THE CHOICE OF AN OPTIMAL CONSUMER PLANNING HORIZON

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ABSTRACT

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By

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New and improved analytic treatment has recently been given to optimal consumer behavior within an intertemporal framework. Consumer plans involving the optimal choice of a consumption path, a leisure path, a demand for money path, and a bequest have been studied. First-order and second-order conditions have been derived for intertemporal utility maximization, properties of optimal paths have been determined, and, to a certain extent, uncertainty has been introduced into the consumer's environment. An analytic tool much used in this recent work has been the calculus of variations, a branch of mathematics serving as the classic foundation of modern optimal control theory.

This recent work on intertemporal consumer theory has taken the interval of time over which plans are to be formulated as being given to the consumer. Typically, the consumer's remaining life-span is this interval of time where, with a single exception, the remaining life-span is assumed to be known with certainty. Even in the uncertain lifetime case, the planning interval is given or fixed for the consumer.

The fundamental contribution of this thesis is to consider the interval of time over which a consumer constructs an optimal plan as

being another decision variable for the consumer. That is, the consumer, in maximizing intertemporal utility, chooses the length of his planning interval as well as the paths of economic activity defined over that interval.

This thesis considers consumer planning both in a "certainty" environment (among other things, the consumer knows when he will die) and in an "uncertainty" environment (the date of death is a random variable with a known probability density function). In the simpler certainty case, first-order and second-order conditions for utility maximization are derived along with some comparative static properties (Chapters II and III), while in the uncertainty case, complications introduced by uncertainty are discussed and some necessary conditions are derived for expected utility maximization (Chapter IV). Elementary variational calculus is used to set up and solve the planning problems presented in this thesis.

The fundamental analytic results center around conditions necessary for intertemporal utility maximization or expected intertemporal utility maximization. Especially important are those conditions which must be satisfied by the optimal planning horizon, the end-point of the consumer's planning interval. Stated simply, the key condition which must be met at the optimal planning horizon, given that this point is not some last possible date of death, is that the planned path of nonhuman assets must be declining with the rate of decline equalling the marginal rate of substitution between expanding the planning interval and nonhuman assets.

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By

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CHAPTER I

INTRODUCTION

The problem of intertemporal utility maximization by a consumer is one which economists have considered for years and for a variety of reasons. At least forty years ago, Irving Fisher [6] incorporated intertemporal utility notions in his well-known theory of the interest rate. Roughly forty years ago, F. P. Ramsey [18] pioneered the use of the calculus of variations to solve intertemporally for the optimal rate of saving for a community and for an individual with a fixed remaining life-span. A bit later Gerhard Tintner [22] directly approached the problem of maximizing consumer intertemporal utility and derived intra-temporal and intertemporal necessary conditions. More recently, James Duesenberry [5], Modigliani and Brumberg [16], and Milton Friedman [8] used intertemporal analyses in separate attempts to reconcile analytically estimated short-run consumption functions with estimated long-run consumption functions.

Even more recently, a small but growing body of literature has addressed itself to a wide range of interrelated consumer decisions. These efforts have centered around the choice of optimal time paths for consumer economic variables, of which the universally included one is the optimal lifetime consumption path. R. H. Strotz [21] tackled the question of whether a consumer would follow,over time, a previously selected optimal consumption path. Strotz found that the consumer would follow a previously selected consumption path if the discount function attached to future utility is logarithmically linear with respect to the distance between any future date and the present date. Menahem E. Yaari presented a series of articles on consumer behavior. In one [25], he derived existence conditions for optimal plans. In another [24], he incorporated the bequest in the consumer's utility functional, derived marginal utility conditions for the optimal consumption path, and determined the properties of that path. In the third [26], he considered optimal consumption and savings behavior in the face of uncertainty regarding the amount of remaining lifetime with and without life insurance or annuity streams. Pesek and Saving [17] used an intertemporal approach to determine, among other things, the optimal time path of real money holdings for a consumer. Nils Hakansson [11] very recently published a discrete-time model generating for a particular class of utility functions optimal consumption, investment, and borrowing-lending strategies in a world of risky investment opportunities.

The quality and wide range of work done by this impressive group of economists indicates that intertemporal consumer theory is an interesting and important area of economic theory. This thesis will attempt to contribute to this area.

The Basic Purpose of This Thesis

This thesis will focus attention on the choice by a consumer of the terminal point for his planning interval. The planning interval is the period of time for which the consumer chooses paths of various economic variables. In this thesis, the terminal point of the planning interval will alternatively be called the planning horizon, while the date of death will alternatively be called the horizon.[1]

The common assumption made in intertemporal consumer models is that the consumer plans over his entire remaining life. With the exception of Yaari's paper on uncertain lifetime, these models take or assume the remaining lifetime as being certain to the consumer, occasionally treating it as infinite in duration. In a world of total certainty, it may be reasonable for many purposes to consider the consumer's planning horizon as being his date of death. However, it may also be of interest not to constrain a consumer to plan completely to a known or assumed known death date. One problem investigated by this thesis is the determination of conditions necessary for utility maximization given that the planning horizon falls short of the horizon.

