ASYMPTOTIC BAYES SEQUENTIAL TESTS OF THE HYPOTHESIS THAT THE DRIFT OF A WIENER PROCESS IS ZERO

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THURMAN JOSEPH BROWN, Jr.

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This is to certify that the

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ABSTRACT

ASYMPTOTIC BAYES SEQUENTIAL TESTS OF THE HYPOTHESIS THAT THE DRIFT OF A WIENER PROCESS IS ZERO

By

Thurman Joseph Brown, Jr.

Let $\{X_t: t > 0\}$ be a Wiener process plus a drift μ. This paper is concerned with approximating the optimal sequential procedure for testing H_0: $\mu = 0$ vs. H_1: $\mu \neq 0$ for the large sample case when the prior distribution for the alternative is approximately Lebesgue and the loss is approximately proportional to $\|\mu\|^k$. The fixed sample size problem was treated by Rubin and Sethuraman (Sankhyā, A, Vol. 27, 1965, pp. 347-356). The solution is similar to that of Chernoff (Sequential Tests for the Mean of a Normal Distribution, Proc. Fourth Berkeley Symp. Math. Stat. Prob. 1, 79-91, University of California Press). It consists of two strictly increasing functions $a_0(t)$ defined on $[T_0,\infty)$ and $a_1(t)$ defined on $[T_1,\infty)$, with $0 \leq T_1 < T_0$ and $a_0(t) < a_1(t)$ on $[T_0,\infty)$, which determine the following sequential procedure. Suppose observation begins at time $t_s \geq 0$. Observe $\|X_{t_s}\|$. If $a_0(t_s) < \|X_{t_s}\| < a_1(t_s)$, continue sampling until $\|X_t\| = a_i(t)$. If $i = 0$, accept: if $i = 1$, reject. The asymptotic nature of the solution is derived, and standard numerical procedures are used to approximate the regions and the risk. Rubin and Sethuraman's work has shown that the general asymptotic testing problem may be reduced to the above case.

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By

Thurman Joseph Brown, Jr.

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SECTION 1

Suppose $\{X_t: t > 0\}$ is an n-dimensional Wiener process with covariance matrix $\sigma^2 I_n$, σ^2 known, plus a drift $\mu = (\mu_1,\ldots,\mu_n)$. That is:

$$X_t = \mu t + Z_t = \mu t + (Z_{1t},\ldots,Z_{nt})$$

$$\text{with } E(Z_t) = 0 \text{ and } E(Z_{it}Z_{jt}) = \begin{cases} 0 & \text{if } i \neq j \\ \sigma^2 t & \text{if } i = j \end{cases}$$

We want to test sequentially $H_0: \mu = 0$ vs. $H_1: \mu \neq 0$ for the case in which the apriori measure $\xi_{0,0}$ is proportionately normal on the alternative. That is,

$$\xi_{0,0}(E) = \epsilon I_E(0) + (1-\epsilon)\int_E \Phi(\mu,\mu_0,\sigma_0^2)\,d\mu$$

where μ_0 represents the mean vector, σ_0^2 is a constant, and

$$\Phi(\mu,\mu_0,\sigma_0^2) = \prod_{i=1}^{n} \Phi(\mu_i,\mu_{0i},\sigma_0^2) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{\frac{n}{2}} e^{-\frac{\Sigma(\mu_i-\mu_{0i})^2}{2\sigma_0^2}} .$$

The dependence on n will be suppressed. Throughout this paper unindexed summation signs will go from 1 to n. The cost of sampling per unit time is c, and the losses of type I and type II errors are W_0 and $W_1\|\mu\|^k$ respectively. The fixed sample size problem was treated by Rubin and Sethuraman [1]. Chernoff [2] treated the problem of testing sequentially $H_0: \mu \le 0$ vs. $H_1: \mu > 0$. Bickel and Yahav [5] showed his procedure to be robust. Lindley [6] and others have shown that for large samples the exact form of the prior distribution is irrelevant and that maximum likelihood estimation is optimum.

We now normalize the problem to one in which we test $H_0: \mu = 0$ vs. $H_1: \mu \neq 0$, but with apriori measure placing mass one at $\mu = 0$ and Lebesgue on the alternative: $\rho_{0,0}(E) = I_E(0) + \nu(E)$, ν Lebesgue, and with W_0 and σ^2 replaced by ones. Although the apriori measure is not a probability measure, it gives rise to a posterior probability measure.

Three steps are employed to accomplish the normalization.

Step 1) Let

$$\mu^* = \alpha\mu \quad ,$$
$$t^* = \beta t \quad .$$

Then $X_t = \frac{\mu^* t^*}{\alpha\beta} + Z^*_{t^*}$ with $Z^*_{t^*} \sim N(0, \frac{\sigma^2}{\beta} I_n t^*)$ where $N(a,B)$ denotes the normal distribution with mean a and covariance matrix B.

If 1) $\frac{1}{\alpha\beta} = 1$ and 2) $\frac{\sigma^2}{\beta} = 1$ then X_t has drift $\mu^* = (\mu^*_1,\ldots,\mu^*_n)$ and covariance matrix I_n per unit time measured in t^*.

The Bayes risk for a given procedure in the original problem is given by

$$\mathcal{B} = \epsilon(cE[T|0] + \gamma(0)W_0) + (1-\epsilon)\int(cE[T|\mu] + W_1\|\mu\|^k\gamma(\mu))\Phi(\mu,\mu_0,\sigma_0^2)\,d\mu$$

where $\gamma(\mu)$ is the probability of error given μ, and T is the sampling time.

$$\mathcal{B} = W_0\{\epsilon(\frac{c}{W_0\beta}E[T^*|0] + \gamma^*(0)) + (1-\epsilon)\int(\frac{c}{W_0\beta}E[T^*|\mu^*] + \gamma^*(\mu^*)\frac{W_1}{W_0\alpha^k}$$
$$\|\mu^*\|^k)\Phi(\mu^*,\alpha\mu_0,\alpha^2\sigma_0^2)\,d\mu^*\}$$

where $\gamma^*(\mu^*) = \gamma(\mu)$ and $\frac{1}{\beta} E[T^*|\mu^*] = E[T|\mu]$.

So letting $c^* = \frac{c}{W_0\beta}$, $\mu_0^* = \alpha\mu_0$, $\sigma_0^{*2} = \alpha^2\sigma_0^2$, and $W_1^* = \frac{W_1}{W_0\alpha^k}$,

minimizing $\mathcal{B}$ is equivalent to minimizing

$$\mathcal{B}^* = \varepsilon(c^*E[T^*|0] + \gamma^*(0)) + (1-\varepsilon)\int(c^*E[T^*|\mu^*] + W_1^*\|\mu^*\|^k\gamma^*(\mu^*))$$
$$\Phi(\mu^*,\mu_0^*,\sigma_0^{*2})\,d\mu^* .$$

The solution to equations 1) and 2) is

$$\alpha = \frac{1}{\sigma^2}$$

$$\beta = \sigma^2 .$$

So
$$c^* = \frac{c}{W_0\sigma^2}$$

$$W_1^* = \frac{W_1(\sigma^2)^k}{W_0} .$$

Step 2) We now show that testing $H_0: \mu = 0$ vs. $H_1: \mu \neq 0$ with apriori measure ε at $\mu = 0$ and $(1-\varepsilon)N(\mu_0,\sigma_0^2 I_n)$ on the alternative is equivalent to testing the same hypothesis against the same alternative, but with apriori measure placing mass one at $\mu = 0$, and $b\nu$, b constant and ν Lebesgue, on the alternative, and starting observation at some positive time (discounting sampling cost accordingly).

The posterior measure given $X_t = x$ is now computed for the normal apriori problem.

$$\xi_{x,t}(0) = P(\mu=0|X_t=x) = \frac{\epsilon\Phi(x,0,t)}{\epsilon\Phi(x,0,t)+(1-\epsilon)\int\Phi(x,\mu t,t)\ \Phi(\mu,\mu_0,\sigma_0^2)\,d\mu}$$

$$= \frac{\epsilon\Phi(x,0,t)}{\epsilon\Phi(x,0,t)+(1-\epsilon)\Phi(x,\mu_0 t,t(1+t\sigma_0^2))}$$

$$= [1+(\frac{1-\epsilon}{\epsilon})(\frac{1}{1+t\sigma_0^2})^{\frac{n}{2}} e^{\frac{\Sigma x_i^2}{2t} - \frac{\Sigma(x_i-\mu_{0i}t)^2}{2t(1+t\sigma_0^2)}}]^{-1}$$

$$d\xi_{x,t}(\mu\neq 0) = \frac{(\frac{1-\epsilon}{\epsilon})(\frac{1}{1+t\sigma_0^2})^{\frac{n}{2}} e^{\frac{\Sigma x_i^2}{2t} - \frac{\Sigma(x_i-\mu_{0i}t)^2}{2t(1+t\sigma_0^2)}}\ \Phi(\mu,\frac{x+\mu_0/\sigma_0^2}{t+1/\sigma_0^2},\frac{1}{t+1/\sigma_0^2})}{1+(\frac{1-\epsilon}{\epsilon})(\frac{1}{1+t\sigma_0^2})^{\frac{n}{2}} e^{\frac{\Sigma x_i^2}{2t} - \frac{\Sigma(x_i-\mu_{0i}t)^2}{2t(1+t\sigma_0^2)}}}\,d\mu .$$

Letting $X'_{t'} = X_t + \frac{\mu_0}{\sigma_0^2}$, $x' = x + \frac{\mu_0}{\sigma_0^2}$, and $t' = t + \frac{1}{\sigma_0^2}$

$$\xi_{x',t'}(0) = P(\mu=0|X'_{t'} = x') = [1 + (\frac{1-\epsilon}{\epsilon})(\frac{1}{t'\sigma_0^2})^{\frac{n}{2}} e^{\frac{\Sigma x_i'^2}{2t'} - \frac{\Sigma\mu_{0i}^2}{2\sigma_0^2}}]^{-1}, \quad (1)$$

$$d\xi_{x',t'}(\mu\neq 0) = \frac{(\frac{1-\epsilon}{\epsilon})(\frac{1}{t'\sigma_0^2})^{\frac{n}{2}} e^{\frac{\Sigma x_i'^2}{2t'} - \frac{\Sigma\mu_{0i}^2}{2\sigma_0^2}}\ \Phi(\mu,\frac{x'}{t'},\frac{1}{t'})}{1+(\frac{1-\epsilon}{\epsilon})(\frac{1}{t'\sigma_0^2})^{\frac{n}{2}} e^{\frac{\Sigma x_i'^2}{2t'} - \frac{\Sigma\mu_{0i}^2}{2\sigma_0^2}}} . \quad (1)$$

For the uniform alternative problem

$$\rho_{x,t}(0) = \frac{\Phi(x,0,t)}{\Phi(x,0,t) + b\int(\frac{1}{t})^n\Phi(\mu,\frac{x}{t},\frac{1}{t})\,d\mu}$$

$$= [1 + b(\frac{2\pi}{t})^{\frac{n}{2}} e^{\frac{\Sigma x_i^2}{2t}}]^{-1} , \quad (2)$$

$$d\rho_{x,t}(\mu\neq 0) = \frac{b(\frac{2\pi}{t})^{\frac{n}{2}} e^{\frac{\Sigma x_i^2}{2t}} \Phi(\mu,\frac{x}{t},\frac{1}{t})}{[1 + b(\frac{2\pi}{t})^{\frac{n}{2}} e^{\frac{\Sigma x_i^2}{2t}}]} d\mu \quad . \tag{2}$$

Equations (1) and (2) are equivalent for the choice

$$b = b(\epsilon,\mu_0,\sigma_0^2) = (\frac{1-\epsilon}{\epsilon})(\frac{1}{2\pi\sigma_0^2})^{\frac{n}{2}} e^{-\frac{\Sigma\mu_{0i}^2}{2\sigma_0^2}} \quad .$$

Step 3) For testing $H_0: \mu = 0$ vs. $H_1: \mu \neq 0$ with apriori mass one at $\mu = 0$ and $b\nu$, b constant and ν Lebesgue, on the alternative, the Bayes risk of a given procedure is

$$\mathcal{B} = (cE[T|0] + \gamma(0)) + \int (cE[T|\mu] + W_1\|\mu\|^k\gamma(\mu))b\, d\mu \quad .$$

Only procedures for which the risk is finite are considered.

Let $X^*_{t^*} = \delta X_t$,

$\mu^* = \alpha\mu$,

$t^* = \beta t$.

Then $X^*_{t^*} = \frac{\delta}{\alpha\beta}\mu^* t^* + Z^*_{t^*}$, $Z^*_{t^*} \sim N(0, \frac{\delta^2}{\beta} I_n t^*)$,

so for 1) $\frac{\delta}{\alpha\beta} = 1$ and 2) $\frac{\delta^2}{\beta} = 1$, $X^*_{t^*}$ has drift μ^* and covariance matrix I_n per unit time measured in t^*.

In terms of the starred variables

$$\mathcal{B} = (\frac{c}{\beta} E[T^*|0] + \gamma^*(0)) + \int(\frac{c}{\beta} E[T^*|\mu^*] + \frac{W_1}{\alpha^k}\|\mu^*\|^k\gamma^*(\mu^*))\frac{b}{\alpha^n}\, d\mu^*$$

with $\gamma^*(\mu^*) = \gamma(\mu)$ and $\frac{1}{\beta}E[T^*|\mu^*] = E[T|\mu]$.

So for 3) $\frac{b}{\alpha^n} = 1$, minimizing $\mathcal{B}$ is equivalent to minimizing

$$\mathcal{B}^* = (c^* E[T^*|0] + \gamma^*(0)) + \int (c^* E[T^*|\mu^*] + W_1^* \|\mu^*\|^k \gamma^*(\mu^*)) d\mu^*$$

where $c^* = \frac{c}{\beta}$ and $W_1^* = \frac{W_1}{\alpha^k}$.

The solution to the three equations is

$$\alpha = b^{\frac{1}{n}}$$

$$\beta = b^{-\frac{2}{n}}$$

$$\delta = b^{-\frac{1}{n}} \quad .$$

The total effect of the normalization is to make the normal alternative problem with constants c, W_0, W_1, k, σ^2, and apriori constants ϵ, μ_0, and σ_0^2 equivalent to the Lebesgue alternative problem (mass one at $\mu = 0$) with constants

$$W_0^* = 1 \quad ,$$

$$\sigma^{*2} = 1 \quad ,$$

$$c^* = \frac{c}{W_0} \left(\frac{1-\epsilon}{\epsilon}\right)^{\frac{2}{n}} \left(\frac{\sigma^2}{2\pi\sigma_0^2}\right) e^{-\frac{\Sigma\mu_{0i}^2}{n\sigma_0^2}} \quad ,$$

$$W_1^* = \frac{W_1}{W_0} \left(\frac{\epsilon}{1-\epsilon}\right)^{\frac{k}{n}} (2\pi\sigma_0^2)^{\frac{k}{2}} e^{\frac{k\Sigma\mu_{0i}^2}{2n\sigma_0^2}} \quad ,$$

$$k^* = k \quad ,$$

with observation of the linear functions of X_t and t

$$X^*_{t^*} = \left(\frac{\epsilon}{1-\epsilon}\right)^{\frac{1}{n}} \left(2\pi \frac{\sigma_0^4}{\sigma^2}\right)^{\frac{1}{2}} e^{\frac{\Sigma\mu_{0i}^2}{2n\sigma_0^2}} \left(X_t + \frac{\sigma^2 \mu_0}{\sigma_0^2}\right) \quad ,$$

$$t^* = \left(\frac{\epsilon}{1-\epsilon}\right)^{\frac{2}{n}} \left(2\pi \frac{\sigma_0^2}{\sigma^2}\right) e^{\frac{\Sigma\mu_{0i}^2}{n\sigma_0^2}} \left(t + \frac{\sigma^2}{\sigma_0^2}\right) \quad ,$$

so that observation begins at the point

$$\left(\left(\frac{\epsilon}{1-\epsilon}\right)^{\frac{1}{n}}\left(\frac{2\pi}{\sigma_0^2}\right)^{\frac{1}{2}} e^{\frac{\Sigma\mu_{0i}^2}{2n\sigma_0^2}}\mu_0,\ \left(\frac{\epsilon}{1-\epsilon}\right)^{\frac{2}{n}} 2\pi\, e^{\frac{\Sigma\mu_{0i}^2}{n\sigma_0^2}}\right).$$

Equivalence of the two classes is in the sense of what might be termed lens equivalence. That is, the normalized problem views linear functions of the process X_t and the time t of the original problem. Thus, if $X^*_{t^*} = aX_t + b$, and $t^* = ct + d$, and if the optimal boundaries in the normalized problem are $a_i^*(t^*)$, then the optimal boundaries in the original problem are $a_i(t) = \frac{a_i^*(t^*)-b}{a}$.

Considering the normalized problem, the posterior probability measure given $X_t = x$ agrees with (2), but with $b = 1$. Letting

$$U(x,t) = \left(\frac{2\pi}{t}\right)^{\frac{n}{2}} e^{\frac{\Sigma x_i^2}{2t}},$$

suppressing its dependence on n, we have

$$\rho_{x,t}(0) = [1 + U(x,t)]^{-1}$$

$$d\rho_{x,t}(\mu\neq 0) = [1 + U(x,t)]^{-1}U(x,t)\phi(\mu,\tfrac{x}{t},\tfrac{1}{t})d\mu \ .$$

Two strictly increasing continuous functions a_0 defined on $[T_0,\infty)$ and a_1 defined on $[T_1,\infty)$, with $0 \le T_1 < T_0$, $a_0(T_0) = a_1(T_1) = 0$, and $a_0 < a_1$ on $[T_0,\infty)$ determine a sequential procedure in the following way. The two functions partition quadrant I into three regions:

$$D^0 = [(y,t): y \le a_0(t),\ t \ge T_0] \qquad \text{"stop and accept"}$$

$$D^1 = [(y,t): t < T_1 \text{ or } y \ge a_1(t),\ t \ge T_1] \qquad \text{"stop and reject"}$$

$$B = [(y,t): y < a_1(t),\ T_1 < t < T_0, \text{ or}$$

$$a_0(t) < y < a_1(t),\ t \ge T_0] \qquad \text{"continue sampling".}$$

Suppose the first observation $X_{t'}$ on the process is at time t'. The procedure is: observe $\|X_{t'}\|$. If $(\|X_{t'}\|,t') \in D^i$, $i = 0,1$, stop. If $(\|X_{t'}\|,t') \in B$, continue observation and stop at the first $t > t'$ such that $\|X_t\| = a_i(t)$. If $i = 0$, accept; if $i = 1$, reject.

A procedure of the type just described will be denoted by (a_0,a_1,T_1,T_0). Since the procedure depends on norm and to avoid plathoria of notation, D^0 will interchangeably denote the subset of quadrant I given above, and the subset of $n+1$ space given by $[(x,t); (\|x\|,t) \in D^0]$, and similarly for D^1 and B. The meaning in each case will be clear from the context. In general, the domain and range of the functions that map (x,t) into $(\|x\|,t)$ will not be distinguished.

We now define $R(x,t,\mu)$ to be the conditional risk given $X_t = x$ at initial time t, for a procedure of the above type which observes a process with drift μ, and with sampling cost ct incurred at the onset of the observation.

On D^1 the conditional risk is given by

$$R^1(x,t,\mu) = \begin{cases} ct & \text{if } \mu \neq 0 \\ ct + 1 & \text{if } \mu = 0 \ . \end{cases}$$

On D^0 the conditional risk is given by

$$R^0(x,t,\mu) = ct + W_1\|\mu\|^k .$$

We restrict consideration to boundary functions $a_i(t)$ for which the conditional expected sampling time T, given $X_{t'} = x'$, for a process with drift μ, is finite for all $(x',t') \in B$, and μ.*

$$E(T|X_{t'} = x',\mu) < \infty . \qquad (A)$$

Theorem.** On the set B the conditional risk has continuous second partial derivatives in x_i and t for every μ, and satisfies the partial differential equation

$$R_t^B(x,t,\mu) + \Sigma\, \mu_i R_i^B(x,t,\mu) + \frac{1}{2}\Sigma\, R_{ii}^B(x,t,\mu) = 0, \text{ for each } \mu \qquad (3)$$

with subscript i denoting differentiation with respect to x_i.

The proof of (3) is based on Theorem 2.1 of Doob's paper [7].

We now fix $\mu \neq 0$, and consider an arbitrary initial position $X_{t'} = x'$. On the boundary $\bigcup_{i=1,2}\ \bigcup_{t \geq T_i} (a_i(t),t)$ we define a continuous function g^u by

$$g^\mu(a_0(t),t) = ct + W_1\|\mu\|^k ,$$

$$g^\mu(a_1(t),t) = ct .$$

Define the stopping time $T = \inf_{t>t'}\{t: X_t \text{ hits the boundary}\}$. Note that T is the exit time from the continuation region. The conditional risk R^B in (3) can then be considered as the expectation

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* Later in the chapter we make further restrictions on the boundary functions.

** Thanks to Dr. Vaclav Fabian for pointing out the application of Doob's work to this result to me.

of $g^{\mu}(X_T, T)$ with initial position $X_{t'} = x'$ at time t'.

The conditional distribution of $\{X_t : t \geq t'\}$, given $X_{t'} = x'$, is the same as that of $\{x' + \mu(t - t') + \frac{1}{\sqrt{2}} \xi_{t-t'};$ $t \geq t'\}$, where ξ_t is Brownian motion with covariance matrix $2tI_n$. We may thus consider ξ_t instead of $X_{t'}$.

In terms of ξ_t, the stopping time T is the first time

$$(x' + \mu(t-t') + \frac{1}{\sqrt{2}} \xi_{t-t'}, t) \in \bigcup_{i=1,2} (a_i(t), t);\ t \geq t'.$$

The stopping time T is equivalently determined by the first time

$$(\sqrt{2}\ (x'-\mu t') + \xi_{t-t'}, t) \in \bigcup_{i=1,2} (\sqrt{2}\ (a_i(t) - \mu t), t);\ t \geq t',$$

which in terms of ξ_t can be considered the exit time from the continuation region defined by the boundary $(\sqrt{2}\ (a_i(t) - \mu t), t)$ $(i = 0,1)$ with initial position $\sqrt{2}(x' - \mu t')$ at time t'.

