APPLICATION OF THE THEORY OF ROULETTES TO THE SYNTHESIS OF MECHANISMS

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ABSTRACT

APPLICATION OF THE THEORY OF ROULETTES TO THE SYNTHESIS OF MECHANISMS

by William H. Bussell

Four bar mechanisms are desirable machine elements because of their simplicity. However, because the properties of such a mechanism are changed when the relative lengths of its links are changed, cut and try methods of designing them are time consuming. Graphical methods are useful and help to give a feeling for the mechanism. Analytical methods, however, provide means of programming a digital computer for a numerical solution.

The approach used here in devising an analytical solution, is that of instantaneous motion of the coupler bar plane and as such belongs to the class of analyses based on infinitesimal displacements. The theory of roulettes, which treats of the paths of points in the plane of a curve rolling without slipping on another, is used. This supplies a means of applying the concept of stationary curvature of point paths in obtaining a numerical solution to a mechanism synthesis problem. Since any plane motion can be reduced to the motion of a curve rolling on another curve, part of the problem is one of determining a suitable rolling curve pair.

The instant center concept is used to obtain the rolling curve pair for a given path or function generation problem. The equations of the rolling curve pair are then used with principles from the Calculus to determine the location of all points in the moving plane which, during the instantaneous motion chosen, move in paths of stationary curvature. Any two of these points are used as hinge joints at one end of a pair of links joining the moving plane to the fixed plane. The links are joined to the fixed plane at the centers of curvature of the pair of points chosen. This forms a four bar mechanism. The plane motion of the coupler bar of this mechanism will closely approx imate the motion of the moving curve plane over a small range of displacement.

This method is useful in devising mechanisms in which a point on the coupler bar traces a portion of some required continuous curve. It is also useful, by means of mechanism inversion, for devising function generator mechanisms. If the function generator can be made with a pair of rolling curves, a portion of the motion of the rolling curve can be generated with a four bar mechanism.

An analytical method of determining the output angle of the function generator of this mechanism is devised so that a computer can be programmed to test possible solutions of a given problem. The method does not supply the dimensions of the best linkage arrangement, so there remains the problem of testing a finite number of possible mechanisms in order to obtain one which will satisfy the problem.

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PREFACE

The synthesis of mechanisms has received much attention for many years. Before World War II almost all of the methods used were graphical. During the war years the increased use of mechanical analog computers and function generators focused more attention to the need for more accurate methods of synthesis. Since then, and particularly during the last ten years, various analytical methods have been developed for the synthesis of the basic four bar linkage.

While the four link mechanism is simple in appearance in that there are only three moving links, the analysis of the motion of the linkage is not simple. There are many theories and techniques in use. One of these, referred to later as the inflection circle concept, has been in use for many years and a special terminology has been built up around it. However, there seems to be no strictly analytical method of synthesis based on the theory underlying this method.

The object of this investigation was to develop a procedure for applying the theory of roulettes to two kinds of synthesis problems: mechanisms for tracing curves and mechanisms to generate functions. The inflection circle concept originates in the theory of roulettes.

The writer wishes to express his thanks to Dr. G. H. Martin of the Department of Mechanical Engineering for encouragement and suggestions while this work was in preparation.

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The four link mechanism. The four link mechanism is an assembly of four links pivoted together at their end points to form a closed chain. It has been studied extensively in the past and with good reason. It is the simplest linkage device having constrained motion, does not require expensive machining to produce, and can be used in an endless variety of applications. Such a mechanism can be used to produce plane motion or some input-output crank angular position, velocity, or acceleration relationship. The plane motion referred to here is the moticn of the coupler bar, link b, in Figure 1. The input-output motion is that of cranks a and c. For the linkage to be considered a mechanism, one link must be fixed.[1]

Other mechanism elements, such as rolling curves and cams, can be used to provide input-output relationships.[2,3] They can be designed to meet exact position requirements over a given range, but ease of construction,

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[1]Rolland T. Hinkle, Kinematics of Machines (2d ed.; Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1960) p. 7.

[2]Hinkle, p. 170.

[3]Alexander Cowie, Kinematics and Design of Mechanisms (Scranton, Penna.: The International Textbook Co., 1961) p. 368.

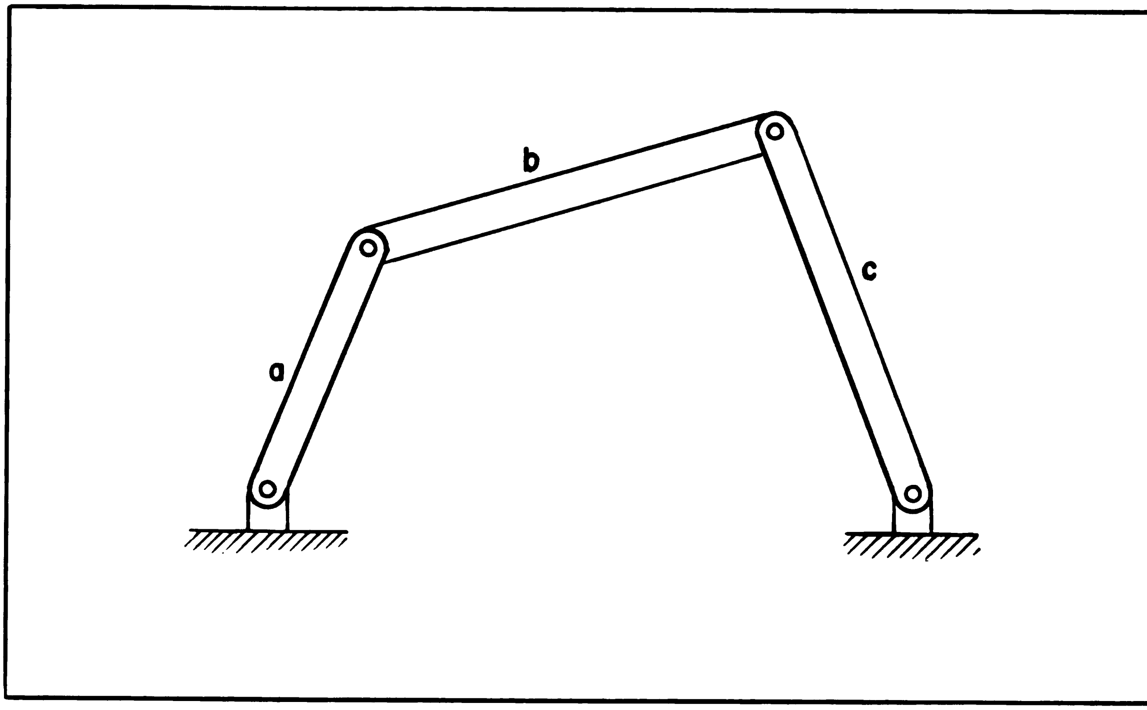

Fig. 1.-- The four link mechanism.

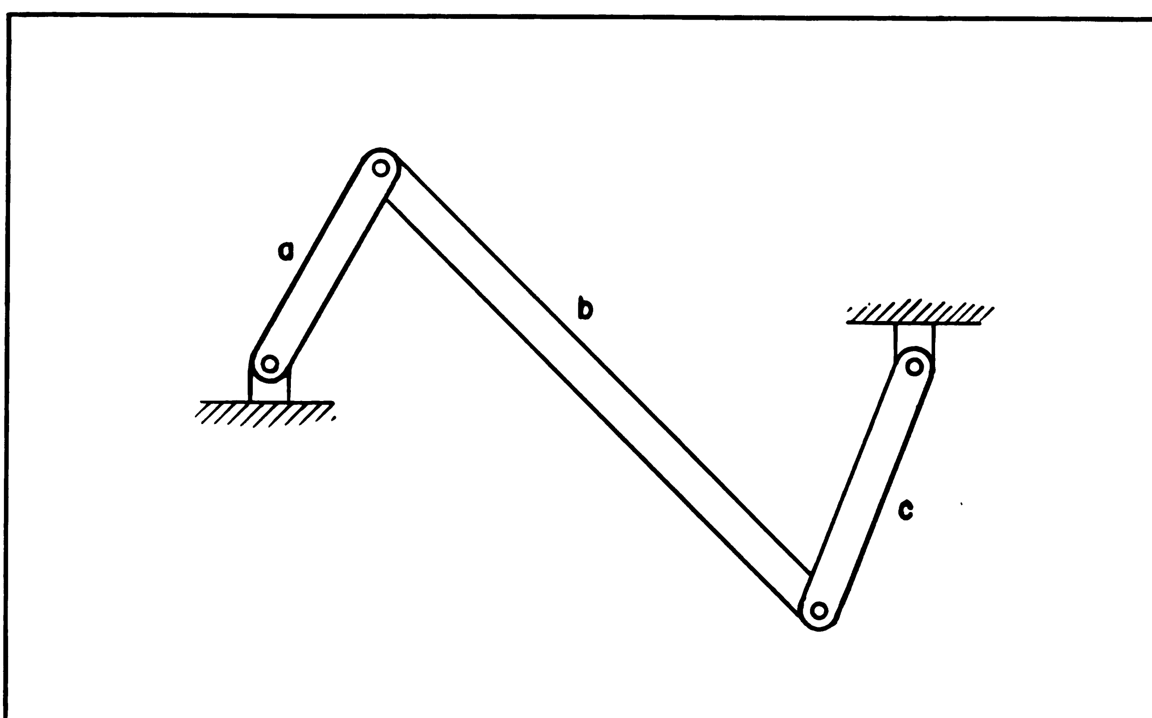

Fig. 2.-- The crossed four link mechanism.

good wear properties, positive constraint and perhaps other virtues make the four link mechanism the most attractive when it can be designed to provide motion similar to that of a pair of rolling curves. The procedure begins with a pair of rolling curves, one being fixed and the other moving, which are synthesized from the motion requirements. The writer has found no reference to a procedure in which a pair of rolling curves is synthesized and these curves used to develop an approximating four link mechanism.

Only a few functional relationships between input and output cranks can be satisfied exactly by a four link mechanism.[4] There are, however, many applications where a close approximation to a given functional relationship over a limited range is all that is desired. Such requirements can usually be satisfied with a four-link mechanism.

The design of a four link mechanism to satisfy some coupler bar or output crank motion for a complete cycle of the mechanism is beyond the scope of this work. Attention is directed toward the problem of devising mechanisms which approximate a given required motion over a limited range. The problem can be divided into two categories: path generation and function generation.

Path and Function Generation. Path generation is

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[4] B. W. Shaffer and I. Cochin, "Synthesis of the Four Bar Mechanism when the Position of two Members is Prescribed," Transactions of the ASME, v. 76, (Oct. 1954), p. 1137.

the case in which some point in the plane of the coupler bar (link b of Figures 1 and 2) traces a portion of some prescribed path. There are practical applications in machining surfaces and providing special motions in machines.[5,6] In the case of function generation, crank c has some particular motion relationship to crank a. The prescribed motion may satisfy requirements for position, velocity, or acceleration. There are useful applications in the fields of control mechanisms and computing devices.[7]

There are two basic approaches to both cases. The first, as applied to function generators, consists of choosing several values of the independent variable and computing the corresponding values of the dependent variable. A mechanism is then devised such that the output crank passes through several angular positions representing the dependent variable during the same phases in which the input crank is in angular positions corresponding to the independent variable values.[8,9,10]

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[5] James C. Wolford and Donald C. Haack, "Applying the Inflection Circle Concept," Transactions of the Fifth Conference on Mechanisms (Cleveland: The Penton Publishing Company, 1958) p. 232.

[6] Joseph S. Beggs, Mechanism (New York: McGraw Hill Book Company, Inc., 1955) p. 200.

[7] Hinkle, p. 293.

[8] Ferdinand Freudenstein and George N. Sandor, "Synthesis of Path Generating Mechanisms by Means of a Programmed Digital Computer," American Society of Mechanical Engineers Paper No. 58-A-85.

[9] Ferdinand Freudenstein, "Approximate Synthesis of Four Bar Mechanisms," Transactions of the ASME, v. 77, (August, 1955) p. 853.

[10] Hinkle, p. 267.

There are, mathematically at least, an unlimited number of possibilities in devising mechanisms to match a given motion requirement.

The second type of solution is based on the geometry of the plane motion of the coupler bar and is usually referred to as the inflection circle concept.[11,12] It is based on the theory of roulettes, which treats of the paths of points in the planes of curves as they roll without slipping on other curves.[13] This theory can be used to develop the Euler-Savary Equation which is used in applying the inflection circle concept to determine the center of curvature of the path of a point in a moving plane. In application, the inflection circle can be used to synthesize mechanisms which have motions matching a given requirement over a finite range of displacement. There are many examples in the literature.[14,15] As with the precision point method, there are infinitely many mechanisms obtainable from this method which will satisy a motion requirement over a small range.

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[11]Allen S. Hall, Jr., "Inflection Circle and Polode Curvature," <u>Transactions of the Fifth Conference on Mechanisms</u> (Cleveland: The Penton Publishing Company, 1958) p.207.

[12]Allen S. Hall, Jr., <u>Kinematics and Linkage Design</u> (Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1961),p. 65.

[13]Benjamin Williamson, An Elementary Treatise on the Differential Calculus, (London: Longmans Green and Co., LTD, 1927), p. 335.

[14]Wolford and Haak, p. 233.

[15]Hall, p. 94.

Synthesis methods using the inflection circle concept are not well suited to strictly analytical methods. Since the programmed digital computer can do numerical work with such rapidity, the use of such a device for mechanism synthesis seems very attractive. The development of such methods however, necessitated a reconsideration of the underlying mathematical theory.

CHAPTER

I

THEORETICAL DEVELOPMENT

I

THEORETICAL DEVELOPMENT

Roulettes

<u>The roulette</u>. A curve generated by some point invariably connected to a curve which rolls without slipping on another curve is a roulette.[16] Two well known examples are cycloids and trochoids. In Figure 3, the curves C_1 and C_3 are trochoids while curve C_2 is a cycloid. If the coordinate system in Figure 3 is located so that the origin is at O and the radius of the circle is r, then the parametric equations for the location of a point n, which lies in the moving plane, are:

$$X_n = r\theta - \overline{Qn} \sin\theta \tag{1}$$

$$Y_n = r - \overline{Qn} \cos\theta \tag{2}$$

Equations (1) and (2) can be used to devise a mechanism having four rigid, hinged links, one link of which will closely approximate, over a small range, the motion of the circle when rolling without slipping on the straight line. To do this, one locates a pair of points in the plane of the circle which have "stationary" or unchanging curvature.[17] Since the general point must be moving

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[16] Williamson, p. 335.

[17] Hall, p. 97.

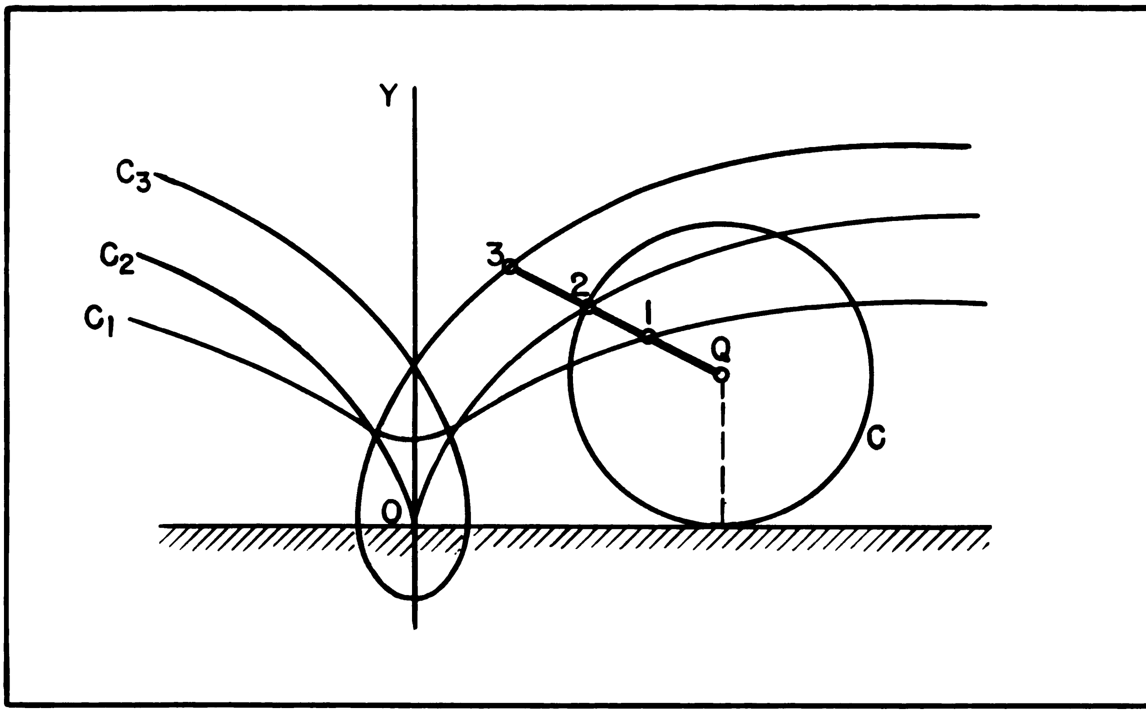

Fig. 3.-- Three roulettes: The cycloid C_2 and the trochoids C_1 and C_3.

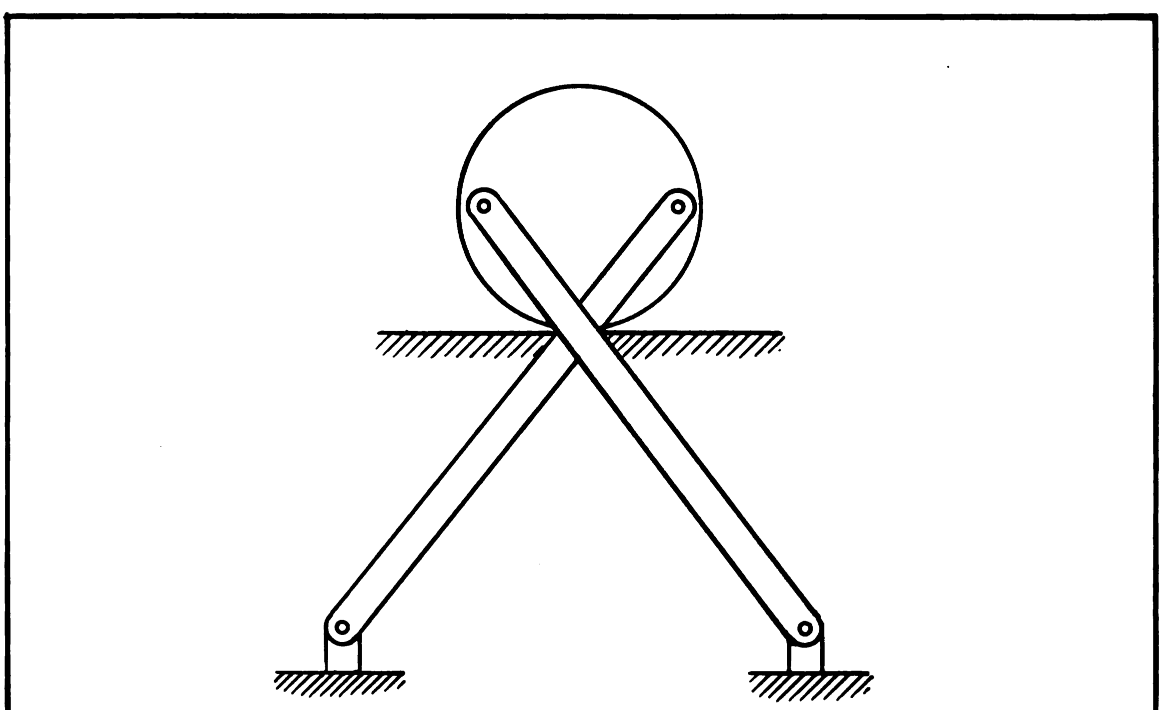

Fig. 4.--A four link mechanism to replace rolling motion.

along a trochoidal path with respect to the straight line, the curvature at every point is changing more or less rapidly. The points having the greatest range of stationary curvature are best. After the two points are selected, their centers of curvature are determined. The straight line is considered to be a line scribed in a fixed plane, and the two points chosen in the moving plane are joined by means of rigid links to their centers of curvature located in the fixed plane. See Figure 4.

This method of synthesis will apply to any pair of rolling curves. A more general expression than equations (1) and (2) will be required; one which will include the rolling curves used. Since the shape of the rolling curves is not generally known in the beginning, some method must be devised to determine the expressions of the curves as an intermediate step. This is done in the following consideration of plane motion.

Plane Motion and the Rolling Curve Pair

Plane motion. The plane motion of a plane may be some combination of translation and rotation. Whatever the motion, it may be considered to be composed of a number of small rotations about different instantaneous centers. Hence, any plane motion is the equivalent of the rolling of one curve on another.[18,19]

The given motion can be reduced to a rolling curve pair by using the concept of the instant center in the manner shown in the following development. Refer to Figure 5. A moving point P is located in a left-hand coordinate system x-y, which moves in a fixed right-hand system X-Y.

The location of P in the X-Y system in terms of the parameter ϕ, the angular displacement of x-y with respect to X-Y is:

$$X_P = X_Q + x_P \cos\phi - y_P \sin\phi \tag{3}$$

$$Y_P = Y_Q + x_P \sin\phi + y_P \cos\phi \tag{4}$$

X_Q and Y_Q locate the origin of the moving set in the fixed set.

The location of P in the x-y system is made similarly (refer to Figure 6.):

$$x_P = x_O + X_P \cos\phi + Y_P \sin\phi \tag{5}$$

$$y_P = y_O - X_P \sin\phi + Y_P \cos\phi \tag{6}$$

X_Q AND Y_Q are related to x_O AND y_O BY:

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[18]Williamson, p. 363.

[19]Edwin Bidwell Wilson, Advanced Calculus (Boston: Ginn and Company, 1911), p. 73.