In Yaari's treatment of a random horizon, the assumption is made that the consumer plans to his final possible date of life. In a world of annuity streams, one might argue that such a constraint is not unreasonable. However, in general, it may be unreasonable. For one thing, annuities are not free goods. The consumer may choose not to buy one. Moreover, even if an annuity contract is purchased, that alone provides no reason for the consumer to plan to any fixed future date, let alone the last possible date of life. Also, planning typically involves the expenditure of current resources. Time and perhaps money must be spent in gathering information about an uncertain future and in forming expectations about the future. In general, these current costs

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[1]In the literature of intertemporal consumer theory, the term "horizon" has meant a number of things. "Horizon" refers sometimes to a point in time [Strotz (21, p.170); Yaari (24, p.304); Arrow and Kurz (2, p.155)], sometimes to an interval of time [Henderson and Quandt (12, p.298), Hadar (10, pp.209-249)], and occasionally to either [Pesek and Saving (17, p.309)].

of planning will increase as the planning interval increases because of increased uncertainty the further one tries to forecast. Thus, the presence of "planning costs" may be expected to influence the choice of a planning horizon. A second problem investigated by this thesis, then, is to determine conditions necessary for utility maximization for the case of a random horizon.

In short, the basic feature of this thesis is to consider the planning horizon as another decision variable for the consumer. The basic purpose is to determine conditions which must be met for utility to be maximized with emphasis placed upon those conditions which must be met by the planning horizon.

The Analytic Approach of This Thesis

An infinite dimensional, intertemporal approach will be used. Intertemporal analyses have been used successfully by others, and infinite dimensional ones have provided convenient ways of looking at complex economic problems. The infinite dimension aspect enters by considering time to be a continuous variable, and allows the use of the calculus of variations to set up and solve various consumer problems.

The basic assumption made with respect to consumer behavior is that the consumer is an intertemporal utility maximizing unit subject to an intertemporal wealth constraint. By using an infinite dimensional approach, it is assumed that a consumer derives utility from the paths of economic variables over time. Specifically, utility is a functional; i.e., a correspondence that assigns a real number to each function or curve belonging to some class.[2] The utility functional to be maximized will be taken to be dependent upon certain points in time, certain economic paths over time, and certain variables or functionals. The intertemporal wealth constraint will involve the consumer's initial asset position, his paths of consumption and human income, and his bequest position or, more generally, his path of nonhuman assets.

The interval of time involved for the paths of economic activity clearly must depend upon the consumer's expected remaining life-span. Concerning his own economic activity, the consumer would not plan beyond any known date of death or beyond any last possible date of life.[3] The planning horizon consequently must depend upon the consumer's mortality function, the function describing his chance of dying at any future date. This thesis will first consider the case of a known or assumed known death date and then will consider the death date as a random variable. One of the goals of the following models is to show how aging and uncertainty regarding death may be expected to influence currently planned consumer behavior. To achieve this goal, the effect of this uncertainty on the planning horizon must be taken into account.

The only source of uncertainty that will be considered in this thesis is that connected with remaining lifetime. Uncertainties

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[2]The above definition of a functional is taken from Gelfand and Fomin [9, p.1]. For a concise description of the nature of a functional, see R. G. D. Allen [1, p.521]. Integrals are the only kind of functional dealt with in this thesis.

[3]The case where a consumer leaves a bequest with the stipulation that his heirs may consume only the interest and not the principal may be looked at as a case where the consumer is planning part of his heirs' economic activity, but not part of his own. The size of the principal would be the only value of interest to the models that follow. As an aside, any spiritual planning beyond death would be considered religious activity, not economic activity.

regarding future income parameters, future rates of return on savings, or future investment opportunities will be bypassed.

It should be pointed out that the analysis will be based on the classic calculus of variations. In particular, the choice of an optimal consumer plan defined on an optimal planning horizon will be solved as a variable end-point problem in the calculus of variations. Almost entirely, interior solutions will be assumed to exist.

Outline of the Thesis

Chapter II will present a "certainty" model, one for which the date of death is known. The model presented follows closely that of Pesek and Saving. It differs from the Pesek-Saving model in that the time path of real money holdings and transactions time are completely disregarded. More importantly, the bequest left at death rather than a gift-expenditure function of time is used as an argument of the utility functional.[4] First- and second-order utility maximizing conditions are derived, and some comparative static properties of the solution are derived. A numerical example is presented with its solution.

Chapter III will consider the possibility of the planning interval falling short of the known remaining life-span in a certainty world. Necessary conditions regarding the end point of the planning interval will be stressed, an example and solution will be given, and the existence of planning costs will be touched upon.

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[4]It may be noted that towards the end of their analysis, Pesek and Saving allow the time path of gifts to become a true bequest variable in that they assume this path to be zero except for a small interval at the end of the consumer's remaining life; see Pesek and Saving [17, p.352].

Chapter IV will present an "uncertainty" model. Here the horizon will be taken to be a random variable. The effect of an uncertain death date on the utility functional will be discussed along with alternative meanings of utility maximization in an uncertainty environment. The expected utility hypothesis will be considered. Necessary conditions for utility maximization will be presented and compared with those of the certainty case. The role played by the mortality density function as it bears upon the utility functional will be stressed. Simple consumer planning problems under various environments will be given along with the solutions. Again, planning costs will be mentioned but not formally made part of the model.

