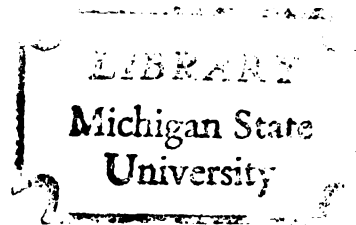


LOCAL JORDAN ALGEBRAS

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This is to certify that the

thesis entitled

"Local Jordan Algebras"

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ABSTRACT

LOCAL JORDAN ALGEBRAS

By

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The purpose of this thesis is to study local Jordan algebras and their completions. A local Jordan algebra J is a quadratic Jordan algebra over a commutative associative ring Φ with identity such that $\mathcal{R}(J)$ is the unique maximal ideal of J , $J/\mathcal{R}(J)$ satisfies the minimum condition, and $\bigcap_{k=1}^{\infty} \mathcal{R}(J)^{(k)} = 0$, where $\mathcal{R}(J)$ is the Jacobson radical of J and $\mathcal{R}(J)^{(1)} \equiv \mathcal{R}(J)$, $\mathcal{R}(J)^{(k+1)} \equiv \mathcal{U}_{\mathcal{R}(J)^{(k)}}(\mathcal{R}(J)^{(k)})$, $k \in \mathbb{N}$. The main results of this thesis are:

Theorem. If J is a local Jordan algebra, then the completion of J is a local Jordan algebra.

Theorem. If J is a complete local Jordan algebra over a field of characteristic not 2, then either

- (1) J is a complete, completely primary local Jordan algebra,
- (2) J is the vector space direct sum of two completely primary local Jordan algebras and a subspace of J , or
- (3) J is isomorphic to a Jordan matrix algebra $\mathfrak{S}(\mathfrak{A}_n, j_a)$ of order $n \geq 3$, where (i) $(\mathfrak{A}, j)$ is an alternative algebra with involution and identity such that $\mathfrak{S}(\mathfrak{A}, j) \subseteq N(\mathfrak{A})$, the Smiley radical $\mathcal{K}(\mathfrak{A})$ is the unique maximal ideal of $(\mathfrak{A}, j)$, and $\mathfrak{A}$ is complete in a topology induced by the topology

on J , or $(\mathcal{A}, j)$ is an associative algebra with involution such that $\mathcal{R}(\mathcal{A})$, the Jacobson radical of $\mathcal{A}$, is the unique maximal ideal of $(\mathcal{A}, j)$ and $\mathcal{A}$ is a complete semi-local algebra.

If J is either a complete, completely primary Jordan algebra, or J is one of the algebras in (3), then J is a complete local Jordan algebra.

LOCAL JORDAN ALGEBRAS

By

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INTRODUCTION

The importance of the theory of (commutative, associative) local rings in algebraic geometry and the theory of commutative rings is well-known (see [8], [12]). Cohen [2] provided the first structure theory for complete Noetherian local rings in his now classic paper. In 1957, Batho [1] in developing a theory of noncommutative, associative, semi-local rings proved that any noncommutative complete local ring is isomorphic to a full matrix ring over a complete, completely primary, local ring.

If A is any associative ring with identity, then A is an algebra over the ring $\mathbb{Z}$ of integers in the sense of §1.1. Thus A^+ is a quadratic Jordan algebra, and when A is a local ring, much of the structure of A can be seen to carry over to A^+ . Hence it seems reasonable to ask the following question: is it possible under a suitable definition to develop a structure theory for complete local Jordan algebras? In this thesis, an affirmative answer is given for a Jordan algebra over a field of characteristic not 2. This is accomplished through the concept of a completion of a local Jordan algebra.

Chapters 1 and 2 contain the basic concepts with which we will be dealing throughout the thesis. In Chapter 1, algebras with involution are considered with emphasis on the simple ones. In Chapter 2, both quadratic and linear Jordan algebras are defined as well as the

Jacobson radical, Jordan matrix algebras, and the concept of isotopy. A sketch of the structure theory of certain simple Jordan algebras over a field of characteristic not 2 is included. Many of the proofs which are omitted in the first two chapters can be found in either [5] or [6].

Chapter 3 deals with the completion of a local (quadratic) Jordan algebra, beginning with an outline of completions of modules. For a more extensive treatment of the material on modules, the reader is referred to Zariski and Samuel [12]. The main result of this chapter is the following: the completion of any local Jordan algebra is a local Jordan algebra.

In Chapter 4, the main structure theory is developed through the use of infinite series and an idempotent lifting property. Completions of local Jordan algebras are classified according to the capacity of their residue class algebras modulo the (Jacobson) radical. When this capacity is greater than or equal to 3, a completion is shown to be a Jordan matrix algebra over a certain type of alternative algebra or over a semi-local associative algebra with involution.

CHAPTER I

ALTERNATIVE AND ASSOCIATIVE ALGEBRAS

1.1 Algebras over Rings

Let Φ be an arbitrary commutative associative ring with identity element 1, and let $\mathfrak{U}$ and $\mathfrak{B}$ be unital left Φ -modules. A mapping $f : \mathfrak{U} \times \mathfrak{U} \rightarrow \mathfrak{B}$ is bilinear, if $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$, $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$, and $f(\alpha x, y) = \alpha f(x, y) = f(x, \alpha y)$, for all $x_1, x_2, x, y_1, y_2, y \in \mathfrak{U}$, and $\alpha \in \Phi$. For the special case $\mathfrak{B} = \Phi$, f is called a bilinear form on $\mathfrak{U}$. $\mathfrak{U}$ is a (unital) algebra over Φ , if $\mathfrak{U}$ is equipped with a bilinear multiplication $(x, y) \rightarrow xy$, $x, y \in \mathfrak{U}$ and $\mathfrak{U}$ has an identity. The mapping $\mathfrak{U} \times \mathfrak{U} \times \mathfrak{U} \rightarrow \mathfrak{U}$ defined by $[x, y, z] = (xy)z - x(yz)$, $x, y, z \in \mathfrak{U}$ is called the associator, and it is linear in each component. $\mathfrak{U}$ is alternative, if $[x, x, y] = 0 = [y, x, x]$, for every $x, y \in \mathfrak{U}$, and $\mathfrak{U}$ is associative, if $[x, y, z] = 0$, for every $x, y, z \in \mathfrak{U}$. For an arbitrary algebra $\mathfrak{U}$ we define the nucleus of $\mathfrak{U}$ to be the set $N(\mathfrak{U}) = \{n \in \mathfrak{U} : [n, x, y] = [x, n, y] = [x, y, n] = 0, \text{ for all } x, y \in \mathfrak{U}\}$. The center of $\mathfrak{U}$ is the set $C(\mathfrak{U}) = \{c \in N(\mathfrak{U}) : cx = xc, \text{ for all } x \in \mathfrak{U}\}$. $N(\mathfrak{U})$ is an associative algebra and $C(\mathfrak{U})$ a commutative associative algebra.

Let $\mathfrak{U}$ be an algebra over Φ . A subalgebra K is a submodule of $\mathfrak{U}$ such that for all $x, y \in K$, $xy \in K$. An ideal of $\mathfrak{U}$ is a submodule K such that xa and $ax \in K$ for all $a \in \mathfrak{U}$,

$x \in K$. Right and left ideals are defined in a similar fashion. The concepts of residue class ring, homomorphism, kernel, etc., are taken to be the standard ones (see [9]). $(\mathfrak{U}, j)$ is an algebra with involution, if $\mathfrak{U}$ is an algebra and j is an anti-automorphism of $\mathfrak{U}$ such that $j^2 = \text{id}$. A subalgebra (ideal) K of $(\mathfrak{U}, j)$ is a subalgebra (ideal) of $\mathfrak{U}$ such that $K^j = K$. If $(\mathfrak{B}, k)$ is also an algebra with involution, then a homomorphism from $(\mathfrak{U}, j)$ into $(\mathfrak{B}, k)$ is a homomorphism $\theta : \mathfrak{U} \rightarrow \mathfrak{B}$ such that $\theta j = k\theta$. If $a \in \mathfrak{U}$ satisfies $a^j = a$, then a is called a symmetric element of $\mathfrak{U}$ and the set of all symmetric elements of $(\mathfrak{U}, j)$ is denoted by $\mathfrak{S}(\mathfrak{U}, j)$. If no confusion can arise, this set will simply be denoted by $\mathfrak{S}(\mathfrak{U})$. Now in general an algebra $\mathfrak{U}$ is simple, if $\mathfrak{U}$ has no ideals other than 0 and $\mathfrak{U}$, and $\mathfrak{U}^2 \neq 0$. An algebra with involution $(\mathfrak{U}, j)$ is a simple algebra with involution, if $\mathfrak{U}$ contains no j -invariant ideals other than 0, and $\mathfrak{U}$, and $\mathfrak{U}^2 \neq 0$.

If $\mathfrak{U}$ and $\mathfrak{B}$ are (unital) algebras over $\mathbb{F}$, the direct sum of $\mathfrak{U}$ and $\mathfrak{B}$ is the set $\mathfrak{U} \times \mathfrak{B}$ together with componentwise addition, scalar multiplication, and multiplication. The direct sum is denoted by $\mathfrak{U} \oplus \mathfrak{B}$, and $\mathfrak{U}$ and $\mathfrak{B}$ are isomorphic to the ideals $\{(a, 0) \in \mathfrak{U} \oplus \mathfrak{B} : a \in \mathfrak{U}\}$ and $\{(0, b) \in \mathfrak{U} \oplus \mathfrak{B} : b \in \mathfrak{B}\}$, respectively. Hereafter we shall identify $\mathfrak{U}$ and $\mathfrak{B}$ with these ideals. K is an ideal of $\mathfrak{U} \oplus \mathfrak{B}$ if and only if $K = K_1 \oplus K_2$, where K_1 is an ideal of $\mathfrak{U}$ and K_2 is an ideal of $\mathfrak{B}$. It is clear that if $\mathfrak{U}$ and $\mathfrak{B}$ are associative, then so is $\mathfrak{U} \oplus \mathfrak{B}$.

Let $\mathfrak{U}$ be an associative algebra and let $\mathfrak{B}_1, \mathfrak{B}_2$ be ideals of $\mathfrak{U}$. It is well-known that, if $\mathfrak{B}_1 \cap \mathfrak{B}_2 = 0$, and $\mathfrak{U} = \mathfrak{B}_1 + \mathfrak{B}_2 = \{b_1 + b_2 : b_1 \in \mathfrak{B}_1, b_2 \in \mathfrak{B}_2\}$, then $\mathfrak{U} \cong \mathfrak{B}_1 \oplus \mathfrak{B}_2$. In this case we

suppress the ordered pair notation and write $b_1 + b_2$ for (b_1, b_2) . Also we will write $\mathfrak{U} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$.

Let $\mathfrak{U}$ be a (unital) associative algebra and define a new product on $\mathfrak{U} \times \mathfrak{U}$ by $x^\circ y = yx$, for all $x, y \in \mathfrak{U}$. The opposite algebra of $\mathfrak{U}$ denoted by $\mathfrak{U}^\circ$ is the algebra determined by using the $\mathfrak{U}$ -module structure of $\mathfrak{U}$ together with the product $\circ$. $\mathfrak{U}^\circ$ is an associative algebra which is anti-isomorphic to $\mathfrak{U}$. Let $\mathfrak{B} = \mathfrak{U} \oplus \mathfrak{U}^\circ$. Hence, if $(x_1, y_1), (x_2, y_2) \in \mathfrak{B}$, then $(x_1, y_1)(x_2, y_2) = (x_1 x_2, y_1^\circ y_2) = (x_1 x_2, y_2 y_1)$, and $\mathfrak{B}$ is an associative algebra. If 1 is the identity of $\mathfrak{U}$, then 1 is the identity of $\mathfrak{U}^\circ$ and $(1, 1)$ is the identity for $\mathfrak{B}$. Now define the mapping $j : \mathfrak{B} \rightarrow \mathfrak{B}$ by $(x, y)^j = (y, x)$, for all $(x, y) \in \mathfrak{B}$. Then j is an involution called the exchange involution and so $(\mathfrak{B}, j)$ is an algebra with involution. It is clear that $\mathfrak{Q}(\mathfrak{B}) = \{(a, a) : a \in \mathfrak{U}\}$. If K is an ideal of $\mathfrak{B}$, then $K = K_1 \oplus K_2$, where K_1 is an ideal of $\mathfrak{U}$ and K_2 is an ideal of $\mathfrak{U}^\circ$. Suppose $(o, y) \in K_2$. Then $(o, y) \in K$ and $K^j = K$ implies $(y, o) = (o, y)^j \in K$. Thus $(y, o) \in K_1$ and $(o, y) = (y, o)^j$, so $(o, y) \in K_1^j$. Conversely, if $(w, z) \in K_1^j$, then $(w, z) = (x, o)^j$, for some $(x, o) \in K_1$. Hence $(w, z) = (o, x)$ and $(w, z) \in K$, so $(w, z) \in K_2$. Thus $K_1^j \subseteq K_2$ and hence equality holds; i.e., $K = K_1 \oplus K_1^j$, K_1 an ideal of $\mathfrak{U}$. On the other hand, if K_1 is an ideal of $\mathfrak{U}$, then $K_1 \oplus K_1^j$ is an ideal of $(\mathfrak{B}, j)$. Therefore there is a bijection from the set of ideals of $(\mathfrak{B}, j)$ onto the set of ideals of $\mathfrak{U}$.

Finally, note that, if $(\mathfrak{U}, j)$ is an algebra with involution such that $\mathfrak{U} = \mathfrak{B} \oplus \mathfrak{B}^j$, $\mathfrak{B}$ an ideal of $\mathfrak{U}$, then $\mathfrak{B}^j \cong \mathfrak{B}^\circ$ under j . This follows since $(b_1^\circ b_2)^j = (b_2 b_1)^j = b_1^j b_2^j$.

1.2 Simple Artinian Algebras with Involution

Let $\mathfrak{U}$ be an associative algebra over a field Φ . $\mathfrak{U}$ satisfies the minimum condition for right ideals, if $\mathfrak{U}$ has no infinite descending chain of right ideals. An algebra $\mathfrak{U}$ which satisfies the minimum condition for right ideals is called (right) Artinian. By the Wedderburn-Artin theorem, an Artinian associative algebra over a field Φ is simple if and only if $\mathfrak{U} \cong \Delta_n$, an algebra of $n \times n$ matrices with entries from some division algebra Δ . A simple Artinian algebra with involution $(\mathfrak{U}, j)$ is a simple associative algebra with involution such that $\mathfrak{U}$ is Artinian. The structure of these algebras will be determined next.

Let $(\mathfrak{U}, j)$ be a simple associative algebra with involution. If $\mathfrak{U}$ is not itself simple, then there exists an ideal $\mathfrak{B}$ of $\mathfrak{U}$ such that $\mathfrak{B} \neq 0, \mathfrak{U}$. Clearly $\mathfrak{B}^j$ is also an ideal of $\mathfrak{U}$, so that $\mathfrak{B} \cap \mathfrak{B}^j$ is an ideal of $(\mathfrak{U}, j)$ such that $\mathfrak{B} \cap \mathfrak{B}^j \subseteq \mathfrak{B} \subset \mathfrak{U}$. By the simplicity of $(\mathfrak{U}, j)$, we must have $\mathfrak{B} \cap \mathfrak{B}^j = 0$. Similarly, $\mathfrak{B} + \mathfrak{B}^j$ is an ideal of $(\mathfrak{U}, j)$ and $0 \subset \mathfrak{B} \subseteq \mathfrak{B} + \mathfrak{B}^j$. Thus $\mathfrak{U} = \mathfrak{B} + \mathfrak{B}^j$ and hence $\mathfrak{U} = \mathfrak{B} \oplus \mathfrak{B}^j$. If B is an ideal of $\mathfrak{B}$, then B is an ideal of $\mathfrak{U}$, and thus $B + B^j$ is an ideal of $(\mathfrak{U}, j)$. Hence, $B + B^j = 0$ or $B + B^j = \mathfrak{U}$, so that either $B = 0$ or $B = \mathfrak{B}$. Therefore $\mathfrak{B}$ is a simple algebra.

Now assume $(\mathfrak{U}, j)$ is a simple Artinian algebra with involution. If $\mathfrak{U}$ is simple, then $\mathfrak{U} \cong \Delta_n$, Δ an associative division algebra with involution. If $\mathfrak{U}$ is not simple, then $\mathfrak{U} = \mathfrak{B} \oplus \mathfrak{B}^j$, where $\mathfrak{B}$ is a simple ideal of $\mathfrak{U}$. Since $\mathfrak{U}$ is Artinian, $\mathfrak{B}$ is Artinian, and thus $\mathfrak{B} \cong \Delta_n$, Δ an associative division algebra. It is clear that $\mathfrak{B}^j \cong (\Delta^\circ)_n$, so that $\mathfrak{U} \cong \Delta_n \oplus (\Delta^\circ)_n$, with j the exchange involution.

1.3 Alternative Algebras

Let $\mathfrak{U}$ be an alternative algebra over a field Φ . Then $[x, x, y] = 0$, for all $x, y \in \mathfrak{U}$, and $[y, x, x] = 0$, for all $x, y \in \mathfrak{U}$. These statements are equivalent to the left and right alternative laws: $x^2 y = x(xy)$, $yx^2 = (yx)x$, for all $x, y \in \mathfrak{U}$. By the theorem of Artin, these laws are equivalent to the statement that every subalgebra of $\mathfrak{U}$ generated by two elements is associative. The associator in an alternative algebra has the following property: if σ is any permutation of $\{1, 2, 3\}$, then $[x_{1\sigma}, x_{2\sigma}, x_{3\sigma}] = (\text{sgn } \sigma)[x_1, x_2, x_3]$, for all $x_1, x_2, x_3 \in \mathfrak{U}$. The Moufang identities also hold in any alternative algebra $\mathfrak{U}$:

$$(1) \quad (xyx)z = x[y(xz)]$$

$$(2) \quad z(xyx) = [(zx)y]x$$

$$(3) \quad (xy)(zx) = x(yz)x,$$

for every $x, y, z \in \mathfrak{U}$ [9].

Assume that $\mathfrak{U}$ is alternative with identity 1. $x \in \mathfrak{U}$ is invertible, if there exists $y \in \mathfrak{U}$ such that $xy = yx = 1$. If x is invertible with inverse y , then $[x, y, z] = 0$, for all $z \in \mathfrak{U}$ and y is unique, (see [9], p. 38). If x is invertible, the unique inverse of x will be denoted by x^{-1} . Suppose $x, y, z \in \mathfrak{U}$ are such that $yx = 1 = xz$. Then by the second Moufang identity, $1 = yx = y(xzx) = [(yx)z]x = zx$. Hence x is invertible with inverse $x^{-1} = z = y$. Suppose $x, y \in \mathfrak{U}$ are invertible. Then $0 = [x^{-1}, xy, y^{-1}] = [xy, y^{-1}, x^{-1}] = [(xy)y^{-1}]x^{-1} - (xy)(y^{-1}x^{-1}) = 1 - (xy)(y^{-1}x^{-1})$. Thus $(xy)(y^{-1}x^{-1}) = 1$. Similarly, $(y^{-1}x^{-1})(xy) = 1$, so xy is invertible, and $(xy)^{-1} = y^{-1}x^{-1}$. Next suppose $x, y \in \mathfrak{U}$ are such that x and xy are invertible. Then

$y = (x^{-1}x)y = x^{-1}(xy)$, is invertible.

Definition 1.3.1. Let $\mathfrak{U}$ be a (unital) alternative algebra. $x \in \mathfrak{U}$ is quasi-invertible with quasi-inverse y , if $1 - x$ is invertible with inverse $1 - y$. An ideal K of $\mathfrak{U}$ is a quasi-invertible ideal, if every element of K is quasi-invertible.

Proposition 1.3.1. Let K be a quasi-invertible ideal of $\mathfrak{U}$. If $u \in \mathfrak{U}$ is invertible and $x \in K$, then $u - x$ is invertible.

Proof. $u^{-1}(u - x) = 1 - u^{-1}x$ is invertible, since $u^{-1}x \in K$ implies $u^{-1}x$ is quasi-invertible. Thus $u - x$ is invertible.

Proposition 1.3.2. If A and B are quasi-invertible ideals of $\mathfrak{U}$, then $A + B$ is a quasi-invertible ideal of $\mathfrak{U}$.

Proof. If $a \in A$, $b \in B$, then $1 - (a + b) = (1 - a) - b$ is invertible by Proposition 1.3.1.

Theorem 1.3.1. Let $\mathfrak{U}$ be an alternative algebra with 1 over a field Φ . Then $\mathfrak{U}$ contains a unique maximal quasi-invertible ideal $\mathcal{R}(\mathfrak{U})$ which contains every quasi-invertible ideal of $\mathfrak{U}$. Moreover, $\mathfrak{U}/\mathcal{R}(\mathfrak{U})$ contains no nonzero quasi-invertible ideals.

Proof. Since 0 is a quasi-invertible ideal, the set of all quasi-invertible ideals is nonempty, and Zorn's lemma may be applied to find a maximal quasi-invertible ideal $\mathcal{R}(\mathfrak{U})$. If A is any quasi-invertible ideal of $\mathfrak{U}$, then by Proposition 1.3.2, $A + \mathcal{R}(\mathfrak{U})$ is a quasi-invertible ideal of $\mathfrak{U}$. Hence $A + \mathcal{R}(\mathfrak{U}) = \mathcal{R}(\mathfrak{U})$, and $A \subseteq \mathcal{R}(\mathfrak{U})$. Thus the uniqueness of $\mathcal{R}(\mathfrak{U})$ is clear. Now if $a + \mathcal{R}(\mathfrak{U})$ is invertible in $\mathfrak{U}/\mathcal{R}(\mathfrak{U})$, then there exists $b \in \mathfrak{U}$ such that $ab - 1 \in \mathcal{R}(\mathfrak{U})$. Hence $ab = 1 - z$, $z \in \mathcal{R}(\mathfrak{U})$ and, so ab is invertible. Similarly ba is invertible in $\mathfrak{U}$. Thus

$ab^2a = (ab)(ba)$ is invertible in $\mathfrak{U}$. Let $c \in \mathfrak{U}$ be the inverse of ab^2a . Then $a[b^2(ac)] = (ab^2a)c = 1 = c(ab^2a) = [(ca)b^2]a$, and a is invertible in $\mathfrak{U}$. Therefore, if $x + \mathcal{R}(\mathfrak{U})$ is quasi-invertible in $\mathfrak{U}/\mathcal{R}(\mathfrak{U})$, x is quasi-invertible in $\mathfrak{U}$, and if K is a quasi-invertible ideal of $\mathfrak{U}/\mathcal{R}(\mathfrak{U})$, the complete inverse image of K in $\mathfrak{U}$ under the natural homomorphism is a quasi-invertible ideal of $\mathfrak{U}$ and thus is contained in $\mathcal{R}(\mathfrak{U})$. Hence $K = 0$.

Definition 1.3.2. The unique quasi-invertible ideal $\mathcal{R}(\mathfrak{U})$ of $\mathfrak{U}$ is called the Smiley radical of $\mathfrak{U}$ [10].

We now turn our attention to alternative algebras with involution, and first consider the defining conditions for a quaternion algebra Q over a field Φ of characteristic not 2. Recall that Q is a noncommutative associative algebra with identity 1, and generators i, j satisfying $i^2 = \lambda 1$, $j^2 = \mu 1$, and $ij = -ji$, where $\lambda, \mu \in \Phi$ are nonzero. The set $\{1, i, j, k \equiv ij\}$ constitutes a basis for Q , and the following are also satisfied: $k^2 = -\lambda\mu 1$; $jk = -kj = -\mu i$; $ki = -ik = -\lambda j$. If $a = \alpha_1 1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$, then $a \rightarrow \tilde{a} = \alpha_1 1 - \alpha_2 i - \alpha_3 j - \alpha_4 k$ is an involution called the standard involution, and the norm of $a \in Q$, $n(a)$, is defined by $a \tilde{a} = n(a)1 = \tilde{a} a$. The norm is a quadratic form on Q whose associated symmetric bilinear form, $n(a, b) = n(a + b) - n(a) - n(b) = a\tilde{b} + b\tilde{a}$, is nondegenerate.

Let $\mathcal{O} = Q \oplus Q$, where the sum is a vector space direct sum, and define multiplication in $\mathcal{O}$ by

$$(a, b)(c, d) = (ac + v\tilde{d}b, da + b\tilde{c}), \quad v \neq 0 \in \Phi.$$

$\mathcal{O}$ is a (not associative) alternative algebra over Φ called the algebra of octonions defined by $\mathcal{O}$ and ν . $\mathcal{O}$ has identity $(1,0)$ and the subset $\{(a,0) : a \in Q\}$ is an isomorphic copy of Q . If we write a for $(a,0)$, and let $\ell = (0,1)$, then every element of $\mathcal{O}$ can be written uniquely as $a + b\ell$, $a, b \in Q$. If $x = a + b\ell \in \mathcal{O}$, the mapping $x \rightarrow \tilde{x} = \bar{a} - b\ell$ is an involution in $\mathcal{O}$ again called the standard involution. As with Q , the quadratic form n defined by $x \tilde{x} = n(x)1 = \tilde{x} x$, $x \in \mathcal{O}$ is called the norm, and the associated bilinear form is nondegenerate. Q and $\mathcal{O}$ are examples of simple alternative algebras.

Definition 1.3.3. An algebra with involution $(\mathcal{A}, j)$ is called a composition algebra, if $\mathcal{A}$ is alternative and for all $x \in \mathcal{A}$, $xx^j = Q(x)1 = x^j x$, where Q is a quadratic form whose associated bilinear form $Q(x, y) \equiv Q(x + y) - Q(x) - Q(y)$ is nondegenerate. $(\mathcal{A}, j)$ is a split composition algebra, if $\mathcal{A}$ is not a division algebra.

Now any composition algebra $(\mathcal{A}, j)$ is finite dimensional and is isomorphic to a two-dimensional commutative associative algebra with basis $\{1, q\}$ where $q^j = -q$, a quaternion algebra with standard involution, or an octonian algebra with standard involution [5]. In addition, we state a result in which the determination of certain simple alternative algebras with involution is given. This result as well as a more detailed discussion of the topics in this section can be found in [5].

Proposition 1.3.3. Let $(\mathcal{A}, j)$ be a simple alternative algebra with involution and identity element such that every nonzero symmetric

element is invertible in the nucleus, and let Γ be the subset of $\mathcal{A}$ of symmetric elements in $C(\mathcal{A})$. Then Γ is a subfield of $\mathcal{A}$ and the following possibilities occur for $(\mathcal{A}, j)$: I. $\mathcal{A} = \Delta \oplus \Delta^\circ$, Δ an associative division algebra, j the exchange involution; II. $(\mathcal{A}, j)$ an associative division algebra with involution; III. a split quaternion algebra over Γ isomorphic to Γ_2 ; IV. an algebra of octonions over Γ with standard involution. Conversely, any algebra in one of the classes I-IV satisfies the given conditions.

CHAPTER 2

JORDAN ALGEBRAS

2.1 Quadratic Jordan Algebras

Quadratic Jordan algebras were first introduced by McCrimmon [7], although the presentation given in this section follows that of Jacobson [6].

Definition 2.1.1. Let Φ be an arbitrary commutative associative ring with 1 and let M, N be left Φ -modules. Then a mapping $Q : M \rightarrow N$ is called quadratic, if (1) for all $\alpha \in \Phi$, $a \in M$, $Q(\alpha a) = \alpha^2 Q(a)$, and (2) $Q(a, b) \equiv Q(a + b) - Q(a) - Q(b)$ is a Φ -bilinear mapping from $M \times M$ into N .

If P is a commutative associative ring extension of Φ in the sense that P is a (unital) algebra over Φ , let $M_P = P \otimes_{\Phi} M$, M a unital Φ -module. If $Q : M \rightarrow \text{End}_{\Phi} M$ is a quadratic mapping, then there exists a unique quadratic mapping $\tilde{Q} : M_P \rightarrow \text{End}_P M_P$ such that the following diagram:

$$\begin{array}{ccc}
 M & \xrightarrow{Q} & \text{End}_{\Phi} M \\
 \downarrow \nu & & \downarrow \tilde{\nu} \\
 M_P & \xrightarrow{\tilde{Q}} & \text{End}_P M_P
 \end{array}$$

is commutative, where $\nu(a) = 1 \otimes a$, $\tilde{\nu}(A) = 1 \otimes A$, and $1 \otimes A(p \otimes a) = p \otimes A(a)$ [6].