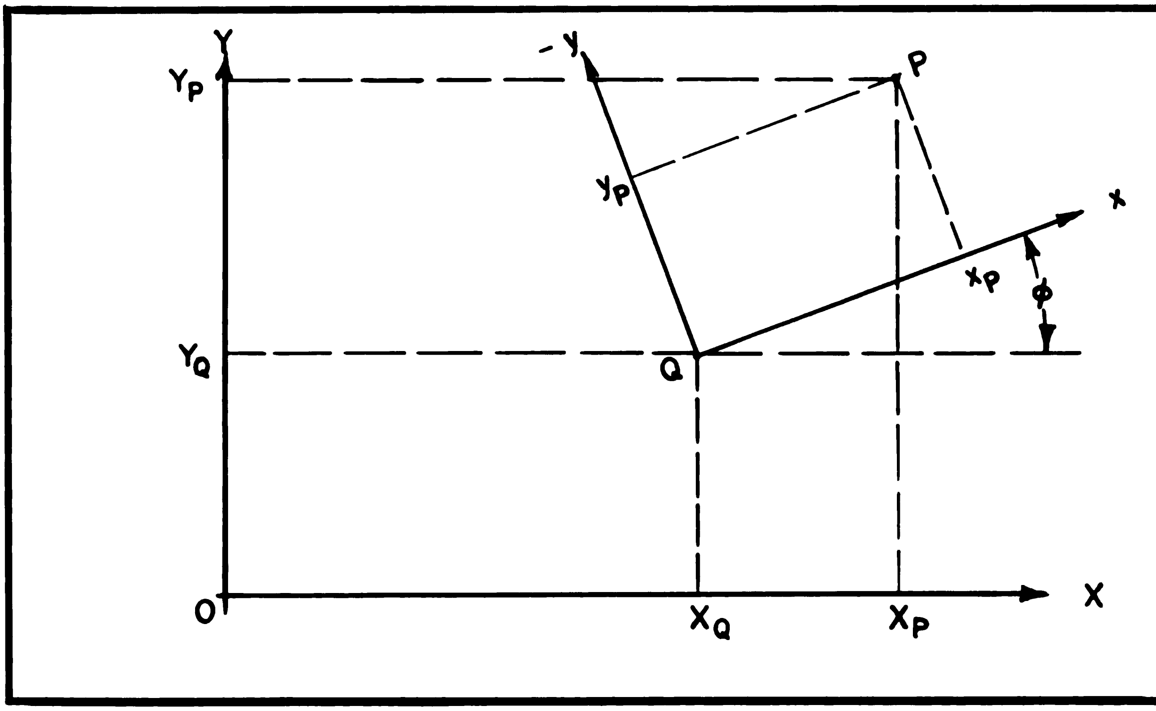

Fig. 5.--The fixed and moving coordinate systems.

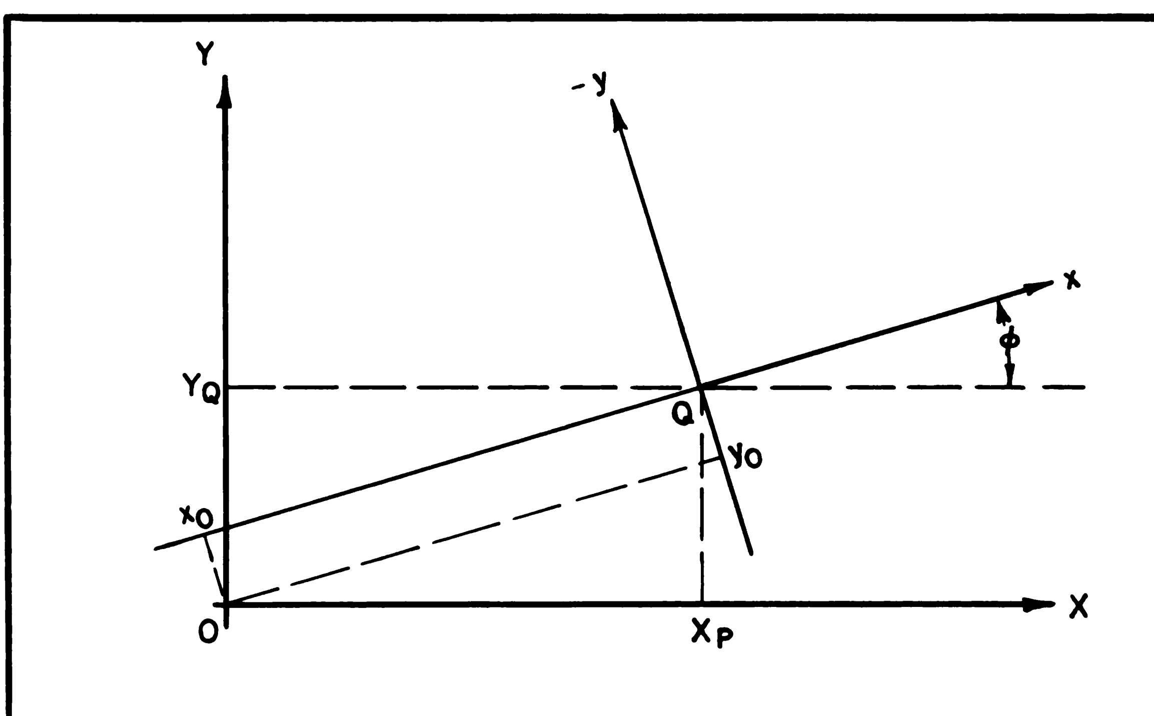

Fig. 6.-- The location of the fixed coordinate system relative to the moving system.

$$x_O = -(X_Q \cos\phi + Y_Q \sin\phi) \tag{7}$$

$$y_O = X_Q \sin\phi - Y_Q \cos\phi . \tag{8}$$

The fixed curve. The point P can be made to be the point of contact of the rolling curve pair. The requirement for pure rolling is satisfied if the velocity of the point of contact zero with respect to both systems: $\dot{x}_P = \dot{y}_P = \dot{X}_P = \dot{Y}_P = 0$. Here the dot notation is used to represent derivatives with respect to time. The imposition of the condition for pure rolling at point P on the time derivatives of equations (3) and (4) results in:

$$0 = \dot{X}_Q - x_P\dot{\phi}\sin\phi - y_P\dot{\phi}\cos\phi$$

$$0 = \dot{Y}_Q + x_P\dot{\phi}\cos\phi - y_P\dot{\phi}\sin\phi$$

which can be rearranged as:

$$\frac{\dot{X}_Q}{\dot{\phi}} = x_P \sin\phi + y_P \cos\phi \tag{9}$$

$$\frac{\dot{Y}_Q}{\dot{\phi}} = -(x_P \cos\phi - y_P \sin\phi) . \tag{10}$$

The parametric equations for the fixed curve, which is in the fixed coordinate system, are obtained by substituting equations (9) and (10) into equations (3) and (4). They are:

$$X_P = X_Q - \frac{\dot{Y}_Q}{\dot{\phi}} \tag{11}$$

$$Y_P = Y_Q + \frac{\dot{X}}{\dot{\phi}} \tag{12}$$

<u>The Moving Curve.</u> The equations of the moving curve are obtained in the following manner. Equations (7) and (8) are substituted into equations (5) and (6). The results are:

$$x_P = (X_P - X_Q)\cos\phi + (Y_P - Y_Q)\sin\phi$$
$$y_P = -(X_P - X_Q)\sin\phi + (Y_P - Y_Q)\cos\phi$$

Equations (11) and (12) are rearranged and substituted into the two equations above. The parametric expressions for the moving curve result.

$$x_P = \frac{1}{\dot\phi}(\dot X_Q \sin\phi - \dot Y_Q \cos\phi) \tag{13}$$
$$y_P = \frac{1}{\dot\phi}(\dot X_Q \cos\phi + \dot Y_Q \sin\phi)\,. \tag{14}$$

P is the point of contact between the rolling curves.

It is expected that the path of Q will be expressed as some function of X_Q. That is:

$$Y_Q = f(X_Q)$$

Then equations (11), (12), (13) and (14) will be expressions written as functions of X_Q and ϕ. Obviously, it will be helpful if ϕ is also a function of X_Q. For a particular path of point Q, Y_Q will be expressed as a particular function of X_Q. However, because the curve is a point path, the relationship between ϕ and X_Q is independent of the curve and any relationship may be used. Further, there will be a different rolling curve pair for every $\phi = g(X_Q)$. This is not particularly important if the path generator is to consist of a pair of rolling curves with the tracing point fixed in the moving curve, but if a four bar function generator with the tracing point fixed to the coupler bar is to be synthesized, the best function for ϕ will of necessity be determined by trial. This is so because the motion of the tracing point can be made to trace portions of some curves more accurately than others.

A rolling curve example. Some examples will be needed as the development proceeds so that principles can be illustrated. A simple one illustrating rolling curve development follows. The path to be generated is a straight line inclined upward to the right at an angle $\pi/4$ with the following specifications for the plane motion:

$$Y_Q = X_Q \,, \qquad \phi = K X_Q$$

in which K is some assumed constant. From equations (11) and (12):

$$X_P = X_Q - \frac{1}{K}$$

$$Y_P = Y_Q + \frac{1}{K}\,.$$

This may be rewritten as:

$$Y_P = X_P + \frac{2}{K}\,.$$

The fixed curve is seen to be a straight line inclined upward to the right at the angle $\pi/4$ from the X-axis. The line crosses the Y-axis at 2/K.

The equations for the moving curve are:

$$x_P = (1/K)(\sin\phi - \cos\phi)$$

$$y_P = -(1/K)(\sin\phi + \cos\phi)$$

After squaring both equations and adding, an equation of a circle about the origin having a radius of $\sqrt{2}/K$ is obtained.

$$x_P^2 + y_P^2 = 2/K$$

The tracing point is the center of a circle which will follow the line desired. See Figure 7.

The Point Path Traced in the Fixed Plane

Equations of the point. With the rolling curve pair expressed mathematically, the paths on the fixed plane traced by points fixed in the moving plane can be determined. Figure 8 shows the rolling curve pair and the point path. Q is the origin of the moving coordinate set, and ζ is the moving point, which is located by:

$$\overline{O\zeta} = \overline{OQ} + \overline{Q\zeta} \tag{15}$$

In order to write the X and Y components of equation (15), a radius vector angle α is defined. This angle, a position angle in the moving set, is measured counterclockwise from the negative y-axis. The X and Y components of the position of ζ are written:

$$X_\zeta = X_Q + \overline{Q\zeta}\sin(\alpha - \phi) \tag{16}$$

$$Y_\zeta = Y_Q + \overline{Q\zeta}\cos(\alpha - \phi)\,. \tag{17}$$

The distance Q can be expressed as the product of some number m, to be determined, and the distance $\overline{QP}$ at the initial position of the rolling curve pair.

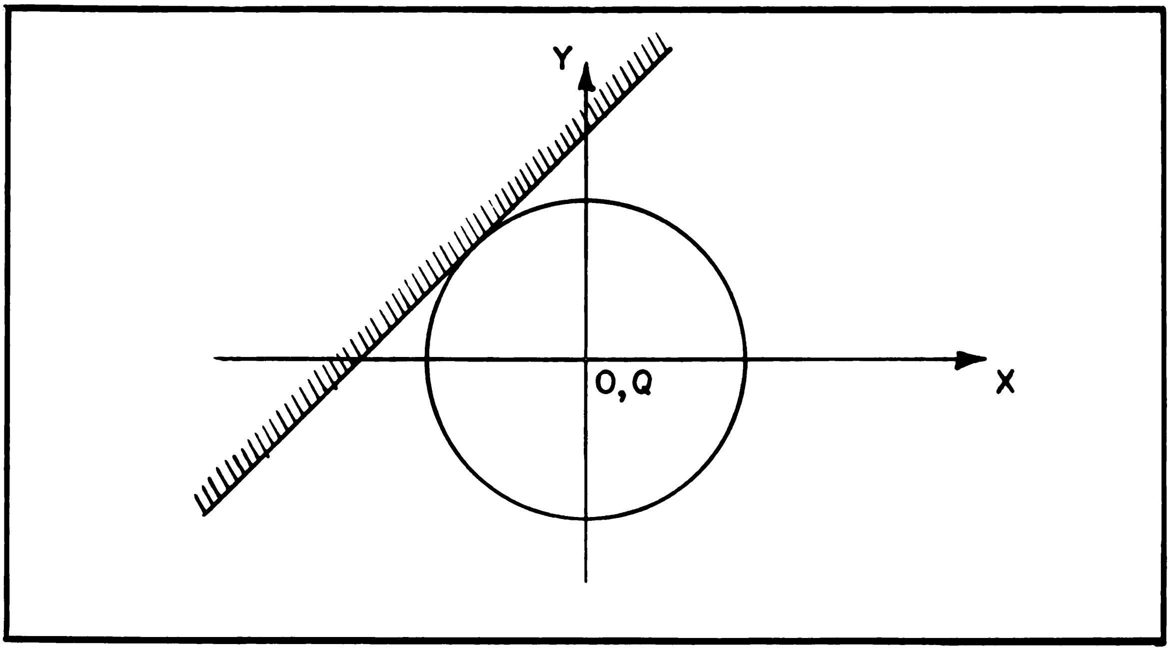

Fig. 7.-- The disc and straight line rolling pair.

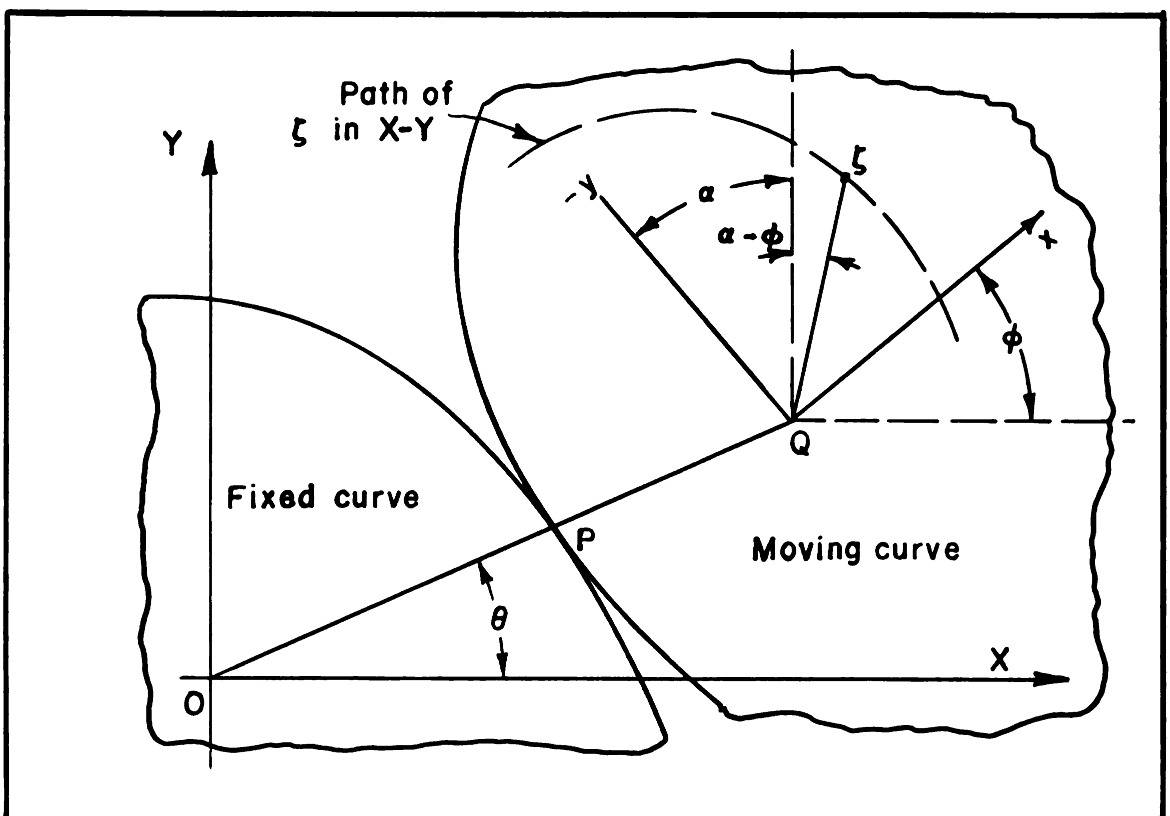

Fig. 8.--The general tracing point and the rolling curve pair.

$$\overline{Q\zeta} = m\,\overline{QP} = m\sqrt{x_P^2)_{t=0} + y_P^2)_{t=0}} \tag{18}$$

The expression under the radical in equation (18) can be represented by γ.

$$\gamma = \sqrt{x_P^2)_{t=0} + y_P^2)_{t=0}} \tag{19}$$

Equations (16) and (17) can now be written:

$$X_\zeta = X_Q + m\gamma \sin(\alpha - \phi) \tag{20}$$

$$Y_\zeta = Y_Q + m\gamma \cos(\alpha - \phi)\,. \tag{21}$$

Points having stationary curvature. The points in the moving plane which trace paths having momentarily stationary curvature can be located by means of expressions for the radius of curvature. Radius of curvature expressions can be obtained from any textbook on the Calculus.[20] The procedure is to equate the derivative of the radius of curvature, with respect to the position angle α, to zero and solve the resulting equation for m. Equations (20) and (21) can then be used to locate the stationary curvature points. The centers of curvature are located by using expressions obtained from the Calculus. (See equations (25) and (26)). There is, of course, a solution for m for every chosen value of α.

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[20]William Anthony Granville, Percy F. Smith, and William Raymond Longley, Elements of the Differential and Integral Calculus (Boston: Ginn and Company, 1941), p.152.

The expression for the radius of curvature, taken from the Calculus, is:

$$R_\zeta = \frac{\left[1 + \left(\frac{dY_\zeta}{dX_\zeta}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2 Y_\zeta}{dX_\zeta^2}} \tag{22}$$

where

$$\frac{dY_\zeta}{dX_\zeta} = \frac{dY_\zeta}{dt} \Big/ \frac{dX_\zeta}{dt} \tag{23}$$

and

$$\frac{d^2 Y_\zeta}{dX_\zeta^2} = \frac{\frac{dX_\zeta}{dt}\frac{d^2 Y_\zeta}{dt^2} - \frac{dY_\zeta}{dt}\frac{d^2 X_\zeta}{dt^2}}{\left(\frac{dX_\zeta}{dt}\right)^3} \tag{24}$$

The expressions for the centers of curvature are:[21]

$$CX = X_\zeta - \frac{dY_\zeta}{dX_\zeta}\,\frac{1 + \left(\frac{dY_\zeta}{dX_\zeta}\right)^2}{\frac{d^2 Y_\zeta}{dX_\zeta^2}} \tag{25}$$

$$CY = Y_\zeta + \frac{1 + \left(\frac{dY_\zeta}{dX_\zeta}\right)^2}{\frac{d^2 Y_\zeta}{dX_\zeta^2}} \tag{26}$$

CX and CY are the X and Y coordinates of the center of curvature.

Equations (20) and (21) are now put into equation (22). The derivatives are determined, using equations (23) and (24): (The dot notation is used for time derivatives.)

[21]Granville, Smith, and Longley, p. 157.

$$\frac{dY_\zeta}{dX_\zeta} = \frac{\dot{Y}_Q + m\gamma\dot{\phi}\ \sin(\alpha-\phi)}{\dot{X}_Q - m\gamma\dot{\phi}\ \cos(\alpha-\phi)} \tag{27}$$

$$\frac{d^2Y_\zeta}{dX_\zeta^2} = \frac{\dot{X}_Q\ddot{Y}_Q - \dot{Y}_Q\ddot{X}_Q + m\gamma[M\sin(\alpha-\phi) + N\cos(\alpha-\phi)] + m^2\gamma^2\dot{\phi}^2}{[\dot{X}_Q - m\gamma\dot{\phi}\cos(\alpha-\phi)]^3}. \tag{28}$$

M and N are defined as:

$$M = \dot{X}_Q\ddot{\phi} - \ddot{X}_Q\dot{\phi} + \dot{Y}_Q\dot{\phi}^2 \tag{29}$$

$$N = \dot{Y}_Q\ddot{\phi} - \ddot{Y}_Q\dot{\phi} - \dot{X}_Q\dot{\phi}^2. \tag{30}$$

The expression for the radius of curvature can now be written:

$$R_\zeta = \frac{\left\{\dot{X}_Q^2 - \dot{Y}_Q^2 + 2m\gamma[\dot{Y}_Q\sin(\alpha-\phi) - \dot{X}_Q\cos(\alpha-\phi)] + m^2\gamma^2\dot{\phi}^2\right\}^{3/2}}{\dot{X}_Q\ddot{Y}_Q - \dot{Y}_Q\ddot{X}_Q + m\gamma[M\sin(\alpha-\phi) + N\cos(\alpha-\phi)] + m^2\gamma^2\dot{\phi}^3}. \tag{31}$$

The process of writing the derivative of R_ζ with respect to α can be shortened by writing R_ζ symbolically as:

$$R_\zeta = \frac{A^{3/2}}{B}.$$

The derivative is then:

$$\frac{dR_\zeta}{d\alpha} = \frac{\left(\frac{3}{2}\right)BA^{1/2}\frac{dA}{d\alpha} - A^{3/2}\frac{dB}{d\alpha}}{B^2}.$$

If $\frac{dR_\zeta}{d\alpha} = 0$, then

$$3B\frac{dA}{d\alpha} = 2A\frac{dB}{d\alpha}. \tag{32}$$

Since

$$A = \dot{X}_Q^2 + \dot{Y}_Q^2 + 2m\gamma\dot{\phi}[\dot{Y}_Q\sin(\alpha-\phi) - \dot{X}_Q\cos(\alpha-\phi)] + m^2\gamma^2\dot{\phi}^2$$

$$B = \dot{X}_Q\ddot{Y}_Q - \dot{Y}_Q\ddot{X}_Q - m\gamma[M\sin(\alpha-\phi) + N\cos(\alpha-\phi)] + m^2\gamma^2\dot{\phi}^3$$

then

$$\frac{dA}{d\alpha} = 2m\gamma\dot{\phi}\left[\dot{Y}_Q \cos(\alpha-\phi) + \dot{X}_Q \sin(\alpha-\phi)\right]$$

$$\frac{dB}{d\alpha} = m\gamma\left[M \cos(\alpha-\phi) - N \sin(\alpha-\phi)\right]$$

When values of A, B, $\frac{dA}{d\alpha}$, and $\frac{dB}{d\alpha}$ are put into equation (32) and simplified, a quadratic equation in m is obtained as follows:

$$m^2\gamma^2\dot{\phi}^2\left[3\dot{\phi}^2 - Z\right] + m\gamma\dot{\phi}\left[(3M - 2\dot{Y}_Q Z)\sin(\alpha-\phi) + (3N + 2\dot{X}_Q Z)\cos(\alpha-\phi)\right] - 3\dot{\phi}(\dot{Y}_Q\ddot{X}_Q - \dot{X}_Q\ddot{Y}_Q) - Z(\dot{X}_Q^2 + \dot{Y}_Q^2) = 0 \quad (33)$$

in which

$$Z = \frac{M\cos(\alpha-\phi) - N\sin(\alpha-\phi)}{\dot{Y}_Q\cos(\alpha-\phi) + \dot{X}_Q\sin(\alpha-\phi)} \quad (34)$$

Equation (33) completes the derivation of the equations locating the points in the moving plane having stationary curvature and their centers of curvature. The values of m obtained from a solution of equation (33) are put into equations (20), (21), (25), and (26). For every value of α chosen there are two values of m, and it follows that along any line drawn through point Q and making the angle α with the -y axis, there are two points having stationary curvature.

