CASE STUDIES OF HIGH SCHOOL STUDENTS USING PHYSICAL MODELS TO STUDY MATHEMATICAL SYSTEMS

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ABSTRACT

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by

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Purpose

The purpose of this study was to examine in detail the cognitive outcomes of high school students using physical models to study the structure and nature of mathematical systems. By mathematizing some physical representation of a phenomena and studying that model, the study was designed to explore how well the students could deal effectively with mathematical models, understand the nature of an axiomatic system and the process of logically deducing propositions for investigations.

Procedure

The students chosen for this study were selected from intermediate algebra classes taught at Suncoast High School, Riviera Beach, Florida. During the first semester of the 1972-73 school term, students were introduced to the operations and properties of the real numbers in an algebraic setting with emphasis on the postulational procedure which included operations and properties of a group. At the beginning of the second semester, ten highly motivated and above

average students were selected to complete a ten-week schedule of activities instead of their regular class work. All sessions were recorded on audio tapes as the students worked in small groups or individually. Ten activity-based exercises, most of which were adapted from Laboratory Manual for Elementary Mathematics by Fitzgerald, et al., were the source of techniques and procedures. The results of the study were reported using the case study procedure.

Findings

The results of the research demonstrates that within the constraints of normal classroom conditions, high school students can achieve a higher level of understanding the entire nature of: (1) model building both physical and abstract, (2) the axiomatic process, and (3) the process by which propositions are logically deduced from other assumptions and proved.

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By

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A great debt of gratitude is due the members of my doctoral committee: Dr. William M. Fitzgerald, chairman, Dr. John Wagner, Dr. Joe Byers and Dr. Glen Anderson. Each contributed tremendously in his own manner along the way to the results realized in this report.

I owe a special debt of gratitude to generous and audacious students at Suncoast High School who participated in this study over the two year period.

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CHAPTER I

INTRODUCTION

Ideas in mathematics once thought to be too difficult for high school students to study are now commonplace in many secondary school curricula. Many of the ideas fundamental to understanding the nature of mathematical structure are no longer reserved for advanced courses or specialized enrichment programs. On matrix theory, Davis makes the following observation:

> A generation ago, the subject was taught as an intermediate or first-year graduate course in college and was taken by majors in mathematics and theoretical physics. The School Mathematics Study Group (SMSC) has been instrumental in introducing matrix theory into the secondary curriculum.[1]

Fuller found analytic geometry suitable for above average students when he wrote:

> Inasmuch as this particular course is taught almost exclusively in colleges, this publication is designed for college freshmen. This book finds ready application in high school which provide for such a study for their mathematically inclined students.[2]

Brumfiel implies the postulation method to correct logical deficiencies in high school geometry when he cited:

> It is common belief that plane geometry was completed by Euclid 2000 years ago and that nothing has been added to it or

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[1]Philip J. Davis, The Mathematics of Matrices: A First Book of Matrix Theory and Linear Algebra (Boston, Mass.: Ginn and Company, 1965) pp. vi-vii.

[2]Gordon Fuller, Analytic Geometry (Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1964) p. v.

> taken from it since. This is simply not true. That these are logical gaps in Euclid's presentation has been known for a long time. Means to remedy these deficiencies have been known for about sixty years, but strangely enough a mathematically adequate and yet elementary treatment of plane geometry in the spirit of Euclid has not appeared in print. This text represents an earnest effort to do this. The current interest in improvement of the secondary school curriculum makes this an appropriate time for such an attempt.[3]

These ideas have been incorporated into the high school curriculum as a result of the efforts put forth by mathematicians, mathematics educators, classroom teachers, administrators, national curricula study groups and committees. This change in the curriculum has given rise to a challenge in which experimentation and exploration with ideas and activities can be realized. Those ideas and activities when properly implemented, can lead the high school student to a greater understanding of the nature of mathematical structure and the deductive process therein. Mathematics should be presented in a manner which is conducive to developing critical and creative thinking at all levels.

Rationale for the Study

The purpose of this study is to observe and report in detail the cognitive attainments of high school students as they study the mathematical properties which are embodied in physical models. This writer believes that such an examination will reveal how well individual high school students can in fact deal with the ideas of mathematical systems. By mathematizing some physical presentation of a phenomena then constructing a mathematical model of the phenomena and studying that model. This study will explore the students' ability to deal effectively with mathe-

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[3]Charles F. Brumfiel, Robert E. Eicholz, and Merrill E. Shanks, Geometry (Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1960) p. ix.

matical models, the nature of an axiomatic system, and the process of logically deducing propositions for investigation. Blattner makes the following observation concerning models to support this notion:

> In the beginning of an axiomatic study of a mathematical system, it is instructive to examine models of the situation, for models have an unexcelled power of clarifying concepts and suggest proper questions for investigations.[4]

The high school student can play the "mathematical game" of setting certain rules and understanding certain assumptions as they are related to the idealization of an axiomatic system by means of a physical model.

Mathematical systems are characterized by certain operations and properties; this study makes full use of physical models in the real world to represent those operations and properties so that in the final process, abstract properties of the mathematical system can be understood. This notion is supported by the Cambridge Conference which reported that:

> . . . Every application of mathematics depends on a model, and the value of the deduction is more an attribute of the model than it is of the mathematics. We believe that students can be made aware of the distinction between the real world and its various mathematical models.[5]

Concrete operations can be executed by the student with a physical model and related to an analogous operation in a mathematical system. Physical models can be used to formulate properties which can be related to analogous properties in a mathematical system. This process of mathematizing a physical phenomena leads to a fuller understanding of the

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[4]John W. Blattner, Projective Plane Geometry (San Francisco: Holden-Day, Inc., 1968) p. 17.

[5]The Report of the Cambridge Conference on School Mathematics, The Goals for School Mathematics (Boston, Mass.: Houghton Mifflin Company, 1963) p. 47.

relationship between physical models and mathematical systems as models was expressed by Meserve when he wrote:

> In recent years the emphasis throughout mathematics has shifted to thinking of the mathematical system as a model of the physical situation. The interrelation between a physical model (representation, application) of the mathematical system and the mathematical model (system, representation, abstraction) are being recognized as a basic aspect of mathematics. The trend toward thinking of the mathematical system as a model is based, at least in part, on the use of a variety of mathematical models to represent different aspects of and different approximations of the same-physical situation. The process of model building is used at all levels and in all branches of mathematics.[6]

The simplicity employed in formulating the operations and properties of physical models enhances the high school student's ability to transfer a few basic operations and properties that can characterize abstract mathematical systems. Although the models will vary with physical representation, an abstract pattern can be formulated which will identify many different and revealing consequences which are familiar to the student from his previous experiences in the study of mathematics. The importance of this simplicity is recognized by Anderson who made the following observation:

> One striking characteristic of a mathematical model is its simplicity. In designing a mathematical model, we try to focus our attention on the important ideas and ignore the irrelevent ones.[7]

Since the operations and properties of the physical models are basic and simple, they can be changed at will and a variety of physical representations can be employed. The basic operations and properties of mathematical systems can likewise be altered. By making changes in

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[6]Bruce E. Meserve, "An Improved Year of Geometry," The Mathematics Teacher, LXV (February, 1972) pp. 177-78.

[7]Richard Anderson, Jack Garon, and Joseph Gremillian, School Mathematics Geometry (Boston, Mass.: Houghton Mifflin Company, 1966) Chapter 12.

certain postulates of a mathematical model, new systems can be derived. Lick points out the following:

> Mathematics as a study of deductive systems allows great, even unlimited flexibility of individual innovations, inventions, and creativity. In no other discipline is this true. By slightly altering some of the axioms, or definitions, one can create whole new systems. For example, changing a few of the basic axioms of high school geometry, can lead to new and different geometries (i.e. Euclidean versus non-Euclidean).[8]

It is the opinion of this writer that lateral transfers of concrete ideas of operations and properties acquired during the study of physical models can be made to study the abstract structure of mathematical systems. The ultimate realization is one in which students use such acquired knowledge and understanding to reveal the structure and nature of the deductive mathematical systems which are analogs of the physical models. The Cambridge Conference reported the following: " . . . It is only when the model is fully formulated that the purely deductive methods of mathematics takes over."[9] Lick summarizes the challenge proposed in this study when he stated:

> Mathematics is abstract, mathematics is not nature. However, the key to the study of nature and natural phenomena is the concept of the mathematical model. That is, a mathematical system can be chosen that its terms and assumptions have some meaningful relation to the physical world and so may be a model for a physical situation. This is another beautiful aspect of mathematics; in one instance it may be used as a tool with application to models representing physical phenomena, and in another it may be an abstract discipline in and of itself . . . whether pure (i.e. completely abstract) or applied (i.e. having application to physical phenomena), the creation of a mathematical system or model, the study of this entity, and the derivation of consequences and results from it are

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[8]Dale Lick, "Why Not Mathematics," The Mathematics Teacher, LXIV (January, 1971) p. 85.

[9]The Report of the Cambridge Conference on School Mathematics, The Goals for School Mathematics (Boston, Mass.: Houghton Mifflin Company, 1963) p. 47.

steps in a process than can be extremely challenging and stimulating (as well as frustrating at times). This process can and should take place at every level of mathematical training and study.[10]

Pilot Study

In order to realize some of the aforementioned suggestions dealing with constructing a mathematical model of a physical situation this investigator conducted a pilot study during the 1971-72 school year. The student population consisted of five classes of first year geometry at Suncoast High School, Riviera Beach, Florida.