We are now ready to define a quadratic Jordan algebra.

Definition 2.1.2. A (unital) quadratic Jordan algebra over a commutative associative ring with identity is a triple $(J, U, 1)$, where J is a unital left Φ -module, 1 a distinguished element of J , and $U : J \rightarrow \text{End } J$ ($a \rightarrow U_a$) is a quadratic mapping such that

$$\text{QJ1} \quad U_1 = \text{Id}_J$$

$$\text{QJ2} \quad \text{for every } x, y \in J, U_x U_y U_x = U_{U_x(y)}$$

$$\text{QJ3} \quad \text{If } U_{x,y} = U_{x+y} - U_x - U_y \text{ is the associated symmetric bilinear mapping, and } V_{x,y} \in \text{End } J \text{ is defined by } V_{x,y}(z) = U_{z,y}(x), \text{ for all } z \in J, \text{ then}$$

$$V_{x,y} U_y = U_y V_{y,x} = U_{U_y(x), y}.$$

$$\text{QJ4} \quad \text{If } P \text{ is any commutative associative algebra over } \Phi \text{ and } \tilde{U} \text{ is the extension of } U \text{ to } J_P, \text{ then } \tilde{U} \text{ satisfies QJ1 - QJ3}$$

By a linearization of QJ2 and QJ3, the following identities are obtained:

$$\text{QJ5} \quad U_x U_y U_{x,z} + U_{x,z} U_y U_x = U_{U_x(y), U_{x,z}(y)}$$

$$\text{QJ6} \quad U_x U_y U_z + U_z U_y U_x + U_{x,z} U_y U_{x,z} = U_{U_x(y), U_z(y)} + U_{U_x(y), z}$$

$$\begin{aligned} \text{QJ7} \quad U_x U_y U_{z,w} + U_{x,z} U_y U_{x,w} + U_{z,w} U_y U_x + U_{x,w} U_y U_{x,z} \\ = U_{U_{x,z}(y), U_{x,w}(y)} + U_{U_x(y), U_{z,w}(y)} \end{aligned}$$

$$\text{QJ8} \quad V_{x,y} U_{y,z} + V_{x,z} U_y = U_{y,z} V_{y,x} + U_y V_{z,x}$$

In the presence of QJ1 - QJ3, QJ4 is equivalent to QJ5 - QJ8. Hence QJ1 - QJ3 together with QJ5 - QJ8 constitute an intrinsic set of defining conditions for a quadratic Jordan algebra [6].

Definition 2.1.3. For $x, y \in J$, (i) $x^2 \equiv U_x(1)$, (ii) $x \circ y \equiv U_{x,y}(1)$, (iii) $V_x \equiv V_{1,x}$.

It follows immediately from Definition 2.1.3 that $V_x(y) = V_{1,x}(y) = U_{x,y}(1) = U_{y,x}(1) = y \circ x$, and $\circ$ is a symmetric bilinear composition. We now list identities which are necessary for this thesis [6].

$$\text{QJ9} \quad V_x = V_{x,1} = U_{x,1}$$

$$\text{QJ9}' \quad V_1 = 2\text{Id}_J; 1 \circ x = 2x$$

$$\text{QJ10} \quad U_x^2 = U_x^2$$

$$\text{QJ11} \quad U_x U_y + U_y U_x = U_{U_x(y),y} + U_{x \circ y} - V_x U_y V_x$$

$$\text{QJ12} \quad U_x V_y + V_y U_x = U_{x,x \circ y}$$

$$\text{QJ13} \quad 2U_x = (V_x)^2 - V_x^2$$

$$\text{QJ14} \quad U_{x,y} + V_{x,y} = V_y V_x$$

$$\text{QJ14}' \quad U_{x,y}(z) + U_{z,y}(x) = (z \circ x) \circ y$$

Definition 2.1.4. If $x \in J$, let $x^0 = 1$, $x^1 = x$, $x^2 = U_x(1)$ and for $n \geq 2$, $x^n = U_x(x^{n-2})$.

From Definition 2.1.4, we have

$$\text{QJ15} \quad U_x^n = U_x^n$$

$$\text{QJ16} \quad (x^m)^n = x^{mn}$$

$$\text{QJ17} \quad x^m \circ x^n = 2x^{m+n} \quad [6]$$

For any nonnegative integer n , $U_x(x^n) = x^{n+2}$. If $U_{x^m}(x^n) = x^{n+2m}$, m a nonnegative integer, then $U_{x^{m+1}}(x^n) = U_x^{m+1}(x^n) = U_x U_x^m(x^n) = U_x(x^{n+2m}) = x^{n+2m+2} = x^{n+2(m+1)}$.

Therefore by induction on m we have

QJ18 $U_{x^m}(x^n) = x^{n+2m}$, m, n nonnegative integers.

2.2 Inner and Outer Ideals

Let $(J, U, 1)$ be a quadratic Jordan algebra. If $A, B \subseteq J$, let $U_B(A)$ be the set of all ϕ -linear combinations of elements of the form $U_b(a)$, $b \in B$, $a \in A$.

Definition 2.2.1. Let $(J, U, 1)$ be a quadratic Jordan algebra. A subset K of J is an inner ideal of J , if K is a submodule of J and $U_K(J) \subseteq K$. A subset K is an outer ideal, if K is a submodule of J and $U_J(K) \subseteq K$. A subset K is an ideal of J , if K is both an inner and an outer ideal of J . A subset K is a subalgebra of J , if K is a submodule, $1 \in K$ and $U_K(K) \subseteq K$.

If K is an inner ideal of J and $x \in J$, then $U_x(K)$ is an inner ideal of J , so in particular $U_x(J)$ is an inner ideal called the principal inner ideal determined by x . $U_x(J)$ need not contain x , so the inner ideal generated by x is $\phi x + U_x(J)$ [11].

For outer ideals we have the following:

Proposition 2.2.1. Let $(J, U, 1)$ be a quadratic Jordan algebra and K an outer ideal of J . If $x, y \in J$ and $k \in K$, then

(i) $U_{x,y}(k) = V_{k,y}(x) \in K$, (ii) $V_k(x) = V_x(k) = x \circ k \in K$, so $J \circ K \subseteq K$, and (iii) $U_{x,k}(y) = U_{k,x}(y) = V_{y,k}(x) = V_{y,x}(k) \in K$.

Proof. (i) $U_{x,y}(k) = U_{x+y}(k) - U_x(k) - U_y(k) \in U_J(K) \subseteq K$.

(ii) By QJ9, $x \circ k = V_x(k) = U_{x,1}(k) \in K$ by (i).

(iii) By QJ14', $U_{k,x}(y) = (y \circ k) \circ x - U_{x,y}(k) \in K$ by (i) and (ii).

Definition 2.2.2. Let $(J, U, 1)$ and $(\tilde{J}, \tilde{U}, \tilde{1})$ be quadratic Jordan algebras. A mapping $\theta : J \rightarrow \tilde{J}$ is a (Jordan) homomorphism, if θ is a linear mapping and

$$(1) \quad \theta(1) = \tilde{1}$$

$$(2) \quad \theta(U_x(y)) = \tilde{U}_{\theta(x)}(\theta(y)), \text{ for all } x, y \in J.$$

Isomorphism, endomorphism and automorphism all have the usual meanings.

It is clear from Definition 2.2.2, that the kernel of any homomorphism θ is an ideal of J . If K is any ideal of J , then $(J/K, \bar{U}, \bar{1})$ is a quadratic Jordan algebra, where

$$\bar{U}_{x+K}(y+K) = U_x(y) + K \quad \text{and} \quad \bar{1} = 1 + K. \quad \text{The natural } \phi\text{-module}$$

homomorphism $\theta : J \rightarrow J/K$ is a Jordan homomorphism and if

$(\tilde{J}, \tilde{U}, \tilde{1})$ is any homomorphic image of $(J, U, 1)$, then $\tilde{J} \cong J/\ker \theta$.

If K is any inner (outer) ideal of J , then $\theta(K)$ is an inner

(outer) ideal of $\tilde{J}$, for if $x \in J, k \in K, \tilde{U}_{\theta(k)}(\theta(x)) =$

$\theta(U_k(x)) \in \theta(K)$ ($\tilde{U}_{\theta(k)}(\theta(k)) = \theta(U_x(k)) \in \theta(K)$). Conversely, if

$\tilde{K}$ is any inner (outer) ideal of $\tilde{J}$, then $K = \theta^{-1}(\tilde{K})$ is an inner

(outer) ideal of J and $\ker \theta \subseteq K$, for $x \in J, k \in K$ implies

$\tilde{U}_{\theta(k)}(\theta(x)) \in \tilde{K}$, so $\theta(U_k(x)) \in \tilde{K}$, and $U_k(x) \in K(\theta(U_x(k)) =$

$\tilde{U}_{\theta(k)}(\theta(x)) \in \tilde{K}$ implies $U_x(k) \in K$). Thus the (inner, outer)

ideals of $\tilde{J}$ are in one-to-one correspondence with the (inner, outer) ideals of J which contain $\ker \theta$.

Proposition 2.2.2. If $(J, U, 1)$ is a quadratic Jordan algebra and

K is an outer ideal of J , then for all $x, y \in J$ such that

$x - y \in K, x^n - y^n \in K$, for all $n \in \mathbb{N}$.

Proof. The case $n = 1$ holds by hypothesis. Let $x - y = w \in K$,

and assume $x^k - y^k \in K$ for $k < n$. Then

$$\begin{aligned}
x^n - y^n &= U_x(x^{n-2}) - U_y(y^{n-2}) \\
&= U_{y+w}(x^{n-2}) - U_y(y^{n-2}) \\
&= U_{y,w}(x^{n-2}) + U_y(x^{n-2}) + U_w(x^{n-2}) - U_y(y^{n-2}) \\
&= U_{y,w}(x^{n-2}) + U_y(x^{n-2} - y^{n-2}) + U_w(x^{n-2}) \in K
\end{aligned}$$

by Proposition 2.2.1 and the induction hypothesis.

Definition 2.2.3. Let $(J, U, 1)$ be a quadratic Jordan algebra and K be an ideal of J . Then $K^{(0)} \equiv J$, $K^{(1)} \equiv K$, and for $n \geq 1$, $K^{(n+1)} = U_{K^{(n)}}(K^{(n)})$.

Proposition 2.2.3. If $(J, U, 1)$ is a quadratic Jordan algebra and K is an ideal of J , then $K^{(n)}$ is an ideal of J for $n = 0, 1, 2, \dots$.

Proof. Clearly $K^{(0)}, K^{(1)}$ are ideals of J . The proof follows by induction on n , if $K^{(2)} = U_K(K)$ is an ideal. Let $x \in J$, $k_1, k_2 \in K$. Then $U_{U_{k_1}(k_2)}(x) = U_{k_1} U_{k_2} U_{k_1}(x) = U_{k_1} [U_{k_2}(U_{k_1}(x))]$ $\in U_K(U_K(J)) \subseteq U_K(K)$, since K is an inner ideal. Thus $K^{(2)}$ is an inner ideal. To show that $U_K(K)$ is an outer ideal first note that, if $k \in K$, then $U_{k_1, k_2}(k) = U_{k_1+k_2}(k) - U_{k_1}(k) - U_{k_2}(k) \in U_K(K)$. Thus by QJ11 and QJ12,

$$\begin{aligned}
U_x U_{k_1}(k_2) &= U_{U_x(k_1), k_1}(k_2) + U_{x \circ k_1}(k_2) - U_{k_1} U_x(k_2) - V_x U_{k_1} V_x(k_2) \\
&= U_{U_x(k_1), k_1}(k_2) + U_{x \circ k_1}(k_2) - U_{k_1} U_x(k_2) - (U_{k_1, x \circ k_1} \\
&\quad - U_{k_1} V_x) V_x(k_2) \\
&= U_{U_x(k_1), k_1}(k_2) + U_{x \circ k_1}(k_2) - U_{k_1} [U_x(k_2)] - \\
&\quad U_{k_1, x \circ k_1}(x \circ k_2) + U_{k_1}(x \circ (x \circ k_2)) \\
&\in U_K(K).
\end{aligned}$$

Thus $K^{(2)}$ is an ideal, and the proposition follows as indicated.

Proposition 2.2.4. Let $(J, U, 1)$ be a quadratic Jordan algebra and K an ideal of J . If $x \in K$, then $x^m \in K^{(k)}$ for all $m \geq 3^k$, $k \in \mathbb{N}$.

Proof. We first show that $x^{3^{k-1}} \in K^{(k)}$, $k \in \mathbb{N}$ by induction on

k . If $k = 1$, then $x \in K^{(1)}$, and if $k = 2$, then

$x^3 = U_x(x) \in U_K(K) = K^{(2)}$. Suppose the property holds for $r < k$, where $k \geq 2$. Then by QJ18, $x^{3^{k-1}} = U_{x^{3^{k-2}}}(x^{3^{k-2}}) \in U_{K^{(k-1)}}(K^{(k-1)})$

$= K^{(k)}$. Now, if $m \geq 3^k$, then $x^m = U_{x^{3^{k-1}}}(x^{m-2 \cdot 3^{k-1}}) \in K^{(k)}$, since $K^{(k)}$ an ideal.

2.3 Invertible and Quasi-invertible Elements

Definition 2.3.1. Let $(J, U, 1)$ be a quadratic Jordan algebra.

$x \in J$ is invertible, if there exists $y \in J$ such that

(i) $U_x(y) = x$, and (ii) $U_x(y^2) = 1$. y is called the inverse of x and we write $y = x^{-1}$.

Proposition 2.3.1 (Theorem on Inverses). 1. The following are equivalent: (i) x is invertible; (ii) U_x is invertible in $\text{End } J$; (iii) $1 \in U_x(J)$.

2. If x is invertible, then (i) x^{-1} is unique and $x^{-1} = U_x^{-1}(x)$, (ii) $U_{x^{-1}} = U_x^{-1}$, (iii) x^{-1} is invertible and $(x^{-1})^{-1} = x$, (iv) $x \circ x^{-1} = 2$, (v) $x^2 \circ x^{-1} = 2x$, (vi) $V_{x^{-1}} = V_x U_x^{-1} = U_x^{-1} V_x$.

3. $U_x(y)$ is invertible if and only if x and y are invertible, and $[U_x(y)]^{-1} = U_{x^{-1}}(y^{-1})$. [6]

By 1(ii) and 1(iii), if U_x is onto for $x \in J$, then U_x is one-to-one. However, the converse need not be true. If U_x is not onto, then x is called a zero divisor of J . Thus $x \in J$ is a zero divisor if and only if there exists $y \neq 0 \in J$ such that $U_x(y) = 0$. A quadratic Jordan algebra which has no zero divisors other than zero is called a Jordan integral domain. Similarly, a quadratic Jordan algebra in which every nonzero element is invertible is called a Jordan division algebra. $z \in J$ is an absolute zero divisor, if $U_z = 0$. J is (strongly) nondegenerate, if J has no nonzero absolute zero divisors. The ideal, $\text{zer } J$, generated by the set of all absolute zero divisors of J is a nil ideal in the usual sense that $z \in \text{zer } J$ implies $z^n = 0$, for some $n \in \mathbb{N}$, and hence,

if K is an ideal of J such that $\text{zer } J \subseteq K$, then J/K is non-degenerate [6].

Definition 2.3.2. Let $(J, U, 1)$ be a quadratic Jordan algebra. $z \in J$ is called quasi-invertible, if $1 - z$ is invertible. If the inverse of $1 - z$ is denoted by $1 - w$, then w is called the quasi-inverse of z .

Let z be quasi-invertible with quasi-inverse w . Then the following properties are immediate from the definition (see [6]).

$$(1) \quad U_{1-z}(1-w) = 1 - z, \quad U_{1-z}[(1-w)^2] = 1$$

$$(2) \quad w + z - z^2 - w \circ z + U_z(w) = 0$$

$$(3) \quad 2w - z^2 + U_z(2w) - U_z(w^2) + 2z - 2w \circ z - w^2 + w^2 \circ z = 0$$

$$(4) \quad w = U_{1-z}^{-1}(z^2 - z)$$

$$(5) \quad z \circ w = 2(z + w)$$

$$(6) \quad w + z + U_z(1 - w) = 0.$$

From (6), it is clear that any ideal which contains z must also contain w . Also note that w is quasi-invertible with quasi-inverse z .

An (inner, outer) ideal is quasi-invertible, if every element is quasi-invertible. The following useful lemma is stated without proof (see [6]).

Lemma 2.3.1. Let $(J, U, 1)$ be a quadratic Jordan algebra and let K be a quasi-invertible ideal of J . If $u \in J$ is invertible and $z \in K$, then $u + z$ is invertible.

If K_1 and K_2 are quasi-invertible ideals of J , and $x \in K_1$, $y \in K_2$, then $1 - x$ is invertible, and thus, by Lemma 2.3.1, $1 - (x + y) = (1 - x) + y$ is invertible. Hence $K_1 + K_2$ is a quasi-invertible ideal. This fact together with Zorn's lemma implies the existence of a unique maximal quasi-invertible ideal $\mathcal{R}(J)$, which contains every quasi-invertible ideal of J . $\mathcal{R}(J)$ is called the Jacobson radical of J and J is (Jacobson) semisimple, if $\mathcal{R}(J) = 0$. Since the homomorphic image and the complete inverse image of a quasi-invertible ideal is a quasi-invertible ideal, $J/\mathcal{R}(J)$ contains no nonzero quasi-invertible ideals. Finally, since $\mathcal{R}(J)$ contains every nil ideal of J , in particular $\text{zer } J \subseteq \mathcal{R}(J)$, and so $J/\mathcal{R}(J)$ is nondegenerate [6].

We conclude this section with a brief discussion of isotopes. For more details, see [6].

Definition 2.3.3. Let $(J, U, 1)$ be a quadratic Jordan algebra and $c \in J$ invertible. Define $U^{(c)} : J \rightarrow \text{End } J$ by $U_x^{(c)} = U_x U_c$, $x \in J$, and let $1^{(c)} = c^{-1}$. Then the quadratic Jordan algebra $J^{(c)} = (J, U^{(c)}, 1^{(c)})$ is called the c-isotope of $(J, U, 1)$.

If $d \in J$ is also invertible, then $(J^{(c)})^{(d)} = J^{(U_c^{(d)})}$.

Thus if $c^{-2} \equiv (c^{-1})^2$, $(J^{(c)})^{(c^{-2})} = J^{(U_c^{(c^{-2})})} = J^{(1)} = J$. Now if K is an ideal of J and $x \in J$, $k \in K$, then $U_k^{(c)}(x) = U_k U_c(x) \in K$ and $U_x^{(c)}(k) = U_x U_c(k) \in K$. Thus K is an ideal of $J^{(c)}$. Similarly, if K is an ideal of $J^{(c)}$, then K is an ideal of $(J^{(c)})^{(c^{-2})} = J$. Thus the ideals of J and any isotope coincide. In particular, $\mathcal{R}(J^{(c)}) = \mathcal{R}(J)$ (see [6]).

2.4 Linear Jordan Algebras

Linear Jordan algebras play an important role in Chapter 4, where the only algebras considered will be linear Jordan algebras over fields of characteristic not 2.

Definition 2.4.1. A (unital) linear Jordan algebra over a commutative associative ring Φ (with 1) containing $\frac{1}{2}$ is a triple $(J, R, 1)$ such that J is a (unital) left Φ -module and R is a mapping from $J \rightarrow \text{End } J$ such that R is a Φ -homomorphism, and if the image of $x \in J$ is denoted by R_x , then

$$J1 \quad R_1 = \text{Id}_J$$

$$J2 \quad R_{R_x(x)} R_x = R_x R_{R_x(x)}$$

$$J3 \quad \text{If } L_x \text{ is defined by } L_x(y) = R_y(x), \text{ then } L_x = R_x.$$

Let $(J, R, 1)$ be a linear Jordan algebra and for $x, y \in J$ define $x \cdot y = R_y(x)$. Then $\cdot$ is a bilinear composition, by J1 $x \cdot 1 = x$, by J3 $x \cdot y = y \cdot x$, and by J2, $(y \cdot x) \cdot x^2 = (y \cdot x^2) \cdot x$, where $x^2 = x \cdot x$. Conversely, let J be a unital left Φ -module, $\frac{1}{2} \in \Phi$, equipped with a symmetric bilinear composition $\cdot$ such that 1 is an identity for $\cdot$ and $(y \cdot x) \cdot x^2 = (y \cdot x^2) \cdot x$ for all $x, y \in J$. By defining R by $R_x(y) = y \cdot x$, R satisfies J1 - J3 and $(J, R, 1)$ is a linear Jordan algebra. Thus we have the more familiar alternate definition which is equivalent to Definition 2.4.1.

Definition 2.4.1'. A (unital) linear Jordan algebra J over the commutative associative ring Φ (with 1) containing $\frac{1}{2}$ is a unital left Φ -module with a bilinear composition $\cdot$ and $1 \in J$ such that (i) $1 \cdot x = x = x \cdot 1$, $x \in J$, (ii) $x \cdot y = y \cdot x$, for $x, y \in J$,

(iii) (Jordan identity) $(y.x).x^2 = (y.x^2).x$, for all $x, y \in J$.

If $(J, R, 1)$ is any linear Jordan algebra over Φ ($\frac{1}{2} \in \Phi$) and P is any commutative associative algebra over Φ , then there exists a unique extension $\tilde{R}$ of R to $J_P = P \otimes_{\Phi} J$ so that $(J_P, \tilde{R}, 1 \otimes 1)$ is a linear Jordan algebra [6].

Finally, if $(J, R, 1)$ is a linear Jordan algebra, define $U : J \rightarrow \text{End } J$ by $U_x = 2R_x^2 - R_{\frac{1}{2}x}$. Then U is quadratic and $(J, U, 1)$ is a quadratic Jordan algebra. Conversely, if $(J, U, 1)$ is a quadratic Jordan algebra over Φ , where $\frac{1}{2} \in \Phi$, then define $R : J \rightarrow \text{End } J$ by $R_x = \frac{1}{2} U_x$. In this case $(J, R, 1)$ is a linear Jordan algebra. In fact there is a category isomorphism between the category of linear Jordan algebras with homomorphisms as morphisms and quadratic Jordan algebras over rings Φ such that $\frac{1}{2} \in \Phi$ with homomorphisms as morphisms [6].

For the remainder of this section let $(J, U, 1)$ be a quadratic Jordan algebra over the unital commutative associative ring Φ and assume $\frac{1}{2} \in \Phi$. Then we will write J for the triple $(J, U, 1)$ and the triple $(J, R, 1)$, since J is also a linear Jordan algebra in the sense of Definition 2.4.1' with $x.y = \frac{1}{2} x \circ y$. First note that K is an ideal of J if and only if K is an outer ideal of J . Since one direction is clear, assume K is an outer ideal of J . Then for $k \in K, x \in J, U_k(x) = 2R_k^2(x) - R_{\frac{1}{2}k}(x) = 2(x.k).k - x.k^2 = \frac{1}{2} (x \circ k) \circ k - \frac{1}{2} x \circ k^2 \in K$ by Proposition 2.2.1. Hence K is also an inner ideal and therefore an ideal. Also it is clear that K is an ideal of J if and only if K is a submodule of J and $x.k \in K$ for all $x \in J, k \in K$.

It is also possible to state the concepts of inverse and quasi-inverse in terms of $\cdot$. Suppose $x \in J$ is invertible with inverse y . By Proposition 2.3.1, $x \cdot y = \frac{1}{2} x \circ y = \frac{1}{2} 2 = 1$ and $x^2 \cdot y = \frac{1}{2} x^2 \circ y = \frac{1}{2} 2x = x$. Conversely, if for $x \in J$ there exists y such that $x \cdot y = 1$ and $x^2 \cdot y = x$, then we show first that $x^2 \cdot y^2 = 1$ and $x \cdot y^2 = y$. Since $\frac{1}{2} \in \Phi$, the linearized form of the Jordan identity holds; i.e., for $a, b, c, d \in J$, $[(b \cdot c) \cdot a] \cdot d + [(c \cdot d) \cdot a] \cdot b + [(d \cdot b) \cdot a] \cdot c = (a \cdot b) \cdot (c \cdot d) + (c \cdot b) \cdot (a \cdot d) + (d \cdot b) \cdot (a \cdot c)$. By taking $c = d = x$ and $a = b = y$, we get $x^2 \cdot y^2 = 1$. Then $x \cdot y^2 = (x^2 \cdot y) \cdot y^2 = (x^2 \cdot y^2) \cdot y = y$. Hence $U_x(y) = 2R_x^2(y) - R_x^2(y) = 2(y \cdot x) \cdot x - y \cdot x^2 = 2x - x = x$, and $U_x(y^2) = 2R_x^2(y^2) - R_x^2(y^2) = 2(y^2 \cdot x) \cdot x - y^2 \cdot x^2 = 2y \cdot x - 1 = 2 - 1 = 1$, and so x is invertible with inverse y . By a straightforward computation it can be shown that $x \in J$ is quasi-invertible with quasi-inverse y if and only if $x + y - x \cdot y = 0$ and $x^2 \cdot y - x \cdot (x \cdot y) = 0$.

By QJ3 with $y = 1$, $U_x V_x = V_x U_x$. Thus since $x^0 = 1$, $x^1 = x$, $x^2 = U_x(1) = 2R_x^2(1) - R_x^2(1) = 2x \cdot x - x^2$, so that $x^2 = x \cdot x$, we have $x^{n+1} = U_x(x^{n-1}) = \frac{1}{2} U_x V_x(x^{n-2}) = \frac{1}{2} V_x U_x(x^{n-2}) = x^n \cdot x$ by induction. Hence when $\frac{1}{2} \in \Phi$, powers can be defined in terms of $\cdot$, and $x^n \cdot x^m = \frac{1}{2} x^n \circ x^m = x^{n+m}$ implies that $(J, \cdot, 1)$ is power-associative.

We finish this section with a special result for linear Jordan algebras concerning $K^{(n)}$, $n \in \mathbb{N}$.

Proposition 2.4.1. Let J be a linear Jordan algebra ($\frac{1}{2} \in \Phi$) and K an ideal of J . Then $K^{(n+1)} = (K^{(n)})^3$, $n \in \mathbb{N}$, where $A^3 \equiv A \cdot (A \cdot A)$.

Proof. Since $K^{(n+1)} = U_{K^{(n)}}(K^{(n)})$, the result will follow by induction, if $U_K(K) = K^3$. If $k_1, k_2 \in K_1$ then

$U_{k_1}(k_2) = 2R_{k_1}^2(k_2) - R_{k_1}^2(k_2) = 2(k_2 \cdot k_1) \cdot k_1 - k_2 \cdot k_1^2 \in K^3$. Conversely, suppose $k_1, k_2, k_3 \in K$. Then by QJ14', $4(k_1 \cdot k_2) \cdot k_3$
 $= (k_1 \circ k_2) \circ k_3 = U_{k_1, k_3}(k_2) + U_{k_2, k_3}(k_1) = U_{k_1+k_3}(k_2) - U_{k_1}(k_2)$
 $- U_{k_3}(k_2) + U_{k_2+k_3}(k_1) - U_{k_2}(k_1) - U_{k_3}(k_1) \in U_K(K)$.

2.5 Constructions of Jordan Algebras

I. Let $\mathfrak{U}$ be an associative algebra with 1 over the commutative associative ring Φ with identity. Define $U : \mathfrak{U} \rightarrow \text{End } \mathfrak{U}$ by $x \rightarrow U_x$, where $U_x(y) = xyx$, $y \in \mathfrak{U}$. Then U is a quadratic mapping and $\mathfrak{U}^+ = (\mathfrak{U}, U, 1)$ is a quadratic Jordan algebra (see [7]). The associated bilinear map is determined by $U_{x,y}(z) = xzy + yzx$. If $\frac{1}{2} \in \Phi$, then $x \cdot y = \frac{1}{2} x \circ y = \frac{1}{2} U_x(y) = \frac{1}{2} U_{x,1}(y) = \frac{1}{2} (xy + yx)$. Conversely, if $\mathfrak{U}$ is an associative algebra and $\frac{1}{2} \in \Phi$, then $\mathfrak{U}$ together with $\cdot$ defined by $x \cdot y = \frac{1}{2} (xy + yx)$ is a linear Jordan algebra.

