \ker \beta$ has an elementary representation. In fact, it will be shown that $f(\overline{C}_g) = \overline{C}_g$ for all $g \in G$; the result then follows by applying Theorem 1.1.8 with $f_1 = f$ and f_2 = the identity element of $\mathrm{Aut}(G)$. Suppose $f(\overline{C}_g) = \overline{C}_{g_1}$ for some $g_1 \in G$. $f(\overline{C}_g) = \overline{C}_{f(g)} = \overline{C}_{g_1}$ so, as in the proof of Lemma 1.3.1, $f(g) = g_1 \in \Lambda$. By Proposition 1.1.1, $f(g)$ sums to 1 on C_{g_1} and to 0 on all other conjugacy classes of G. But $f \in \ker \beta$ implies that $f(g) - g \in \Delta(G')\Delta(G)$ and so, by Lemma 1.3.3, $f(g)$

sums to 1 on C_g and to 0 on all other conjugacy classes of G. Thus g_1 is a conjugate of g and $f(\bar{C}_g) = \bar{C}_g$, as was claimed.

If G' is abelian and the conclusion of Lemma 1.3.3 is true for G, then one can obtain an $\mathcal{S}.\mathcal{R}.$ for the elements of $\mathcal{N}\mathcal{U}$ by repeating the proof of Theorem 1.3.4. The next lemma gives another collection of groups for which the conclusion of Lemma 1.3.3 is valid.

Lemma 1.3.5 Suppose G has at most one non-linear irreducible representation over C. If $h \in H$, $g \in G$ are such that $h \equiv g \bmod \Delta(G')\Delta(G)$ then h sums to 1 on C_g and to 0 on all other conjugacy classes of G.

Proof: By remarks in [13], G' is abelian. Suppose, by Theorem 1.1.4, that $\bar{C}_h = \bar{C}_{g_1}$ for some $g_1 \in G$. By Lemma 1.3.1, we will be done if g_1 is a conjugate of g. Let ρ be any linear representation of G over C and χ^ρ its character. We can extend ρ and χ^ρ to a ring homomorphism of Z(G) into C. $\bar{C}_h = \bar{C}_{g_1}$ implies, as in the proof of Lemma 1.3.1, that $h-g_1 \in \Lambda \subseteq \Delta(G')$. Recalling the form for elements of $\Delta(G')$, we see that $\rho(h) = \rho(g_1)$ since G' is in the kernel of ρ. By hypothesis, $h-g \in \Delta(G')\Delta(G) \subseteq \Delta(G')$ so that $\rho(h) = \rho(g)$. Thus $\chi^\rho(g) = \chi^\rho(g_1)$ for any linear representation ρ of G. If $g_1 = 1$ then $h = 1$ and so $g = 1$, by Theorem 1.1.7. If $g = 1$ then $h = 1$, by the same theorem, so that $g_1 = 1$. Thus we may assume that neither g nor g_1 is 1. By an orthogonality relation for complex characters, $0 = \sum_\rho \chi^\rho(g)\chi^\rho(1) = \sum_\rho \chi^\rho(g_1)\chi^\rho(1)$ where the sum is over all irreducible representations of G. Since $\chi^\rho(g) = \chi^\rho(g_1)$ if ρ is linear and G has at most 1 non-linear representation, we see that $\chi^\rho(g) = \chi^\rho(g_1)$ for all irreducible representations ρ of G. This certainly implies that g, g_1 are conjugate.

Corollary 1.3.6 If G satisfies the hypothesis of Lemma 1.3.5, then each $f \in \mathfrak{N}\mathfrak{a}$ has an elementary representation.

Proof: See the remarks immediately preceding Lemma 1.3.5.

We note that the alternating group on 4 symbols satisfies the hypothesis of Lemma 1.3.5.

Section 4. Order of Elements is Preserved by Normalized Automorphisms; An Application. The main result of this section is Corollary 1.4.3; this corollary lends some motivation to the search for an $\mathcal{E}.\mathcal{R}$, for elements of $\mathfrak{N}\mathfrak{a}$. As a first application of this corollary, we prove Theorem 1.4.5. Unless explicitly stated we do not assume G' abelian in this section. The following lemma is proved much like Proposition 2 of [9].

Lemma 1.4.1 If $\bar{C}_h = \bar{C}_g$ for some $h \in H$, $g \in G$ then $\bar{C}_{h^p} = \bar{C}_{g^p}$ for any prime p.

Proof: As in the proof of Lemma 1.3.1, $h - g \in \Lambda \subseteq \Lambda_p$. By Proposition 1.1.3, $h^p - g^p \in \Lambda_p$. Theorem 1.1.4 says that $\bar{C}_{h^p} = \bar{C}_{g_1}$ for some $g_1 \in G$ and so $h^p - g_1 \in \Lambda_p$, as was the case with h and g. Thus $g^p - g_1 \in \Lambda_p$ and, by Proposition 1.1.2, g_1 and g^p are conjugate. Whence $\bar{C}_{h^p} = \bar{C}_{g_1} = \bar{C}_{g^p}$.

Corollary 1.4.2 Under the hypothesis of Lemma 1.4.1, $\bar{C}_{h^n} = \bar{C}_{g^n}$ for any positive integer n.

Proof: The proof is by induction on n; if $n = 1$ the result is true by hypothesis. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ be the decomposition of n into prime powers. By induction, we have that

$\overline{C}_{h^{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t - 1}}} = \overline{C}_{g^{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t - 1}}}$ and by Lemma 1.4.1 that

$\overline{C}_{h^n} = \overline{C}_{g^n}$.

<u>Corollary 1.4.3</u> Suppose $f \in \mathcal{NA}$ and that $f(\overline{C}_g) = \overline{C}_{g_1}$ for $g, g_1 \in G$. Then $|g| = |g_1|$ and $|C_g| = |C_{g_1}|$.

<u>Proof</u>: $|C_g| = |C_{g_1}|$ can be seen by applying ℓ to the equality $f(\overline{C}_g) = \overline{C}_{g_1}$. Since $f(G)$ is a group basis for $Z(G)$ and $\overline{C}_{f(g)} = \overline{C}_{g_1}$, the result $|g| = |g_1|$ follows from Corollary 1.4.2 by noting that $|f(g)| = |g|$.

From Corollary 1.4.3 we note that if $f \in \mathcal{A}$ and $f(\overline{C}_g) = \pm \overline{C}_{g_1}$ for some $g, g_1 \in G$, then $|g| = |g_1|$. For if $\hat{f}$ is the element of $\mathcal{NA}$ derived from f as in Section 2, we have $\hat{f}(\overline{C}_g) = \ell(f(g))f(\overline{C}_g) = \overline{C}_{g_1}$.

<u>Lemma 1.4.4</u> Suppose that G' is abelian and that $f \in \ker \beta = M$. If $f(\overline{C}_g) = \overline{C}_{g_1}$ for some $g, g_1 \in G$, then $g_1 \in gG'$. (Hence, an $f \in \ker \beta$ permutes the class sums corresponding to elements inside the various cosets of G' in G.)

<u>Proof</u>: By Lemma 1.3.2, $f(g) - g_* \in \Delta(G')\Delta(G)$ for some $g_* \in g_1G'$. The uniqueness statement in Theorem 1.1.7 and the fact that $f(g) - g \in \Delta(G')\Delta(G)$ implies that $g_* = g$. Hence $g_1 \in gG'$.