Chapter V will serve as a review and summary of the main analytic conclusions. In addition, further problems of consumer planning which might be handled by models similar to those in this thesis will be suggested but no applications or solutions will be carried out.

CHAPTER II

A CERTAINTY MODEL OF CONSUMER BEHAVIOR

In the model in this chapter, the consumer maximizes a utility functional subject to certain constraints and in which he knows or assumes to know when he will die. As an initial step, the case is considered where the consumer plans over his entire remaining lifetime.

The initial question to be asked regards the form of the utility functional. The utility functional will be taken to be an integral whose limits of integration are the end-points of the consumer's remaining life-span and whose integrand has the path of consumption, the path of leisure, the bequest, and certain points in time as arguments. Such utility functionals typically have involved additive intertemporal utility (Hadar [10, pp.228-249], Pesek and Saving [17, pp.308-312], Yaari [24,25,26]); that is, the rate of utility at any point t in the consumer's planning interval has been taken to be a function of the rates of consumption, leisure, etc., all at time t only. Alternatively, the rate of utility derived from the rates of economic activity at any point t has been taken to be independent of the rates of economic activity at any other point t + k. Using a functional with this property destroys some desirable generality. In particular, this approach eliminates any intertemporal complementarity or substitutability that might exist between consumption, leisure, and assets. The utility functional used in the model in this chapter will be somewhat more general in that the rate of utility at any point t will not be taken to be dependent only on the rates of economic activity at t. This element of increased generality will be discussed later in connection with the form of the integrand.

With respect to the arguments of the integrand, consider first the time path of real consumption. By real consumption is meant expenditures made on nondurable commodities and services plus the imputed rental values of any consumer durable goods held by the consumer, all expressed in terms of constant dollars. Purchasing a durable good is not part of current consumption. Consumption is taken as the aggregate of expenditures on individual commodities and services.

The second argument is the time path of leisure. Clearly, leisure is a good which yields utility to the consumer and, in general, is a decision variable for any consumer. Directly associated with the choice of an optimal leisure path is the choice of an optimal work path which determines the consumer's human income path and his present wealth constraint. With the exception of the Pesek-Saving model, leisure has been neglected in the previous literature. Instead, the consumer has been viewed as having a fixed stock of wealth to allocate to consumption over time; or, he has been viewed as facing a fixed human income path. The insertion of the leisure path provides an avenue for considering retirement decisions and allows the consumer discretionary power over the size of his present wealth constraint.

The third argument is the consumer's bequest, alternatively called his "terminal assets." Presumably, a consumer, especially the head of a household, will derive utility from knowing that his family or some other heir will receive a stock of nonhuman assets at his death.

Some previous models have included the bequest as a decision variable, others have omitted it.

Finally, time and in particular specific points in time will be included as arguments. One would expect that as a consumer grows older, the subjective criteria on which he partially bases economic decisions would change. That is, the utility function can be expected to shift with age. For example, the subjective evaluation of a quantity of leisure relative to a quantity of consumption can be expected to increase as the age of a consumer approaches his age at death. Consequently, the present value of the marginal utility of consumption at a fixed future date t relative to that of leisure at t can be expected to change as the present point in time approaches t. (By the present value of marginal utility of consumption at some future date t is meant the marginal utility of consumption at t currently discounted back to the present point in time τ by use of a subjective rate of discount. For convenience, call this the present marginal utility of consumption at t.) The insertion of time in general, the present point in time, and the expected date of death provide for proper discounting of future events and allow marginal rates of substitution to change with age.

Given the above arguments, the integrand of the utility functional can be written as

$$\mu[c(t),l(t),a(T),\tau,T,t], \qquad \text{(i)}$$

where c(t) is the time path of real consumption, l(t) is the time path of leisure, a(T), nonhuman assets at the point in time T, is the bequest in real terms, τ is the present point in time, T is the date of death, and t is time. Some authors (Strotz [21], Yaari [24,26]) have chosen to write the integrand as the product of a discount function and a utility function. The discount function has been taken to be a function of time, in particular, the distance between any date and the present date, with a particular subjective rate of discount being assumed. The utility function has been taken to be independent of the present point in time. Under this approach, μ would appear as either

$$\alpha(\tau,T,t)g[c(t),l(t),a(T)], \text{ or} \tag{iia}$$

$$\alpha(\tau,T,t)g[c(t),l(t),a(T),t], \tag{iib}$$

where α is the discount function, and g is the utility function. Also, the utility function g has typically been additive. That is, Yaari, in considering the utility to be derived from consumption and from a bequest, merely added together two utility functions, each with an associated discount function. Under this approach, μ would appear as either

$$\alpha(\tau,T,t)\upsilon[c(t),l(t)] + \beta(\tau,T)\nu[a(T)], \text{ or} \tag{iiia}$$

$$\alpha(\tau,T,t)\upsilon[c(t),l(t),t] + \beta(\tau,T)\nu[a(T),T], \tag{iiib}$$

where α and β are discount functions, and υ and ν are utility functions.