Clearly every ideal of $\mathfrak{U}$ is also an ideal of $\mathfrak{U}^+$. The following theorem due to Herstein has an interesting consequence. Let (x) denote the associative ideal of $\mathfrak{U}$ generated by x .

Theorem 2.5.1 (Herstein). Let $\mathfrak{U}$ be an associative algebra (with 1) and $0 \neq K$ an ideal of $\mathfrak{U}^+$. Then either (1) $b^2 = 0$ for every $b \in K$, and if $b \neq 0$, then $(b) \neq 0$ but $(b)^3 = 0$, or (2) there exists $b \in K$, $b^2 \neq 0$, and $0 \neq (b^2) \subseteq K$.

Proof. First note that $x^2 = x1x = U_x(1)$ for all $x \in J$ and $x \circ y = U_{x,y}(1) = xy + yx$, for all $x, y \in \mathfrak{U}$. Secondly, note that $bxby$ and $xb^2y \in K$ for all $b \in K$, $x, y \in \mathfrak{U}$, since $bxby = (bx)(by) + (by)(xb) - b(yx)b = U_{b,by}(x) - U_b(yx)$, and $xb^2y = (xb)(by) + (bx)(by) + (by)(xb) + (by)(bx) - bxby - b(yx)b - bybx = (x \circ b) \circ by - U_b(yx) - bxby - bybx$.

(1) Assume $b^2 = 0$ for every $b \in K$. Then for all $a, c \in K$, $ac + ca = a \circ c = U_{a,c}(1) = U_{a+c}(1) - U_a(1) - U_c(1) = (a+c)^2 - a^2 - c^2 = 0$, so $ac = -ca$. Now, if $b \neq 0$, then $b \in (b) = \mathfrak{U}b\mathfrak{U} \neq 0$. For

$x, y \in \mathfrak{U}$, $bxbyb = (bxby)b = -b(bxby) = -b^2xby = 0$, so
 $(b)^3 = 0$.

(2) If $b \in K$ and $b^2 \neq 0$, then $b^2 \in (b^2)$, so $0 \neq (b^2)$
 $= \mathfrak{U}b^2\mathfrak{U} \subseteq K$.

Definition 2.5.1. A quadratic Jordan algebra $(J, U, 1)$ is simple, if $J \neq 0$ and the only ideals of J are 0 and J .

We now derive a corollary to Herstein's theorem which provides many examples of simple Jordan algebras. Recall that an associative algebra $\mathfrak{U}$ is simple, if $\mathfrak{U}^2 \neq 0$, and 0 and $\mathfrak{U}$ are the only ideals of $\mathfrak{U}$. Since $\mathfrak{U}^2$ is an ideal of $\mathfrak{U}$, $\mathfrak{U}^2 = \mathfrak{U}$ when $\mathfrak{U}$ is simple.

Corollary 2.5.1. $\mathfrak{U}$ is a simple associative algebra with identity if and only if $\mathfrak{U}^+$ is a simple quadratic Jordan algebra.

Proof. If $\mathfrak{U}^+$ is simple, then $\mathfrak{U}$ is simple since every ideal of $\mathfrak{U}$ is an ideal of $\mathfrak{U}^+$. Hence assume $\mathfrak{U}$ is simple. Then $\mathfrak{U}^+ \neq 0$, so let K be a nonzero ideal of $\mathfrak{U}^+$. If $b \in K$ and $b^2 \neq 0$, then $0 \neq (b^2) \subseteq K \subseteq \mathfrak{U}$. Since $\mathfrak{U}$ is simple $(b^2) = \mathfrak{U}$ and hence $K = \mathfrak{U}^+$. On the other hand, if $b^2 = 0$ for all $b \in K$, then $(b) \neq 0$, so $(b) = \mathfrak{U}$. But by Theorem 2.5.1, $0 = (b)^3 = \mathfrak{U}^3$, and thus $\mathfrak{U}^2 = 0$, a contradiction. Thus the only nonzero ideal of $\mathfrak{U}^+$ is $\mathfrak{U}^+$ itself, and $\mathfrak{U}^+$ is simple.

Proposition 2.5.1. $x \in \mathfrak{U}$ is invertible with inverse y if and only if x is invertible in $\mathfrak{U}^+$ with inverse y . x is quasi-invertible in $\mathfrak{U}$ with quasi-inverse y if and only if x is quasi-invertible in $\mathfrak{U}^+$ with quasi-inverse y . Let $x \cdot^n$ denote powers of x in $\mathfrak{U}^+$. Then $x \cdot^n = x^n$, where x^n denotes powers in $\mathfrak{U}$.

Proof. If $xy = 1 = yx$, then $U_x(y) = xyx = x$ and $U_x(y^2) = xy^2x = (xy)(yx) = 1^2 = 1$. Conversely, if $U_x(y) = x$ and $U_x(y^2) = 1$, then $x(y^2x) = 1 = (xy^2)x$, so x is invertible in $\mathfrak{U}$, and $y^2x = xy^2 = x^{-1}$. Since $x = U_x(y) = xyx$, and x invertible, $y = x^{-1}xyxx^{-1} = x^{-1}xx^{-1} = x^{-1}$. If x is quasi-invertible in $\mathfrak{U}$, then $1 - x$ is invertible in $\mathfrak{U}$ and hence invertible in $\mathfrak{U}^+$. Thus x is quasi-invertible in $\mathfrak{U}^+$, and conversely. If x has quasi-inverse y in $\mathfrak{U}$, then $1 - x$ and $1 - y$ are inverses in $\mathfrak{U}^+$, and conversely. Thus the second statement is clear.

Clearly $x \cdot 1 = x^1$ and $x \cdot 2 = U_x(1) = x1x = x^2$, so for $n > 2$, $x \cdot n = U_x(x^{n-2}) = xx^{n-2}x = x^n$.

It can be shown that the nil radicals and Jacobson radicals of $\mathfrak{U}$ and $\mathfrak{U}^+$ coincide. Since only the result concerning the Jacobson radicals is needed for this thesis, it is the only one which will be proved. However, the proofs in the two cases are nearly identical, both results being corollaries to Herstein's Theorem (2.5.1).

Corollary 2.5.2. Let $\mathfrak{U}$ be an associative algebra (with 1), and let $\mathcal{R}(\mathfrak{U})$ be the Jacobson radical of $\mathfrak{U}$. Then $\mathfrak{U}$ is (Jacobson) semi-simple if and only if $\mathfrak{U}^+$ is (Jacobson) semi-simple.

Proof. By Proposition 2.5.1, $\mathcal{R}(\mathfrak{U})$ is a quasi-invertible ideal of $\mathfrak{U}^+$ so that $\mathcal{R}(\mathfrak{U}) \subseteq \mathcal{R}(\mathfrak{U}^+)$. Now suppose $\mathfrak{U}^+$ is semi-simple. Then $\mathcal{R}(\mathfrak{U}) \subseteq \mathcal{R}(\mathfrak{U}^+) = 0$ implies $\mathcal{R}(\mathfrak{U}) = 0$; i.e., $\mathfrak{U}$ is semi-simple.

Conversely, assume $\mathfrak{U}$ is semi-simple. If $x^2 = 0$ for all $x \in \mathcal{R}(\mathfrak{U}^+)$, then $(x)^3 = 0$, and $(x) \subseteq \mathcal{R}(\mathfrak{U}) = 0$. Thus $x = 0$ and $\mathfrak{U}^+$ is semi-simple. If there exists $x \in \mathcal{R}(\mathfrak{U}^+)$ such that $x^2 \neq 0$,

then $0 \neq (x^2) \subseteq R(\mathfrak{U}^+)$. But then (x^2) is a nonzero quasi-invertible ideal of $\mathfrak{U}$, a contradiction. Thus $\mathfrak{U}^+$ is semi-simple.

Corollary 2.5.3. $R(\mathfrak{U}) = R(\mathfrak{U}^+)$, $\mathfrak{U}$ as in Corollary 2.5.2.

Proof. Since $R(\mathfrak{U}) \subseteq R(\mathfrak{U}^+)$, it remains only to show the reverse inclusion. Let $\theta : \mathfrak{U} \rightarrow \mathfrak{U}/R(\mathfrak{U})$. Then clearly θ is a Jordan homomorphism from $\mathfrak{U}^+ \rightarrow (\mathfrak{U}/R(\mathfrak{U}))^+$ with kernel $R(\mathfrak{U})$. Hence $\mathfrak{U}^+/R(\mathfrak{U}) \cong (\mathfrak{U}/R(\mathfrak{U}))^+$, where the latter is semi-simple by Corollary 2.5.2. Since $R(\mathfrak{U}^+)/R(\mathfrak{U})$ is a quasi-invertible ideal of $\mathfrak{U}^+/R(\mathfrak{U})$, $R(\mathfrak{U}^+)/R(\mathfrak{U}) = 0$ so $R(\mathfrak{U}^+) \subseteq R(\mathfrak{U})$.

Definition 2.5.2. A quadratic Jordan algebra $(J, U, 1)$ is special, if there exists a (Jordan) monomorphism from J into $\mathfrak{U}^+$, where $\mathfrak{U}$ is an associative algebra with identity. Otherwise $(J, U, 1)$ is exceptional.

Clearly, if $\mathfrak{U}$ associative, then $\mathfrak{U}^+$ is special.

II. The construction in (I) can be generalized to $\mathfrak{U}$ an alternative algebra with identity. Again $U_x(y) \equiv xyx$, and the resulting quadratic Jordan algebra is denoted by $\mathfrak{U}^+$. If $\frac{1}{2} \in \Phi$, $x \cdot y \equiv \frac{1}{2}(xy + yx)$ yields $\mathfrak{U}^+$ as a linear Jordan algebra. Since in this case $\mathfrak{U}^+ \cong \text{Hom}_{\Phi}(\mathfrak{U}, \mathfrak{U})^+$, $\mathfrak{U}^+$ is clearly special. Finally, it is again clear that every ideal of $\mathfrak{U}$ is also an ideal of $\mathfrak{U}^+$.

III. Let $(\mathfrak{U}, j)$ be an associative algebra with identity and involution j , and let $\mathfrak{S}(\mathfrak{U}, j) = \{x \in \mathfrak{U} : x^j = x\}$ be the set of j -symmetric elements of $(\mathfrak{U}, j)$. Since $\mathfrak{S}(\mathfrak{U}, j)$ is clearly a submodule of $\mathfrak{U}$, $1^j = 1$, and $U_x(y)^j = (xyx)^j = x^j y^j x^j = xyx = U_x(y)$ for all $x, y \in \mathfrak{S}(\mathfrak{U}, j)$, $\mathfrak{S}(\mathfrak{U}, j)$ is a subalgebra of $\mathfrak{U}^+$. If

$\theta : (\mathfrak{U}, j) \rightarrow (\tilde{\mathfrak{U}}, \tilde{j})$ is a homomorphism of algebras with involution, then $\theta|_{\mathfrak{S}(\mathfrak{U}, j)}$ is a (Jordan) homomorphism from $\mathfrak{S}(\mathfrak{U}, j) \rightarrow \mathfrak{S}(\tilde{\mathfrak{U}}, \tilde{j})$.

Now let $\mathfrak{U}^\circ$ be the opposite algebra of $\mathfrak{U}$, $\mathfrak{B} = \mathfrak{U} \oplus \mathfrak{U}^\circ$ and j the exchange involution. Then we have seen that $\mathfrak{B}$ is an associative algebra with involution and identity $(1, 1)$. Hence $\mathfrak{S}(\mathfrak{B}) = \{(a, a) : a \in \mathfrak{U}\}$ is a Jordan subalgebra of $\mathfrak{B}^+$. Since $\mathfrak{U}^+ \cong \mathfrak{S}(\mathfrak{B}, j)$ under the mapping $a \mapsto (a, a)$, we have a second equivalent construction of $\mathfrak{U}^+$, $\mathfrak{U}$ associative.

IV. Let V be a vector space over a field Φ which is equipped with a symmetric bilinear form f , and let $J = \Phi 1 \oplus V$ be the vector space direct sum of V with the one-dimensional vector space $\Phi 1$ with basis $\{1\}$. We define a product on J by

$$(\alpha 1 + x) \cdot (\beta 1 + y) = (\alpha\beta + f(x, y))1 + (\beta x + \alpha y).$$

Then if Φ does not have characteristic 2, $(J, \cdot, 1)$ is a linear Jordan algebra. Let $V^\perp = \{x \in V : f(x, y) = 0 \text{ for all } y \in V\}$. Then f is nondegenerate, if $V^\perp = 0$. $V^\perp$ is a subspace of J which is properly contained in J , and if $\alpha 1 + x \in J$, $y \in V^\perp$, then $(\alpha 1 + x) \cdot (0 1 + y) = f(x, y)1 + \alpha y = \alpha y \in V^\perp$. Thus $V^\perp$ is an ideal of J , and so if J is simple, $V^\perp = 0$, and f is nondegenerate. Conversely, if f is nondegenerate and $\dim V > 1$, then J is simple.

2.6 Maximal Ideals

Let J be a quadratic Jordan algebra. Since 1 is quasi-invertible if and only if $J = 0$, we assume throughout this section that $J \neq 0$. Hence $1 \notin \mathcal{R}(J)$ so that $\mathcal{R}(J) \subset J$.

Definition 2.6.1. An ideal M of a quadratic Jordan algebra J is a maximal ideal, if $M \neq J$ and $M \subseteq N$, N an ideal of J , implies $N = M$ or $N = J$. An ideal P is prime, if A and B ideals of J such that $U_B(A) \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ (for equivalent conditions see [11]).

Before considering properties of maximal ideals, note that K an ideal of J implies $K \cap \{x \in J : x \text{ is invertible}\} = \emptyset$ or $K = J$. For if $x \in K$ is invertible, then $1 = U_x(x^{-2}) \in K$ so that $y = U_1(y) \in K$ for all $y \in J$.

Proposition 2.6.1. Let J be a quadratic Jordan algebra with Jacobson radical $\mathcal{R}(J)$. Then:

- (1) Every maximal ideal of J is prime.
- (2) M is a maximal ideal of J if and only if J/M is simple.
- (3) Every ideal properly contained in J is contained in a maximal ideal.
- (4) $\mathcal{R}(J)$ is contained in every maximal ideal of J , and if $\mathcal{R}(J)$ is itself a maximal ideal, then $\mathcal{R}(J)$ is the only maximal ideal of J .
- (5) If M is the unique maximal ideal of J , then every ideal properly contained in J is contained in M , and conversely.
- (6) $J/\mathcal{R}(J)$ is a division algebra if and only if $\mathcal{R}(J) = \{x \in J : x \text{ is not invertible}\}$. In this case $\mathcal{R}(J)$ is the unique

maximal ideal of J .

Proof. (1) Let M be a maximal ideal of J , and assume A and B are ideals of J such that $A \not\subseteq M$ and $B \not\subseteq M$. Then $M \subset A + M$ and $M \subset B + M$, so $A + M = B + M = J$. Hence $J = U_1(J) \subseteq U_J(J) \subseteq U_{B+M}(A + M)$. Let $a \in A$, $b \in B$, and $x, y \in M$. Then $U_{b+x}(a + y) = U_{b,x}(a + y) + U_b(a + y) + U_x(a + y) = U_{b,x}(a + y) + U_b(a) + U_b(y) + U_x(a + y) \in U_B(A) + M$. Thus $J \subseteq U_{B+M}(A + M) \subseteq U_B(A) + M$, which implies that $U_B(A) \not\subseteq M$, since $M \neq J$. Hence M is prime.

(2) K is an ideal of J such that $M \subset K$ if and only if K/M is a nonzero ideal of J/M . Thus the result is immediate.

(3) Let K be an ideal of J such that $K \neq J$. The set of all ideals containing K and properly contained in J is nonempty, since this set contains K . Thus Zorn's lemma can be used to imply the existence of a maximal element M of the set. That is $M \neq J$ is an ideal, $K \subseteq M$, and if $K \subseteq N$, N an ideal of J , then $N \subseteq M$ or $N = J$. But any ideal of J containing M must necessarily contain K , and so M is actually a maximal ideal of J .

(4) Suppose M is a maximal ideal of J and $\mathcal{R}(J) \not\subseteq M$. Then $M \subset \mathcal{R}(J) + M$, so that $\mathcal{R}(J) + M = J$. Hence there exists $z \in \mathcal{R}(J)$ and $x \in M$ such that $z + x = 1$, and so $x = 1 - z$ is invertible. This is a contradiction, and so $\mathcal{R}(J) \subseteq M$. If $\mathcal{R}(J)$ is a maximal ideal, then it is clear that $\mathcal{R}(J)$ is the only maximal ideal of J .

(5) If M is the only maximal ideal of J and K is any ideal different from J , then $K \subseteq M$ by (3).

(6) One direction is clear. Hence assume $J/\mathcal{R}(J)$ is a Jordan division algebra. If $x \notin \mathcal{R}(J)$, then $\theta(x) \neq 0$, where θ is the natural homomorphism from J onto $J/\mathcal{R}(J)$. Thus $\theta(x)$ is invertible

and there exists $y \in J$ such that $U_x(y) + \mathcal{R}(J) = 1$. Hence there exists $z \in \mathcal{R}(J)$ such that $U_x(y) = 1 - z$, which implies that $U_x(y)$ is invertible. By Proposition 2.3.1, x is invertible. The maximality of $\mathcal{R}(J)$ follows immediately.

We next turn to the relationship between maximal ideals of an associative algebra $\mathfrak{U}$ and the maximal ideals of $\mathfrak{U}^+$.

Proposition 2.6.2. Let $\mathfrak{U}$ be an associative algebra with 1. M is a maximal ideal of $\mathfrak{U}$ if and only if M is a maximal ideal of $\mathfrak{U}^+$. If M is the unique maximal ideal of $\mathfrak{U}^+$, then M is the unique maximal ideal of $\mathfrak{U}$. If $M = \mathcal{R}(\mathfrak{U})$ is a maximal ideal of $\mathfrak{U}$, then it is the unique maximal ideal of both $\mathfrak{U}$ and $\mathfrak{U}^+$.

Proof. Recall that $\mathfrak{U}^+/M \cong (\mathfrak{U}/M)^+$. By Corollary 2.5.1, $\mathfrak{U}/M$ is simple if and only if $(\mathfrak{U}/M)^+$ is simple. Thus $\mathfrak{U}/M$ is simple if and only if $\mathfrak{U}^+/M$ is simple and the first result follows from Proposition 2.6.1 (2). If M is the unique maximal ideal of $\mathfrak{U}^+$, then by Proposition 2.6.1 (5), M contains every ideal of $\mathfrak{U}^+$ which is properly contained in $\mathfrak{U}^+$. In particular maximal ideals of $\mathfrak{U}$ are such ideals, so M is the only maximal ideal of $\mathfrak{U}$. If $M = \mathcal{R}(\mathfrak{U})$ is maximal in $\mathfrak{U}$, then M is the unique maximal ideal of $\mathfrak{U}$, since $\mathcal{R}(\mathfrak{U})$ is contained in every maximal ideal of $\mathfrak{U}$. Since $\mathcal{R}(\mathfrak{U}) = \mathcal{R}(\mathfrak{U}^+)$ by Corollary 2.5.3, M is also the unique maximal ideal of $\mathfrak{U}^+$ by Proposition 2.6.1 (4).

Proposition 2.6.2 allows one to construct many examples of quadratic Jordan algebras with unique maximal ideals. Let $\mathfrak{U} = \{a/b \in \mathbb{Q} : b \not\equiv 0 \pmod{p}, p \text{ a fixed odd prime in } \mathbb{Z}\}$. Then $\mathfrak{U}$ is a unital algebra over $\mathbb{Z}$ with unique maximal ideal

$\mathcal{R}(\mathfrak{U}) = \{a/b \in \mathfrak{U} : a \equiv 0 \pmod{p}\}$. Now $\mathfrak{U}$ itself is a quadratic Jordan algebra, but we can derive a less trivial one by forming $(\mathfrak{U}_n)^+$, $n \geq 2$. Then $(\mathfrak{U}_n)^+$ has unique maximal ideal $\mathcal{R}(\mathfrak{U}_n^+)$
 $= \mathcal{R}(\mathfrak{U}_n) = \mathcal{R}(\mathfrak{U})_n$. Of course any associative algebra $\mathfrak{U}$ with identity such that $\mathcal{R}(\mathfrak{U})$ is a maximal ideal would do as well.

2.7. Linear Jordan Algebras over Fields

In this final section of Chapter 2, it is assumed that J is a linear Jordan algebra (with 1) over a field Φ of characteristic not 2. None of the proofs will be given for results whose proofs can be found in [5].

We first consider the associativity of certain finitely generated subalgebras of J . Recall that J is special, if there exists a monomorphism $\sigma : J \rightarrow \mathfrak{U}^+$, $\mathfrak{U}$ an associative algebra. Whenever this occurs in what follows, J will be identified with J^σ and 1 will be assumed to be the identity of $\mathfrak{U}$ (see [5], p. 10). For the sake of completeness, the fact that J is power-associative is restated in another form. Note: $\mathcal{B}$ a subalgebra of J implies $1 \in \mathcal{B}$.

Proposition 2.7.1. For every $x \in J$, the subalgebra generated by $\{1, x\}$ is associative.

Proposition 2.7.2. Let $R_J(\mathcal{B}) = \{R_b \in \text{Hom}_\Phi(J, J) : b \in \mathcal{B}\}$. If $\mathcal{B}$ is a subalgebra of J and X a set of generators of $\mathcal{B}$ containing 1, then the subalgebra of $\text{Hom}_\Phi(J, J)$ generated by $R_J(\mathcal{B})$ and Id_J is generated by $\{U_{x,y} : x, y \in X\}$.

Proposition 2.7.3. (Shirshov-Cohn). Any Jordan algebra (with 1) generated by two elements (and 1) is special.

Proposition 2.7.4. Let K be the subalgebra of J generated by $\{1, x, y\}$, where J is a subalgebra of $\mathfrak{U}^+$, $\mathfrak{U}$ associative. If $xy = yx$, then K is associative.

Proof. Note that J is a subalgebra of $\mathfrak{U}^+$, $\mathfrak{U}$ associative, by Proposition 2.7.3. We first show that $xy = yx$ implies R_x, R_y and $R_{x,y}$ commutes. For

$$\begin{aligned}
R_x R_y(z) &= (z \cdot y) \cdot x = \frac{1}{4}(zyx + yzx + xzy + xyz) \\
&= \frac{1}{4}(zxy + yzx + xzy + yxz) \\
&= (z \cdot x) \cdot y = R_y R_x(z),
\end{aligned}$$

and

$$\begin{aligned}
R_{x \cdot y} R_x(z) &= (z \cdot x) \cdot (x \cdot y) = \frac{1}{8}(zx^2y + zxyx + xzxy + xzyx + xyzx \\
&\quad + xyxz + yxz x + yx^2z) \\
&= \frac{1}{8}(zxyx + zyx^2 + xyzx + yxz x + xzxy + xzyx + x^2yz + xyxz) \\
&= [z \cdot (x \cdot y)] \cdot x = R_x R_{x \cdot y}(z) .
\end{aligned}$$

Similarly $R_{x \cdot y} R_y = R_y R_{x \cdot y}$. Now since $U_{a,b} = R_a R_b + R_b R_a - R_{a \cdot b}$, the set $\{U_{a,b} : a,b \in \{1,x,y\}\}$ is a commutative subset of $\text{Hom}_{\mathbb{F}}(J,J)$. Since the subalgebra of $\text{Hom}_{\mathbb{F}}(J,J)$ generated by $R_J(K)$ is generated by $\{U_{a,b} : a,b \in \{1,x,y\}\}$, this associative subalgebra is commutative, call it $\mathcal{B}$.

Suppose $a,b,c \in K$. Then $R_b, R_c \in \mathcal{B}$, and so $(a \cdot b) \cdot c = R_c R_b(a) = R_c R_a(b) = R_a R_c(b) = (b \cdot c) \cdot a = a \cdot (b \cdot c)$. Thus K is associative.

Proposition 2.7.5. Let K be the subalgebra of J generated by $\{1,x,y\}$. Then K is associative, if (i) x is invertible and $y = x^{-1}$, or (ii) x is quasi-invertible and y is the quasi-inverse of x .

Proof. Let $\mathfrak{U}$ be the associative algebra such that J is a subalgebra of $\mathfrak{U}^+$. By Proposition 2.7.4, it is sufficient to show that $xy = yx$ in each case. In case of (i) or (ii), x

and y are inverses (quasi-inverses) in $\mathfrak{U}$ by Proposition 2.5.1. Thus $xy = 1 = yx$, if (i) holds or $xy = x + y = yx$, if (ii) holds.

We next consider idempotents in J .

Definition 2.7.1. Let $(J, U, 1)$ be a quadratic Jordan algebra (no restriction on Φ). An element $e \in J$ is idempotent, if $e^2 = e$. Two idempotents e and f are orthogonal ($e \perp f$), if $e \circ f = U_e(f) = U_f(e) = 0$. A set of orthogonal idempotents $\{e_1, e_2, \dots, e_n\}$ is supplementary, if $\sum_{i=1}^n e_i = 1$.

It is immediate from the definition that e an idempotent implies $e^n = e$ for all $n \in \mathbb{N}$. Now if $\frac{1}{2} \in \Phi$, then e and f orthogonal implies $e \cdot f = \frac{1}{2} e \circ f = 0$. Conversely, if $e \cdot f = 0$, then $e \circ f = 2e \cdot f = 0$ and $U_e(f) = \frac{1}{2} [V_e^2(f) - V_{e^2}(f)] = \frac{1}{2} ((f \circ e) \circ e - f \circ e) = 0$; and similarly $U_f(e) = 0$. Thus $e \perp f$.

If e is an idempotent of J , then $U_e(J)$ is an inner ideal of J , since $x, y \in J$ implies $U_{U_e(x)}(y) = U_e U_x U_e(y) = U_e[U_x U_e(y)] \in U_e(J)$. Moreover, if $\tilde{U}_a = U_a|_{U_e(J)}$, $a \in U_e(J)$, then $(U_e(J), \tilde{U}, e)$ is a quadratic Jordan algebra. Since QJ2-QJ3 and QJ5-QJ8 are inherited by $\tilde{U}$ from U , all that needs to be shown is that $\tilde{U}_a$ is an endomorphism of $U_e(J)$ and $\tilde{U}_e = \text{Id}_{U_e(J)}$. But $U_{U_e(x)}[U_e(y)] = U_e U_x U_e U_e(y) = U_e(U_x U_e(y)) \in U_e(J)$ and $U_e U_e(x) = U_e(x)$ proves this is so. Thus if $\frac{1}{2} \in \Phi$, $U_e(J)$ is a linear Jordan algebra under $\cdot$. In particular, this also implies $x^2 = U_x(e)$ for $x \in U_e(J)$.

Let $\{e_1, e_2, \dots, e_n\}$ be a supplementary set of idempotents in J . Then for $i \neq j$, U_{e_i}, U_{e_i, e_j} form a set of pairwise orthogonal projections and $\text{Id}_J = \sum_{i=1}^n U_{e_i} + \sum_{i < j} U_{e_i, e_j}$. If $J_{ii} = U_{e_i}(J)$ and $J_{ij} = U_{e_i, e_j}(J)$, $i < j$, then we have $J = \sum_{i \leq j} \oplus J_{ij}$, which is called the Pierce decomposition of J relative to $\{e_1, e_2, \dots, e_n\}$. The inner ideals J_{ii} , $i = 1, 2, \dots, n$ are called the Pierce inner ideals determined by the e_i . As before we will be interested in the case where Φ is a field of characteristic not 2, and since the decomposition is a vector space direct sum, any J_{ij} , $i \leq j$ will be called a Pierce space of the decomposition. We have the following characterizations of the Pierce spaces for this case:

- (1) $J_{ii} = \{x \in J : x.e_i = x\}$
- (2) $J_{ij} = \{x \in J : x.e_i = \frac{1}{2}x = x.e_j\}.$

Proposition 2.7.6. Let J be a linear Jordan algebra over a field Φ of characteristic not 2, and let $J = \sum_{i \leq j} \oplus J_{ij}$ be the Pierce decomposition of J relative to the supplementary set of idempotents $\{e_1, e_2, \dots, e_n\}$. Then:

- (i) $J_{ii} \cdot J_{ii} \subseteq J_{ii}$, $i = 1, 2, \dots, n$,
- (ii) $J_{ij} \cdot J_{ii} \subseteq J_{ij}$, if $i \neq j$,
- (iii) $J_{ij} \cdot J_{ij} \subseteq J_{ii} + J_{jj}$, if $i \neq j$,
- (iv) $J_{ii} \cdot J_{jj} = 0$, if $i \neq j$,
- (v) $J_{ij} \cdot J_{jk} \subseteq J_{ik}$, if i, j, k are distinct,
- (vi) $J_{ij} \cdot J_{kk} = 0$, if i, j, k are distinct.