As an application of the results in this section, we have the following theorem.

<u>Theorem 1.4.5</u> Suppose that G' is abelian and that $g_1G' = G'$, $g_2G', \ldots, g_mG'$ are the distinct cosets of G' in G. Further assume that the following holds for each $i = 1,2,\ldots,m$: If $g, g_* \in g_iG'$ and neither of g, g_* is in $\mathcal{Z}(G)$, then either $|C_g| \neq |C_{g_*}|$ or $|g| \neq |g_*|$ or g, g_* are conjugate. Then each $f \in \ker \beta = M$ fixes

each class sum corresponding to elements in G and hence all elements of $\mathfrak{N}\mathfrak{a}$ have an elementary representation.

Proof: Let $f \in \ker \beta$ and $g \in G$. If $g \in \mathcal{Z}(G)$ then $f(\bar{C}_g) = \bar{C}_g$ since $f(g) \in \mathcal{Z}(G)$ and $f(g) - g \in \Delta(G')\Delta(G)$ and we have the uniqueness statement in Theorem 1.1.7. If $g \notin \mathcal{Z}(G)$, let $f(\bar{C}_g) = \bar{C}_{g_*}$ for some $g_* \in G$. By Lemma 1.4.4, $g_* \in gG'$. By our hypothesis, g and g_* must be conjugate. Hence $f(\bar{C}_g) = \bar{C}_{g_*} = \bar{C}_g$.

Corollary 1.4.6 If $|G'| = 1,2$, or 3 then each $f \in \mathfrak{N}\mathfrak{a}$ has an elementary representation.

Proof: One can easily check that the hypothesis of Theorem 1.4.5 is satisfied.

Section 5. Groups with a Cyclic Normal Subgroup of Prime Index.

In this section we exhibit a fairly large class of groups such that every element of $\mathfrak{N}\mathfrak{a}$ has an elementary representation.

Theorem 1.5.1 Suppose G has a cyclic normal subgroup A of index p, for some prime p. Then each $f \in \mathfrak{N}\mathfrak{a}$ has an elementary representation.

Proof: We first make some observations about how the conjugacy classes of G are distributed among the cosets of G' in G and then produce a $\sigma \in \mathrm{Aut}(G)$ which, when extended linearly to $Z(G)$, does the same thing to class sums of elements of G as an element $f \in \ker \beta$; clearly G' is cyclic.

Let A be generated by a and set $|A| = |a| = m$. A is a maximal subgroup of G, so if $b \notin A$ then $G = A\cdot\langle b\rangle = \langle a\rangle\cdot\langle b\rangle$. Since $A \trianglelefteq G$, we have $b^{-1}ab = a^r$ for some positive integer r such that (r,m) = greatest common divisor of r and $m = 1$. By 47.10

of [3], $G' = \langle a^{r-1}\rangle$. Since $[G : A] = p$, $b^p \in A$ and $b^k \notin A$ for any integer k such that $1 \leq k < p$. Set $d = (r-1,m)$. If $Z(G) = G$, then we are done, by Theorem 1.3.4. Thus we may assume that $Z(G) \neq G$. Claim: $Z(G) = \langle a^{m/d}\rangle$. For we must have $Z(G) \leq A$ since A is a maximal subgroup of the non-abelian group G. $a^k \in Z(G)$ iff $b^{-1}a^k b = a^k$. But $b^{-1}a^k b = a^{kr}$ so that $a^k \in Z(G)$ iff $a^k = a^{kr}$ iff $a^{k(r-1)} = 1$ iff m divides $k(r-1)$ iff m/d divides k. Thus $Z(G) = \langle a^{m/d}\rangle$ and hence has order d.

Next we claim that the following pd cosets of G' in G are disjoint. The cosets are $G', aG', a^2G', \ldots, a^{d-1}G'$, $bG', abG', \ldots, a^{d-1}bG'$, $b^2G', ab^2G', \ldots, a^{d-1}b^2G', \ldots, b^{p-1}G', ab^{p-1}G', \ldots, a^{d-1}b^{p-1}G'$. We verify that the first d of these cosets are disjoint and leave the remainder to systematic observation; recall that $b^k \notin A$ if k is an integer such that $1 \leq k < p$. Suppose $a^iG' = a^jG'$ with $0 \leq i,j \leq d-1$ and $i > j$. Thus $a^{i-j} = a^{n(r-1)}$ and $(i-j) \equiv n(r-1) \bmod m$ for some integer n. Hence $(r-1,m) = d$ divides $(i-j)$ and so $i = j$. Since $[G : G'] = \frac{|G|}{|G'|} = \frac{mp}{m/d} = pd$, these pd cosets are the distinct cosets of G' in G.

Finally we claim that a^ib^jG' is the conjugacy class of G containing a^ib^j for any i,j such that $0 \leq i \leq (d-1)$ and $1 \leq j \leq (p-1)$. That is, all except the first d cosets of G' listed above are complete conjugacy classes in G. The argument is as follows. If $g \in A$ then $g \in Z(G)$ or g has exactly p distinct conjugates in G, since the centralizer of g in G is A or G. We are working under the assumption that $Z(G) \subseteq A$, so that there are $|Z(G)| + \frac{|A| - |Z(G)|}{p} = d + \frac{m-d}{p}$ conjugacy classes of G inside A. By Corollary 47.15 of [3], there are $pd + \frac{m-d}{p}$ irreducible representations of G and hence this many conjugacy classes of G. If x is the number of conjugacy classes of G outside A, we must have $x + d + \frac{m-d}{p} = pd + \frac{m-d}{p}$, so that $x = d(p-1)$. Each coset of G' must contain at least one conjugacy class of G and there are $d(p-1)$ cosets of G' outside of A. Hence $a^ib^jG' = C_{a^ib^j}$ as claimed.

If $f \in \ker \beta$, we produce a $\sigma \in \mathrm{Aut}(G)$ such that $f(\overline{C}_g) = \sigma(\overline{C}_g)$ for all $g \in G$. By Lemma 1.4.4, $f(\overline{C}_a) = \overline{C}_{a^s}$ for some $a^s \in aG'$. Set $\sigma(a) = a^s$ and $\sigma(b) = b$ and extend this to all of G by defining $\sigma(a^ib^j) = a^{is}b^j$. By Corollary 1.4.3, $|a| = |a^s|$ so that $\sigma \in \mathrm{Aut}(G)$. By Corollary 1.4.2, $f(\overline{C}_{a^i}) = \overline{C}_{f(a)^i} = \overline{C}_{a^{is}} = \overline{C}_{\sigma(a^i)} = \sigma(\overline{C}_{a^i})$ for any positive integer i. Hence $\sigma(\overline{C}_g) = f(\overline{C}_g)$ if $g \in A$. By Lemma 1.4.4, $f(\overline{C}_{a^ib^j}) = \overline{C}_{a^ib^j}$ for any integers i,j such that $0 \le i \le (d-1)$, $1 \le j \le (p-1)$ since $C_{a^ib^j}$ is the only conjugacy class of G inside a^ib^jG'. The proof will be complete if $\sigma(a^ib^j) \in a^ib^jG'$ for these values of i,j. We have $a^ib^jG' = G'a^ib^j = \{a^{i+k_1(r-1)}b^j \mid k_1 \text{ is a non-negative integer}\}$ and $s \equiv [1 + k(r-1)] \bmod m$ for some non-negative integer k since $a^s \in aG'$. Therefore $\sigma(a^ib^j) = a^{is}b^j = a^{i(1+k(r-1))}b^j \in a^ib^jG'$, as was to be shown.