With the points of stationary curvature, ζ_1 and ζ_2 and their centers of curvature C_1 and C_2 located, a mechanism can now be constructed which will cause the point Q to move along a path which is similar over a small range

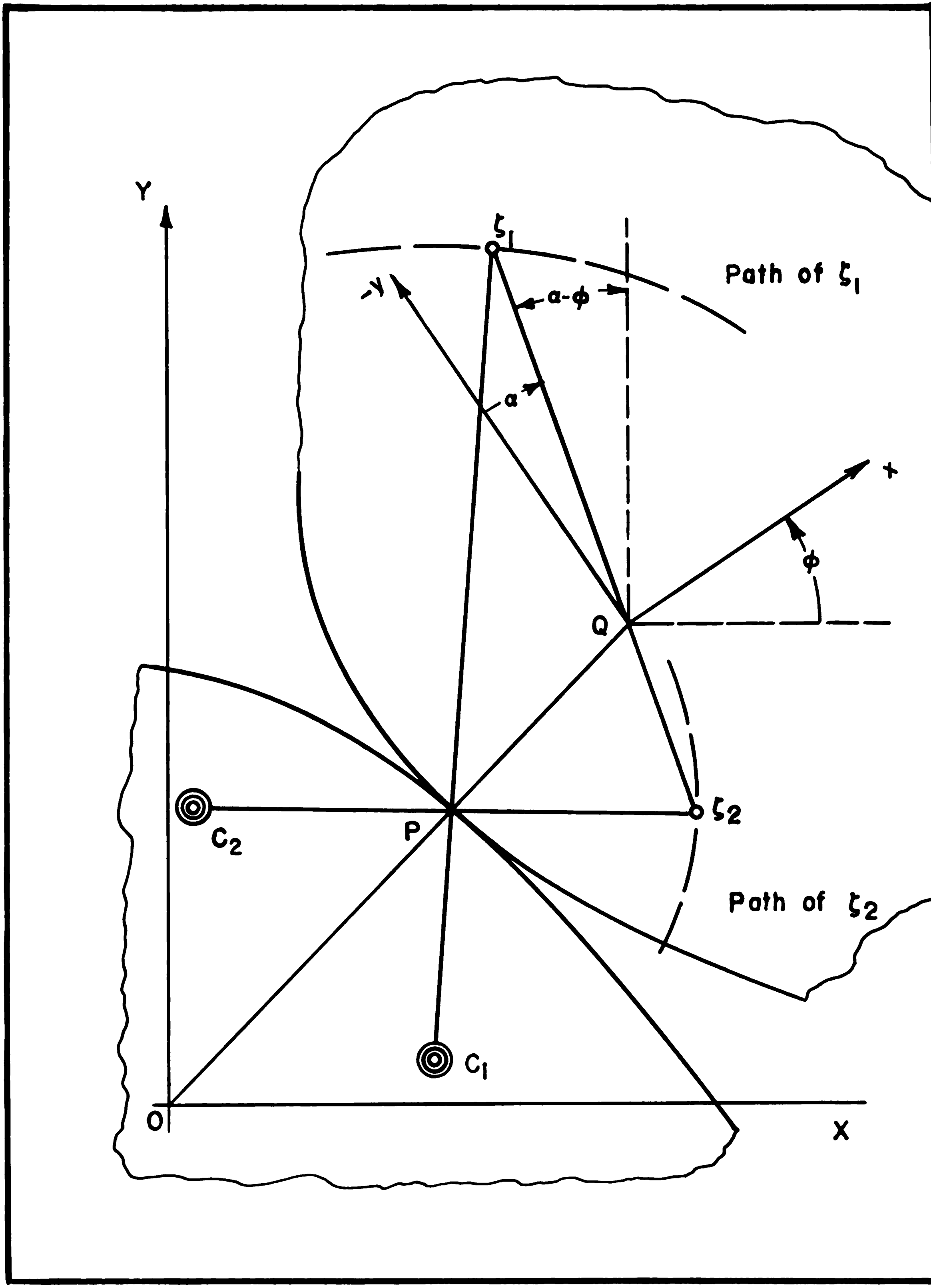

Fig. 9.-- The rolling curve pair and the derived approximating four link mechanism.

to that produced by Q as a point in the moving curve. Points ζ_1 and C_1 become the joints of a link joining the fixed to the moving plane; points ζ_2 and C_2 form another link.

Application. The example presented earlier for rolling curve formation can be used to illustrate mechanism synthesis by this method. Some arbitrarily assumed values which simplify the problem are:

$$\ddot{\phi} = \ddot{Y}_Q = \ddot{X}_Q = 0.$$

Then M and N from equations (29) and (30) are:

$$M = K\dot{\phi}^3; \quad N = -K\dot{\phi}^3$$

also, from equation (34):

$$Z = \dot{\phi}^2.$$

Equation (33) becomes:

$$2m^2\gamma^2\dot{\phi}^4 + mK\gamma\dot{\phi}^4[\sin(\alpha-\phi) - \cos(\alpha-\phi)] - 2K^2\dot{\phi}^4 = 0.$$

By equation (19):

$$\gamma = \frac{\sqrt{2}}{K}.$$

The constant K is a scale factor and the geometry of the mechanism will not be changed by any chosen value. Unity is the most convenient choice and the quadratic equation in m becomes:

$$2m^2 + \frac{m}{\sqrt{2}}[\sin(\alpha-\phi) - \cos(\alpha-\phi)] - 1 = 0. \qquad \text{(A)}$$

Any value of ϕ can be chosen since the character of the motion of a disc on a straight line is independent of the position of the disc. When $\phi = 0$, the solution of equation (A) becomes:

$$m = \frac{\sqrt{2}}{8}\left[\cos\alpha - \sin\alpha \pm \sqrt{17 - \sin 2\alpha}\,\right] \tag{B}$$

Since a different solution of (B) exists for each value of α chosen, there are, mathematically, infinitely many possible mechanisms for drawing this straight line. The locii of the stationary curvature points and their centers can be obtained by computing the positions of the points and their centers for a finite number of values of α between 0° and 360°. The equations needed are listed below.

$$Q\zeta = \frac{1}{4}\left[\cos\alpha - \sin\alpha \pm \sqrt{17 - \sin 2\alpha}\,\right] \tag{C}$$

$$X_\zeta = Q\zeta \sin\alpha \tag{D}$$

$$Y_\zeta = Q\zeta \cos\alpha \tag{E}$$

$$Y'_\zeta = \frac{1 + X_\zeta}{1 - Y_\zeta} \tag{F}$$

$$Y''_\zeta = \frac{Q\zeta\,(X_\zeta - Y_\zeta + Q\zeta)}{(1 + Y_\zeta)^3} \tag{G}$$

$$CX = X_\zeta - \frac{Y'_\zeta\,[1 + (Y'_\zeta)^2]}{Y''_\zeta} \tag{H}$$

$$CY = Y_\zeta + \frac{1 + (Y'_\zeta)^2}{Y''_\zeta} \tag{I}$$

The locii are shown plotted in Figure 10. The numerical results of this program on a digital computer are shown in Table 1.

A single mechanism is constructed by choosing a value of α which is measured clockwise from the negative y-axis. In this case, with $\phi = 0$, the negative y-axis

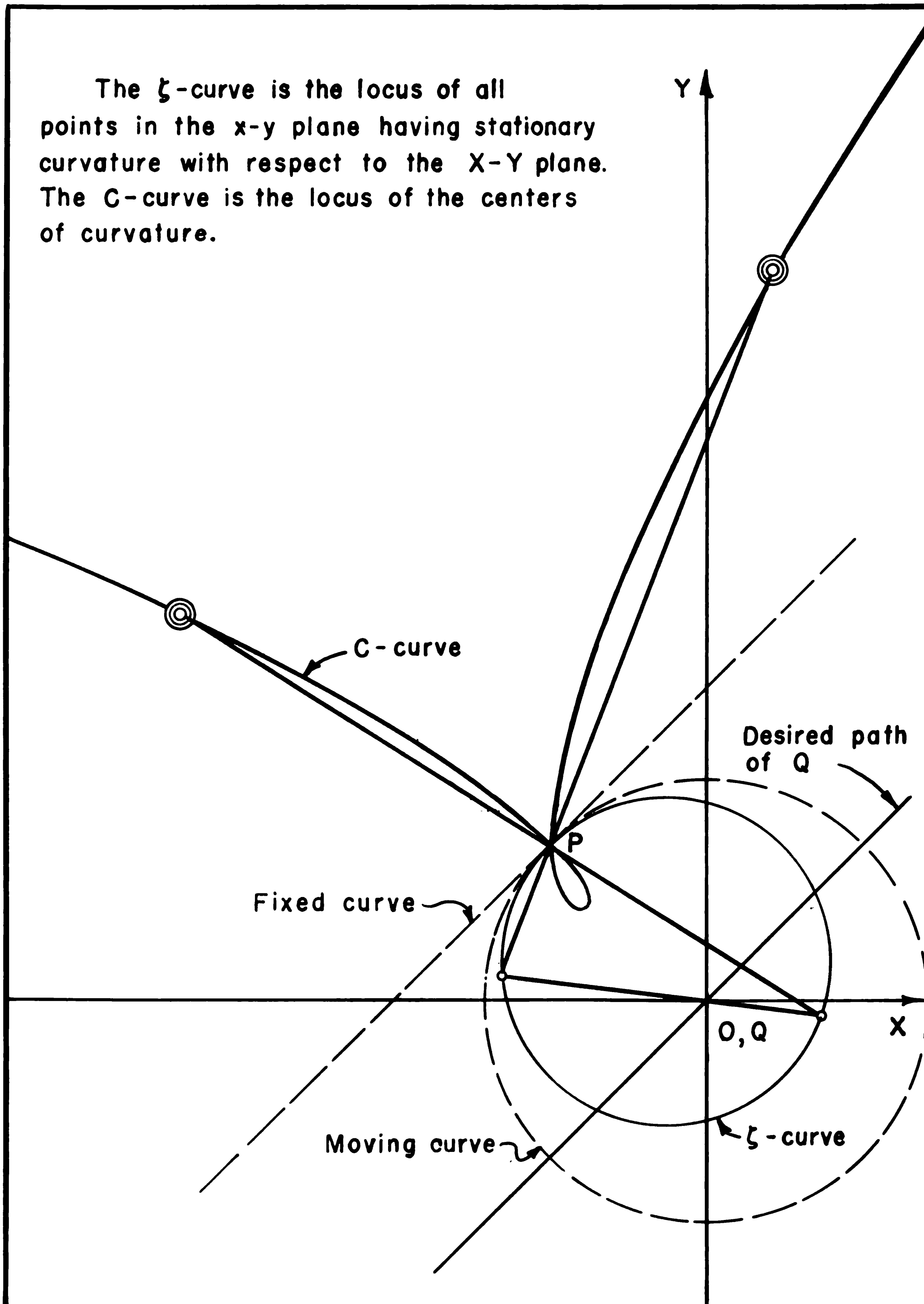

Fig. 10.--The locus curves for the straight line mechanism.

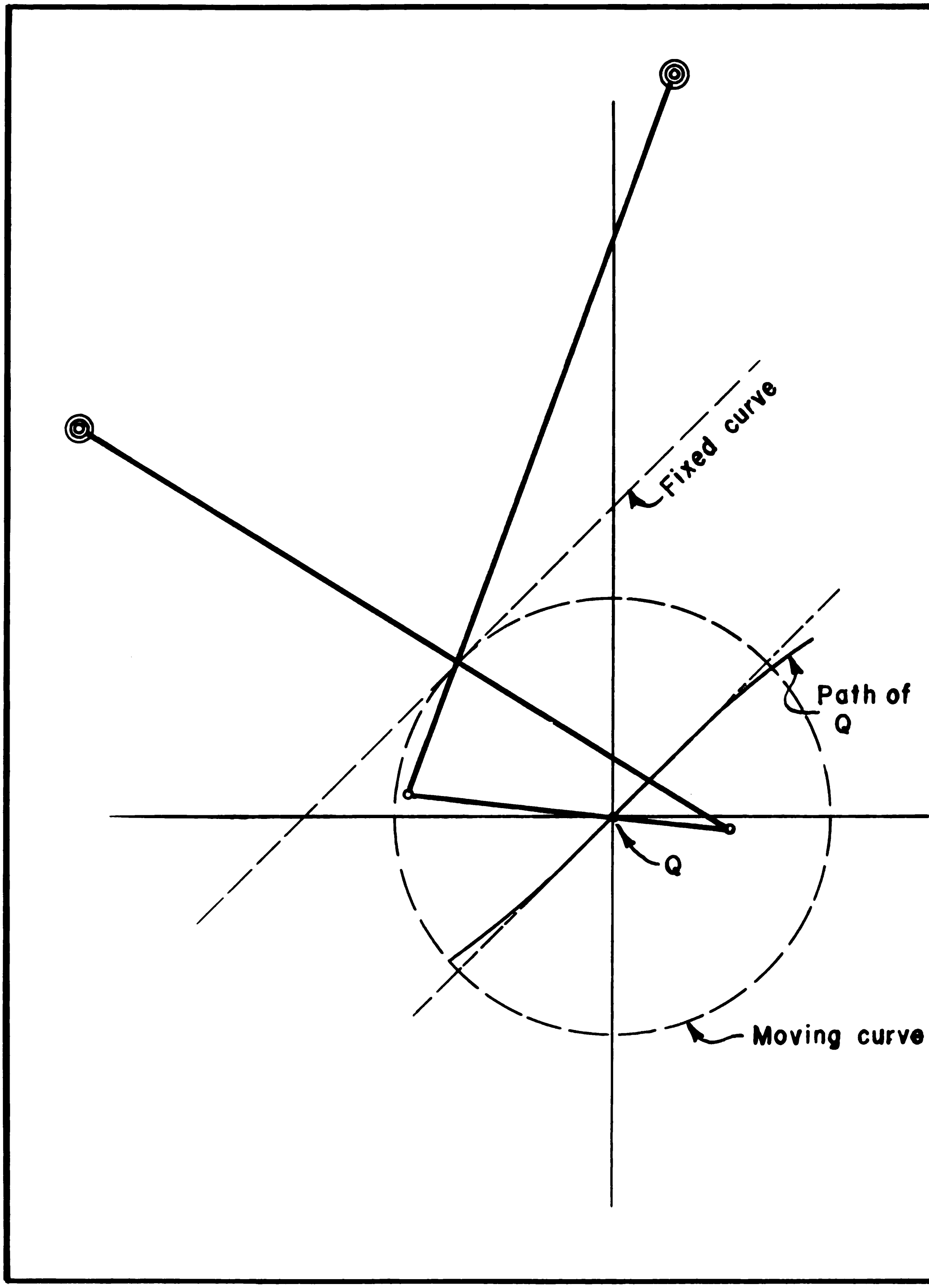

Fig. 11.-- A straight line mechanism obtained from the locus curves.

Table 1. Tabulated values for the X and Y components of stationary curvature and their centers for the straight line mechanism.

A	YP1	YP2	XP1	XP2	CX1	CX2	CY1	CY2
.1745	1.204	-.805	.212	-.141	-317.461	-.807	-52.401	.596
.3490	1.090	-.809	.396	-.294	-1674.825	-.841	-107.540	.592
.5235	.948	-.790	.547	-.456	-5081.727	-.880	168.490	.607
.6980	.790	-.742	.662	-.623	-272.810	-.924	35.304	.649
.8725	.623	-.662	.742	-.789	-73.448	-.966	16.653	.737
1.0470	.456	-.547	.790	-.948	-29.485	-.997	9.646	.909
1.2215	.294	-.397	.809	-1.090	-14.281	-.984	6.174	1.236
1.3960	.142	-.212	.805	-1.204	-7.749	-.857	4.207	1.846
1.5705	0.000	0.000	.780	-1.280	-4.565	-.439	3.001	2.997
1.7450	-.130	.230	.740	-1.310	-2.883	.709	2.223	5.236
1.9195	-.249	.468	.685	-1.287	-1.947	3.867	1.702	9.988
2.0940	-.357	.698	.619	-1.211	-1.411	14.034	1.345	22.463
2.2685	-.454	.907	.542	-1.082	-1.101	77.619	1.095	89.392
2.4430	-.542	1.081	.455	-.908	-.924	-135.189	.920	-118.495
2.6175	-.619	1.210	.357	-.699	-.829	-75.366	.796	-51.157
2.7920	-.685	1.287	.249	-.469	-.785	-76.341	.710	-39.857
2.9665	-.739	1.310	.130	-.231	-.774	-96.508	.652	-37.606
3.1414	-.780	1.280	0.000	0.000	-.784	-148.933	.615	-40.594
3.3155	-.805	1.204	-.141	.211	-.807	-316.365	.596	-52.331
3.4900	-.809	1.091	-.294	.396	-.841	-1658.704	.592	-107.061
3.6645	-.790	.949	-.455	.547	-.880	-5187.238	.607	170.290
3.8390	-.743	.790	-.622	.662	-.924	-274.379	.649	35.417
4.0135	-.663	.624	-.789	.742	-.966	-73.711	.737	16.689
4.1880	-.548	.457	-.948	.790	-.996	-29.565	.909	9.662
4.3625	-.397	.295	-1.089	.809	-.984	-14.313	1.234	6.183

Continued on next page

Table 1--Continued. Tabulated values for the X and Y components of points of stationary curvature and their centers for the straight line mechanism.

A	YP1	YP2	XP1	XP2	CX1	CX2	CY1	CY2
4.5370	-.213	.142	-1.204	.805	-.858	-7.764	1.844	4.212
4.7115	-.001	0.000	-1.280	.780	-.441	-4.573	2.992	3.004
4.8860	.229	-.129	-1.310	.740	.703	-2.887	5.226	2.225
5.0605	.467	-.248	-1.288	.685	3.850	-1.950	9.964	1.704
5.2350	.697	-.357	-1.211	.619	13.971	-1.413	22.390	1.346
5.4095	.906	-.454	-1.082	.542	76.947	-1.102	88.706	1.906
5.5840	1.081	-.541	-.909	.455	-135.875	-.925	-119.201	.920
5.7585	1.210	-.618	-.700	.358	-75.406	-.829	-51.228	.797
5.9330	1.287	-.685	-.470	.250	-76.305	-.785	-39.875	.710
6.1075	1.310	-.739	-.232	.131	-96.401	-.774	-37.604	.652
6.2820	1.281	-.780	-.001	0.000	-148.651	-.783	-40.574	.615

corresponds to the positive Y-axis. A line is drawn through Q at angle α. The intersection of the line with the ζ-curve locates the two points ζ_1 and ζ_2. These are the joints connecting the cranks with the moving plane. Lines are now drawn from ζ_1 and ζ_2 through P to the point of intersection with the C-curve. This determines the length of the cranks and the location of the joints on the fixed plane. The construction is shown in Figure 10 and the mechanism is shown in Figure 11. The portion of the path of Q as a point on the coupler bar is shown to indicate the range of match with the desired curve.

Practicality. The utility of a curve tracing device may not be too obvious. One possible application is its use as a special mechanism to aid in machining surfaces. There is another possibility in replacing gears and rolling curves to provide a particular motion. In the example given, the link which is formed by drawing a line from ζ_1 to ζ_2 has the same motion it would have if it were a line scribed on the disc. This suggests the possibility of replacing a rack and pinion with a four bar when the range of motion required is small.

Angular acceleration of the disc. The specification that $\dot{\phi}$ be constant does not make this a special case, for as long as $X_Q = Y_Q = K\phi$, the rolling curve pair will have the same form and the mechanism will be the same.

CHAPTER

II

APPLICATIONS TO PATH GENERATION

II

PATH GENERATION

Open Curves

Open Curve paths. We shall now consider the problem of synthesis of a four link mechanism having a tracing point Q which traces some portion of an open curve. The general procedure previously described applies. The principal difficulty lies in the selection of the best relationship between ϕ and X_Q. There are no general rules for selecting the relationship so that trial and error will perhaps be necessary.

Parabola. As a first step in the exploration of the method, a mechanism is to be designed to trace a portion of a parabolic curve.

$$Y_Q^2 = 4X_Q \tag{35}$$

The ϕ to X_Q relationship is assumed to be

$$X_Q = \phi. \tag{36}$$

The following equations are derived from equations (35) and (36).

$$\dot{X}_Q = \dot{\phi} \tag{37}$$

$$\ddot{X}_Q = \ddot{\phi} \tag{38}$$

$$\dot{Y}_Q = \dot{\phi}/\sqrt{\phi} \tag{39}$$

$$\ddot{Y}_Q = \ddot{\phi}/\sqrt{\phi} - \dot{\phi}/(\sqrt{\phi})^3. \tag{40}$$

The fixed and moving curves are expressed by:

$$X_P = \phi - 1/\sqrt{\phi} \tag{41}$$

$$Y_P = 2\sqrt{\phi} + 1 \tag{42}$$

$$y_P = \sin\phi - \frac{\cos\phi}{\sqrt{\phi}} \tag{43}$$

$$x_P = -(\cos\phi + \frac{\sin}{\sqrt{\phi}}) \tag{44}$$

The expressions for the roulette are:

$$X_\zeta = \phi + m\gamma\sin(\alpha-\phi) \tag{45}$$

$$Y_\zeta = 2\sqrt{\phi} + m\gamma\cos(\alpha-\phi) \tag{46}$$

in which

$$\gamma = \sqrt{\frac{\phi+1}{\phi}} \; .$$

Before a mechanism can be designed it is necessary to select some values of ϕ and $\dot{\phi}$. By choosing $\phi = \dot{\phi} = 1$ the following series of equations is obtained.

$$\gamma = \sqrt{2} \; ; \quad M = 1 \; ; \quad N = -1$$

$$Z = \frac{\cos(\alpha-\phi) - 0.5\sin(\alpha-\phi)}{\cos(\alpha-\phi) + \sin(\alpha-\phi)} \; .$$

Equation (33) then becomes:

$$2m^2 + \frac{m}{\sqrt{2}} \frac{5 + \sin 2(\alpha-\phi) - 3\cos 2(\alpha-\phi)}{4\cos(\alpha-\phi) + 5\sin(\alpha-\phi)} - \frac{7\cos(\alpha-\phi) + 5\sin(\alpha-\phi)}{4\cos(\alpha-\phi) + 5\sin(\alpha-\phi)} = 0$$

which can be written:

$$2m^2 + \frac{m}{\sqrt{2}} D - E = 0 \tag{J}$$

if

$$D = \frac{5 + \sin 2(\alpha-\phi) - 3\cos 2(\alpha-\phi)}{4\cos(\alpha-\phi) + 5\sin(\alpha-\phi)} \; , \quad E = \frac{7\cos(\alpha-\phi) + 5\sin(\alpha-\phi)}{4\cos(\alpha-\phi) + 5\sin(\alpha-\phi)} \; . \tag{K,L}$$

The following list of equations can be used to make the computations needed to synthesize a mechanism which will trace a portion of the parabola of this example. It was the basis for a computer program used to make a more complete solution.

$$m = \frac{1}{4\sqrt{2}} \quad -D \pm \sqrt{D^2 + 16E} \tag{M}$$

$$X_\zeta = 1 + \sqrt{2} m \sin(\alpha - \phi) \tag{N}$$

$$Y_\zeta = 2 + \sqrt{2} m \cos(\alpha - \phi) \tag{O}$$

$$Y'_\zeta = \frac{1 + \sqrt{2}\ m \sin(\alpha-\phi)}{1 - \sqrt{2}\ m \cos(\alpha-\phi)} \tag{P}$$

$$Y''_\zeta = \frac{-1/2 + \sqrt{2}\, m\, [\sin(\alpha-\phi) - 1/2 \cos(\alpha-\phi) + 2m]}{[1 - \sqrt{2}\, m \cos(\alpha-\phi)]^3} \tag{Q}$$

$$CX = X_\zeta - \frac{Y'_\zeta(1 + Y'^2_\zeta)}{Y''_\zeta} \tag{R}$$

$$CY = Y_\zeta + \frac{(1 + Y'^2_\zeta)}{Y''_\zeta}\ . \tag{S}$$

The Fortran program for this series of equations is in the appendix. The tabulated results are presented in Table 2.. The locus plot of the points and their centers are shown in Figure 12. The performance of one of the possible mechanisms is shown in Figure 13.