An additive integrand as in (iiia) or (iiib) has some disadvantages. Define intratemporal complementarity to be the case where a positive value results for cross-partials of μ evaluated at a point in time, e.g., $\mu_{c(t)l(t)} > 0$. Define intratemporal substitutability similarly but for a resulting negative value, e.g., $\mu_{c(t)l(t)} < 0$. Define intertemporal complementarity to be the case where a positive value results for cross-partials of μ given that the differentiation is evaluated at two separate points in time, e.g., $\mu_{c(t)l(t+k)} > 0$. Define intertemporal substitutability similarly but for a resulting negative value, e.g., $\mu_{c(t)l(t+k)} < 0$. Using these definitions under the formulation given by (iii), no intertemporal complementarity or substitutability may exist for consumption, leisure, and the bequest, e.g., $\mu_{c(t)l(t+k)} \equiv 0$ and $\mu_{c(t)c(t+k)} \equiv 0$, while intratemporal complementarity or substitutability may exist only between leisure and consumption, e.g., $\mu_{c(t)l(t)} \not\equiv 0$.

In order to allow more rather than less generality, the integrand to be used in this chapter will take the form given by (i). Such a general integrand allows, though does not require, the present point in time and the date of death to influence intratemporal marginal rates of substitution and allows some intertemporal complementarity or substitutability to exist among consumption, leisure, and the bequest, e.g., $\mu_{c(t)a(T)} \not\equiv 0$.

Given that the date of death is known with certainty (or assumed to be known) by the consumer and that he plans over his entire remaining lifetime, the utility functional can be written as:

$$U[c,l,a(T);\tau,T] = \int_{\tau}^{T} \mu[c(t),l(t),a(T),\tau,T,t]dt, \tag{1}$$

where U is the present subjective value of total utility (present total utility) to be enjoyed from the bequest and from the paths of leisure and consumption over the remaining life. The integrand μ is the rate of present utility derived from the rates of consumption and leisure at any t and from the bequest. Integrating μ over some interval gives the contribution to present utility made by the rates of consumption and leisure and by the bequest over that interval. Splitting the integral in (1) into n one-year intervals,

$$\int_{\tau}^{T} \mu \, dt = \int_{\tau}^{\tau+1} \mu \, dt + \int_{\tau+1}^{\tau+2} \mu \, dt + \ldots + \int_{T-1}^{T} \mu \, dt,$$

each one-year integral gives the present utility derived from the respective one-year rates of leisure and consumption expenditure and from the size of the bequest. (In a sense, the bequest can be interpreted as weighting the streams of leisure and consumption so that if the bequest were to increase, the utility from consumption and leisure and the marginal utilities of consumption and leisure would increase.) For example, the term $\int_{\tau}^{\tau+1} \mu \, dt$ gives the utility at τ to be derived from the first year's planned leisure and consumption weighted by the bequest. Adding over all years in the remaining life-span gives the total present utility of the bequest and paths of consumption and leisure.

Consider a(t) as the time path of real nonhuman assets, and assume that c(t), l(t), and a(t) are all bounded continuous real functions defined over the interval $[\tau,T]$ that meet the side conditions below; a(T) is a continuous real variable such that $0 \leqq a(T) \leqq a_M$, where a_M is some finite maximum quantity allowed by the consumer's wealth constraint. The side conditions to be met are:

$$\begin{aligned} &c(t) \geqq 0, && \text{for all } t \text{ in } [\tau,T]; \\ &l(t) \geqq 0, && \text{for all } t \text{ in } [\tau,T]; \qquad (2) \\ &a(t) \gtreqless 0, && \text{for all } t \text{ in } [\tau,T); \\ &0 \leqq a(T) \leqq a_M. && \end{aligned}$$

As the utility functional is written in equation (1), the decision variables facing the consumer are the two continuous functions of time, c(t) and l(t), and the single continuous variable, a(T), which

takes on a value at the point in time, T. (Formally, from the lifetime wealth constraint imposed below, a(T) is a functional.) This approach seems to indicate that the bequest variable is somewhat different in nature from the time paths of consumption and leisure. While choosing particular values for the rates of consumption and of leisure throughout his remaining lifetime, the consumer chooses a particular value for nonhuman assets only at his date of death. Negative nonhuman assets short of the date of death provide no disutility to the consumer, only the terminal value counts in his utility considerations. In order to have all the decision variables be members of the same linear space[1] (bounded continuous real functions defined over $[\tau,T]$) as is often the case in variational problems, one could adopt an alternative form for a(T). For every permissible value for a(T), there exists a function A(t) defined over $[\tau,T]$ which is a constant equal in value to a(T) and, of course, is real and continuous. Accordingly, one could use the functions c(t), l(t), and A(t) as the decision functions for the consumer being sure, of course, to modify wherever necessary any constraints on the consumer's behavior. These decision functions would all be members of the same linear space and would be required to meet the nonnegativity requirements everywhere. An advantage of this approach would be that one could use basic concepts of variational calculus such as "continuity of functionals" and "norm of a function" as they are generally defined without any qualification or special note. However, a disadvantage of this approach would be that the *admissible* c(t), l(t), and A(t), viz., those that satisfy the wealth constraint, would not, themselves, form a

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[1]For a definition of a linear space, see Gelfand and Fomin [9, p.5].

linear space. The sum of any two such functions may not belong to the space of all functions satisfying the constraint. As a consequence of this disadvantage introduced by the consideration of a constraint, the model here will use c(t), l(t), and a(T) as the decision variables for the consumer. This approach merely requires that one be careful to define distance between functions, variation of a functional, and other concepts used in variational problems as the analysis develops.[2]