We conclude this Chapter with some remarks about the connection between group bases and elementary representations of elements of $\mathfrak{N}\mathfrak{a}$. Suppose that all elements of $\mathfrak{N}\mathfrak{a}$ have elementary representations and that $H \cong G$ for all group bases H of Z(G). Let H be a fixed group basis for Z(G) and suppose α is an isomorphism of G onto H. α can be extended linearly to an element of $\mathfrak{N}\mathfrak{a}$ which we also denote by α. Let $\alpha = T_u \circ \sigma$ be an elementary representation for α. Then $H = \alpha(G) = T_u \circ \sigma(G) = T_u(G) = uGu^{-1} = G^u$. That is, H can be obtained from G by conjugation by some unit in Q(G).

CHAPTER 2

PRELIMINARY RESULTS ABOUT $\mathfrak{N}\mathcal{A}$

Section 1. Properties of $M \cap I$. An interesting problem, directly related to obtaining an elementary representation for the elements of $\mathfrak{N}\mathcal{A}$ when G' is abelian, is to find when $\ker \beta \subseteq I$. We have seen that $\ker \beta \subseteq I$ for the groups in Theorem 1.3.4, Lemma 1.3.5 and Theorem 1.4.5. Set $N = \ker \beta \cap I$. Since $[\mathcal{A} : I]$ is finite and $\frac{\ker \beta}{N} = \frac{\ker \beta}{\ker \beta \cap I} \simeq \frac{I \cdot \ker \beta}{I}$, we see that $[\ker \beta : N]$ is finite. In connection with this problem, we also have the following proposition.

Proposition 2.1.1 If G' is abelian and $N = \ker \beta \cap I$ is trivial then $\ker \beta$ is trivial.

Proof: Let $f \in \ker \beta$ and $g \in G$. We need to show that $f(g) = g$. $T_{g^{-1}f(g)}$ defined by $T_{g^{-1}f(g)}(x) = g^{-1}f(g)xf(g^{-1})g$ for all $x \in Z(G)$ is an element of $\mathfrak{N}\mathcal{A}$. We claim that $T_{g^{-1}f(g)}$ is in $\ker \beta$. If $g_* \in G$ then, since $f \in \ker \beta$, $f(g_*) = g_* + \Delta_{g_*}$ for some $\Delta_{g_*} \in \Delta(G')\Delta(G)$. If $x \in Z(G)$ then $T_{g^{-1}f(g)}(x) =$
$g^{-1}f(g)xf(g^{-1})g = g^{-1}(g + \Delta_g)x(g^{-1} + \Delta_{g^{-1}})g = (1 + g^{-1}\Delta_g)x(1 + \Delta_{g^{-1}}g)$
$= x + x\Delta_{g^{-1}}g + g^{-1}\Delta_g x + g^{-1}\Delta_g x\Delta_{g^{-1}}g \equiv x \bmod \Delta(G')\Delta(G)$ since $\Delta(G')\Delta(G)$ is an ideal of $Z(G)$. Thus $T_{g^{-1}f(g)} \in \ker \beta \cap I$, so $g^{-1}f(g) \in \mathcal{Z}(Z(G))$, by hypothesis. Set $f(g) = gz$ for some $z \in \mathcal{Z}(Z(G))$. Since $|f(g)| = |g|$, z is a unit of finite order in $\mathcal{Z}(Z(G))$. By

Theorem 2.1 of [2], z is an element of $\mathcal{Z}(G)$. But this implies, by Theorem 1.1.7, that $f(g) = g$ since $f \in \ker \beta$. This completes the proof.

By Theorem 1.1.8, any $f \in I$ can be written as $f = T_u$ for some unit u in Q(G). The next proposition shows that conjugation by certain units cannot be an element of $\ker \beta$.

<u>Proposition 2.1.2</u> Suppose G is nilpotent and G' is abelian. Let $u \in Z(G)$ be a unit of finite order with $T_u \in N = \ker \beta \cap I$. Then $u = \pm g_1$ for some $g_1 \in \mathcal{Z}(G)$ so that T_u is the identity of $\mathcal{A}$.

<u>Proof</u>: $T_u \in \ker \beta$ says that $ugu^{-1} - g \in \Delta(G')\Delta(G)$ for all $g \in G$. By Lemma 8 of [6], $u \equiv \pm g_* \mod \Delta(G')\Delta(G)$ for some $g_* \in G$. Thus $g_* g g_*^{-1} \equiv g \mod \Delta(G')\Delta(G)$ for all $g \in G$ and $g^{-1}g_* g g_*^{-1} \equiv 1 \mod \Delta(G')\Delta(G)$. By the results of [11], $g^{-1}g_* g g_*^{-1} = 1$ so that $g_* \in \mathcal{Z}(G)$. Hence $u(\pm g_*^{-1}) \equiv 1 \mod \Delta(G')\Delta(G)$ and $u(\pm g_*^{-1})$ is a unit of finite order in Z(G). Again by [11], $u(\pm g_*^{-1}) = 1$ so that $u = \pm g_*$. This completes the proof.

<u>Section 2</u>. <u>When is $\mathcal{N}\mathcal{A}$ = Aut(G)?</u> G' is not necessarily abelian in this section. First, we consider when $\mathcal{N}\mathcal{A}$ = Aut(G). The next proof follows the ideas of Theorems 9 and 10 in [5].

<u>Proposition 2.2.1</u> If $\mathcal{N}\mathcal{A}$ = Aut(G) then every subgroup of G is normal.

<u>Proof</u>: Let $1 \neq g_1$ be any element of G. It suffices to show that $g_2 g_1 g_2^{-1}$ is a power of g_1 for any $g_2 \in G$. Set $P = g_2(1-g_1)$ and $Q = 1 + g_1 + g_1^2 + \ldots + g_1^{n-1}$, the sum of the distinct powers of g_1. Since $PQ = 0$, $(1-3QP)(1 + 3QP) = 1$ so that $1-3QP$ is a unit in

$Z(G)$ with inverse $1 + 3QP$. Thus conjugation by $1-3QP$ is an automorphism of $Z(G)$ which is clearly in $\mathcal{N}\mathcal{A}$. By the hypothesis, $(1-3QP)g_1(1 + 3QP)$ is an element of G. But

$$\begin{aligned}(1-3QP)g_1(1 + 3QP) &= (1-3QP)(g_1 + 3QP) \quad \text{since } g_1Q = Q \\ &= g_1 + 3QP - 3QPg_1 \quad \text{since } PQ = 0 \\ &= g_1 + 3(QP - QPg_1).\end{aligned}$$

$[g_1 + 3(QP - QPg_1)] \in G$ says that $QP - QPg_1 = 0$ since we are working in $Z(G)$. $QP = g_2 + g_1g_2 + g_1^2g_2 + \ldots + g_1^{n-1}g_2 - g_2g_1 - g_1g_2g_1 - g_1^2g_2g_1 - \ldots - g_1^{n-1}g_2g_1$. We may suppose that no summand of QP with a minus sign is g_2 since $g_2 = g_2g_1$ implies $g_1 = 1$ and $g_2 = g_1^ig_2g_1$ implies $g_1^{-i} = g_2g_1g_2^{-1}$. Expanding QPg_1 we have $QPg_1 = g_2g_1 + g_1g_2g_1 + g_1^2g_2g_1 + \ldots + g_1^{n-1}g_2g_1 - g_2g_1^2 - g_1g_2g_1^2 - \ldots - g_1^{n-1}g_2g_1^2$. The equality of QP and QPg_1 implies that $g_2 = g_1^jg_2g_1$ for some $j = 1,\ldots,n-1$. Whence $g_1^{-j} = g_2g_1g_2^{-1}$ and we are done.