Letting $s' = -t'$ and $\tau = t-t'$, the stopping time is equivalently determined by the first time

$$(\sqrt{2}\ (x' + \mu s') + \xi_\tau : s'-\tau) \in \bigcup_{i=1,2} (\sqrt{2}(a_i(\tau - s') - \mu(\tau - s')), s'-\tau).$$

Thus we are observing a trajectory process in the sense of [7, p. 256]. Let $\mathcal{T} = T - t'$, and define

$$h^{\mu}(\sqrt{2}(a_i(\mathcal{T}-s') - \mu(\mathcal{T}-s')),\ s'-\mathcal{T}) = g^{\mu}(a_i(T), T)\ .$$

If $\eta(x,s)$ is the conditional expectation of h^{μ} given the initial value $\xi_s = x$ at the initial time s, then under assumption (A) the conditions of Doob's theorem [7, Theorem 2.1] are satisfied, and we get

$$\frac{\partial}{\partial s}\eta = \Sigma \frac{\partial^2}{\partial x_i^2}\eta .$$

The theorem now follows since for each fixed μ

$$R^B(x,t,\mu) = \eta(\sqrt{2}(x + \mu s),s)$$

where $s = -t$.

The proof for the case $\mu = 0$ is similar, the only difference being in the definition of g^μ.

Now we define the transformed risk

$$H(x,t) = \int R(x,t,\mu)e^{\Sigma\mu_i x_i - \frac{t}{2}\Sigma\mu_i^2}\, d\rho_{0,0}(\mu) .$$

On D^1, H is given by

$$H^1(x,t) = (ct+1) + U(x,t)\int ct\Phi(\mu,\tfrac{x}{t},\tfrac{1}{t})d\mu = ct(1 + U(x,t)) + 1 .$$

On D^0

$$H^0(x,t) = ct + U(x,t)\int(ct + W_1\|\mu\|^k)\Phi(\mu,\tfrac{x}{t},\tfrac{1}{t})d\mu$$

$$= ct + U(x,t)\left[ct + W_1\sum_{j=0}^{\infty}\frac{(\frac{\Sigma x_i^2}{2t})^j e^{-\frac{\Sigma x_i^2}{2t}}}{j!}\left(\frac{2}{t}\right)^{\frac{k}{2}}\frac{\Gamma(j + \frac{n+k}{2})}{\Gamma(j + \frac{n}{2})}\right]$$

$$= ct(1 + U(x,t)) + W_1U(x,t)\sum_{j=0}^{\infty}\frac{(\frac{\Sigma x_i^2}{2t})^j e^{-\frac{\Sigma x_i^2}{2t}}}{j!}$$

$$\left(\frac{2}{t}\right)^{\frac{k}{2}}\frac{\Gamma(j + \frac{n+k}{2})}{\Gamma(j + \frac{n}{2})}$$

$$= ct(1 + U(x,t)) + F^0(x,t) .$$

Furthermore, for nonnegative integer m

$$\int \|\mu\|^{2m} \Phi(\mu,\frac{x}{t},\frac{1}{t})\,d\mu = \sum_{L=0}^{m} c_n(L,m)\,\frac{(\Sigma x_i^2)^L}{t^{L+m}} ,$$

with

$$c_n(L,m) = \binom{m}{L} \prod_{j=1}^{m-L} (n+2m-2j) \quad \text{for} \quad L < m: \; c_n(m,m) = 1 .$$

For $\Sigma x_i^2 = 0$, the sum in H^0 is

$$\left(\frac{2}{t}\right)^{\frac{k}{2}} \frac{\Gamma(\frac{n+k}{2})}{\Gamma(\frac{n}{2})} ,$$

so that the smallest value of t for which $H^0(x,t)$ could equal $H^1(x,t)$, namely that value of t for which $H^0(0,t) = H^1(0,t)$, call it t_z, is

$$t_z = 2\left(W_1 \frac{\Gamma(\frac{n+k}{2})}{\Gamma(\frac{n}{2})}\right)^{\frac{2}{k+n}} \pi^{\frac{n}{k+n}} > 0 .$$

We now investigate the function $H(x,t)$ on the region B, which we call $H^B(x,t)$. From (3), we get $R^B(x,t,\mu)$ has continuous second partial derivatives in x_i and t and satisfies the partial differential equation

$$R_t^B(x,t,\mu) + \Sigma\mu_i R_i^B(x,t,\mu) + \tfrac{1}{2}\Sigma R_{ii}^B(x,t,\mu) = 0 \quad \text{for each fixed } \mu.$$

Since $S(x,t,\mu) = \exp[\Sigma x_i\mu_i - \frac{(\Sigma\mu_i^2)}{2} t]$ satisfies the equation

$$S_t(x,t,\mu) + \tfrac{1}{2}\Sigma S_{ii}(x,t,\mu) = (-(\Sigma\mu_i^2)/2)S(x,t,\mu) + \tfrac{1}{2}(\Sigma\mu_i^2)S(x,t,\mu) = 0$$

for each μ,

we get that $T(x,t,\mu) = R^B(x,t,\mu)S(x,t,\mu)$ satisfies the equation

$$T_t(x,t,\mu) + \tfrac{1}{2}\Sigma T_{ii}(x,t,\mu) = 0 \quad \text{for each } \mu,$$

in view of the following calculations.

$$T_t(x,t,\mu) + \tfrac{1}{2}\Sigma T_{ii}(x,t,\mu) = (R_t^B S + R^B S_t) + \tfrac{1}{2}(\Sigma R_{ii}^B S + 2\Sigma R_i^B S_i +$$

$$\Sigma R^B S_{ii}) = (R_t^B + \Sigma\mu_i R_i^B + \tfrac{1}{2}\Sigma R_{ii}^B)S + R^B(S_t + \tfrac{1}{2}\Sigma S_{ii}) = 0 ,$$

since $S_i = \mu_i S$. It should be noted that (3) was used in the above derivation.

Now

$$H^B(x,t) = R^B(x,t,0) + \int T(x,t,\mu)d\mu = R^B(x,t,0) + H'^B(x,t).$$

From now on we restrict the consideration of boundaries to those for which

$$E[T|x,t,\mu] < t + m \text{ where } m \text{ is finite}^*. \qquad \text{(B)}$$

For such boundaries we shall show that $H^B(x,t)$ satisfies a differential equation. Let $L = \frac{\partial}{\partial t} + \frac{1}{2}\Sigma \frac{\partial^2}{\partial x_i^2}$ be a differential operator and $C_c^\infty(B)$ be the space of infinitely differentiable functions with compact support contained in B, and endowed with the topology of Schwartz [8]. We observe that H'^B is bounded on compact sets. Indeed,

$$H'^B(x,t) < \int(c(t + m) + W_1\|\mu\|^k)S(x,t,\mu)d\mu = U(x,t)c(t + m) + F^0(x,t)$$

where m as given in (B) is the bound on the expected continued sample time. Hence,

$$\Pi(\psi) = \iint H'^B(x,t)\psi(x,t)dxdt$$

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* This does not severely restrict the problem as can be seen from Sections 6 and 7.

exists for all $\psi \in C_c^\infty(B)$. Clearly, Π is linear and continuous in the topology of $C_c^\infty(B)$. This implies that Π is a distribution. Using the definition of the differential operator on the space of distributions as in ([9], p. 250) we shall prove the following theorem.

Theorem. Let L and Π be as above, then $L\Pi = 0$.

Proof. By ([9], Def. 23.4) we should show that $\Pi(L^*\psi) = 0$ for all $\psi \in C_c^\infty(B)$. Noting that ([9], p. 249) $L^*\psi \in C_c^\infty(B)$, we get

$$\Pi(L^*\psi) = \iint H'^B(x,t)L^*\psi(x,t)\,dxdt = \iint[\int T(x,t,\mu)\,d\mu]L^*\psi(x,t)\,dxdt,$$

and by Fubini's theorem we may write

$$\Pi(L^*\psi) = \int[\iint T(x,t,\mu)L^*\psi(x,t)\,dxdt]d\mu \ .$$

For each μ, $T(x,t,\mu)$ is locally integrable and hence by [9, p. 249, 250]

$$\Pi(L^*\psi) = \int[\iint LT(x,t,\mu)\psi(x,t)\,dxdt]d\mu = \iint[\int LT(x,t,\mu)\psi(x,t)\,d\mu]dxdt$$

since the transpose of $L^* = L$. But for each μ, $LT(x,t,\mu) = 0$ giving $\Pi(L^*\psi) \equiv 0$. Thus the theorem holds.

(C) Assume $H'^B_{n+1}(x,t)$ and $H'^B_{ii}(x,t)$ are continuous. Then we get that $H^B_{n+1}(x,t)$ and $H^B_{ii}(x,t)$ for $(x,t) \in B$ are continuous by the definition of H^B and property of $R(x,t,0)$. Under the assumptions (B) and (C) we get

Corollary 1. $H'^B_t(x,t) + \frac{1}{2}\Sigma\, H'^B_{ii}(x,t) = 0$. For $\mu = 0$ we get from (3) that $R^B(x,t,0)$ satisfies the equation $R^B_t(x,t,0) + \frac{1}{2}\Sigma\, R^B_{ii}(x,t,0) = 0$. Hence by definition of $H^B(x,t)$ we get

Corollary 2. $H^B_t(x,t) + \frac{1}{2}\Sigma\, H^B_{ii}(x,t) = 0$.

Remark. In [2], it is claimed that the analogue of $H^B(x,t)$ for his procedure and the problems satisfies the equation of Corollary 2. However the conditions for the validity of the equation are not stated precisely.

The procedures for which the analogue of Corollary 2 is valid for [2] are not clear to this author. In the following example we show that there do exist procedures for which assumptions of our Corollary 2 are valid.

We now demonstrate that for a sequential probability ratio test (SPRT), in fact $H'^B(x,t)$ does have continuous derivatives of the first order in t and of the second order in x, so that on B,

$$H_2'^B(x,t) + \tfrac{1}{2} H_{11}'^B(x,t) = 0 .$$

Let $a_0 < 0 < a_1$ and $\frac{M}{2}$ = slope define a SPRT for testing $H_0\colon \mu = 0$ vs. $H_1\colon \mu = M > 0$ with cost of sampling c per unit time and loss of acceptance (loss of hitting the lower boundary) given by $W_1|\mu|^k$. Let $R^B(x,t,\mu)$ denote the conditional risk of starting observation at the point (x,t), having incurred cost ct at the onset of observation, and continuing observation of the process with drift μ until the boundary is contacted, with additional loss of $W_1|\mu|^k$ if the lower boundary is contacted.

Using the notation:

$b_i(t) = a_i + \frac{M}{2} t,\ i = 0,1$,

$W(x,t,\mu) = (M - 2\mu)(b_1 - x)$ and $L(x,t,\mu) = (M - 2\mu)(b_0 - x)$,

$h = a_1 - a_0 = b_1 - b_0$,

then on $[t > 0,\ b_0 < x < b_1] = B$, and $-\infty < \mu < \infty$,

$$R^B(x,t,\mu) = ct + [e^W - e^L]^{-1}[e^W - 1]W_1|\mu|^k + c(2(W(e^L - 1) +$$

$$L(1 - e^W))/[(e^W - e^L)(M - 2\mu)^2]) = ct + V(x,t,\mu)W_1|\mu|^k + cQ(x,t,\mu),$$

where V corresponds to the probability of hitting the lower boundary, and Q corresponds to the expected continued sampling time. This calculation is an application of the results of Section 3.11 of Lehmann [4]. ($R^B(x,t,\frac{M}{2})$ is a limit.) Now on B

$$R_1^B = [e^W - e^L]^{-1}[-W_1|\mu|^k(M - 2\mu)] + [\frac{2c}{M - 2\mu} - \frac{2ch}{(e^W - e^L)}] ,$$

$$R_{11}^B = [e^W - e^L]^{-1}[-W_1|\mu|^k(M - 2\mu)^2 - 2ch(M - 2\mu)] ,$$

$$R_2^B = c + [e^W - e^L]^{-1}[W_1|\mu|^k \frac{M}{2} (M - 2\mu) + \frac{2c}{M - 2\mu} (\frac{M}{2} (e^L - e^W) + \frac{hM}{2} (M - 2\mu))],$$

so that using the fact that

$$\frac{M}{2} (M - 2\mu) - \mu(M - 2\mu) - \frac{(M - 2\mu)^2}{2} = 0 ,$$

it is easy to see that

$$R_2^B + \frac{1}{2} R_{11}^B + \mu R_1^B = 0 .$$

Before proceding, we show that $Q(x,t,\mu) < m$, a constant independent of x,t, and μ, by extending the definition to B closure,

$$Q(b_0(t),t,\mu) = Q(b_1(t),t,\mu) = 0 ,$$

and utilizing the facts

a) $Q(x + \frac{M}{2}\tau,\ t + \tau,\mu) = Q(x,t,\mu)$

b) $\sup_{\substack{b_0 \le x \le b_1 \\ -\infty < \mu < \infty}} Q(x,t,\mu) < m$ for fixed t .

The proof of b) follows from the fact that if $\mu < \frac{M}{2}$, then $e^W \geq 1$ and $e^L \leq 1$, so

$$Q(x,t,\mu) = 2[(M - 2\mu)(e^W - e^L)]^{-1}[(b_1 - x)(e^L - 1) + (b_0 - x)(1 - e^W)] \leq 2h[M - 2\mu]^{-1}$$

and so there exists a k_ϵ^+ such that for $\mu > k_\epsilon^+$,

$$\sup_{\substack{b_0 \leq x \leq b_1 \\ \mu > k_\epsilon^+}} Q(x,t,\mu) < \epsilon .$$

Similarly k_ϵ^- may be chosen, and letting $k_\epsilon = \max [k_\epsilon^+, |k_\epsilon^-|]$,

$$\sup_{\substack{b_0 \leq x \leq b_1 \\ |\mu| > k_\epsilon}} Q(x,t,\mu) < \epsilon ,$$

a fact that is conceptually evident.

But $Q(x,t,\mu)$ is continuous $(Q(x,t,\frac{M}{2}) = (b_1 - x)(b_0 - x))$ in x,t, and μ, so that on the compact set $[b_0 \leq x \leq b_1, |\mu| \leq k]$ it is bounded. Thus Q satisfies b), and indeed, if m is the bound on Q,

$$H'^B(x,t) = \int R^B(x,t,\mu)S(x,t,\mu)\,d\mu \leq \int (c(t + m) + W_1|\mu|^k)S(x,t,\mu)\,d\mu$$
$$= U(x,t)c(t + m) + F^0(x,t) .$$

Now for arbitrary $(x',t') \in B$, we let

$$\bar{R} = [t_0 \leq t \leq t_1, b_0'(t) = a_0' + \frac{M}{2} t \leq x \leq b_1'(t) = a_1' + \frac{M}{2} t,\ a_0 < a_0' < a_1' < a_1]$$

define a rectangle containing (x',t') and contained in B, and show that

$$\sup_{(x,t) \in \bar{R}} |T_2(x,t,\mu)| \leq Y(\mu) \quad \text{integrable } d\mu .$$

Then 3^*, p. 126 of Loève [10] says

$$H_2'^{B}(x,t) = \int T_2(x,t,\mu)\,d\mu \qquad (x,t) \in \bar{R}. \tag{4}$$

Now

$$|T_2| = |(R_2^B - \mu^2 R^B)S| < [|R_2| + \mu^2|R^B|]S \ ,$$

and

$$R_2^B(x,t,\mu) = c + W_1|\mu|^k V_2(x,t,\mu) + \alpha Q_2(x,t,\mu) \ .$$

But

$$|V_2(x,t,\mu)| = |[e^W - e^L]^{-1} \tfrac{M}{2}[M - 2\mu]|$$

and

$$|Q_2(x,t,\mu)| = |2[(M - 2\mu)(e^W - e^L)]^{-1}(\tfrac{M}{2}(e^L - e^W) + \tfrac{hM}{2}(M - 2\mu))|$$

satisfy the conditions a) and b), where the sup in condition b) reads $b_0' \le x \le b_1'$. Thus on $\bar{R}$

$$|R_2^B| \le c(1 + m) + W_1|\mu|^k\, m.$$

On $\bar{R}$,

$$S(x,t,\mu) = U(x,t)\Phi(\mu,\tfrac{x}{t},\tfrac{1}{t}) \le U(b_1'(t_1),t_0)g(\mu)$$

where

$$g(\mu) = \begin{cases} (t_1/2\pi)^{\frac{1}{2}}\exp[\frac{t_0}{2}(\mu - (b_0'(t_0)/t_1))^2] & \mu < b_0'(t_0)/t_1 \\ (t_1/2\pi)^{\frac{1}{2}} & b_0'(t_0)/t_1 < \mu < b_1'(t_1)/t_0 \\ (t_1/2\pi)^{\frac{1}{2}}\exp[\frac{t_0}{2}(\mu - (b_1'(t_1)/t_0))^2] & \mu > b_1'(t_1)/t_0 \ . \end{cases}$$

Although g is not a probability measure, it has the property that all of its "moments" exist. Thus

$$|T_2| \le U(b_1'(t_1),t_0)(|R_2^B| + \mu^2|R^B|)g(\mu) \quad \text{integrable} \quad d\mu,$$

and so (4) follows. The corresponding proof that $H_{11}'^B$ exists is similar. The continuity of the derivatives is clear.

We now return to the problems at hand. The remainder of the paper will concern itself with numerical approximations to the optimal boundaries for problems characterized by the triple (c,k,W_1).

From this point on, it will be assumed that the boundaries are such that

$$LH'^B = 0,$$

so that

$$LH^B = H_t^B(x,t) + \tfrac{1}{2}\Sigma H_{ii}^B(x,t) = 0 \quad \text{on} \quad B.$$

We also note that $F^1(x,t) = 1$ and $F^0(x,t)$ satisfy the condition $LF^i = 0$, $i = 0,1$, since

$$F^0(x,t) = \int W_1\|\mu\|^k S(x,t,\mu)\,d\mu = \int S(x,t,\mu)\,d\Lambda(\mu) \quad ,$$

and the integrand satisfies the condition $LS = 0$.

Note that $\hat{R}(x,t)$, the untransformed risk, is given by

$$\hat{R}(x,t) = \int R(x,t,\mu)\,d\rho_{x,t}(\mu)$$

$$= \frac{R(x,t,0)}{(1+U(x,t))} + \frac{U(x,t)}{(1+U(x,t))}\int R(x,t,\mu)\,\phi(\mu,\tfrac{x}{t},\tfrac{1}{t})\,d\mu$$

$$= \frac{1}{(1+U(x,t))}\left[R(x,t,0) + \int R(x,t,\mu)e^{\Sigma\mu_i x_i - \frac{t}{2}\Sigma\mu_i^2}d\mu\right]$$

$$= \frac{H(x,t)}{(1+U(x,t))} \quad .$$

Further, letting $r(x_1,\ldots,x_n) = \|x\| = (\Sigma x_i^2)^{\frac{1}{2}}$, $U(r,t) = (\frac{2\pi}{t})^{\frac{n}{2}} e^{\frac{r^2}{2t}}$, and $G(r,t) = ct(1 + U(r,t))$, we have

$$H^1(r,t) = G(r,t) + 1 = G(r,t) + F^1(r,t),$$

$$H^0(r,t) = G(r,t) + W_1U(r,t) \sum_{j=0}^{\infty} \frac{(\frac{r^2}{2t})^j e^{-\frac{r^2}{2t}}}{j!} (\frac{2}{t})^{\frac{k}{2}} \frac{\Gamma(j + \frac{n+k}{2})}{\Gamma(j + \frac{n}{2})}$$

$$= G(r,t) + F^0(r,t) ,$$

$$H_t^B(r,t) + \frac{1}{2} H_{rr}^B(r,t) + (\frac{n-1}{2r})H_r^B(r,t) = 0 ,$$

and $F^0(r,t)$ and $F^1(r,t)$ satisfy the same differential equation.

The last equation follows from

$$H_t^B + \frac{1}{2} \Sigma H_{ii}^B = 0 = H_t^B + \frac{1}{2} \Sigma[H_{rr}^B x_i^2 (\Sigma x_i^2)^{-1} + H_r^B((\Sigma x_i^2)^{-\frac{1}{2}} - x_i^2(\Sigma x_i^2)^{-\frac{3}{2}})]$$

$$= H_t^B + \frac{1}{2} [H_{rr}^B + H_r^B(\Sigma(\frac{1}{r} - \frac{x_i^2}{r^3}))] = H_t^B + \frac{1}{2} H_{rr}^B + (\frac{n-1}{2r})H_r^B .$$

The risk function $H(r,t)$ for an arbitrary procedure $(a_0(t),a_1(t),T_1,T_0)$ is given by

$$H(r,t) = \begin{cases} H^1(r,t) & \text{on } D^1 \\ H^0(r,t) & \text{on } D^0 \\ H^B(r,t) & \text{on } B . \end{cases}$$

$H(r,t)$ is continuous, of course. Now the general properties of the problem have been displayed.

Let $(\bar{a}_0(t),\bar{a}_1(t), \bar{T}_1,\bar{T}_0)$ denote the optimal procedure, depending of course on the triple (c,W_1,k) and the dimension n. The risk function $H(r,t)$ will satisfy the following free boundary condition.

$$\lim_{\substack{r\in B \\ r\to \bar{a}_i(t)}} H_r(r,t) = \lim_{\substack{r\in D^i \\ r\to \bar{a}_i(t)}} H_r(r,t) \qquad i = 0,1$$

That is: the r-derivatives "match up" on the boundaries.

The free boundary condition is determined as by Chernoff in [2]. The posterior distribution has nothing to do with the proof. We now essentially repeat his hueristic proof. Clarifying remarks are given in [2].