(vii) $J_{ij} \cdot J_{kl} = 0$, if i, j, k, l are distinct.

Proposition 2.7.7. Let J be a linear Jordan algebra with Pierce decomposition $J = \sum_{i \leq j} \oplus J_{ij}$ with respect to the supplementary set of idempotents $\{e_1, e_2, \dots, e_n\}$, and let K be an ideal of J . If $\theta : J \rightarrow J/K$ is the natural homomorphism and for $S \subseteq J$ we write $\theta(S) = \bar{S}$, then $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ is a supplementary set of idempotents of $\bar{J}$ and $\bar{J} = \sum_{i \leq j} \oplus \bar{J}_{ij}$ is the Pierce decomposition of $\bar{J}$ relative to this set of idempotents.

Proof. Since $\bar{e}_i^2 = \overline{e_i^2} = \bar{e}_i$ and $\bar{e}_i \cdot \bar{e}_j = \overline{e_i \cdot e_j} = \bar{0}$, for $i \neq j$, $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ is a set of orthogonal idempotents of $\bar{J}$. This set is supplementary, since $\sum_{i=1}^n \bar{e}_i = \overline{\sum_{i=1}^n e_i} = \bar{1}$. If $\bar{x} \in (\bar{J})_{ii} = \bar{U}_{\bar{e}_i}(\bar{J})$, then $\bar{x} = \bar{U}_{\bar{e}_i}(\bar{y})$, $y \in J$. Hence $\bar{x} = U_{e_i}(y) + K \in \bar{J}_{ii}$. Conversely, if $\bar{x} \in \bar{J}_{ii}$, then $\bar{x} = U_{e_i}(y) + K = \bar{U}_{\bar{e}_i}(\bar{y}) \in \bar{U}_{\bar{e}_i}(\bar{J}) = (\bar{J})_{ii}$, $y \in J$. Hence $(\bar{J})_{ii} = \bar{J}_{ii}$. Now if $\bar{x} \in (\bar{J})_{ij} = \bar{U}_{\bar{e}_i, \bar{e}_j}(\bar{J})$, $i \neq j$, then $\bar{x} = \bar{U}_{\bar{e}_i, \bar{e}_j}(\bar{y}) = U_{e_i, e_j}(y) + K$, $y \in J$. Thus $\bar{x} \in \overline{U_{e_i, e_j}(J)} = \bar{J}_{ij}$. Conversely, if $\bar{x} \in \bar{J}_{ij}$, then $\bar{x} = U_{e_i, e_j}(y) + K = \bar{U}_{\bar{e}_i, \bar{e}_j}(\bar{y}) \in (\bar{J})_{ij}$. Hence $(\bar{J})_{ij} = \bar{J}_{ij}$.

Definition 2.7.2. Two nonzero orthogonal idempotents $e_i, e_j \in J$ are called connected, if there exists $u_{12} \in U_{e_1, e_2}(J)$ which is invertible in $U_{e_1+e_2}(J)$. e_1 and e_2 are strongly connected, if there exists an element $u_{12} \in U_{e_1, e_2}(J)$ such that $u_{12}^2 = e_1 + e_2$.

The relation of connectedness is transitive.

Proposition 2.7.8. Let e_1, e_2, e_3 be nonzero orthogonal idempotents such that e_1 and e_2 are connected (strongly connected) and e_2 and e_3 are connected (strongly connected). Then e_1 and e_3 are connected (strongly connected).

The usefulness of connected idempotents will become apparent after Jordan matrix algebras have been defined. Let $\mathcal{A}$ be an algebra over a field Φ of characteristic not 2 with identity element 1 and involution $j : d \rightarrow d^j = \hat{d}$. Let $\mathcal{A}_n$ be the algebra of $n \times n$ matrices with entries in $\mathcal{A}$. Let $a_i \in N(\mathcal{A}) \cap \mathfrak{S}(\mathcal{A})$, $i = 1, 2, \dots, n$ ($N(\mathcal{A})$ is the nucleus of $\mathcal{A}$ and $\mathfrak{S}(\mathcal{A}) = \{d \in \mathcal{A} : \hat{d} = d\}$), and assume a_i is invertible in $N(\mathcal{A})$. If $a = \text{diag} \{a_1, a_2, \dots, a_n\}$, then we define an involution on $\mathcal{A}_n$ called a canonical involution by $j_a : X \rightarrow a^{-1} \hat{X}^t a$, $X \in \mathcal{A}_n$, where $\hat{X}^t$ is the conjugate transpose of X under j . If $a = I_n$, then the involution is called standard. Let $\mathfrak{S}(\mathcal{A}_n, j_a)$ denote the set of symmetric elements of $\mathcal{A}_n$. If $\mathcal{A}_n^+$ is the algebra which has the same underlying vector space structure as $\mathcal{A}_n$, but multiplication defined by $X.Y = \frac{1}{2}(XY + YX)$, then $\mathfrak{S}(\mathcal{A}_n, j_a)$ is a subalgebra of $\mathcal{A}_n^+$. We will write $\mathfrak{S}(\mathcal{A}_n)$ for $\mathfrak{S}(\mathcal{A}_n, j_{I_n})$.

The following notation will be used for elements of $\mathcal{A}_n$ and $\mathfrak{S}(\mathcal{A}_n, j_a)$ wherever these algebras are encountered. Let e_{ij} , $i, j = 1, 2, \dots, n$ be the element of $\mathcal{A}_n$ with 1 in the (i, j) position and 0 elsewhere, and if $x \in \mathcal{A}$, identify x with $\sum_{i=1}^n x e_{ii} = \text{diag} \{x, x, \dots, x\}$. Then $x e_{ij}$ is the matrix with x in the (i, j) position and 0 elsewhere, and for $x \in \mathcal{A}$ we put

$$(JMA0) \quad x[ij] = x e_{ij} + (x e_{ij})^{j_a} = x e_{ij} + (a_j^{-1} \hat{x} a_i) e_{ji}, \quad i, j = 1, 2, \dots, n.$$

Since the characteristic of Φ is not 2, $\mathfrak{Q}(\mathfrak{A}_n, j_a) = \{X + X^{j_a} : X \in \mathfrak{A}_n\}$, and hence every element of $\mathfrak{Q}(\mathfrak{A}_n, j_a)$ is a sum of elements $x[ij]$, $x \in \mathfrak{A}$, $i, j = 1, 2, \dots, n$. If $\mathfrak{Q}_{ij} = \{x[ij] : x \in \mathfrak{A}\}$, then $\mathfrak{Q}_{ij} = \mathfrak{Q}_{ji}$ and $\mathfrak{Q}(\mathfrak{A}_n, j_a) = \sum_{i \leq j=1}^n \oplus \mathfrak{Q}_{ij}$. The following multiplication rules hold in $\mathfrak{Q}(\mathfrak{A}_n, j_a)$:

$$(JMA1) \quad 2 x[ij].y[jk] = xy[ik], \text{ if } i, j, k \text{ are distinct};$$

$$(JMA2) \quad 2 x[ii].y[ij] = (xy + (a_i^{-1} \hat{x} a_i)y)[ij], \text{ if } i \neq j;$$

$$(JMA3) \quad 2 x[ij].y[ji] = xy[ii] + yx[jj], \text{ if } i \neq j;$$

$$(JMA4) \quad 2 x[ii].y[ii] = (x + a_i^{-1} \hat{x} a_i).(y + a_i^{-1} \hat{y} a_i)[ii].$$

Also we have $x[ij] = (a_j^{-1} \hat{x} a_j)[ji]$, and $x[ij].y[kl] = 0$, if $\{i, j\} \cap \{k, l\} = \emptyset$.

The next proposition determines when $\mathfrak{Q}(\mathfrak{A}_n, j_a)$ is a (linear) Jordan algebra for $n \geq 3$.

Proposition 2.7.9. $\mathfrak{Q}(\mathfrak{A}_n, j_a)$ for $n \geq 3$ is Jordan if and only if $(\mathfrak{A}, j)$ is associative or $n = 3$ and $(\mathfrak{A}, j)$ is alternative with symmetric elements in the nucleus.

For $n = 2$, $\mathfrak{Q}(\mathfrak{A}_n, j_a)$ is clearly Jordan if $(\mathfrak{A}, j)$ is associative (see Section 2.5), and for $n = 1$, nothing more can be said than to state the Jordan conditions for $\mathfrak{Q}(\mathfrak{A}_n, j_a)$. In any case, whenever $\mathfrak{Q}(\mathfrak{A}_n, j_a)$ is Jordan, $\mathfrak{Q}(\mathfrak{A}_n, j_a)$ is called a Jordan matrix algebra of order n .

Proposition 2.7.10. Let $\mathfrak{Q} = \mathfrak{Q}(\mathfrak{A}_n, j_a)$ be a Jordan matrix algebra of order $n \geq 3$ defined by the canonical involution j_a in $\mathfrak{A}_n$ such that $a_1 = 1$. Then the mapping $\mathcal{B} \rightarrow \mathcal{B}_n \cap \mathfrak{Q}$ is a lattice isomorphism of the lattice of subalgebras $\mathcal{B}$ of $(\mathfrak{A}, j)$ containing

$a_i, a_i^{-1}, i = 1, 2, \dots, n$ onto the lattice of subalgebras of $\mathfrak{Q}$ containing the elements $1[i, j], i, j = 1, 2, \dots, n$. Also the mapping $\mathcal{B} \rightarrow \mathcal{B}_n \cap \mathfrak{Q}$ is a lattice isomorphism of the lattice of ideals $\mathcal{B}$ of $(\mathcal{A}, j)$ onto the lattice of ideals of $\mathfrak{Q}$, and $\mathcal{B} = \{d \in \mathcal{A} : d[ij] \in \mathcal{B}_n \cap \mathfrak{Q} \text{ for every } (i, j) \in \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}\}$. In either the ideal case or subalgebra case, $\mathcal{B}$ is characterized by $\mathcal{B} = \{d \in \mathcal{A} : d \text{ is an entry of a matrix in } \mathcal{B}_n \cap \mathfrak{Q}\}$.

Note that there is no loss of generality in assuming $a_1 = 1$ in the preceeding proposition (see [5], p. 128).

Proposition 2.7.11. Let $\mathfrak{Q}(\mathcal{A}_n, j_a)$ and $\mathfrak{Q}(\mathcal{B}_n, k_b)$ be Jordan matrix algebras of order $n \geq 3$ determined by canonical involutions j_a and k_b respectively where the first is defined by an involution j in $\mathcal{A}$ and a diagonal matrix a with $a_1 = 1$ and the second by an involution k in $\mathcal{B}$ and a diagonal matrix b with $b_1 = 1$.

If η is a homomorphism of $(\mathcal{A}, j)$ into $(\mathcal{B}, k)$ such that $a_i^\eta = b_i, i = 1, \dots, n$, then the restriction σ to $\mathfrak{Q}(\mathcal{A}_n, j_a)$ of the mapping $(d_{ij}) \rightarrow (d_{ij}^\eta)$ of $\mathcal{A}_n$ is a homomorphism of $\mathfrak{Q}(\mathcal{A}_n, j_a)$ into $\mathfrak{Q}(\mathcal{B}_n, k_b)$ such that $1[ij]^\sigma = 1\{ij\}$, where $d[ij]$ and $e\{ij\}$ are defined by (JMA0) in $\mathfrak{Q}(\mathcal{A}_n, j_a)$ and $\mathfrak{Q}(\mathcal{B}_n, k_b)$ respectively.

Conversely, if σ is a homomorphism of $\mathfrak{Q}(\mathcal{A}_n, j_a)$ into $\mathfrak{Q}(\mathcal{B}_n, k_b)$ such that $1[ij]^\sigma = 1\{ij\}, i, j = 1, \dots, n$, then there exists a homomorphism η of $(\mathcal{A}, j)$ into $(\mathcal{B}, k)$ such that $a_i^\eta = b_i, i = 1, 2, \dots, n$ and σ is the restriction to $\mathfrak{Q}(\mathcal{A}_n, j_a)$ of the mapping $(d_{ij}) \rightarrow (d_{ij}^\eta)$ of $\mathcal{A}_n$.

Corollary 2.7.1. Let $\mathfrak{Q} = \mathfrak{Q}(\mathfrak{A}_n, j_a)$ be a Jordan matrix algebra of order $n \geq 3$ determined by a canonical involution j_a with $a_1 = 1$, and let $K = \mathfrak{Q}_n \cap \mathfrak{Q}$ be an ideal of $\mathfrak{Q}$. Then $\mathfrak{Q}/K \cong \mathfrak{Q}(\mathfrak{A}/\mathfrak{Q})_n, j_{\bar{a}})$, where $(\mathfrak{A}/\mathfrak{Q}, j)$ is the image of $(\mathfrak{A}, j)$ under the natural homomorphism η and $\bar{a} = \text{diag} \{a_1^\eta, a_2^\eta, \dots, a_n^\eta\}$.

Proof. By Proposition 2.7.11, there exists a homomorphism σ from $\mathfrak{Q}(\mathfrak{A}_n, j_a)$ into $\mathfrak{Q}((\mathfrak{A}/\mathfrak{Q})_n, j_{\bar{a}})$ which is the restriction of the homomorphism $(d_{ij}) \rightarrow (d_{ij}^\eta)$. Since η onto implies that $(d_{ij}^\eta) \rightarrow (d_{ij}^\eta)$ is onto, σ is clearly onto. Let $\theta : \mathfrak{A}_n \rightarrow \mathfrak{A}_n/\mathfrak{Q}_n \cong (\mathfrak{A}/\mathfrak{Q})_n$ be the natural map. Then $\ker \sigma = \ker \theta \cap \mathfrak{Q} = \mathfrak{Q}_n \cap \mathfrak{Q} = K$, and hence $J/K \cong \mathfrak{Q}((\mathfrak{A}/\mathfrak{Q})_n, j_{\bar{a}})$.

Proposition 2.7.12. (Coordinatization Theorem). Let J be a linear Jordan algebra over a field Φ of characteristic not 2, let $\{e_1, e_2, \dots, e_n\}$ be a supplementary set of nonzero idempotents of J , $n \geq 3$, and let $J = \sum_{i \leq j} \oplus J_{ij}$ be the corresponding Pierce decomposition. If for $j = 2, 3, \dots, n$ there exists $u_{1j} \in J_{1j}$ which is invertible in $J_{11} + J_{jj} + J_{1j}$, then there exists a Jordan matrix algebra $\mathfrak{Q}(\mathfrak{A}_n, j_a)$ and an isomorphism ζ of J onto $\mathfrak{Q}(\mathfrak{A}_n, j_a)$ such that $e_i^\zeta = \frac{1}{2}[ii]$ and $u_{1j}^\zeta = 1[1j]$, $j = 2, 3, \dots, n$. If in addition $u_{1j}^2 = e_1 + e_j$, $j = 2, 3, \dots, n$, then the involution j_a is standard and $J \cong \mathfrak{Q}(\mathfrak{A}_n)$.

Since supplementary connected idempotents satisfy the first hypothesis and supplementary strongly connected idempotents the latter in addition, it follows immediately that if J has a supplementary set of connected (strongly connected) idempotents $\{e_1, e_2, \dots, e_n\}$, $n \geq 3$, then J is isomorphic to a Jordan matrix

algebra $\mathfrak{Q}(\mathfrak{A}_n, j_a)$ ($\mathfrak{Q}(\mathfrak{A}_n)$).

Lemma 2.7.1. Let $(J, U, 1)$ be a quadratic Jordan algebra and $e \neq 0$ an idempotent in J . If $K = U_e(J)$, then $\mathcal{R}(J) \cap K \subseteq \mathcal{R}(K)$.

Proof. If $U_e(x) \in \mathcal{R}(J) \cap K$, then for any $U_e(y) \in K$, $U_{U_e(x)}(U_e(y)) = U_e U_x U_e(y) \in \mathcal{R}(J) \cap K$, and $U_{U_e(y)}(U_e(x)) = U_e U_y U_e(x) \in \mathcal{R}(J) \cap K$, since $\mathcal{R}(J)$ is an ideal and $K = U_e(J)$. Thus $\mathcal{R}(J) \cap K$ is an ideal of K . Now $U_e(x) \in \mathcal{R}(J) \cap K$ implies $U_e(x)$ is quasi-invertible in J with quasi-inverse y . Then $y = U_{1-y}[(U_e(x))^2 - U_e(x)] = U_{1-y}U_e(U_x(e) - x)$, since $(U_e(x))^2 = U_{U_e(x)}(e) = U_e U_x U_e(e) = U_e U_x(e)$. Thus $U_e(y) = U_e U_{1-y}U_e(U_x(e) - x) = U_e U_{1-y}U_e^2(U_x(e) - x) = U_{U_e(1-y)}(U_e U_x(e) - U_e(x)) = U_{e-U_e(y)}[(U_e(x))^2 - U_e(x)]$. Therefore $U_e(x)$ has quasi-inverse $U_e(y) \in K$, and $\mathcal{R}(J) \cap K$ is a quasi-invertible ideal of K . Thus $\mathcal{R}(J) \cap K \subseteq \mathcal{R}(K)$.

The next proposition plays a fundamental role in obtaining the results of Chapter 4.

Proposition 2.7.13. Let J be a linear Jordan algebra, K a quasi-invertible ideal of J such that $\bar{J} \equiv J/K \cong \mathfrak{Q}(\mathfrak{A}_n, j_b)$, a Jordan matrix algebra of order $n \geq 3$ determined by a canonical involution, and let $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ be a supplementary set of connected idempotents such that $\bar{e}_i \rightarrow \frac{1}{2}[ii]$ under the isomorphism of $\bar{J}$ onto $\mathfrak{Q}(\mathfrak{A}_n, j_b)$ for $i = 1, 2, \dots, n$. If J contains a supplementary set of connected idempotents $\{f_1, f_2, \dots, f_n\}$ satisfying $\bar{f}_i = \bar{e}_i$ for $i = 1, 2, \dots, n$, then

- (1) $J \cong \mathfrak{Q}(\mathfrak{A}_n, j_a)$, a Jordan matrix algebra,
- (2) $(\mathcal{G}, j) \cong (\mathfrak{A}/\mathfrak{F}, j)$, where $\mathfrak{F}$ is the ideal of $\mathfrak{A}$ such that

$K \leftrightarrow \mathfrak{F}_n \cap \mathfrak{Q}(\mathfrak{A}_n, j_a)$ under the isomorphism of J onto

$\mathfrak{Q}(\mathcal{D}_n, j_a)$, and $b = \text{diag} \{a_1 + \mathfrak{F}, a_2 + \mathfrak{F}, \dots, a_n + \mathfrak{F}\}$.

Proof. Without loss of generality we may let $e_i = f_i$, $i = 1, 2, \dots, n$.

Thus, if $J = \sum_{i \leq j} \oplus J_{ij}$ is the Pierce decomposition of J with respect to the set $\{e_1, e_2, \dots, e_n\}$, then by Proposition 2.7.6,

$\bar{J} = \sum_{i \leq j} \oplus \bar{J}_{ij}$ is the Pierce decomposition of $\bar{J}$ with respect

to $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$. Let τ be the given isomorphism of $\bar{J}$ onto

$\mathfrak{Q}(\mathcal{D}_n, j_b)$. Then $\tau(\bar{e}_i) = \frac{1}{2} \{ii\}$ (as defined in (JMA0)), $i = 1, 2, \dots, n$.

Also for every $j = 2, 3, \dots, n$, there exists $\bar{u}_{1j} \in \bar{J}_{1j}$ such that

$\tau(\bar{u}_{1j}) = \bar{l}\{1j\}$, and $\bar{u}_{1j}$ is invertible in $\bar{J}_{11} + \bar{J}_{1j} + \bar{J}_{jj}$
 $= \bar{u}_{\bar{e}_1 + \bar{e}_j}(\bar{J})$. Let $\bar{e} = \bar{e}_1 + \bar{e}_j$. Then there exists $v \in J$ such
 that $U_{u_{1j}}(v) = e + z$, $z \in \mathcal{R}(J)$ (since $K \subseteq \mathcal{R}(J)$). Hence by

Lemma 2.3.1, $U_{u_{1j}}(v)$ is invertible in $J_{11} + J_{1j} + J_{jj} = U_e(J)$,

and so u_{1j} is invertible in $J_{11} + J_{1j} + J_{jj}$, by Proposition

2.3.1. For $j = 2, 3, \dots, n$, let u_{j1} be the inverse of u_{1j} in
 $J_{11} + J_{1j} + J_{jj}$. Then $\tau(\bar{u}_{j1}) = \bar{l}\{j1\}$.

Now by hypothesis the e_i are connected and supplementary,

so by Proposition 2.7.12, there exists a Jordan matrix algebra

$\mathfrak{Q}(\mathcal{D}_n, j_a)$ and an isomorphism ζ of J onto $\mathfrak{Q}(\mathcal{D}_n, j_a)$ such that

$\zeta(e_i) = \frac{1}{2} [ii]$, $i = 1, 2, \dots, n$, and $\zeta(u_{1j}) = l[1j]$, $j = 2, 3, \dots, n$.

Let $\nu : J \rightarrow \bar{J}$ be the natural homomorphism. Then $\sigma = \tau \circ \nu \circ \zeta^{-1}$

is a homomorphism of $\mathfrak{Q}(\mathcal{D}_n, j_a)$ onto $\mathfrak{Q}(\mathcal{D}_n, j_b)$ such that

$$(i) \quad \sigma\left(\frac{1}{2}[ii]\right) = \tau \circ \nu \circ \zeta^{-1}\left(\frac{1}{2}[ii]\right) = \tau \circ \nu(e_i) = \tau(\bar{e}_i) = \frac{1}{2}\{ii\},$$

$$(ii) \quad \sigma(l[1j]) = \tau \circ \nu \circ \zeta^{-1}(l[1j]) = \tau \circ \nu(u_{1j}) = \tau(\bar{u}_{1j}) = \bar{l}\{1j\},$$

$$(iii) \quad \sigma(l[j1]) = \bar{l}\{j1\}, \text{ as in (ii).}$$

Hence by Proposition 2.7.11, there exists a homomorphism η of $(\mathcal{A}, j)$ into $(\mathcal{B}, j)$ satisfying $\eta(a_i) = b_i$, $i = 1, 2, \dots, n$ and σ is the restriction to $\mathfrak{Q}(\mathcal{A}_n, j_a)$ of $(d_{ij}) \rightarrow (d_{ij}^\eta)$. Since σ is onto, η is onto, and $(\mathcal{B}, j) \cong (\mathcal{A}, j)/\ker \eta$. Let $\mathfrak{F} = \ker \eta$. Hence $\zeta(K) = \ker \sigma = (\ker \eta)_n \cap \mathfrak{Q}(\mathcal{A}_n, j_a) = \mathfrak{F}_n \cap \mathfrak{Q}(\mathcal{A}_n, j_a)$, since σ is the restriction of the homomorphism $(d_{ij}) \rightarrow (d_{ij}^\eta)$.

We next identify the Jacobson radical of a Jordan matrix algebra for $n \geq 3$, and $\mathcal{A}$ associative.

Proposition 2.7.14. Let $(\mathcal{A}, j)$ be an associative algebra with involution and identity over a field $\mathfrak{F}$ of characteristic not 2. Then $\mathfrak{Q} = \mathfrak{Q}(\mathcal{A}_n, j_a)$, j_a a canonical involution and $n \geq 3$, satisfies $R(\mathfrak{Q}) = R(\mathcal{A})_n \cap \mathfrak{Q}$.

Proof. By Proposition 2.7.10, there exists an ideal $\mathcal{G}$ of $(\mathcal{A}, j)$ such that $R(\mathfrak{Q}) = \mathcal{G}_n \cap \mathfrak{Q}$. Now $\mathcal{A}$ associative with 1 implies $R(\mathcal{A})_n = R(\mathcal{A}_n)$, and hence $R(\mathcal{A})_n \cap \mathfrak{Q} = R(\mathcal{A}_n) \cap \mathfrak{Q}$ is a quasi-invertible ideal of $\mathfrak{Q}$. Thus $R(\mathcal{A})_n \cap \mathfrak{Q} \subseteq R(\mathfrak{Q})$. Suppose $x \in \mathcal{G}$. Then $x[12] \in \mathcal{G}_n \cap \mathfrak{Q} = R(\mathfrak{Q})$. Also $u = 1[12] + \sum_{i=3}^n e_{ii} \in \mathfrak{Q}$ and $u^2 = I_n$, so u is invertible in $\mathcal{A}_n$ and hence in $\mathfrak{Q}$. Since $x[12]$ is an element of the quasi-invertible ideal $R(\mathfrak{Q})$, $u - x[12] = (1 - x)[12] + \sum_{i=3}^n e_{ii}$ is invertible in $\mathfrak{Q}$. One verifies by direct computation that the inverse of $u - x[12]$ has the form $(1 - y)[12] + \sum_{i=3}^n e_{ii}$, where $1 - y = (1 - x)^{-1}$. Therefore x is quasi-invertible in $\mathcal{A}$ and $\mathcal{G}$ is a quasi-invertible ideal. Hence $\mathcal{G} \subseteq R(\mathcal{A})$ and $R(\mathfrak{Q}) = \mathcal{G}_n \cap \mathfrak{Q} \subseteq R(\mathcal{A})_n \cap \mathfrak{Q}$.

Finally, we consider linear Jordan algebras which satisfy minimum conditions on inner ideals.

Definition 2.7.3. Let J be a linear Jordan algebra over a field Φ of characteristic not 2. An inner ideal B of J is called a minimal inner ideal, if $B \neq 0$ and C an inner ideal of J such that $0 \subseteq C \subseteq B$ implies $C = 0$ or $C = B$. J satisfies the minimum condition (for inner ideals), if (i) there exist no properly decreasing sequence $U_{e_1}(J) \supset U_{e_2}(J) \supset \dots, e_i^2 = e_i$, and (ii) every inner ideal $U_e(J)$, $e^2 = e \neq 0$, contains a minimal inner ideal.

Jordan algebras will now be considered which satisfy the following axioms:

- (i) J has an identity;
- (ii) J is nondegenerate; i.e., J has no nonzero absolute zero divisors;
- (iii) J satisfies the minimum condition .

Recall that $J/\mathcal{R}(J)$ is nondegenerate, and therefore, if J is a linear Jordan algebra such that $J/\mathcal{R}(J)$ satisfies the minimum condition , then $J/\mathcal{R}(J)$ satisfies the axioms (i)-(iii). This case becomes of prime importance in Chapter 4.

Definition 2.7.4. Let J be a linear Jordan algebra. An idempotent $e \in J$ is primitive, if $e \neq 0$ and e cannot be written as $e = e_1 + e_2$, where e_1 and e_2 are nonzero orthogonal idempotents. An idempotent $e \in J$ is completely primitive, if $e \neq 0$ and $U_e(J)$ is a division algebra. J has (finite) capacity n , if J contains a supplementary set $\{e_1, e_2, \dots, e_n\}$ of completely primitive orthogonal idempotents and n is minimal with respect to this property.

Note that a nonzero idempotent e is primitive if and only if $U_e(J)$ contains no idempotents $\neq 0, e$, and J has capacity 1 if and only if J is a division algebra.

Proposition 2.7.15. Any Jordan algebra J satisfying axioms (i)-(iii) has a finite capacity.