As a partial converse, we have the following result.

Proposition 2.2.2 If G is abelian or is a 2-group in which every subgroup is normal, then $\mathcal{N}\mathcal{A} = \text{Aut}(G)$.

Proof: We use the following result from [1]. If G is abelian or is a 2-group in which every subgroup is normal and $u \in Z(G)$ is a unit of finite order, then $u = \pm g_1$ for some $g_1 \in G$. Now if $f \in \mathcal{N}\mathcal{A}$ and $g \in G$, we have, by this result, that $f(g) \in G$. Thus if $f \in \mathcal{N}\mathcal{A}$ then f restricted to G, denoted $f|_G$, is such that $f|_G \in \text{Aut}(G)$. Hence $\mathcal{N}\mathcal{A} = \text{Aut}(G)$.

Section 3. What is $f(\Delta(K))$? Here we consider $f(\Delta(K))$ where $f \in \mathcal{A}$ has an elementary representation and $K \trianglelefteq G$.

Proposition 2.3.1 Suppose $f \in \mathcal{A}$ has an elementary representation, say $f = T_u \circ \sigma$ with $\sigma \in \mathrm{Aut}(G)$ and $u \in Q(G)$. Then $f(\Delta(K)) = \Delta(K)$ iff $\sigma(K) = K$ where $K \trianglelefteq G$.

Proof: One easily checks that $T_u \in \mathcal{A}$. It is clear also that $T_u \in \mathcal{NA}$. Thus $T_u(G)$ is a group basis for $Z(G)$.

If Φ is the isomorphism between the lattice of normal subgroups of $T_u(G)$ and the lattice of normal subgroups of G described in Section 1 of Chapter 1, we claim that $\Phi(T_u(K)) = K$. Suppose $\Phi(T_u(K)) = L \trianglelefteq G$. If $\ell_1, \ell_2, \ldots, \ell_t$ are representatives of the conjugacy classes of G inside L and $k_1, k_2, \ldots, k_t$ are representatives of the conjugacy classes of G inside K, then $\bar{C}_{\ell_1} + \bar{C}_{\ell_2} + \ldots + \bar{C}_{\ell_t} = \bar{C}_{uk_1u^{-1}} + \bar{C}_{uk_2u^{-1}} + \ldots + \bar{C}_{uk_tu^{-1}} = u\bar{C}_{k_1}u^{-1} + u\bar{C}_{k_2}u^{-1} + \ldots + u\bar{C}_{k_t}u^{-1} = \bar{C}_{k_1} + \bar{C}_{k_2} + \ldots + \bar{C}_{k_t}$. But G is a basis for $Z(G)$, so $L = K$. By Theorem 1.1.5, $\Delta(uKu^{-1}) = \Delta(T_u(K)) = \Delta(K)$.

Now we can show that $\sigma(K) = K$ implies $f(\Delta(K)) = \Delta(K)$. To show that $f(\Delta(K)) \subseteq \Delta(K)$, it suffices to show that $f(k-1) \in \Delta(K)$ for each $k \in K$ since $\Delta(K)$ is an ideal of $Z(G)$. $f(k-1) = f(k) - 1 = u\sigma(k)u^{-1} - 1 = uk_1u^{-1} - 1$ for some $k_1 \in K = \sigma(K)$. Thus $f(k-1) \in \Delta(uKu^{-1}) = \Delta(K)$. To show $\Delta(K) \subseteq f(\Delta(K))$, it suffices to show that $uku^{-1} - 1 \in f(\Delta(K))$ for each $k \in K$ since $f(\Delta(K))$ is an ideal of $Z(G)$ and $\Delta(uKu^{-1}) = \Delta(K)$. But $f(\sigma^{-1}(k)) = T_u\sigma(\sigma^{-1}(k)) = uku^{-1}$ implies that $f(\sigma^{-1}(k)-1) = uku^{-1} - 1$, so $\Delta(uKu^{-1}) = \Delta(K) \subseteq f(\Delta(K))$ and we have $f(\Delta(K)) = \Delta(K)$.

For the reverse implication, we suppose $f(\Delta(K)) = \Delta(K)$ and show $\sigma(K) = K$. It suffices, since $|K|$ is finite, to show $\sigma(k) \in K$ for all $k \in K$. $f(k-1) = u\sigma(k)u^{-1} - 1$ is in $\Delta(K) = \Delta(uKu^{-1})$, by

hypothesis. Thus, $u\sigma(k)u^{-1} - 1 = \sum_{k_1 \in K} (uk_1u^{-1} - 1)t(k_1)$ with $t(k_1) \in Z(G)$ for all $k_1 \in K$. So $\sigma(k) - 1 = \sum_{k_1 \in K} u^{-1}(uk_1u^{-1} - 1)t(k_1)u$ $= \sum_{k_1 \in K} u^{-1}(uk_1u^{-1} - 1)uu^{-1}t(k_1)u = \sum_{k_1 \in K} (k_1-1)t'(k_1)$ with $t'(k_1) = u^{-1}t(k_1)u \in Z(G)$ since $u^{-1}Z(G)u = u^{-1}(uZ(G)u^{-1})u = Z(G)$. Thus $\sigma(k) - 1 \in \Delta(K)$. If π denotes the canonical ring homomorphism of $Z(G)$ onto $Z(G/K)$, $\pi(\sigma(k)) - \bar{1} = \sigma(k)K - \bar{1} = \bar{0}$ since $\Delta(K)$ is the kernel of π. Thus $\sigma(k) \in K$ and the proof is complete.

<u>Section 4</u>. <u>How is $\mathcal{N}\mathcal{A}$ like Aut(G)</u>? Some of the properties of Aut(G) enjoyed by $\mathcal{A}$ and $\mathcal{N}\mathcal{A}$ have been detailed previously. The next two propositions give additional information along these lines.

<u>Proposition 2.4.1</u> If $f \in \mathcal{A}$ and $f(\Delta(K)) \subseteq \Delta(K)$ for $K \trianglelefteq G$, then f permutes the class sums of elements of G inside K; hence $f(\bar{K}) = \bar{K}$ where $\bar{K} = \sum_{k \in K} k$.