One significant feature of geometry is that it can be characterized as a mathematical model of physical space. The pilot study was designed to expose the students to some activities that developed a meaningful relation between the terms (undefined, defined) and assumptions (postulates, theorems) of a mathematical model and physical objects in the real world; and the nature of an axiomatic system in which the process of logically deducing propositions was applied. The study included topics in finite geometries based on a mathematical model of a finite number of points and the process of deducing propositions from other assumptions of the model. To emphasize the importance of the postulational method, a non-Euclidean geometry was introduced and studied. This non-Euclidean geometry was characterized basically by a simple alteration of a postulate and some pertinent definitions of a "kind" of geometry studied earlier. The results obtained from the pilot study served as a guide in assisting this investigator to organize and complete the study for this report.

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[10]Dale Lick, "Why Not Mathematics," The Mathematics Teacher, LXIV (January, 1971) p. 85.

Finally, the most important significance of this study lies in the fact that it observes a situation which affords students almost unlimited opportunities to do some critical and creative thinking in terms of relating that which is concrete (operations and properties) and derived from the physical models to that which is abstract (operations and properties) and applied to the mathematical system. To assess the significance of this study, the following nonstatistical hypothesis is stated.

Hypothesis

> When selected, bright high school students are placed in a circumstance where they can study mathematical models which are embodied in physical models, they will demonstrate their abilities to deal effectively with mathematical models; the nature of an axiomatic system, and the process of logically deducing propositions for investigations.

The case study procedure is the technique used to determine how well the students were able to realize the expectations stated in the hypothesis.

Organization of the Final Report

Chapter I includes an introduction, rationale for the study, a description of the pilot study, and the assessment hypothesis. Chapter II includes an introduction, a review of the literature and a summary. Chapter III contains the procedure used in the study. An introduction is included; a description of the activities, a description of the students, and a summary of selected test results of students participating in the study. Chapter IV includes an introduction, a case study report of each activity as listed in Chapter III of this report and a summary. Chapter V includes the summary, discussion recommendations for further research and the conclusions.

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CHAPTER II

A REVIEW OF THE LITERATURE

Introduction

To deal effectively with the kind of report investigated in this study, an out of the ordinary type of assessment was used to evaluate how well the individuals involved in this study performed. The assessment procedure used to evaluate the results obtained from this investigation is the case study approach.

A close examination of individual, and/or small group performance is of utmost importance to the appropriate evaluation of the results obtained, and it was felt that the case study approach would serve the best purpose in that from the point of view of research, this approach uses an intensive investigation of the activity, individual and/or group under observation. Good reports that the most important step in the case study is to identify the unit for investigation in the form of some aspect of an observed behavior or recorded activity.[1] This study deals with the activities of students investigating specific activity-based concepts which are observable and recorded. The literature however, did not contain any major studies at the secondary level similar in nature to the study in this report.

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[1] Carter V. Good, "The Individual and Case Study: Diagnosis and Therapy," Essentials of Educational Research (New York: Appleton-Crofts, 1966) p. 313.

A major problem at the time of this writing is the limitations imposed upon this investigator due to the almost complete absence of similar studies. Shaughnessy makes mention of the work advanced by Zoltan Dienes who advocates the use of physical materials and games in a manipulative manner before moving gradually to formal mathematical symbols and abstract systems.[2] Though most of Dienes work has been done on the primary and intermediate levels, his results have implications for all levels of mathematics.

Review of the Literature

As was pointed out earlier in this report, research on activity-based mathematical studies at the secondary level are virtually non-existent; and none have been found that reported their results using the case study method. This, however, does not mean that there are no studies dealing with the basic ideas of mathematical systems or models; the nature of axiomatic systems or the process of logically deducing propositions from assumptions for investigations. The pilot study (Chapter I) for this report dealt specifically with the process of characterizing geometry as a mathematical model of physical space by relating physical objects of the real world to components of a mathematical model, and deducing propositions from certain assumptions in the model for investigation. Adler makes the following point concerning the nature of geometry as a mathematical model:

> Students who understand the nature of deductive reasoning and of inductive reasoning can be led to understand what is

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[2]J. Michael Shaughnessy, "Research in Laboratory Approaches to Mathematics at the Secondary and College Levels" (unpublished paper presented at Michigan State University, 1973).

meant when we say Euclidean geometry is a mathematical model of physical space because it has these four characteristics:

1. Each undefined term and each defined term in Euclidean geometry is associated with some physical object.
2. The axioms of Euclidean geometry express certain assumed relationships of these undefined terms.
3. Theorems of Euclidean geometry are deductions from these axioms that express other relationships among the undefined terms and/or the defined terms.
4. If the axioms and theorems of Euclidean geometry are interpreted to be assertions about the physical objects associated with its terms, then to the extent that these assertions have been tested by experiment they have been found to be approximately true.[3]

The four characteristics mentioned above are readily adaptable to the three major aspects under investigation in this study which are: (1) the study of physical models with concrete operations and properties and are adaptable to analogous abstract operations and properties of mathematical systems; (2) the nature of axiomatic systems derived from the study of the models; and (3) the process involved when propositions are logically deduced from assumptions of the systems and investigations. The combined realization of these aspects should provide opportunities for secondary school students to do some critical and creative thinking.

Several studies reviewed in the literature were similar in nature to this study in that they dealt with one or more of the above mentioned aspects or some combination of them. Lewis demonstrated that teaching a course in high school geometry based on, or in part on, the components of a mathematical model showed evidence to support the fact that students developed in reflective thinking in non-mathematical areas far greater than in either the traditional course in the subject or no

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[3]Irvin Adler, "What Shall We Teach in High School Geometry?", The Mathematics Teacher, LXI (March, 1968) pp. 227-28.

exposure to the subject at all.[4] When geometry is organized for the specific purpose to further the ability to think critically, when materials are developed to focus upon this aim, when the teaching method was directed to this end, then the students would come out of the course competent in their ability to do plane geometry and more competent in their analysis of non-mathematical issues than if they had been exposed to the traditional course.

The most influential and widely distributed study dealing with the nature of a mathematical system and the inference of proofs was conducted by Fawcett.[5] In the study, students developed through class discussion a set of undefined terms, definitions and assumptions as the foundation on which they erected their geometric edifice. During the investigation, it was assumed that a student understood the nature of a deductive proof when the following was accomplished: (1) the place and significance of undefined concepts in proving any conclusion, (2) the necessity for clearly defined terms and their effect on the conclusion, (3) the necessity for assumptions or unproved propositions, (4) that no demonstration proves anything that is not implied by the assumptions. The investigator concluded in the report that:

> Mathematical methods illustrated by a small number of theorems yield a control of the subject matter of geometry at least equal to that obtained from the usual formal course by following (appropriate) procedure . . . it is possible to improve the reflective thinking of secondary school pupils . . . This improvement is general in character and transfers to a variety of situations.[6]

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[4]Harry Lewis, "An Experiment in Developing Critical Thinking Through the Teaching of Plane Demonstrative Geometry," (unpublished doctoral thesis, New York University, 1950).

[5]Harold P. Fawcett, The Nature of Proof, Thirteenth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1938).

[6]*Ibid*.

The investigation conducted in this report relies greatly on the reflective thinking of high school students and their ability to transfer knowledge acquired to a variety of familiar as well as unfamiliar situations. Ulmer conducted a major study in which he pointed out that it is possible to cultivate reflective thinking under normal classroom conditions without sacrificing an understanding of geometric relationships; and that students at all levels are capable of profiting from a method when definite provisions are made to study methods of thinking as an important end in itself.[7] Another study that investigated the role of logical proofs and critical thinking was conducted by Platt.[8] In the process of the study, an evaluation was made of the effect of the use of mathematical logic in high school geometry on: (1) the achievement of students in high school geometry, (2) achievement in reasoning in geometry, (3) critical thinking of students, and (4) attitude of students toward logic, deduction, and proofs in mathematics. The investigator's analysis of the results appeared to support the following conclusions: (1) mathematical logic is an appropriate area of study well within the capability of successful high school students, (2) there is no loss of achievement in geometry caused by devoting time to the study of mathematical logic even in the traditional course, (3) including instruction in mathematical logic appears to produce a more effective treatment of high school geometry with high achieving students in its effect upon student achievement in reasoning in geometry.

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[7]Gilbert Ulmer, "Teaching Geometry to Cultivate Reflecting Thinking: An Experimental Study with 1239 High School Pupils," Journal of Experimental Education (September, 1939).

[8]John L. Platt, "The Effect of the Use of Mathematical Logic in High School Geometry: An Experimental Study" (unpublished doctoral thesis, Colorado State College, 1967).

The investigator reported, however, that the inclusion of mathematical logic in high school geometry does not result in a course which is significantly superior to traditional courses in its over-all effect upon student achievement in reasoning in geometry, deductive thinking and proof in mathematics.

The results of this report deal with a variety of physical models that were used to derive unfamiliar mathematical systems and the procedures used to deduce some theorems (propositions) from the assumptions of systems and prove them. The pilot study dealt with unfamiliar mathematical systems by studying a non-Euclidean and finite geometries as mathematical systems and using the components (i.e. undefined terms, definitions, assumptions, theorems and laws of logic) of the system to deduce propositions and prove them.

Two studies reviewed in the literature that dealt with the unfamiliar approach to studying high school geometry as axiomatic systems were formulated by Keezer[9] and Beard.[10] Keezer formulated an axiomatic system based on three primitive notions related to the following: (1) "point," (2) a four-termed interior relation among points, and (3) a six-termed equi-distance relation among points. The axioms of the system were divided into five groups which were concerned with: (1) set relations, (2) interior relation or betweenness, (3) the equi-distance relation (congruence), (4) continuity and parallelism. According to the author, the axioms characterizing the interior relation, logically

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[9]Sr. J. M. Keezer, "An Axiom System for Plane Euclidean Geometry" (unpublished doctoral thesis, St. Louis University, 1965).