Proposition 2.7.16. (Second Structure Theorem). The following conditions on a Jordan algebra J are equivalent: (1) J is a simple algebra satisfying axioms (i)-(iii); (2) J is either a division algebra, a Jordan algebra of a nondegenerate symmetric bilinear form in a vector space V over an extension field Γ with $\dim V/\Gamma > 1$ (only if the capacity of $J = 2$), a Jordan matrix algebra $\mathfrak{S}(\mathfrak{A}_n, j_a)$ where $n \geq 2$ and $(\mathfrak{A}, j)$ is either $\Delta \oplus \Delta^\circ$ with Δ an associative division algebra and j the exchange involution, an associative division algebra with involution, a split quaternion algebra over an extension field ($n = 2$ only), an algebra of octonians over an extension field with standard involution (only if $n = 3$); (3) J is either a division algebra, a Jordan algebra of a nondegenerate symmetric bilinear form in a vector space V over an extension field Γ with $\dim V/\Gamma > 1$, a Jordan matrix algebra $\mathfrak{S}(\mathfrak{A}_3, j_Y)$ where $\mathfrak{A}$ is an octonian algebra over an extension field with standard involution and j_Y is a canonical involution, or an algebra $\mathfrak{S}(\mathfrak{U}, j)$ where $(\mathfrak{U}, j)$ is simple Artinian with involution.

CHAPTER 3

COMPLETIONS OF QUADRATIC JORDAN ALGEBRAS

3.1 Topological Modules

Let Φ be a commutative associative ring with identity and J a unital Φ -module. If $J = K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$ is any decreasing sequence of submodules of J , then it is well-known that the set $\{x + K_i : i = 0, 1, 2, \dots \text{ and } x \in J\}$ forms a basis for a topology on J . If S is any subset of J , then

$$cl_J S = \bigcap_{i=0}^{\infty} (S + K_i) \text{ so that } S \text{ is closed if and only if}$$

$$S = \bigcap_{i=0}^{\infty} (S + K_i). \text{ In particular every open subset of } J \text{ is closed.}$$

Clearly the topology is l^0 countable, and it can be shown that the topology is Hausdorff if and only if $\bigcap_{i=0}^{\infty} K_i = 0$ (see [12]).

Let $\mathcal{T}$ be the topology induced by a decreasing sequence of submodules of J as above, and assume $\bigcap_{i=0}^{\infty} K_i = 0$. Since $x, y \in J$ are distinct if and only if $x - y \notin \bigcap_{i=0}^{\infty} K_i$, $x \neq y$ implies there exists $k \in \{0, 1, 2, \dots\}$ such that $x - y \in K_k - K_{k+1}$. Let e be any fixed real number greater than 1, and define $d : J \times J \rightarrow \mathbb{R}$ by (i) $d(x, y) = 0$, if $x = y$, (ii) $d(x, y) = e^{-k}$, if $x \neq y$ and $x - y \in K_k - K_{k+1}$. Then it is well-known that (J, d) is a metric space and $\mathcal{T}(d) = \mathcal{T}$. We now establish some useful inequalities for d .

Proposition 3.1.1. (i) For every $x, y, z \in J$, $d(x, z)$

$$\leq \max \{d(x, y), d(y, z)\}.$$

(ii) For every $x, w, y, z \in J$, $d(x + y, w + z)$

$$\leq \max \{d(x, w), d(y, z)\}.$$

(iii) For every $x, y \in J$ and $\alpha \in \Phi$, $d(\alpha x, \alpha y) \leq d(x, y)$.

Proof. (i) If $x = z$, then $d(x, z) = 0 \leq d(x, y) = d(z, y) = d(y, z)$.

If $x = y$, then $d(x, z) = d(y, z)$ and $d(x, y) = 0 \leq d(y, z)$.

Similarly if $y = z$, and (i) holds for these special cases. Now assume x, y, z are distinct. Then $d(x, y) = e^{-k_1}$, $d(y, z) = e^{-k_2}$, and $d(x, z) = e^{-k_3}$, where $x - y \in K_{k_1} - K_{k_1+1}$, $y - z \in K_{k_2} - K_{k_2+1}$ and $x - z \in K_{k_3} - K_{k_3+1}$. Thus $x - z = (x - y) + (y - z)$

$\in K_{\min\{k_1, k_2\}}$, so that $K_{k_3} \subseteq K_{\min\{k_1, k_2\}}$. Hence $k_3 \geq k_1$ or $k_3 \geq k_2$ and thus $d(x, z) = e^{-k_3} \leq \max \{e^{-k_1}, e^{-k_2}\} = \max \{d(x, y), d(y, z)\}$.

(ii) If $x = w$, then $(x + y) - (w + z) = y - z$ and so

$$d(x + y, w + z) = d(y, z) \leq \max \{d(x, w), d(y, z)\}.$$

Similarly, if $y = z$, the result holds. Hence assume $x \neq w$ and $y \neq z$. Then

$$d(x, w) = e^{-k_1}, d(y, z) = e^{-k_2}, \text{ where } x - w \in K_{k_1} - K_{k_1+1} \text{ and}$$

$$y - z \in K_{k_2} - K_{k_2+1}. \text{ So } (x + y) - (w + z) = (x - w) + (y - z)$$

$\in K_{\min\{k_1, k_2\}}$ and the result follows as in (i).

(iii) If $x = y$, then $d(\alpha x, \alpha y) = 0 = d(x, y)$. If $x \neq y$, then

$$d(x, y) = e^{-k}, \text{ where } x - y \in K_k - K_{k+1}. \text{ Hence } \alpha x - \alpha y = \alpha(x - y) \in K_k, \text{ and } d(\alpha x, \alpha y) \leq e^{-k} = d(x, y).$$

From Proposition 3.1.1 (ii), it follows easily that the mapping $(x, y) \rightarrow x + y$ from $J \times J \rightarrow J$ is (uniformly) continuous.

The mapping $(\alpha, x) \rightarrow \alpha x$ from $\Phi \times J \rightarrow J$ is also continuous when Φ has a suitable topology: in particular, if Φ is given one of the trivial topologies. Since we are primarily concerned with a ring theoretic structure, the continuity of the module operation plays no role. Thus a module with a continuous addition will be called a topological module in this thesis.

The notions of sequence, Cauchy sequence, series, and convergence will be taken to be the usual ones associated with metric spaces. Since $\mathcal{T}(d) = \mathcal{T}$ it is convenient to have these ideas stated in terms of the topology $\mathcal{T}$.

Proposition 3.1.2. Let $J = K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$ be a decreasing sequence of submodules of J such that $\bigcap_{i=0}^{\infty} K_i = 0$. Then:

- (1) If $\{x_n\}$ is a sequence in J , then $\lim_n x_n = x$ if and only if for every nonnegative integer k there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \equiv x \pmod{K_k}$.
- (2) $\{x_n\}$ is a Cauchy sequence if and only if for every $k \in \{0, 1, 2, \dots\}$ there exists $N \in \mathbb{N}$ such that $n, m \geq N$ implies $x_n \equiv x_m \pmod{K_k}$.

- (3) If every Cauchy sequence in J converges in J , then $\sum_{i=0}^{\infty} x_i \equiv \lim_n \left(\sum_{i=0}^n x_i \right)$ converges if and only if $\lim_i x_i = 0$.

Proof. (1) $\lim_n x_n = x$ if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $d(x_n, x) < \epsilon$. Suppose $\lim_n x_n = x$ and let k be given. Choose $\epsilon > 0$ so that $\epsilon \leq e^{-k}$. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $d(x_n, x) < \epsilon$, which implies $d(x_n, x) < e^{-k}$. Hence for $n \geq N$, $x_n \equiv x \pmod{K_k}$. Conversely, let $\epsilon > 0$ be given. Then for some k , $e^{-k} < \epsilon$.

Hence for this k there exists $N \in \mathbb{N}$ so that $x_n \equiv x \pmod{K_k}$, if $n \geq N$; i.e., if $n \geq N$, $d(x_n, x) \leq e^{-k} < \epsilon$. Hence $\lim_n x_n = x$.

(2) Proof is similar to (1).

(3) All that remains to be shown is that $\lim_i x_i = 0$ implies $\sum_{i=0}^{\infty} x_i$ converges. This will be accomplished by showing that the sequence of partial sums is a Cauchy sequence. Thus let k be any nonnegative integer. Since $\lim_i x_i = 0$, there exists $N \in \mathbb{N}$ such that $i \geq N$ implies $x_i \equiv 0 \pmod{K_k}$. Thus for $n \geq m \geq N$, $\sum_{i=0}^n x_i - \sum_{i=0}^m x_i = \sum_{i=m+1}^n x_i \equiv 0 \pmod{K_k}$, and therefore the sequence of partial sums is Cauchy by (2).

Using the standard metric space arguments, there exists a metric space (J^*, d^*) which is the completion of (J, d) in that (i) (J^*, d^*) is a complete metric space, (ii) d^* coincides with d on $J \times J$, (iii) $\text{cl}_{J^*} J = J^*$; i.e., J is a dense subset of J^* , and (iv) (J^*, d^*) is unique to within isometry. Addition and scalar multiplication are defined on J^* by $x^* + y^* = \lim_n (x_n + y_n)$ and $\alpha x^* = \lim_n \alpha x_n$, where $\alpha \in \Phi$, $\lim_n x_n = x^*$, $\lim_n y_n = y^*$ and $\{x_n\}, \{y_n\}$ are sequences on J . Of course every element of J^* is the limit of a sequence on J , and moreover, if we let $S^* = \text{cl}_{J^*} S$ for $S \subseteq J$, then $x^* \in S^*$ if and only if there exists a sequence $\{x_n\}$ in S such that $\lim_n x_n = x^*$. We now establish some important properties of J^* .

Proposition 3.1.3. Let (J^*, d^*) be the completion of (J, d) as above. Then:

(1) If $\lim_n x_n = x^*$, $\lim_n y_n = y^*$, $\{x_n\}, \{y_n\}$ sequences on J ,

- then $d^*(x^*, y^*) = \lim_n d(x_n, y_n)$.
- (2) J^* is a unital Φ -module.
- (3) $J^* = K_0^* \supseteq K_1^* \supseteq K_2^* \supseteq \dots$ is a decreasing sequence of submodules of J^* with $\bigcap_{i=0}^{\infty} K_i^* = 0$ and thus induces a metric ρ on J^* .
- (4) $\mathcal{T}(d^*) = \mathcal{T}(\rho)$
- (5) There is a one-to-one correspondence between the open submodules A of J and the open submodules of J^* given by $A \leftrightarrow A^*$, and $A = J \cap A^*$. In particular, $K_i = J \cap K_i^*$ for $i = 0, 1, 2, \dots$.
- (6) If A is any open submodule of J , then $J/A \cong J^*/A^*$ as Φ -modules.

Proof. (1) Since $\lim_n x_n = x^*$ and $\lim_n y_n = y^*$, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $d^*(x_n, x^*) < \epsilon/2$ and $d^*(y_n, y^*) < \epsilon/2$. Thus for every $n \geq N$,

$$\begin{aligned} d(x_n, y_n) - d^*(x^*, y^*) &= d^*(x_n, y_n) - d^*(x^*, y^*) \leq d^*(x_n, x^*) + d^*(x^*, y_n) \\ &\quad - d^*(x^*, y^*) \\ &\leq d^*(x_n, x^*) + d^*(x^*, y^*) + d^*(y^*, y_n) - d^*(x^*, y^*) \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

and

$$\begin{aligned} d^*(x_n, y_n) - d^*(x^*, y^*) &\geq d^*(x_n, y_n) - d^*(x^*, y_n) - d^*(y_n, y^*) \\ &\geq d^*(x_n, y_n) - d^*(x^*, x_n) - d^*(x_n, y_n) - d^*(y_n, y^*) \\ &> -\epsilon/2 - \epsilon/2 = -\epsilon. \end{aligned}$$

Therefore, $|d(x_n, y_n) - d^*(x^*, y^*)| < \epsilon$, and $\lim_n d(x_n, y_n) = d^*(x^*, y^*)$.

- (2) It will be shown first that the definitions $x^* + y^* = \lim_n (x_n + y_n)$

and $\alpha x^* = \lim_n (\alpha x_n)$, where $\alpha \in \Phi$ and $\{x_n\}, \{y_n\}$ are sequences in J such that $\lim_n x_n = x^* \in J^*$, $\lim_n y_n = y^* \in J^*$, are independent of representations of x^*, y^* . First suppose $x^* = \lim_n x_n$, $x^* = \lim_n x_n'$, and $y^* = \lim_n y_n$, $y^* = \lim_n y_n'$. Then by Proposition 3.1.1, $d(x_n + y_n, x_n' + y_n') \leq \max \{d(x_n, x_n'), d(y_n, y_n')\}$ and $d(\alpha x_n, \alpha x_n') \leq d(x_n, x_n')$, for every $n \in \mathbb{N}$. But it is clear that if $\{w_n\}$ is a sequence in J and $\{w_n'\}$ is a sequence in J , then $\lim_n w_n = w^*$, $\lim_n w_n' = w^*$ if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that $d(w_n, w_n') < \epsilon$, for all $n \geq N$. Hence $\lim_n (x_n + y_n) = \lim_n (x_n' + y_n')$ and $\lim_n (\alpha x_n) = \lim_n (\alpha x_n')$ follows immediately from the above inequalities. Note that the fact that $\{x_n + y_n\}$, and $\{\alpha x_n\}$ are Cauchy sequences also follows from these inequalities. Thus the operations are well-defined and clearly closed. That J^* is an Abelian semigroup follows easily from the corresponding properties in J . Also it is clear that the constant sequence $\{0\}$ is an identity since $x^* + 0 = \lim_n (x_n + 0) = \lim_n x_n = x^*$. Finally $-x^* = \lim_n (-x_n)$, since $x^* + \lim_n (-x_n) = \lim_n (x_n + (-x_n)) = \lim_n 0 = 0$, and $d(-x_n, -x_m) = d(x_n, x_m)$ so that $\{-x_n\}$ is Cauchy. Thus $(J^*, +)$ is an Abelian group. Now, if $\alpha, \beta \in \Phi$, we have

- i) $\alpha(x^* + y^*) = \lim_n [\alpha(x_n + y_n)] = \lim_n (\alpha x_n + \alpha y_n) = \alpha x^* + \alpha y^*$,
- ii) $(\alpha + \beta)x^* = \lim_n (\alpha + \beta)x_n = \lim_n (\alpha x_n + \beta x_n) = \alpha x^* + \beta x^*$,
- iii) $(\alpha\beta)x^* = \lim_n (\alpha\beta)x_n = \lim_n \alpha(\beta x_n) = \alpha(\beta x^*)$,
- iv) $1x^* = \lim_n (1x_n) = \lim_n x_n = x^*$,

since J is a unital Φ -module. Hence J^* is a unital Φ -module.

(3) Since $J = K_0 \supseteq K_1 \supseteq \dots$, $J^* = K_0^* \supseteq K_1^* \supseteq \dots$, and K_i^* is a Φ -submodule of J^* exactly as in (2). Hence let $x^* \in \bigcap_{i=0}^{\infty} K_i^*$. Then for each $i = 0, 1, 2, \dots$, $x^* \in K_i^*$, and since K_i^* is closed, there is a sequence $\{x_{in}\}$ on K_i such that $\lim_n x_{in} = x^*$. Now let $\epsilon > 0$ be arbitrary, and choose i so that $e^{-i} < \epsilon/2$. Since $x_{in} \in K_i$, for every $n \in \mathbb{N}$, $d^*(x_{in}, 0) = d(x_{in}, 0) < e^{-i} < \epsilon/2$. Also since $\lim_n x_{in} = x^*$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $d^*(x^*, x_{in}) < \epsilon/2$. Thus, in particular $d^*(x^*, 0) \leq d^*(x^*, x_{iN}) + d^*(x_{iN}, 0) < \epsilon/2 + \epsilon/2 = \epsilon$, and so $d^*(x^*, 0) = 0$. Hence $x^* = 0$.

The second statement about the existence of ρ follows immediately from the definition of the metric determined by the submodules $J^* \supseteq K_0^* \supseteq K_1^* \supseteq K_2^* \supseteq \dots$.

(4) To show $\mathcal{T}(d^*) = \mathcal{T}(\rho)$, it is sufficient to show that the two metrics are equivalent. If $\epsilon > 0$ is given and $x^* = y^*$, then $d^*(x^*, y^*) = 0 = \rho(x^*, y^*)$, and $d^*(x^*, y^*) < \epsilon$ if and only if $\rho(x^*, y^*) < \epsilon$. Hence we may assume $x^* \neq y^*$. First assume $\rho(x^*, y^*) < \epsilon/2$. Then $\rho(x^*, y^*) = e^{-k}$, where $x^* - y^* \in K_k^* - K_{k+1}^*$ and $e^{-k} < \epsilon/2$. Since K_k^* is closed, there exists a sequence $\{z_n\}$ in K_k such that $\lim_n z_n = x^* - y^*$. Thus there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $d^*(x^* - y^*, z_n) < \epsilon/2$. Also $z_n \in K_k$ for all $n \in \mathbb{N}$ implies $d^*(z_n, 0) = d(z_n, 0) \leq e^{-k} < \epsilon/2$. Hence

$$\begin{aligned} d^*(x^*, y^*) &= \lim_n d(x_n, y_n) = \lim_n d(x_n - y_n, 0) = d^*(x^* - y^*, 0) \\ &\leq d^*(x^* - y^*, z_n) + d^*(z_n, 0) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Conversely, assume $d^*(x^*, y^*) < \epsilon/2$. By (1) $d^*(x^*, y^*) = \lim_n d(x_n, y_n)$, and hence there exists $N \in \mathbb{N}$ such that when $n \geq N$, $|d(x_n, y_n) - d^*(x^*, y^*)| < \epsilon/2$; i.e., $d(x_n, y_n) < d^*(x^*, y^*) + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon$ for all $n \geq N$. Thus if $n \geq N$, $x_n - y_n \in K_{k_n} - K_{k_n+1}$, where $e^{-k_n} < \epsilon$ or $x_n - y_n = 0 \in \bigcap_{i=0}^{\infty} K_i$. Let k be the smallest positive integer k_n , $n \geq N$. Then $K_{k_n} \subseteq K_k$ for all $n \geq N$ and thus $x_n - y_n \in K_k$ for all $n \geq N$, where $e^{-k} < \epsilon$. Since K_k^* is closed, $x^* - y^* = \lim_n (x_n - y_n) \in K_k^*$, and $\rho(x^*, y^*) \leq e^{-k} < \epsilon$. Therefore the two metrics are equivalent.

(5) By (4) convergence, etc. in (J^*, d^*) is characterized by Proposition 3.1.2. Also since $d^*|_{J \times J} = d$, the topology on (J, d) coincides with the subspace topology inherited from (J^*, d^*) .

Let A be an open submodule of J and define φ on the set of open submodules of J by $\varphi(A) = A^*$. First, A open and $0 \in A$ implies that there exists $k \in \mathbb{N}$ such that $K_k \subseteq A \subseteq A^*$. Thus $K_k^* \subseteq A^*$ and for any $x^* \in A^*$, $x^* + K_k^* \subseteq A^*$. Hence A^* is open and φ is a map into the set of open ideals of J^* . Since $A_1 = A_2$ implies $A_1^* = A_2^*$, φ is well-defined. Next let B be any open submodule of J^* . Then $A = B \cap J$ is an open submodule of J and $A \subseteq B$. Now B open implies B closed in J^* so that $A^* \subseteq B^* = B$. Suppose $x^* \in B$. Then there exists a sequence $\{x_n\}$ in J such that $\lim_n x_n = x^*$. Hence for every $k \in \mathbb{N}$ there exists $N(k) \in \mathbb{N}$ such that if $n \geq N(k)$, then $x_n - x^* \in K_k^*$. Also B is open in J^* and $0 \in B$ so there exists $m \in \mathbb{N}$ such that $K_m^* \subseteq B$, which implies $K_i^* \subseteq B$ for all $i \geq m$. Define a new sequence $y_n = x_{N(m)+n-1}$. Clearly $\lim_n y_n = x^*$, and since

$x^* \in B$ and $x_{N(m)+n-1}^* - x^* \in K_m^* \subseteq B$. Thus $y_n \in B \cap J = A$ for all n , and so $x^* \in A^*$. Thus $B \subseteq A^*$, and equality holds. From this the remainder of (5) can be deduced. First, this certainly implies that φ is onto the set of open submodules of J^* .

Secondly, if A is open in J , then $A = B \cap J$ where B is open in J^* which implies that $A = A^* \cap J$ by what was just shown.

Finally φ is one-to-one, for if $A^* = B^*$ where A and B are open in J , then $A = J \cap A^* = J \cap B^* = B$.

(6) Let A be any open submodule of J and define $\varphi : J \rightarrow J^*/A^*$ by $\varphi(x) = x + A^*$ for $x \in A$. If $\alpha, \beta \in \Phi$, $x, y \in J$, then $\varphi(\alpha x + \beta y) = \alpha x + \beta y + A^* = \alpha x + A^* + \beta y + A^* = \alpha(x + A^*) + \beta(y + A^*) = \alpha\varphi(x) + \beta\varphi(y)$. Thus φ is a Φ -module homomorphism. Suppose $x^* + A^* \in J^*/A^*$. Then there exists a sequence $\{x_n\}$ in J such that $\lim_n x_n = x^*$. Also A is open, so A^* is open by (5) and there exists $k \in \mathbb{N}$ such that $K_k^* \subseteq A^*$. Hence there exists $N \in \mathbb{N}$ such that $x_n \equiv x^* \pmod{K_k^*}$ for all $n \geq N$. Hence $x_N \equiv x^* \pmod{A^*}$, and $\varphi(x_N) = x_N + A^* = x^* + A^*$. Thus φ is onto, and $J^*/A^* \cong J/\ker \varphi$. But $x \in A$ implies $x \in A^*$ so that $\varphi(x) = x + A^* = A^*$ and $x \in \ker \varphi$, and $x \in \ker \varphi$ implies $A^* = \varphi(x) = x + A^*$ so that $x \in J \cap A^* = A$. Hence $\ker \varphi = A$ and (6) follows.

3.2. Topological Quadratic Jordan Algebras

Let $(J, U, 1)$ be a quadratic Jordan algebra, and consider the ideals $J = K^{(0)}, K^{(1)}, K^{(2)}, \dots$ as defined in Definition 2.2.3. Since for $n \geq 1$, $K^{(n+1)} = U_{K^{(n)}}(K^{(n)})$, clearly $K^{(0)} \supseteq K^{(1)} \supseteq K^{(2)} \supseteq \dots$.

Definition 3.2.1. An ideal K of J is called a nucleus for J , if $\bigcap_{n=0}^{\infty} K^{(n)} = 0$.

Since J and $K^{(n)}$, $n = 0, 1, 2, \dots$, are Φ -modules, the considerations in §3.1 apply. Hence if K is a nucleus for J , K induces a topology in J called the K -topology. We have the metric d as defined in §3.1, and of course (J, d) is a metric space whose topology coincides with the K -topology.

As we have previously seen, addition and scalar multiplication are continuous (the latter under suitable assumptions about Φ). Next consider the mapping $(x, y) \rightarrow U_x(y)$ from $J \times J \rightarrow J$. We shall now prove that this operation is also continuous. Let $U_x(y) + K^{(k)}$ be any basic open set of $\mathcal{T}(d)$ which contains $U_x(y)$. Clearly $(x + K^{(k)}) \times (y + K^{(k)})$ is open in the product topology on $J \times J$ and $(x, y) \in (x + K^{(k)}) \times (y + K^{(k)})$. If $a \in K^{(k)}$ and $b \in K^{(k)}$, then $x + a \in x + K^{(k)}$ and $y + b \in y + K^{(k)}$ so that $(x + a, y + b) \in (x + K^{(k)}) \times (y + K^{(k)})$. Also $U_{x+a}(y + b) = U_x(y + b) + U_a(y + b) + U_{x,a}(y + b) = U_x(y) + U_x(b) + U_a(y + b) + U_{x,a}(y + b) \in U_x(y) + K^{(k)}$, since $K^{(k)}$ is an ideal and $a, b \in K^{(k)}$ implies $U_{x,a}(y + b) \in K^{(k)}$ by Proposition 2.2.1. Thus $(x, y) \rightarrow U_x(y)$ is continuous and J is a topological Jordan algebra according to

Definition 3.2.2. A quadratic (linear) Jordan algebra J is called a topological Jordan algebra, if J is a topological space and the mappings $(x,y) \rightarrow x + y$, $(x,y) \rightarrow U_x(y)$ are continuous.

Before considering completions of topological Jordan algebras with respect to the metric topology determined by a nucleus K , the following useful inequality is established.

Lemma 3.2.1. Let K be a nucleus for J and d the metric of the K -topology. Then $d(U_x(y), U_w(z)) \leq \max \{d(x,w), d(y,z)\}$ for all $x, y, w, z \in J$.

Proof. Suppose $x = w$ so that $d(x,w) = 0$. If $y = z$, then $U_x(y) - U_w(z) = U_x(y - z) = U_x(0) = 0$, and the inequality holds. If $y \neq z$, then $d(y,z) = e^{-k}$ where $y - z \in K^{(k)} - K^{(k+1)}$. Hence $U_x(y) - U_w(z) = U_x(y - z) \in K^{(k)}$ implies $d(U_x(y), U_w(z)) \leq e^{-k} = d(y,z)$. Now assume $x \neq w$ so that $d(x,w) = e^{-k}$ where $x - w = a \in K^{(k)} - K^{(k+1)}$. If $y = z$, then $U_x(y) - U_w(z) = (U_x - U_{x-a})(y) = (U_x - U_x - U_a + U_{x,a})(y) = U_{x,a}(y) - U_a(y) \in K^{(k)}$. Thus in this instance $d(U_x(y), U_w(z)) \leq e^{-k} = d(x,w) = \max \{d(x,w), d(y,z)\}$. Finally assume $y \neq z$ so that $d(y,z) = e^{-m}$ where $y - z = b \in K^{(m)} - K^{(m+1)}$. Then we have

$$\begin{aligned} U_x(y) - U_w(z) &= U_x(y) - U_{x-a}(y - b) = U_x(y) - U_x(y - b) \\ &\quad - U_a(y - b) + U_{x,a}(y - b) \\ &= U_x(y) - U_x(y) + U_x(b) - U_a(y - b) + U_{x,a}(y - b) \\ &= U_x(b) - U_a(y - b) + U_{x,a}(y - b) \in K^{(\ell)}, \end{aligned}$$

where $\ell = \min \{k, m\}$, since $K^{(k)} \subseteq K^{(\ell)}$ and $K^{(m)} \subseteq K^{(\ell)}$.

Thus $d(U_x(y), U_w(z)) \leq e^{-l} = \max \{d(x, w), d(y, z)\}$.

Definition 3.2.3. Let J be a quadratic Jordan algebra with nucleus K and let d be the metric of the K -topology. Then a quadratic Jordan algebra (J^*, U^*, l) is a completion of (J, U, l) with respect to d , if

- i) (J^*, d^*) is a complete topological space,
- ii) (J, d) is a dense subspace of (J^*, d^*) ,
- iii) J^* is a topological Jordan algebra,
- iv) J is a subalgebra of J^* .

Theorem 3.2.1. If (J, U, l) has nucleus K with metric d , then there exists a completion (J^*, U^*, l) which is unique to within isomorphism.

Proof. By the considerations in §3.1 there exists a unital Φ -module J^* with metric d^* such that among other things, J is a dense subspace of J^* , $d^*|_{J \times J} = d$, and J^* is complete with respect to the metric d^* . Recall that addition and scalar multiplication were defined by $x^* + y^* = \lim_n (x_n + y_n)$, $\alpha x^* = \lim_n \alpha x_n$, $\alpha \in \Phi$ and $x^* = \lim_n x_n$, $y^* = \lim_n y_n \in J^*$ where $\{x_n\}, \{y_n\}$ are sequences in J . In a similar fashion define U^* by

$$U_{x^*}^*(y^*) = \lim_n U_{x_n}(y_n) .$$

Suppose $\lim_n x_n = x^*$, $\lim_n x'_n = x^*$, $\lim_n y_n = y^*$, $\lim_n y'_n = y^*$. Then for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$d(x_n, x'_n) < \epsilon$ and $d(y_n, y'_n) < \epsilon$. Hence by Lemma 3.2.1,

$d(U_{x_n}(y_n), U_{x'_n}(y'_n)) \leq \max \{d(x_n, x'_n), d(y_n, y'_n)\} < \epsilon$ for all $n \geq N$.