In maximizing his utility functional, the consumer faces the following basic wealth constraint which can be looked upon as a rewriting of the definition of the bequest,

$$a(\tau) + \int_{\tau}^{T} e^{r(\tau-t)} y(t)dt = \int_{\tau}^{T} e^{r(\tau-t)} c(t)dt + e^{r(\tau-T)} a(T), \qquad (3)$$

where $a(\tau)$ is the real value of present nonhuman assets, r is the rate of interest assumed to be constant over $[\tau,T]$, and y(t) is the time path of real human income. Writing the constraint this way implies that borrowing-lending opportunities are unlimited short of the date of death and involve the same rate of interest for borrowing as for lending, the market rate of interest. The only institutional constraint placed upon the consumer as a borrower is that included in side-conditions (2), namely, that he not die with a negative net worth. Also, constraint (3) implies that all nonhuman assets held earn the market rate of interest.[3]

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[2]Gelfand and Fomin [9, p.8] remark that the solution to a variational problem does not require that one deal with a linear space, only that concepts associated with such a space such as continuity of functionals are still relevant.

[3]Other analyses of this type have taken all nonhuman assets to be held in the form of real bonds which always yield the market rate of interest, see Yaari [24, p.304] and Pesek and Saving [17, p.311]. The only serious difficulty with this approach is that the consumer might hold noninterest-bearing money; this complication will be neglected in

Further, note the basic time constraint that any period of time, $t_1 - t_0$, must be spent entirely in leisure and work, or

$$\int_{t_0}^{t_1} [l(t)+w(t)]dt = t_1 - t_0, \tag{4}$$

where w(t) is the time path of work, a bounded continuous nonnegative real function defined over $[\tau,T]$. But (4) implies the following identity,

$$l(t) + w(t) \equiv 1. \tag{5}$$

Considering the human income function to be a function of the time path of work and of time itself (in order to allow the wage rate to change over time), $y(t) = f[w(t),t]$, one obtains from (5),

$$y(t) = F[l(t),t], \tag{6}$$

where it is assumed that $F_l = -f_w < 0$, $F_{ll} = f_{ww} = 0$, and $F(1,t) = f(0,t) = 0$, for all t in $[\tau,T]$. Upon substituting (6) into (3) and solving for $a(T)$, one obtains the following expression for terminal assets,

$$a(T) = e^{r(T-\tau)}a(\tau) + \int_{\tau}^{T} e^{r(T-t)}\{F[l(t),t]-c(t)\}dt. \tag{7}$$

From identity (5) and the nonnegativity of w(t), $l(t) \leqq 1$, which implies $F[l(t),t] \geqq 0$, for all t in $[\tau,T]$; i.e., over the remaining lifetime, the rate of human income must be nonnegative.

The consumer's utility maximization problem can be summarized in

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this thesis. Pesek and Saving later in their model take money holdings into account and work out some of the implications.

the following way:

$$\max_{c,l,a(T)} U[c,l,a(T);\tau,T] = \int_\tau^T \mu[c(t),l(t),a(T),\tau,T,t]dt,$$

subject to:

(a) $a(T) = e^{r(T-\tau)}a(\tau) + \int_\tau^T e^{r(T-t)}\{F[l(t),t]-c(t)\}dt,$

(b) $c(t) \geq 0,$ for all t in $[\tau,T]$;

$l(t) \geq 0,$ for all t in $[\tau,T]$;

$F[l(t),t] \geq 0,$ for all t in $[\tau,T]$;

$a(T) \geq 0.$

Necessary Conditions for a Solution

Assume a solution exists, and let $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$ be that solution; i.e., $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$ are respectively the optimal consumption path, the optimal leisure path, and the optimal bequest. What conditions must be met by the solution?

First, rewrite the bequest constraint to place the bequest inside the integral,

$$I[c,l,a(T);\tau,T] \equiv \int_\tau^T \left(e^{r(T-t)}\{c(t)-F[l(t),t]\} + \left(\frac{1}{T-\tau}\right)a(T)\right)dt \qquad (8)$$

$$= e^{r(T-\tau)}a(\tau).$$

By the "isoperimetric theorem,"[4] if $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$ are extremals of U,[5] but not of I, then $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$ are extremals of

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[4]Gelfand and Fomin [9, pp.42-46]; Hestenes [13, pp.83-87].

[5]An extremal (or extremaloid) is a curve which satisfies Euler's equation which is a necessary condition for a functional to have an extremum.

$$U^*[c,l,a(T);\tau,T,\lambda] = \int_\tau^T (\mu+\lambda g)dt, \tag{9}$$

where λ is a nonzero constant, and g is the integrand of I. By the "boundary arc theorems,"[6] if $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$ are extremals of U^*, then $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$ are extremals of

$$U^{**}[c,l,a(T);\tau,T,\lambda,\eta_c,\eta_{l_0},\eta_a(T)] = \int_\tau^T \{[\mu+\lambda g] \tag{10}$$

$$+ [\eta_c(t)c(t)+\eta_{l_0}(t)l(t)+\eta_{l_1}(t)F[l(t),t]+\eta_a(T)\left(\frac{1}{T-\tau}\right)a(T)]\}dt,$$

where $\eta_c(t)$, $\eta_{l_0}(t)$, $\eta_{l_1}(t)$, and $\eta_a(T)$ are continuous nonnegative multipliers, and where $\eta_c(t)$, $\eta_{l_0}(t)$, $\eta_{l_1}(t)$, and $\eta_a(T)$ equal zero anytime the respective nonnegativity constraints on c(t), l(t), F[l(t),t], and a(T) are not binding.