[10]Earl M. Beard, "An Axiom System for High School Geometry" (unpublished doctoral thesis, The University of Wisconsin, 1968).

implied theorems analogous to the betweenness property of point on a Euclidean line. The six-termed equi-distance relation among points is used to define congruence relation on certain point sets; this development of congruence relation is then used to define equality between line segments. Theorems are proved analogous to congruence properties of line segments and triangles in Euclidean geometry. The investigator assumed a two-dimensional Cartesian space based on the undefined set of points and reported that the set of axioms proved to be consistent in a Cartesian model for the system.

Finally, the author demonstrated that the system characterized plane Euclidean geometry by proving that all the plane axioms of a known categorical system for Euclidean geometry are logical consequences of the given system.

Beard formulated an axiomatic system for high school geometry based on the principles of isometries. The purpose of the study was to seek answers to the following questions: (1) Is it feasible to develop a course beginning with fundamental ideas suitable for use in the high school that used transformations as the basis for the development of plane geometry? (2) If so, what could be a sequence of fundamental theorems? (3) What comparisons could be made between such a course and the standing or existing geometry course? (4) What special characteristics might such a course have that would be useful to other facets of the mathematics curriculum? The content comprised the usual materials found in the first year course with some modifications to accommodate the major premises of the thesis; such as: the geometry of the triangle, similar figures, circles and parallels. The author reported that the concept of area theory was omitted since a transformational

approach does not simplify the topic. The study developed a geometry from fundamentals which meant that no background in Euclidean geometry was necessary. A suggested sequence of theorems for standard topics in high school geometry is given along with some proofs involving transformations for comparative purposes. In the final analysis, the author made a comparison of the proposed and existing geometry courses with respect to: degree of rigor and intuition possible, types of proofs available and concepts of congruence.

Summary

The studies reviewed in the literature lends support to the basic ideas expressed in the hypothesis of this study in that high school students can understand the nature and structure of mathematics based on models. That implied relationships between objects found in the physical world and components of an abstract mathematical system can be determined. The studies also imply that is is well within the capabilities of high school students to deal effectively with familiar and unfamiliar mathematical systems which they can study and logically deduce propositions from assumptions in these systems and prove them, thus developing reflective, critical and creative thinking in secondary school mathematics.

CHAPTER III

PROCEDURE

Introduction

The primary source of techniques, ideas, and methodology for individual and group activities for this study were adapted from Laboratory Manuel for Elementary Mathematics by Fitzgerald, et al.[1] The activities described in the procedure were chosen from: Unit 3 Relations (reflexivity, symmetry, and transitivity); Unit 4 Functions (Tower Puzzle); Unit 8 Mathematical Systems (clock arithmetic, equate heights, zigzag, rectangle: flips and turns, and equilateral triangle: flips and turns); Unit 11 Topology (Beans and Brussel Sprouts). Two other sources included were: Circle[2] (counterclockwise rotations) and Instant Insanity.[3] These activities provided a situation in which physical models could be studied and mathematical systems could be developed. The resulting mathematized systems can then be studied, and in fact, be extended to finite and infinite abstract systems having properties and operations embedded in those physical models.

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[1]William M. Fitzgerald, et al., Laboratory Manuel for Elementary Mathematics, 2nd ed. (Boston, Mass.: Prindle Weber, and Schmidt, Inc., 1973).

[2]Mary P. Dolciani, et al., Modern School Mathematics Algebra and Trigonometry (Boston, Mass.: Houghton-Mifflin Company, 1968) p. 249.

[3]Commercial Purchase (Salem, Mass.: Parker Bros., 1967).

Description of Activities

During the school term 1972-73, this investigator taught geometry and intermediate algebra classes at Suncoast High School, Riviera Beach, Florida. The students chosen for this study were selected from the intermediate algebra classes. During the first semester, students in the algebra classes were introduced to the operations and properties of the real number system in an algebraic setting with emphasis on the postulational system. Operations and properties of a group were studied as well.

Before planning activities using physical models, students were instructed on the nature and importance of the properties and operations of a mathematical system. At the beginning of the second semester, this writer planned a ten-week schedule of activities for ten students who were above average performers and highly motivated. The students were directed to complete the scheduled activities instead of regular class work. Each activity was planned to last a period of one week. All sessions were recorded on audio tapes as the students worked through the weekly activity. Written records were kept of each activity and retained in a folder prepared for each student. During the course of the study, students worked in groups and as individuals in their regularly scheduled class periods.

There were two students in one class, five in a second class and three in another. The activities were scheduled and completed in the following sequence:

1. Equilateral triangle: Flips and Turns
2. Circle: Turns
3. Clock Arithmetic: Modulus 8

4. Equate the Height
5. Zigzag
6. Relations
7. Rectangle: Flips and Turns
8. Beans and Brussel Sprouts
9. Tower Puzzle
10. Instant Insanity

1. Equilateral Triangle: Flips and Turns

The students were given an equilateral triangle model prepared from cardboard and a background sheet of paper labeled as in figure 3.1.

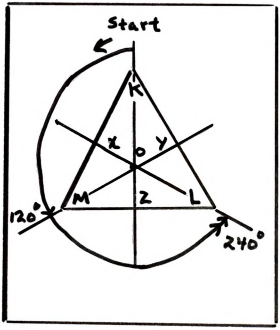

Figure 3.1

The equi-lateral triangle was attached to the sheet using a thumbtack at the center 0 of the triangle. The students were instructed to consider as elements of the group turns (rotations) and flips of the triangle into itself from the start position as follows:

r_1 = rotation about 0 counterclockwise 120^o

r_2 = rotation about 0 counterclockwise 240^o

f_1 = reflection about Kz 180^o (pick the triangle up and turn it over)

f_2 = reflection about My 180^o

f_3 = reflection about Lx 180^o

e = the identity element (no turn at all)

The symbol $\oplus$ meant "followed by" which denoted $r_1 \oplus r_1 = r_2$ indicated that the triangle was first rotated (r_1) counterclockwise 120^o about the center 0 from the start position, then "followed by" r_1 or a second counterclockwise rotation (r_1) of 120^o. This resulted in 240^o which was r_2. Other elements of the group were demonstrated based on the conditions for rotations or flips. Tables were constructed for first operations using e_1, r_1, r_2, f_1, f_2, and f_3 "followed by" second operations e_1, r_1, r_2, f_1, f_2, and f_3. The students used the results in the table to investigate five basic properties: closure, identity, inverses, associativity and commutativity which are the properties of an Abelian group. Other properties of the equilateral triangle investigated and identified were the order of the group, the generator of the group and whether or not the group was cyclic. Subgroups of the group were investigated for all of the above mentioned properties of the group.

2. Circle: Turns

To investigate the circle, each student was given a circle model prepared from cardboard and a background sheet of paper labeled as shown in figure 3.2.

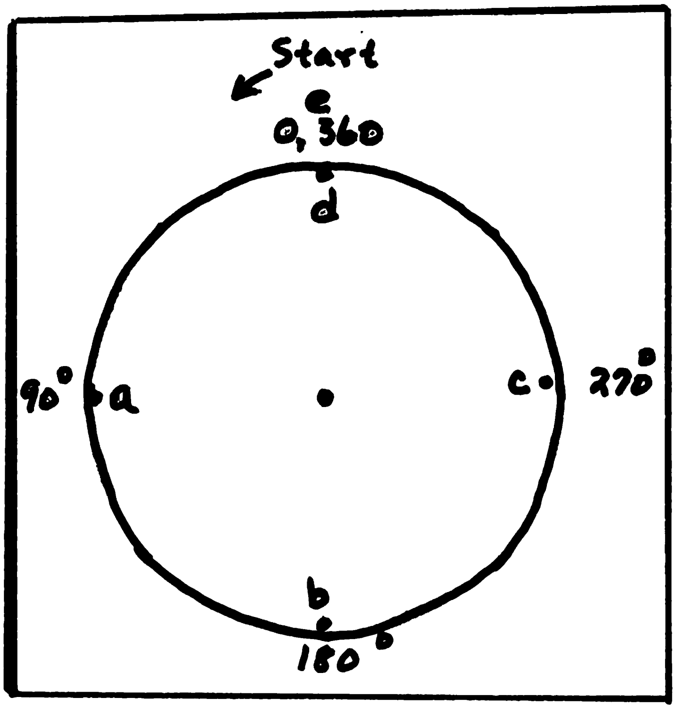

Figure 3.2

The circle was attached to the sheet using a thumbtack which allowed the circle to rotate about its center. The students were instructed to consider three counterclockwise rotations about the center of the circle. Those rotations were designated from the start position as follows:

a = rotation about the center counterclockwise 90^{o}

b = rotation about the center counterclockwise 180^{o}

c = rotation about the center counterclockwise 270^{o}

e = rotation about the center counterclockwise 360^{o} or no rotation at all

As in the first exercise, $\oplus$ represented the operation "followed by." Using the elements of the model and the operation, the five basic properties of an Abelian group were investigated as well as the properties of the order of the group, the generator of the group and whether or not the group was cyclic. Students were instructed to investigate the

physical model for a subgroup and all properties of the subgroup analogous to the group.