Thus $\lim_n U_{x_n}(y_n) = \lim_n U_{x_n}(y'_n)$ and so the operation is independent of representation. Similarly, it follows that $U_{x,y}^*(y^*) \in J^*$. Hence $U_{x,y}^* \in \text{End}_{\Phi} J^*$. Also for every $\alpha \in \Phi$, $U_{x,y}^*(y^*) = \lim_n U_{\alpha x_n}(y_n) = \lim_n \alpha^2 U_{x_n}(y_n) = \alpha^2 U_{x,y}^*(y^*)$. Therefore U^* is a quadratic mapping from J^* into $\text{End}_{\Phi} J^*$. If $U_{x,y}^*$ is the associated bilinear mapping, then

$$\begin{aligned} U_{x,y}^*(z^*) &\equiv U_{x+y}^*(z^*) - U_x^*(z^*) - U_y^*(z^*) \\ &= \lim_n [U_{x_n+y_n}(z_n) - U_{x_n}(z_n) - U_{y_n}(z_n)] \\ &= \lim_n U_{x_n,y_n}(z_n). \end{aligned}$$

Similarly $V_{x,y}^*(z^*) = \lim_n V_{x_n,y_n}(z_n)$ and $V_x^*(y^*) = \lim_n V_{x_n}(y_n)$, so that $x^* \circ y^* \equiv V_{x,y}^*(y^*) = \lim_n V_{x_n}(y_n) = \lim_n x_n \circ y_n$, where the definitions of the operators are as in §2.1. It can easily be seen that the fact that $(J, U, 1)$ satisfies QJ1 - QJ3 and QJ5 - QJ8 implies that these hold for $(J^*, U^*, 1)$ also. For example, by QJ3 in $(J, U, 1)$,

$$\begin{aligned} U_{x,y}^* V_{x,y}^*(z^*) &= \lim_n U_{x_n} V_{y_n,x_n}(z_n) = \lim_n V_{x_n,y_n} U_{x_n}(z_n) \\ &= V_{x,y}^* U_x^*(z^*), \end{aligned}$$

and thus QJ3 holds for $(J^*, U^*, 1)$. Therefore $(J^*, U^*, 1)$ is a quadratic Jordan algebra with $(J, U, 1)$ as a subalgebra.

By Proposition 3.1.3 (1), $d^*(x^*, y^*) = \lim_n d(x_n, y_n)$, if $\lim_n x_n = x^*$ and $\lim_n y_n = y^*$. Thus $d^*(x^* + y^*, w^* + z^*)$

$\leq \max \{d^*(x^*, w^*), d^*(y^*, z^*)\}$ and $d^*(U_{*x}^*(y^*), U_{*w}^*(z^*))$
 $\leq \max \{d^*(x^*, w^*), d^*(y^*, z^*)\}$. Let ρ^* be the metric induced on
 the product $J^* \times J^*$ where $\rho^*((x^*, y^*), (w^*, z^*))$
 $= \max \{d^*(x^*, w^*), d^*(y^*, z^*)\}$. Then for any $\epsilon > 0$, $\rho^*((x^*, y^*), (w^*, z^*))$
 $< \epsilon$ implies $d^*(x^* + y^*, w^* + z^*) < \epsilon$ and $d^*(U_{*x}^*(y^*), U_{*w}^*(z^*)) < \epsilon$.
 Hence the operations are uniformly continuous, and $(J^*, U^*, 1)$
 is a topological Jordan algebra.

If $(\hat{J}, \hat{U}, 1)$ is another completion of J , then there
 exists an isometry $\varphi : J^* \rightarrow \hat{J}$ such that φ is onto and $\varphi(x) = x$
 for all $x \in J$. Since $d^*(x^*, y^*) = \hat{d}(\varphi(x^*), \varphi(y^*))$ for all
 $x^*, y^* \in J^*$, if $x^* = \lim_n x_n$, $y^* = \lim_n y_n$, then $\varphi(x^*) = \lim_n \varphi(x_n)$,
 $\varphi(y^*) = \lim_n \varphi(y_n)$. Hence $\varphi(\alpha x^* + \beta y^*) = \varphi[\lim_n (\alpha x_n + \beta y_n)]$
 $= \alpha \varphi(x^*) + \beta \varphi(y^*)$, and $\varphi(U_{*x}^*(y^*)) = \varphi(\lim_n U_{*x_n}(y_n)) = \hat{U}_{\varphi(x^*)}(\varphi(y^*))$.
 Thus φ is a Jordan homomorphism and $J^* \cong \hat{J}$.

Since a completion of J^* of J is unique to within
 isomorphism, J^* will be called the completion of J . Also since
 $\mathcal{T}(d^*) = \mathcal{T}(\rho)$, where ρ is the metric induced by the decreasing
 sequence of modules $J^* = K^{(0)*} \supseteq K^{(1)*} \supseteq K^{(2)*} \supseteq \dots$, ρ will
 be used to denote the metric for J^* . Some important properties
 of J^* are considered next.

Lemma 3.2.4. If J has nucleus K and J^* is the completion
 of J , then:

- (i) $K^{*(n)} \subseteq (K^{(n)})^*$, $n = 0, 1, 2, \dots$.
- (ii) $\bigcap_{n=0}^{\infty} (K^*)^{(n)} = 0$, and if $(K^{(n)})^* \subseteq K^{*(n)}$ for every
 $n = 0, 1, 2, \dots$, then K^* is a nucleus for (J^*, ρ) .

- (iii) If A is an ideal of J , then A^* is an ideal of J^* .
- (iv) If A is an open ideal of J , then $A = A^* \cap J$ and $A \leftrightarrow A^*$ is a one-to-one correspondence between the open ideals of J and the open ideals of J^* .
- (v) For any open ideal A of J , $J/A \cong J^*/A^*$ as quadratic Jordan algebras.

Proof. (i) Certainly $K^{*(0)} = J^* = (K^{(0)})^*$ and $K^{*(1)} = K^* = (K^{(1)})^*$. Proceed by induction. For $n \geq 1$ assume $K^{*(n)} \subseteq (K^{(n)})^*$. If $z^* \in K^{*(n+1)}$, then z^* is a Φ -linear combination of elements of the form $U_{x^*}^*(y^*)$, $x^*, y^* \in K^{*(n)}$. Since $(K^{(n+1)})^*$ is closed under such sums, it suffices to show $U_{x^*}^*(y^*) \in (K^{(n+1)})^*$ for every $x^*, y^* \in K^{*(n)}$. By the induction hypothesis, $x^*, y^* \in K^{*(n)}$ implies $x^*, y^* \in (K^{(n)})^*$. Hence there exist sequences $\{x_k\}, \{y_k\}$ in $K^{(n)}$ such that $x^* = \lim_k x_k$, $y^* = \lim_k y_k$ so that $U_{x^*}^*(y^*) = \lim_k U_{x_k}^*(y_k)$. Now $U_{x_k}^*(y_k) \in U_{K^{(n)}}(K^{(n)}) = K^{(n+1)}$ for every $k = 1, 2, \dots$, so $U_{x^*}^*(y^*) \in (K^{(n+1)})^*$.

(ii) $\bigcap_{n=0}^{\infty} K^{*(n)} \subseteq \bigcap_{n=0}^{\infty} (K^{(n)})^* = 0$, by Proposition 3.1.3 (3). Thus K^* is a nucleus for a topology for J^* . If $(K^{(n)})^* \subseteq K^{*(n)}$, then $(K^{(n)})^* = K^{*(n)}$, and the topology coincides with the ρ -metric topology.

(iii) Let A be an ideal of J and $x^* \in A^*, y^* \in J^*$. Then there exist sequences $\{x_n\}, \{y_n\}$ in A and J respectively such that $\lim_n x_n = x^*, \lim_n y_n = y^*$. Thus for all $n \in \mathbb{N}$, $U_{x_n}^*(y_n), U_{y_n}^*(x_n) \in A$. Hence $U_{x^*}^*(y^*) = \lim_n U_{x_n}^*(y_n)$,

$U_y^*(x^*) = \lim_n U_{y_n}(x_n) \in A^*$. Since A^* is a submodule of J^* by Proposition 3.1.3 (2), A^* is an ideal of J^* .

(iv) Let A be any open ideal in J . By Proposition 3.1.3 (5), $A = A^* \cap J$ and A^* is open in J^* . Thus the map $\varphi(A) = A^*$ is a one-to-one map of the set of open ideals of J into the set of open ideals of J^* . If B is any open ideal of J^* , then $A = B \cap J$ is an open ideal of A and $B = A^*$. Thus φ is onto.

(v) From the proof of Proposition 3.1.3 (6) we have, if A is an open ideal of J , then $\varphi : J \rightarrow J^*/A^*$ defined by $\varphi(x) = x + A^*$, $x \in J$, is a module epimorphism with kernel A . Now $\varphi(1) = 1 + A^*$ and $1 + A^*$ is the unit of J^*/A^* . Let $\tilde{U}^*$ denote the quadratic mapping for J^*/A^* induced by U^* . Then $\varphi(U_x(y)) = U_x(y) + A^* = U_x^*(y) + A^* = \tilde{U}_{x+A^*}^*(y + A^*) = \tilde{U}_{\varphi(x)}^*(\varphi(y))$ for all $x, y \in J$.

Hence φ is a Jordan homomorphism so that $J/A \cong J^*/A^*$ as quadratic Jordan algebras.

3.3. Local Jordan Algebras

If J is a quadratic Jordan algebra, then $\mathcal{R}(J)$ denotes the Jacobson radical of J (see §2.3).

Definition 3.3.1. A quadratic Jordan algebra J is a local Jordan algebra, if (i) $\mathcal{R}(J)$ is the unique maximal ideal of J , (ii) $\mathcal{R}(J)$ is a nucleus for J , and (iii) $J/\mathcal{R}(J)$ satisfies the minimum condition.

Let $\mathfrak{U}$ be a commutative associative ring with 1. If $\mathfrak{U}$ has a unique maximal ideal, then this ideal is $\mathcal{R}(\mathfrak{U})$. Since $\mathcal{R}(\mathfrak{U})^{(k)} \subseteq \mathcal{R}(\mathfrak{U})^{3^{k-1}}$ is clear by induction on k , if $\bigcap_{n=0}^{\infty} \mathcal{R}(\mathfrak{U})^n = 0$, then $\bigcap_{k=0}^{\infty} \mathcal{R}(\mathfrak{U})^{(k)} = 0$ and $\mathcal{R}(\mathfrak{U})$ is a nucleus for $\mathfrak{U}$ as a quadratic Jordan algebra over $\mathbb{Z}$. $\mathfrak{U}/\mathcal{R}(\mathfrak{U})$ is a field and hence contains no idempotents other than 0,1. Thus $\mathfrak{U}/\mathcal{R}(\mathfrak{U})$ trivially satisfies the minimum condition. Therefore any not-necessarily Noetherian, commutative, associative, local ring is a local Jordan algebra.

We now proceed to consider the completion of a local Jordan algebra.

Lemma 3.3.1. If K is an ideal of J and J/K is semi-simple, then $\mathcal{R}(J) \subseteq K$.

Proof. Let $\varphi : J \rightarrow J/K$ be the natural homomorphism. Then $\varphi(\mathcal{R}(J))$ is a quasi-invertible ideal of J/K , and thus $\varphi(\mathcal{R}(J)) = 0$. Hence $\mathcal{R}(J) \subseteq K$.

Lemma 3.3.2. Let J be a local Jordan algebra with completion J^* , let $\alpha_n \in \Phi$, $n \in \mathbb{N}$, and $z \in \mathcal{R}(J)^*$. Then $\sum_{n=1}^{\infty} \alpha_n z^n$ converges in J^* .

Proof. Let $k \in \mathbb{N}$ and $N = 3^k$. By Proposition 2.2.4, $n \geq N$ implies $z^n \in \mathcal{R}(J)^{*(k)} \subseteq \mathcal{R}(J)^{(k)*}$, and hence $\alpha_n z^n \in \mathcal{R}(J)^{(k)*}$ for $n \geq N$.

Thus $\lim_n \alpha_n z^n = 0$, and $\sum_{n=1}^{\infty} \alpha_n z^n$ converges by Proposition 3.1.2 (3).

Lemma 3.3.3. Let J be a local Jordan algebra with completion J^* , e an idempotent in J^* , and $K = U_e(J^*)$ closed in J^* . If $z \in \mathcal{R}(J)^* \cap K$ has quasi-inverse $w \in K$, then $-w = \sum_{n=1}^{\infty} z^n$.

Proof. Recall that K is a quadratic Jordan algebra with quadratic mapping $\tilde{U}^* = U^*|_K$ and unit e (§2.7, p. 38). Since $\tilde{U}^* = U^*$ on K we shall use U^* to denote both mappings. Note that for all $x, y \in K$, $x^2 = U_x^*(e)$ and $x \circ y = U_{x,y}^*(e) = U_{x,e}^*(y)$, the latter following since $U_{x,y}^*(e) = (x+y)^2 - x^2 - y^2$. Thus the operations in J^* coincide with those in K . Hence we have for any $m \in \mathbb{N}$,

$$\begin{aligned} U_{e-z}^*(e + \sum_{n=1}^m z^n) &= U_e^*(e + \sum_{n=1}^m z^n) + U_z^*(e + \sum_{n=1}^m z^n) - U_{e,z}^*(e + \sum_{n=1}^m z^n) \\ &= U_e^*(e) + \sum_{n=1}^m U_e^*(z^n) + U_z^*(e) + \sum_{n=1}^m U_z^*(z^n) - U_{e,z}^*(e) - \sum_{n=1}^m U_{e,z}^*(z^n) \\ &= e + \sum_{n=1}^m z^n + z^2 + \sum_{n=1}^m z^{n+2} - e \circ z - \sum_{n=1}^m z \circ z^n \\ &= e - 2z + \sum_{n=1}^m z^n + \sum_{n=2}^{m+2} z^n - 2 \sum_{n=1}^m z^{n+1} \\ &= e - z - z^{m+1} + z^{m+2}. \end{aligned}$$

Given any $k \in \mathbb{N}$, if $m \geq 3^k$, then $z^{m+2} - z^{m+1} \in \mathcal{R}(J)^{*(k)} \subseteq \mathcal{R}(J)^{(k)*}$, and thus it is clear that $\lim_m U_{e-z}^*(e + \sum_{n=1}^m z^n) = e - z$. By Lemma 3.3.2, $\sum_{n=1}^{\infty} z^n = u$ exists in J^* , and hence, since K is closed, $u \in K$. Now

$$U_{e-z}^*(e+u) = \lim_m U_{e-z}^*(e + \sum_{n=1}^m z^n) = e - z = U_{e-z}^*(e - w),$$

since w is the quasi-inverse of z in K . By the invertibility of U^* (on K), $-w = u$.

Lemma 3.3.4. If J is a local Jordan algebra and $\sum_{i=1}^{\infty} x_i = x$, $\sum_{i=1}^{\infty} z_i = z$, then $\sum_{i=1}^{\infty} (x_i + z_i) = x + z$.

Proof. For any nonnegative integer k , there exists $N \in \mathbb{N}$ such that when $m \geq N$, $x - \sum_{i=1}^m x_i$ and $z - \sum_{i=1}^m z_i$ are in $\mathcal{R}(J)^{(k)}$. Thus for all $m \geq N$ $(x + z) - \sum_{i=1}^m (x_i + z_i) = (x - \sum_{i=1}^m x_i) + (z - \sum_{i=1}^m z_i) \in \mathcal{R}(J)^{(k)}$.

Lemma 3.3.5. If J is a local Jordan algebra with completion J^* , then $\mathcal{R}(J)^* = \mathcal{R}(J^*)$.

Proof. By Lemma 3.2.4, $J/\mathcal{R}(J) \cong J^*/\mathcal{R}(J)^*$, and hence $J^*/\mathcal{R}(J)^*$ is semi-simple. By Lemma 3.3.1, $\mathcal{R}(J^*) \subseteq \mathcal{R}(J)^*$. Also by Lemma 3.2.4, $\mathcal{R}(J)^*$ is an ideal of J^* , and thus the result will follow, if every element of $\mathcal{R}(J)^*$ is quasi-invertible. If $x^* \in \mathcal{R}(J)^*$, then there exists a sequence on $\mathcal{R}(J)$ such that $\lim_n x_n = x^*$. Thus for each $n \in \mathbb{N}$ there exists $y_n \in \mathcal{R}(J)$ which is the quasi-inverse of x_n . We will first show that the sequence $\{y_n\}$ is Cauchy. Now for any $k = 0, 1, 2, \dots$, there exists $N \in \mathbb{N}$ such that $n, m \geq N$ implies $x_n - x_m \in \mathcal{R}(J)^{(k)}$. By Proposition 2.2.2, $(x_n)^i - (x_m)^i \in \mathcal{R}(J)^{(k)}$ for every $i \in \mathbb{N}$. By Lemma 3.3.3, $-y_n = \sum_{i=1}^{\infty} (x_n)^i$ and $-y_m = \sum_{i=1}^{\infty} (x_m)^i$, and so $y_n - y_m = \sum_{i=1}^{\infty} (x_m)^i - \sum_{i=1}^{\infty} (x_n)^i = \sum_{i=1}^{\infty} [(x_m)^i - (x_n)^i] \in \mathcal{R}(J)^{(k)*}$ for all $n, m \geq N$. Therefore $\{y_n\}$ is a Cauchy sequence on $\mathcal{R}(J)$, so that there exists $y^* \in \mathcal{R}(J)^*$ such that $y^* = \lim_n y_n$.

Since $U_{1-x}^*(1-y^*) = \lim_n U_{1-x_n}(1-y_n) = \lim_n (1-x_n) = 1-x^*$, and $U_{1-x}^*[(1-y^*)^2] = \lim_n U_{1-x_n}[(1-y_n)^2] = \lim_n 1 = 1$, y^* is the quasi-inverse of x^* in J^* .

Theorem 3.3.1. If J is a local Jordan algebra with completion J^* , then J^* is a local Jordan algebra.

Proof. By Lemma 3.3.5, $\mathcal{R}(J)^* = \mathcal{R}(J^*)$, and so $J/\mathcal{R}(J) \cong J^*/\mathcal{R}(J)^* = J^*/\mathcal{R}(J^*)$. Since $\mathcal{R}(J)$ is the unique maximal ideal of J , $J/\mathcal{R}(J)$ is simple by Proposition 2.6.1 (2). Hence $J^*/\mathcal{R}(J^*)$ is simple and so $\mathcal{R}(J^*)$ is a maximal ideal of J^* . By Proposition 2.6.1 (4), $\mathcal{R}(J^*)$ is the unique maximal ideal of J^* . Finally $\bigcap_{n=0}^{\infty} \mathcal{R}(J^*)^{(n)} = 0$ by Lemma 3.2.4, and $J^*/\mathcal{R}(J^*)$ satisfies the minimum condition since $J/\mathcal{R}(J)$ does. Thus J^* is a local Jordan algebra.

The last result of this section is concerned with properties of power series in J^* which will be useful in Chapter 4.

Proposition 3.3.1. Let J be a local Jordan algebra with completion J^* , $z \in \mathcal{R}(J^*)$, $\alpha_n, \beta_n \in \Phi$, $n = 0, 1, 2, \dots$. Then:

- (i) Any regrouping or rearrangement of $\sum_{n=0}^{\infty} \alpha_n z^n$ gives a convergent series (not necessarily a power series), which converges to $\sum_{n=0}^{\infty} \alpha_n z^n$.
- (ii) If $y = \sum_{n=0}^{\infty} \alpha_n u^n$, and $u = \sum_{n=1}^{\infty} \beta_n z^n$, then $y = \sum_{n=0}^{\infty} \gamma_n z^n$, where $\gamma_0 = \alpha_0 1$, $\gamma_1 = \alpha_1 \beta_1$, $\gamma_2 = \alpha_1 \beta_2 + \alpha_2 \beta_1^2$, $\gamma_3 = \alpha_1 \beta_3 + 2\alpha_2 \beta_1 \beta_2 + \alpha_3 \beta_1^2$, $\gamma_4 = \alpha_1 \beta_4 + 2\alpha_2 \beta_1 \beta_3 + \alpha_2 \beta_3^2 + 3\alpha_3 \beta_1^2 \beta_2 + \alpha_4 \beta_1^4, \dots$.

Proof. (i) First note that by Lemma 3.3.2, every series of the form $\sum_{n=1}^{\infty} \alpha_n z^n$, $z \in \mathcal{R}(J^*)$, $\alpha_n \in \Phi$, converges in J^* . Trivially

then, so does any series of the form $\sum_{n=0}^{\infty} \alpha_n z^n = \alpha_0 1 + \sum_{n=1}^{\infty} \alpha_n z^n$.

Now any series formed from $\sum_{n=0}^{\infty} \alpha_n z^n$ by regrouping (adding parentheses) will have its sequence of partial sums as a subsequence of the sequence of partial sums of $\sum_{n=0}^{\infty} \alpha_n z^n$, and will therefore converge to $\sum_{n=0}^{\infty} \alpha_n z^n$.

Next let $a_n = \alpha_n z^n$ and let π be any permutation of $\mathbb{N} \cup \{0\}$. Let $b_n = a_{\pi(n)}$ and $B_m = \sum_{n=0}^m b_n$. Then letting $a = \sum_{n=0}^{\infty} \alpha_n z^n = \sum_{n=0}^{\infty} a_n$, and $A_m = \sum_{n=0}^m a_n$, we have $a = \lim_m A_m$. Now given $m \in \mathbb{N}$, let $r(m) = \max \{\pi(i) : 0 \leq i \leq m\}$. Then for any $m \in \mathbb{N}$, $B_m = \sum_{n=0}^m b_n = \sum_{n=0}^m a_{\pi(n)} = A_{r(m)} - \sum_{\substack{\pi(n) \leq r(m) \\ n > m}} a_{\pi(n)}$. Let $k \in \mathbb{N}$ be given and let $h = \max \{n : \pi(n) \leq 3^k\}$. Then for every $n \geq h + 1$, $\pi(n) > 3^k$ so that $a_{\pi(n)} = \alpha_{\pi(n)} z^{\pi(n)} \in \mathcal{R}(J^*)^{(k)} \subseteq \mathcal{R}(J)^{(k)*}$. Since $\sum_{n=0}^{\infty} a_n = a$, there exists $N_1 \in \mathbb{N}$ such that $j \geq N_1$, implies $a - A_j \in \mathcal{R}(J^*)^{(k)} \subseteq \mathcal{R}(J)^{(k)*}$. Now let $p \in \mathbb{N}$ satisfy $\pi(p) = N_1$, and let $N_2 = \max \{p, h\}$. If $m \geq N_2$, then (i) $r(m) \geq \pi(n)$, for all $n \leq m$ implies $N_1 = \pi(p) \leq r(m)$, so that $a - A_{r(m)} \in \mathcal{R}(J^*)^{(k)} \subseteq \mathcal{R}(J)^{(k)*}$, and (ii) $m \geq N_2 \geq h$ implies for $n > m$, $n \geq h + 1$ so $a_{\pi(n)} \in \mathcal{R}(J^*)^{(k)} \subseteq \mathcal{R}(J)^{(k)*}$. Thus for $m \geq N_2$, $a - B_m = a - A_{r(m)} + \sum_{\substack{\pi(n) \leq r(m) \\ n > m}} a_{\pi(n)} \in \mathcal{R}(J^*)^{(k)} \subseteq \mathcal{R}(J)^{(k)*}$. Hence $\sum_{n=0}^{\infty} b_n = \lim_m B_m = a$.

(iii) Let $w = \sum_{n=0}^{\infty} \gamma_n z^n$ be the series derived from $\sum_{n=0}^{\infty} \alpha_n u^n$ by substituting $u = \sum_{n=1}^{\infty} \beta_n z^n$ for u and collecting terms. Now $\sum_{n=0}^{\infty} \gamma_n z^n$ is derived from the series $\alpha_0 + \alpha_1 \beta_1 z + \alpha_1 \beta_2 z^2 + \alpha_2 \beta_1^2 z^2 + \alpha_1 \beta_3 z^3 + \dots$ by regrouping and if this series is denoted by $\sum_{i=0}^{\infty} b_i z^i$, then $\lim_i b_i = 0$, since there are only finitely many terms of a fixed power of z .

Thus $\sum_{i=0}^{\infty} b_i$ converges by (i) and $w = \sum_{i=0}^{\infty} b_i$.

Next consider the following matrix in which the entry in the m^{th} row and n^{th} column is denoted by c_{mn} .

$$\begin{array}{ccccccccc}
 \alpha_0 & 0 & 0 & 0 & 0 & . & \dots \\
 0 & \alpha_1 \beta_1 z & \alpha_1 \beta_2 z^2 & \alpha_1 \beta_3 z^3 & \alpha_1 \beta_4 z^4 & . & \dots \\
 0 & 0 & \alpha_2 \beta_1^2 z^2 & 2\alpha_2 \beta_1 \beta_2 z^3 & (2\alpha_2 \beta_1 \beta_3 + \alpha_2 \beta_2^2) z^4 & . & \dots \\
 0 & 0 & 0 & \alpha_3 \beta_1^3 z^3 & 3\alpha_3 \beta_1^2 \beta_2 z^4 & . & \dots \\
 0 & 0 & 0 & 0 & \alpha_4 \beta_1^4 z^4 & . & \dots \\
 0 & 0 & 0 & 0 & 0 & . & \dots \\
 . & \dots & . & \dots & . & \dots & . & \dots
 \end{array}$$

Since each row constitutes a power series in $z \in \mathcal{R}(J)^*$, with the first m terms 0, each row series converges. Let $c_m = \sum_{n=0}^{\infty} c_{mn}$, $m = 0, 1, 2, \dots$. Also let $B_r = \sum_{i=0}^r b_i$ and let $k \in \mathbb{N}$ be given.

Then there exists $N_1 \in \mathbb{N}$ such that for all $r \geq N_1$, $w - B_r \in \mathcal{R}(J^*)^{(k)} \subseteq \mathcal{R}(J)^{(k)*}$. Now B_r is a finite sum of entries from the above matrix, so let $m'(r) = \max \{\text{row subscripts of terms of } B_r\}$

and $n'(r) = \max \{\text{column subscripts of terms of } B_r\}$. Then $w - B_r = w - \sum_{m=0}^{m'(r)} \left(\sum_{n=0}^{n'(r)} c_{mn} \right) \in \mathcal{R}(J^*)^{(k)} \subseteq \mathcal{R}(J)^{(k)*}$, for every $r \geq N_1$.

Also, since $c_m = \sum_{n=0}^{\infty} c_{mn}$, there exists $N_2 \in \mathbb{N}$ such that $p \geq N_2$

implies $c_m - \sum_{n=0}^p c_{mn} = x(p) \in \mathcal{R}(J^*)^{(k)} \subseteq \mathcal{R}(J)^{(k)*}$. Hence for

$$n'(r) \geq N_2, w - B_r = w - \sum_{m=0}^{m'(r)} [c_m - x(p)] = w - \sum_{m=0}^{m'(r)} c_m + m'(r)x(p),$$

and if $r \geq N_1$, $w - B_r \in \mathcal{R}(J^*)^{(k)} \subseteq \mathcal{R}(J)^{(k)*}$ implies $w - \sum_{m=0}^{m'(r)} c_m + m'(r)x(p) \in \mathcal{R}(J^*)^{(k)} \subseteq \mathcal{R}(J)^{(k)*}$, so that $w - \sum_{m=0}^{m'(r)} c_m \in \mathcal{R}(J^*)^{(k)} \subseteq \mathcal{R}(J)^{(k)*}$. Hence if $Q = m'(N_1)$, for any $q \geq Q$, there exists

$r \geq N_1$ such that $B_r = \sum_{m=0}^{q=m'(r)} \left(\sum_{n=0}^{n'(r)} c_{mn} \right)$ and therefore the above
 computations give $w - \sum_{m=0}^q c_m \in \mathcal{R}(J^*)^{(k)} \subseteq \mathcal{R}(J)^{(k)*}$. Hence
 $w = \lim_q \sum_{m=0}^q c_m = \sum_{m=0}^{\infty} c_m$. Finally, this gives $\sum_{n=0}^{\infty} \gamma_n z^n = w = \sum_{m=0}^{\infty} c_m$
 $= \sum_{n=0}^{\infty} \alpha_n u^n = y$.