Closed Curves

Closed curve paths. The circle. The next problem is to explore the synthesis of a mechanism having a coupler bar point which moves in a circular path. It is not to be expected that the resulting mechanism will generate a circle, but that it will approximate the curve over a limited range. The problem is depicted in Figure 14.

The equation for the path of Q is:

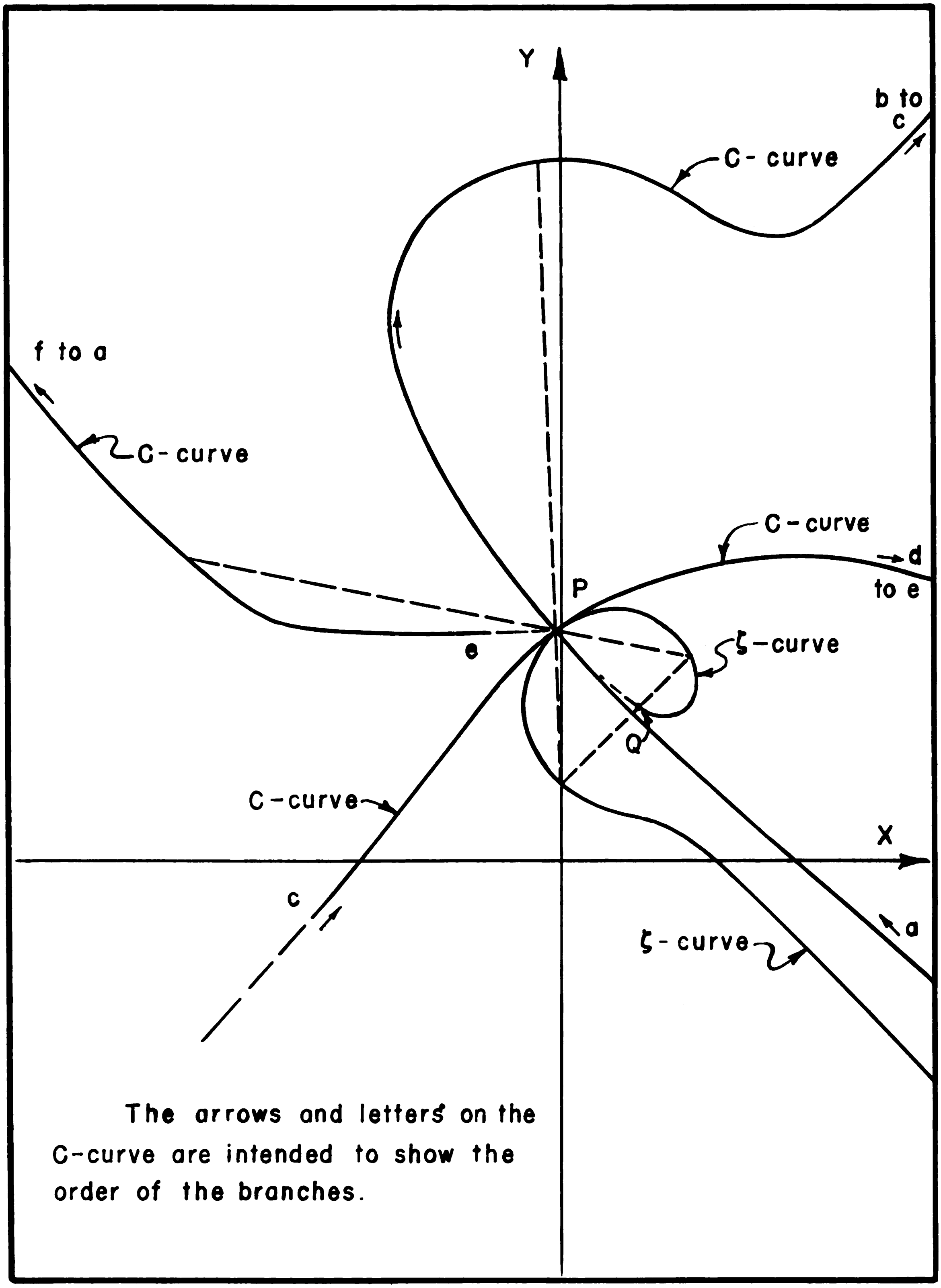

Fig. 12.-- The locii of the ζ and C- points for the parabolic path.

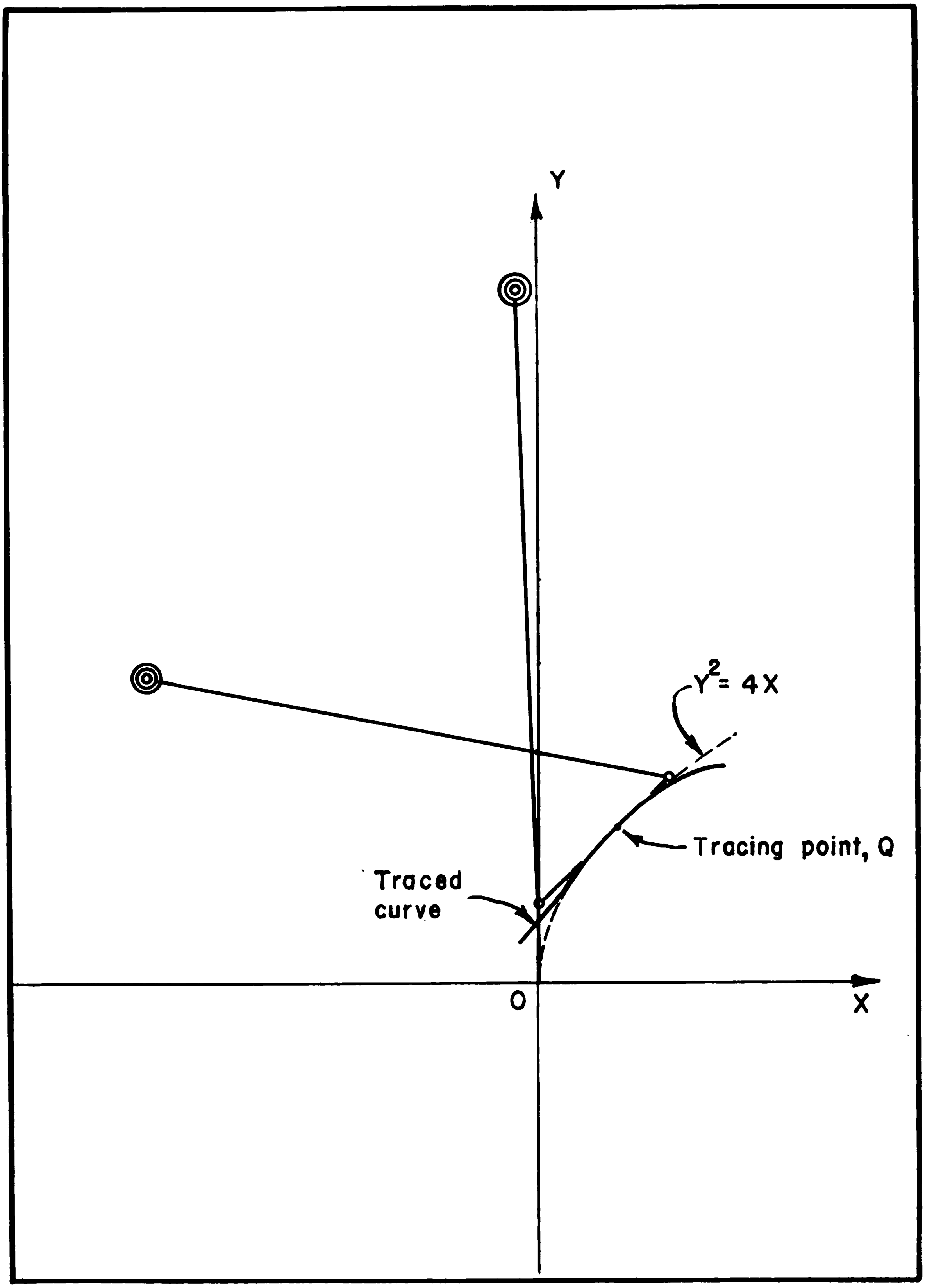

Fig. 13.--One parabolic path mechanism.

Table 2. Results for the Parabolic path program.

A	XP1	YP1	CX1	CY1	XP2	YP2	CX2	CY2
.174	-.042	2.962	-.056	2.950	.424	2.531	.657	2.273
.349	.161	3.101	.376	3.236	8.284	-7.562	-1.280	4.632
.523	.384	3.193	- 1.170	3.588	2.060	-.053	-1.974	5.926
.698	.614	3.236	2.153	3.827	1.510	.362	-2.189	6.823
.892	.842	3.232	3.228	3.890	1.193	.493	-2.146	7.509
1.047	1.055	3.185	4.327	3.758	.932	.566	-1.935	8.049
1.221	1.247	3.100	5.400	3.436	.692	.634	-1.600	8.467
1.396	1.412	2.987	-3.585	3.032	.463	.717	-1.172	8.771
1.570	1.546	2.851	-4.247	3.407	.246	.825	-.675	8.959
1.745	1.647	2.701	-4.880	3.884	.045	.964	-.134	9.029
1.919	1.714	2.544	-5.491	4.459	-.133	1.136	.428	8.983
2.094	1.748	2.386	-6.102	5.141	-.281	1.337	.988	8.828
2.268	1.750	2.233	-6.760	5.959	-.392	1.565	1.527	8.586
2.443	1.719	2.092	-7.572	6.996	-.457	1.812	2.045	8.305
2.617	1.655	1.969	-8.822	8.493	-.473	2.068	2.595	8.102
2.792	1.551	1.876	-11.613	11.414	-.437	2.323	3.495	8.403
2.966	1.391	1.836	-28.412	26.756	-.350	2.564	10.405	15.923
3.141	1.121	1.921	9.656	-6.280	-.217	2.780	-1.435	1.548
3.515	.428	2.526	.668	2.261	-.043	2.961	-.057	2.949
3.490	8.461	-7.783	-1.277	4.627	.160	3.100	.374	3.235
3.664	2.063	-.056	-1.973	5.923	.383	3.192	1.166	3.587
3.839	1.511	.361	-2.189	6.821	.614	3.236	2.150	3.826
4.013	1.194	.493	-2.147	7.507	.841	3.224	3.224	3.890
4.188	.933	.566	-1.936	8.047	1.055	3.185	4.323	3.759
4.362	.693	.634	-1.601	8.466	1.247	3.101	5.397	3.438

Table 2--Continued. Results for the Parabolic path program.

A	XP1	YP1	CX1	CY1	XP2	YP2	CX2	CY2
4.537	.464	.717	-1.174	8.770	1.412	2.987	-3.584	3.031
4.711	.247	.825	-.677	8.958	1.545	2.851	-4.244	3.406
4.886	.046	.964	-.136	9.029	1.646	2.702	-4.877	3.882
5.060	-.132	1.135	.426	8.983	1.714	2.544	-5.489	4.457
5.235	-.281	1.337	.986	8.829	1.748	2.387	-6.100	5.138
5.409	-.391	1.565	1.525	8.587	1.750	2.234	-6.758	5.956
5.584	-.457	1.811	2.044	8.306	1.719	2.092	-7.569	6.992
5.758	-.473	2.068	2.593	8.102	1.655	1.969	-8.816	8.487
5.933	-.437	2.322	3.490	8.399	1.551	1.876	-11.597	11.398
6.107	-.351	2.563	10.291	15.789	1.392	1.836	-28.209	26.575
6.282	-.217	2.779	-1.447	1.534	1.123	1.921	9.738	-6.354

$$(X_Q - a)^2 + (Y_Q - b)^2 = C^2 \tag{48}$$

in which C is the circular path radius.

The angle ψ is measured counterclockwise from the X-axis to the radius of the circle locating point Q, and is expressed by:

$$\tan\psi = \frac{Y_Q - b}{X_Q - a}. \tag{49}$$

The angular displacement of the moving coordinate system is:

$$\phi = f(\psi). \tag{50}$$

The equations for X_Q and Y_Q are written by inspection of Figure 14.

$$X_Q = a + C\cos\psi \tag{51}$$

$$Y_Q = b + C\sin\psi. \tag{52}$$

Their time derivatives are:

$$\dot{X}_Q = -C\dot{\psi}\sin\psi \tag{53}$$

$$\dot{Y}_Q = C\dot{\psi}\cos\psi \tag{54}$$

$$\ddot{X}_Q = -C(\ddot{\psi}\sin\psi + \dot{\psi}^2\cos\psi) \tag{55}$$

$$\ddot{Y}_Q = C(\ddot{\psi}\cos\psi - \dot{\psi}^2\sin\psi). \tag{56}$$

The parametric equations for the fixed curve are:

$$X_P = a + C(1 - \frac{\dot{\psi}}{\dot{\phi}})\cos\psi \tag{57}$$

$$Y_P = b + C(1 - \frac{\dot{\psi}}{\dot{\phi}})\sin\psi, \tag{58}$$

and for the moving curve:

$$x_P = -C(\dot{\psi}/\dot{\phi})\cos(\psi - \phi) \tag{59}$$

$$y_P = -C(\dot{\psi}/\dot{\phi})\sin(\psi - \phi). \tag{60}$$

Equations (20), (21), (25), and (26) are used to

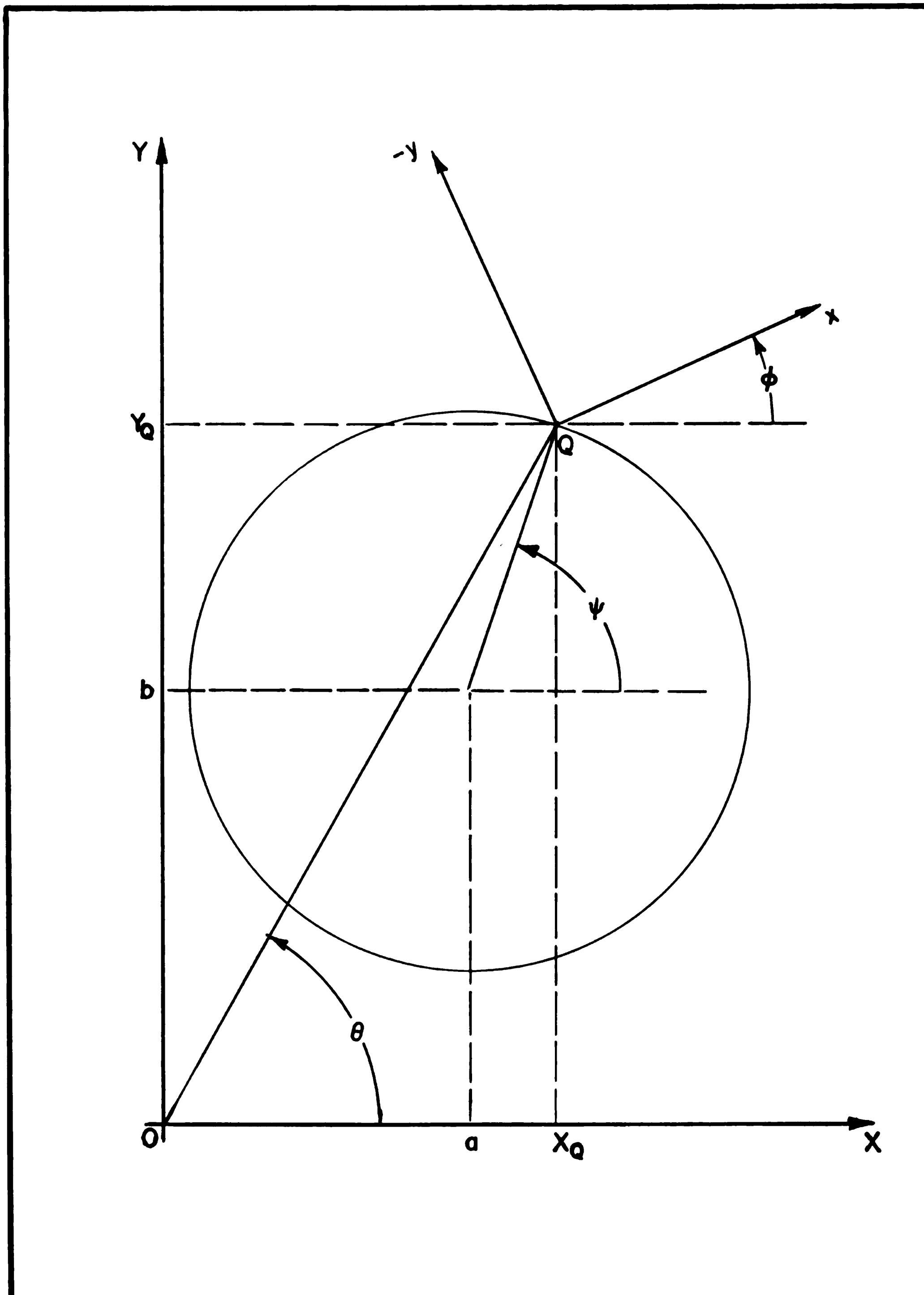

Fig. 14.-- The circular path problem.

locate the joints of the approximating mechanism. Equations (29), (30), and (34) are written as follows:

$$M = C\left[(\ddot{\psi}\dot{\phi} - \dot{\psi}\ddot{\phi})\sin\psi + (\dot{\psi}^2\dot{\phi} + \dot{\psi}\dot{\phi}^2)\cos\psi\right] \tag{61}$$

$$N = -C\left[(\ddot{\psi}\dot{\phi} - \dot{\psi}\ddot{\phi})\cos\psi + (\dot{\psi}^2\dot{\phi} + \dot{\psi}\dot{\phi}^2)\sin\psi\right] \tag{62}$$

$$Z = \frac{(\ddot{\psi}\dot{\phi} - \dot{\psi}\ddot{\phi})\sin(\psi + \alpha - \phi) + (\dot{\psi}^2\dot{\phi} + \dot{\psi}\dot{\phi}^2)\cos(\psi + \alpha - \phi)}{\dot{\psi}\cos(\psi + \alpha - \phi)}. \tag{63}$$

The value of γ is:

$$\gamma = \sqrt{x_p^2 + y_p^2} = C\frac{\dot{\psi}}{\dot{\phi}}. \tag{64}$$

After putting equations (61), (62), (63), and (64) into equation (33) the result, after some simplifying, is:

$$m^2 + \frac{m}{2}\left[\frac{(\dot{\phi}\dot{\psi}^2 + \dot{\psi}\dot{\phi}^2)\sin 2\Theta - (\ddot{\psi}\dot{\phi} - \dot{\psi}\ddot{\phi})(5 + \cos 2\Theta)}{(2\dot{\psi}\dot{\phi}^2 - \dot{\phi}\dot{\psi}^2)\cos\Theta - (\ddot{\psi}\dot{\phi} - \dot{\psi}\ddot{\phi})\sin\Theta}\right]$$
$$- \left[\frac{(\dot{\psi}\dot{\phi}^2 - 2\dot{\psi}^2\dot{\phi})\cos\Theta + (\ddot{\psi}\dot{\phi} - \dot{\psi}\ddot{\phi})\sin\Theta}{(2\dot{\psi}\dot{\phi}^2 - \dot{\phi}\dot{\psi}^2)\cos\Theta - (\ddot{\psi}\dot{\phi} - \dot{\psi}\ddot{\phi})\sin\Theta}\right] = 0 \tag{65}$$

in which $\Theta = (\psi + \alpha - \phi)$.

The four link mechanism which is to approximate this motion will be designed so that the tracing point Q fits at one point and values of ϕ, $\dot{\phi}$, $\ddot{\phi}$, ψ, $\dot{\psi}$, and $\ddot{\psi}$ will be put into equation (65). This will simplify the equation. The coupler bar (attached to the moving plane) hinge joints are located at $(X_{\zeta 1}, Y_{\zeta 1})$, $(X_{\zeta 2}, Y_{\zeta 2})$ and the hinge joints on the fixed plane are located at (CX_1, CY_1) and (CX_2, CY_2). The points and the derivatives of the path curves are:

$$X_\zeta = a + C[\cos\psi + m(\dot{\psi}/\dot{\phi})\sin(\alpha - \phi)] \tag{66}$$

$$Y_\zeta = b + C[\sin\psi + m(\dot{\psi}/\dot{\phi})\cos(\alpha - \phi)] \tag{67}$$

$$Y'_\zeta = -\frac{\dot\psi\cos\psi + m[\dot\psi\sin(\alpha-\phi) + \xi\cos(\alpha-\phi)]}{\dot\psi\sin\psi + m[\dot\psi\cos(\alpha-\phi) - \xi\sin(\alpha-\phi)]} \tag{68}$$

where $$\xi = \frac{\dot\phi\ddot\psi - \dot\psi\ddot\phi}{\dot\phi^2} \tag{69}$$

$$Y''_\zeta = -\frac{\{\ddot\psi + m[P\cos\Theta + \Upsilon\sin\Theta] + \Lambda m^2\}}{C\{\dot\psi\sin\psi + m[\dot\psi\cos(\alpha-\phi) - \xi\sin(\alpha-\phi)]\}^3} \tag{70}$$

where $$P = \frac{(\dddot\psi - \dot\psi\ddot\psi\dot\phi + \dot\psi^2\ddot\psi)}{\dot\phi} - \frac{\ddot\phi\dot\psi^2}{\dot\phi^2} \tag{71}$$

$$\Upsilon = \frac{\dot\phi}{\dot\phi^2}(\dot\psi\dddot\phi - \dot\phi\dddot\psi) + \dot\psi^2\left(\dot\phi + \dot\psi - \frac{2\ddot\phi}{\dot\phi^3}\right) + \ddot\psi\left(\dot\psi\frac{\ddot\phi}{\dot\phi^2} + \frac{\ddot\psi}{\dot\phi}\right) \tag{72}$$

and

$$\Lambda = \frac{\dot\phi\dot\psi(\dot\psi\dddot\phi - 3\ddot\phi\ddot\psi - \dot\phi\dddot\psi) + \dot\psi(\dot\psi\dot\phi^4 + 2\ddot\phi\ddot\psi) + 3(\dot\psi\ddot\phi)^2 + 2(\dot\phi\ddot\psi)^2}{\dot\phi^3}. \tag{73}$$

The centers of curvature are determined by putting equations (66), (67), (68), and (70) into equations (25) and (26).