$a_0(t)$ and $a_1(t)$ defined on $[t_0,\infty)$ determine the transformed Bayes risk $H(r,t)$ for all (r,s) with $s > t_0$. It is desired to extend $a_i(t)$ backwards to uniformly minimize

$$H(t) = \int_0^{a_0} H^0(r,t)\,dr + \int_{a_0}^{a_1} H^B(r,t)\,dr + \int_{a_1}^{\infty} H^1(r,t)\,dr$$

for $t < t_0$. On the boundary $H^i = H^B$, so

$$\frac{dH}{dt} = \int_0^{a_0} H_t^0\,dr + \int_{a_0}^{a_1} H_t^B\,dr + \int_{a_1}^{\infty} H_t^1\,dr \quad .$$

If

$$H_t^B(a_i,t) \neq H_t^i(a_i,t)$$

an increase in a_0 if $H_t^0 > H_t^B$, and similar adjustments for other possibilities, would increase $\frac{dH}{dt}$ and thereby decrease H for $t < t_0$.

If the optimal boundary has finite slope, we must then have on the boundary

$$H_t^i = H_t^B \quad .$$

Differentiating $H^i(\bar{a}_i(t),t) = H^B(\bar{a}_i(t),t)$ along the boundary, we get

$$H_r^i \frac{d\bar{a}_i}{dt} + H_t^i = H_r^B \frac{d\bar{a}_i}{dt} + H_t^B \quad .$$

So

$$H_r^i(r,t) = H_r^B(r,t)$$

along the boundary.

Since t_z is the smallest value of t for which we could possibly stop and accept

$$\bar{T}_1 < t_z < \bar{T}_0 \quad .$$

For t small,

$$\rho_{x,t}(0) \le \rho_{0,t}(0) = [1 + (\frac{2\pi}{t})^{\frac{n}{2}}]^{-1}$$

is small, which would lead one to believe $\bar{T}_1$ strictly greater than zero. This can be shown easily for sufficiently large values of c. That is,

$$\bar{T}_1 > t_z - \frac{1}{c}$$

for otherwise the sampling cost of getting to the acceptance region is larger than type I error = 1.

However, the author has not shown in general that $\bar{T}_1 > 0$. More will be said about this in Sections 6 and 7.

SECTION 2

The method we propose to use to approximate $\bar{a}_0(t)$ and $\bar{a}_1(t)$ is the following. For large fixed t, we approximate $\bar{a}_0(t)$ and $\bar{a}_1(t)$, using the known functions H^0 and H^1 and a polynomial function to approximate H^B. Then finite difference techniques are used to "fill in" the boundaries backwards in time. This section deals with approximating $\bar{a}_0(t)$ and $\bar{a}_1(t)$ for large t.

Bars in general will denote optimal quantities, and primes will denote approximations to optimal quantities. Also, an arbitrary procedure will be condensed to $(a_0(t),a_1(t))$: T_1 and T_0 being understood to be the values of t for which $a_1(t)$ and $a_0(t)$ equal 0.

We note that the notation $H(r,t)$ hides the procedure $(a_0(t),a_1(t))$. Also, the risk function associated with a procedure depends on the functions $a_0(t)$ and $a_1(t)$, and not just on the values of the function at a fixed value of t. That is, let $HA(r,t)$ be the risk function corresponding to the procedure $(a_0(t),a_1(t))$ and $HB(r,t)$ be the risk function corresponding to the procedure $(b_0(t),b_1(t))$. Then $a_0(t_0) = b_0(t_0)$ and $a_1(t_0) = b_1(t_0)$ by no means implies $HA^B(r,t_0) = HB^B(r,t_0)$.

Let $\bar{H}(r,t)$ denote the optimal risk function: ie, the risk function associated with the optimal procedure $(\bar{a}_0(t),\bar{a}_1(t))$. Throughout this section $i = 0,1$.

Suppose the optimal procedure were known. Then differentiating the equations $\overline{H}^B(\bar{a}_i(t),t) = \overline{H}^i(\bar{a}_i(t),t)$ and recalling the free boundary condition

$$\overline{H}_1^B(\bar{a}_i(t),t) = \overline{H}_1^i(\bar{a}_i(t),t) \tag{1}$$

we have

$$\overline{H}_2^B(\bar{a}_i(t),t) = \overline{H}_2^i(\bar{a}_i(t),t) , \tag{2}$$

and differentiating equations (1) and (2), we get

$$\overline{H}_{11}^B(\bar{a}_i(t),t)\frac{d\bar{a}_i}{dt} + \overline{H}_{12}^B(\bar{a}_i(t),t) = \overline{H}_{11}^i(\bar{a}_i(t),t)\frac{d\bar{a}_i}{dt} + \overline{H}_{12}^i(\bar{a}_i(t),t),$$

$$\overline{H}_{12}^B(\bar{a}_i(t),t)\frac{d\bar{a}_i}{dt} + \overline{H}_{22}^B(\bar{a}_i(t),t) = \overline{H}_{12}^i(\bar{a}_i(t),t)\frac{d\bar{a}_i}{dt} + \overline{H}_{22}^i(\bar{a}_i(t),t),$$

from which it follows

$$(\overline{H}_{11}^B(\bar{a}_i(t),t) - \overline{H}_{11}^i(\bar{a}_i(t),t))\ (\overline{H}_{22}^B(\bar{a}_i(t),t) - \overline{H}_{22}^i(\bar{a}_i(t),t)) =$$
$$(\overline{H}_{12}^B(\bar{a}_i(t),t) - \overline{H}_{12}^i(\bar{a}_i(t),t))^2 .$$

The expression $\overline{H}_{11}^B - \overline{H}_{11}^i$ may be reduced.

$$\overline{H}_{11}^B(\bar{a}_i(t),t) - \overline{H}_{11}^i(\bar{a}_i(t),t) = -2\overline{H}_2^B(\bar{a}_i(t),t) - \frac{n-1}{\bar{a}_i(t)}\overline{H}_1^B(\bar{a}_i(t),t)$$

$$- \overline{H}_{11}^i(\bar{a}_i(t),t) = -2\overline{H}_2^i(\bar{a}_i(t),t) - \frac{n-1}{\bar{a}_i(t)}\overline{H}_1^i(\bar{a}_i(t),t) - \overline{H}_{11}^i(\bar{a}_i(t),t)$$

$$= -2[G_2(\bar{a}_i(t),t) + (\frac{n-1}{2\bar{a}_i(t)})G_1(\bar{a}_i(t),t) + \frac{1}{2}G_{11}(\bar{a}_i(t),t)]$$

$$- 2[F_2^i(\bar{a}_i(t),t) + (\frac{n-1}{2\bar{a}_i(t)})F_1^i(\bar{a}_i(t),t) + \frac{1}{2}F_{11}^i(\bar{a}_i(t),t)] .$$

The second expression in brackets is zero, while the first term is

$$-2[c(1+U(\bar{a}_i(t),t))] - 2ct[U_2(\bar{a}_i(t),t) + (\frac{n-1}{2\bar{a}_i(t)})U_1(\bar{a}_i(t),t)$$

$$+ \frac{1}{2} U_{11}(\bar{a}_i(t),t)] .$$

The second term in this expression is also zero. Hence

$$\vec{H}^B_{11}(\bar{a}_i(t),t) - \vec{H}^i_{11}(\bar{a}_i(t),t) = -2c(1 + U(\bar{a}_i(t),t)) .$$

However, $\vec{H}^B_{12}(\bar{a}_i(t),t)$ and $\vec{H}^B_{22}(\bar{a}_i(t),t)$ are unknown (even assuming $\bar{a}_i(t)$ known).

The following set of calculations holds for an arbitrary procedure, so the bar will be dropped.

Differentiating the equation

$$H^B_t(r,t) + (\frac{n-1}{2r}) H^B_r(r,t) + \frac{1}{2} H^B_{rr}(r,t) = 0$$

twice with respect to r and once with respect to t, we have

$$H^B_{rt} + (\frac{n-1}{2r}) H^B_{rr} - (\frac{n-1}{2r^2}) H^B_r + \frac{1}{2} H^B_{rrr} = 0 ,$$

$$H^B_{rrt} + (\frac{n-1}{2r}) H^B_{rrr} - 2(\frac{n-1}{2r^2}) H^B_{rr} + 2(\frac{n-1}{2r^3}) H^B_r + \frac{1}{2} H^B_{rrrr} = 0,$$

$$H^B_{tt} + (\frac{n-1}{2r}) H^B_{rt} + \frac{1}{2} H^B_{rrt} = 0 ,$$

from which it follows

$$H^B_{rt}(r,t) = -\frac{1}{2} H^B_{rrr}(r,t) - (\frac{n-1}{2r}) H^B_{rr}(r,t) + (\frac{n-1}{2r^2}) H^B_r(r,t) ,$$

$$H^B_{tt}(r,t) = \frac{1}{4} H^B_{rrrr}(r,t) + (\frac{n-1}{2r}) H^B_{rrr}(r,t) ,$$

$$+ [(\frac{n-1}{2})^2 - (\frac{n-1}{2})](\frac{1}{r^2}) H^B_{rr}(r,t) + [(\frac{n-1}{2}) - (\frac{n-1}{2})^2](\frac{1}{r^3}) H^B_r(r,t) .$$

We now consider t fixed and large, say T, assume $(\bar{a}_0(t),\bar{a}_1(t))$ known, approximate $\vec{H}^B(r,T)$ by a 5^{th} degree polynomial,

compute its third and fourth derivatives, and substitute them for $\vec{H}^B_{111}(\bar{a}_i(T),T)$ and $\vec{H}^B_{1111}(\bar{a}_i(T),T)$. This in turn enables us to approximate $\vec{H}^B_{12}(\bar{a}_i(T),T)$ and $\vec{H}^B_{22}(\bar{a}_i(T),T)$.

We find a 5^{th} degree polynomial $f_T(r)$ subject to the following conditions.

$$f_T(\bar{a}_i(T)) = \vec{H}^B(\bar{a}_i(T),T) \;,$$

$$f'_T(\bar{a}_i(T)) = \vec{H}^B_1(\bar{a}_i(T),T) \;,$$

$$f''_T(\bar{a}_i(T)) = \vec{H}^B_{11}(\bar{a}_i(T),T) \;.$$

Suppose polynomials $P_0(r)$, $P_1(r)$, $P_2(r)$, $Q_0(r)$, $Q_1(r)$, and $Q_2(r)$ satisfy the following conditions.

$$P_0(\bar{a}_0(T)) = 1,\; P_0(\bar{a}_1(T)) = P'_0(\bar{a}_0(T)) = P'_0(\bar{a}_1(T)) = P''_0(\bar{a}_0(T)) = P''_0(\bar{a}_1(T)) = 0,$$

$$P'_1(\bar{a}_0(T)) = 1,\; P_1(\bar{a}_0(T)) = P_1(\bar{a}_1(T)) = P'_1(\bar{a}_1(T)) = P''_1(\bar{a}_0(T)) = P''_1(\bar{a}_1(T)) = 0,$$

$$P''_2(\bar{a}_0(T)) = 1,\; P_2(\bar{a}_0(T)) = P_2(\bar{a}_1(T)) = P'_2(\bar{a}_0(T)) = P'_2(\bar{a}_1(T)) = P''_2(\bar{a}_1(T)) = 0,$$

$$Q_0(\bar{a}_1(T)) = 1,\; Q_0(\bar{a}_0(T)) = Q'_0(\bar{a}_0(T)) = Q'_0(\bar{a}_1(T)) = Q''_0(\bar{a}_0(T)) = Q''_0(\bar{a}_1(T)) = 0,$$

and similarly for Q_1 and Q_2.

Then $f_T(r) = \vec{H}^0(\bar{a}_0(T),T)P_0(r) + \vec{H}^0_1(\bar{a}_0(T),T)P_1(r) + [-2c(1 + U(\bar{a}_0(T),T)) + \vec{H}^0_{11}(\bar{a}_0(T),T)]\, P_2(r) + \vec{H}^1(\bar{a}_1(T),T)\, Q_0(r) + \vec{H}^1_1(\bar{a}_1(T),T)\, Q_1(r) + [-2c(1 + U(\bar{a}_1(T),T)) + \vec{H}^1_{11}(\bar{a}_1(T),T)]\, Q_2(r)$ will satisfy the above conditions.

A solution in terms of P_0 thru Q_2 is the following.
Let $\lambda = \bar{a}_1(T) - \bar{a}_0(T)$, $s(r) = r - \bar{a}_0(T)$, and $v(r) = \bar{a}_1(T) - r$.

$$P_0(r) = \frac{(v(r))^3}{\lambda^3}[1 + 3\frac{s(r)}{\lambda} + 6\frac{(s(r))^2}{\lambda^2}],$$

$$P_1(r) = \frac{(v(r))^3}{\lambda^3}[s(r) + 3\frac{(s(r))^2}{\lambda}],$$

$$P_2(r) = \frac{(v(r))^3}{\lambda^3}\frac{(s(r))^2}{2},$$

$$Q_0(r) = \frac{(s(r))^3}{\lambda^3}[1 + 3\frac{v(r)}{\lambda} + 6\frac{(v(r))^2}{\lambda^2}],$$

$$Q_1(r) = \frac{(s(r))^3}{\lambda^3}[-v(r) - 3\frac{(v(r))^2}{\lambda}],$$

$$Q_2(r) = \frac{(s(r))^3}{\lambda^3}\frac{(v(r))^2}{2}.$$

We now compute the third and fourth derivatives and evaluate at $\bar{a}_i(T)$.

$$f_T'''(\bar{a}_0(T)) = \frac{60}{\lambda^3}(\vec{H}^B(\bar{a}_1(T),T) - \vec{H}^B(\bar{a}_0(T),T)) + \frac{1}{\lambda^2}[-36\vec{H}_1^B(\bar{a}_0(T),T)$$
$$-24\vec{H}_1^B(\bar{a}_1(T),T)] + \frac{1}{\lambda}[-9\vec{H}_{11}^B(\bar{a}_0(T),T) + 3\vec{H}_{11}^B(\bar{a}_1(T),T)],$$

$$f_T'''(\bar{a}_1(T)) = \frac{60}{\lambda^3}(\vec{H}^B(\bar{a}_1(T),T) - \vec{H}^B(\bar{a}_0(T),T)) + \frac{1}{\lambda^2}[-24\vec{H}_1^B(\bar{a}_0(T),T)$$
$$-36\vec{H}_1^B(\bar{a}_1(T),T)] + \frac{1}{\lambda}[-3\vec{H}_{11}^B(\bar{a}_0(T),T) + 9\vec{H}_{11}^B(\bar{a}_1(T),T)],$$

$$f_T''''(\bar{a}_0(T)) = -\frac{360}{\lambda^4}(\vec{H}^B(\bar{a}_1(T),T) - \vec{H}^B(\bar{a}_0(T),T)) + \frac{1}{\lambda^3}[192\,\vec{H}_1^B(\bar{a}_0(T),T)$$
$$+ 168\,\vec{H}_1^B(\bar{a}_1(T),T)] + \frac{1}{\lambda^2}[36\,\vec{H}_{11}^B(\bar{a}_0(T),T) - 24\,\vec{H}_{11}^B(\bar{a}_1(T),T)],$$

$$f_T''''(\bar{a}_1(T)) = \frac{360}{\lambda^4}(\vec{H}^B(\bar{a}_1(T),T) - \vec{H}^B(\bar{a}_0(T),T)) + \frac{1}{\lambda^3}[-168\,\vec{H}_1^B(\bar{a}_0(T),T)$$
$$-192\,\vec{H}_1^B(\bar{a}_1(T),T)] + \frac{1}{\lambda^2}[-24\,\vec{H}_{11}^B(\bar{a}_0(T),T) + 36\,\vec{H}_{11}^B(\bar{a}_1(T),T)].$$

In view of the foregoing calculations, for large fixed t, say T, we find two constants a_0' and a_1', which simultaneously satisfy the equations

$$[-2c(1 + U(a_1',T)][H_{22}^b(a_i',T) - H_{22}^i(a_i',T)] = [H_{12}^b(a_i',T) - H_{12}^i(a_i',T)]^2 ,$$

where

$$H_{12}^b(a_i',T) = -\frac{1}{2} f_T'''(a_i') - \left(\frac{n-1}{2a_i'}\right)H_{11}^b(a_i',T) + \left(\frac{n-1}{2(a_i')^2}\right)H_1^b(a_i',T),$$

$$H_{22}^b(a_i',T) = \frac{1}{4} f_T''''(a_i') + \left(\frac{n-1}{2a_i'}\right)f_T'''(a_i') + \left[\left(\frac{n-1}{2}\right)^2 - \left(\frac{n-1}{2}\right)\right]\left(\frac{1}{(a_i')^2}\right) ,$$

$$H_{11}^b(a_i',T) + \left[\left(\frac{n-1}{2}\right) - \left(\frac{n-1}{2}\right)^2\right]\left(\frac{1}{(a_i')^3}\right) H_1^b(a_i',T),$$

$$H^b(a_i',T) = H^i(a_i',T) ,$$

$$H_1^b(a_i',T) = H_1^i(a_i',T) ,$$

$$H_{11}^b(a_i',T) = -2c(1 + U(a_i',T)) + H_{11}^i(a_i',T),$$

and with $\lambda = a_1' - a_0'$,

$$f_T'''(a_0') = \frac{60}{\lambda^3} (H^b(a_i',T) - H^b(a_0',T)) + \frac{1}{\lambda^2}[-36\, H_1^b(a_0',T) - 24\, H_1^b(a_1',T)]$$

$+ \frac{1}{\lambda}[-9\, H_{11}^b(a_0',T) + 3\, H_{11}^b(a_1',T)]$, and similarly for $f_T'''(a_1')$, $f_T''''(a_0')$, and $f_T''''(a_1')$.

These values a_i' are approximations to $\bar{a}_i(T)$, and $f_T(r)$ is the approximation to $\bar{H}(r,T)$, $a_0' < r < a_1'$. Of course $H^1(r,T)$ serves as the approximation to $\bar{H}(r,T)$ and is exact for $r \geq \max(\bar{a}_1(T),a_1')$, while $H^0(r,T)$ serves as the approximation to $\bar{H}(r,T)$, and is exact for $r \leq \min(\bar{a}_0(T),a_0')$.

We note that just as $H(r,t)$ hides the procedure, the notation for the spanning polynomial $f_t(r)$ hides the endpoints and the function value and first two derivatives at the endpoints.

SECTION 3

Suppose now for large t, say T, we have $a_0'(T)$ approximating $\bar{a}_0(T)$, $a_1'(T)$ approximating $\bar{a}_1(T)$, and $f_T(r)$ approximating $\bar{H}(r,T)$, $a_0'(T) < r < a_1'(T)$. Now a mesh Δr, Δt is chosen and we consider the grid points (r_j,T) between $a_0'(T)$ and $a_1'(T)$. That is, we define integer

$$K(a,\Delta r) = [\frac{a}{\Delta r}]$$

where $[x]$ is the largest integer $\leq x$, and consider the points (r_j,T) with

$$r_j = j \cdot \Delta r \qquad K(a_0'(T),\Delta r) + 1 \leq j \leq K(a_1'(T),\Delta r) .$$

Δr will always be chosen so that the number of grid points, $K(a_1'(T),\Delta r) - K(a_0'(T),\Delta r) \geq 3$. At these grid points, we define

$$H^b(r_j,T) = f_T(r_j) \quad .$$

Throughout this section $i = 0,1$.

We now employ an iterative procedure, supposing at t we have $a_i'(t)$ as approximations to $\bar{a}_i(t)$, and $H^b(r_j,t)$ as approximations to $\bar{H}(r_j,t)$. First we find $a_i'(t - \Delta t)$, the approximations to $\bar{a}_i(t - \Delta t)$, then use Taylor series expansions to approximate $\bar{H}$ at grid points near the boundaries, and finally employ finite difference techniques to approximate $\bar{H}$ at grid points away from the boundaries.

From the equations

$$\overrightarrow{H}^B_{11}(\bar{a}_i(t),t)\ \frac{d\bar{a}_i}{dt} + \overrightarrow{H}^B_{12}(\bar{a}_i(t),t) = \overline{H}^i_{11}(\bar{a}_i(t),t)\ \frac{d\bar{a}_i}{dt} + \overline{H}^i_{12}(\bar{a}_i(t),t)$$

it follows that

$$\frac{d\bar{a}_i}{dt} = \frac{(\overrightarrow{H}^B_{12}(\bar{a}_i(t),t) - \overline{H}^i_{12}(\bar{a}_i(t),t))}{(\overline{H}^i_{11}(\bar{a}_i(t),t) - \overrightarrow{H}^B_{11}(\bar{a}_i(t),t))} = \frac{(\overrightarrow{H}^B_{12}(\bar{a}_i(t),t) - \overline{H}^i_{12}(\bar{a}_i(t),t))}{2c(1 + U(\bar{a}_i(t),t))} .$$

$\overrightarrow{H}^B_{12}$ is the only unknown function of $(\bar{a}_i(t),t)$.
However, since

$$\overrightarrow{H}^B_{12}(\bar{a}_i(t),t) = -\frac{1}{2}\overrightarrow{H}^B_{111}(\bar{a}_i(t),t) - (\frac{n-1}{2\bar{a}_i(t)})\overrightarrow{H}^B_{11}(\bar{a}_i(t),t)$$

$$+ (\frac{n-1}{2(\bar{a}_i(t))^2})\overrightarrow{H}^B_{1}(\bar{a}_i(t),t) ,$$

estimates of $\overrightarrow{H}^B_{111}(\bar{a}_i(t),t)$ would provide us with estimates of $\overrightarrow{H}^B_{12}(\bar{a}_i(t),t)$, and in turn of $\frac{d\bar{a}_i}{dt}$ (and $\bar{a}_i(t - \Delta t)$). Accordingly, letting $m(0) = K(a_0'(t),\Delta r) + 2$, and $m(1) = K(a_1'(t),\Delta r) - 1$ (or some such), we consider a Taylor series expansion of $\overrightarrow{H}^B(r_{m(i)},t)$ around $(\bar{a}_i(t),t)$, and define

$$H^b_{111}(a_i'(t),t) = (\frac{6}{(r_{m(i)} - a_i'(t))^3})[H^b(r_{m(i)},t) - H^i(a_i'(t),t) -$$

$$(r_{m(i)} - a_i'(t))H^i_1(a_i'(t),t) - (\frac{(r_{m(i)} - a_i'(t))^2}{2})(-2c(1 + U(a_i'(t),t))$$

$$+ H^i_{11}(a_i'(t),t))] ,$$

$$H^b_{12}(a_i'(t),t) = \frac{1}{2}H^b_{111}(a_i'(t),t) - (\frac{n-1}{2a_i'(t)})(-2c(1 + U(a_i'(t),t))$$

$$+ H^i_{11}(a_i'(t),t)) + (\frac{n-1}{2(a_i'(t))^2})\ H^i_1(a_i'(t),t) ,$$

and approximate $\bar{a}_i(t - \Delta t)$ by

$$a_i'(t - \Delta t) = a_i'(t) - \Delta t \, \frac{H^b_{12}(a_i'(t),t) - H^i_{12}(a_i'(t),t)}{2c(1 + U(a_i'(t),t))} .$$

It is assumed that $a_i'(t - \Delta t) \le a_i'(t)$. At the grid points close to the boundaries, we use a Taylor series expansion to approximate $\overline{H}$. At $j = K(a_1'(t - \Delta t),\Delta r)$ we let

$$H^b(r_j,t - \Delta t) = H^1(a_1'(t-\Delta t),t-\Delta t) - (a_1'(t-\Delta t) - r_j)\, H^1_1(a_1'(t-\Delta t),t-\Delta t)$$

$$+ \frac{(a_1'(t-\Delta t) - r_j)^2}{2} [-2c(1 + U(a_1'(t-\Delta t),t-\Delta t)) + H^1_{11}(a_1'(t-\Delta t),t-\Delta t)] -$$

$$\frac{(a_1'(t-\Delta t) - r_j)^3}{6} H^b_{111}(a_1'(t),t) ,$$

and at the points near the lower boundary, namely $K(a_0'(t-\Delta t),\Delta r) + 1 \le j \le K(a_0'(t),\Delta r) + 1$, we define

$$H^b(r_j,t-\Delta t) = H^0(a_0'(t-\Delta t),t-\Delta t) + (r_j - a_0'(t-\Delta t))\, H^0_1(a_0'(t-\Delta t),t-\Delta t)$$

$$+ \frac{(r_j - a_0'(t-\Delta t))^2}{2} [-2c(1 + U(a_0'(t-\Delta t),t-\Delta t)) + H^0_{11}(a_0'(t-\Delta t),t-\Delta t)]$$

$$+ \frac{(r_j - a_0'(t-\Delta t))^3}{6} H^b_{111}(a_0'(t),t) .$$

$H^b_{111}(a_i'(t),t)$, lag computations, have already been computed, while $H^b_{111}(a_i'(t-\Delta t),t-\Delta t)$ would (possibly) require values which are not as yet computed.