CHAPTER 4

STRUCTURE OF COMPLETIONS OF LOCAL JORDAN ALGEBRAS

4.1. Basic Concepts

Throughout Chapter 4 all Jordan algebras will be assumed to be linear Jordan algebras over a field of characteristic different from 2 unless it is stated otherwise. Recall that such an algebra is equipped with a commutative bilinear product which satisfies the Jordan identity. If $\mathcal{J}$ is a local Jordan algebra with completion J , then, in the notation of Chapter 3, $U_x^*|_{\mathcal{J}} = U_x$ for all $x \in \mathcal{J}$, and so we will write U for both the quadratic mapping on $\mathcal{J}$ and the quadratic mapping on J . Likewise the stars will be omitted from the remaining operators, but we will still denote $\mathcal{C}_J S$ by S^* . In particular, we have $J = \mathcal{J}^*$. Let $\{x_n\}, \{y_n\}$ be Cauchy sequences on $\mathcal{J}$ such that $\lim_n x_n = x$ and $\lim_n y_n = y$. Then $x, y \in J$, and $x \cdot y \equiv \frac{1}{2} x \circ y = \lim_n (\frac{1}{2} x_n \circ y_n) = \lim_n x_n \cdot y_n$. Thus the product $\cdot$ satisfies the usual defining condition for a product, and the continuity of U clearly implies the continuity of $\cdot$.

In a linear Jordan algebra J , the subalgebra generated by 1 and an arbitrary element $x \in J$ is associative; i.e., J is power-associative. If $\mathcal{J}$ is a local Jordan algebra with completion J , then the set of convergent power series in $x \in J$ with coefficients in Φ is an associative subalgebra of J . We now prove this in

a somewhat limited form.

Proposition 4.1.1. Let $\mathcal{J}$ be a local Jordan algebra with completion J , let $z \in \mathcal{R}(J)$ be fixed, and $\mathfrak{U}_z = \{ \sum_{n=0}^{\infty} \alpha_n z^n : \alpha_n \in \Phi, n = 0, 1, 2, \dots \}$. Then $\mathfrak{U}_z$ is an associative subalgebra of J .

Proof. Note that by Lemma 3.3.2, all elements of $\mathfrak{U}_z$ are convergent series. Suppose $\sum_{n=0}^{\infty} \alpha_n z^n, \sum_{n=0}^{\infty} \beta_n z^n \in \mathfrak{U}_z$ and $\gamma \in \Phi$.

Then $\sum_{n=0}^{\infty} \alpha_n z^n + \sum_{n=0}^{\infty} \beta_n z^n = \lim_m (\sum_{n=0}^m \alpha_n z^n + \sum_{n=0}^m \beta_n z^n) = \lim_m (\sum_{n=0}^m (\alpha_n + \beta_n) z^n) = \sum_{n=0}^{\infty} (\alpha_n + \beta_n) z^n \in \mathfrak{U}_z$, and $\gamma \sum_{n=0}^{\infty} \alpha_n z^n = \lim_m (\gamma \sum_{n=0}^m \alpha_n z^n) = \lim_m (\sum_{n=0}^m (\gamma \alpha_n) z^n) = \sum_{n=0}^{\infty} (\gamma \alpha_n) z^n \in \mathfrak{U}_z$. Hence $\mathfrak{U}_z$ is a vector subspace of J . Also

$$\begin{aligned} (\sum_{n=0}^{\infty} \alpha_n z^n) \cdot (\sum_{n=0}^{\infty} \beta_n z^n) &= \lim_m [(\sum_{n=0}^m \alpha_n z^n) \cdot (\sum_{n=0}^m \beta_n z^n)] = \lim_m \sum_{n=0}^m \sum_{j=0}^m \alpha_n \beta_j z^{n+j} \\ &= \lim_m [\sum_{n=0}^m (\sum_{j=0}^n \alpha_j \beta_{n-j}) z^n + \sum_{n=m+1}^{2m} (\sum_{j=n-m}^{2m-n+1} \alpha_j \beta_{n-j}) z^n] \\ &= \lim_m \sum_{n=0}^m (\sum_{j=0}^n \alpha_j \beta_{n-j}) z^n + \lim_m \sum_{n=m+1}^{2m} (\sum_{j=n-m}^{2m-n+1} \alpha_j \beta_{n-j}) z^n \\ &= \sum_{n=0}^{\infty} (\sum_{j=0}^n \alpha_j \beta_{n-j}) z^n \in \mathfrak{U}_z. \end{aligned}$$

Finally, the associativity of $\mathfrak{U}_z$ follows from the associativity of $\Phi(z)$; for if $\sum_{n=0}^{\infty} \alpha_n z^n, \sum_{n=0}^{\infty} \beta_n z^n, \sum_{n=0}^{\infty} \gamma_n z^n \in \mathfrak{U}_z$, then

$$\begin{aligned} [(\sum_{n=0}^{\infty} \alpha_n z^n) \cdot (\sum_{n=0}^{\infty} \beta_n z^n)] \cdot \sum_{n=0}^{\infty} \gamma_n z^n &= \lim_m [(\sum_{n=0}^m \alpha_n z^n) \cdot (\sum_{n=0}^m \beta_n z^n)] \cdot \sum_{n=0}^m \gamma_n z^n \\ &= \lim_m \sum_{n=0}^m \alpha_n z^n \cdot [(\sum_{n=0}^m \beta_n z^n) \cdot (\sum_{n=0}^m \gamma_n z^n)] \\ &= \sum_{n=0}^{\infty} \alpha_n z^n \cdot [\sum_{n=0}^{\infty} (\beta_n z^n) \cdot \sum_{n=0}^{\infty} \gamma_n z^n]. \end{aligned}$$

Clearly $1 = 11 + \sum_{n=1}^{\infty} 0 z^n \in \mathfrak{U}_z$ and the proposition follows.

For the last proposition of this section recall that a set of idempotents $\{e_1, e_2, \dots, e_n\}$ is supplementary, if they are pair-wise orthogonal and $\sum_{i=1}^n e_i = 1$.

Proposition 4.1.2. Let J be a quadratic Jordan algebra over a commutative, associative ring Φ with identity, and let K be a nucleus for J . If $J = \sum_{i \leq j} \oplus J_{ij}$ is the Pierce decomposition of J relative to a set of supplementary idempotents $\{e_1, e_2, \dots, e_n\}$, then J_{ij} is closed for $i \leq j$.

Proof. We have that $J_{ii} = \{x \in J : U_{e_i}(x) = x\}$,
 $J_{ij} = \{x \in J : U_{e_i, e_j}(x) = x\}$, $i < j$. Thus if $x \in \mathcal{C}_J J_{ii}$, then there exists a sequence $\{x_k\}$ on J_{ii} such that $x = \lim_k x_k$. Hence $U_{e_i}(x) = \lim_k U_{e_i}(x_k) = \lim_k x_k = x$, and so $x \in J_{ii}$. Thus J_{ii} is closed. Since $U_{e_i, e_j} = U_{e_i + e_j} - U_{e_i} - U_{e_j}$, J_{ij} , $i < j$, is closed in a similar manner.

In particular, if $\mathcal{J}$ is a local Jordan algebra, then the Pierce spaces of $\mathcal{J}$ and $\mathcal{J}^*$ are closed in the $\mathcal{R}(\mathcal{J})$ and $\mathcal{R}(\mathcal{J}^*)$ -topologies, respectively, and therefore those of $\mathcal{J}^*$ are closed in the completion topology.

4.2. Idempotent Lifting

If $\mathcal{Q}$ is a quadratic Jordan algebra over a ring Φ and K is an inner ideal of $\mathcal{Q}$, then for any $x \in K$, $x^1 \in K$ and $x^n = \mathbf{U}_x(x^{n-2}) \in K$ for all $n > 1$.

Lemma 4.2.1. Let $\mathcal{Q}$ be a local Jordan algebra with completion J , and let K be any closed inner ideal of J . Then for every $y \in \mathcal{R}(J) \cap K$, there exists $x \in \mathcal{R}(J) \cap K$ such that $x^2 - x = y$.

Proof. Define a sequence $\{\alpha_n\}$ as follows: let $\alpha_1 = 1 \in \Phi$ and $\alpha_{n+1} = \sum_{i=1}^n \alpha_i \alpha_{n-i+1}$ for all $n \geq 1$. Clearly, $\alpha_n \in \Phi$ for every $n \in \mathbb{N}$. Since $y \in \mathcal{R}(J) \cap K$, $-y \in \mathcal{R}(J) \cap K$, and so $\sum_{n=1}^{\infty} \alpha_n (-y)^n$ converges. If $x = \sum_{n=1}^{\infty} \alpha_n (-y)^n$, then $x \in \mathcal{R}(J) \cap K$, since each is closed in J . Now from the proof of Proposition

4.1.1, $x^2 = \sum_{n=2}^{\infty} (\sum_{i=1}^{n-1} \alpha_i \alpha_{n-i}) (-y)^n = \sum_{n=2}^{\infty} \alpha_n (-y)^n$. Thus

$$x^2 - x = \sum_{n=2}^{\infty} \alpha_n (-y)^n - \sum_{n=1}^{\infty} \alpha_n (-y)^n = y.$$

Lemma 4.2.2. Let $\mathcal{Q}$ be a local Jordan algebra with completion J , and let K be a closed inner ideal of J . If $v \in \mathcal{R}(J) \cap K$, $w = \sum_{n=1}^{\infty} \gamma_n v^n$, $\gamma_n \in \Phi$, and $u^2 - u = v$, then

$$(i) \quad (u.w)^2 = u^2 . w^2$$

$$(ii) \quad u.(u.w) = u^2 . w$$

$$(iii) \quad (u.w).w = u.w^2$$

Proof. Let $W_m = \sum_{n=1}^m \gamma_n v^n$, $m \in \mathbb{N}$. Since $v = u^2 - u$, $W_m \in \Phi(u)$

for all $m \in \mathbb{N}$, and $\Phi(u)$ associative yields that for all $m \in \mathbb{N}$,

$$(u.W_m)^2 = u^2 . W_m^2, \quad u.(u.W_m) = u^2 . W_m, \quad (u.W_m).W_m = u.W_m^2. \quad \text{Hence}$$

$$\begin{aligned}
w = \lim_m W_m, \text{ implies } (u.w)^2 &= \lim_m (u.W_m)^2 = \lim_m u^2.W_m^2 = u^2.w^2, \\
u.(u.w) &= \lim_m u.(u.W_m) = \lim_m u^2.W_m = u^2.w, \text{ and } (u.w).w = \lim_m (u.W_m).W_m \\
&= \lim_m u.W_m^2 = u.w^2.
\end{aligned}$$

Theorem 4.2.1. Let $\mathcal{J}$ be a local Jordan algebra with completion J , and let K be a closed inner ideal of J . Let $\theta : J \rightarrow \bar{J} = J/\mathcal{R}(J)$ be the natural homomorphism and let $\bar{K} = \theta(K)$. If $\bar{f} \in \bar{K}$ is idempotent, then there exists $e \in K$ such that $e^2 = e$ and $\bar{e} = \bar{f}$.
Proof. $\bar{f}^2 - \bar{f} = \bar{0}$ and $\bar{f} \in \bar{K}$ implies that $f^2 - f = z \in \mathcal{R}(J) \cap K$.

Now $-4z \in \mathcal{R}(J) \cap K$ implies $1 + 4z$ is invertible, so by Lemma 3.3.3, $(1 + 4z)^{-1} = \sum_{n=0}^{\infty} (-4z)^n$. Thus $-z.(1 + 4z)^{-1} = \sum_{n=1}^{\infty} (-4)^{n-1} z^n$ $\in \mathcal{R}(J) \cap K$, since $\mathcal{R}(J) \cap K$ is closed. By Lemma 4.2.1, there exists $x = \sum_{n=1}^{\infty} \alpha_n [-z.(1 + 4z)^{-1}]^n$, $\alpha_1 = 1 \in \Phi$, $\alpha_{n+1} = \sum_{i=1}^n \alpha_i \alpha_{n-i+1}$, and $x^2 - x = -z.(1 + 4z)^{-1}$. By Proposition 3.3.1, $x = \sum_{n=1}^{\infty} \beta_n z^n$, $\beta_n \in \Phi$. Since $z = f^2 - f$, Lemma 4.2.2 implies $(f.x)^2 = f^2.x^2$, $f.(f.x) = f^2.x$, and $(f.x).x = f.x^2$.

Next let $h = 1 - 2x$ and $g = x$. Then $h^2 + 2g.h = (1 - 2x)^2 + 2x.(1 - 2x) = 1 - 4x + 4x^2 + 2x - 4x^2 = 1 - 2x = h$, and $g^2 + h^2.z = x^2 + (1 - 2x)^2.z = x - z.[(1 + 4z)^{-1} - 1 + 4z.(1 + 4z)^{-1}] = x + z - z.[(1 + 4z).(1 + 4z)^{-1}] = x + z - z = x = g$. Now let $e = f.h + g$. Since $f, x \in K$, $e = f.h + g = f - 2f.x + x \in K$, and since $x \in \mathcal{R}(J)$, $-2f.x + x \in \mathcal{R}(J)$, so $\bar{e} = \bar{f}$. It remains to show that e is idempotent. Since $(f.h)^2 = f^2 - 4f.(f.x) + 4(f.x)^2 = f^2 - 4f^2.x + 4f^2.x^2 = f^2.h^2$, and $(f.h).g = f.x - 2(f.x).x = f.x - 2f.x^2 = f.(h.g)$, we have

$$\begin{aligned}
e^2 &= (f.h)^2 + 2(f.h).g + g^2 = (f+z).h^2 + 2f.(h.g) + g^2 \\
&= f.h^2 + z.h^2 + 2f.(h.g) + g^2 = f.(h^2 + 2h.g) + g^2 + h^2.z \\
&= f.h + g = e.
\end{aligned}$$

For the next theorem, recall that if J is a linear Jordan algebra over Φ and $\frac{1}{2} \in \Phi$, then for all $x \in J$, $U_x = 2R_x^2 - R_x^2$.

Theorem 4.2.2. Let $\mathcal{J}$ be a local Jordan algebra with completion J . If $\{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n\}$ is a set of nonzero orthogonal idempotents in $\bar{J} = J/\mathcal{R}(J)$, then there exists a set $\{e_1, e_2, \dots, e_n\}$ of nonzero idempotents in J such that $\bar{e}_i = \bar{f}_i$, $i = 1, 2, \dots, n$. Furthermore if $\{\bar{f}_1, \dots, \bar{f}_n\}$ is supplementary, then $\{e_1, \dots, e_n\}$ is a supplementary set.

Proof. First note that if $\{e_1, \dots, e_n\}$ is a set of orthogonal idempotents such that $\bar{e}_i = \bar{f}_i$, $i = 1, 2, \dots, n$, then the e_i are nonzero. The proof is by induction on n . The case $n = 1$ follows from Theorem 4.2.1, where we take $K = J$. Hence assume for $1 \leq r < n$, orthogonal idempotents $e_1, e_2, \dots, e_r$ exist with $\bar{e}_j = \bar{f}_j$, $j = 1, 2, \dots, r$. Let $u_1 = \sum_{j=1}^r e_j$ and $u_2 = 1 - u_1$. Then $\{u_1, u_2\}$ is a supplementary set of idempotents. Let $J = J_{11} \oplus J_{12} \oplus J_{22}$ be the Pierce decomposition of J relative to $\{u_1, u_2\}$. Since for $j = 1, 2, \dots, r$, $u_1.e_j = (\sum_{i=1}^r e_i).e_j = e_j$, $e_j \in J_{11}$ for all $j = 1, 2, \dots, r$. Now put $f = U_{u_2}(f_{r+1})$, so that $f \in J_{22}$. Since $f = U_{u_2}(f_{r+1}) = 2R_{u_2}^2(f_{r+1}) - R_{u_2}^2(f_{r+1}) = 2(f_{r+1}.u_2).u_2 - f_{r+1}.u_2$
 $= 2[f_{r+1}.(1 - u_1)].(1 - u_1) - f_{r+1}.(1 - u_1)$
 $= 2f_{r+1}.(1 - u_1) - 2(f_{r+1}.u_1).(1 - u_1) - f_{r+1}.(1 - u_1)$

$$= f_{r+1} - f_{r+1} \cdot u_1 - 2(f_{r+1} \cdot u_1) \cdot (1 - u_1),$$

$$\text{and } \bar{f}_{r+1} \cdot \bar{u}_1 = \sum_{i=1}^r \bar{f}_{r+1} \cdot \bar{e}_i = \sum_{i=1}^n \bar{f}_{r+1} \cdot \bar{f}_i = \bar{0}, \bar{f} = \bar{f}_{r+1} \neq \bar{0}.$$

Now by Proposition 4.1.2, J_{22} is closed, and J_{22} is an inner ideal of J . Also $\bar{f}^2 = \bar{f} \in \bar{J}_{22}$, so by Theorem 4.2.1, there exists $e \in J_{22}$ such that $e^2 = e$ and $\bar{e} = \bar{f}$. Let $e_{r+1} = e$. Then e_{r+1} is an idempotent in J and $\bar{e}_{r+1} = \bar{e} = \bar{f} = \bar{f}_{r+1}$. Now for any $i = 1, 2, \dots, r$, $e_i \in J_{11}$, so $e_i \cdot e_{r+1} \in J_{11} \cdot J_{22} = 0$, and $\{e_1, e_2, \dots, e_{r+1}\}$ is a nonzero set of orthogonal idempotents of J such that $\bar{e}_i = \bar{f}_i$, $i = 1, 2, \dots, n$.

Finally, assume $\sum_{i=1}^n \bar{f}_i = \bar{1}$, and let $e = \sum_{i=1}^n e_i$. Then $\bar{e} = \sum_{i=1}^n \bar{e}_i = \sum_{i=1}^n \bar{f}_i = \bar{1}$, and hence $e = 1 + z$, $z \in \mathcal{R}(J)$. Now

$$(1 + z)^2 = e^2 = e = 1 + z, \text{ so } 1 + 2z + z^2 = (1 + z)^2 = 1 + z.$$

$$\text{Thus } z \cdot (1 + z) = z + z^2 = 0, \text{ and } [z \cdot (1 + z)] \cdot (1 + z)^{-1} = 0.$$

$$\text{But } z, 1 + z, (1 + z)^{-1} \in \mathfrak{U}_2, \text{ so } z = 0.$$

Lemma 4.2.3. Let $\mathcal{Q}$ be a local Jordan algebra with completion J , e a nonzero idempotent of J , and $K = U_e(J)$. Then:

- (i) If $\bar{u} \in \bar{K}$ is invertible in $\bar{K}$, then u is invertible in K .
- (ii) If $u = e - 4z$, $z \in \mathcal{R}(J) \cap K$, then there exists $w = \sum_{n=1}^{\infty} \alpha_n z^n$ such that $(e - 2w)^2 = u = e - 4z$.
- (iii) If $u \in K$ and $u^2 - e \in \mathcal{R}(J) \cap K$, then there exists $v \in K$ such that $v^2 = e$ and $\bar{v} = \bar{u}$.

Proof. (i) Recall that K is a linear Jordan algebra and $\mathcal{R}(J) \cap K \subseteq \mathcal{R}(K)$. If $\bar{u}$ is invertible in $\bar{K}$, then there exists $\bar{x} \in \bar{K}$ such that $\bar{U}_{\bar{u}}(\bar{x}) = \bar{e}$; i.e., $U_u(x) - e = z \in \mathcal{R}(J) \cap K$. Hence $U_u(x) = e + z$, and since $-z \in \mathcal{R}(J) \cap K \subseteq \mathcal{R}(K)$, $e + z$ is

invertible in K . Thus $U_u(x)$ is invertible in K and so u is invertible in K by Proposition 2.3.1.

(ii) Since $K = U_e(J)$ is a Pierce space from the Pierce decomposition of J relative to $\{e, 1 - e\}$, K is closed. Hence by Lemma 4.2.1, there exists $w \in \mathcal{R}(J) \cap K$ such that $w^2 - w = z$, where $w = \sum_{n=1}^{\infty} \alpha_n z^n$, $\alpha_n \in \Phi$. Now $(e - 2w)^2 = e^2 - 4e.w + 4w^2 = e - 4w + 4w^2 = e - 4z$.

(iii) Since $u^2 - e \in \mathcal{R}(J) \cap K$, let $u^2 = e - 4z$, $z \in \mathcal{R}(J) \cap K$.

Then by (ii) there exists $w = \sum_{n=1}^{\infty} \alpha_n z^n$ such that $(e - 2w)^2 = e - 4z$.

Since $\mathcal{R}(J) \cap K$ is closed, $2w \in \mathcal{R}(J) \cap K \subseteq \mathcal{R}(K)$, so $2w$ is quasi-invertible in K . Hence $e - 2w$ is invertible in K . Let

$$v = u.(e - 2w)^{-1}. \text{ Clearly } v \in K. \text{ Now by Lemma 3.3.3, } (e - 2w)^{-1} = e + \sum_{n=1}^{\infty} (2w)^n, \text{ so } v = u.(e - 2w)^{-1} = u.e + u. \sum_{n=1}^{\infty} (2w)^n = u + u. \sum_{n=1}^{\infty} (2w)^n.$$

Since $2w \in \mathcal{R}(J)$ and $\mathcal{R}(J)$ is closed in J , $\sum_{n=1}^{\infty} (2w)^n \in \mathcal{R}(J)$, and

so $u. \sum_{n=1}^{\infty} (2w)^n \in \mathcal{R}(J)$. Thus $\bar{v} = \bar{u}$. Finally, since $w = \sum_{n=1}^{\infty} \alpha_n z^n$,

$(e - 2w)^{-1} = e + \sum_{n=1}^{\infty} \beta_n z^n$, by Proposition 3.3.1. Thus, since

$$(e - 2w)^{-1} = \lim_m (e + \sum_{n=1}^m \beta_n z^n) = \lim_m [e + \sum_{n=1}^m \frac{\beta_n}{4^n} (e - u^2)^n], \text{ and}$$

$e + \sum_{n=1}^m \frac{\beta_n}{4^n} (e - u^2)^n$ is in the subalgebra of K generated by

$$e \text{ and } u, v^2 = [u.(e - 2w)^{-1}]^2 = u^2.(e - 2w)^{-2} = (e - 2w)^2.(e - 2w)^{-2} = e.$$

Theorem 4.2.3. Let $\mathcal{J}$ be a local Jordan algebra with completion J and let e_1, e_2 be nonzero orthogonal idempotents in J . Then e_1 and e_2 are (strongly) connected if and only if $\bar{e}_1$ and $\bar{e}_2$ are strongly connected in $\bar{J} = J/\mathcal{R}(J)$.

Proof. If e_1 and e_2 are (strongly) connected, then there exists $u \in U_{e_1, e_2}(J)$ such that u is invertible in $U_{e_1 + e_2}(J)$ ($u^2 = e_1 + e_2$). Hence $\bar{u}$ is invertible in $\bar{U}_{\bar{e}_1 + \bar{e}_2}(\bar{J})$ ($\bar{u}^2 = \bar{e}_1 + \bar{e}_2$), and $\bar{u} \in \bar{U}_{\bar{e}_1, \bar{e}_2}(\bar{J})$.

Hence assume $\bar{e}_1, \bar{e}_2$ are connected in $\bar{J}$. Then there exists $\bar{u}_{12} \in \bar{U}_{\bar{e}_1, \bar{e}_2}(\bar{J})$ such that $\bar{u}_{12}$ is invertible in $\bar{U}_{\bar{e}}(\bar{J})$, where $\bar{e} = \bar{e}_1 + \bar{e}_2$. Let $v_{12} = U_{e_1, e_2}(u_{12})$. Then $v_{12} \in U_{e_1, e_2}(J)$, and $\bar{v}_{12} = \bar{U}_{\bar{e}_1, \bar{e}_2}(\bar{u}_{12}) = \bar{u}_{12}$, so $\bar{v}_{12}$ is invertible in $\bar{U}_{\bar{e}}(\bar{J})$. By Lemma 4.2.3, v_{12} is invertible in $U_e(J)$. Thus e_1 and e_2 are connected.

If, moreover $\bar{e}_1$ and $\bar{e}_2$ are strongly connected, then we may assume that $\bar{u}_{12}^2 = \bar{e}$. Let v_{12} be as above. Then $v_{12} \in U_{e_1, e_2}(J)$, and $\bar{v}_{12}^2 = \bar{e}$. By Lemma 4.2.3 (iii), there exists $v \in U_e(J)$ such that $v^2 = e$ and $\bar{v} = \bar{v}_{12}$. Even more can be said from the proof of the lemma, since

$v = \lim_m [v_{12} + v_{12} \cdot \sum_{n=1}^m \frac{\beta_n}{4^n} (e - v_{12}^2)^n]$. But $v_{12} + v_{12} \cdot \sum_{n=1}^m \frac{\beta_n}{4^n} (e - v_{12}^2)^n \in U_{e_1, e_2}(J)$, for all $m \in \mathbb{N}$, and $U_{e_1, e_2}(J)$ is closed. Therefore $v \in U_{e_1, e_2}(J)$, and e_1, e_2 are strongly connected.

The importance of Theorem 4.2.3 lies in the relationship between (strongly) connected idempotents and Jordan matrix algebras. In essence, it allows us to infer the structure of a completion from the known structure of $J/R(J)$.

4.3. Structure of Completions

If $\mathfrak{U}$ is any algebra, A an ideal of $\mathfrak{U}$ and $\theta : \mathfrak{U} \rightarrow \mathfrak{U}/A$ the natural homomorphism, then for $S \subseteq \mathfrak{U}$ and $x \in S$ we will write $\theta(x) = \bar{x}$ and $\theta(S) = \bar{S}$ throughout this section.

Definition 4.3.1. Let J be a Jordan algebra with radical $\mathcal{R}(J)$. J is said to have (finite) radical capacity n , if $J/\mathcal{R}(J)$ has finite capacity n . J is completely primary, if J has radical capacity 1.

Lemma 4.3.1. Let J be a Jordan algebra. Then the following are equivalent:

- (1) J is completely primary.
- (2) $\bar{J} = J/\mathcal{R}(J)$ is a division algebra.
- (3) $\mathcal{R}(J)$ is the set of noninvertible elements of J .

Proof. See p. 49 and Proposition 2.6.1.

If J is the completion of the local Jordan algebra $\mathfrak{J}$, then J is a local Jordan algebra (Theorem 3.3.1). Consequently, $\bar{J} = J/\mathcal{R}(J)$ is simple (Proposition 2.6.1), $\bar{J}$ satisfies axioms (i)-(iii), and there exists $n \in \mathbb{N}$ such that J has radical capacity n (Proposition 2.7.15). Thus, the structure of $\bar{J}$ is completely determined by the Second Structure Theorem (Proposition 2.7.16). The several possibilities are listed according to the radical capacity n of J .

I ($n = 1$). $\bar{J}$ is a Jordan division algebra.

II ($n = 2$). $\bar{J}$ is isomorphic to a Jordan algebra of a nondegenerate symmetric bilinear form in a vector space V over an extension field $\Gamma/\mathbb{F}$ with $\dim V/\Gamma > 1$, or a Jordan matrix algebra

$\mathfrak{S}(\mathcal{A}_2, j_a)$, where $(\mathcal{A}, j)$ is an associative division algebra with involution or $\mathcal{A} \cong \Delta \oplus \Delta^\circ$, Δ an associative division algebra and j the exchange involution.

III ($n = 3$). $\bar{J}$ is isomorphic to a Jordan matrix algebra

$\mathfrak{S}(\mathcal{A}_3, j_a)$, where $(\mathcal{A}, j)$ is an associative division algebra with involution, $\mathcal{A} \cong \Delta \oplus \Delta^\circ$, Δ an associative division algebra and j the exchange involution, or an Octonian algebra with standard involution over an extension field $\Gamma/\mathbb{F}$.

IV ($n \geq 4$). J is isomorphic to a Jordan matrix algebra $\mathfrak{S}(\mathcal{A}_n, j_a)$, where $(\mathcal{A}, j)$ is an associative division algebra with involution or $\mathcal{A} \cong \Delta \oplus \Delta^\circ$, Δ an associative division algebra and j the exchange involution.

Since J has finite radical capacity n , $\bar{J}$ contains a supplementary set of (strongly) connected idempotents $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$.