<u>Circles rolling on circles.</u> The foregoing procedure may be used to synthesize mechanisms which approximate the motion of circles rolling on circles. In this case, the fixed center O is shifted to the center of the circular path. See Figure 15. Here,

$$C = \frac{D + d}{2} \tag{74}$$

$$\psi \equiv \theta.$$

The moving circle motion can begin at $Y_Q = 0$, $X_Q = C$. After an interval, $\overline{OQ}$ rotates an angle ψ and the moving circle rotates through angle λ with respect to the line $\overline{OQ}$. The moving system, attached to the moving circle, turns through the angle $\lambda + \psi$:

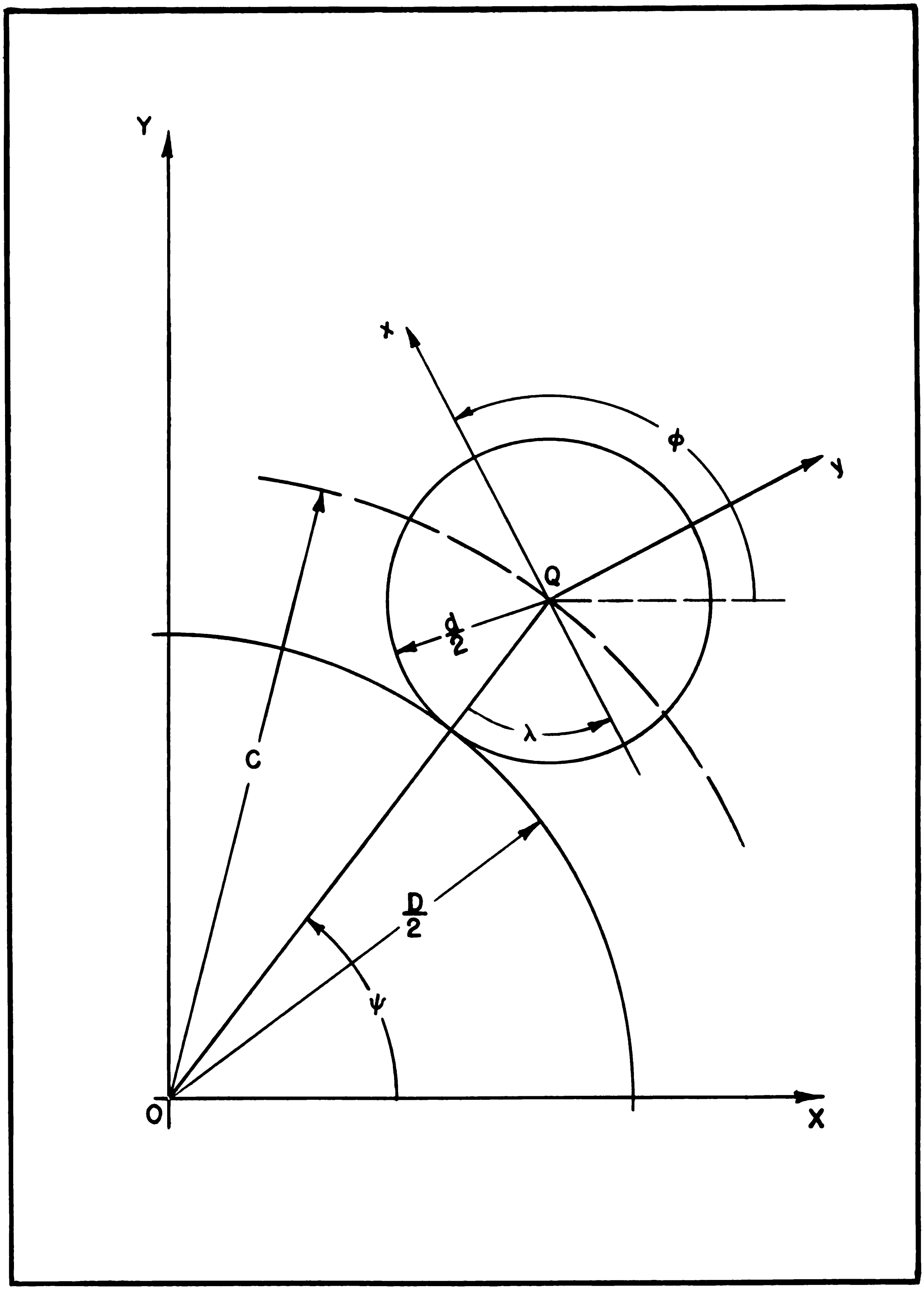

Fig. 15. -- The disc rolling on a disc.

$$\phi = \lambda + \psi. \tag{75}$$

From Figure 14:

$$\lambda d = \psi D \tag{76}$$

and

$$\phi = \psi \frac{D+d}{d}. \tag{77}$$

By substituting $n = \frac{D+d}{d}$, $\phi = n\psi$, $\dot{\phi} = n\dot{\psi}$, and $\ddot{\phi} = n\ddot{\psi}$

into equation (65), the resulting quadratic in m becomes:

$$m^2 + 2m\frac{n+1}{2n-1}\sin\Theta - \frac{n-2}{2n-1} = 0 \tag{78}$$

in which $\Theta = \psi - \phi + \alpha$

$$\gamma = C\frac{\dot{\psi}}{\dot{\phi}} = \frac{D+d}{2n}.$$

The points of stationary curvature are:

$$X_\zeta = \frac{nd}{2}\left[\cos\psi + \frac{m}{n}\sin(\alpha - \phi)\right] \tag{79}$$

$$Y_\zeta = \frac{nd}{2}\left[\sin\psi + \frac{m}{n}\cos(\alpha - \phi)\right]. \tag{80}$$

Expressions (69), (71), (72), and (73) become:

$$\xi = 0$$

$$P = \frac{\ddot{\psi}}{n\dot{\psi}}(1 - n\dot{\psi}^2)$$

$$\Upsilon = \dot{\psi}^2\left[\dot{\psi}(n+1) - 2\frac{\ddot{\psi}}{n\dot{\psi}^3}\right] + \frac{2\ddot{\psi}}{n\dot{\psi}}$$

$$\Lambda = \frac{n^4\dot{\psi}^6 + 2n\dot{\psi}\ddot{\psi}^2 + 2n^2\dot{\psi}^2\ddot{\psi}^2}{n^3\dot{\psi}^3}.$$

The geometry of such a mechanism would not be altered by taking $\dot{\psi} = 1$, $\ddot{\psi} = \dddot{\psi} = 0$, so that

$$P = 0$$

$$\Upsilon = n + 1$$

$$\Lambda = n$$

and the derivatives can be written:

$$Y'_\zeta = -\frac{\cos\psi + m\sin(\alpha - \phi)}{\sin\psi + m\cos(\alpha - \phi)} \tag{81}$$

$$Y''_{\zeta} = -\frac{2}{D+d}\,\frac{1 + m(n+1)\sin\Theta + nm^2}{[\sin\psi + m\cos(\alpha-\phi)]^3}\,. \tag{82}$$

<u>Application.</u> A mechanism is designed by putting values into equations (78), (79), (80), (81), and (82). As an example, for $d = 2$, $n = 4$, $\psi = \pi/2$, $\phi = 2\pi$ and equation (78) becomes:

$$m^2 + \frac{5}{7} m \cos\alpha - \frac{2}{7} = 0$$

for $\alpha = 90°$, $m_1 = 0.535$, $m_2 = -0.535$. The coordinates of the joints are:

$X_{\zeta 1} = 0.535$; $Y_{\zeta 1} = 4.0$; $CX_1 = -0.744$; $CY_1 = 1.608$

$X_{\zeta 2} = -0.535$; $Y_{\zeta 2} = 4.0$; $CX_2 = 0.744$; $CY_2 = 1.608$

The mechanism is constructed in Figure 16.

<u>Reduction of the case of the disc rolling on a disc to that of a disc rolling on a straight line.</u> In the extension of this case to that of the disc rolling on a straight line, the diameter D becomes infinitely large. As

$$D \rightarrow \infty; \quad n \rightarrow \infty; \quad s = \frac{1}{n} \rightarrow 0\ ,$$

Then, replacing n by 1/s, equation (78) becomes:

$$m^2 + \left[\frac{s+1}{2-s}\right] m \sin\Theta - \frac{1-2s}{2-s} = 0\ . \tag{83}$$

By putting s = 0, equation (82) becomes:

$$m^2 + \frac{m}{2}\sin\Theta - \frac{1}{2} = 0\ . \tag{84}$$

Now, by using the trigonometric identity:

$$\sin\Theta = \sin(\psi + \alpha - \phi) = \sin\psi\cos(\alpha-\phi) + \cos\psi\sin(\alpha-\phi)\ ,$$

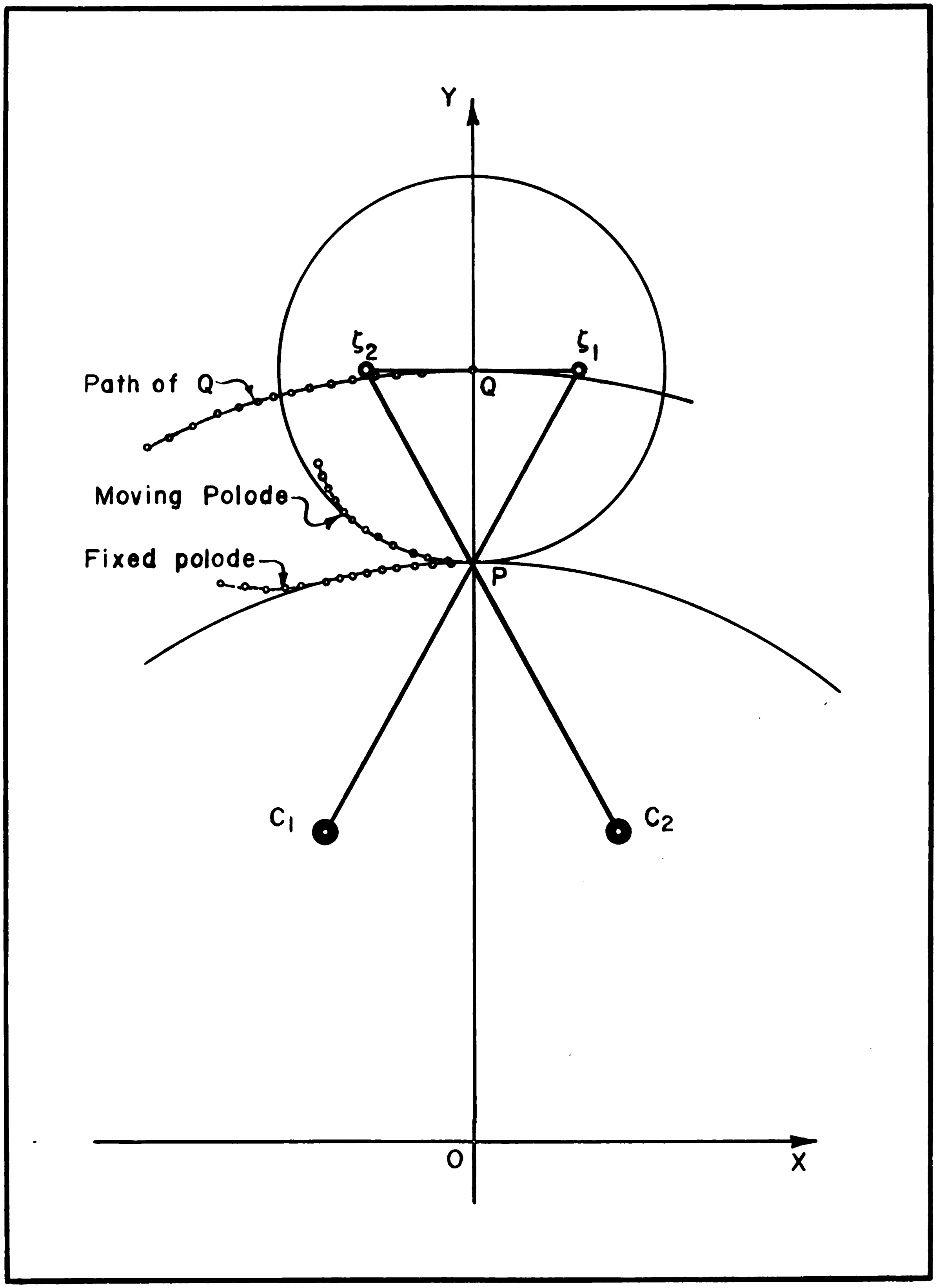

Fig. 16.-- The disc rolling on disc mechanism for the case $\alpha = 0$.

and taking, for a horizontal line below the disc, the value $\psi = \pi/2$,

$$\sin\theta = \cos(\alpha - \phi)$$

and equation (82) becomes:

$$m^2 + \frac{m}{2}\cos(\alpha - \phi) - \frac{1}{2} = 0. \tag{85}$$

Now, if the value for ψ is taken as $-\pi/4$ in the first case considered, in which the disc rolls on a straight line above the disc and inclined upward to the right,

$$\sin\theta = -\frac{\sqrt{2}}{2}\cos(\alpha - \phi) + \frac{\sqrt{2}}{2}\sin(\alpha - \phi)$$

Equation (82) becomes:

$$m^2 + \frac{m}{2\sqrt{2}}\left[\sin(\alpha - \phi) - \cos(\alpha - \phi)\right] - \frac{1}{2} = 0 \tag{86}$$

which, when multiplied by 2 is identical to equation (A) of the first example. Thus, the development of the mechanism to generate a portion of a straight line path is a special case of the more general curved path case.

It is to be noted that the mechanism development begins with a different equation for m for each differently inclined line, and for each line there is a very large number of possible mechanisms.

CHAPTER

III

FUNCTION GENERATION

III

FUNCTION GENERATION

Function Generators

Four link function generators. The relation between the input and output link angular displacements of a mechanism is a geometrical property. A mechanism so designed that the input-output relationship satisfies a particular mathematical function between two quantities is a function generator. Function generators may be used as components in control systems, instruments, or as mechanical analog computing elements.

Mechanisms which match any given function relationship exactly can be constructed from rolling curves.[22,23] Such devices can be difficult to machine and when the input output requirements are not exact or a small range of motion is required, a four bar function generator may suffice. For the purpose of this study, the four bar function generator is a mechanism constructed of bars or links, the lengths of which are such that the crank angles correspond to the variation of some dependent

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[22]H. E. Golber, "Rollcurve Gears," Transactions of the ASME, v. 61, (1939) p. 223.

[23]Beggs, p. 74.

variable and the variation of its independent variable. The devices considered here will be approximate function generators.

Function generators can be synthesized by seeking a mechanism such that the output link will be in certain positions when the input link is in corresponding positions, the positions being obtained from several numerical solutions of the desired functional relationship.[24,25] There are an infinite number of possibilities mathematically, and a very large number of different mechanisms can be obtained from one set of precision points. The mechanism positions between precision points are in error and part of the problem is that of locating the precision points so as to minimize the error.[26]

Mechanism synthesis based on the inflection circle concept is a different method.[27] As pointed out earlier, that method and the method presented here are based on the same fundamental theory, which is one of matching the mechanism performance to the function over a small range.

<u>The application of the roulette method</u>. The direct

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[24]Ferdinand Freudenstein, "Approximate Synthesis of Four-Bar Linkages," Trans. ASME, v. 77, (Aug., 1955), p. 853.

[25]Hinkle, p. 267.

[26]Ferdinand Freudenstein, "Structural Error Analysis in Plane Kinematic Synthesis," Trans. ASME, v.81, ser.B, n.1, (Feb.1959), p.15.

[27]Hall, p. 106.

application of the method of roulettes to function generation requires beginning with a rolling curve pair. The curves will be expressed in parametric form as discussed previously.

The procedure for designing rolling curves is well known.[28] Any pair of rolling curve function generators which can be designed by Golber's method can be used as a basis for four bar function generator design. The procedure for synthesizing a circular path generating mechanism is combined with an inversion of the rolling curve mechanism about one curve, preferably the input curve.

The rolling curve function generator is depicted in Figure 17. Angle θ is the input. Figure 18 shows the inversion of the mechanism about the input link and is the basis of this develcpment. Since the rolling curve mechanism is usually designed with fixed centers, the path of the ground joint of the output link is a circle. The output angle, ψ, when added to the input angle, becomes ϕ, the displacement angle of the moving coordinate system. The case of the circular path generating mechanism with the circle center at the fixed origin applies. The function generating mechanism resulting from this synthesis, however, is quite different from the path generating mechanism, as shall be seen. The moving system displacement angle is:

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[28]Golber

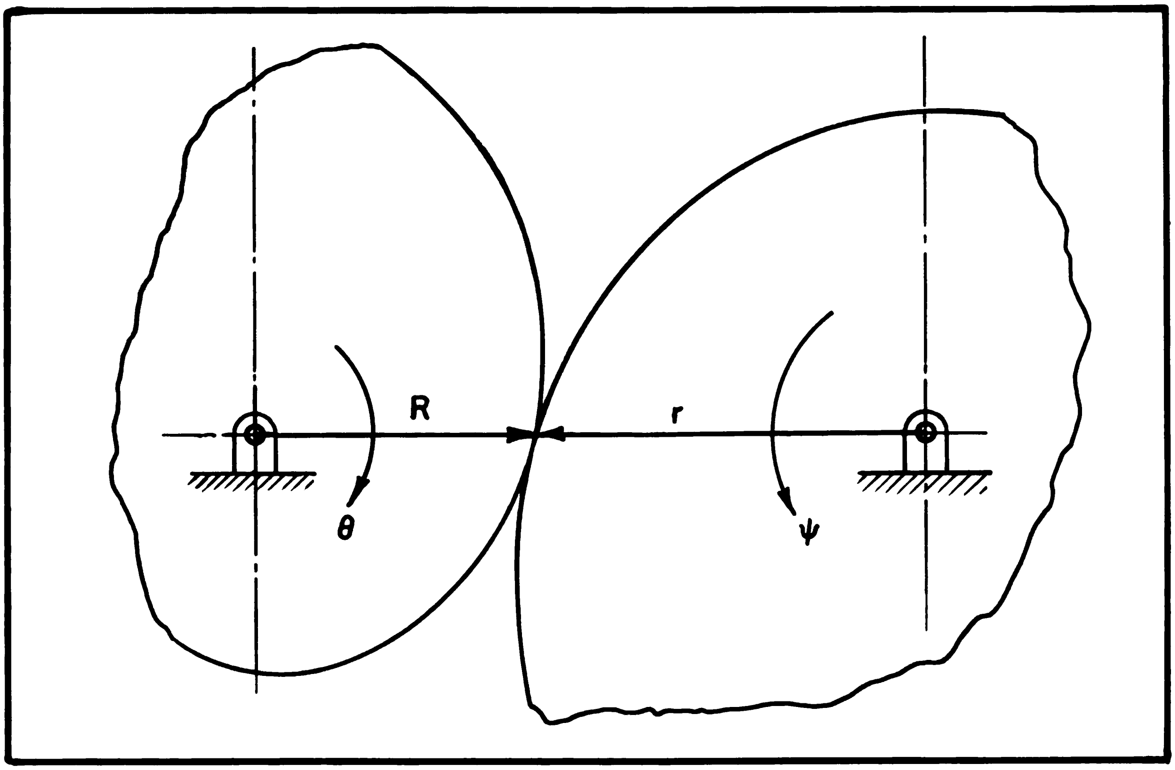

Fig. 17. -- The rolling curve function generator.