<u>Proof</u>: Since $f(k) - 1 \in \Delta(K)$ for all $k \in K$, $\ell(f(k)) = 1$ for all $k \in K$. If $k \in K$, we know that $f(\bar{C}_k) = \pm \bar{C}_g$ for some $g \in G$. We must choose the plus sign by the preceding remark. We need to show that $g \in K$. Since $f(k) - k = f(k-1) - (k-1) \in \Delta(K)$, by hypothesis, $f(\bar{C}_k) - \bar{C}_k = \bar{C}_g - \bar{C}_k \in \Delta(K)$. Let π again denote the canonical ring homomorphism of $Z(G)$ onto $Z(G/K)$. Applying π to $\bar{C}_g - \bar{C}_k$ we obtain $gK + x_2gx_2^{-1}K + \ldots + x_ngx_n^{-1}K - |C_k| \cdot K = \bar{0}$, where $\{g, x_2gx_2^{-1}, \ldots, x_ngx_n^{-1}\}$ is the set of all conjugates of g in G. Whence $gK = K$ and $g \in K$.

<u>Corollary 2.4.2</u> If G' is abelian and K is a characteristic subgroup of G with $K \leq G'$ or $G' \leq K$, then any $f \in \mathcal{N}\mathcal{A}$ permutes the class sums of elements of G inside K.

Proof: By Lemma 1.2.2, $f(\Delta(K)) = \Delta(K)$ and the result follows from Proposition 2.4.1.

Proposition 2.4.3 If $f \in \mathcal{A}$ and $f(g) = g_1$ for $g, g_1 \in G$ then $f(\overline{C}_g) = \overline{C}_{g_1}$.

Proof: We know that $f(\overline{C}_g) = \pm \overline{C}_{g_2}$ for some $g_2 \in G$. For any $x \in G$, $\ell(f(xgx^{-1})) = \ell(f(g)) = \ell(g_1) = 1$ so we must choose the plus sign. Thus $f(\overline{C}_g) = \overline{C}_{g_2}$ and $f(g) - g_2 = g_1 - g_2 \in \Lambda$ as in the proof of Lemma 1.3.1. By Proposition 1.1.1, g_1 and g_2 are conjugate so $\overline{C}_{g_1} = \overline{C}_{g_2} = f(\overline{C}_g)$.

Section 5. The Center of $\mathcal{NA}$. This section gives some information about the center of $\mathcal{A}$.

Proposition 2.5.1 Let G be arbitrary. If $f \in \mathcal{NA}$ commutes with T_g for all $g \in G$ then f restricted to G, denoted $f_{|G}$, takes G onto G. Moreover, $f_{|G}$ is a central automorphism of G.

Proof: We recall that $\sigma \in \mathrm{Aut}(G)$ is central if $\sigma(g) \in g\,\mathfrak{Z}(G)$ for all $g \in G$. By hypothesis $f \circ T_g = T_g \circ f$ for all $g \in G$. If $g_1 \in G$ then $f \circ T_g(g_1) = f(g)f(g_1)f(g^{-1}) = T_g \circ f(g_1) = g\, f(g_1)g^{-1}$ so $g^{-1}f(g)f(g_1) = f(g_1)g^{-1}f(g)$. Hence $g^{-1}f(g)$ is in $\mathfrak{Z}(Z(G))$. Let $f(g) = gz$ for some $z \in \mathfrak{Z}(Z(G))$. Since $|f(g)| = |g|$, z is a unit of finite order with $\ell(z) = 1$. By Theorem 2.1 of [2], $z \in \mathfrak{Z}(G)$. Thus $f(g) = gz \in G$. This shows that $f_{|G}$ is a central automorphism of G and completes the proof.

Corollary 2.5.2 If G is arbitrary with $\mathfrak{Z}(G) = 1$ then $\mathcal{NA}$ contains no non-trivial elements which centralize the group $\{T_g | g \in G\}$. As a result, $\mathcal{NA}$ contains no non-trivial elements in the center of

$\mathcal{N}\mathcal{A}$ is trivial.

Proof: This follows from Proposition 2.5.1 and the fact that if $Z(G) = 1$ then there are no non-trivial central automorphisms of G.

CHAPTER 3

A METHOD FOR STUDYING $\mathfrak{N}\mathfrak{A}$ AND SOME EXAMPLES

Section 1. Introduction. In this chapter, we examine a method for dealing with the question of when an elementary representation exists which is particularly useful in case the character table of the group is available, or partially available, and one suspects that $I = \mathfrak{N}\mathfrak{A}$. The method is to extend $f \in \mathfrak{N}\mathfrak{A}$ linearly to an automorphism of $C(G)$ and see what f does to the identities of the simple components of $C(G)$. This method is then used to study $\mathfrak{N}\mathfrak{A}$ when $G = S_n$, the symmetric group on n symbols, for small values of n. We find that if $n \leq 10$ and $G = S_n$ then any $f \in \mathfrak{N}\mathfrak{A}$ has an elementary representation. Thus it is possible that all elements of $\mathfrak{N}\mathfrak{A}$ can have elementary representations without G' being abelian.

Section 2. The Method. Let G be an arbitrary finite group. If $f \in \mathfrak{N}\mathfrak{A}$, we can extend f linearly to a ring homomorphism of $C(G)$ into $C(G)$ which we also denote by f. Actually f is an automorphism of $C(G)$. For, let $\sum_{g \in G} \alpha_g g$ be an arbitrary element of $C(G)$ with $\alpha_g \in C$. For each $g \in G$, $g = f(x_g)$ for some $x_g \in Z(G)$ so that $\Sigma \alpha_g g = \Sigma \alpha_g f(x_g) = \Sigma f(\alpha_g x_g) = f(\Sigma \alpha_g x_g)$. This shows that f maps $C(G)$ onto $C(G)$ and since $C(G)$ is a finite dimensional vector space over C, f is a monomorphism of $C(G)$ onto $C(G)$. Whence f is an automorphism of $C(G)$. Let $\{S_i\}_{i=1}^n$ denote the finite set of simple components of $C(G)$ such that $C(G) = S_1 \oplus \ldots \oplus S_n$

and let e_i be the multiplicative identity of S_i for $i = 1,2,\ldots,n$. Since f effects a permutation of the set $\{S_i\}_{i=1}^n$, f also gives rise to a permutation of the set $\{e_i\}_{i=1}^n$. By Theorem 33.8 of [3], we have that $e_j = \frac{\chi^j(1)}{|G|} \sum_{i=1}^{n} \overline{\chi^j(g_i)}\, \overline{C}_{g_i}$ where χ^j is the irreducible character of G afforded by a minimal left ideal of S_j and $g_1 = 1, g_2, \ldots, g_n$ is a set of conjugacy class representatives for G. We record a series of results framed in this setting; the usefulness of these results will be demonstrated in the next section. Throughout the remainder of this chapter, f will denote the linear extension of an element of $\mathfrak{M}$ to $C(G)$ as described above.

<u>Proposition 3.2.1</u> Suppose $f(\overline{C}_{g_3}) = \overline{C}_{g_2}$ and $f(e_i) = e_j$. Then $\chi^j(g_2) = \chi^i(g_3)$.

<u>Proof</u>: $f(e_i) = f\left(\frac{\chi^i(1)}{|G|} \sum_{k=1}^{n} \overline{\chi^i(g_k)}\, \overline{C}_{g_k}\right) = \frac{\chi^i(1)}{|G|} \sum_{k=1}^{n} \overline{\chi^i(g_k)}\, f(\overline{C}_{g_k})$.