3. Clock Arithmetic: Modulus 8

The physical model used in this activity was a circular piece of cardboard with the numerals 0, 1, 2, 3, 4, 5, 6 and 7 painted on the face of the circle and a pointer which could be rotated about the center of the cardboard circle as shown in figure 3.3

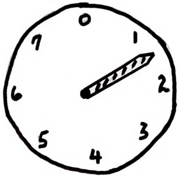

Figure 3.3

To build a mathematical system, the numerals on the face of the clock were considered elements, an operation addition was denoted by $\oplus$. The elements and the operation were related as follows: $x \oplus y = q$, where q was found by starting the pointer at 0 then moving it clockwise x hours, followed by moving it clockwise y hours. Q would represent the "sum" of x and y. Using the numerals 0, 1, 2, 3, 4, 5, 6 and 7 as left and right addends, a table was constructed for the operation $\oplus$. The system was investigated for the following properties: closure, an identity element, inverses of elements, associativity and commutativity. Multiplication for the system was denoted by $\odot$ where physical model for a subgroup and all properties of the subgroup analogous · the group.

$x \odot y = k$ and k was the number obtained by starting the pointer at 0 and moving it clockwise y hours x times. Using the procedure described above, a table was constructed for $\odot$ and the following properties were investigated: closure, commutativity, associativity, identity element, an inverse element and the distributive property of $\odot$ over $\oplus$.

4. Equate the Height

The physical model mathematized and studied in this activity consisted of colored rods and several square pieces of paper which had the same area as one of the square surfaces of any of the colored rods. Each square piece of paper was identified by writing the word "plane" on each piece. The colored rods varied in heights and were used as elements of the set for this system. The following names were noted: orange, blue, brown, black, dark green, yellow, purple, light green, red, white and plane. An operation "circle-times" denoted by $\otimes$ on the set resulted in the following: place the first mentioned rod on its square surface, place a second mentioned rod on its square surface along side of the first rod. The result was the element (rod) which was placed upon the shorter of the two rods to make them the same height. See figure 3.4 for an example of brown $\otimes$ red = dark green.

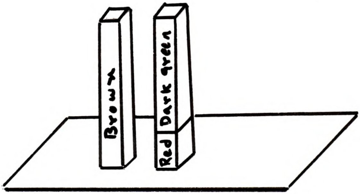

Figure 3.4

Using the elements plane, white, red, light green, purple, yellow, dark green, black, brown, blue and orange as left factors and right factors, a table was constructed for ⓧ. The system was investigated to determine which of the following properties were exhibited by ⓧ: closure, commutativity, associativity, identity element, and inverses of each element. After the system was investigated, its properties were compared with the system of natural numbers under addition, and the set of integers under the operation of subtraction.

5. Zigzag

This system did not consist of a physical model, but was adapted from a series of "Table System" exercises. The operation "zigzag" was defined on members of set R = $\{0, 1, 2, 3\}$ by the table below:

TABLE 3.1

ZIGZAG

ϟ	0	1	2	3
0	3	0	1	2
1	1	2	3	0
2	0	1	2	3
3	2	3	0	1

The system was investigated for the following properties: closure, commutativity, associativity, an identity element and inverses of each element.

6. Relations

This activity was designed to use a physical model to investigate the abstract notion of a relation using colored rods. Due to the nature of the objectives of the activity, definitions were provided for

the following:

1. Cartesian cross product
2. Relation R on a Cartesian cross product
3. Reflexivity, symmetry, and transitivity of a relation R
4. An equivalence relation
5. An equivalence class

Considering x and y as elements of a relation R, a table was constructed to investigate the properties of reflexivity, symmetry and transitivity with respect to eight possible combinations based on the following:

(x, y) $\in$ R if and only if:

1. x "has the same length as" y
2. x "is shorter than" y
3. x "has a different length than" y
4. the length of x exceeds the length of y by an amount equal to the length of the white rod
5. neither x nor y are purple
6. the difference between the lengths of x and y is less than the length of the light green rod
7. x "is shorter than, or has the same length as" y
8. x and y have the same color, or if the length of x exceeds the length of y by an amount less than the length of the yellow rod

TABLE 3.2

EQUIVALENCE RELATION

	Reflexive	Symmetric	Transitive
1			
2			
3			
4			
5			
6			
7			
8			

The equivalence relations as defined on the set of colored rods was investigated to determine the following:

(x, y) $\in$ R if and only if:

1. the length of x differs from the length of y by an amount equal to a multiple of the length of the red rod

2. the names for the colors of x and y begin with the same letter

Finally seven non-physical abstract relations were investigated to determine which property (ies) was exhibited by each relation, and whether any of the relations was an equivalence relation; where any relation was an equivalence relation, its equivalence class was listed

1. R = {(1, 2), (2, 1)}, as defined on {1, 2 .}
2. R = {(1, 1), (2, 2), (2, 1)}, as defined on {1, 2}.
3. R = {(1, 1), (2, 2), (1, 2), (2, 1) , as defined on 1, 2}.
4. R = {(1, 2), (2, 3), (3, 4)}, as defined on {1, 2, 3, 4}.
5. R = {(1, 2), (2, 3), (1, 3)}, as defined on {1, 2, 3}.
6. R = {(0, 0), (1, 1), (2, 2), (1, 2), (2, 1), (0, 1), (1, 0)}, as defined on {0, 1, 2}.
7. (x, y) $\in$ R if and only if the last name of x and y begin with the same letter, as defined on the set of students in Suncoast High School.

7. Rectangle: Flips and Turns

The physical model used in this activity was prepared from a piece of cardboard, called a frame, by cutting a rectangular hole in it and fitting a rectangular piece of cardboard, with an asterisk placed in the upper left corner, in the hole as shown in figure 3.5.

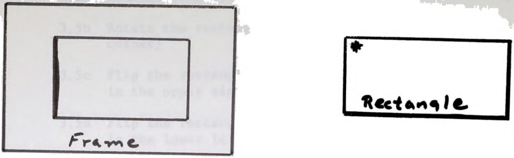

Figure 3.5

The rectangle with the asterisk was moved in such a way that the asterisk was in a different place as the rectangle was fitted into the hole of the frame. Four ways were determined in which the rectangle could fit by starting with the asterisk in the upper corner of the rectangle. See figures 3.5a, 3.5b, 3.5c, and 3.5d.

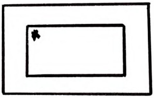

Figure 3.5a

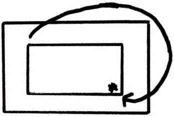

Figure 3.5b

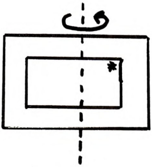

Figure 3.5c

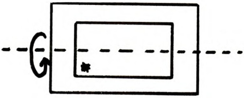

Figure 3.5d

3.5a Removed from the frame and replaced in the same way

3.5b Rotate the rectangle 180° (asterisk in lower right corner)

3.5c Flip the rectangle about its vertical axis (asterisk in the upper right corner)

3.5d Flip the rectangle about its horizontal axis (asterisk in the lower left corner)

These flips and turns resulted in a set of four transformations of the rectangle denoted by $\{a, b, c, d\}$. An operation was defined on the set by performing one of the transformations "and then" another. The operation "and then" was denoted by ⓐ. A table was constructed as usual and the system was investigated to determine the following properties: closure, commutativity, associativity, an identity element and inverses for each element, paring elements and their inverses and listing those elements having no inverses; and finally a comparative study of the properties of Rectangle with the Clock Arithmetic System.

8. Beans and Brussel Sprouts

Beans and Brussel Sprouts was a topological game for two players. To start the game, an arbitrary number of dots were marked on a piece of paper; players took turns connecting any two dots or connecting a dot to itself. When arcs were drawn, new dots were marked on the arc; no arc was crossed with a connecting line, and no dot was the endpoint of more than three arcs. The following questions were considered by players:

1. Does the game always end?

2. Is there always a winner?

3. Beginning with two dots, what is the maximum number of arcs that can be drawn before the game ends? What is the minimum number of arcs that can be drawn to end the game? Can you predict who will be the winner?

4. Apply questions 1, 2, and 3 for a game beginning with three dots; four dots; and five dots.

9. Tower Puzzle

The tower puzzle pieces for this study were made from construction paper using the following: 4 squares were cut from a piece of construction paper of one color with the following dimensions - 1" x 1", 2" x 2", 3" x 3" and 4" x 4". Using a different color of construction paper, 3 squares were cut with the following dimensions - $1\frac{1}{2}$" x $1\frac{1}{2}$", $2\frac{1}{2}$" x $2\frac{1}{2}$", and $3\frac{1}{2}$" x $3\frac{1}{2}$". These seven squares represented the seven discs. A piece of construction paper of dimensions 6" x 18" with three circles drawn five inches apart along the length of the paper was used instead of spindles. The circles represented the spindles for this game. The object of the game was to start with a pile of squares in descending size, the largest square on the bottom and the smallest square on the top, at one spindle and move the pile to another spindle. Moves were made according to the following conditions:

1. Only one square could be moved at a time
2. A larger square could never be placed on top of a smaller square

Students had to determine the least number of moves necessary to transfer the entire pile of squares from one spindle to another using 2 squares, 3 squares, 4 squares, 5 squares, 6 squares and 7 squares; this information was recorded in a table with the number of squares in one column and the number of moves in another. Students were asked to generalize their finding to a pile of n square based on the apparent results in the table. The following questions were posed for study by the students:

1. Why do you think the squares are of alternating colors?

2. Suppose that the squares were all the same color. Could you develop a strategy for determining which square should be moved at each turn?