Next, assume that μ has continuous partial derivatives up to at least order two with respect to all its arguments, and consider the increment in the functional U^{**} resulting from arbitrary increments given to $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$. Initially,

$$\Delta U^{**} = \int_\tau^T \{\mu[\hat{c}(t)+h_c(t),\hat{l}(t)+h_l(t),\hat{a}(T)+\delta a(T),\tau,T,t] \tag{11}$$

$$+ \lambda[e^{r(T-t)}\left(\hat{c}(t)+h_c(t)-F[\hat{l}(t)+h_l(t),t]\right)+\left(\frac{1}{T-\tau}\right)\left(\hat{a}(T)+\delta a(T)\right)]$$

$$+ \eta_c(t)[\hat{c}(t)+h_c(t)] + \eta_{l_0}(t)[\hat{l}(t)+h_l(t)] + \eta_{l_1}(t)F[\hat{l}(t)+h_l(t),t]$$

$$+ \eta_a(T)\left(\frac{1}{T-\tau}\right)[\hat{a}(T)+\delta a(T)]\}dt$$

$$- U^{**}[\hat{c},\hat{l},\hat{a}(T);\tau,T,\lambda,\eta_c,\eta_{l_0},\eta_{l_1},\eta_a(T)]\}$$

where $h_c(t)$ is the increment given to the consumption path, $h_l(t)$ is the

[6] Hestenes [13, pp.93-97].

increment given to the leisure path, and $\delta a(T)$ is the increment given to the bequest. Using a Taylor expansion around $\mu[\hat{c}(t),\hat{l}(t),\hat{a}(T),\tau,T,t]$ and evaluating the partial derivatives at $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$, one obtains

$$\begin{aligned}
\Delta U^{**} = \int_{\tau}^{T} \{ & [\mu_c(t)h_c(t)+\mu_l(t)h_l(t)+\mu_a(t)\delta a(T)] + \tfrac{1}{2}[\mu_{cc}(t)h_c^2(t) \\
& + \mu_{ll}(t)h_l^2(t) + \mu_{aa}(t)\delta^2 a(T) + 2\mu_{cl}(t)h_c(t)h_l(t) + 2\mu_{ca}(t)h_c(t)\delta a(T) \\
& + 2\mu_{la}(t)h_l(t)\delta a(T)] + \varepsilon + \lambda[e^{r(T-t)}\left(h_c(t)-F_l(t)h_l(t)\right) \\
& + \left(\frac{1}{T-\tau}\right)\delta a(T)] + \eta_c(t)h_c(t) + \eta_{l_0}(t)h_l(t) + \eta_{l_1}(t)F_l(t)h_l(t) \\
& + \eta_a(T)\left(\frac{1}{T-\tau}\right)\delta a(T)\}dt,
\end{aligned} \tag{12}$$

where $\varepsilon = \varepsilon_1 h_c^2(t) + \varepsilon_2 h_l^2(t) + \varepsilon_3 \delta^2 a(T) + \varepsilon_4 h_c(t)h_l(t) + \varepsilon_5 h_c(t)\delta a(T) +$ $\varepsilon_6 h_l(t)\delta a(T)$. Expressing U^{**} as a sum of integrals, one can write

$$\begin{aligned}
\Delta U^{**} = \int_{\tau}^{T} \{ & [\mu_c(t)h_c(t)+\mu_l(t)h_l(t)+\mu_a(t)\delta a(T)] + \lambda[e^{r(T-t)}\left(h_c(t)\right. \\
& \left. - F_l(t)h_l(t)\right)+\left(\frac{1}{T-\tau}\right)\delta a(T)] + \eta_c(t)h_c(t) + \eta_{l_0}(t)h_l(t) \\
& + \eta_{l_1}(t)F_l(t)h_l(t) + \eta_a(T)\left(\frac{1}{T-\tau}\right)\delta a(T)\}dt + \tfrac{1}{2}\int_{\tau}^{T}[\mu_{cc}(t)h_c^2(t) \\
& + \mu_{ll}(t)h_l^2(t) + \mu_{aa}(t)\delta^2 a(T) + 2\mu_{cl}(t)h_c(t)h_l(t) \\
& + 2\mu_{ca}(t)h_c(t)\delta a(T) + 2\mu_{la}(t)h_l(t)\delta a(T)]dt + \int_{\tau}^{T} \varepsilon\, dt.
\end{aligned} \tag{13}$$