Since $f(e_i) = e_j$, we have the equation

$$(*) \qquad \frac{\chi^i(1)}{|G|} \sum_{k=1}^{n} \overline{\chi^i(g_k)}\, f(\overline{C}_{g_k}) = \frac{\chi^j(1)}{|G|} \sum_{k=1}^{n} \overline{\chi^j(g_k)}\, \overline{C}_{g_k} .$$

Recall that $\{\overline{C}_{g_k}\}_{k=1}^n$ forms a C-basis for $\mathcal{Z}(C(G))$ and that $f(\overline{C}_{g_1}) = \overline{C}_{g_1}$ since $g_1 = 1$. Thus, from (*), we obtain the equality $\frac{\chi^i(1)}{|G|} \overline{\chi^i(1)} = \frac{\chi^j(1)\overline{\chi^j(1)}}{|G|}$ so that $\chi^i(1) = \chi^j(1)$.

Comparing the coefficients of $\overline{C}_{g_2}$ on the left and right side of (*), it is clear that $\frac{\chi^i(1)}{|G|} \overline{\chi^i(g_3)} = \frac{\chi^j(1)}{|G|} \overline{\chi^j(g_2)}$. Therefore, $\chi^i(g_3) = \chi^j(g_2)$ as was to be shown.

<u>Remark</u>. The above proof shows that if $f(e_i) = e_j$ then $\chi^i(1) = \chi^j(1)$.

<u>Corollary 3.2.2</u> If $f(\overline{C}_{g_3}) = \overline{C}_{g_3}$ and $f(e_i) = e_j$ then $\chi^j(g_3) = \chi^i(g_3)$.

Proof: Take $g_2 = g_3$ in Proposition 3.2.1.

Corollary 3.2.3 If $f(\overline{C}_{g_3}) = \overline{C}_{g_2}$ and $f(e_i) = e_i$ then $\chi^i(g_2) = \chi^i(g_3)$.

Proof: Take $i = j$ in Proposition 3.2.1.

Corollary 3.2.4 Suppose G has only one conjugacy class C_g with $|C_g|$ elements. If $\chi^j(g) \neq \chi^i(g)$ then $f(e_i) \neq e_j$.

Proof: By Corollary 1.4.3, $f(\overline{C}_g) = \overline{C}_g$. Since $\chi^j(g) \neq \chi^i(g)$, $f(e_i) \neq e_j$ by Corollary 3.2.2.

Corollary 3.2.5 Suppose G has only one irreducible complex character χ^i of degree $\chi^i(1)$. If $\chi^i(g_2) \neq \chi^i(g_3)$ then $f(\overline{C}_{g_3}) \neq \overline{C}_{g_2}$.

Proof: By the Remark preceding Corollary 3.2.2, $f(e_i) = e_i$. Since $\chi^i(g_2) \neq \chi^i(g_3)$, $f(\overline{C}_{g_3}) \neq \overline{C}_{g_2}$ by Corollary 3.2.3.

Lemma 3.2.6 Suppose G has exactly 2 irreducible complex characters of degree 1. Denote the non-trivial character of degree 1 by χ^2. If $\chi^2(g_2) \neq \chi^2(g_3)$ then $f(\overline{C}_{g_3}) \neq \overline{C}_{g_2}$.

Proof: If e_1 is the idempotent of $C(G)$ associated with the trivial character of G, then $f(e_1) = e_1$ since f permutes the class sums of elements of G. Hence $f(e_2) = e_2$ by the Remark preceding Corollary 3.2.2, where e_2 is the idempotent of $C(G)$ associated with χ^2. By Corollary 3.2.3, $f(\overline{C}_{g_3}) \neq \overline{C}_{g_2}$ as was to be shown.

Corollary 3.2.7 Let $G = S_n$. Any $f \in \mathfrak{N}\mathfrak{a}$ must take a class sum of elements of G inside A_n, the alternating group on n symbols, to a class sum of elements of G inside A_n.

Proof: Since $A_n = S_n'$ and $[S_n : A_n] = 2$, S_n has exactly 2 irreducible complex characters of degree 1. If χ^2 denotes the non-trivial linear character of S_n then $\chi^2(g) = 1$ if $g \in A_n$ and -1 if $g \notin A_n$. Let $g_3 \in A_n$ be arbitrary and g_2 be any element of S_n

not in A_n. Clearly $\chi^2(g_2) \neq \chi^2(g_3)$. Whence, by Lemma 3.2.6, $f(\overline{C}_{g_3}) \neq \overline{C}_{g_2}$.

Section 3. The Examples. This section is devoted to showing that if $G = S_n$, $n = 2,3,\ldots,10$, then each $f \in \mathfrak{N}\mathfrak{a}$ has an elementary representation. The methods utilize Corollary 1.4.3 and the results of Section 2, Chapter 3. All information about the character tables of these groups is in [7].

Theorem 3.3.1 If $G = S_n$ for any $n = 2,3,\ldots,10$ then each $f \in \mathfrak{N}\mathfrak{a}$ has an elementary representation.

Proof: (1) If $G = S_2$ then $\mathfrak{N}\mathfrak{a} = \mathrm{Aut}(G)$ by Proposition 2.2.2.

(2) If $G = S_3$ then each $f \in \mathfrak{N}\mathfrak{a}$ fixes each class sum by Corollary 1.4.3 and the fact that no 2 conjugacy classes of S_3 have the same number of elements. Hence, each $f \in \mathfrak{N}\mathfrak{a}$ is of the form T_u for some $u \in Q(S_3)$.

(3) Suppose $G = S_4$. Conjugacy class representatives for S_4 may be listed as follows: $g_1 = 1$, $g_2 = (12)$, $g_3 = (123)$, $g_4 = (1234)$, $g_5 = (12)(34)$. The number of elements in each class is 1,6,8,6,3 respectively and the order of the elements in each class is 1,2,3,4,2 respectively. Since there is exactly one class containing 1,8 or 3 elements, f fixes $\overline{C}_{g_1}$, $\overline{C}_{g_3}$, $\overline{C}_{g_5}$, by Corollary 1.4.3. Since the elements of C_{g_2} and C_{g_4} are not of the same order, f also fixes $\overline{C}_{g_2}$ and $\overline{C}_{g_4}$, by Corollary 1.4.3. Hence $f = T_u$ for some $u \in Q(S_4)$.

(4) Suppose $G = S_5$. As in the case of S_4, one can examine the number and order of the elements in the conjugacy classes of S_5 and use Corollary 1.4.3 to conclude that $f = T_u$ for some $u \in Q(S_5)$.

(5) Suppose $G = S_6$. The character table for G is given in Table 1. By Corollary 1.4.3, f fixes $\overline{C}_{g_1}$, $\overline{C}_{g_5}$, $\overline{C}_{g_7}$. By Corollary 3.2.7, f also fixed $\overline{C}_{g_9}$ and $\overline{C}_{g_4}$ since $g_9 \in A_6$ and $g_4 \notin A_6$. We now consider two cases.