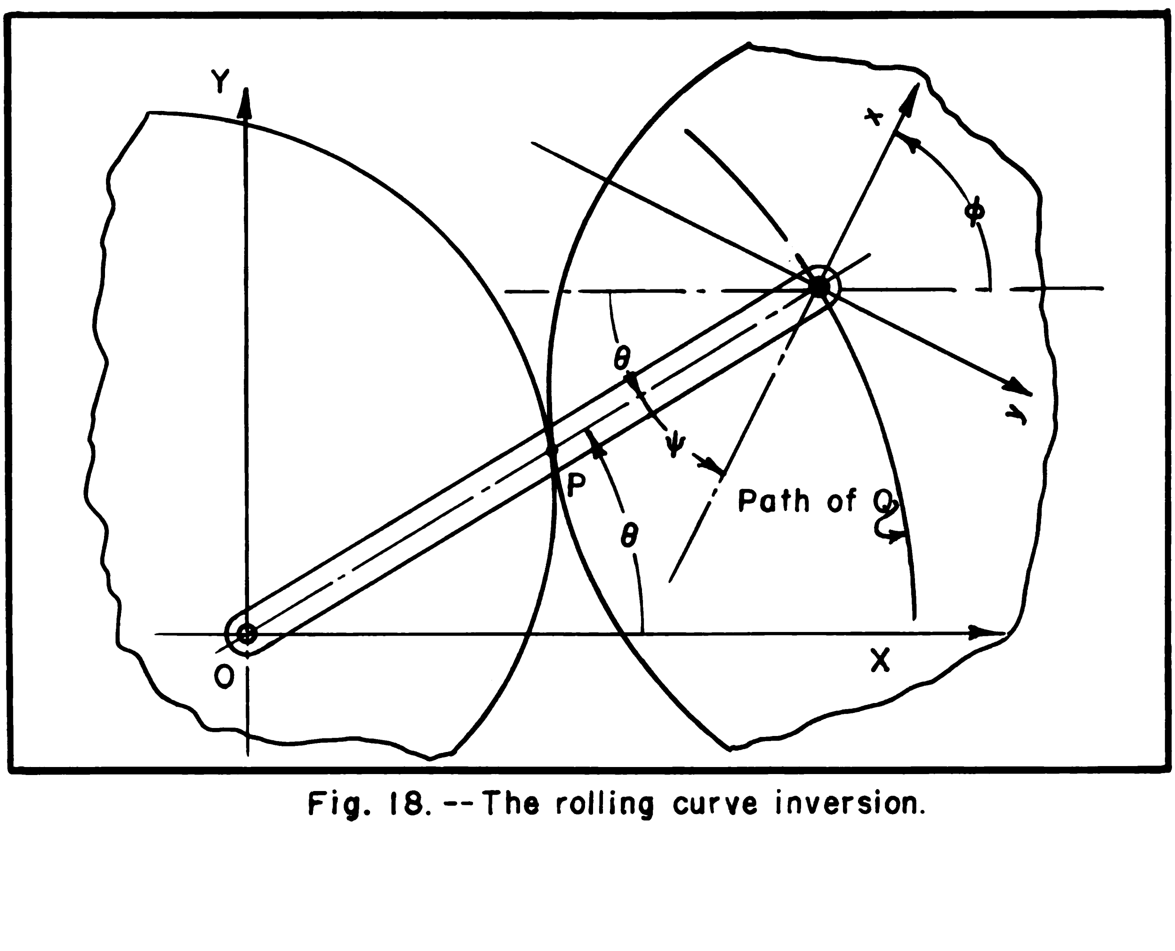

Fig. 18. -- The rolling curve inversion.

$$\phi = \theta + \psi \text{ .} \tag{87}$$

<u>Rolling curve design.</u> It is to be assumed that some function is to be generated by a pair of rolling curves such as

$$\psi = f(\theta) \tag{88}$$

and that the curves have a fixed center distance so that

$$R + r = C \tag{89}$$

R and r are associated with input and output angles, θ and ψ respectively. The condition for pure rolling is satisfied by:

$$R\,d\theta = r\,d\psi. \tag{90}$$

By rearranging equation (90)

$$\frac{R}{r} = \frac{d\psi}{d\theta} \tag{91}$$

so that

$$r = R\,\frac{1}{d\psi/d\theta} \tag{92}$$

and finally:

$$R = \frac{C(d\psi/d\theta)}{(d\psi/d\theta) + 1} \tag{93}$$

and

$$r = C\,\frac{1}{(d\psi/d\theta) + 1} \text{ .} \tag{94}$$

Equations (93) and (94) express R and r as functions of the derivative of the desired function, which is in turn a function of the independent variable, θ.

<u>Rolling curves using instant centers.</u> The rolling curve development by the method of this thesis can be compared with the method of Golber referred to earlier.

The path of Q is now taken to be a circle about the fixed origin and its equation is:

$$X_Q^2 + Y_Q^2 = C^2 \tag{95}$$

Here, $C = \overline{OQ}$.

Since

$$\theta = \tan^{-1} \frac{Y_Q}{X_Q} \tag{96}$$

then

$$X_Q = C \cos \theta \tag{97}$$

$$Y_Q = C \sin \theta \, . \tag{98}$$

The time derivatives are;

$$\dot{X}_Q = -C\dot{\theta} \sin \theta \tag{99}$$

$$\dot{Y}_Q = C\dot{\theta} \cos \theta \, . \tag{100}$$

The parametric equations for the rolling curve pair are obtained by putting equations (97), (98), (99), and (100) into equations (11), (12), (13), and (14).

$$X_P = C[1 - (\dot{\theta}/\dot{\phi})] \cos \theta \tag{101}$$

$$Y_P = C[1 - (\dot{\theta}/\dot{\phi})] \sin \theta \tag{102}$$

$$x_P = -C(\dot{\theta}/\dot{\phi}) \cos(\theta - \phi) \tag{103}$$

$$y_P = C(\dot{\theta}/\dot{\phi}) \sin(\theta - \phi) \, . \tag{104}$$

<u>Comparison of results with Golber's method</u>. To compare this with Golber's method, it is noted that:

$$R = \sqrt{X_P^2 + Y_P^2} \quad \text{and} \quad r = \sqrt{x_P^2 + y_P^2} \, . \tag{105}$$

Then

$$R = C[1 - (\dot{\theta}/\dot{\phi})] \tag{106}$$

$$r = C(\dot{\theta}/\dot{\phi}) \, . \tag{107}$$

Replacing $\dot{\phi}$ with $\dot{\psi} + \dot{\theta}$:

$$R = \frac{C\ \dot{\psi}/\dot{\theta}}{(\dot{\psi}/\dot{\theta}) + 1} \tag{108}$$

and
$$r = \frac{C}{(\dot{\psi}/\dot{\theta}) + 1} \tag{109}$$

since $\dot{\psi}/\dot{\theta} = \frac{d\psi}{d\theta}$, it is seen that equations (108) and (109) are the same as equations (93) and (94).

Synthesis of the Four Link Function Generator.

Basic equations. With a=b=0 in equations (65) and (66) and with ψ replaced by θ, the equations of the point on the roulette are:

$$X_\zeta = C[\cos\theta + m(\dot{\theta}/\dot{\phi})\sin(\alpha - \phi)] \tag{110}$$

$$Y_\zeta = C[\sin\theta + m(\dot{\theta}/\dot{\phi})\cos(\alpha - \phi)]. \tag{111}$$

The location of the points of stationary curvature is made as before, that is, by solving equation (65) for m and putting the values of m into equations (110) and (111). Equation (65) can be simplified by noting that the geometry of the four link mechanism will not be changed if the input link has constant velocity, that is,

$$m^2[\dot{\phi}(2\dot{\phi} - \dot{\theta})\cos\Theta + \ddot{\phi}\sin\Theta] + \frac{m}{2}[\dot{\phi}(\dot{\phi} + \dot{\theta})\sin 2\Theta + \ddot{\phi}(5 + \cos 2\Theta)] -$$

$$[\dot{\phi}(\dot{\phi} - 2\dot{\theta})\cos\Theta + \ddot{\phi}\sin\Theta] = 0. \tag{112}$$

In this equation, $\Theta = \theta - \phi + \alpha$.

The mechanism is constructed to fit the desired function exactly at one point. This point can be selected in the center of the desired operating range with the expectation that good approximation will extend an equal amount on either side of the matching point. Whether or not the mechanism will satisfy the requirement over the range desired cannot be determined since the range of best fit cannot be found at this time.

When the matching point of the mechanism is chosen, values of θ, $\dot{\theta}$, ϕ, $\dot{\phi}$, $\ddot{\phi}$, and $\dddot{\phi}$ will be fixed. This will simplify equation (112). The points of stationary curvature are determined by inserting values of m from equation (112) into equations (110) and (111). It must be remembered that a pair of values of m are obtained from equation (112) by fixing all variables, so that the same variables must be used in any given solution of equations (110) and (111). Equations (25) and (26) are to be used for the centers of curvature, using the following derivatives:

$$Y'_{\zeta} = -\frac{\cos\theta + m\left[\sin(\alpha-\phi) - (\ddot{\phi}/\dot{\phi}^2)\cos(\alpha-\phi)\right]}{\sin\theta + m\left[\cos(\alpha-\phi) + (\ddot{\phi}/\dot{\phi}^2)\sin(\alpha-\phi)\right]} \tag{113}$$

$$Y''_{\zeta} = -\frac{\dot{\theta} + m\left[\left(\frac{\dot{\theta}\dot{\phi} + \dot{\phi}^2 + \dot{\phi}\dddot{\phi} - 2\ddot{\phi}^2}{\dot{\phi}}\right)\sin\Theta - \frac{(\dot{\theta}+\dot{\phi})\ddot{\phi}}{\dot{\phi}^2}\cos\Theta\right] + \frac{m^2}{\dot{\phi}}\left[(\dot{\phi}^2 + \dot{\phi}\dddot{\phi} - 2\ddot{\phi}^2) - \frac{\ddot{\phi}}{\dot{\phi}^2}\right]}{C\dot{\theta}\left\{\sin\theta + m\left[\cos(\alpha-\phi) + \frac{\ddot{\phi}}{\dot{\phi}^2}\sin(\alpha-\phi)\right]\right\}^3} \tag{114}$$

where $\Theta = \theta - \phi + \alpha$.

<u>Function generator linkage construction.</u> The de-

rived mechanism for path generation is illustrated in Figure 19. The point Q is intended to describe a segment of a circle. ζ_1 and ζ_2 are the points of stationary curvature and points C_1 and C_2 are their centers of curvature. This is the inversion of the function generator. It is to be noted that in the origional motion, the X-Y system is to rotate about the point O through θ while the x-y system is to rotate about the fixed point Q. The distance $\overline{OQ}$ then is required to be fixed and to be made so by using a link which becomes the fixed link of the mechanism. Next a link is used to attach points ζ_1 to C_1 and the two planes, X-Y and x-y have constrained motion with respect to each other. These two planes can be reduced to links $\overline{OC_1}$ and $\overline{Q\zeta_1}$ and the result is the four link function generator. A second possibility is mechanism $OC_2\zeta_2Q$. See Figure 20.

Application. An easily followed procedure for numerical synthesis can be devised by putting the solution of equation(112) in a more easily handled form. If the mechanism is a position function generator, i. e., velocity of the input link is not specified, then $\dot{\theta} = 1$ can be inserted into the solution. The result is:

$$m = \frac{1}{4}\left(-D \pm \sqrt{D^2 + 16E}\right) \tag{115}$$

where

$$D = \frac{\dot{\phi}(\dot{\phi}+1)\sin 2\theta + \ddot{\phi}(5+\cos 2\theta)}{\dot{\phi}(2\dot{\phi}-1)\cos\theta + \ddot{\phi}\sin\theta} \tag{116}$$

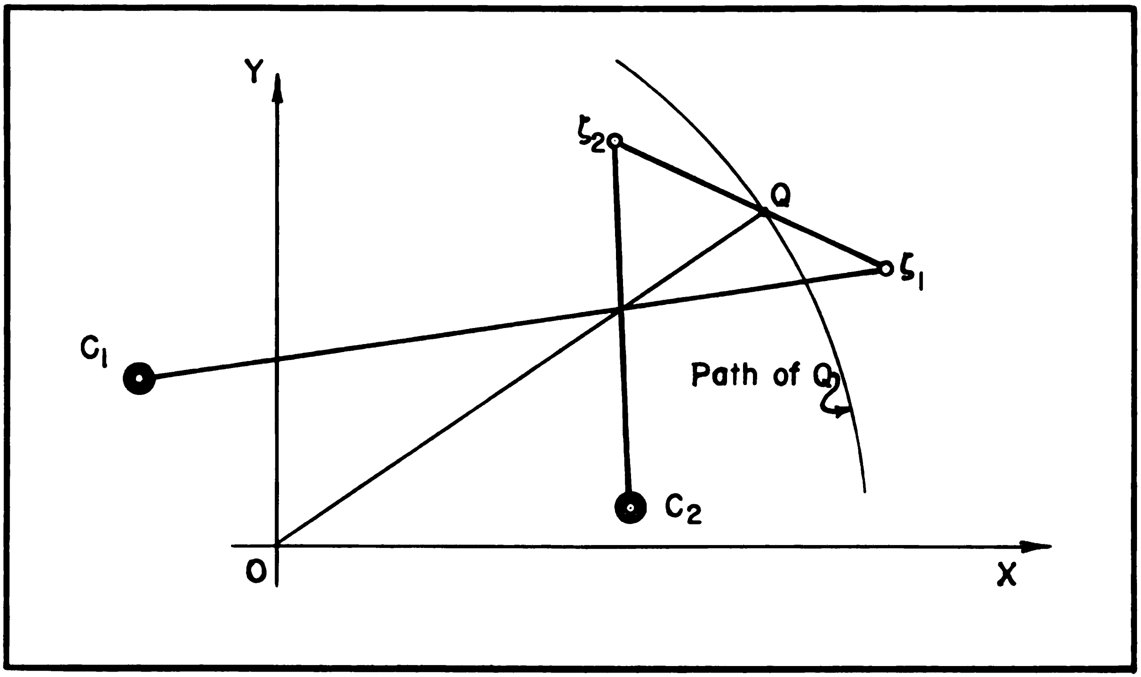

Fig. 19.--The derived path tracing mechanism.

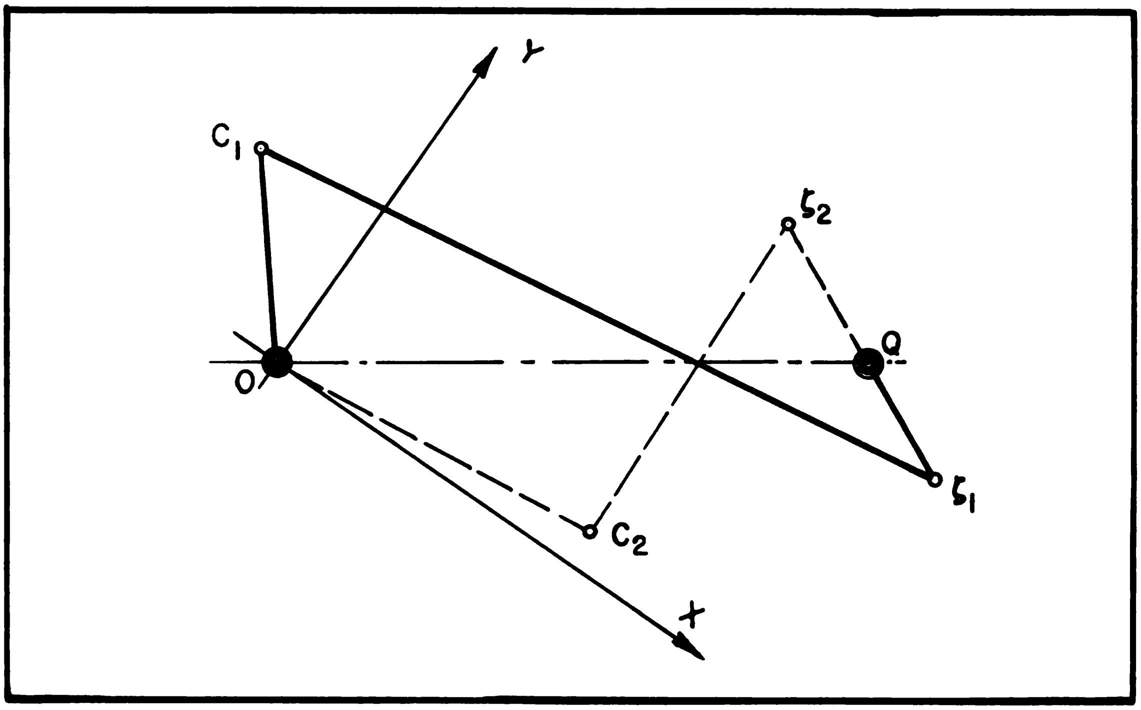

Fig. 20.-- The inversion for constructing the function generator mechanism.

and

$$E = \frac{\dot{\phi}(\dot{\phi}+2)\cos\theta - \ddot{\phi}\sin\theta}{\dot{\phi}(2\dot{\phi}-1)\cos\theta + \ddot{\phi}\sin\theta}\,. \tag{117}$$

Example. An arbitrary function is chosen to illustrate the procedure.

$$\psi = 3\theta^{1.2}$$

This is put into the form to be used in this synthesis method by replacing ψ by $\phi - \theta$: (This accomplishes the inversion)

$$\phi = 3\theta^{1.2} + \theta$$

with time derivatives:

$$\dot{\phi} = (3.6\,\theta^{.2} + 1)\,\dot{\theta}$$

$$\ddot{\phi} = (0.72\,\theta^{-.8})\,\dot{\theta}^2$$

$$\dddot{\phi} = (-0.576\,\theta^{-1.8})\,\dot{\theta}^3$$

For $\theta = 1$, $\dot{\theta} = 1$, $\phi = 4$, $\dot{\phi} = 4.6$, $\ddot{\phi} = 0.72$, $\dddot{\phi} = -0.576$. ($\alpha = 0$)

$$D = -0.319, \quad E = 0.314$$

$$m = \frac{1}{4}\left[0.319 \pm \sqrt{(0.319)^2 + 16(0.314)}\,\right]$$

so that $m_1 = 0.642$, $m_2 = -0.488$. The following values are obtained:

$X_{\zeta 1} = 0.645$, $Y_{\zeta 1} = 0.736$, $CX_1 = 0.065$, $CY_1 = 0.505$

$X_{\zeta 2} = 0.460$, $Y_{\zeta 2} = 0.907$, $CX_2 = 0.358$, $CY_2 = 0.173$.

The constant C is a scale factor and can be taken as unity. The derived mechanism is shown in Figure 21. The function generators resulting from the inversion are shown in Figures 22 and 24. The performance curves, graphically determined, are shown in Figures 23 and 25. The locus plot from a computer solution is shown in Figure 26.

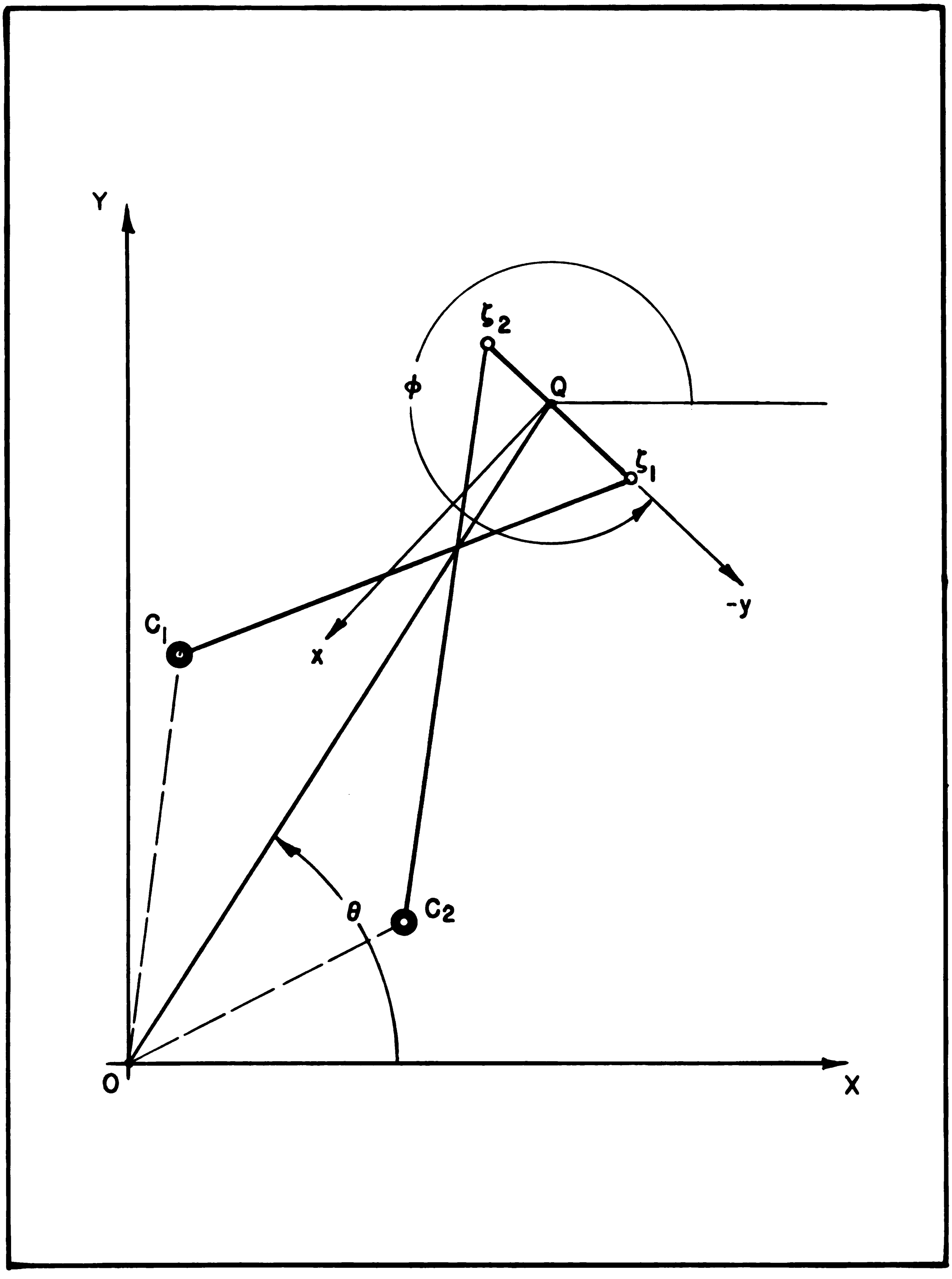

Fig. 21.-- The derived path generator mechanism for the function $\psi = 3\theta^{1.2}$

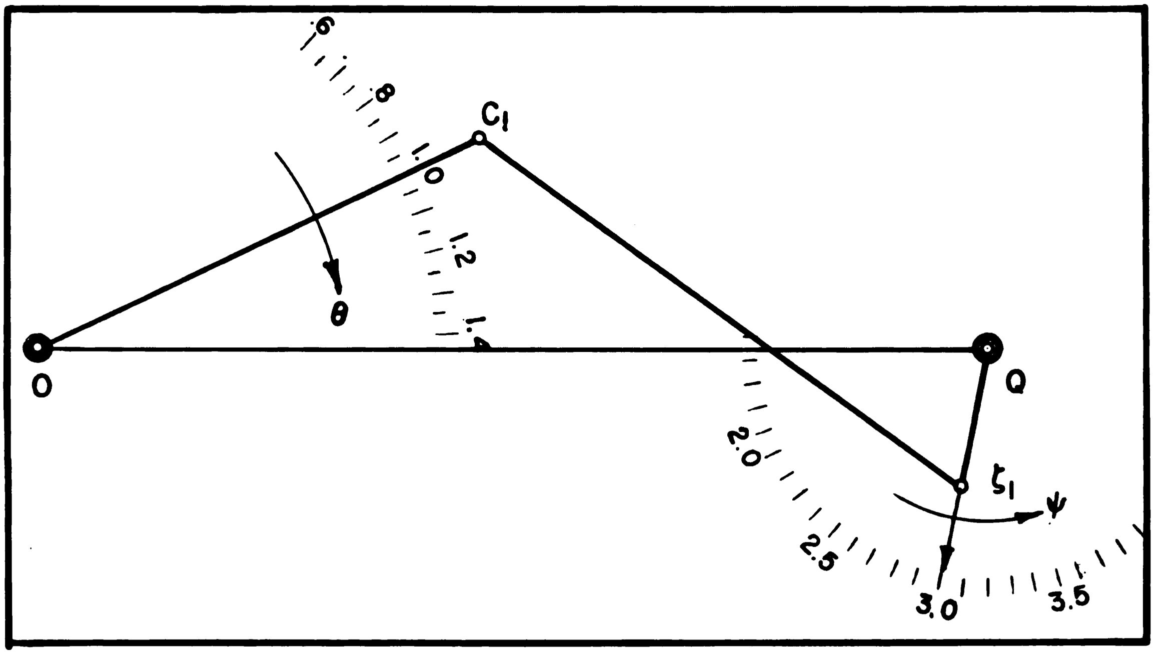

Fig. 22.-- The function generator mechanism derived for $\psi = 3\,\theta^{1.2}$ and obtained from m_1 at $a = 0$.