Define the distance between the consumption path $\hat{c}(t)$ and the consumption path $\hat{c}(t) + h_c(t)$ by $\|h_c(t)\| = \max_{t \text{ in } [\tau,T]} |h_c(t)|$; define the distance between $\hat{l}(t)$ and $\hat{l}(t) + h_l(t)$ similarly by $\|h_l(t)\| =$ $\max_{t \text{ in } [\tau,T]} |h_l(t)|$; and define the distance between $\hat{a}(T)$ and $\hat{a}(T) + \delta a(T)$

by $\|\delta a(T)\| = |\delta a(T)|$. Next, define $\|h\| = \|h_c(t)\| + \|h_l(t)\| + \|\delta a(T)\|$ as a means of defining closeness of functions within the set of admissible variations for the consumption path, the leisure path, and the bequest. If μ is taken to be a function with continuous partial derivatives only up to order two with respect to all its arguments, all higher order partials being zero, then $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6 = 0$, and $\varepsilon = 0$. If μ is taken to be a function with continuous partial derivatives up to at least order three with respect to all its arguments, then $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6 \to 0$ as $\|h\| \to 0$. Therefore, $\varepsilon \to 0$ as $\|h\| \to 0$, and ε is an infinitesimal of order higher than two relative to $\|h\|^2$. The first integral on the right-hand side of (13), the linear expression in $h_c(t)$, $h_l(t)$, and $\delta a(T)$ which differs from ΔU^{**} by an infinitesimal of order higher than one relative to $\|h\|$, is defined to be the first variation of U^{**} and is denoted δU^{**}. The second integral, the quadratic expression in $h_c(t)$, $h_l(t)$, and $\delta a(T)$, is defined to be the second variation of U^{**} and is denoted $\delta^2 U^{**}$. Necessary conditions for U^{**} to have a maximum for $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$ are that $\delta U^{**} = 0$, for all admissible $h_c(t)$, $h_l(t)$, and $\delta a(T)$, and that $\delta^2 U^{**} \leqq 0$, for all admissible $h_c(t)$, $h_l(t)$, and $\delta a(T)$ which satisfy $\delta I = 0$.[7]

First-order conditions for utility maximization can be found by setting

$$\begin{aligned}\delta U^{**} = &\int_\tau^T [\mu_c(t)+\lambda e^{r(T-t)}+\eta_c(t)]h_c(t)dt \\ &+ \int_\tau^T [\mu_l(t)-\lambda e^{r(T-t)}F_l(t)+\eta_{l_0}(t)+\eta_{l_1}(t)F_l(t)]h_l(t)dt \\ &+ \int_\tau^T [\mu_a(t)+\lambda\left(\frac{1}{T-\tau}\right)+\eta_a(T)\left(\frac{1}{T-\tau}\right)]\delta a(T)dt = 0.\end{aligned}$$

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[7]Hestenes [13, pp.84-85].

Since $h_c(t)$, $h_1(t)$, and $\delta a(T)$ are now arbitrary, independent increments, one is free to set any two of them equal to zero everywhere. Consequently, if $\delta U^{**} = 0$, for any continuous $h_c(t)$, $h_1(t)$, and $\delta a(T)$, then each of the integrals in (14) must equal zero separately; i.e.,

$$\int_{\tau}^{T} [\mu_c(t)+\lambda e^{r(T-t)}+\eta_c(t)]h_c(t)dt = 0, \qquad (15a)$$

$$\int_{\tau}^{T} [\mu_1(t)-\lambda e^{r(T-t)}F_1(t)+\eta_{1_0}(t)+\eta_{1_1}(t)F_1(t)]h_1(t)dt = 0, \qquad (15b)$$

$$\int_{\tau}^{T} [\mu_a(t)+\left(\lambda+\eta_a(T)\right)\left(\frac{1}{T-\tau}\right)]\delta a(T)dt = 0. \qquad (15c)$$

Since $\mu_c(t)$, $\lambda e^{r(T-t)}$, and $\eta_c(t)$ are all continuous, a necessary condition for the integral in (15a) to equal zero for any arbitrary, continuous, nonzero increment $h_c(t)$ is that the coefficient $h_c(t)$ equals zero for all t in $[\tau,T]$.[8] Similarly, for the equality to hold in (15b)

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[8]This proposition is easily demonstrated by modifying the proof of a similar proposition for the case of a fixed-end-point problem in Gelfand and Fomin [9, p.9].

If $\phi(t)$ is continuous in $[\tau,T]$, and if $\int_{\tau}^{T} \phi(t)h_c(t)dt = 0$, for every function $h_c(t)$ belonging to the class of continuous functions defined over $[\tau,T]$, then $\phi(t) = 0$, for all t in $[\tau,T]$.

Proof. Suppose $\phi(t) \neq 0$, say is positive, at some t_0 in (τ,T). By the continuity of $\phi(t)$, $\phi(t)$ is positive in some interval $[t_1,t_2]$ contained in (τ,T); assume $\phi(t) = 0$ elsewhere. Arbitrarily set $h_c(t) = h_0$ = constant > 0 over $[\tau,T]$. Then $\int_{\tau}^{T} \phi(t)h_c(t)dt = h_0 \int_{t_1}^{t_2} \phi(t)dt > 0$, contrary to the conditions of the proposition. The supposition that $\phi(t) \neq 0$, for some t_0 in (τ,T), therefore, is false. Now suppose $\phi(t) \neq 0$ at one of the end points, say is positive at T. By the continuity of $\phi(t)$, $\phi(t)$ is positive in some interval $[t_3,T]$ contained in $[\tau,T]$; assume $\phi(t) = 0$ elsewhere. Setting $h_c(t) = h_0$ = constant > 0 over $[\tau,T]$, one obtains $\int_{\tau}^{T} \phi(t)h_c(t)dt = h_0 \int_{t_3}^{T}\phi(t)dt > 0$, contrary to the conditions of the proposition. Therefore, $\phi(t)$ must be zero at the end points as well as at all the interior points of $[\tau,T]$.