The values $m(i)$ are chosen with two thoughts in mind. First, it is desirable to choose the points as close to the boundaries as possible in order to reduce the residual error in estimating the third derivatives. Secondly, as we have just seen, the values of the points close to the boundaries are filled in using Taylor series expansions around $a_i'(t)$. To then expand around these points (very often) in order to estimate $a_i'(t-\Delta t)$ is inappropriate. Inappropriate in the sense that the partial differential equation aspect of the problem is lost.

At the other grid points away from the boundaries, namely those points $(r_j, t-\Delta t)$ with $K(a_0'(t), \Delta r) + 2 \le j \le K(a_1'(t-\Delta t), \Delta r) - 1$, our approximations to $\overline{H}(r_j, t-\Delta t)$ are the solutions $H^b(r_j, t-\Delta t)$ to the simultaneous equations

$$\frac{H^b(r_j,t-\Delta t) - H^b(r_j,t)}{\Delta t} = \epsilon \Big[\frac{(H^b(r_{j+1},t-\Delta t) - 2H^b(r_j,t-\Delta t) + H^b(r_{j-1},t-\Delta t))}{2\Delta r^2}$$

$$+ \left(\frac{n-1}{2r_j}\right)\left(\frac{H^b(r_{j+1},t-\Delta t) - H^b(r_{j-1},t-\Delta t)}{2\Delta r}\right)\Big]$$

$$+ (1-\epsilon)\Big[\frac{(H^b(r_{j+1},t) - 2H^b(r_j,t) + H^b(r_{j-1},t))}{2\Delta r^2}$$

$$+ \left(\frac{n-1}{2r_j}\right)\left(\frac{H^b(r_{j+1},t) - H^b(r_{j-1},t)}{2\Delta r}\right)\Big] ,$$

where only schemes for which $0 \le \epsilon \le 1$ are considered.

The choice $\epsilon = 0$ gives the four point explicit scheme, and the solution is simply

$$H^b(r_j,t-\Delta t) = H^b(r_j,t) + \frac{\Delta t}{2\Delta r^2} (H^b(r_{j+1},t) - 2H^b(r_j,t) + H^b(r_{j-1},t))$$

$$+ \left(\frac{n-1}{2r_j}\right) \frac{\Delta t}{2\Delta r} (H^b(r_{j+1},t) - H^b(r_{j-1},t)) .$$

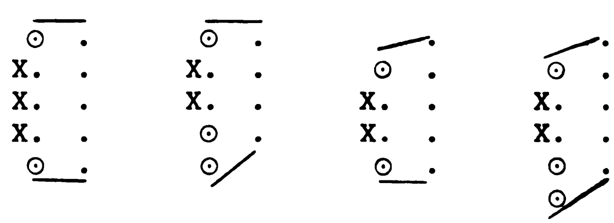
For $0 < \epsilon \le 1$ we have implicit finite difference schemes involving six points, unless $\epsilon = 1$, in which case only four points are concerned. The solution of the simultaneous equations in the implicit case is made easier by the tri-diagonal feature of the matrix [3].

Note that for the explicit scheme, the order in which the values approximated using Taylor series expansions, and those involving the finite difference equations are calculated, is immaterial.

The choice of ϵ is dictated by time considerations and conditions discussed in Section 5.

The following graphs indicate the four most frequent cases; circles indicate points whose approximating values involve Taylor series expansions, and X's points at which finite difference techniques are employed.

Table **A**



The iterative process is continued until the lower boundary crosses the axis. Suppose T_{00} such that

$$a_0'(T_{00}) > 0 \geq a_0'(T_{00} - \Delta t) .$$

Then we let T_0', the approximation to $\bar{T}_0$, be

$$T_0' = T_{00} - \Delta t \left(\frac{a_0'(T_{00})}{a'_0(T_{00}) - a_0'(T_{00} - \Delta t)}\right) = T_{00} - \Delta t_0 ,$$

and fill in the values H^b at the grid points

$$(0,T_0'),(\Delta r,T_0')\ldots\ldots\ldots\ldots(K(a_1'(T_0'),\Delta r)\cdot\Delta r,T_0')$$

with

$$a_1'(T_0') = a_1'(T_{00}) - \frac{\Delta t_0}{\Delta t}(a_1'(T_{00}) - a_1'(T_{00} - \Delta t))$$

as follows.

$$H^b(0,T_0') = H^0(0,T_0') ,$$

and Taylor series expansions are employed to calculate

$H^b(r_j,T_0')$ for $1 \le j \le K(a_0'(T_{00}),\Delta r) + 1$ and for $j = K(a_1'(T_0'),\Delta r)$.

For $K(a_0'(T_{00}),\Delta r) + 2 \le j \le K(a_1'(T_0'),\Delta r) - 1$, a finite difference scheme is used with Δt_0 substituted for Δt.

From T_0', the iterative procedure is continued in time steps of Δt, except for the initial step of length $\Delta t - \Delta t_0$, until $a_1'(t) < 2\Delta r$. The purpose of the first step is to be able to compare different schemes after the lower boundary crosses the axis. That is, T_0' will vary from scheme to scheme. The procedure is similar, but with the following changes due to the special role played by $r = 0$.

Defining in the obvious way

$$H_1^B(0,t) = \lim_{r\downarrow 0} \frac{H^B(r,t) - H^B(0,t)}{r} = \lim_{r\downarrow 0} H_1^B(r,t)$$

and similarly for all partial derivatives evaluated at $r = 0$, then since $H_1^B(0,t) = 0$, and in fact all odd partials evaluated at $r = 0$ vanish,

$$\lim_{r\downarrow 0}[H_2^B(r,t) + \frac{1}{2} H_{11}^B(r,t) + (\frac{n-1}{2}) \frac{H_1^B(r,t)}{r}] = H_2^B(0,t) + \frac{n}{2} H_{11}^B(0,t) = 0 .$$

For the explicit scheme, we approximate $\overline{H}(0,t-\Delta t)$ by

$$H^b(0,t-\Delta t) = H^b(0,t) + \frac{n\Delta t}{\Delta r^2} (H^b(\Delta r,t) - H^b(0,t)),$$

and if an implicit scheme is used, the equation involved is

$$-\epsilon\left(\frac{n\Delta t}{\Delta r^2}\right) H^b(\Delta r,t-\Delta t) + \left(1 + \epsilon \frac{n\Delta t}{\Delta r^2}\right) H^b(0,t-\Delta t) = (1-\epsilon) \frac{n\Delta t}{\Delta r^2} H^b(\Delta r,t)$$

$$+ \left(1 - (1 - \epsilon) \frac{n\Delta t}{\Delta r^2}\right) H^b(0,t) .$$

It will be shown in Section 5 that an implicit scheme is necessarily employed when n is large.

Suppose T_{11} to be the value of t such that $a_1'(t) < 2\Delta r$. Then there are obvious estimates of $\overline{T}_1$, including T_{11} itself.

SECTION 4

We now find the asymptotic width of the region B. As t gets large, the density of $\|\mu\|$ under the alternative hypothesis becomes concentrated at $\frac{r}{t} = \frac{\|x\|}{t}$, as is easily seen by noting that in one dimension

$$\frac{|x|}{t} = |E\mu| < E|\mu| < E^{\frac{1}{2}}|\mu|^2 = \left(\frac{x^2}{t^2} + \frac{1}{t}\right)^{\frac{1}{2}} ,$$

and since $\dfrac{\Gamma(j + \frac{n}{2} + \frac{k}{2})}{\Gamma(j + \frac{n}{2})}$ is an increasing function of n, $E\|\mu\|$ is an increasing function of n for fixed $\|x\|$. So

$$\frac{\|x\|}{t} < E\|\mu\| < E^{\frac{1}{2}}\|\mu\|^2 = \left(\frac{\|x\|^2}{t^2} + \frac{n}{t}\right)^{\frac{1}{2}} .$$

Now let us consider the general problem of testing the simple hypothesis H_0: $\mu = 0$ vs. the simple alternative H_1: $\mu = M > 0$, with apriori probabilities ρ and $1 - \rho$, losses $L(\text{rej}|H_0) = W_0$, $L(\text{acc}|H_1) = W_1$, and cost of sampling per unit time c.

z_0 and z_1, with $z_0 \leq 0 \leq z_1$, determine a sequential probability ratio test (SPRT). That is, one which has boundaries $z_i + \frac{M}{2} t$. We observe $(X_t: t > 0)$ until $X_t = z_i + \frac{M}{2} t$. If $i = 0$, accept; if $i = 1$, reject.

Instead of considering the risk as a function of z_0 and z_1, let us consider the risk to be a function of λ_1 and h, with $\lambda_1 = z_1 M$, $h = \lambda_1 - \lambda_0$, $\lambda_0 = z_0 M$. In terms of λ_1 and h, we sample as long as $\frac{\lambda_1 - h}{M} + \frac{M}{2} t < X_t < \frac{\lambda_1}{M} + \frac{M}{2} t.$

Now

$$R(\lambda_1,h) = \frac{1}{(e^{\lambda_1} - e^{\lambda_1 - h})} [\rho((1-e^{\lambda_1 - h})W_0 + k(\lambda_1(e^{\lambda_1 - h} - 1) + (\lambda_1 - h)(1-e^{\lambda_1})))$$

$$+ (1-\rho)(e^{\lambda_1 - h}(e^{\lambda_1} - 1)W_1 + k(\lambda_1 e^{\lambda_1}(1-e^{\lambda_1 - h}) + (\lambda_1 - h)(e^{\lambda_1} - 1)))]$$

where $k = \frac{2c}{M^2}$.

λ_1 or $\lambda_1 - h$ may be zero, and in fact, when ρ is small or large, the optimal procedure will be to reject or accept without sampling.

We use the results and notation displayed in Lehmann, Chapter 3, to obtain the expression of the above risk function. The terms e^{λ_0} and e^{λ_1} represent the bounds on the likelihood ratio function $L(x,t)$.

$$L(x,t) = \frac{p(X_t = x|H_1)}{p(X_t = x|H_0)} = \frac{(1/2\pi t)^{\frac{1}{2}} e^{-\frac{(x-Mt)^2}{2t}}}{(1/2\pi t)^{\frac{1}{2}} e^{-\frac{(x)^2}{2t}}} = e^{M(x - \frac{M}{2} t)} ,$$

and $\lambda_0 < M(x - \frac{M}{2} t) < \lambda_1$ iff $e^{\lambda_0} < e^{M(x - \frac{M}{2} t)} < e^{\lambda_1}$.

The probability of hitting the upper boundary, given drift μ, is given by

$$\beta(\mu) = \frac{1-e^{\lambda_0(1 - \frac{2\mu}{M})}}{e^{\lambda_1(1 - \frac{2\mu}{M})} - e^{\lambda_0(1 - \frac{2\mu}{M})}} .$$

Then the probabilities of mistakes are given by

$$\beta(0) = \frac{1-e^{\lambda_0}}{e^{\lambda_1}-e^{\lambda_0}} \quad \text{and} \quad 1-\beta(M) = 1-\left(\frac{1-e^{-\lambda_0}}{e^{-\lambda_1}-e^{-\lambda_0}}\right) = \frac{e^{\lambda_0}(e^{\lambda_1}-1)}{e^{\lambda_1}-e^{\lambda_0}} .$$

The expected sampling time $E_\mu(T)$ given drift μ, is given by

$$E_\mu(T) = \frac{\ln e^{\lambda_1}\beta(\mu) + \ln e^{\lambda_0}(1-\beta(\mu))}{E_\mu(\ln L(x,1))} .$$

$L(x,1)$ is the likelihood ratio evaluated at $t = 1$. $L(x,1) = e^{M(x - \frac{M}{2})}$. So

$$E_0(T) = \frac{1}{\frac{-M^2}{2}}\left[\lambda_1\left(\frac{1-e^{\lambda_0}}{e^{\lambda_1}-e^{\lambda_0}}\right) + \lambda_0\left(\frac{e^{\lambda_1}-1}{e^{\lambda_1}-e^{\lambda_0}}\right)\right]$$

$$= \frac{2}{M^2(e^{\lambda_1}-e^{\lambda_0})}\left[\lambda_1(e^{\lambda_0}-1) + \lambda_0(1-e^{\lambda_1})\right] ,$$

and

$$E_M(T) = \frac{1}{\frac{M^2}{2}}\left[\lambda_1\left(\frac{e^{\lambda_1}-e^{\lambda_1}e^{\lambda_0}}{e^{\lambda_1}-e^{\lambda_0}}\right) + \lambda_0\left(\frac{e^{\lambda_0}(e^{\lambda_1}-1)}{e^{\lambda_1}-e^{\lambda_0}}\right)\right]$$

$$= \frac{2}{M^2(e^{\lambda_1}-e^{\lambda_0})}\left[\lambda_1(1-e^{\lambda_0})e^{\lambda_1} + \lambda_0(e^{\lambda_1}-1)e^{\lambda_0}\right] .$$

In order to find λ_1 and h which minimizes $R(\lambda_1,h)$, we set the first partial derivatives equal to zero.

$$\frac{\partial R(\lambda_1,h)}{\partial\lambda_1} = 0 \quad \text{iff} \quad \rho[(W_0 - kh) + ke^{\lambda_1}(1-e^{-h})]$$

$$= (1-\rho)[(W_1 - kh)e^{2\lambda_1-h} + ke^{\lambda_1}(1-e^{-h})] . \tag{1}$$

$$\frac{\partial R(\lambda_1,h)}{\partial h} = 0 \quad \text{iff} \quad \rho[(W_0 - kh) - k(1-e^{h})]$$

$$= (1-\rho)\ e^{\lambda_1}[(W_1 - kh) + k(1-e^{-h})] \ . \qquad (2)$$

Dividing (1) by (2), we obtain

$$[(W_0 - kh) + ke^{\lambda_1}(1-e^{-h})][(W_1 - kh) + k(1-e^{-h})]$$

$$= [(W_0 - kh) - k(1-e^{h})][(W_1 - kh)e^{\lambda_1 - h} + k(1-e^{-h})] \ ,$$

which reduces to

$$(W_0 - kh)(W_1 - kh) = k^2(e^{h} + e^{-h} - 2) \ . \qquad (3)$$

This equation involves only h.

The LHS of (3) is a decreasing function of h for $kh < \min(W_0,W_1)$ and is $\le k^2h^2 <$ RHS for $kh > \min(W_0,W_1)$. The RHS is an increasing function of h. For $h = 0$, $W_0W_1 > 0$. Thus there exists a unique $\bar{h}$ satisfying (3).

$$W_0W_1 - (W_0 + W_1)k\bar{h} + k^2\bar{h}^2 = k^2(\bar{h}^2 + \frac{2\bar{h}^4}{4!} + \frac{2\bar{h}^6}{6!} \ldots\ldots\ldots) \ .$$

The approximation h' of $\bar{h}$ used here is obtained by neglecting terms of order greater than two in $\bar{h}$.

$$W_0W_1 - (W_0 + W_1)kh' + k^2h'^2 = k^2h'^2 \ ,$$

$$h' = \frac{1}{k}\,\frac{W_0W_1}{(W_0 + W_1)} \ .$$

The approximation h' serves as an upper bound for $\bar{h}$.

$$\bar{h} \le h' = \frac{M^2}{2c}\,\frac{W_0W_1}{(W_0 + W_1)} \ .$$

For a given h, λ_1 must satisfy (1) in order to be optimal. h' serves as an approximation to $\lambda_1 - \lambda_0$, so $\frac{h'}{M}$ serves as an approximation to $z_1 - z_0$.

Returning to our problem, we define $r_I(t)$ to be the "indifference" point, that is, the value of r for which the risk of stopping and accepting equals the risk of stopping and rejecting. $r_I(t)$ is in the continuation region B.

For the $(c,W_1,2k)$ problem in n dimensions, $r_I(t)$ is the solution of the equation

$$1 = [W_1 \sum_{L=0}^{k} C_n(L,k)\frac{r^{2L}}{t^{k+L}}]\ (2\pi/t)^{\frac{n}{2}}\ e^{\frac{r^2}{2t}} \quad .$$

We note that $r_I(t)$ does not depend on the cost of sampling c. By the definition of t_z,

$$r_I(t_z) = 0 \ .$$

Considering the problem of testing the simply hypothesis $H_0: \mu = 0$ vs. the simple alternative $H_1: \mu = \frac{r_I(t)}{t}$ with

$$\beta = (t/2\pi)^{\frac{n}{2}}\ e^{-\frac{(r_I(t))^2}{2t}} \Big/ \Big(1 + (t/2\pi)^{\frac{n}{2}}\ e^{-\frac{(r_I(t))^2}{2t}}\Big) \ ,$$

$$= \frac{1}{(1 + U(r_I(t),t))} \quad ,$$

$$L(\text{rej}|H_0) = 1 \ ,$$

$$L(\text{acc}|H_1) = W_1\ E(\|\mu\|^{2k}|r_I(t),H_1) = \frac{1}{U(r_I(t),t)} \quad ,$$

we employ the above results, and for large t, we approximate

$$\bar{a}_1(t)-\bar{a}_0(t) \ \text{ by } \ \frac{(\frac{r_I(t)}{t})}{2c}\ [\frac{1(U(r_I(t),t))^{-1}}{1+(U(r_I(t),t))^{-1}}] = \frac{r_I(t)}{2ct}\ (\frac{1}{1+U(r_I(t),t)}) \ .$$

SECTION 5

In Section 3 it is stated that a choice of Δr, Δt, and ϵ is made, and this section investigates criteria for making that choice. The general (at most) six point schemes are analyzed here.