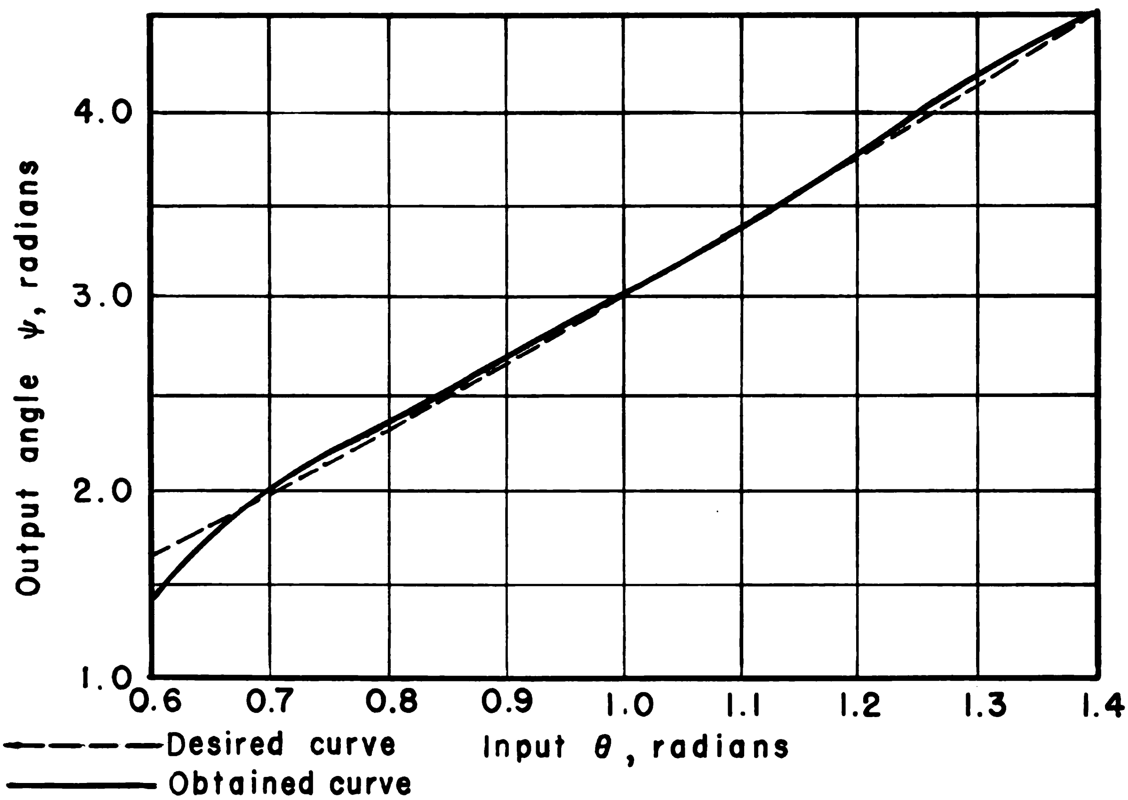

Fig. 23.--The performance curve for the mechanism of Fig.22.

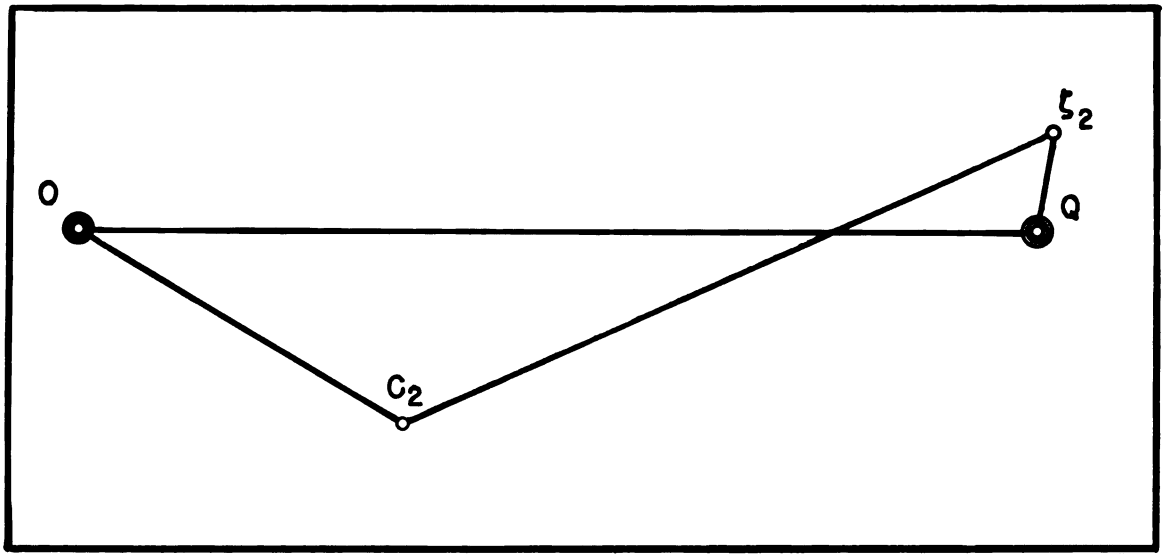

Fig. 24.-- The function generator linkage for $\psi = 3\theta^{1.2}$ obtained from m_2 at $\alpha = 0$.

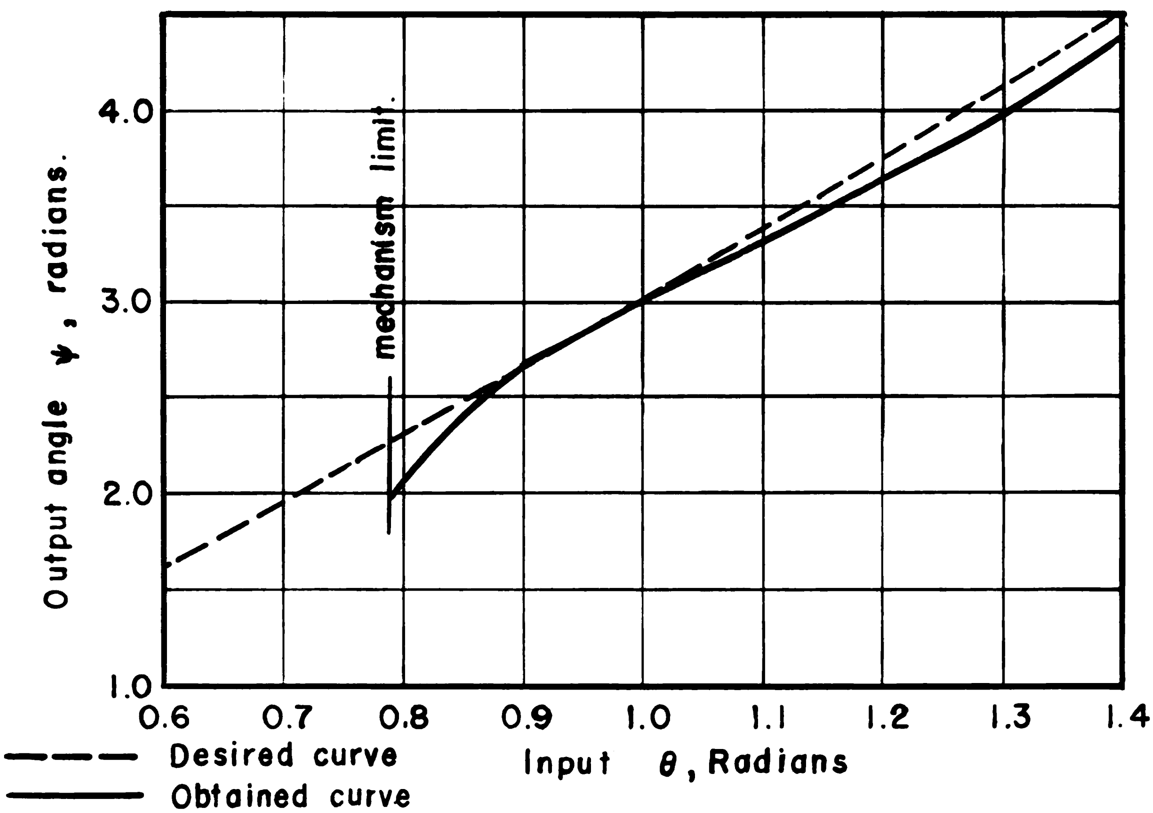

Fig. 25.-- The performance curve for the mechanism of Fig. 24.

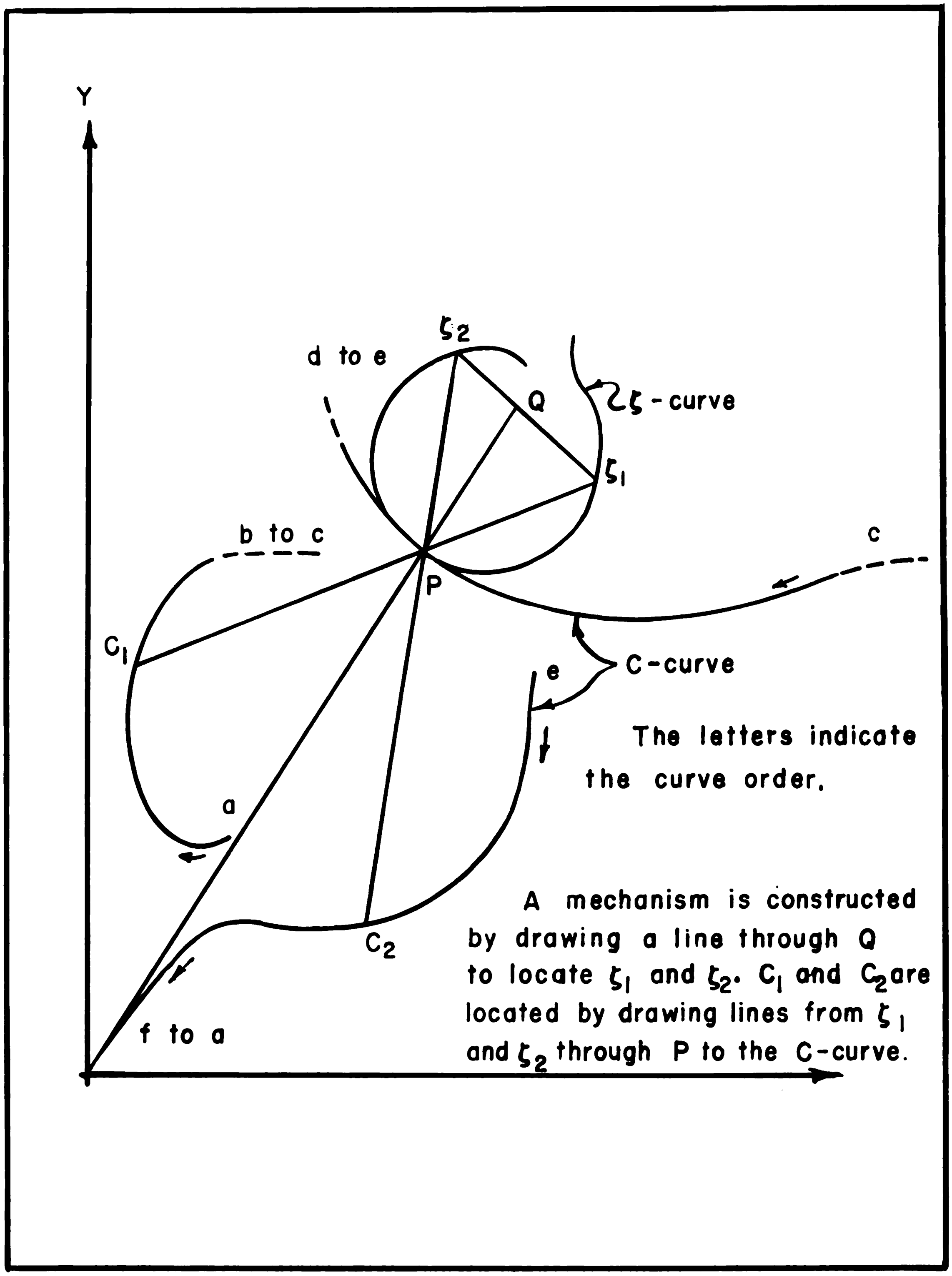

Fig. 26.-- The locii of the points of stationary curvature and their centers of curvature for the equation $\psi = 3\,\theta^{1.2}$

Table 3. Results for the function generator program.

A	XP1	YP1	CX1	CY1	XP2	YP2	CX2	CY2
.174	.637	.722	.071	.548	.480	.915	.322	.183
.349	.622	.695	.098	.594	.498	.916	.288	.181
.523	.599	.670	.137	.635	.514	.914	.258	.182
.698	.571	.650	.955	.632	.529	.911	.230	.185
.872	.537	.637	.830	.586	.541	.905	.204	.188
1.047	.500	.633	.696	.573	.551	.897	.178	.190
1.221	.463	.639	.565	.593	.556	.883	.143	.178
1.396	.428	.654	.461	.635	.522	.810	-.490	-.721
1.570	.399	.678	.274	.787	.615	.928	.171	.290
1.745	.377	.708	.563	.502	.609	.898	.138	.282
1.919	.364	.743	.559	.463	.615	.883	.117	.291
2.094	.360	.778	.555	.407	.622	.870	.100	.306
2.268	.365	.813	.540	.351	.629	.855	.084	.326
2.443	.377	.843	.514	.300	.636	.840	.070	.351
2.617	.395	.869	.479	.258	.643	.821	.059	.381
2.792	.416	.888	.440	.226	.647	.800	.052	.416
2.966	.438	.902	.399	.204	.648	.776	.050	.456
3.141	.459	.910	.359	.190	.645	.750	.056	.501
3.315	.480	.915	.322	.183	.637	.722	.071	.548
3.490	.498	.916	.288	.181	.622	.695	.098	.594
3.664	.514	.914	.258	.182	.599	.670	.137	.635
3.839	.528	.911	.230	.185	.571	.650	.955	.632
4.013	.541	.905	.204	.188	.537	.637	.831	.587
4.188	.551	.897	.178	.190	.500	.633	.697	.573
4.362	.556	.883	.144	.178	.463	.639	.565	.592

Table 3--Continued. Results for the function generator program.

A	XP1	YP1	CX1	CY1	XP2	YP2	CX2	CY2
4.537	.522	.811	-.450	-.661	.429	.654	.461	.635
4.711	.615	.928	.171	.290	.399	.678	.278	.783
4.886	.609	.898	.138	.282	.377	.708	.563	.502
5.060	.615	.883	.117	.291	.364	.743	.559	.463
5.235	.622	.870	.100	.306	.360	.778	.555	.407
5.409	.629	.856	.084	.326	.365	.813	.540	.351
5.584	.636	.840	.070	.351	.377	.843	.514	.300
5.758	.643	.821	.059	.381	.395	.869	.479	.258
5.933	.647	.800	.052	.416	.416	.888	.440	.226
6.107	.648	.776	.050	.456	.438	.902	.399	.204
6.282	.645	.750	.056	.501	.459	.910	.359	.190

The mechanism is expected to represent the function to be generated over only a small range of values. It will be necessary to decide on the range of values of the function the mechanism is to generate. In the example just given the scale divisions of the function and the angular displacement of the cranks in radians corresponded. In another case it might be desirable to expand or to compress the function scale. In the absence of any specific information on the mechanism, a limitation of something less than 180° rotation of the output link can be imposed when the range of values of the function to be generated is greater than that of its independent variable. This particular mechanism will probably be a crank and rocker mechanism which is driven by the rocker.

To expand the range of the function to be generated, in this case, a revision of the equation relating the input crank angle to the output crank angle is necessary. By rewriting the function, i. e., renaming the variables;

$$y = 3x^{1.2}$$

The relationship between the input angle range and the independent variable range is:

$$\Delta\theta = K\Delta x$$

For the two variables to have equal values, K equals unity. For $\Delta\theta = 1$, and $\Delta x = 2$, $K = 1/2$

then $\quad x = 2\theta, \quad y = 2\psi$

and the function to be generated can be determined by substitution into the function statement:

$$(2\psi) = 3(2\theta)^{1.2}$$

or

$$\psi = 3.445\,\theta^{1.2}.$$

After replacing ψ by $\phi - \theta$ and solving for ϕ:

$$\phi = 3.445\,\theta^{1.2} + \theta$$

$$\dot{\phi} = (4.14\,\theta^{.2} + 1)\dot{\theta}$$

$$\ddot{\phi} = (0.828\,\theta^{-.8})\dot{\theta}^2$$

$$\dddot{\phi} = (-0.662\,\theta^{-1.8})\dot{\theta}^3$$

for $x = 1$, $\theta = 0.5$, $\dot{\theta} = 1$, $\phi = 2.0$, $\dot{\phi} = 4.6$, $\ddot{\phi} = 1.443$, $\dddot{\phi} = -1.757$. Choosing $\alpha = 0$, the calculated values are: $D = -0.875$, $E = 0.40$, $m_1 = 0.887$, $m_2 = -0.451$. Then:

$X_{\zeta 1} = 0.365$, $Y_{\zeta 2} = 0.762$, $CX_1 = 0.916$, $CY_1 = -0.440$

$X_{\zeta 2} = 0.629$, $Y_{\zeta 2} = 0.883$, $CX_2 = 0.240$, $CY_2 = 0.467$.

The stationary curvature points and their centers are plotted in Figure 27. The derived mechanism for path generation is shown. The function generating mechanism, constructed using points O, C_1, ζ_1, and Q and inversion about link OC_1 is shown in Figure 28. Note, however, that the input range has only been extended a small amount and that the degree of fit is not as good as that of the first mechanism constructed. (Figure 24) However, there is probably still a better mechanism and it can be obtained by taking another value of α to be put into the equations. A locus plot is an interesting study to make.

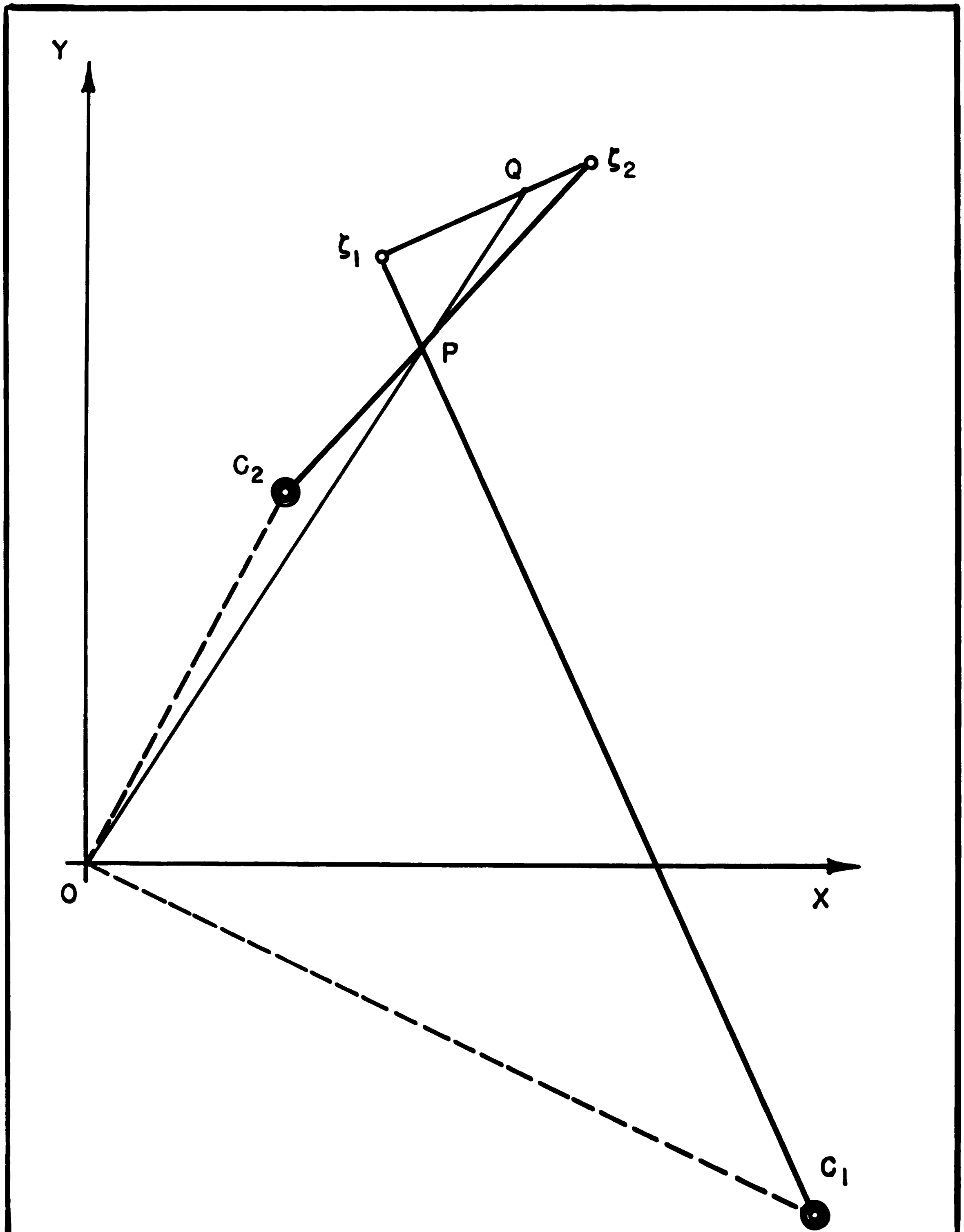

Fig. 27-- Equivalent path mechanism for $\psi = 3.445\theta^{1.2}$ at $\alpha = 0$

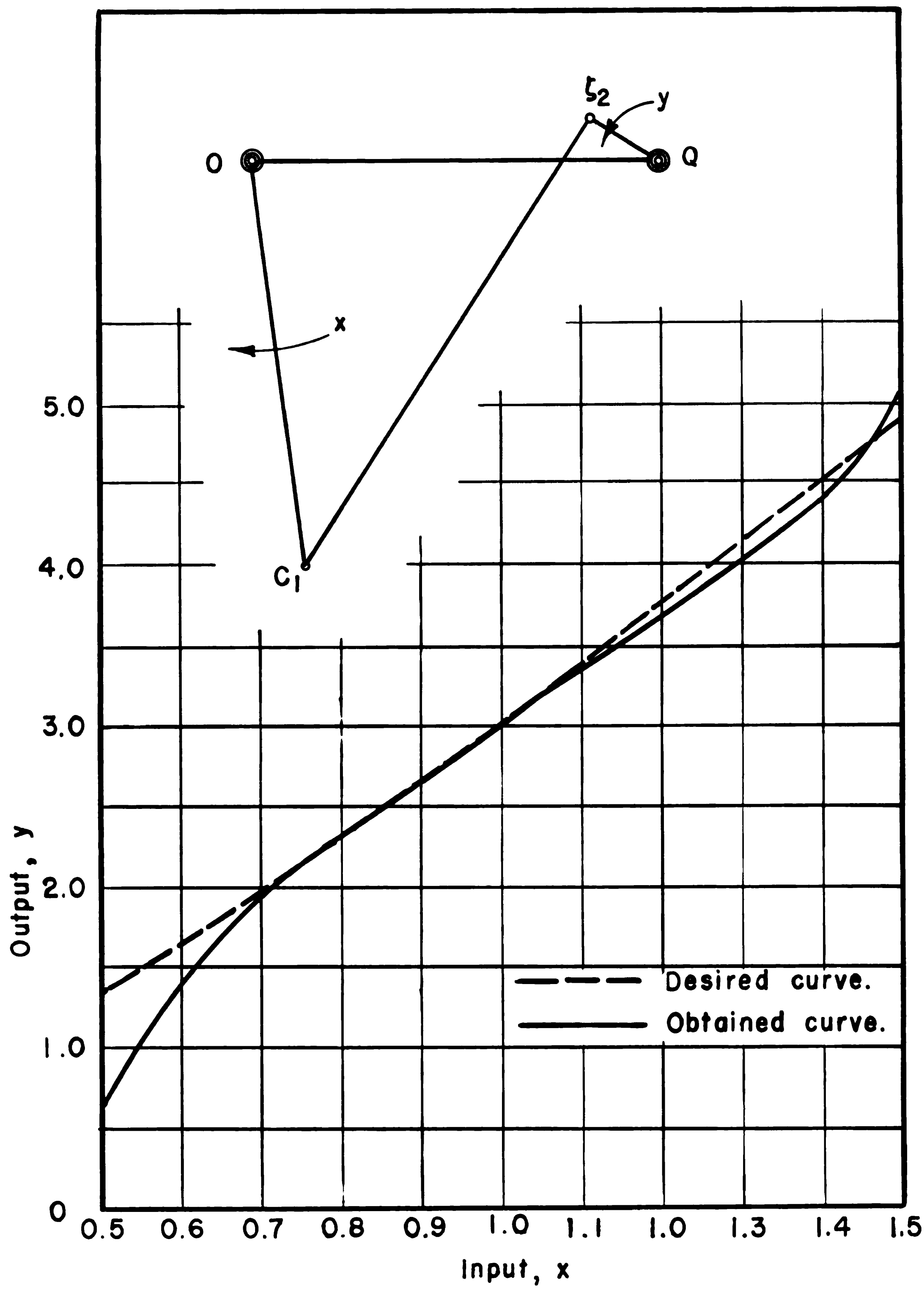

Fig. 28.--The mechanism and the performance curve based on the solution of Fig. 27.