By Theorem 4.2.2 and Theorem 4.2.3, we may assume that

$\{e_1, e_2, \dots, e_n\}$ is a supplementary set of (strongly) connected idempotents in J . If $n = 1$, then J is a completely primary local Jordan algebra (Lemma 4.3.1). If $n = 2$, then

$J = J_{11} \oplus J_{22} \oplus J_{12}$ (Pierce decomposition relative to $\{e_1, e_2\}$),

where J_{11} and J_{22} are completely primary local Jordan algebras

and J_{12} is a subspace of J . This follows, since $\{\bar{e}_1, \bar{e}_2\}$ a set of completely primitive idempotents implies that $\bar{J}_{ii}$ is a

division algebra, $i = 1, 2$, and thus, $\bar{J}_{ii} = J_{ii}/J_{ii} \cap R(J)$ is

semi-simple, which in turn implies $J_{ii} \cap R(J) = R(J_{ii})$ (Lemma

2.7.1 and Lemma 3.3.1). There does not seem to be more that can

be said for this case, because of the lack of a coordinatization

theorem for $n = 2$.

If $n \geq 3$, then $J \cong \mathfrak{J}(\mathfrak{A}_n, j_a)$, a Jordan matrix algebra, and if $\bar{\mathfrak{A}} = \mathfrak{A}/\mathfrak{M}$, where $\mathcal{R}(\mathfrak{J}(\mathfrak{A}_n, j_a)) = \mathfrak{M}_n \cap \mathfrak{J}(\mathfrak{A}_n, j_a)$, then $\bar{J} \cong \mathfrak{J}(\bar{\mathfrak{A}}_n, j_{\bar{a}})$ with $\bar{a} = \text{diag}\{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n\}$ for $a = \text{diag}\{a_1, a_2, \dots, a_n\}$ (Proposition 2.7.13). Of course $(\bar{\mathfrak{A}}, j)$ is characterized by III and IV. Also, since J is a local Jordan algebra, $\mathcal{R}(\mathfrak{J}(\mathfrak{A}_n, j_a))$ is the unique maximal ideal of $\mathfrak{J}(\mathfrak{A}_n, j_a)$. Therefore, by Proposition 2.7.10, $\mathfrak{M}$ must be the unique maximal ideal of $(\mathfrak{A}, j)$.

Since $\mathfrak{J}(\mathfrak{A}_n, j_a)$ is Jordan and $n \geq 3$, $(\mathfrak{A}, j)$ is associative or $n = 3$ and $(\mathfrak{A}, j)$ is alternative with $\mathfrak{J}(\mathfrak{A}, j) \subseteq N(\mathfrak{A})$ (Proposition 2.7.9). Hence, if $(\mathfrak{A}, j)$ is associative, then $\mathfrak{M} = \mathcal{R}(\mathfrak{A})$ (Proposition 2.7.14). Suppose $(\mathfrak{A}, j)$ is alternative. If $\mathcal{R}(\mathfrak{A})$ is the Smiley radical of $\mathfrak{A}$, then $\mathfrak{M} \subseteq \mathcal{R}(\mathfrak{A})$ by the same proof as in Proposition 2.7.14. Since $\mathcal{R}(\mathfrak{A})^j = \mathcal{R}(\mathfrak{A})$ and $\mathfrak{M}$ is the unique maximal ideal of $(\mathfrak{A}, j)$, we have $\mathfrak{M} = \mathcal{R}(\mathfrak{A})$.

Following the next two lemmas, we will be in a position to establish the main structure theorem.

Lemma 4.3.2. If J is a subalgebra of $\mathfrak{U}^+$, $\mathfrak{U}$ an associative algebra, and K is an ideal of J , then $K^{(k)} \subseteq K^{3^{k-1}}$, $k = 1, 2, \dots$. Moreover, if $\mathfrak{U}$ is commutative, then $K^{(k)} = K^{3^{k-1}}$, $k = 1, 2, \dots$.

Proof. If $x, y \in K$, then $U_x(y) = xyx \in K^3$. Hence, $K^{(2)} = U_K(K) \subseteq K^3$. Since $K^{(k+1)} = U_{K^{(k)}}(K^{(k)})$, the first inclusion follows by induction. Now assume $\mathfrak{U}$ is commutative, and let $x, y, z \in K$. Then $xyz = \frac{1}{2}(xyz + zyx) = \frac{1}{2}U_{x,z}(y) = \frac{1}{2}[U_{x+z}(y) - U_x(y) - U_z(y)] \in K^{(2)}$. Thus, $K^3 \subseteq K^{(2)}$. The remaining inclusions follow by induction as in the first part.

Lemma 4.3.3. Let $J = \mathfrak{J}(\mathfrak{D}_n, j_a)$ be a Jordan matrix algebra of order $n \geq 3$. If $(\mathfrak{D}, j)$ is associative, then $\mathcal{R}(J)^{(k)} = \mathfrak{M}^{3^{k-1}}_n \cap J$ for all $k \in \mathbb{N}$ and $\bigcap_{k=1}^{\infty} \mathcal{R}(J)^{(k)} = 0$ if and only if $\bigcap_{k=1}^{\infty} \mathfrak{M}^k = 0$.

If $(\mathfrak{D}, j)$ is alternative and $\mathcal{R}(J)^{(k)} = A(k)_n \cap J$ for all $k \in \mathbb{N}$, then $\bigcap_{k=1}^{\infty} \mathcal{R}(J)^{(k)} = 0$ if and only if $\bigcap_{k=1}^{\infty} A(k) = 0$.

Proof. Since $\mathcal{R}(J)^{(k)}$ is an ideal of $J = \mathfrak{J}(\mathfrak{D}_n, j_a)$ for all $k = 1, 2, \dots$, (Proposition 2.2.3), for each $k \in \mathbb{N}$ there exists an ideal $A(k)$ of $(\mathfrak{D}, j)$ such that $\mathcal{R}(J)^{(k)} = A(k) \cap J$ (Proposition 2.7.10). In this notation $A(1) = \mathfrak{M}$, and in either case we have $\bigcap_{k=1}^{\infty} \mathcal{R}(J)^{(k)} = \bigcap_{k=1}^{\infty} [A(k)_n \cap J] = [\bigcap_{k=1}^{\infty} A(k)_n] \cap J = [\bigcap_{k=1}^{\infty} A(k)]_n \cap J$. Now, if B is an ideal of $(\mathfrak{D}, j)$ and $0 \neq b \in B$, then $b[12] \neq 0 \in B_n \cap J$. Thus $B_n \cap J = 0$ if and only if $B_n = 0$. Thus it follows that $\bigcap_{k=1}^{\infty} \mathcal{R}(J)^{(k)} = 0$ if and only if $\bigcap_{k=1}^{\infty} A(k) = 0$.

Assume that $(\mathfrak{D}, j)$ is associative. First note that $\mathfrak{M}$ an ideal of $(\mathfrak{D}, j)$ and j an involution implies that $\mathfrak{M}^k$ is an ideal of $(\mathfrak{D}, j)$ for all $k \in \mathbb{N}$. By Proposition 2.7.10, $A(k) = \{d \in \mathfrak{D} : d[ij] \in \mathcal{R}(J)^{(k)}, i, j = 1, 2, \dots, n\}$ for each $k \in \mathbb{N}$. If $d \in A(k)$, then $d[12] \in A(k)_n \cap J = \mathcal{R}(J)^{(k)} \subseteq \mathcal{R}(J)^{3^{k-1}} = \mathfrak{M}^{3^{k-1}}_n \cap J$ by Lemma 4.3.2. Thus $d[12]$ is equal to a finite sum of products of 3^{k-1} matrices each of whose entries are in $\mathfrak{M}$, and hence each entry of $d[12]$ is a finite sum of products of 3^{k-1} elements of $\mathfrak{M}$. In particular then, the $(1,2)$ entry of $d[12]$, which is d , is in $\mathfrak{M}^{3^{k-1}}$. Hence $A(k) \subseteq \mathfrak{M}^{3^{k-1}}$.

We will use induction on k to establish the reverse inclusion. If $k = 1$, then $A(1) = \mathfrak{M}$. Assume that $\mathfrak{M}^{3^{k-1}} \subseteq A(k)$ for $k \geq 1$. Let $d_1, \dots, d_{p_1}, d_{p_1+1}, \dots, d_{p_2}, d_{p_2+1}, \dots, d_{p_3} \in \mathfrak{M}$, where $p_1 = 3^{k-1}$, $p_2 = 2 \cdot 3^{k-1}$, and $p_3 = 3^k$. Let $x = d_1 \dots d_{p_1}$, $y = d_{p_1+1} \dots d_{p_2}$, and $z = d_{p_2+1} \dots d_{p_3}$. Then $x, y, z \in \mathfrak{M}^{3^{k-1}}$, and therefore, by the induction hypothesis, $x, y, z \in A(k)$. By JMA1 (p. 42), and Proposition 2.4.1, $xyz[13] = 2xy[12].z[23] = 4(x[13].y[32]).z[23] \in (\mathcal{R}(J)^{(k)}) \cdot 3 = \mathcal{R}(J)^{(k+1)}$, since $x[13], y[32], z[23] \in A(k)_n \cap J = \mathcal{R}(J)^{(k)}$. Hence $d_1 d_2 \dots d_{p_3} = xyz \in A(k+1)$, and so $\mathfrak{M}^{3^k} \subseteq A(k+1)$. By induction $\mathfrak{M}^{3^{k-1}} \subseteq A(k)$ for all $k \in \mathbb{N}$, and so $A(k) = \mathfrak{M}^{3^{k-1}}$ for all $k \in \mathbb{N}$.

From the first part we now have that $\bigcap_{k=1}^{\infty} \mathcal{R}(J)^{(k)} = 0$ if and only if $\bigcap_{k=1}^{\infty} \mathfrak{M}^{3^{k-1}} = 0$. Since $\bigcap_{k=1}^{\infty} \mathfrak{M}^k = 0$ if and only if $\bigcap_{k=1}^{\infty} \mathfrak{M}^{3^{k-1}} = 0$, the remainder of the lemma follows.

It is immediate from the setting of Lemma 4.3.3, that if $(\mathcal{B}, j)$ is associative, then $\bigcap_{k=1}^{\infty} \mathcal{R}(J)^{(k)} = 0$ if and only if $\bigcap_{k=1}^{\infty} \mathcal{R}(\mathcal{B})^k = 0$, since $\mathfrak{M} = \mathcal{R}(\mathcal{B})$ by Proposition 2.7.14.

We can now establish the main structure theorem. Note that the two cases previously considered are also included for the sake of completeness.

Theorem 4.3.1. Let $\mathcal{J}$ be a local Jordan algebra with completion J , and let J have radical capacity $n \in \mathbb{N}$.

- (1) If $n = 1$, then J is a completely primary local Jordan algebra, and $\mathcal{R}(J)$ is the set of noninvertible elements

of J .

- (2) If $n = 2$, then $J = J_1 \oplus J_2 \oplus S$, where J_1 and J_2 are completely primary local Jordan algebras with identities e_1 and e_2 respectively such that $1 = e_1 + e_2$, and S is a subspace of J .
- (3) If $n \geq 3$, then $J \cong \mathfrak{S}(\mathcal{B}_n, j_a)$, a Jordan matrix algebra, and $J/R(J) = \bar{J} \cong \mathfrak{S}(\bar{\mathcal{B}}_n, j_a) = \mathfrak{S}((\mathcal{B}/\mathcal{M})_n, j_a)$, where $R(\mathfrak{S}(\mathcal{B}_n, j_a)) = \mathcal{M}_n \cap \mathfrak{S}(\mathcal{B}_n, j_a)$. $(\mathcal{B}, j)$ admits the following possibilities:
- (i) $(\mathcal{B}, j)$ is a (not associative) alternative algebra with involution and identity over an extension field of Φ such that $\mathfrak{S}(\mathcal{B}, j) \subseteq N(\mathcal{B})$. $\mathcal{M} = R(\mathcal{B})$ is the Smiley radical of $\mathcal{B}$, and $(\bar{\mathcal{B}}, j)$ is either an Octonian algebra with standard involution or a simple Artinian algebra with involution. In the first instance $\bar{\mathcal{B}}$ is simple and $R(\mathcal{B})$ is the unique maximal ideal of $\mathcal{B}$. In either case $R(\mathcal{B})$ is the unique maximal ideal of $(\mathcal{B}, j)$.
- (ii) $(\mathcal{B}, j)$ is an associative algebra with involution such that $\mathcal{B}$ is a semi-local algebra and $R(\mathcal{B})$ is the unique maximal ideal of $(\mathcal{B}, j)$.

If $n \geq 3$ and $R(\mathcal{B})^{(k)*} = A(k)_n \cap \mathfrak{S}(\mathcal{B}_n, j_a)$ for $k \in \mathbb{N}$, then $\mathcal{B}$ is complete in the topology induced by the decreasing sequence of ideals $R(\mathcal{B}) = A(1) \supseteq A(2) \supseteq \dots$, where we define $A(0) = \mathcal{B}$.

Proof. (1), (2) and the first part of (3) were established in the discussion preceeding Lemma 4.3.2. In (3-i) we have not yet shown that $\bar{\mathcal{B}}$ simple implies that $R(\mathcal{B})$ is the unique maximal ideal of $\mathcal{B}$. Since the proof is identical to the Jordan case (Proposition 2.6.1),

it will not be repeated here. From (3-ii), it remains to be shown that if $n \geq 3$ and $(\mathcal{B}, j)$ is associative, then $\mathcal{B}$ is a semi-local algebra. By a semi-local algebra $\mathcal{U}$ we mean an associative algebra with identity such that (a) $\mathcal{U}$ has only a finite number of maximal ideals, (b) $\mathcal{R}(\mathcal{U})$ is a nucleus for $\mathcal{U}$; i.e., $\bigcap_{k=1}^{\infty} \mathcal{R}(\mathcal{U})^k = 0$, and (c) $\mathcal{U}/\mathcal{R}(\mathcal{U})$ is Artinian. For $\mathcal{B}$, (b) follows from Lemma 4.3.3 and the fact that $\bigcap_{k=1}^{\infty} \mathcal{R}(J)^{(k)} = 0$, and (c) has previously been proved. By III and IV (p. 83), $\bar{\mathcal{B}} = \Delta$, an associative division algebra with involution or $\bar{\mathcal{B}} = \Delta \oplus \Delta^{\circ}$, Δ an associative division algebra and j the exchange involution. From the discussion in §1.2, it is clear that in either case $\bar{\mathcal{B}}$ has at most two ideals. Thus $\mathcal{B}$ has at most two ideals which contain $\mathcal{R}(\mathcal{B})$. Since every maximal ideal of $\mathcal{B}$ must contain $\mathcal{R}(\mathcal{B})$, $\mathcal{B}$ has at most two maximal ideals. Note that this also yields that $\mathcal{B}$ is a local algebra (not necessarily right Noetherian) if and only if $\mathcal{B}$ is completely primary in the sense of Jacobson [4].

We will now show that $\mathcal{B}$ is complete in the topology induced by the ideals $A(k)$, $k = 0, 1, 2, \dots$. In what follows J is identified with $\mathfrak{S}(\mathcal{B}_n, j_a)$. By Lemma 4.3.3, $\bigcap_{k=1}^{\infty} A(k) = 0$, since $\bigcap_{k=1}^{\infty} \mathcal{R}(\mathcal{B})^{(k)*} = 0$. Thus $\mathcal{B}$ is actually a metric space. Let $\{d_i\}$ be a Cauchy sequence in $\mathcal{B}$, and let $k \in \mathbb{N}$ be given. Then there exists $N \in \mathbb{N}$ such that $i_1, i_2 \geq N$ implies $d_{i_1} - d_{i_2} \in A(k)$. Since $A(k)$ is an ideal of $(\mathcal{B}, j)$, $d_{i_1}^j, d_{i_2}^j \in A(k)$. Now for any $x \in \mathcal{B}$, $x[12] \in \mathfrak{S}(\mathcal{B}_n, j_a)$, and hence for $i_1, i_2 \geq N$, $d_{i_1}[12] - d_{i_2}[12] = d_{i_1} e_{12} + a_2^{-1} d_{i_1}^j a_1 e_{21} - d_{i_2} e_{12} - a_2^{-1} d_{i_2}^j a_1 e_{21} \in \mathcal{R}(\mathcal{B})^{(k)*}$, since

$d_{i_1}, d_{i_2}, a_2^{-1} d_{i_1}^j a_1, a_2^{-1} d_{i_2}^j a_1 \in A(k)$. Thus $\{d_i[12]\}$ is a Cauchy sequence in $\mathfrak{S}(\mathfrak{D}_n, j_a) = J$, and since J is complete relative to the topology induced by the ideals $\mathcal{R}(J)^{(k)*}$, there exists $x_{pq} \in \mathfrak{D}$, $p, q = 1, 2, \dots, n$, such that $\lim_i d_i[12] = \sum_{p,q} x_{pq} e_{pq}$. Therefore given any $k \in \mathbb{N}$, there exists $N' \in \mathbb{N}$ such that $i \geq N'$ implies $d_i[12] - \sum x_{pq} e_{pq} \in \mathcal{R}(J)^{(k)*}$, and so $d_i - x_{12} \in A(k)$. Thus $\lim_i d_i = x_{12} \in \mathfrak{D}$ and $\mathfrak{D}$ is complete.

4.4. Complete Local Jordan Algebras

Definition 4.4.1. A quadratic Jordan algebra J over a commutative associative ring with identity is said to be complete, if $\mathcal{R}(J)$ is a nucleus and J is a complete metric space with respect to the $\mathcal{R}(J)$ -topology.

If J is a complete local Jordan algebra, and J^* is the completion of J , then $J = J^*$, and $\mathcal{R}(J)^{(k)} = \mathcal{R}(J)^{(k)*}$ for all $k \in \mathbb{N}$.

Hence we can apply Theorem 4.3.1 to obtain

Theorem 4.4.1. Let J be a complete local Jordan algebra of radical capacity n . Then:

- (1) If $n = 1$, J is a complete, completely primary local Jordan algebra and $\mathcal{R}(J)$ is the set of noninvertible elements of J .
- (2) If $n = 2$, then $J = J_1 \oplus J_2 \oplus S$, where J_1, J_2, S are as described in Theorem 4.3.1.
- (3) If $n \geq 3$, then $J \cong \mathfrak{S}(\mathfrak{A}_n, j_a)$, a Jordan matrix algebra such that $\mathcal{R}(\mathfrak{S}(\mathfrak{A}_n, j_a)) = \mathcal{R}(\mathfrak{A})_n \cap \mathfrak{S}(\mathfrak{A}_n, j_a)$, and
 - (i) $(\mathfrak{A}, j)$ is a (not associative) alternative algebra as in Theorem 4.3.1 ($n = 3$ only) or
 - (ii) $(\mathfrak{A}, j)$ is an associative algebra with involution which is a complete semi-local algebra such that $\mathcal{R}(\mathfrak{A})$ is the unique maximal ideal of $(\mathfrak{A}, j)$.

Conversely, if J is a complete, completely primary Jordan algebra, or $J = \mathfrak{S}(\mathfrak{A}_n, j_a)$, a Jordan matrix algebra of order $n \geq 3$ with $(\mathfrak{A}, j)$ as in (3), then J is a complete local Jordan algebra.

Proof. (1), (2) and (3-i) are immediate from Theorem 4.3.1.

(3-ii) is immediate once it is noted that in the notation of Theorem 4.3.1, $A(k) = \mathcal{R}(\mathcal{A})^{3^{k-1}}$ by Lemma 4.3.3.

For the converse, first assume that J is a complete, completely primary Jordan algebra. Then $\mathcal{R}(J)$ is a nucleus for J , $J/\mathcal{R}(J)$ is a Jordan division algebra so that $\mathcal{R}(J)$ is the unique maximal ideal of J , and $J/\mathcal{R}(J)$ trivially satisfies the minimum condition. Thus J is a complete local Jordan algebra.

Next let $n \geq 3$, $J = \mathfrak{A}(\mathcal{A}_n, j_a)$, and $\mathcal{R}(J)^{(k)} = A(k) \cap J$, where $A(1) = \mathcal{R}(\mathcal{A})$. We will prove the completeness of J for $(\mathcal{A}, j)$ associative only, but it will be clear how the proof can be adjusted for the alternative case. So assume $\mathcal{A}$ is an associative semi-local algebra which is complete in the $\mathcal{R}(\mathcal{A})$ -topology, and $\mathcal{R}(\mathcal{A})$ is the unique maximal ideal of $(\mathcal{A}, j)$. Then $\mathcal{R}(J) = \mathcal{R}(\mathcal{A})_n \cap J$ is the unique maximal ideal of J and $\mathcal{R}(\mathcal{A})$ a nucleus for $\mathcal{A}$ implies $\mathcal{R}(J)$ is a nucleus for J (Lemma 4.3.3). Since $(\mathcal{A}/\mathcal{R}(\mathcal{A}), j)$ is a simple Artinian algebra with involution, and $J/\mathcal{R}(J) \cong \mathfrak{A}(\bar{\mathcal{A}}_n, j_a)$ by Corollary 2.7.1, $J/\mathcal{R}(J)$ satisfies the minimum condition by the Second Structure Theorem (Proposition 2.7.16). Thus J is a local Jordan algebra.

Let $\{X_r\}$ be a Cauchy sequence in J . Then for each $r \in \mathbb{N}$, $X_r = \sum_{i,j} x_{ij}^{(r)} e_{ij}$, $x_{ij}^{(r)} \in \mathcal{A}$. If k is given, then there exists $N \in \mathbb{N}$ such that $r_1, r_2 \geq N$ implies $X_{r_1} - X_{r_2} \in \mathcal{R}(J)^{(k)} = A(k)_n \cap J = \mathcal{R}(\mathcal{A})_n^{3^{k-1}} \cap J$ (Lemma 4.3.3). Thus when $r_1, r_2 \geq N$, $x_{ij}^{(r_1)} - x_{ij}^{(r_2)} \in \mathcal{R}(\mathcal{A})^{3^{k-1}}$. Therefore, if $p \in \mathbb{N}$, choose k such that $3^{k-1} \geq p$. Then for $r_1, r_2 \geq N$, $x_{ij}^{(r_1)} - x_{ij}^{(r_2)} \in \mathcal{R}(\mathcal{A})^{3^{k-1}} \subseteq \mathcal{R}(\mathcal{A})^p$;

i.e., for each pair $i, j = 1, 2, \dots, n$, $\{x_{ij}^{(r)}\}$ is a Cauchy sequence

in $\mathcal{A}$. Thus for each (i, j) there exists $x_{ij} \in \mathcal{A}$ such that

$\lim_r x_{ij}^{(r)} = x_{ij}$, and hence for any $p \in \mathbb{N}$ there exists $N_{ij} \in \mathbb{N}$

such that if $r \geq N_{ij}$, then $x_{ij} - x_{ij}^{(r)} \in \mathcal{R}(\mathcal{A})^p$. If $k \in \mathbb{N}$,

let $p = 3^{k-1}$. Then for $r \geq \max \{N_{ij} : i, j = 1, 2, \dots, n\}$,

$x_{ij} - x_{ij}^{(r)} \in \mathcal{R}(\mathcal{A})^p = \mathcal{R}(\mathcal{A})^{3^{k-1}}$ for every $i, j = 1, 2, \dots, n$. Thus,

if $X = \sum_{i,j} x_{ij} e_{ij}$, then $X - X_r \in \mathcal{R}(J)^{(k)}$, and $\lim_r X_r = X \in \mathcal{A}_n$.

It remains to be shown that $X \in \mathfrak{S}(\mathcal{A}_n, j_a) = J$. Now for all $r \in \mathbb{N}$,

$X_r \in \mathfrak{S}(\mathcal{A}_n, j_a)$, and thus $X_r = \sum_{i < j} x_{ij}^{(r)} [ij] + \frac{1}{2} \sum_i x_{ii}^{(r)} [ii]$. Since

$$\sum_{i < j} x_{ij} [ij] + \frac{1}{2} \sum_i x_{ii} [ii] - X_r = \sum_{i < j} (x_{ij} - x_{ij}^{(r)}) [ij] + \frac{1}{2} \sum_i (x_{ii} - x_{ii}^{(r)}) [ii],$$

$X = \lim_r X_r = \sum_{i < j} x_{ij} [ij] + \frac{1}{2} \sum_i x_{ii} [ii] \in \mathfrak{S}(\mathcal{A}_n, j_a)$. Therefore J

is a complete local Jordan algebra.

If $n = 3$ and $(\mathcal{A}, j)$ is alternative with $\mathcal{R}(\mathcal{A})$ the unique maximal ideal of $(\mathcal{A}, j)$ and $\mathcal{R}(\mathcal{A})_n \cap J = \mathcal{R}(J)$, then $\mathcal{R}(J)$ is the

unique maximal ideal of J . If $\bigcap_{k=1}^{\infty} A(k) = 0$, then $\bigcap_{k=1}^{\infty} \mathcal{R}(J)^{(k)} = 0$,

so that $\mathcal{R}(J)$ is a nucleus for J . If $\mathcal{A}$ is complete in the

topology induced by the $A(k)$, $k = 0, 1, 2, \dots$, then J is complete

in the $\mathcal{R}(J)$ topology as in the associative case. Since $J/\mathcal{R}(J)$

$\cong \mathfrak{S}(\bar{\mathcal{A}}_n, j_a)$, where $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{R}(\mathcal{A})$, and $(\bar{\mathcal{A}}, j)$ is either an Octonian

algebra with standard involution or a simple Artinian algebra

with involution, $J/\mathcal{R}(J)$ satisfies the minimum condition by the

Second Structure Theorem. Thus J is a complete local Jordan

algebra.

The concepts of commutative associative algebra with identity and associative (linear) Jordan algebra coincide over a field of

characteristic not 2. If $\mathfrak{U}$ is such an algebra, then $\mathcal{R}(\mathfrak{U})^{(k)} = \mathcal{R}(\mathfrak{U})^{3^{k-1}}$, $k \in \mathbb{N}$, by Lemma 4.3.2. Thus $\mathfrak{U}$ is a not-necessarily Noetherian local associative algebra if and only if $\mathfrak{U}$ is a local Jordan algebra. $\mathfrak{U}$ is a metric space with respect to both the associative and Jordan $\mathcal{R}(\mathfrak{U})$ topologies. Suppose $\{x_n\}$ is a Cauchy sequence in $\mathfrak{U}$ relative to the associative $\mathcal{R}(\mathfrak{U})$ -topology and assume $\mathfrak{U}$ is complete with respect to the Jordan $\mathcal{R}(\mathfrak{U})$ -topology. Then for any $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies $x_m - x_n \in \mathcal{R}(\mathfrak{U})^k$. Hence given $k' \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies $x_m - x_n \in \mathcal{R}(\mathfrak{U})^{3^{k'-1}} = \mathcal{R}(\mathfrak{U})^{(k')}$, and $\{x_n\}$ is Cauchy in the Jordan topology. Thus there exists $x \in \mathfrak{U}$ such that $\lim_n x_n = x$ in this topology. That is for any $k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x - x_n \in \mathcal{R}(\mathfrak{U})^{(k)} = \mathcal{R}(\mathfrak{U})^{3^{k-1}}$. Thus if $p \in \mathbb{N}$, and $3^{k-1} \geq p$, $x - x_n \in \mathcal{R}(\mathfrak{U})^{3^{k-1}} \subseteq \mathcal{R}(\mathfrak{U})^p$ for all $n \geq N$. Hence $\lim_n x_n = x$ in the associative topology, and $\mathfrak{U}$ is complete in this topology. The converse is equally clear. Therefore $\mathfrak{U}$ is a complete local (associative) algebra if and only if $\mathfrak{U}$ is a complete local Jordan algebra. Note that in this case $\mathfrak{U}$ has capacity 1.

We close this section with several questions which remain open.

1. Is there a corresponding local algebra theory for alternative algebras? The difficulty here lies in finding ideals corresponding to A^k in the associative case and $A^{(k)}$ in the Jordan case.
2. Can the structure for radical capacity 2 be more completely determined? This problem seems to be

laden with difficulties.

3. Under what conditions is the completion J of a local Jordan algebra $\mathcal{J}$ itself complete? A trivial condition is that $\mathcal{R}(\mathcal{J}^*)^{(k)} = \mathcal{R}(\mathcal{J})^{(k)*}$.

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