Case I: $f(\overline{C}_{g_3}) \neq \overline{C}_{g_{11}}$. By Corollary 1.4.3, $f(\overline{C}_{g_3}) = \overline{C}_{g_3}$ and $f(\overline{C}_{g_{11}}) = \overline{C}_{g_{11}}$. It will be shown that f fixes all other class sums of elements of G. Since $f(\overline{C}_{g_4}) = \overline{C}_{g_4}$, Corollary 3.2.2 says that $f(e_2) = e_2$ or $f(e_2) = e_7$. Since $f(\overline{C}_{g_3}) = \overline{C}_{g_3}$, Corollary 3.2.2 requires that $f(e_2) = e_2$ or $f(e_2) = e_{10}$. Thus $f(e_2) = e_2$. Since $\chi^2(g_2) \neq \chi^2(g_{10})$ and $f(e_2) = e_2$, $f(\overline{C}_{g_2}) = \overline{C}_{g_2}$ by Corollary 3.2.3. Hence $f(\overline{C}_{g_{10}}) = \overline{C}_{g_{10}}$. Since $\chi^2(g_6) \neq \chi^2(g_8)$ and $f(e_2) = e_2$, $f(\overline{C}_{g_6}) = \overline{C}_{g_6}$ by Corollary 3.2.3. Hence $f(\overline{C}_{g_8}) = \overline{C}_{g_8}$. Thus f fixes all class sums of elements of G and hence $f = T_u$ for some $u \in Q(S_6)$.

Case II. $f(\overline{C}_{g_3}) = \overline{C}_{g_{11}}$. By 11.4.3 of [10], there is an element $\sigma \in \mathrm{Aut}(S_6)$ such that $\sigma(\overline{C}_{g_3}) = \overline{C}_{g_{11}}$. We emphasize the fact that $f \in \mathfrak{N}\mathfrak{a}$ is arbitrary with $f(\overline{C}_{g_3}) = \overline{C}_{g_{11}}$ so that f may in fact be σ. It is claimed that f interchanges $\overline{C}_{g_2}$ and $\overline{C}_{g_{10}}$ and interchanges $\overline{C}_{g_6}$ and $\overline{C}_{g_8}$. Once this claim is established, it will be clear that $f(\overline{C}_g) = \sigma(\overline{C}_g)$ for all $g \in S_6$ so that $f = T_u \circ \sigma$ for some $u \in Q(S_6)$.

Why is the claim true? We know $f(\overline{C}_{g_4}) = \overline{C}_{g_4}$. If $f(e_2) = e_5$ then, by Corollary 3.2.2, $\chi^5(g_4) = \chi^2(g_4)$. Since this is not the case, $f(e_2) \neq e_5$. If $f(e_2) = e_{10}$ then, by Corollary 3.2.2, $\chi^{10}(g_4) = \chi^2(g_4)$. Since this is not the case, $f(e_2) \neq e_{10}$. If $f(e_2) = e_2$, then $\chi^2(g_3) = \chi^2(g_{11})$ since $f(\overline{C}_{g_3}) = \overline{C}_{g_{11}}$, by Corollary 3.2.3. Since $\chi^2(g_3) \neq \chi^2(g_{11})$, $f(e_2) \neq e_2$. Thus

TABLE 1: Character Table for S_6

Conjugacy Class Representative g	g_1 1	g_2 (12)	g_3 (123)	g_4 (1234)	g_5 (12)(34)	g_6 (12)(345)	g_7 (12345)	g_8 (123456)	g_9 (12)(3456)	g_{10} (12)(34)(56)	g_{11} (123)(456)
$\lvert C_g \rvert$	1	15	40	90	45	120	144	120	90	15	40
χ^1	1	1	1	1	1	1	1	1	1	1	1
χ^2	5	3	2	1	1	0	0	-1	-1	-1	-1
χ^3	9	3	0	-1	1	0	-1	0	1	3	0
χ^4	10	2	1	0	-2	-1	0	1	0	-2	1
χ^5	5	1	-1	-1	1	1	0	0	-1	-3	2
χ^6	16	0	-2	0	0	0	1	0	0	0	-2
χ^7	5	-1	-1	1	1	-1	0	0	-1	3	2
χ^8	10	-2	1	0	-2	1	0	-1	0	2	1
χ^9	9	-3	0	1	1	0	-1	0	1	-3	0
χ^{10}	5	-3	2	-1	1	0	0	1	-1	1	-1
χ^{11}	1	-1	1	-1	1	-1	1	-1	1	-1	1

$f(e_2) = e_7$ by the Remark preceding Corollary 3.2.2. Since $f(e_2) = e_7$ and $\chi^7(g_2) \neq \chi^2(g_2)$, $f(\bar{C}_{g_2}) \neq \bar{C}_{g_2}$ by Corollary 3.2.2. Thus $f(\bar{C}_{g_2}) = \bar{C}_{g_{10}}$ by Corollary 1.4.3. Since $f(e_2) = e_7$ and $\chi^7(g_6) \neq \chi^2(g_6)$, $f(\bar{C}_{g_6}) \neq \bar{C}_{g_6}$ by Corollary 3.2.2. Hence $f(\bar{C}_{g_6}) = \bar{C}_{g_8}$ by Corollary 1.4.3. This establishes the claim.

(6) Suppose $G = S_7$. We will show that any $f \in \mathfrak{N}\mathfrak{a}$ fixes all class sums and hence has an elementary representation. Exactly the same statement can be made if $G = S_8$, S_9 or S_{10} and the method of proof is so much like the case $G = S_7$ that these 3 cases are left to the reader. Now to proceed with $G = S_7$. One notices that the class of transpositions in S_7 has 21 elements and no other class has exactly 21 elements. We will apply Corollary 3.2.4 to show that $f(e_i) = e_i$ for each idempotent e_i, $i = 1,2,\ldots,15$. Since $\{e_i\}_{i=1}^{15}$ is a C-basis for $\mathcal{Z}(C(S_7))$, it is clear that f fixes all class sums of elements in S_7. We reproduce a part of the character table for S_7 from [7] in Table 2. Clearly $f(e_1) = e_1$ and by the Remark preceding Corollary 3.2.2, $f(e_{15}) = e_{15}$. This Remark also shows that $f(e_8) = e_8$ and that $f(e_2) = e_2$ or e_{14}. But $\chi^{14}(g_2) \neq \chi^2(g_2)$ so that $f(e_2) = e_2$, by Corollary 3.2.4. Hence $f(e_{14}) = e_{14}$. Similarly $f(e_4) = e_4$, $f(e_{12}) = e_{12}$ and $f(e_6) = e_6$, $f(e_{10}) = e_{10}$ and $f(e_7) = e_7$, $f(e_9) = e_9$. By Corollary 3.2.4, $f(e_3)$ cannot be e_5, e_{11}, or e_{13} so $f(e_3) = e_3$. Similarly $f(e_5) = e_5$, $f(e_{11}) = e_{11}$ and $f(e_{13}) = e_{13}$. Thus f fixes all of the e_i's and we are done.