10. Instant Insanity

This game contained four cubes with surfaces painted in the following colors: red, white, blue and green. The purpose of the game was to note the arrangement of the colors on the lateral sides of the cubes, before the package was opened: white, blue, green and red on one side and green, white, red and blue on the other and similar arrangements on the other two sides. The cubes were mixed very thoroughly and ten reassembled in a manner in which they were first observed. Students were asked to formulate any generalization in the arrangements which were successful.

Students

The population of this study consisted of ten students selected from three intermediate algebra classes taught by this investigator. All ten students were considered to be above average or superior in their class performances. Seven of the students had been part of the student population of the pilot study conducted during the previous school year. Two of the students in the population were seniors, seven were juniors and one a sophomore.

A review and study was made of the mathematical performances (i.e. total scores, percentile rank, etc.) of students in the population on the following tests: Florida State-Wide Ninth Grade Test, the mathematics section of this test is a balance between traditional and

contemporary topics. The more contemporary materials include such topics as: number sentences, inequalities, primes, absolute values, clock and remainder arithmetic, set notation, closure, multiplicative and additive inverses, the identity elements for addition and mulitplication, and the commutative, associative and distributive principles. The test does not emphasize the abstract symbolism of logic and sets; School and College Ability Tests (SCAT, Tenth grade), Series II was designed to provide estimates of basic verbal and mathematical ability; Differential Aptitude Test (DAT, Eleventh grade) using diagrams, the abstract reasoning tests measure how easily and clearly students can reason when problems are presented in terms of size, or shape, or position, or quantity or other non-verbal, non-numerical forms; Preliminary Scholastic Aptitude Test (PSAT, Tenth or Eleventh grade) and Scholastic Aptitude Test (SAT, Eleventh or Twelfth grade); in form and content, the PSAT and the SAT are parallel. The mathematics sections measure the ability to reason with numbers and other mathematical symbols; the sections also contain various kinds of problems to be solved, stress reasoning ability rather than knowledge of specific college preparatory course in mathematics.

The Florida State-Wide Twelfth Grade Test, the mathematics section covers both traditional and modern topics. It includes materials on the number system, set theory, coordinate geometry, data interpretation, algebra and geometry. The table below is a summary of the tests and student's reports by percentiles, and total scores on the PSAT and SAT.

TABLE 3.3

SUMMARY OF TEST RESULTS

Student-Grade	A-11	B-11	C-11	D-12	E-11	F-11	G-11	H-12	J-10	K-11
Florida State-Wide Ninth Grade	97	--	90	81	93	98	96	79	99	60
School and College Ability Test (10)	94	73	85	57	99	96	86	87	99	83
Differential Aptitude Test (11)	99	70	95	45	99	60	60	95	--	95
Preliminary Scholastic Aptitude Test (10, 11)	71	--	--	--	61	59	43	44	--	47
Scholastic Aptitude Test (11, 12)	730	510	--	520	--	530	510	540	--	--
Florida Twelfth Grade Test				71				80		

CHAPTER IV

CASE STUDIES RESULTS

Introduction

In order to assess the results of this study, the case study procedure is used for each activity as outlined in Chapter III. The case study procedure permitted this investigator to concentrate on a particular activity of an individual student performing an activity of a small group of students working on an activity. The results of the activities are reported in table form, where possible, and based on the following: (1) what all students were expected to report and did (marked with an asterisk*); (2) what a particular student or students should have reported but didn't (marked with a hypen -), at this point, where possible, comments are given to explain such actions; and (3) what a particular student or students reported that was not required but is most unusual and very pertinent to the activity under investigation. Such reports are marked with double asterisk (**) and pertinent comments are offered here also. Results which are true for the cases tested but not necessarily valid for generalized cases are marked with #.

Activity 1

Equilateral Triangle: Flips and Turns

Consider an equilateral triangle with its altitudes x,y, and z intersecting at a point 0 in the center of the triangle. Build a

flips and turn system for the triangle using as elements of a group the following:

e = identity (no rotation)

r_1 = rotation about 0 counterclockwise 120°

r_2 = rotation about 0 counterclockwise 240°

f_1 = reflection about Kz 180° (pick the triangle up and flip it over)

f_2 = reflection about My 180°

f_3 = reflection about Lx 180°

Using the operation "followed by" denoted by $\oplus$, (i.e. $r_1 + r_2$ means rotation r_1 "followed by" r_2) and the elements of the group (e, r_1, r_2, f_1, f_2, f_3) investigate the five basic properties: (1) closure, (2) identity, (3) inverses, (4) associativity and (5) commutativity. Consider the subgroup $G'(e, r_1, r_2)$ and investigate it using the same properties. Determine the generator(s) of the subgroup, the order of the subgroup and whether or not the subgroup is cyclic.

TABLE 4.1

ACTIVITY SUMMARY: EQUILATERAL TRIANGLE

Students	1	2	3	4	5	6	7
A	*	*	*	*	*	*	**
B	*	*	*	*	*	*	**
C	*	*	–	–	–	–	**
D	*	*	*	*	*	*	**
E	*	*	–	–	–	–	
F	*	*	–	–	–	–	
G	*	*	–	–	–	–	
H	*	*	*	*	*	*	**
J	*	*	*	*	*	*	
K	*	*	*	*	*	*	

1. The equilateral triangle characterized the properties of a group.

2. The equilateral triangle did not characterize an Abelian group.

3. The subgroup G' characterized the properties of an Abelian group.

4. Determined that r_1 and r_2 were generators of the subgroup.

5. Determined that the order of the subgroup was 3 .

6. Demonstrated that the subgroup was cyclic.

7. Supplementary activities.

Student A(**)

This student formulated the following conjecture:

Using any of the flip elements (f_1, f_2, f_3) with the operation $\oplus$ defined as in the activity, if one of the flips was used once it results in the inverse of that element; if the element was used twice, it results in the identity element.

An even number of f's resulted in the identity element because we showed that used twice, it results in the identity and the definition of an even number is 2 used as a factor n times; using f_1 twice results in the identity and $e \oplus e$ any number of times is equal to e .

Student B(**)#

This student studied a series of rotations and flips and formulated the following for selected subsets:

Flips

even: $f_n \oplus f_n = e$

odd: $f_n \oplus f_n \oplus f_n = f_n$

Rotations

even: $r_1 \oplus r_1 = r_2$

odd: $r_1 \oplus r_1 \oplus r_1 = e$

even: $r_2 \oplus r_2 = r_1$

odd: $r_2 \oplus r_2 \oplus r_2 = e$

Student C(**)

This student reported that the set of integers under addition was found to be an Abelian group. This investigation could possibly account for the fact that no report on the subgroup was submitted.

Student D(**)

This student formulated two theorems without proofs:

Theorem 1: Flips
Using any flip element, if one of the flips is used once (or odd number of times) it results in the inverse of that element, if used twice (or an even number of times), then the result is the identity element of the set.

Theorem 2: Turns
Using the elements r_1 or r_2, if the sums of these were used three or a multiple of three times, the result would be the identity.

Students E, F, and G (-)

There were no records to indicate that these students had investigated the subgroup for any properties.

Student H(**)

This student concluded that the properties of the subgroup and the integers under addition satisfied the properties of a group.

Activity 2

Circle: Turns

Using the circle with center 0, let a, b, c and e represent counterclockwise rotations about the center with the following notations:

a = rotation of 90^o c = rotation of 270^o

b = rotation of 180^o e = no rotation or 360^o

Using the operation "followed by" denoted by $\oplus$ and the elements a, b, c, and e as a set, show the following:

1. That S constitutes a group.
2. That S is Abelian.
3. That a is of order 4 in S.
4. That S is cyclic.
5. That (b, e) is a subgroup of S.
6. That the subgroup is cyclic and its generator is b and is of order 2.
7. Supplementary activities.

TABLE 4.2

ACTIVITY SUMMARY: CIRCLE

Students	1	2	3	4	5	6	7
A	*	*	*	*	*	*	**
B	*	*	*	*	*	*	**
C	*	*	*	*	*	*	
D	*	*	*	*	*	*	**
E	*	*	*	*	*	*	
F	*	*	*	*	*	*	
G	*	*	*	*	*	*	**
H	*	*	*	*	*	*	
J	*	*	*	*	*	*	**
K	*	*	*	*	*	*	

Student A(**)

In addition to reporting that c was a generator of S also with order 4, this student submitted two theorems based on the following condition: "If each element in the set is divided by 10, e = 36, a = 9, b = 18 and c = 27; and given that e and b are even elements; a and c are odd elements."

Theorem 1

The inverse of the even element is that element, i.e. $e^{-1} = e$, $b^{-1} = b$; the inverse of the odd element is the other odd element, $a^{-1} = c$, and $c^{-1} = a$.

Theorem 2

$e^{-1} \oplus c^{-1} = b^{-1} \oplus a^{-1}$ (corollary to D's theorem)

Student B(**)

This student submitted three theorems with proofs.

Theorem 1

In a circular plane choose elements moving counterclockwise and using using the operation "followed by" with the clockwise operation of the element's successor is always C.

Proof

$$a \oplus b^{-1} = c$$

$$b \oplus c^{-1} = c$$

$$c \oplus e^{-1} = c$$

$$e \oplus a^{-1} = c$$

Theorem 2

If you take one operation and "followed by" to the inverse of the operation before the original operation, then you get a.

Proof

$a \oplus e^{-1} = a$

$b \oplus a^{-1} = a$

$c \oplus b^{-1} = a$

$e \oplus c^{-1} = a$

Theorem 3

The operation $\oplus$ produces a generator in S if an element and its clockwise or counterclockwise inverse is used.

Proof

Theorem 1 counterclockwise, c

Theorem 2 clockwise, a

Student D(**)

This student submitted the following theorem to which Student A formulated a corallary.