We first investigate the truncation error for the six point scheme. Expanding the appropriate terms around the point (r,t), we write:

$$\frac{H(r,t) - H(r,t-\Delta t)}{\Delta t} = H_2(r,t) - \frac{\Delta t}{2} H_{22}(r,t) + O(\Delta t^2)$$

$$\frac{\epsilon}{2}\left[\frac{H(r+\Delta r,t-\Delta t) - 2H(r,t-\Delta t) + H(r-\Delta r,t-\Delta t)}{\Delta r^2}\right] = \frac{\epsilon}{2}[H_{11}(r,t-\Delta t)$$

$$+ \frac{\Delta r^2}{12} H_{1111}(r,t-\Delta t) + O(\Delta r^4)] = \frac{\epsilon}{2}[H_{11}(r,t) - \Delta t H_{112}(r,t) + \frac{\Delta r^2}{12} H_{1111}(r,t)$$

$$+ O(\Delta t^2) + O(\Delta t \Delta r^2) + O(\Delta r^4)] ,$$

$$\left(\frac{1-\epsilon}{2}\right)\left[\frac{H(r+\Delta r,t) - 2H(r,t) + H(r-\Delta r,t)}{\Delta r^2}\right] = \left(\frac{1-\epsilon}{2}\right)[H_{11}(r,t)$$

$$+ \frac{\Delta r^2}{12} H_{1111}(r,t) + O(\Delta r^4)] ,$$

$$\left(\frac{n-1}{2r}\right)\epsilon\left[\frac{H(r+\Delta r,t-\Delta t) - H(r-\Delta r,t-\Delta t)}{2\Delta r}\right] = \left(\frac{n-1}{2r}\right)\epsilon[H_1(r,t-\Delta t) + \frac{\Delta r^2}{6} H_{111}(r,t-\Delta t)$$

$$+ O(\Delta r^4)] = \left(\frac{n-1}{2r}\right)\epsilon[H_1(r,t) - \Delta t H_{12}(r,t) + \frac{\Delta r^2}{6} H_{111}(r,t) + O(\Delta t^2) + O(\Delta t \Delta r^2) + O(\Delta r^4)] ,$$

$$\left(\frac{n-1}{2r}\right)(1-\epsilon)\left[\frac{H(r+\Delta r,t) - H(r-\Delta r,t)}{2\Delta r}\right] = \left(\frac{n-1}{2r}\right)(1-\epsilon)[H_1(r,t) + \frac{\Delta r^2}{6} H_{111}(r,t)$$

$$+ O(\Delta r^4)] ,$$

so that adding these five equations, on the region B we have,

$$\frac{H^B(r,t) - H^B(r,t-\Delta t)}{\Delta t} + \epsilon[(\frac{H^B(r+\Delta r,t-\Delta t) - 2H^B(r,t-\Delta t) + H^B(r-\Delta r,t-\Delta t)}{2\Delta r^2})$$

$$+ (\frac{n-1}{2r})(\frac{H^B(r+\Delta r,t-\Delta t) - H^B(r-\Delta r,t-\Delta t)}{2\Delta r})]$$

$$+ (1-\epsilon)[(\frac{H^B(r+\Delta r,t) - 2H^B(r,t) + H^B(r-\Delta r,t)}{2\Delta r^2}) + (\frac{n-1}{2r})(\frac{H^B(r+\Delta r,t) - H^B(r-\Delta r,t)}{2\Delta r})]$$

$$= -\frac{\Delta t}{2} H^B_{22}(r,t) + \frac{\Delta r^2}{24} H^B_{1111}(r,t) + (\frac{n-1}{12r})\Delta r^2 H^B_{111}(r,t) - \frac{\Delta t}{2} \epsilon H^B_{112}(r,t)$$

$$- (\frac{n-1}{2r})\epsilon\Delta t H^B_{12}(r,t) + O(\Delta t^2) + O(\Delta t\Delta r^2) + O(\Delta r^4). \qquad (1)$$

Using formulas derived in Section 2, page , we may write

$$H^B_{12}(r,t) = -\frac{1}{2} H^B_{111}(r,t) - (\frac{n-1}{2r}) H^B_{11}(r,t) + (\frac{n-1}{2r^2}) H^B_1(r,t) ,$$

$$H^B_{112}(r,t) = -\frac{1}{2} H^B_{1111}(r,t) - (\frac{n-1}{2r}) H^B_{111}(r,t) + 2(\frac{n-1}{2r^2}) H^B_{11}(r,t)$$

$$- 2(\frac{n-1}{2r^3}) H^B_1(r,t) ,$$

$$H^B_{22}(r,t) = \frac{1}{4} H^B_{1111}(r,t) + (\frac{n-1}{2r}) H^B_{111}(r,t) + [(\frac{n-1}{2})^2 - (\frac{n-1}{2})]\frac{1}{r^2} H^B_{11}(r,t)$$

$$+ [(\frac{n-1}{2}) - (\frac{n-1}{2})^2]\frac{1}{r^3} H^B_1(r,t) .$$

Hence, the right hand side of (1) may be written

$$RHS = -\frac{\Delta t}{8} H^B_{1111}(r,t) - (\frac{n-1}{4r})\Delta t\, H^B_{111}(r,t) - [(\frac{n-1}{2})^2 - (\frac{n-1}{2})]\frac{\Delta t}{2r^2} H^B_{11}(r,t)$$

$$- [(\frac{n-1}{2}) - (\frac{n-1}{2})^2]\frac{\Delta t}{2r^3} H^B_1(r,t) + \frac{\Delta r^2}{24} H^B_{1111}(r,t) + (\frac{n-1}{12r}) \Delta r^2 H^B_{111}(r,t)$$

$$+ \frac{\Delta t}{4} \epsilon H^B_{1111}(r,t) + \Delta t\, \epsilon(\frac{n-1}{4r}) H^B_{111}(r,t) - \Delta t\, \epsilon(\frac{n-1}{2r^2}) H^B_{11}(r,t)$$

$$+ \Delta t\, \epsilon(\frac{n-1}{2r^3}) H^B_1(r,t) + \Delta t\, \epsilon(\frac{n-1}{4r}) H^B_{111}(r,t) + \Delta t\, \epsilon(\frac{n-1}{2})^2 \frac{1}{r^2} H^B_{11}(r,t)$$

$$- \Delta t\, \epsilon(\frac{n-1}{2})^2 \frac{1}{r^3} H^B_1(r,t) + O(\Delta t^2) + O(\Delta t\Delta r^2) + O(\Delta r^4)$$

$$= [-\frac{\Delta t}{8} + \frac{\Delta r^2}{24} + \frac{\Delta t}{4}\,\epsilon]\, H^B_{1111}(r,t) + [-(\frac{n-1}{4r})\,\Delta t + (\frac{n-1}{12r})\Delta r^2 + \Delta t\,\epsilon(\frac{n-1}{2r})]$$

$$H^B_{111}(r,t) + [(\frac{n-1}{2}) - (\frac{n-1}{2})^2](\frac{\Delta t}{2r^2} - \frac{\Delta t}{r^2}\,\epsilon)\, H^B_{11}(r,t) + [(\frac{n-1}{2})^2 - (\frac{n-1}{2})]$$

$$(\frac{\Delta t}{2r^3} - \frac{\Delta t}{r^3}\,\epsilon)\, H^B_1(r,t) + O(\Delta t^2) + O(\Delta t \Delta r^2) + O(\Delta r^4)\ .$$

If $\epsilon = 0$, corresponding to the four point explicit scheme, the RHS of (1) is given by

$$\text{RHS} = O(\Delta t) + O(\Delta r^2)\ .$$

For dimension $n = 1$ or 3, the choice $\Delta t = \frac{\Delta r^2}{3}$ reduces the error term to

$$\text{RHS} = O(\Delta t^2) = O(\Delta r^4)\ ,$$

and for other dimensions, the terms involving the third and fourth derivatives are cancelled.

For $0 < \epsilon \leq 1$, corresponding to implicit finite difference schemes, the RHS of (1) is reduced by the choice

$$\epsilon = \frac{1}{2} - \frac{\Delta r^2}{6\Delta t}$$

in the case $n = 1$ or 3 to

$$\text{RHS} = O(\Delta t^2) + O(\Delta t \Delta r^2) + O(\Delta r^4)\ ,$$

and for other values of n to terms not involving the third and fourth derivatives.

Similar calculations yield

$$\frac{H^B(0,t) - H^B(0,t-\Delta t)}{\Delta t} + n\epsilon\left(\frac{H^B(\Delta r,t-\Delta t) - H^B(0,t-\Delta t)}{\Delta r^2}\right)$$

$$+ n(1-\epsilon)\left(\frac{H^B(\Delta r,t) - H^B(0,t)}{\Delta r^2}\right) = \left[-\frac{n^2\Delta t}{8} + \frac{n\Delta r^2}{24} + \frac{n^2\Delta t}{4}\epsilon\right] H^B_{1111}(0,t)$$

$$+ 0(\Delta t^2) + 0(\Delta t\Delta r^2) + 0(\Delta r^4) ,$$

utilizing the fact that all odd partial derivatives with respect to r evaluated at $r = 0$ vanish. Only in $n = 1$ dimension can this error be reduced to

$$0(\Delta t^2) + 0(\Delta t\Delta r^2) + 0(\Delta r^4)$$

by choosing

$$\epsilon = \frac{1}{2} - \frac{\Delta r^2}{6\Delta t} .$$

The following table gives the number of calculations necessary at each grip point for the four point explicit and six point implicit schemes.

	multiplications and divisions	additions and subtractions (non-integer)
explicit, one dimension	1	4
implicit, one dimension	5	4
explicit, multi-dimensions	4	4
implicit, multi-dimensions	10	10

The free boundary condition makes the analysis of stability virtually impossible for the author. It appears that the stability condition certainly depends on the dimension n, and quite possibly on the power k. The best that can be hoped for in one dimension is that the stability condition is the same as the fixed boundary case:

$$\frac{\Delta t}{\Delta r^2} \le \frac{1}{1-2\epsilon} \qquad 0 \le \epsilon < .5 \ ,$$

unconditional stability $.5 \le \epsilon \le 1$.

For the four point explicit scheme, this condition is

$$\frac{\Delta t}{\Delta r^2} \le 1 \ .$$

In higher dimensions, these conditions must be strengthened. For the explicit scheme the coefficient of $H^b(r_{j-1},t)$ is

$$\frac{\Delta t}{2\Delta r^2} - \frac{(n-1)\Delta t}{4r_j\Delta r} \ ,$$

which is negative for small values of r_j, the smallest of which is Δr (for which this coefficient is used), if

$$n \ge 4 \ .$$

At $r = 0$, the coefficient of $H^b(0,t)$ is

$$1 - \frac{n\Delta t}{\Delta r^2}$$

which is negative for appropriate values of Δt, Δr, and n.

The existence of negative coefficients insures instability, and in general the explicit scheme will be dropped once the lower boundary crosses zero (if not before) if n is large. The larger n, the larger the value of ϵ chosen, because the computations show that small choices of ϵ give instable schemes. However, no instability problems have been encountered by the author with the choice $\epsilon = 1$. If for large n, it is stated that the explicit scheme is used, it is understood that at or before T_0', an implicit scheme is substituted.

In general Δt is chosen $\leq \Delta r^2$, because it is demonstrated by the computations that in any dimensions large values of the ratio $\frac{\Delta t}{\Delta r^2}$ give instable results, regardless of the value of ϵ.

SECTION 6

Computations were carried out for the constant loss and squared loss problems. This section deals with the constant loss problem.

For the $(c,W_1,0)$ problem in n dimensions,

$$(W_1 \sum_{L=0}^{0} C_n(L,0) \frac{r^{2L}}{t^{0+L}}) = W_1 \quad ,$$

and consequently the indifference function is given explicitly by

$$r_I(t) = [t(n\ln(\frac{t}{2\pi}) - 2\ln W_1)]^{\frac{1}{2}} \qquad \text{for}$$

$$t > t_z = 2\pi(W_1^{\frac{2}{n}}) \quad ,$$

for then

$$W_1 U(r_I(t),t) = 1 \ .$$

The approximate width of the region B, call it $AWB(t)$, as derived in Section 4, is given by

$$\bar{a}_1(t) - \bar{a}_0(t) \cong AWB(t) = \frac{[t(n\ln(\frac{t}{2\pi}) - 2\ln W_1)]^{\frac{1}{2}}}{2ct} (\frac{W_1}{1 + W_1}) .$$

$AWB(t)$ is maximized at

$$t_m = 2\pi e(W_1^{\frac{2}{n}}) = et_z$$

as is seen by setting the derivative with respect to t equal to zero.

Although there is no reason to believe that the approximation AWB(t), based on an approximation that is an upper bound in a simple vs. simple testing case, would serve as an upper bound in this simple vs. composite case, it does in fact turn out to be an upper bound (for $a_1'(t) - a_0'(t)$ and presumably for $\bar{a}_1(t) - \bar{a}_0(t)$) in the constant loss case whenever $t > t_m$.

For fixed W_1 and t, the approximation improves with increasing c, in the sense that the ratio

$$\frac{a_1'(t) - a_0'(t)}{\text{AWB}(t)} \leq 1$$

is an increasing function of c. For fixed c, and a fixed distance beyond t_m (t_z depends on W_1, but not c) the approximation improves with increasing W_1 for small fixed distances beyond t_m, and contrarily improves with decreasing values of W_1 at large fixed distances beyond t_m. For c, W_1, and t fixed, the approximation gets progressively worse with increasing dimension n. Table 6.a and 6.b illustrate these facts.

Table 6.a displays the ratio $\frac{a_1'(t) - a_0'(t)}{\text{AWB}(t)}$ at different values of t, corresponding to varying values of c for two problems and procedures outlines. Starting value denotes the value of t at which the polynomial procedure was applied. Only major changes in the procedure would affect the ratio to any significant degree. We note that the value of ϵ after the crossing would certainly not affect the ratio at values of t before the crossing (which all of these are). The procedure is included primarily for the sake of completeness.

Table 6.a

Problem: $(c,1,0)$ in $n = 1$ dimension

$c = .02, .01, .005, .0025, .00125$

Procedure: $\Delta t = \Delta r^2 = .25^2$

Implicit, $\varepsilon = \frac{1}{2} - \frac{\Delta r^2}{6\Delta t} = \frac{1}{3}$

Starting value $T = 150$

The ratio $\frac{a_1'(t) - a_0'(t)}{AWB(t)}$ at $t = 100$ and $t = 50$

	$t = 100$	$t = 50$
$c = .02$	.992	.982
$c = .01$	.972	.938
$c = .005$	.911	.831
$c = .0025$	.767	.656
$c = .00125$	.547	.455

Problem: $(c,1,0)$ in $n = 10$ dimensions

$c = .02, .01, .005$

Procedure: $\Delta t = \Delta r^2 = .25^2$

Implicit, $\varepsilon = \frac{1}{3}$ before T_0'

$\varepsilon = 1$ after T_0'

Starting value $T = 100$

The ratio at $t = 70$ and $t = 30$

	$t = 70$	$t = 30$
$c = .02$	.765	.649
$c = .01$	.563	.448
$c = .005$	.368	.285

Table 6.b gives the ratio for varying values of W_1. The starting value T is 200, and 195 is considered to be the largest value of t that reflects an accurate estimate of $a_i'(t)$. That is to say, the procedure is given 80 Δt steps to "settle down": to counterbalance the inaccuracies inherent in the initial polynomial approximation. Since for $W_1 = .02$, $t_m = 68.3$, $t_m + 127$ was chosen in order to make its value no larger than 195. If the starting value had been chosen large enough, we would see the ratio for $W_1 = 1$ surpass the ratio for $W_1 = 2$.

Table 6.b

Problem: $(.01,W_1,0)$ in $n = 1$ dimension Procedure: $\Delta t = \Delta r^2 = .25^2$

$W_1 = 2,1,.5,.25$ Implicit, $\epsilon = \frac{1}{3}$

Starting value $T = 200$

t_m and the ratio $\dfrac{a_1'(t) - a_0'(t)}{AWB(t)}$ at $t_m + 5$ and $t_m + 127$

	t_m	$t_m + 5$	$t_m + 127$
$W_1 = 2$	68.3	.988	.993
$W_1 = 1$	17.1	.890	.982
$W_1 = .5$	4.3	.709	.984
$W_1 = .25$	1.1	.649	.991

The question of convergence has been deferred from Section 5. There are two types of convergence to be considered. One is the convergence of the values $H^b(r_j,t)$ to $\bar{H}(r_j,t)$ at grid points common to all meshes, and the other is of the approximations $a_i'(t)$ to $\bar{a}_i(t)$. The convergence (in L_0 norm) rate of an explicit finite difference solution with $\frac{\Delta t}{\Delta r^2} = \frac{1}{3}$ in a fixed boundary case (necessarily convergence at grid points) to a function satisfying the heat equation $H_2 = \frac{1}{2} H_{11}$ with an analytic initial function (certainly $\bar{H}^B(r,T)$ as a function of r is analytic) is $O(\Delta t^2) = O(\Delta r^4)$. The corresponding rate for any other value of the ratio $\frac{\Delta t}{\Delta r^2}$ (but ≤ 1 to insure stability) is $O(\Delta t)$. Thus in the problems at hand, the most that can be hoped for is $O(\Delta t^2)$, and the best chance of that happening is in the one-dimensional case.

Table 6.c shows values of a) $H^b(r_j,t)$ at selected grid points common to all meshes, b) $a_i'(t)$ at selected values of t, and c) T_0' for the explicit scheme outlined below. The first three

values correspond to Δr = .5,.25,.125 respectively, the next two to successive differences, and the number offset to the right to the ratio of the first difference to the second. The first difference may subtract the first number from the second, or visa versa, but whichever way is followed through for the second difference.

Table 6.c

Problem: (.01,1,0) in n = 1 dimension   Procedure: Explicit, $\frac{\Delta t}{\Delta r^2} = \frac{1}{3}$

Format: value when Δr = .5 Δr = .5,.25,.125

value when Δr = .25   Starting value T = 100

value when Δr = .125

first difference

second difference   ratio of differences

$a_0'(60)$		$H^b(10,60)$		$H^b(12,60)$		$H^b(14,60)$		$a_1'(60)$	
9.405498		1.78500146		2.16200199		2.594289809		14.004402	
9.405357		499944		199828		771		564	
9.405344		33		02		66		82	
.000141	11	.00000202	19	.00000371	14	.000000038	7	.000162	9
.000013		11		26		5		18	
$a_0'(20)$		$H^b(3,20)$		$H^b(5,20)$		$H^b(7,20)$		$a_1'(20)$	
2.342487		1.03437318		1.26730696		1.56816093		7.672784	
1974		6641		29513		3728		4108	
875		597		437		502		423	
.000513	5	.00000577	13	.00001183	16	.00002365	10	.001324	4
99		44		76		226		315	
		$H^b(0,5)$		$H^b(1,5)$		$H^b(2,5)$		$a_1'(5)$	
		1.05880985		1.07461814		1.12015429		2.839361	
		70206		54217		39970		795260	
		69505		3754		41125		0645	
		.00010779	15	.00007597	16	.00024541	21	.044101	10
		701		464		1155		4615	
T_0'				$H^b(0,3.5)$		$H^b(.5,3.5)$		$a_1'(3.5)$	
10.225519				1.08088505		1.08313990		1.243785	
53206				106705		59393		0.778694	
8385				9361		3347		0.558537	
.027687	5			.00018200	7	.00045403	-8	.465091	2
5179				2656		- 5954		.220157	

Considering first of all convergence at grid points, if the rate of convergence were $0(\Delta t^2) = 0(\Delta r^4)$, the absolute value of the ratio of successive differences would be about 16. It should be noted that defining convergence in terms of L_0 norm would seem to make more sense in the fixed boundary problem than in the present free boundary problem. It does not seem to be clear from the computations whether the fastest convergence (in absolute value) would occur near the boundaries or near the middle of the continuation region. And in fact, there may be a difference in the boundaries. It is possible that convergence is more meaningfully defined in terms of the average ratio or the minimum ratio. Without defining convergence, ratios of successive differences are displayed in Table 6.d for a one-dimensional problem, and three different procedures.

Table 6.d

Problem: (.01,1.0) in n = 1 dimensions   Procedure: Δr = .5,.25,.125

Format: ratio for $\frac{\Delta t}{\Delta r^2} = \frac{1}{3}$, explicit   Starting value T = 50*

ratio for $\frac{\Delta t}{\Delta r^2} = \frac{2}{3}$, explicit

ratio for $\Delta t = \Delta r^2$, $\epsilon = \frac{1}{3}$

$a_0'(20)$	$H^b(3,20)$	$H^b(5,20)$	$H^b(7,20)$	$a_1'(20)$
5	13	17	10	4
5	2	3	2	10
11	20	17	11	4
	$H^b(0,5)$	$H^b(1,5)$	$H^b(2,5)$	$a_1'(5)$
	15	16	21	10
	15	11	-1	11
	19	14	6	1
T_0'		$H^b(0,3.5)$	$H^b(.5,3.5)$	$a_1'(3.5)$
5		3	5	2
7		3	2	1
6		8	-22	2

---

* Different starting value than Table 6.c

Two things should be pointed out. First of all, the ratio is not consistent in the sense that three different values (like $\Delta r = .25,.125,.0625$) would give substantially different ratios at some grid points. The difference could probably not be explained solely by round off error. If the ratio were consistent, it should be greater than one in absolute value to insure convergence. Secondly, the ratio of successive differences for $\Delta r = .5,.25,.125$, as shown in these tables does not necessarily provide the criterion for the best procedure (if the ratio is not consistent), for all three values using one procedure may be closer to the true value than the corresponding values for another procedure -- even though the ratio for the first set is smaller.

In higher dimensions, as would be suspected, the convergence rate, however it might be defined, becomes progressively slower. Table 6.e gives the ratio at selected points for a given procedure when $n = 1$ and 10 dimensions.

Table 6.e

Problem: (.01,1,0) in n = 1,10 dimensions

Format: ratio for n = 1

ratio for n = 10

Procedure: $\Delta t = \Delta r^2$

$\Delta r = .5, .25, .125$

$\epsilon = \frac{1}{3}$ before crossing

$\epsilon = \frac{1}{3}$ after if n = 1

$\epsilon = 1$ after if n = 10

Starting value T = 100

	$a_0'(60)$	$H^b(10,60)$	$H^b(12,60)$	$H^b(14,60)$	$a_1'(60)$
n = 1	11	19	14	7	9
		$H^b(34,60)$	$H^b(36,60)$	$H^b(38,60)$	
n = 10	7	-5	-3	-9	8
	$a_0'(20)$	$H^b(3,20)$	$H^b(5,20)$	$H^b(7,20)$	$a_1'(20)$
n = 1	5	13	16	10	4
		$H^b(13,20)$	$H^b(15,20)$	$H^b(17,20)$	
n = 10	-25	-1	-2	-9	1
		$H^b(0,5)$	$H^b(1,5)$	$H^b(2,5)$	$a_1'(5)$
n = 1		15	16	21	10
n = 10		3	3	2	2

If the values corresponding to $\Delta r = .5, .25, .125$ (or any other set of three numbers, the last two each one half of the preceding) are monotone, there is an obvious estimate of the true value. Call the values corresponding to $\Delta r = .5, .25$ and $.125$; a, b, and c respectively. Suppose the ratio of successive differences to be $r > 1$. Then an estimate of the true value is

$$\text{true value} = c + \frac{(c - b)}{r - 1}$$

It was stated in Section 5 that as the ratio $\frac{\Delta t}{\Delta r^2}$ increases beyond one, the results become increasingly less reliable, whether due to instability or round off error. Table 6.f presents data for three different schemes, the last having ratio 2. ϵ is chosen optimally in all three instances.

Table 6.f

Problem: (.01,1,0) in n = 1 dimension   Starting value T = 50

		$H^b(3,20)$	$H^b(5,20)$	$H^b(7,20)$	
	.5	1.0343739	1.2673100	1.5681627	
$\Delta t = \frac{1}{3}\Delta r^2$, $\varepsilon = 0$, $\Delta r =$	.25	672	2982	391	
	.125	67	75	68	
	.5	1.0343774	1.2673118	1.5681637	
$\Delta t = \Delta r^2$, $\varepsilon = \frac{1}{3}$, $\Delta r =$	.25	672	2983	390	
	.125	67	75	68	
	.5	----	----	----	*
$\Delta t = 2\Delta r^2$, $\varepsilon = \frac{5}{12}$, $\Delta r =$	.25	1.034340	1.267282	1.568196	
	.125	64	95	39	

* For $\Delta t = .5$ and $\Delta r = .5$, $a_1'(20) = 2.901$, so grid points (3,20), (5,20), and (7,20) were not in continuation region. (Indicating extensive round off error.)