TABLE 2: Partial Character Table for S_7

	$g_1 = 1$	$g_2 = (12)$ Transpositions
χ^1	1	1
χ^2	6	4
χ^3	14	6
χ^4	15	5
χ^5	14	4
χ^6	35	5
χ^7	21	1
χ^8	20	0
χ^9	21	-1
χ^{10}	35	-5
χ^{11}	14	-4
χ^{12}	15	-5
χ^{13}	14	-6
χ^{14}	6	-4
χ^{15}	1	-1

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APPENDIX

APPENDIX

The purpose of this appendix is to prove Propositions 1.1.1, 1.1.2, 1.1.3. The source of these proofs is primarily page 596 of [3]; this material is therein credited to Brauer.

Proposition 1.1.1 $\lambda \in \Lambda$ iff λ sums to 0 on all conjugacy classes of G.

Proof: ($\Rightarrow$) If $\lambda \in \Lambda$, then λ is a finite Z-linear combination of elements of the form $ab-ba$ with $a,b \in Z(G)$. It thus suffices to show that each element $ab-ba$ sums to 0 on all conjugacy classes of G. But this is clear because g_1g_2 and g_2g_1 are conjugate in G for any $g_1,g_2 \in G$.

($\Leftarrow$) Suppose $\alpha \in Z(G)$ sums to 0 on all conjugacy classes of G. If $g_1 = 1,g_2,\ldots,g_n$ is a set of conjugacy class representatives for G then $\alpha = X_1 + X_2 +\ldots+ X_n$ where $X_i = \sum_{g\in C_{g_i}} a_g g$ for $i = 1,2,\ldots,n$ and $a_g \in Z$. Let $g_i, x_2g_ix_2^{-1},\ldots,x_{m_i}g_ix_{m_i}^{-1}$ be the conjugates of g_i in G. Since $\sum_{g\in C_{g_i}} a_g = 0$, we can write

$$X_i = a_1g_i - a_1g_i + a_2x_2g_ix_2^{-1} - a_2g_i +\ldots+ a_{m_i}x_{m_i}g_ix_{m_i}^{-1} - a_{m_i}g_i$$

$$= a_2(x_2g_ix_2^{-1} - g_i) +\ldots+ a_{m_i}(x_{m_i}g_ix_{m_i}^{-1} - g_i),$$ which is clearly an element of Λ. Thus $\alpha \in \Lambda$.

Proposition 1.1.2 $\lambda \in \Lambda_p$ iff λ sums to elements of pZ on the conjugacy classes of G.

Proof: (⇒) If $\lambda \in \Lambda_p$ then $\lambda = x + y$ with $x \in \Lambda$, $y \in pZ(G)$. The result now follows from Proposition 1.1.1 since elements of $pZ(G)$ sum to elements of pZ on the conjugacy classes of G.

(⇐) Let $g_1 = 1, g_2, \ldots, g_t$ be a set of conjugacy class representatives for G. If λ sums to elements of pZ on the conjugacy classes of G, then $\lambda = X_1 + X_2 + \ldots + X_t$ where $X_i = \sum_{g \in C_{g_i}} a_g g$ with $\sum_{g \in C_{g_i}} a_g = pn_i$ for some $n_i \in Z$. Thus, $\lambda = X_1 - pn_1 + X_2 - pn_2 + \ldots + X_t - pn_t + p(n_1 + n_2 + \ldots + n_t)$ which is an element of Λ_p since $X_i - pn_i \in \Lambda$ by Proposition 1.1.1.

Proposition 1.1.3 If $x_1, x_2 \in Z(G)$ are such that $x_1 - x_2 \in \Lambda_p$, then $x_1^p - x_2^p \in \Lambda_p$.

Proof: First, we discuss statements (A) and (B) and then complete the proof.

(A) If $x, y \in Z(G)$ then $(x + y)^p \equiv x^p + y^p \bmod \Lambda_p$. Why? The details of this calculation are left to the reader.

(B) If $\lambda \in \Lambda$ then $\lambda^p \in \Lambda_p$. Why? We first show that $(\alpha\beta-\beta\alpha)^p \in \Lambda_p$ for any $\alpha, \beta \in Z(G)$. By (A), $(\alpha\beta-\beta\alpha)^p \equiv (\alpha\beta)^p + (-1)^p(\beta\alpha)^p \bmod \Lambda_p$. If $p = 2$, then $(\beta\alpha)^p \equiv -(\beta\alpha)^p \bmod \Lambda_p$ and if p is odd, then $(-1)^p = -1$. Hence $(\alpha\beta-\beta\alpha)^p \equiv (\alpha\beta)^p - (\beta\alpha)^p \bmod \Lambda_p$. If $\gamma = \beta(\alpha\beta)^{p-1}$ then $\alpha\gamma-\gamma\alpha = \alpha\beta(\alpha\beta)^{p-1} - \beta(\alpha\beta)^{p-1}\alpha = (\alpha\beta)^p - (\beta\alpha)^p$ so that $(\alpha\beta-\beta\alpha)^p \equiv (\alpha\beta)^p - (\beta\alpha)^p = \alpha\gamma-\gamma\alpha \equiv 0 \bmod \Lambda_p$. Thus $(\alpha\beta-\beta\alpha)^p \in \Lambda_p$, as was to be shown. The proof of (B) can be completed by induction on the number of summands in $\lambda \in \Lambda$. For suppose $n_i \in Z$ and $\alpha_i, \beta_i \in Z(G)$ for $i = 1,2,\ldots,k$. Then

$$[n_1(\beta_1\alpha_1-\alpha_1\beta_1) + n_2(\beta_2\alpha_2-\alpha_2\beta_2) +\ldots+ n_k(\beta_k\alpha_k-\alpha_k\beta_k)]^p$$

$$\equiv [n_1(\beta_1\alpha_1-\alpha_1\beta_1) +\ldots+ n_{k-1}(\beta_{k-1}\alpha_{k-1}-\alpha_{k-1}\beta_{k-1})]^p$$

$$+ n_k^p(\beta_k\alpha_k-\alpha_k\beta_k)^p \bmod \Lambda_p,$$ by (A). But we have

$n_k^p(\beta_k\alpha_k-\alpha_k\beta_k)^p \in \Lambda_p$ and by the induction hypothesis $[n_1(\beta_1\alpha_1-\alpha_1\beta_1) +\ldots+ n_{k-1}(\beta_{k-1}\alpha_{k-1}-\alpha_{k-1}\beta_{k-1})]^p \in \Lambda_p$. Thus $[n_1(\beta_1\alpha_1-\alpha_1\beta_1) +\ldots+ n_k(\beta_k\alpha_k-\alpha_k\beta_k)]^p \in \Lambda_p$.

To complete the proof of the Proposition, let x_1, x_2 be as in the hypothesis. Then $x_1 - x_2 = \lambda + \lambda_p$ for some $\lambda \in \Lambda$, $\lambda_p \in pZ(G)$. Hence

$$\begin{aligned}(x_1 - x_2)^p = (\lambda + \lambda_p)^p &\equiv \lambda^p + \lambda_p^p \bmod \Lambda_p, \text{ by (A)} \\ &\equiv 0 + \lambda_p^p \bmod \Lambda_p, \text{ by (B)} \\ &\equiv 0 \qquad \bmod \Lambda_p, \text{ since } \lambda_p^p \in pZ(G).\end{aligned}$$

But $(x_1 - x_2)^p \equiv x_1^p - x_2^p \bmod \Lambda_p$, so that $x_1^p - x_2^p \in \Lambda_p$, as was to be shown.