Theorem

Given S = (e, a, b, c) with a as a generator and the operation $\oplus$ closed, then $e \oplus a = b \oplus c$.

Proof

Statements	Reasons
1. e, a, b, c $\in$ S, gen, "a", $\oplus$ closed	1. By hypothesis
2. $e \oplus a = a$	2. Identity (table)
3. $b \oplus c = (a \oplus a) \oplus (a \oplus a \oplus a)$	3. Gen. "a"
4. $b + c = (a \oplus a) \oplus c$	4. Gen. "a", subst. prin.
5. $b \oplus c = a \oplus (a \oplus c)$	5. Associative prop.
6. $a \oplus c = e$	6. Gen. "a", table
7. $a \oplus c = a \oplus e$	7. Step 5, subst. prin.

8. $a \oplus e = a$	8. Subst. and identity
9. $b \oplus c = a$ $e \oplus a = a$	9. Steps 2, 4
10. $e \oplus a = b \oplus c$	10. Transitive prop.

This student concluded that c was a generator of the set S.

<u>Student G(**)</u>

This student submitted the following theorems:

<u>Theorem 1</u>

Given the circular plane, if you go in a counterclockwise direction taking one element $\oplus$ the next element's inverse, its "sum" is always c.

<u>Proof</u>

1. $b \oplus c^{-1} = c$
2. $a \oplus b^{-1} = c$
3. $c \oplus a^{-1} = c$
4. $e \oplus a^{-1} = c$

<u>Corollary 1.1</u>

If you take an element and its consecutive clockwise element with $\oplus$, then the result is point a.

<u>Proof</u>

1. $a \oplus e^{-1} = a$
2. $b \oplus a^{-1} = a$
3. $c \oplus b^{-1} = a$
4. $e \oplus c^{-1} = a$

<u>Theorem 2</u>

An element is a generator "if and only if" it can be generated by adding the next (counterclockwise, clockwise) in the sequence.

<u>Proof</u>

1. Theorem 1
2. Corollary 1.1

Student J(**)

This student reported that the element "e" could exist as a subgroup of the set S. In addition to that statement, this student submitted two theorems and a corollary for every operation "followed by", $\oplus$.

Theorem 1

In a circular plane, if you go in a counterclockwise direction using any element "followed by" its successor's inverse, then the result is c.

Proof

1. $a \oplus b^{-1} = c$
2. $b \oplus c^{-1} = c$
3. $c \oplus e^{-1} = c$
4. $e = a^{-1} = c$

Corollary 1.1

In a circular plane, by taking the inverse of the element of the next clockwise element, the result is a.

Proof

1. $b \oplus a^{-1} = a$
2. $c \oplus b^{-1} = a$
3. $e \oplus c^{-1} = a$
4. $a \oplus e^{-1} = a$

Theorem 2

An element is a generator "if and only if" it can be obtained by using "followed by" or an element next to it.

Proof

1. $a \oplus e = a$
2. $b \oplus c = a$
3. $a \oplus b = c$
4. $c \oplus e = c$

Activity 3

Clock Arithmetic: Modulus 8

This system involved using a circular piece of cardboard with eight numerals from zero to seven painted on the face of the circle. Attached to the center of the circle was a movable pointer. To build a mathematical system, the numerals on the face of the clock $\{0, 1, 2, 3, 4, 5, 6, 7\}$ were considered elements and addition denoted by $\oplus$ was defined as: $x \oplus y = q$ where q was determined by starting the pointers at 0 and moving it x hours clockwise followed by y hours clockwise. To investigate the system for $\oplus$ the students responded positively to the following questions:

1. Is the system closed?
2. Is the system commutative?
3. Is the system associative?
4. Is there an identity element?
5. Does each element have an additive inverse?
6. Supplementary activities.

Table 4.3 contains the results for $\oplus$; Table 4.4 contains the results for $\odot$.

TABLE 4.3

ACTIVITY SUMMARY: MODULUS 8 $\oplus$

Students	1	2	3	4	5	6
A	*	*	*	*	*	
B	*	*	*	*	*	
C	*	*	*	*	*	
D	*	*	*	*	*	**
E	*	*	*	*	*	
F	*	*	*	*	*	**
G	*	*	*	*	*	
H	*	*	*	*	*	
J	*	*	*	*	*	
K	*	*	*	*	*	

Student D(**)

This student reported that the clock arithmetic system under $\oplus$ was an Abelian group.

Student F(**)

This student submitted the following statement of relations:

$$\forall \quad a \oplus b = x \ni a + b \leq 7 \text{ (real numbers)};$$

$$a \oplus b = a + b$$

To investigate the system for multiplication denoted by $\odot$ and defined as: $x \odot y = k$, where k was determined by starting the pointer at 0 and moving clockwise y hours x times. The students responded positively to the following questions:

1. Is the system closed?

2. Is the system commutative?

3. Is the system associative?

4. Is there an identity element?

5. Does each element have a multiplicative inverse?

6. Is there only one multiplicative inverse?

7. Does the distributive law for multiplication over addition hold true in the system?

8. Supplementary activities.

TABLE 4.4

ACTIVITY SUMMARY: MODULUS 8 $\odot$

Students	1	2	3	4	5	6	7	8
A	*	*	*	*	*	*	*	**
B	*	*	*	*	*	*	*	**
C	*	*	*	*	*	*	*	**
D	*	*	*	*	*	*	*	**
E	*	*	*	*	*	*	*	**
F	*	*	*	*	*	*	*	
G	*	*	*	*	*	*	*	
H	*	*	*	*	*	*	*	**
J	*	*	*	*	*	*	*	**
K	*	*	*	*	*	*	*	**

Student A(**)

This student reported that the system was commutative for both $\oplus$ and $\odot$ because the elements in the tables were symmetric about the upper left, lower right diagonal row of elements; and that this system as well as the clock arithmetic system are Abelian groups with respect to $\oplus$ and $\odot$.

Students B and J(**)#

These two students working together concluded that the clock arithmetic system was an Abelian group with respect to $\oplus$ and $\odot$; the identity elements for $\oplus$ and $\odot$ were distinct, and the distributive property of $\odot$ over $\oplus$ was true, therefore the clock arithmetic system characterized a field.

Student C(**)

This student submitted the following conclusive report:

> This system holds all of the properties of a group, even commutativity, therefore it is an Abelian group. For multiplication, if you use an even number an odd number of times, you will come up

with an even number; if you use an odd number an odd number of times, you will get an odd number. This is a finite set. We found that when multiplying 4 by 4, if you use 4 an even number of times, you will get 0; if you use 4 an odd number of times, you will get 4.

Student D(**)

This student prepared an elaborate Table of Remainders for $\oplus$ and $\odot$ based on the following statement: "The real system's + and x corresponds directly to the mod 8 system for a $\oplus$ b = c; c $\div$ 8 and a $\odot$ b = c; c $\div$ 8." The student claimed that the tables of remainders corresponded directly to the results in the table prepared during the investigation of the system using $\oplus$ and $\odot$. The following theorem was submitted with a proof:

Theorem

For the elements of the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$, $\{0, 2, 4, 6\}$ are even elements. If the first even elements of $\{0, 2, 4, 6\}$ is added to the second even element, then the sum is equal to the third even element added to the fourth even element.

Proof

Statements	Reasons
1. $0 \oplus 2 = 2$	1. Definition of $\oplus$ hypothesis
2. $4 \oplus 6 = 2$	2. Definition of $\oplus$
3. $0 \oplus 2 = 4 \oplus 2$	3. Substitution principle
4. $2 = 2$	4. Q. E. D.

Finally this student submitted the following conclusive report:

For the operation $\oplus$: $3 \oplus 6 = 2$, definition of $\oplus$; but if we apply some properties of the real number system, we can see that 4 units and 6 units equals 10 units, simple addition. If we use still another property we can divide 10 by the modulus, which in the other system is 8, 10/8 = 1 with a remainder of 2. But look, the remainder in this system (real number) is equal to the result of $4 \oplus 6$ or 2. Two units in the real

system is equal to two units in the modulus system. For the operation $\odot$: 3 $\odot$ 6 = 2, definition of $\odot$. Again by applying the properties of the real number system we can say that 3 units times 6 units equals 18 units and 18 units divided by the modulus of the system in which we are working . . . 18/8 = 2 with a remainder of 2 which is simple division. Again we see that the results are the same 2 units (mod 8) are equal to the units as we define them in the real number system.

Student E(**)

This student and her associates formulated several theorems in the system for $\oplus$ and $\odot$. This student submitted the following theorem:

For every a and every b, when a · b $>$ 7 (real numbers), a $\odot$ b equals the remainder of (a · b)/8.

The students in this group concluded that their theorems could be used for any modulus.

Student H(**)

This student submitted the following conclusive report:

The system is a finite set which has the properties of an Abelian group. For multiplication, the product of any element and an even number, the product is an even element.

Student K(**)

This student submitted the following theorem: For every a and every b: a · b $\leq$ 7 (real numbers) a $\odot$ b = a · b.

Activity 4

Equate the Height

In order to investigate this system, a set of colored rods were used including a square piece of paper which had the same area as one of

the square surfaces of any of the colored rods, and was called "plane." The colored rods including the square pieces of paper were considered elements of the set. Each element, with the piece of paper, was designated by color which were: orange, blue, brown, black, dark green, yellow, purple, light green, red, white and "plane." An operation "circle times" denoted by $\otimes$ was defined as follows: Place the first mentioned element on its square surface, and place the second mentioned element on its square surface immediately to the right of the first. The result will be the element which should be placed upon the shorter of the two to make them the same height. The system was investigated to determine which properties the operation exhibited using the following questions and statements as a guide:

1. Is $\otimes$ closed on the set?
2. Is $\otimes$ a commutative operation?
3. Is $\otimes$ associative?
4. Does this system have an identity element?
5. Does each element have an inverse?
6. Discuss the difference between this system and the natural numbers under ordinary addition.
7. How does this system differ from the set of integers under the operation subtraction?