This table indicates that the results of the scheme with $\Delta t = 2\Delta r^2$ are unsatisfactory. At the grid points (3,20), (5,20), and (7,20) both of the first two schemes would estimate the true values to be 1.034366, 1.267297, and 1.568136 to seven places, employing the above estimation procedure. For $\Delta r = .25$ and $\Delta t = .125$ the last scheme gives values which are further away from the three estimated values at two of three grid points than the second scheme with $\Delta r = .5$ and $\Delta t = .25$. At the other point the values are essentially equally far away. The same thing happens (two values further away and one equally far away) when the last scheme with $\Delta r = .125$ and $\Delta t = .03125$ is compared with the second scheme with $\Delta r = .25$ and $\Delta t = .0625$. This just should not happen, and it appears that round off error, if not stability in this case, is affected by the ratio of Δt to Δr^2, even when ε is chosen optimally in terms of truncation error. Other programs indicate stability problems when the ratio exceeds one.

Nothing will be said about convergence of the boundary approximations $a_i'(t)$ except to say its rate appears to be slower in $n = 1$ dimensions, but faster in higher dimensions than the convergence rate at grid points.

Graph 6.a plots three sets of boundaries for the (c,1,0) problem in one dimension. The set of unmarked lines correspond to $c = .02$, those marked by +'s to $c = .01$, and those marked by |'s to $c = .005$. The procedure used for this graph is implicit with $\Delta t = \Delta r^2 = .25^2$ and $\varepsilon = \frac{1}{2} - \frac{\Delta t}{\Delta r^2} = \frac{1}{3}$; however, only major changes in the procedure would be detected for plotting purposes. Consequently we will omit the procedure in describing following graphs.

We note that for $c = .005$ it appears that the upper boundary comes in to $r = 0$ at $t = 0$, which may indicate that $\bar{T}_1$ is in fact 0. Evidence from solutions in both the constant and squared loss problems strongly substantiates this uncomfortable possibility.

The slope of the indifference function

$$r_I'(t) = \frac{n + (n\ln(\frac{t}{2\pi}) - 2\ln W_1)}{2\{t(n\ln(\frac{t}{2\pi}) - 2\ln W_1)\}^{\frac{1}{2}}} = \frac{n + (n\ln(\frac{t}{2\pi}) - 2\ln W_1)}{2r_I(t)}$$

is infinite at $t = t_z$, and although it can't be shown by the author, the upper boundary appears to come in to zero more sharply than the indifference function <u>for all values of the parameters and all dimensions</u>. Also the lower boundary comes in less abruptly. That is, for ε positive and sufficiently small

$$\bar{a}_1(\bar{T}_1 + \varepsilon) > r_I(t_z + \varepsilon) > \bar{a}_0(\bar{T}_0 + \varepsilon) .$$

There is reason to believe $\bar{a}_0'(\bar{T}_0)$ is finite. Here are two polynomial functions, A^0 even and A^B satisfying the same partial differentiation equation as $\overline{H}^B$, such that both functions and first derivatives match on $r = s(t) = t$.

$$A^0(r,t) = 3r^2(t-1) - 3t^2 - t^3$$
$$A^B(r,t) = 2(r^3 - 3rt)$$
$$A^0(s(t),t) = 2t^3 - 6t^2 = A^B(s(t),t)$$
$$A^0_1(s(t),t) = 6(t^2 - t) = A^B_1(s(t),t)$$

This is not to imply that $\overline{H}^B$ can necessarily be expanded in a polynomial around $(0,\bar{T}_0)$, let alone one containing odd terms.

Graph 6.b shows the results for the (c,1,0) problem in one dimension, with $c = .005$, .0025, and .00125. Therefore the narrowest set of boundaries in Graph 6.b correspond to the widest set of boundaries in Graph 6.a.

Graph 6.c plots sets of boundaries corresponding to $W_1 = 2,1$, and .5 for the $(.01,W_1,0)$ problem in one dimension. As W_1 becomes large (or c becomes large) the upper boundary comes in very closely to t_z. This graph is slightly misleading in that the boundaries would have to be shown for larger values of t in order to see that the width of the continuation region is asymptotically larger for $W_1 = 2$ than for $W_1 = 1$. (The more costly the errors, the more worthwhile the sampling.)

Graph 6.d shows the effect of increasing dimension. Boundaries are plotted for the (.01,1,0) problem in 1,2, and 3 dimensions.

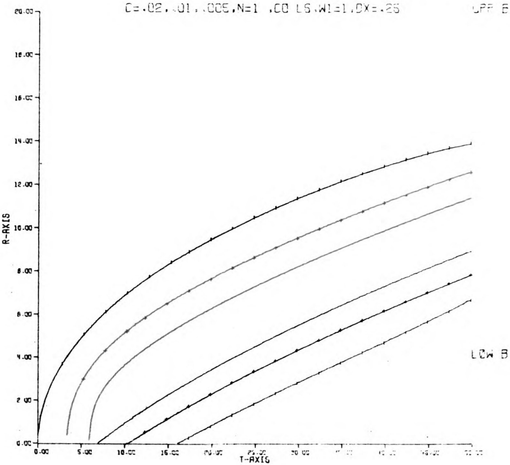
Graph 6.e is similar to 6.a except that $n = 10$.

Graph 6.a

Problem: (c,1,0) in n = 1 dimension

c = .02,.01, .005

t_z = 6.28 and t_m = 17.08 for all three values of c

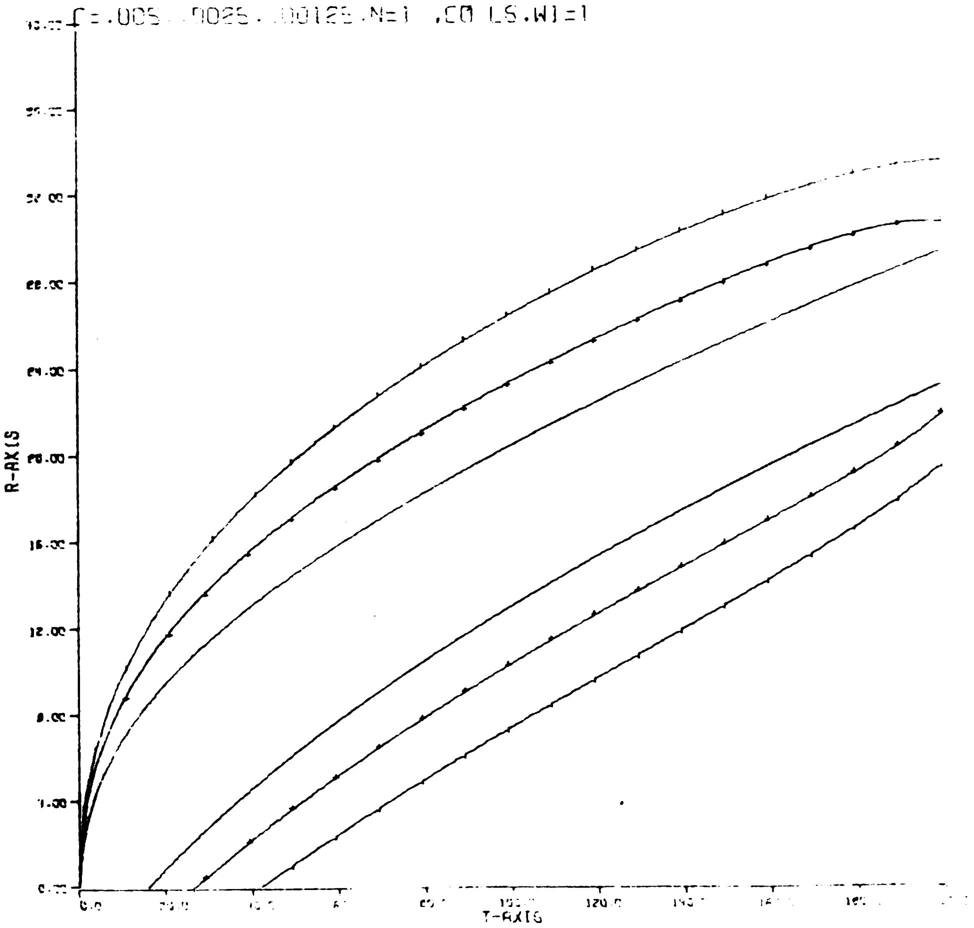


Graph 6.b

Problem: (c,1,0) in n = 1 dimension

$c = .005, .0025, .00125$

$t_z = 6.28$ and $t_m = 17.08$ for all three values of c



RUBIN MH81923

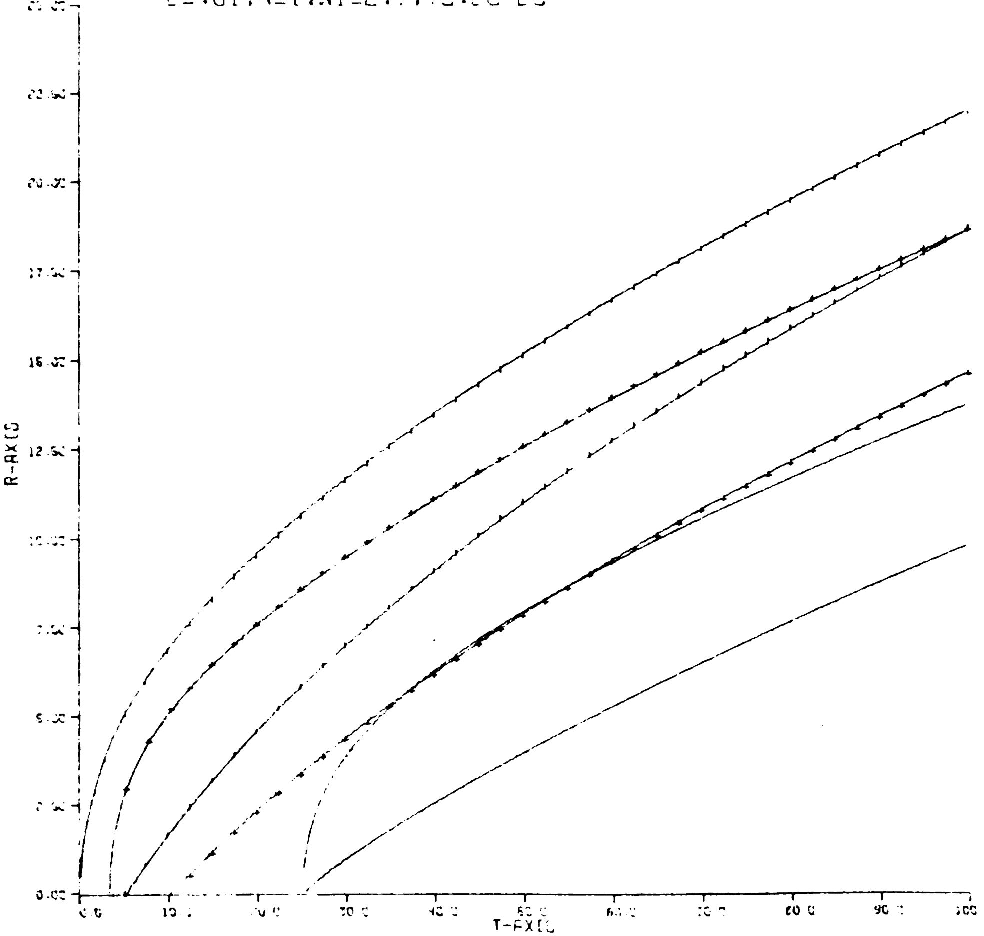
Grpah 6.c

Problem: $(.01, W_1, 0)$ in $n = 1$ dimension

$W_1 = 2, 1, .5$

W_1	t_z	t_m
2	25.13	68.32
1	6.28	17.08
.5	1.57	4.27

C=.01,N=1,W1=2,1,.5,CO LS

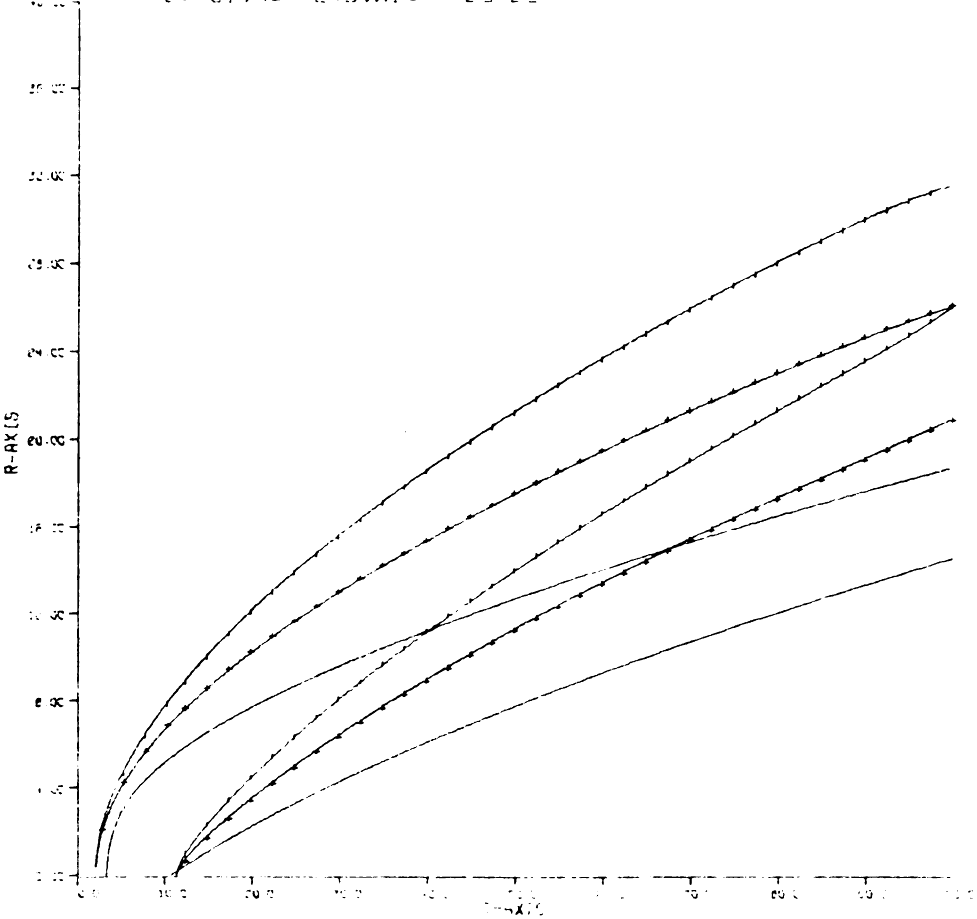


RUBIN MK59308

Graph 6.d

Problem: (.01,1,0) in n = 1,2, and 3 dimensions

$t_z = 6.28$ and $t_m = 17.08$ for all three values of n

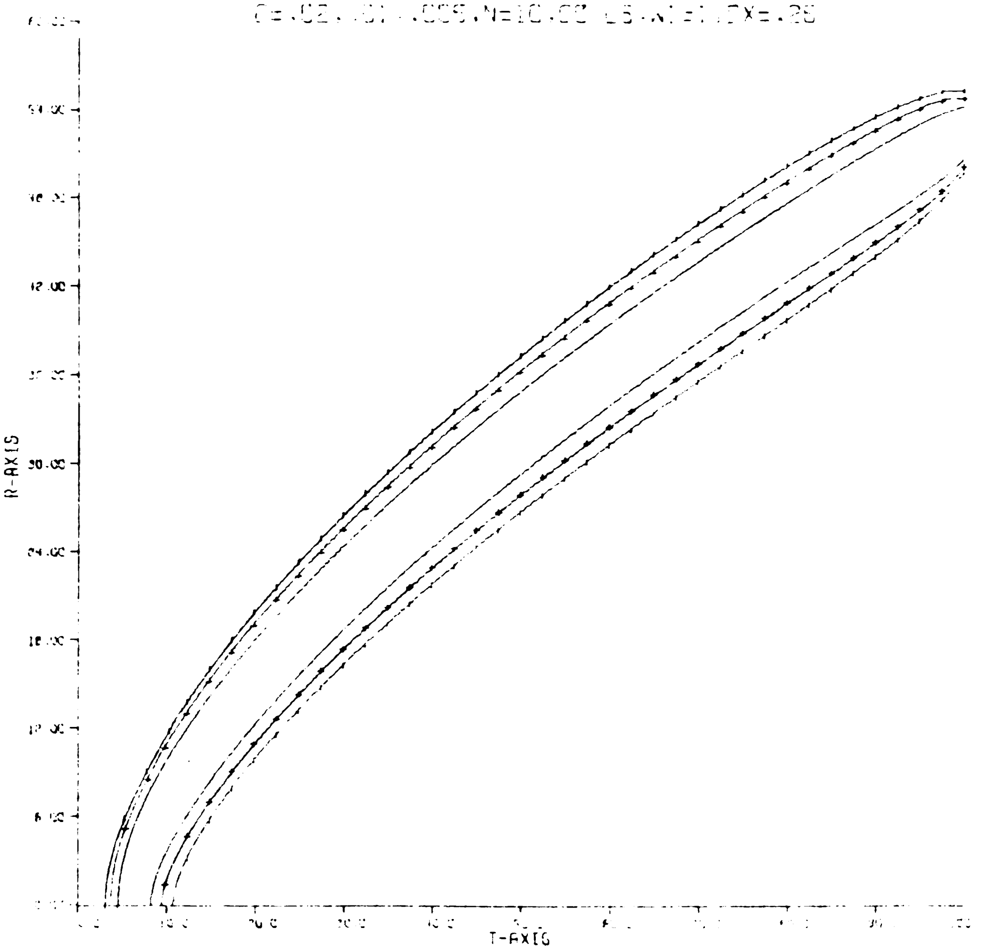


Graph 6.e

Problem: (c,1,0) in n = 10 dimensions

c = .02,.01,.005

t_z = 6.28 and t_m = 17.08 for all three values of c



RUBIN MZ96974

SECTION 7

In this section the squared loss problem is investigated. Graphs will be displayed in the same order as in Section 6.

For the $(c, W_1, 2)$ problem in n dimensions

$$W_1\left(\sum_{L=0}^{1} C_n(L,1)\frac{r^{2L}}{t^{2+L}}\right) = W_1\left(\frac{n}{t} + \frac{r^2}{t^2}\right) ,$$

and $r_I(t)$ is the solution of the equation

$$W_1 U(r,t)\left(\frac{n}{t} + \frac{r^2}{t^2}\right) = 1 \qquad \text{for}$$

$$t > t_z = (nW_1)^{\frac{2}{2+n}} (2\pi)^{\frac{n}{2+n}} .$$

So

$$AWB(t) = \frac{r_I(t)}{2ct}\left(\frac{1}{1+U(r_I(t),t)}\right) = \frac{r_I(t)}{2ct}\left(\frac{W_1\left(\frac{n}{t} + \frac{r_I(t)^2}{t^2}\right)}{\left(1 + W_1\left(\frac{n}{t} + \frac{r_I(t)^2}{t^2}\right)\right)}\right) .$$

However this does not provide an upper bound for the width of the continuation region, and in fact is not nearly as good as the corresponding constant loss approximation.

However, by substituting for $E\|\mu\|^2 = \frac{n}{t} + \frac{r^2}{t^2}$ the quantity

$$\frac{\left(\frac{E\|\mu\|^3}{2}\right)^2}{E\|\mu\|^2} ,$$

with the numerator approximated by expanding $\|\mu\|^3$ around $\frac{r^2}{t^2}$, we obtain a better approximation which does in fact serve as an upper bound in all the situations encountered by the author. Let $v_i = \frac{x_i}{t}$. Then $\mu_i = v_i + Y_i$: Y_i iid $N(0,\frac{1}{t})$. So

$$\|\mu\|^3 = (\Sigma(v_i + Y_i)^2)^{3/2} = (\|v\|^2 + \|Y\|^2 + 2\Sigma\, v_i Y_i)^{3/2}$$

Formally expanding around $\|v\|^2$, we get

$$\|\mu\|^3 = \|v\|^3 + \frac{3}{2}(\|Y\|^2 + 2\Sigma\, v_i Y_i)\|v\| + \frac{3}{8}(\|Y\|^4 + 4\Sigma\, v_i Y_i \|Y\|^2$$

$$+ 4(\Sigma\, v_i^2 Y_i^2 + \sum_{i\neq j}\Sigma\, v_i Y_i v_j Y_j))\frac{1}{\|v\|} \quad \ldots\ldots\ldots\ldots\ldots ,$$

$$E\|\mu\|^3 = \|v\|^3 + \frac{3}{2}\,\frac{(n+1)}{t}\,\|v\| \quad \ldots\ldots\ldots\ldots$$

$$= \frac{r^3}{t^3} + \frac{3}{2}\,(n+1)\frac{r}{t^2} + 0(\frac{1}{rt}) \;.$$

And so if the width of the continuation region is approximated by

$$\bar{a}_1(t) - \bar{a}_0(t) = \frac{r_I(t)}{2ct}\,\left(W_1\left(\frac{\frac{r^3}{t^3} + \frac{3}{2}\,(n+1)\,\frac{r}{t^2}}{\frac{r^2}{t^2} + \frac{n}{t}}\right)^2 \Big/ \left(1 + W_1\left(\frac{\frac{r^3}{t^3} + \frac{3}{2}\,(n+1)\,\frac{r}{t^2}}{\frac{r^2}{t^2} + \frac{n}{t}}\right)^2\right)\right) ,$$

the approximation has properties similar to the corresponding constant loss approximation in Section 6, except that this one does not initially get worse with increasing n. However eventually (in n) it does.

As an approximation which possibly serves as an upper bound in n dimensions for the general (c,W_1,k) problem

$$\left(\frac{E\|\mu\|^{k+1}}{E\|\mu\|^k}\right)^k$$

is suggested.

In section 6 it was shown that $r_I'(t_z)$ is infinite (for $k = 0$). In that section $r_I'(t)$ was given explicitly. Now the slope of the indifference function for general k is found as a function of $r_I(t)$ itself, and shown to be infinite at t_z. $r_I(t)$ depends on W_1, k, and n, but not c.