<u>The Analytical Determination of the Output Crank Angle for the Four Bar Function Generator.</u> With the availability of rapid computational equipment, the determination of the output angle of the derived four bar linkage by numerical rather than graphical methods is desirable. In addition to the speed there is the availability of greater precision afforded by numerical methods.

Figure 23 shows the pair of derived mechanisms which are obtained from a solution using one value of α. An expression relating ψ and θ is to be obtained. The angles of the inverted mechanisms which correspond to the input angle θ of the function generator are η and ν.

The links are: (The subscripts are omitted.)

$$a = OC = \sqrt{(CX)^2 + (CY)^2} \tag{118}$$

$$b = C\zeta = \sqrt{(X - CX)^2 - (Y - CY)^2} \tag{119}$$

$$c = Q\zeta = m\gamma \tag{120}$$

$$d = OQ$$

d is a scale factor. Unity is a convenient value.

Two relations for the output angle ψ result. They are based on the following:

$$\Gamma > \theta \qquad \mu > \pi \qquad \text{Case I}$$

$$\Gamma < \theta \qquad \mu < \pi \qquad \text{Case II}$$

also:

$$\Gamma = \tan^{-1} \frac{CY}{CX} \tag{121}$$

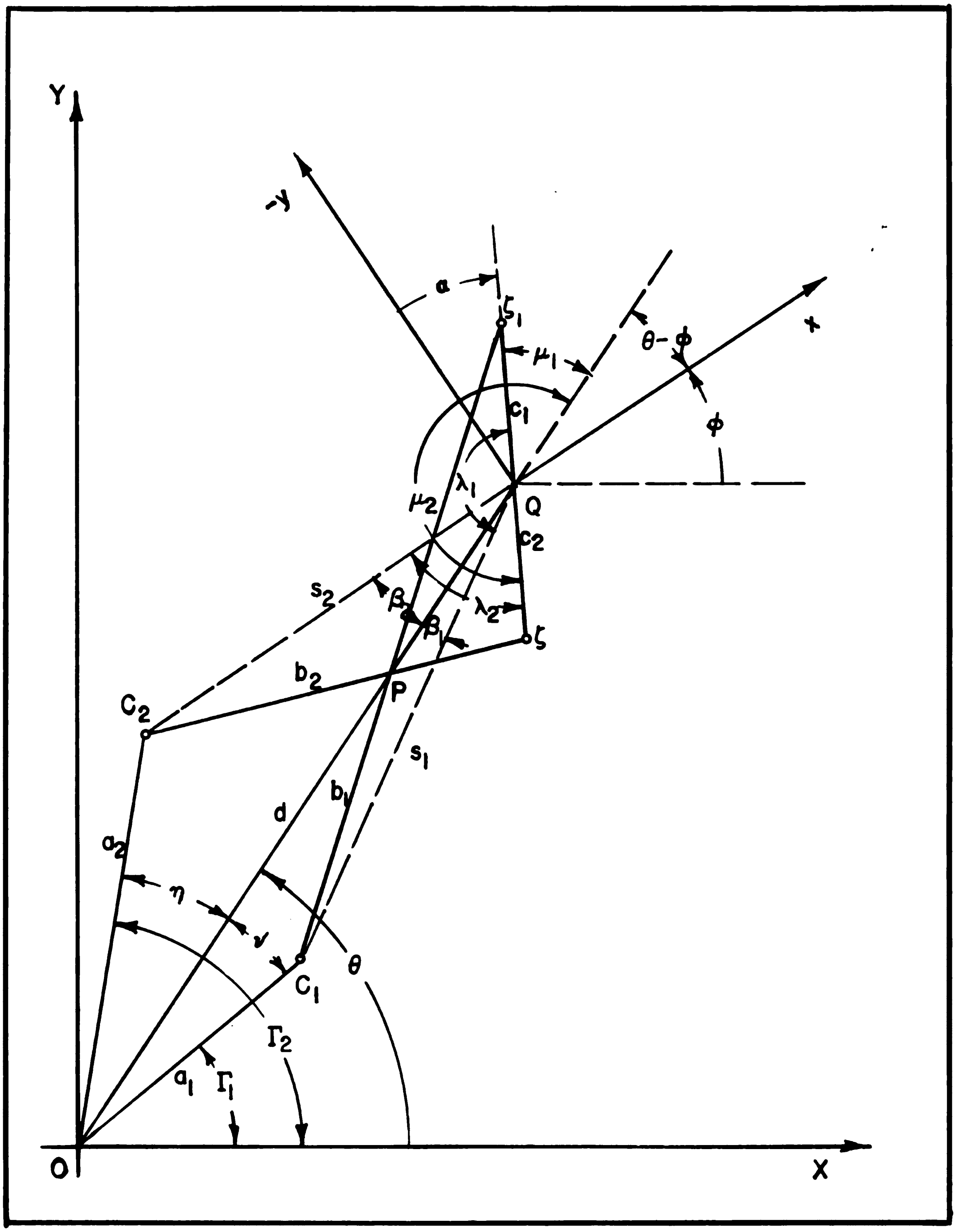

Fig. 29.-- The figure used in deriving the analytical method of determining the output angle.

Case I. For case I, $\eta = \Gamma - \theta$. (122)

The line s = $\overline{CQ}$ is drawn and by the use of the law of cosines:

$$s^2 = a^2 + d^2 - 2ad\cos\eta \tag{123}$$

The two angles β and λ are defined from:

$$\beta = \sin^{-1}\left(\frac{a}{s}\sin\eta\right) \tag{124}$$

$$\lambda = \cos^{-1}\left(\frac{s^2 + c^2 - b^2}{2sc}\right). \tag{125}$$

The two expressions obtained from Figure 29 are:

$$\mu = \frac{3\pi}{2} - (\theta - \phi + a)$$

$$\mu = \pi + (\lambda - \beta)$$

from which:

$$\phi = \theta + a - \frac{\pi}{2} + (\lambda - \beta). \tag{126}$$

By substituting $\phi = \theta + \psi$,

$$\psi = a - \frac{\pi}{2} + (\lambda - \beta). \tag{127}$$

Case II. For case II,

$$\eta = \theta - \Gamma \tag{128}$$

and the expressions for μ are:

$$\mu = \frac{3}{2} + (\theta - \phi + a)$$

$$\mu = \pi + (\lambda - \beta)$$

from which

$$\phi = \theta + a + \frac{\pi}{2} - (\lambda - \beta)$$

and finally:

$$\psi = a + \frac{\pi}{2} - (\lambda - \beta).$$

The computation is carried out in tabular form in Table 4. The output scale is to be regarded as movable, and the position is to be determined by the matching point. In this case, the match is made at $\theta = 1$ and $\psi = 3$. For this reason, the equations for ϕ and ψ are not numbered. In computing the output values it is only necessary to determine $(\lambda - \beta)$ and the difference between the required angle at $\theta = 1$ $(\psi = 3)$, subtract $(\lambda - \beta)$ from this angle and add the difference to $(\lambda - \beta)$ for the other values.

Table 4. Computed output for the function generator of Figure 24.

$\alpha = 0$, $a = 0.509$, $b = 0.627$, $c = 0.142$, $d = 1.000$, $\Gamma = \tan^{-1}\frac{0.505}{0.065} = 86.7°$											
θ (Rad.)	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
θ (deg.)	34.4	40.1	45.8	51.6	57.3	63.1	68.8	74.5	80.2	86.0	91.7
$\eta = (\Gamma - \theta)$	52.3	46.6	40.9	35.1	29.4	23.6	17.9	12.2	6.5	0.7	-5.0
$\cos\eta$	.612	.686	.745	.819	.870	.916	.952	.977	.995	.999	.995
$2ad\cos\eta$	.624	.700	.759	.834	.887	.934	.970	.996	1.015	1.018	1.015
s^2	.635	.559	.500	.424	.372	.325	.289	.263	.244	.241	.244
s	.797	.748	.707	.652	.610	.570	.538	.513	.494	.491	.494
a/s	.637	.680	.718	.781	.833	.908	.945	.990	1.028	1.035	1.028
$\sin\eta$	.791	.727	.555	.576	.491	.400	.308	.211	.113	.012	-.087
$a/s\sin\eta$	.504	.494	.470	.449	.409	.363	.291	.209	.115	.012	-.089
β	30.3	29.6	28.0	26.7	24.2	21.1	16.9	12.1	6.6	.71	-5.14
$s^2 + c^2 - b^2$	.264	.188	.129	.053	.001	-.046	-.082	-.108	-.127	-.130	-.127

Table 4. (Continued) Computed output for the function generator of Figure 24.

(Rad.)	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
2sc	.226	.212	.200	.185	.173	.162	.153	.146	.140	.139	.140
cos λ	1.148	.88	.64	.286	.057	-.28	-.53	-.743	-.902	-.933	-.902
$\lambda - \beta$	--	28.3	50.2	53.4	86.6	91.6	122	138	154	159	154
$\lambda - \beta$	--	-1.3	22.2	46.7	62.4	70.5	105.1	126	147	158	159
ψ (deg.)	--	108	132	157	172	180.	214	235	257	267	268
ψ (rdn.)	--	1.89	2.295	2.740	3.00	3.24	3.73	4.10	4.47	4.66	4.69
ψ req'd		1.95	2.295	2.645	3.00	3.37	3.73	4.11	4.49	4.87	5.27

IV

CLOSURE

CLOSURE

On the preceding pages there has been no attempt to derive any existing equations. The origional intent was the development of a method for optimizing a mechanism synthesized by the inflection circle concept. It was first determined that some expressions of the procedure in an analytical form were necessary. Examination of an out of date calculus book uncovered the theory of roulettes and its use in developing the inflection circle and in deriving the Euler-Savary equation. It seemed to be the necessary theory on which to base an analytical synthesis method. This development was the result.

The basis of this work is the plane motion of a plane: every mechanism synthesis here begins with instantaneous plane motion. The plane motion is described in terms of a pair of rolling curves as an intermediate step in the location of a pair of points which are moving in paths having instantaneously stationary curvature. If the radius of curvature of the path is constant over a considerable displacement of the plane, the coupler bar plane of a linkage having joints located at these points will match the origional plane motion for the same range. Also, if the curvature is not changing rapidly there will be a good match. A rapidly changing radius of curvature means a poorly matching mechanism. In the event of a poor match there are two possibilities: the first and easiest is to select a different position angle, ϕ. The second possibility is to select a different plane position. In the second possibility it will be necessary to rewrite all the equations. In either case it is possible to solve for enough points and centers using a digital computer so that the locii of all points can be plotted. These locus curves can then be used to select the mechanism after a few trials.

A direct method of selecting the best mechanism has not yet been devised. It should be possible, however, to write a computer program that will determine error over some set range for a finite number of the possible mechanisms so that the one with minimum error can be selected.

The path generating method of synthesis involves a selection of the relationship between the moving plane angular displacement and the displacement of the tracing point on the coupler. The best relationship must be obtained by trial since, due to the nature of the problem of path tracing with a point, none will be given in the initial statement. It would seem that this is an area for further work. Obviously, a different rolling curve pair will result from each different angular displacement-path displacement relationship. Once the selection is made, two other selections are necessary: first, the tracing point position so that equations of position may be written and second, the choice of the position angle α. Once the tracing point position is fixed and the equations written, the angle α can be chosen either at random or a systematic exploration can be made using different values of α. As in the case of the function generator, a digital computer program can be written and used to obtain values for plotting the locus curves. These curves are very helpful in selecting the mechanism.

The duplication of rolling curve plane motion as a means of substituting four bar linkages for rolling curves has been considered only briefly. The case included here, that of a disc rolling on a disc, was the most successful of all syntheses attempted in length of match. As pointed out earlier, it has an application where there is an incentive to replace a pair of gears with a four bar linkage. The example shows the application where one gear is fixed. It is also possible to use the linkage to replace a pair of fixed center gears. Such use is a simplification of the function generator problem with constant angular velocity ratio. Other rolling curves in machines may be approximated with this technique.

APPENDIX

COMPUTER PROGRAMS AND RESULTS

The locus curves for the two path generating mechanisms and the function generator were obtained using an IBM 1620 digital computer having a FORTRAN input. Since this is a widely used computer, the programs are included here. However, they are specialized programs and must be altered for other mechanisms. They are arranged to give the X and Y components of the points of stationary curvature for incremental values of α of 10°. Finer increments can be obtained by rewriting the increment statement, A = A + 0.1745 (A is α) to read the desired value. It is not necessary to index α around for 360°, because m_1 for $\alpha = 90°$ becomes m_2 for $\alpha = 90° + 180°$. Thus, if $\Delta\alpha$ (or ΔA) is taken to be one half of 0.1745 then the looping statement, DO 22I = 1,36 will supply enough points.

Synthesis of other mechanisms can be obtained by rewriting the program statements as necessary to suit the new requirements. The form is suitable for specific mechanisms. Specifically, the generation of a different function would require that certain statements of table 7 be changed. These are: Nos. 11 and 12 for a different matching point; 24 through 36 for a different function.

Statements 26 through 36 can be made general by replacing numerical values with alphabetical terms which are then defined earlier in the program.

Table 5

FORTRAN PROGRAM FOR STRAIGHT LINE MECHANISMS

```
C     1 STATIONARY CURVATURE POINTS AND CENTERS=STRAIGHT
        LINE MECHANISM
      2 A=0
      3 DO 22I=1,36
      4 A=A+0.1745
      5 TC=COSF(A)
      6 TS=SINF(A)
      7 TS2=SINF(2.0*A)
      8 QP1=0.25*(TC-TS+SQRTF(17.0-TS2))
      9 QP2=0.25*(TC-TS-SQRTF(17.0-TS2))
     10 YP1=QP1*TC
     11 YP2=QP2*TC
     12 XP1=QP1*TS
     13 XP2=QP2*TS
     14 Y1P1=(1.0+XP1)/(1.0-YP1)
     15 Y1P2=(1.0+XP2)/(1.0-YP2)
     16 Y11P1=(XP1-YP1-QP1**2)/(1.0-YP1**3)
     17 Y11P2=(XP2-YP2-QP2**2)/(1.0-YP2**3)
     18 CX1=XP1-(Y1P1*(1.0+Y1P1**2))/Y11P1
     19 CX2=XP2-(Y1P2*(1.0+Y1P2**2))/Y11P2
     20 CY1=YP1+(1.0+Y1P1**2)/Y11P1
     21 CY2=YP2+(1.0+Y1P2**2)/Y11P2
     22 PUNCH 23,A,YP1,YP2,XP1,XP2,CX1,CX2,CY1,CY2
     23 FORMAT(F7.4,8F9.3)
        END
```

Symbols

A = α, XP1 = $X_{\zeta 1}$, Y1P1 = $Y'_{\zeta 1}$,

Y11P1 = $Y''_{\zeta 1}$, CX1 = CX_1, CY2 = CY_2.

Table 6

FORTRAN PROGRAM FOR THE PARABOLIC PATH MECHANISM

```
C      PATH GENERATING MECHANISM-PARABOLA
     1 A=0.0
     2 DO 24I=1,36
     3 A=A+0.1745
     4 TC=COSF(A-1.0)
     5 TS=SINF(A-1.0)
     6 TC2=COSF(2.0*(A-1.0))
     7 TS2=SINF(2.0*(A-1.0))
     8 D=(5.0+TS2-3.0*TC2)/(4.0*TC+5.0*TS)
     9 E=(7.0*TC+5.0*TS)/(4.0*TC-5.0*TS)
    10 GM1=0.1765*(-D+SQRTF(D**2.0+16.0*E))
    11 GM2=0.1765*(-D-SQRTF(D**2.0+16.0*E))
    12 XP1=1.0+GM1*1.414*TS
    13 XP2=1.0+GM2*1.414*TS
    14 YP1=2.0+GM1*1.414*TC
    15 YP2=2.0+GM2*1.414*TC
    16 Y1P1=(1.0+GM1*1.414*TS)/(1.0-GM1*1.414*TC)
    17 Y1P2=(1.0+GM2*1.414*TS)/(1.0-GM2*1.414*TC)
    18 Y11P1=(-0.50+GM1*1.414*(TS-0.50*TC+1.414*GM1))/
       (1.0-GM1*1.414*TC)**3.0
    19 Y11P2=(-0.50+GM2*1.414*(TS-0.50*TC+1.414*GM2))/
       (1.0-GM2*1.414*TC)**3.0
    20 CX1=XP1-Y1P1*(1.0+Y1P1**2.0)/(Y11P1)
    21 CX2=XP2-Y1P2*(1.0+Y1P2**2.0)/(Y11P2)
    22 CY1=YP1 + (1.0+Y1P1**2.0)/(Y11P1)
    23 CY2=YP2+(1.0+Y1P2**2.0)/(Y11P2)
    24 PUNCH 25,A,XP1,YP1,CX1,CY1,CX2,CY2
    25 FORMAT(3F7.3,2F10.3,2F7.3,2F10.3)
       END
```

Added symbols: GM1 = m_1, GM2 = m_2

Table 7

FORTRAN PROGRAM FOR A FUNCTION GENERATOR - $\psi = 3\theta^{1.2}$

```
C     FUNCTION GENERATING MECHANISM - CASE ONE
    1 A=0.0
    2 B=1.0
    3 FI=4.0
    4 DO 31I=1,36
    5 A=A+0.1745
    6 TH=B-FI+A
    7 TC=COSF(B)
    8 TS=SINF(B)
    9 TCA=COSF(A-FI)
   10 TSA=SINF(A-FI)
   11 TCB=COSF(TH)
   12 TSB=SINF(TH)
   13 TCB2=COSF(2.0*TH)
   14 TSB2=SINF(2.0*TH)
   15 D=(3.6+25.75*TSB2+0.72*TCB2)/(37.75*TCB+0.72*TSB)
   16 E=(11.95*TCB-0.72*TSB)/(37.75*TCB+0.72*TSB)
   17 GM1=0.25*(-D+SQRTF(D**2.0+16.0*E))
   18 GM2=0.25*(-D-SQRTF(D**2.0+16.0*E))
   19 DY1=-(TC+GM1*(TSA-0.034*TCA))/(TS+GM1*(TCA+0.034*TSA))
   20 DY2=-(TC+GM2*(TSA-0.034*TCA))/(TS+GM2*(TCA+0.034*TSA))
   21 DDY1=-(1.0+GM1*(4.79*TSB-0.1905*TCB+3.79*GM1))/
      (TS+GM1*(TCA-0.034*TSA))**3.0
   22 DDY2=-(1.0+GM2*(4.79*TSB-0.1905*TCB+3.79*GM2))/
      (TS+GM2*(TCA-0.034*TSA))**3.0
   23 XP1=TC+GM1*0.2175*TSA
   24 XP2=TC+GM2*0.2175*TSA
   25 YP1=TS+GM1*0.2175*TCA
   26 YP2=TS+GM2*0.2175*TCA
   27 CX1=XP1-DY1*(1.0+DY1**2.0)/DDY1
   28 CX2=XP2-DY2*(1.0+DY2**2.0)/DDY2
   29 CY1=YP1+(1.0+DY1**2.0)/DDY1
   30 CY2=YP2+(1.0+DY2**2.0)/DDY2
   31 PUNCH 32,A,XP1,YP1,CX1,CY1,XP2,YP2,CX2,CY2
   32 FORMAT(3F7.3,2F10.3,2F7.3,2F10.3)
      END
```

Symbols: A = α, B = θ, FI = ϕ, DY1, DY2 = $Y'_{\zeta 1}$, $Y'_{\zeta 2}$

DDY1, DDY2 = $Y''_{\zeta 1}$, $Y''_{\zeta 2}$.

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