Due to the nature of the questions asked and the small group reporting procedure, this activity is not submitted in table form. The reports show how many students are included in the groups.

Students A and D

These students performed the operation $\otimes$ with the elements in

the set and answered the questions as follows: (1) yes, the set was closed because every element $\otimes$ another element results in an element of the set; (2) yes, $\otimes$ was a commutative operation. The table showed that the results were symmetric about the upper left to lower right diagonal of "plane." (3) No, $\otimes$ was not associative because of the following example: (yellow $\otimes$ red) $\otimes$ white $\neq$ yellow $\otimes$ (red $\otimes$ white). (4) Yes, the system had an identity element which was "plane," and (5) the inverse of each element under $\otimes$ was the element itself. These two students reported that the natural numbers have no identity, but this system does; the natural numbers are associative under ordinary addition and this system isn't; and that under subtraction, the integers are not commutative.

Students B, G and J

These students prepared a table using the operation $\otimes$ and the elements of the set. The results in the table provided the following answers: (1) yes, the operation was closed, when $\otimes$ was used with all elements in the set, no new elements were formed; (2) yes, $\otimes$ was commutative: blue $\otimes$ yellow = yellow $\otimes$ blue; (3) no, $\otimes$ was not associative because (W $\otimes$ W) $\otimes$ light green $\neq$ W $\otimes$ (W $\otimes$ light green); (4) yes, the system had an identity represented by "plane" and (5) yes, elements had inverses because each element was its own inverse. These students reported that the natural numbers are associative for addition and is an infinite set whereas this system is not associative for addition and is a finite set; the natural numbers have no identity element for addition and this system does. The students concluded that under subtraction, this system and the integers are the same if the absolute

value is used for the integers.

Students C and H

These students constructed a table for $\otimes$ and the elements in the set. The following answers were provided based on the results found in the table: (1) yes, the operation was closed on the set; (2) yes, $\otimes$ was commutative, verified by blue $\otimes$ light green = (light green $\otimes$ blue; (3) no, the system was not associative because (white $\otimes$ red) $\otimes$ purple $\neq$ white $\otimes$ (red $\otimes$ purple); (4) yes, the system had an identity which was "plane"; and (5) each element had an inverse which was found to be the element itself.

These students reported that this system did not have the properties of a group; the difference between this system and the natural numbers under addition was found to be that the system had an identity element and the natural numbers did not. They reported that the difference between this system and the integers under subtraction was that the system was commutative and the integers were not.

Students E, F, and K

These students performed the operation $\otimes$ with the elements in the set, prepared a table which provided answers to the questions as follows: (1) yes, the set was closed, by the table, a $\otimes$ b = c where a, b, c are elements in the set; (2) yes, $\otimes$ was commutative as the table will show by "mirror images" of elements; (3) no, $\otimes$ was not associative, verified by (blue $\otimes$ red) $\otimes$ yellow $\neq$ blue $\otimes$ (red $\otimes$ yellow); (4) yes, the identity element was "plane"; and (5) each element was its own inverse.

The students reported that this system was not associative, the natural numbers are; this system had an identity element, the natural numbers do not have one under addition. The difference between the integers and this set was that this system demonstrated commutativity under subtraction and the integers did not.

Activity 5

Zigzag

This mathematical system was defined by simply presenting a set and a complete operation table instead of having the students to mathematize some physical model. This system allowed the student to use those ideas which he had learned from the study of previous physical systems (concrete situations) and apply them to analogous abstract systems. The abstract system of zigzag consisted of the operation ϟ (zigzag) defined on the elements of set R = $\{0, 1, 2, 3\}$ by the table:

TABLE 4.5

ZIGZAG

ϟ	0	1	2	3
0	3	0	1	2
1	1	2	3	0
2	0	1	2	3
3	2	3	0	1

and the following statements or questions concerning the system:

1. Complete each of the following to make true statements

    0 ϟ 3 =   3 ϟ 0 =

    1 ϟ 3 =   3 ϟ   = 3

2. For every a of R and for every b or R, is it always true

that

a. (a ϟ b) is an element of R?

b. a ϟ b = b ϟ a?

3. Is ϟ an associative operation?

5. Is there an identity element? Explain. If so, does every element have an inverse?

Students performed this activity in small groups and responded to the questions as a unit.

Students A and D

These students reported the following results based on the table:

1. 0 ϟ 3 = 2 3 ϟ 0 = 2
   1 ϟ 3 = 0 3 ϟ 1 = 3

2. a ϟ b in R was always true.
   a ϟ b = b a was not always true.

3. The operation ϟ was not associative.

4. There was an identity element as long as one condition was satisfied and that was using 2 as a left factor with zigzag. Every element had an inverse as long as a left factor was used with zigzag.

Students B, G, and J

These students reported the following results based on the table:

1. 0 ϟ 3 = 2 3 ϟ 0 = 2
   1 ϟ 3 = 0 3 ϟ 1 = 3

2. a ϟ b was an element in R.
   a ϟ b = b ϟ a wasn't always true.

3. ϟ was not an associative operation.

4. There was an identity element if the first (left factor)

was 2 because: 2 ϟ a = a but a ϟ 2 ≠ a where a was an element in the set; and every element had an inverse.

Students C and H

These students reported the following results based on the results of the table:

1. 0 ϟ 3 = 2   3 ϟ 0 = 2
   1 ϟ 3 = 0   3 ϟ 1 = 3

2. a ϟ b was an element in R.
   a ϟ b = b ϟ a was not always true.

3. The operation ϟ was not associative.

4. There was no constant identity element; each element had a different identity; and every element had a different inverse.

Students E, F, and K

These students submitted the following report based on their study of the results in the table:

1. 0 ϟ 3 = 2   3 ϟ 0 = 2
   1 ϟ 3 = 0   3 ϟ 1 = 3

2. Yes, the system was closed, no, the system was not always commutative.

3. The operation ϟ was not associative: (1 ϟ 2) ϟ 3 ≠ 1 ϟ (2 ϟ 3)

4. There was a number 2 such that: 2 ϟ 0 = 0, 2 ϟ 1 = 1, 2 ϟ 2 = 2 and 2 ϟ 3 = 3 and each element had an inverse.

Activity 6

Relations

This activity was designed to use a physical model to investigate the abstract notion of a relation using colored rods. Due to the nature of the objectives of the activity, definitions were provided for the following: (1) cartesian cross product; (2) a relation R on a cartesian cross product; (3) reflexivity, symmetry, and transivity of a relation R; (4) equivalence relation; and (5) an equivalence class.

The properties, equivalence relations and evaluations are listed in Chapter III of this investigation. For some reason or reasons yet unknown, only two students, A/D, completed part of this activity. Other students reported repeatedly that the exercise did not make sense to them and they couldn't understand it. The only question answered consistently by most students was: How many relations are there on the set of colored rods?

Activity 7

Rectangle: Flips and Turns

To investigate flips and turns of the rectangle as a mathematical system, the physical model was prepared as described in Chapter III of this report.

A set of four transformations involving flips and turns was determined and denoted by $\{a, b, c, d\}$. An operation was defined on the set by performing one of the transformations "and then" another. The operation "and then" was denoted by @ and beginning each time at

the starting position. One operation was performed, b "and then" c which was noted to be the same transformation as d, and placed in the table. Students were to complete the remainder of the table using left transformations "and then" right transformations.

The elements of the system $\{a, b, c, d\}$ and the operation "and then" were investigated to determine the following properties for the model: (1) closure; (2) commutativity; (3) associativity; (4) an identity element; (5) inverses; and (6) how this system compared with the clock arithmetic system (Activity 3).

TABLE 4.6

ACTIVITY SUMMARY: RECTANGLE

Students

	1	2	3	4	5	6
A	*	*	*	*	*	*
B	*	*	*	*	*	*
C	*	*	*	*	*	*
D	*	*	*	*	*	*
E	*	*	*	*	*	*
F	*	*	*	*	*	*
G	*	*	*	*	*	*
H	*	*	*	*	*	*
J	*	*	*	*	*	*
K	*	*	*	*	*	*

The students responded positively to the first five properties listed to be investigated for this activity, however, the responses to statement six was divided into two major groups: Students A, D, E, and K reported that the rectangle system and the clock arithmetic system satisfied the properties of a group, however, the other students, B, C, F, G, H, and J reported that both systems were Albelian groups.

Activity 8

Beans and Brussel Sprouts

Beans and Brussel Sprouts was a topological game for two players. To start the game, an arbitrary number of dots were marked on a piece of paper. Two players alternated turns drawing arcs connecting any two dots or connecting one dot to itself. When an arc was drawn, the player had to mark a new dot on the arc. No arc could be crossed and no dot could be the end point of more than three arcs. The winner was the last player able to draw an arc. The following questions were considered:

1. Does the game always end?

2. Is there always a winner?

3. Beginning with two dots, what is the maximum number of arcs that can be drawn before the game ends? What is the minimum number of arcs that can be drawn to end the game? Can you predict who will be the winner?

4. Answer Questions 1, 2 and 3 for a game beginning with three dots; four dots; five dots.

Students completed this activity with two players performing at a time; the results summarized represents their findings.