For fixed k and t, we define

$$W(j,n,r) = \frac{(\frac{r^2}{2t})^j e^{-\frac{r^2}{2t}}}{j!} \left(\frac{2}{t}\right)^{\frac{k}{2}} \frac{\Gamma(j + \frac{n+k}{2})}{\Gamma(j + \frac{n}{2})} .$$

For fixed W_1, k, and n; $r_I(t)$ satisfies, for $t \geq t_z$

$$W_1 U(r_I(t),t) \sum_{j=0}^{\infty} W(j,n,r_I(t)) = 1 . \qquad (1)$$

Differentiating with respect to t,

$$W_1(U_1 r' + U_2)\left(\sum_{j=0}^{\infty} W(j,n,r)\right) + W_1 U\left\{\sum_{j=0}^{\infty}\left(-\frac{k}{2t}\right)W(j,n,r)\right.$$

$$\left. + \sum_{j=0}^{\infty}\left(\frac{r^2}{2t^2} - \frac{rr'}{t}\right)W(j,n,r) + \sum_{j=1}^{\infty} j\left(\frac{rr'}{t} - \frac{r^2}{2t^2}\right)\frac{W(j,n,r)}{(\frac{r^2}{2t})}\right\} = 0 .$$

Now $U_1 = \frac{r}{t} U$ and $U_2 = \left(-\frac{n}{2t} - \frac{r^2}{2t^2}\right)U$. Also

$$\sum_{j=1}^{\infty} \frac{jW(j,n,r)}{(\frac{r^2}{2t})} = \sum_{j=0}^{\infty} W(j,n+2,r) .$$

So

$$W_1\left(-\frac{n}{2t} - \frac{k}{2t}\right)U \sum_{j=0}^{\infty} W(j,n,r) + W_1 U\left[\frac{rr'}{t} - \frac{r^2}{2t^2}\right] \sum_{j=0}^{\infty} W(j,n+2,r) = 0 .$$

By (1)

$$\frac{k+n}{2t} = W_1 U\left[\frac{rr'}{t} - \frac{r^2}{2t^2}\right] \sum_{j=0}^{\infty} W(j,n+2,r) .$$

So

$$\frac{rr'}{t} = \frac{r^2}{2t^2} + \frac{(k+n)}{2t} \frac{1}{W_1 U \sum_{j=0}^{\infty} W(j,n+2,r)} ,$$

$$r' = \frac{r}{2t} + \frac{(k+n)}{2rW_1 U \sum_{j=0}^{\infty} W(j,n+2,r)} ,$$

and since

$$W_1 U(r_I(t),t) = \frac{1}{\sum_{j=0}^{\infty} W(j,n,r_I(t))} ,$$

the slope of the indifference function may be expressed

$$r_I'(t) = \frac{r_I(t)}{2t} + \frac{(k+n)}{2r_I(t)} \frac{\sum_{j=0}^{\infty} W(j,n,r_I(t))}{\sum_{j=0}^{\infty} W(j,n+2,r_I(t))} .$$

For $k = 0$

$$\sum_{j=0}^{\infty} W(j,n,r) = \sum_{j=0}^{\infty} W(j,n+2,r) = 1 .$$

Therefore

$$r_I'(t) = \frac{t^{\frac{1}{2}}[n\ln(\frac{t}{2\pi}) - 2\ln W_1]^{\frac{1}{2}}}{2t} + \frac{n}{2t^{\frac{1}{2}}[n\ln(\frac{t}{2\pi}) - 2\ln W_1]^{\frac{1}{2}}} ,$$

which reduces to the expression given in Section 6.

Since

$$\lim_{t \downarrow t_z} = \lim_{r_I(t) \downarrow 0} \frac{\sum_{j=0}^{\infty} W(j,n,r_I(t))}{\sum_{j=0}^{\infty} W(j,n+2,r_I(t))} = \frac{\Gamma(\frac{k+n}{2})\ \Gamma(\frac{n+2}{2})}{\Gamma(\frac{n}{2})\ \Gamma(\frac{k+n+2}{2})} = \frac{n}{k+n} ,$$

it follows that, as $t \downarrow t_z$,

$$r_I'(t) = O(\frac{1}{r_I(t)}).$$

In the constant loss problem, the maximum width of the continuation region occurs after $\bar{T}_0$ (or before T_0' is found) in most cases, and in fact if c is large, the maximum width occurs around $t_m = et_z$. In the squared loss problem however, the maximum width occurs at $\bar{T}_0$ unless c is small, and is not predictable otherwise.

Graphs 7.a through 7.e appear in the same order as 6.a-6.e.

Graphs 6.e and 7.e appear similar, partially due to the fact that the two values of t_z are close. This can be explained by the fact that for any k, t_z approaches 2π, as $n \to \infty$.

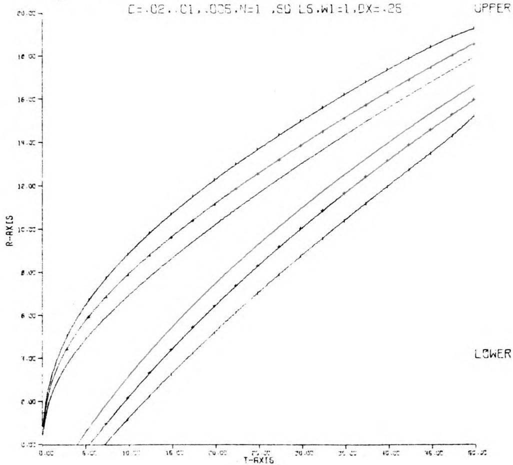
$$t_z = 2\left(W_1 \frac{\Gamma(\frac{k+n}{2})}{\Gamma(\frac{n}{2})}\right)^{\frac{2}{k+n}} \pi^{\frac{n}{k+n}} \to 2\pi \quad \text{as} \quad n \to \infty \quad .$$

Graph 7.a

Problem: (c,1,2) in n = 1 dimension

c = .02,.01,.005

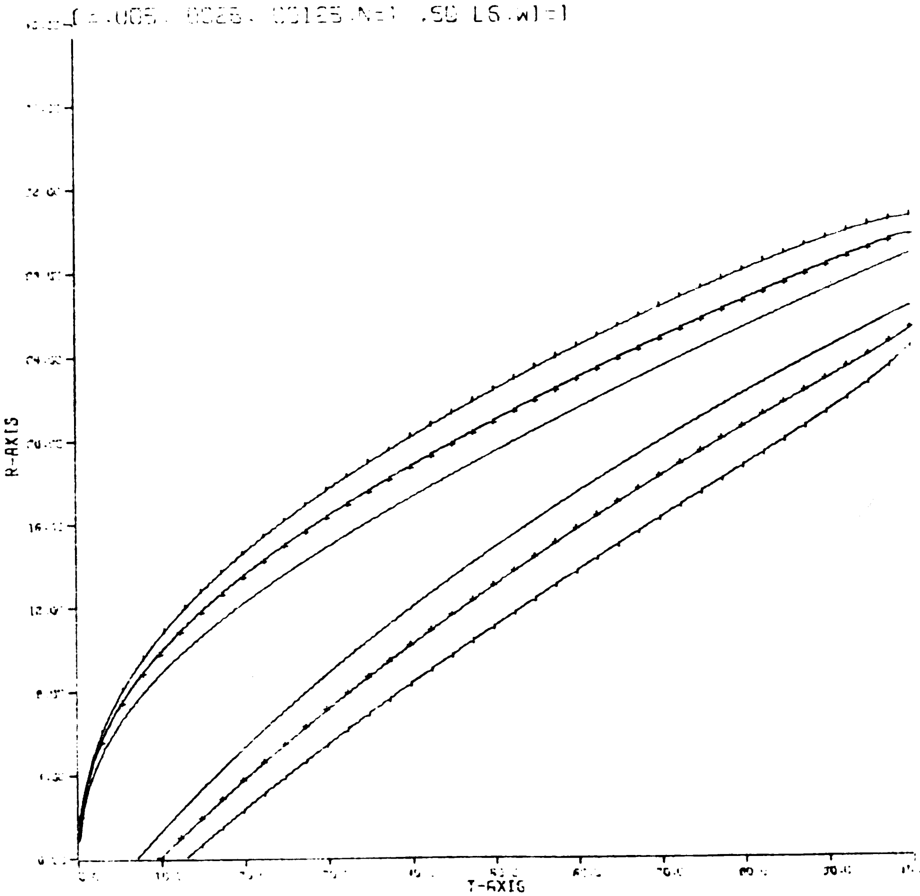
$t_z = 1.85$ for all three values of c



Graph 7.b

Problem: (c,1,2) in n = 1 dimension
c = .005,.0025,.00125

t_z = 1.85 for all three values of c



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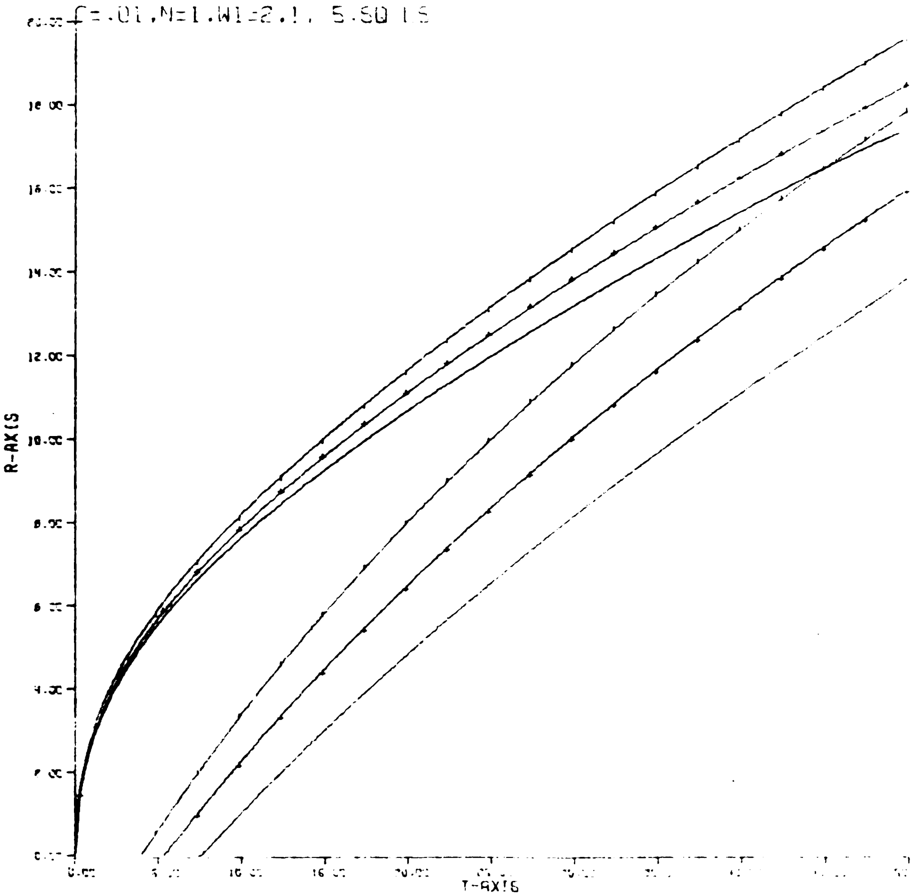
Graph 7.c

Problem: $(.01, W_1, 2)$ in $n = 1$ dimension

$W_1 = 2, 1, .5$

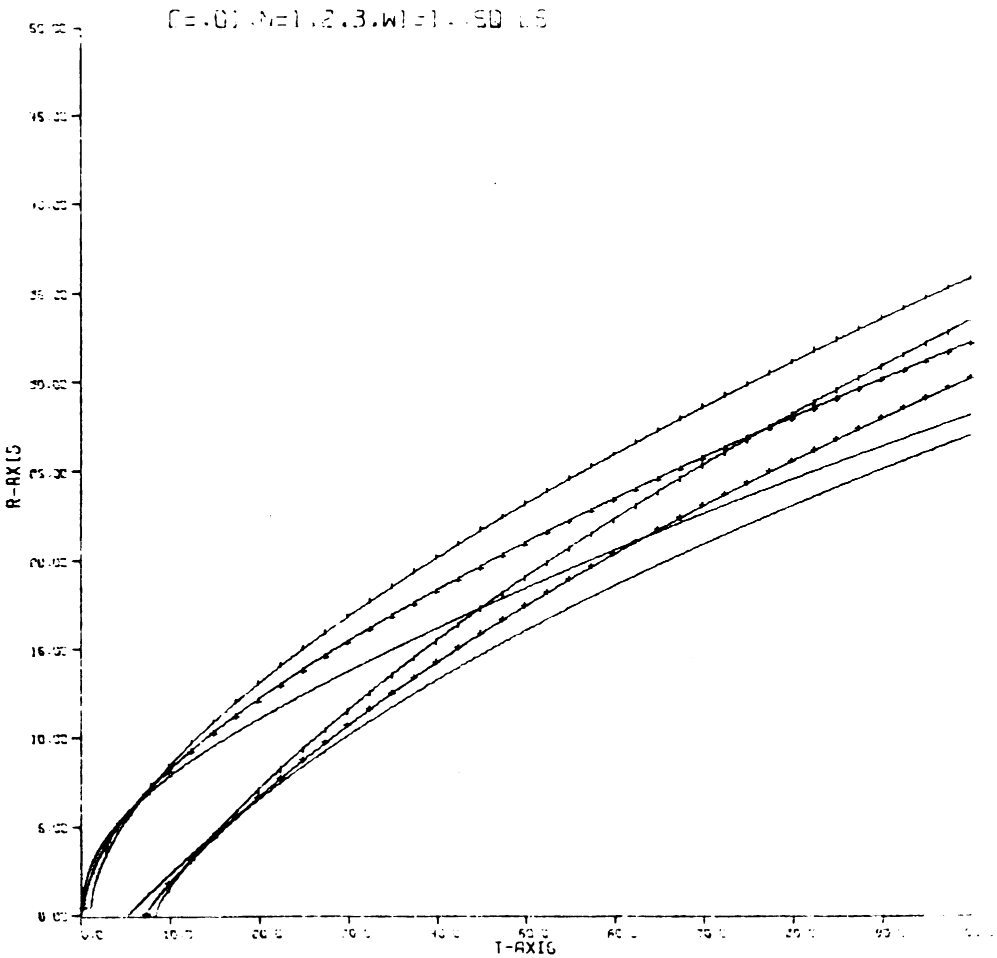
W_1	t_z
2	2.93
1	1.85
.5	1.16



Graph 7.d

Problem: (.01,1,2) in n = 1,2, and 3 dimensions

n	t_z
1	1.85
2	3.54
3	4.67



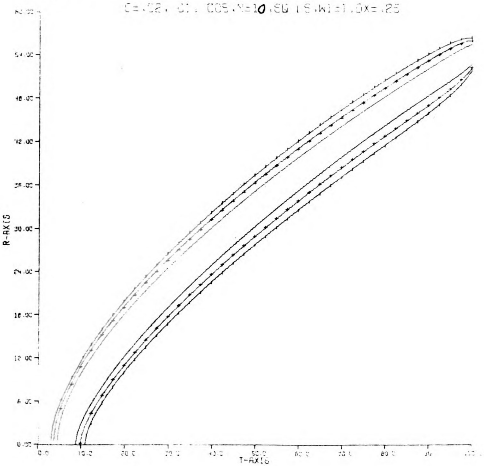
RUBIN MR89670

Graph 7.e

Problem: (c,1,2) in n = 10 dimensions

c = .02,.01,.005

t_z = 6.79 for all three values of c



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APPENDIX

APPENDIX

Printout of program employing the implicit precedure to handle the squared loss problem in any dimensions.

```
      PROGRAM MAIN (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      DIMENSION ST(1000),STF(1000)
      COMMON ID,NL,NLP,NC,C,T,S,V,H,DIM,D1,D3,D4,D5,D6,D7,D8,XL,DX ,STZE
     1,ST,STF,CLA1,CLA2,CRA1,CRA2,BL,BR, FS,FV,FS1,FV1,FS11,FV11
     2,GS12,GV12,DENS,DENV,CUV,CON,Z
C  THIS IS IMPLICIT SQUARED LOSS MULTI DIMENSIONAL
      DO 50 KS=1,3
      ID=100
      W1=1.
      C=.01
      WRITE(6,410) ID
  410 FORMAT(9X8HSQ LS,N=,I3)
      WRITE(6,411) W1
  411 FORMAT(9X3HW1=,F6.4)
      WRITE(6,412) C
  412 FORMAT(9X2HC=,F6.5)
      KLOOK=0
      KERT=10
      DIM=ID
      D5=DIM-1.
      D1=D5*.5
      D3=D1+.5
      D4=D3+1.
      D6=DIM+2.
      D7=DIM+5.
      D8=D6+D6
      TZ=(6.2831853071796**(DIM/D6))*((W1*DIM)**(1./D4))
      WRITE(6,1) TZ
      T=2000.
      CALL PLY(W1)
      DX=2.*(.5**KS)
      D1DX=D1*DX
      DX2=DX+DX
      DXDX=DX*DX
      DELT=DXDX
      U=DELT/DXDX
      G=.5-1./(6.*U)
      CALL CFF(U,G,D1DX)
      WRITE(6,413) DX
  413 FORMAT( 9X3HDX=,F6.4)
      WRITE(6,414) DELT
  414 FORMAT( 9X5HDELT=,F8.6)
      WRITE(6,415) G
  415 FORMAT( 9X15HBEFORE CROSS G=,F8.6)
      GAFT=1.
      KKE=20*(4**(KS-1))
      J=S/DX
      XL=DX*(J+1)
      DEL=XL-S
      DIF=V-XL
```

```
      NCM=DIF/DX
      NC=NCM+1
      DFU=DIF-NCM*DX
      CALL BNDU
      CALL BNDL(W1)
      DO 10 K=1,NC
   10 STF(K)=HP2(XL+DX*(K-1))
      DHT=.5*DFLT
      FONS=1.
   16 DO 15 KK=1,KKF
      D2=DFL+DX
      FST1=(-3./((D2*D2)*D2))*(STF(2)-FS-D2*(FS1-D2*FS11))
      FS12=FST1+D1*(FS11+FS11+FS1/S)/S
      S=GETSL(DHT,FS12,GS12,DENS,S,FONS)
      IF(S) 40,40,41
   41 D2 =DFU+DX
      FVT1=(3./((D2*D2)*D2))*(STF(NCM )-FV+D2*(FV1+D2*FV11))
      FV12=FVT1+D1*(FV11+FV11+CUV)/V
      V=GETSL(DHT ,FV12,GV12,DENV,V,FONS)
      T=T-DFLT
      CALL BNDU
      CALL BNDL(W1)
      NL=(XL-S)/DX+1
      NLP=NL+1
      XL=XL-DX*(NL-1)
      DFL=XL-S
      DIF=V-XL
      NCM=DIF/DX
      IF(NCM-2) 50,803,803
  803 NC=NCM+1
      DFU=DIF-NCM*DX
      DO 11 K=1,NL
      XD=DFL+DX*(K-1)
   11 ST(K)=FS+XD*(FS1-XD*(FS11+XD*FST1/3.))
      XD=DFU
      ST(NC)=FV-XD*(FV1+XD*(FV11-XD*(FVT1)/3.))
      CALL FILL
      DO 13 K=1,NC
   13 STF(K)=ST(K)
   15 CONTINUE
      WRITE(5,2) NC,T
      VMS=V-S
      WRITE(6,1) S,V,VMS
      CALL CUT(W1)
      TOS=H/T
      TOS2=TOS*TOS
      GGT=(TOS2+DIM/T)*W1
      GOOK=TOS*GGT/(2.*C*(1.+GGT))
      RAT=VMS/GOOK
      WRITE(6,1) H,GOOK,RAT
      SIG=DIM/T
      GGT=(TOS2+3.*SIG)/(TOS2+SIG)
      GGT=TOS*GGT
      GGT2=GGT*GGT*W1
      GOOK=TOS*GGT2/(2.*C*(1.+GGT2))
```

```
      RAT=VMS/GOOK
      WRITE(6,1) H,GOOK,RAT
      SAW=HLST-H
      GES=SLST-S
      GEV=VLST-V
      WRITE(6,1) GES,SAW,GEV
      HLST=H
      SLST=S
      VLST=V
      FONS=V/DX
      KLOOK=KLOOK+1
      IF(KLOOK-KERT) 16,253,253
  253 KLOOK=0
      CALL SLK
      GO TO 16
   40 SLST=S+(FS12-GS12)*DHT/DENS
      DHLT=DHT*SLST/(SLST-S)
      D2=DEU+DX
      FVT1=(3./((D2*D2)*D2))*(STF(NCM )-FV+D2*(FV1+D2*FV11))
      FV12=FVT1+D1*(FV11+FV11+CUV)/V
      V=GETSL(DHLT,FV12,GV12,DENV,V,FONS)
      T=(T-DHLT)-DHLT
      CALL BNDU
      STZE=C*T*(1.+Z)+DIM*Z/T
      NC=V/DX
      NCM=NC-1
      XD=V-NC*DX
      ST(NC)=FV-XD*(FV1+XD*(FV11-XD*(FVT1)/3.))
      NL=SLST/DX+1
      NLP=NL+1
      DO 700 K=1,NL
  700 ST(K)=STZE-K*K*DXDX*(C*(1.+Z)-.5*Z*(C+D6/(T*T)))
      UA=(DHLT+DHLT)/DXDX
      GA=.5-1./(6.*UA)
      GA=AMAX1(0.,GA)
      CALL CFF(UA,GA,D1DX)
      CALL FILL
      DO 44 K=1,NC
   44 STF(K)=ST(K)
      WRITE(6,6) NL,V,T
      WRITE(6,2) NC,STZE
      CALL SLK
      DHLT=DHT-DHLT
      D2=XD+DX
      FVT1=(3./((D2*D2)*D2))*(STF(NCM )-FV+D2*(FV1+D2*FV11))
      FV12=FVT1+D1*(FV11+FV11+CUV)/V
      V=GETSL(DHLT,FV12,GV12,DENV,V,FONS)
      T=T-DHLT-DHLT
      CALL BNDU
      NC=V/DX
      NCM=NC-1
      XD=V-DX*NC
      ST(NC)=FV-XD*(FV1+XD*(FV11-XD*(FVT1)/3.))
      UA=(DHLT+DHLT)/DXDX
      GA=.5-1./(6.*UA)
```

```
      GA=AMAX1(0.,GA)
      CALL CFF(UA,GA,D1DX)
      CALL FILL1
      DO 154 K=1,NC
  154 STF(K)=ST(K)
      CALL CFF(U,GAFT,D1DX)
      WRITE(6,416) GAFT
  416 FORMAT(9X14HAFTER CROSS G=,F8.6)
      IF(KK-KKF) 823,824,824
  823 KKFF=KKF-KK
   51 DO 55 KK=1,KKFF
      D2=XD+DX
      FVT1=(3./((D2*D2)*D2))*(STF(NCM )-FV+D2*(FV1+D2*FV11))
      FV12=FVT1+D1*(FV11+FV11+CUV)/V
      V=GETSL(DH T,FV12,GV12,DENV,V,FONS)
      T=T-DELT
      IF(T) 50,50,789
  789 CALL BNDU
      NC=V/DX
      NCM=NC-1
      XD=V-NC*DX
      ST(NC)=FV-XD*(FV1+XD*(FV11-XD*(FVT1)/3.))
      CALL FILL1
      DO 54 K=1,NC
   54 STF(K)=ST(K)
      IF(V-DX2) 50,50,55
   55 CONTINUE
  824 KKFF=KKF/4
      WRITE(6,1) T,V
      WRITE(6,2) NC,STZE
      CALL SLK
      GO TO 51
   50 WRITE(6,6) NL,V,T
  684 STOP
    1 FORMAT(4E30.12)
    2 FORMAT(I5,E30.5)
    6 FORMAT(I5,E85.12,E30.12)
      END
      SUBROUTINE CFF(ZA,ZB,ZC)
      DIMENSION ST(1000),STF(1000)
      COMMON ID,NL,NLP,NC,C,T,S,V,H,DIM,D1,D3,D4,D5,D6,D7,D8,XL,DX ,STZE
     1,ST,STF,CLA1,CLA2,CRA1,CRA2,BL,BR, FS,FV,FS1,FV1,FS11,FV11
      UH=.5*ZA
      CLA1=ZB*UH
      CRA1=UH-CLA1
      CLA2=CLA1*ZC
      CRA2=CRA1*ZC
      BL=1.+CLA1+CLA1
      BR=BL-ZA
      RETURN
      END
      FUNCTION GETSL(ZA,ZB,ZC,ZD,ZE,ZF)
      ST=(ZB-ZC)*ZA/ZD
      ST=AMAX1(0.,ST)
      ST=AMAX1(-ST,-ZF)
```

```
      GETSL=ZE+ST
      RETURN
      END
      SUBROUTINE BNDL(WCON)
      DIMENSION ST(1000),STF(1000)
      COMMON ID,NL,NLP,NC,C,T,S,V,H,DIM,D1,D3,D4,D5,D6,D7,D8,XL,DX ,STZE
     1,ST,STF,CLA1,CLA2,CRA1,CRA2,BL,BR, FS,FV,FS1,FV1,FS11,FV11
     2,GS12,GV12,DENS,DENV,CUV,CON,Z
      SS=S*S*CON
      US=Z*EXP(SS)
      CUS=C*US
      DENS=C+CUS
      FSA=DENS*T
      FS1A=CUS*S
      FS11A=DENS+CUS*(-.5-SS)
      SOT=S/T
      SOT2=SOT*SOT
      FORS1=(-D3-SS)/T
      FORS2=US*SOT
      GS12A=FS1A*FORS1
      US2=US*FORS1
      FSB=-US2-US2
      FAC1=D6/T
      FS1B=FORS2*(FAC1+SOT2)
      FS11B=US*CON*(-FAC1-SOT2*(D7+SS+SS))
      GS12B=FS1B*FORS1+FORS2*(-3.*SOT2-FAC1-FAC1)/T
      FS=FSA+FSB*WCON
      FS1=FS1A+FS1B*WCON
      FS11=FS11A+FS11B*WCON
      GS12=GS12A+GS12B*WCON
      RETURN
      END
      SUBROUTINE BNDU
      DIMENSION ST(1000),STF(1000)
      COMMON ID,NL,NLP,NC,C,T,S,V,H,DIM,D1,D3,D4,D5,D6,D7,D8,XL,DX ,STZE
     1,ST,STF,CLA1,CLA2,CRA1,CRA2,BL,BR, FS,FV,FS1,FV1,FS11,FV11
     2,GS12,GV12,DENS,DENV,CUV,CON,Z
      Z=(6.2831853071796/T)**D3
      CON=.5/T
      VV=V*V*CON
      UV=Z*EXP(VV)
      CUV=C*UV
      DENV=C+CUV
      FV=DENV*T+1.
      FV1=CUV*V
      FV11=DENV+CUV*(-.5-VV)
      GV12=FV1*(-D3-VV)/T
      RETURN
      END
      FUNCTION HP2(X)
      DIMENSION ST(1000),STF(1000)
      COMMON ID,NL,NLP,NC,C,T,S,V,H,DIM,D1,D3,D4,D5,D6,D7,D8,XL,DX ,STZE
     1,ST,STF,CLA1,CLA2,CRA1,CRA2,BL,BR, FS,FV,FS1,FV1,FS11,FV11
      DV=V-X
      DV2=DV*DV
```

```
      DV3=DV2*DV
      DS=X-S
      DS2=DS*DS
      DS3=DS2*DS
      DD=V-S
      D2=DD*DD
      DN=D2*DD
      DP=DV3/DN
      DQ=DS3/DN
      USFS=3.*DS/DD
      USFV=3.*DV/DD
      P0=1.+USFS+6.*DS2/D2
      P1=DS*(USFS+1.)
      Q0=1.+USFV+6.*DV2/D2
      Q1=DV*(-1.-USFV)
      HP2=DP*(P0*FS+P1*FS1-DS2*FS11)+DQ*(Q0*FV+Q1*FV1-DV2*FV11)
    1 FORMAT(6E18.7)
      RETURN
      END
      SUBROUTINE SLK
      DIMENSION ST(1000),STF(1000)
      COMMON ID,NL,NLP,NC,C,T,S,V,H,DIM,D1,D3,D4,D5,D6,D7,D8,XL,DX ,STZE
     1,ST,STF,CLA1,CLA2,CRA1,CRA2,BL,BR, FS,FV,FS1,FV1,FS11,FV11
      N=(NC+7)/8
      DO 1 K=1,N
      K1=8*(K-1)+1
      K2=K1+1
      K3=K1+2
      K4=K1+3
      K5=K1+4
      K6=K1+5
      K7=K1+6
      K8=K1+7
    1 WRITE(6,2) STF(K1),STF(K2),STF(K3),STF(K4),STF(K5),STF(K6)
     1,STF(K7),STF(K8)
    2 FORMAT(8E15.7)
      RETURN
      END
      SUBROUTINE FILL1
      DIMENSION A(1000),Z(1000),ST(1000),STF(1000),D(1000),DEM(1000)
     1,F(1000),E(1000),SF(1000)
      COMMON ID,NL,NLP,NC,C,T,S,V,H,DIM,D1,D3,D4,D5,D6,D7,D8,XL,DX ,STZE
     1,ST,STF,CLA1,CLA2,CRA1,CRA2,BL,BR, FS,FV,FS1,FV1,FS11,FV11
      NK=NC+NC+1
      DO 1 K=1,NC
      M=NK+1-K
      N=NC+1-K
      SF(M)=STF(N)
    1 SF(K)=STF(N)
      NCP=NC+1
      SF(NCP)=STZE
      F(1)=ST(NC)
      E(1)=.0
      X=DX
      NCC=NC-1
```

```
      DO 2 K=1,NCC
      COEFL=CLA2/X
      M=NCP+K
      N=NCP-K
      A(M)=CLA1+COEFL
      Z(M)=CLA1-COEFL
      A(N)=Z(M)
      Z(N)=A(M)
      COFFR=CRA2/X
      X=X+DX
      D(M)=CRA1*(SF(M+1)+SF(M-1))+COEFR*(SF(M+1)-SF(M-1))+BR*SF(M)
    2 D(N)=D(M)
      A(NCP)=DIM*CLA1
      Z(NCP)=A(NCP)
      BLNCP=1.+A(NCP)+A(NCP)
      CRA1N=CRA1*DIM*2.
      D(NCP)=CRA1N*SF(NC)+(1.-CRA1N)*SF(NCP)
      DO 3 K=2,NC
      DEM(K)=BL-Z(K)*E(K-1)
    3 E(K)=A(K)/DEM(K)
      DEM(NCP)=BLNCP-Z(NCP)*E(NC)
      E(NCP)=A(NCP)/DEM(NCP)
      DO 4 K=1,NCC
      M=K+NCP
      DEM(M)=BL-Z(M)*E(M-1)
    4 E(M)=A(M)/DEM(M)
      NKM=NK-1
      DO 5 K=2,NKM
    5 F(K)=(D(K)+Z(K)*F(K-1))/DEM(K)
      DO 6 K=1,NCC
      M=NC-K
      N=NK-K
    6 ST(M)=E(N)*ST(M+1)+F(N)
      STZE=F(NCP)*ST(1)+F(NCP)
    9 FORMAT(2E28.13)
      RETURN
      END
      SUBROUTINE FILL
      COMMON ID,NL,NLP,NC,C,T,S,V,H,DIM,D1,D3,D4,D5,D6,D7,D8,XL,DX ,STZE
     1,ST,STF,CLA1,CLA2,CRA1,CRA2,BL,BR, FS,FV,FS1,FV1,FS11,FV11
      DIMENSION A(1000),Z(1000),ST(1000),STF(1000),D(1000),E(1000)
     1,F(1000),DEM(1000)
      E(1)=0.
      F(1)=ST(NL)
      X=XL+NL*DX
      NCC=NC-NLP
      DO 1 K=1,NCC
      M=K+1
      COEF1=CLA2/X
      A(M)=CLA1+COEF1
      Z(M)=CLA1-COEF1
      DEM(M)=BL-Z(M)*E(K)
      E(M)=A(M)/DEM(M)
      COFFR=CRA2/X
      X=X+DX
```

```
      D(M)=CRA1*(STF(K)+STF(K+2))+COEFR*(STF(K+2)-STF(K))+BR*STF(M)
    1 F(M)=(D(M)+Z(M)*F(K))/DEM(M)
      DO 2 M=1,NCC
      K=NC-M
      MM=K-NLP+2
    2 ST(K)=E(MM)*ST(K+1)+F(MM)
      RETURN
      END
      SUBROUTINE CUT(WCON)
      COMMON ID,NL,NLP,NC,C,T,S,V,H,DIM,D1,D3,D4,D5,D6,D7,D8,XL,DX ,STZE
      CON=.5/T
      Z2=D3*ALOG(6.2831853071796/T)+ALOG(WCON)
      DO 333 K4=1,60
      HH=H*H
      Z1=ALOG((DIM+HH/T)/T)
      Z3=HH*CON
      Y1=Z1+Z2+Z3
      IF(ABS(Y1)-1.E-7) 670,670,335
  335 H=H+.01
      HH=H*H
      Z1=ALOG((DIM+HH/T)/T)
      Z3=HH*CON
      Y2=Z1+Z2+Z3
      H=(H-.01)+Y1*.01/(Y1-Y2)
  333 CONTINUE
  670 RETURN
      END
      SUBROUTINE PLY(WCON)
      DIMENSION FS(2),FV(2),FS1(2),FV1(2),FS11(2),FV11(2),
     1GS12(2),GV12(2),GS22(2),GV22(2),DIF(3),FS12(3),FV12(3),G2(3),G1(3)
     2,DENS(2),DENV(2)
      COMMON ID,NL,NLP,NC,C,T,S,V,H,DIM,D1,D3,D4,D5,D6,D7,D8,XL,DX ,STZE
      DED=1.E-5
      AA=1.E-6
      D11=(2.+D3)*DIM*(1.+D3)
      D12=(3.+D3)*D6+D3*(1.+D3)
      D13=3.+1.5*D3
      H=25.
      CALL CUT(WCON)
      TEM=H*H/T
      SSSS =(DIM*D4+TEM*(D6+.5*TEM))/(H*(D6+TEM))
      VVVV=SSSS
      TOS=H/T
      TOS2=TOS*TOS
      SIG=DIM/T
      GGT=(TOS2+3.*SIG)/(TOS+SIG/TOS)
      GGT2=GGT*GGT*WCON
      Q=GGT2*TOS/(4.*C*(1.+GGT2))
      S=H-Q
      V=H+Q
      SLIM=H-10.*Q
      VLIM=H+10.*Q
      QQ=Q+Q
      ITER=1
      WRITE(6,1) ITER,T,H,QQ
```

```
    1 FORMAT(I5,2E25.12,E65.12)
  405 DO 84 KL=1,ITER
      Z=(6.2831853071796/T)**D3
      CON=.5/T
      DO 183 KM=1,30
      DO 181 N=1,2
      S=S+(N-1)*DFD
      SS=S*S*CON
      US=Z*EXP(SS)
      CUS=C*US
      DENS(N)=C+CUS
      FSA=DENS(N)*T
      FS1A=CUS*S
      FS11A=DENS(N)+CUS*(-.5-SS)
      SOT=S/T
      SOT2=SOT*SOT
      FORS1=(-D3-SS)/T
      FORS2=US*SOT
      GS12A=FS1A*FORS1
      US2=US*FORS1
      FSB=-US2-US2
      FS1B=FORS2*(D6/T+SOT2)
      FS11B=US*.5*(-D6/(T*T)-SOT2*(D7/T+SOT2))
      GS12B=FS1B*FORS1+FORS2*((-3.*SOT2-D8/T)/T)
      FS(N)=FSA+FSB*WCON
      FS1(N)=FS1A+FS1B*WCON
      FS11(N)=FS11A+FS11B*WCON
  181 GS12(N)=GS12A+GS12B*WCON
      DO 182 N=1,2
      V=V+(N-1)*DFD
      VV=V*V*CON
      UV=Z*EXP(VV)
      CUV=C*UV
      DENV(N)=CUV+C
      FV(N)=DENV(N)*T+1.
      FV1(N)=CUV*V
      FV11(N)=DENV(N)+CUV*(-.5-VV)
  182 GV12(N)=FV1(N)*(-D3-VV)/T
      DO 121 K=1,3
      K2=(K+3)/3
      K3=(K+7)/3
      K1=K3-K2
      F=S+(K1-2)*DFD
      F=V+(K2-2)*DFD
      DIF(K)=FV(K2)-FS(K1)
      W=F-F
      W2=W*W
      W3=W2*W
      C21S=60.*DIF(K)/W3
      C22S=(-36.*FS1(K1)-24.*FV1(K2))/W2
      C23S=(18.*FS11(K1)-6.*FV11(K2))/W
      C21V=C21S
      C22V=(-24.*FS1(K1)-36.*FV1(K2))/W2
      C23V=(6.*FS11(K1)-18.*FV11(K2))/W
      FST1=-(C21S+C22S+C23S)/2.
```

```
      FST2=(D1*FS1(K1)/F+D5*FS11(K1))/F
      FS12(K)=FST1+FST2
      FVT1=-(C21V+C22V+C23V)/2.
      FVT2=(D1*FV1(K2)/F+D5*FV11(K2))/F
      FV12(K)=FVT1+FVT2
      FAC2=DENS(K1)*(-2.)
      FAC3=FS12(K)-GS12(K1)
      FAC5=DENV(K2)*(-2.)
      FAC6=FV12(K)-GV12(K2)
      G1(K)=(-FAC3)/FAC2-SSSS
  121 G2(K)=(-FAC6)/FAC5-VVVV
      C1=G1(2)-G1(1)
      C2=G2(2)-G2(1)
      C3=G1(3)-G1(1)
      C4=G2(3)-G2(1)
      D=C1*C4-C2*C3
      A=(C3*G2(1)-C4*G1(1))/D
      B=(C2*G1(1)-C1*G2(1))/D
      S=S+(A-1.)*DFD
      V=V+(B-1.)*DFD
      IF(S-H) 501,501,400
  501 IF(S-SLIM) 400,400,502
  502 IF(V-VLIM) 503,503,400
  503 IF(ABS(G1(1))+ABS(G2(1))-1.E-5) 190,190,183
  183 CONTINUE
  190 DO 83 KK=1,60
      DO 81 N=1,2
      S=S+(N-1)*DFD
      SS=S*S*CON
      US=Z*EXP(SS)
      CUS=C*US
      DENS(N)=C+CUS
      FSA=DENS(N)*T
      FS1A=CUS*S
      FS11A=DENS(N)+CUS*(-.5-SS)
      SOT=S/T
      SOT2=SOT*SOT
      FORS1=(-D3-SS)/T
      FORS2=US*SOT
      GS12A=FS1A*FORS1
      US2=US*FORS1
      FSB=-US2-US2
      FS1B=FORS2*(D6/T+SOT2)
      FS11B=US*.5*(-D6/(T*T)-SOT2*(D7/T+SOT2))
      GS12B=FS1B*FORS1+FORS2*((-3.*SOT2-D8/T)/T)
      FS1(N)=FS1A+FS1B*WCON
      FS(N)=FSA+FSB*WCON
      FS11(N)=FS11A+FS11B*WCON
      GS12(N)=GS12A+GS12B*WCON
      GS22A=US2*C*(1.-D3-SS)+CUS*SS/T
      GS22A=US2*C*(1.-D3-SS)+CUS*SS/T
      GS22B=US*(D11/(T*T*T)+SOT2*(D12/(T*T)+SOT2*(D13/T+.25*SOT2)))
   81 GS22(N)=GS22A+GS22B*WCON
      DO 82 N=1,2
      V=V+(N-1)*DFD
```

```
      VV=V*V*CON
      UV=Z*EXP(VV)
      CUV=C*UV
      DENV(N)=CUV+C
      FV(N)=DENV(N)*T+1.
      FV1(N)=CUV*V
      FV11(N)=DENV(N)+CUV*(-.5-VV)
      GV12(N)=FV1(N)*(-D3-VV)/T
      UV2=UV*(-D3-VV)/T
   82 GV22(N)=UV2*C*(1.-D3-VV)+CUV*VV/T
      DO 21 K=1,3
      K2=(K+3)/3
      K3=(K+7)/3
      K1=K3-K2
      E=S+(K1-2.)*DFD
      F=V+(K2-2.)*DFD
      DIF(K)=FV(K2)-FS(K1)
      W=F-E
      W2=W*W
      W3=W2*W
      W4=W3*W
      C11V=360.*DIF(K)/W4
      C11S=-C11V
      C12S=(192.*FS1(K1)+168.*FV1(K2))/W3
      C13S=(48.*FV11(K2)-FS11(K1)*72.)/W2
      C12V=(-168.*FS1(K1)-192.*FV1(K2))/W3
      C13V=(48.*FS11(K1)-72.*FV11(K2))/W2
      C21S=60.*DIF(K)/W3
      C22S=(-36.*FS1(K1)-24.*FV1(K2))/W2
      C23S=(18.*FS11(K1)-6.*FV11(K2))/W
      C21V=C21S
      C22V=(-24.*FS1(K1)-36.*FV1(K2))/W2
      C23V=(6.*FS11(K1)-18.*FV11(K2))/W
      FST1=-(C21S+C22S+C23S)/2.
      FST2=(D1*FS1(K1)/E+D5*FS11(K1))/E
      FS12(K)=FST1+FST2
      FST3=(C11S+C12S+C13S)/4.
      FST4=-D5*FST1/E+D1*(1.-D1)*(2.*FS11(K1)+FS1(K1)/E)/(E*E)
      FS22=FST3+FST4
      FVT1=-(C21V+C22V+C23V)/2.
      FVT2=(D1*FV1(K2)/F+D5*FV11(K2))/F
      FV12(K)=FVT1+FVT2
      FVT3=(C11V+C12V+C13V)/4.
      FVT4=-D5*FVT1/F+D1*(1.-D1)*(2.*FV11(K2)+FV1(K2)/F)/(F*F )
      FV22=FVT3+FVT4
      FAC1=FS22-GS22(K1)
      FAC2=DENS(K1)*(-2.)
      FAC3=FS12(K)-GS12(K1)
      FAC4=FV22-GV22(K2)
      FAC5=DENV(K2)*(-2.)
      FAC6=FV12(K)-GV12(K2)
      G1(K)=FAC3*FAC3-FAC2*FAC1
   21 G2(K)=FAC6*FAC6-FAC5*FAC4
      C1=G1(2)-G1(1)
      C2=G2(2)-G2(1)
```

```
      C3=G1(3)-G1(1)
      C4=G2(3)-G2(1)
      D=C1*C4-C2*C3
      A=(C3*G2(1)-C4*G1(1))/D
      B=(C2*G1(1)-C1*G2(1))/D
      SLST=S-DFD
      VLST=V-DFD
      S=S+(A-1.)*DFD
      IF(S-SLIM) 400,400,401
  401 V=V+(B-1.)*DFD
      IF(V-VLIM) 402,402,400
  402 IF(ABS(G1(1))+ABS(G2(1))-AA) 90,90,91
   91 IF(ABS(S-SLST)+ABS(V-VLST)-1.E-8) 90,90,83
   83 CONTINUE
   90 GOOK=V-S
      WRITE(6,409) KK,KM,KEEP,T,S,V,GOOK
  409 FORMAT(3I5,4E30.12)
      SLOK=GOOK/QQ
    2 FORMAT(2E40.10 )
      SSSS=(FS12(1)-GS12(1))*.5/DENS(1)
      VVVV=(FV12(1)-GV12(1))*.5/DENV(1)
      S=S-5.*SSSS
      V=V-5.*VVVV
      SLIM=SLIM-SSSS*5.
      VLIM=VLIM-VVVV*5.
   84 T=T-5.
      S=S+2.5*(FS12(1)-GS12(1))/DENS(1)
      V=V+2.5*(FV12(1)-GV12(1))/DENV(1)
      T=T+5.
      KEEP=0
   85 RETURN
  400 KEEP=KEEP+1
      IF(KEEP-10) 403,404,404
  403 FCT=1.-KEEP/10.
      S=H-Q*FCT
      V=H+Q*FCT
      GO TO 405
  404 STOP
      END
```