Students A and D

These two students completed an elaborate network of activities indicating the number of dots and connective conditions; who started; who won and the number of arcs at the end of each game.

	Starting Conditions	Number of Dots	Who Started	Who Won	Number of Arcs
1.	No connection of a dot to itself	2	D	D	10
2.	One connecting itself	2	A	A	10
3.	One connecting itself	2	D	D	10
4.	Two connecting themselves	2	A	A	10
5.	None connecting a dot to itself	2	D	D	10
6.	None connecting a dot to itself	3	A	D	12
7.	One connecting	3	D	D	14
8.	Two connecting	3	A	A	8
9.	Three connecting	3	D	D	14
10.	One connecting	3	A	D	16
11.	No connecting	4	D	A	20
12.	One connecting	4	A	D	20
13.	Three connecting	4	D	A	20
14.	Two connecting	4	A	D	20

The students submitted the following summary:

a. Game always ends

b. Some always win

c. No conclusions

Students B, G, and J

These three students completed the game by playing two at a time and summarized their finding accordingly:

	Started	Finished	Winner
1.	7	25	Second player
2.	11	40	Second player
3.	4	15	First player
4.	6	20	First player

The students submitted the following conclusions:

1. The first player won when there was an even number of dots at the start.

2. The second player won when there was an odd number of dots at the start.

3. The number of dots at the finish was a multiple of five.

The group responded to the questions concerning the exercise accordingly:

1. Yes, the game always ends.

2. Yes, there is a winner.

3. Starting with two dots, five arcs was the maximum number of arcs that could be drawn before the game ends; five arcs was the minimum number of arcs that could be drawn to end a game. The winner can be predicted.

Students C and H

These two students submitted a summary to the questions asked about the activities.

1. No, the game does not end because for every two dots, there is an infinite number of two dots that can be connected to each other.

2. There can be no winner because there are an infinite number of turns that can be taken.

3. Beginning with two dots, there is no way of determining the maximum number of arcs; there is no way to determine the minimum. A winner cannot be determined because there is no winner.

Students E, F, and K

These three students completed the game by playing two at a time and submitted these results:

1. Yes, the game will end.

2. Yes, there will be a winner.

3.

No. of Dots Connected	Max. Arcs	Min. Arcs
2	10	7
3	14	14
4	22	18
5	28	22
6	34	--

Activity 9

Tower Puzzle

The tower puzzle consisted of a piece of construction paper 6" x 18" with circles drawn 5 inches apart which served as spindles, and squares of appropriate sizes. The object of the game was to move a pile of squares from one "spindle" to another maintaining the relative positions of the squares, which was the smallest square on the top, largest square on the bottom. Moves were made according to the following conditions:

1. Only one square could be moved at a time.

2. A larger square could never be placed on top of a smaller square.

Determine the least number of moves necessary to transfer an entire pile of squares from one "spindle" to another. Try playing the game with 2 squares, 3 squares, 4 squares, etc. Once the least number of moves required to transfer the entire pile of squares has been determined, record the information in a table and try to generalize that information to finding a pile of n squares; then determine the least number of moves needed to transfer any pile. Provide answers to the following questions:

1. Why do you think the squares are of alternating color?

2. Suppose that the squares are all the same color. Can you develop a strategy for determining which square should be moved at each turn?

The students worked in their groups for this activity, the results represent their combined efforts.

Students A and D

These two students completed the exercise after many trials and submitted their report which was brief:

> By working with the tower, it can be found that in order to have the right number of movements, you must not place one colored element on another element of the same color. Using different numbers of squares, they worked for the formula 2r+1 where r was the previous number of movements.

Students B, G and J

These three students completed the moves according to the conditions; results were recorded in a table and conclusions drawn

accordingly:

Number of Squares	Number of Turns	"Generalization"
0	0	
1	1	2(0) + 1
2	3	2(1) + 1
3	7	2(3) + 1
4	15	2(7) + 1
5	31	2(15) + 1
6	63	2(31) + 1
7	127	2(63) + 1
.	.	.
.	.	.
.	.	2x + 1

Let y equal the number of consecutive number of squares, the number of turns increase according to the formula 2x + 1 where x is the previous turn for increasing y's. The least number of turns possible was obtained by always placing a square on another of a different color.

Students C and H

These students performed the activities, recorded their results and provided the following:

Number of Squares	Number of Moves
2	3
3	7
4	15
5	31

Number of Squares	Number of Moves
6	63
7	127
n	2x + 1

When the number of squares was increased by 1, the number of moves was increased by twice the preceeding number of moves plus 1. The squares are alternating in color because it's easier to distinguish between them. If all squares were the same color use the smaller of each advancing square.

Students E, F, and K

These students completed the activity as a team; after many trials, the compiled their results in a table with the following conclusions: The least number of moves to transfer one pile to another spindle

Number of Squares	Number of Moves
2	3
3	7
4	15
5	31
6	63
7	127
n	2n + 1

Conjecture: " 2n + 1 is the number of moves where n is the preceding number of moves (for the preceding number of squares.) By demonstration we proved it, therefore it is a theorem. The squares are of alternating colors because they are a guide in working the puzzle. The squares must go together in alternating colors."

Activity 10

Instant Insanity

This game was played with four cubes with their faces colored red, blue, white and green. The purpose of the game was to take note of the arrangement of the colors before the package was opened. At the start of the game, the cubes were mixed very thoroughly and then rearranged in a manner similar to the arrangement before the package was opened. Only one student submitted a report on this activity. Student G reported: "The easiest way to complete the problem is to use color combination, first same color next to each other, or any other combination." Now a diagram of his solution:

	Row 1	Row 2	Row 3	Row 4
Side 1	blue	green	white	red
Side 2	green	blue	red	white
Side 3	white	green	blue	red
Side 4	red	white	green	blue

Row 1 - blue, green, white, red

Row 2 - green, blue, green, white

Row 3 - white, red, blue, green

Row 4 - red, white, red, blue

"There is no sure way to get a solution for it but the information given here might be able to generalize something; block positions may be switched." All of the students had played the game before and their interest in the game was not encouraging as they chose not to

try to generalize a solution.

Summary

The statements, quotes, theorems corollaries and conjectures made by the students and included in this report are exactly as they were made or written. No attempt was made on the part of this investigator to edit or rearrange their thoughts as they perceived them. In several instances, students reported results which were not necessarily valid for generalized cases. These conclusions should be considered conjectures and are marked with a #. The students participating in this investigation were above average and highly motivated. All were considered excellent candidates for further studies in the area of mathematics; they demonstrated time after time what can be accomplished with this kind of high school student. The students, themselves, pointed out many times that what was accomplished could be part of a normal class situation with any student enrolled in an ordinary algebra class.

CHAPTER V

SUMMARY AND CONCLUSION

Summary

The purpose of this study was to examine in detail the cognitive outcomes of high school students using physical models to study the structure and nature of mathematical systems. The study was conducted under normal classroom conditions at Suncoast High School in Riviera Beach, Florida using for the population selected high school students who were considered to be above average and excellent candidates for further studies in mathematics. The hypothesis which stated:

> When selected bright high school students are placed in a circumstance where they can study mathematical models, they will demonstrate their abilities to deal effectively with mathematical models; the nature of an axiomatic system, and the process of logically deducing propositions for investigations

was realized in more than a few instances. The physical model being concrete and familiar in nature, as well as manipulative, provided an excellent means by which the student could develop an understanding of the nature and properties of a mathematical system. The students demonstrated time after time that it was possible for them to do critical and creative thinking in terms of relating that which was concrete and derived from physical models to that which was abstract and applied to a mathematical model. One activity proved to be either too abstract or too unfamiliar for the students and no results were obtained.

The students were familiar with the reflexive, symmetric and transitive properties and the Cartesian cross product concept as they were related to their study of the real number system in a algebraic setting.

It is the opinion of this investigator that since this particular activity was not representative of a physical model to be mathematized the students were unable to determine the significance of the characteristics of an equivalence relation as they relate to physical models. As was pointed out earlier in the report, the only question answered consistently by most students dealt with the Cartesian cross product of A x A of colored rods.

Discussion

The results of this research demonstrates that within the constraints of normal classroom conditions, high school students can achieve a higher level of understanding the nature of: (1) model building both physical and abstract, (2) the axiomatic process, and (3) the truth value of inferential propositions. What was accomplished here is not typical, nearly all courses in high school mathematics at some time or another make mention of physical models of mathematical entities and the properties of mathematical models; but all too often, these topics are treated in isolation. The significance of the model, physical or mathematical, should be emphasized throughout the course as the need arises. The content of the intermediate algebra class, from which the students were selected, was taught based on the axiomatic approach which probably contributed to the results obtained.

Recommendations for Further Research

The following recommendations are made as a direct consequence of this investigator's participation in the pilot and final studies as a classroom teacher.

1. There should be more use of the case study procedure in the elementary and middle school to investigate the content in mathematics using physical models and participatory games to characterize mathematical systems.

2. The physical models mathematized in this study should be used in an attempt to teach high school students about other mathematical systems.

3. That research be done on the effect of the use of these teaching strategies on a wider sample of high school students.

Conclusions

In conclusion, it can be stated that the major aspects as stated in the hypothesis are reasonable and proper goals to work for and attain with high school students.

As was pointed out earlier in this report, models have an unexcelled power to clarify concepts and provide an invaluable means by which difficult and abstract ideas can be made simple and understandable. The students participating in this study submitted fifteen theorems, two corollaries, four conjectures and four generalizations as a testament to that fact.

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