requires the coefficient of $h_1(t)$ to equal zero for all t in $[\tau,T]$. Since $\delta a(T)$ can be factored out of the integral in (15c), the resulting integral coefficient of $\delta a(T)$ must equal zero if the equality in (15c) is to hold for any nonzero increment in the bequest. As a result, the first-order conditions can be written as

$$\mu_c(t) + \lambda e^{r(T-t)} + \eta_c(t) = 0, \qquad \text{for all } t \text{ in } [\tau,T]; \qquad (16a)$$

$$\mu_1(t) - \lambda e^{r(T-t)} F_1(t) + \eta_{1_0}(t) + \eta_{1_1}(t) F_1(t) = 0, \qquad (16b)$$
$$\text{for all } t \text{ in } [\tau,T];$$

$$\int_\tau^T \mu_a(x)dx + \lambda + \eta_a(T) = 0, \qquad (16c)$$

where $\eta_c(t)$, $\eta_{1_0}(t)$, $\eta_{1_1}(t)$, and $\eta_a(T)$ equal zero whenever the non-negativity constraints on $c(t)$, $1(t)$, $F[1(t),t]$, and $a(T)$ respectively are not binding.

Suppose optimal $c(t)$, $1(t)$, and $F[1(t),t]$ are positive everywhere on $[\tau,T]$ and suppose optimal $a(T)$ also is positive. Then

$$\mu_c(t) = -\lambda e^{r(T-t)}, \qquad \text{for all } t \text{ in } [\tau,T]; \qquad (17a)$$

$$\mu_1(t) = \lambda e^{r(T-t)} F_1(t), \qquad \text{for all } t \text{ in } [\tau,T]; \qquad (17b)$$

$$\int_\tau^T \mu_a(x)dx = -\lambda, \qquad (17c)$$

where $-\lambda$ may be interpreted as the present marginal utility of present assets multiplied by the price of the bequest in terms of present assets.[9] Substituting for $-\lambda$ in (17a) and (17b) from (17c) and writing $-\lambda$ as $e^{r(\tau-T)} \frac{\delta U}{\delta a(\tau)}$ in (17c), the first-order conditions (17) appear in ratio form as

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[9]See Appendix, Section A.

$$\frac{\mu_c(t)}{\int_\tau^T \mu_a(x)dx} = e^{r(T-t)}, \qquad \text{for all } t \text{ in } [\tau,T]; \qquad (18a)$$

$$\frac{\mu_l(t)}{\int_\tau^T \mu_a(x)dx} = -e^{r(T-t)}F_l(t), \qquad \text{for all } t \text{ in } [\tau,T]; \qquad (18b)$$

$$\frac{\delta U/\delta a(\tau)}{\int_\tau^T \mu_a(x)dx} = e^{r(T-\tau)}. \qquad (18c)$$

The term $\int_\tau^T \mu_a(x)dx$ is the rate of change in present utility due to a change in the bequest, i.e., the present marginal utility of the bequest. $\frac{\delta U}{\delta a(\tau)}$ is the present marginal utility of present assets. The term $\mu_c(t)$ is the rate of change in present utility due to a change in the planned rate of consumption at t, i.e., the present marginal utility of consumption at t, while $\mu_l(t)$ is the present marginal utility of leisure at t. The terms $e^{r(T-t)}$, $e^{r(T-\tau)}$, and $-F_l(t)$ are rate of exchange terms or prices: $e^{r(T-t)}$ is the price of consumption at t in terms of the bequest, $e^{r(T-\tau)}$ is the price of present assets in terms of the bequest, and $-F_l(t)$ is the price of leisure at t in terms of human income at t.

The first-order conditions can be interpreted in the following manner: if the consumer is maximizing utility, then the present marginal rate of substitution between the rate of consumption at any t in the remaining life-span and the bequest equals the ratio of the price of consumption at t to the price of the bequest; the present marginal rate of substitution between the rate of leisure at any t in the remaining life-span and the bequest equals the ratio of the price of leisure at t to the price of the bequest; and the present marginal rate of substitution between present and terminal assets equals the ratio of the price of present assets to the price of the bequest. (All prices here are expressed in terms of the bequest. The price of the bequest is taken as one.) It can also be seen from (18a) and (18b) that the present marginal rate of substitution between the rate of consumption at any t in $[\tau,T]$ and the rate of leisure at that t must equal the ratio of prices. The model, therefore, gives standard intratemporal (static) first-order conditions for utility maximization.

(17a) and (17b) can be used to obtain intertemporal necessary conditions. Since (17a) and (17b) hold for all t in $[\tau,T]$,

$$\begin{aligned} \mu_c(t) &= -\lambda e^{r(T-t)}, \\ \mu_c(t+k) &= -\lambda e^{r(T-t-k)}, \\ \mu_l(t) &= \lambda e^{r(T-t)} F_l(t), \\ \mu_l(t+k) &= \lambda e^{r(T-t-k)} F_l(t+k), \end{aligned} \tag{19}$$

where t and t + k are both in $[\tau,T]$. The following intertemporal equalities, therefore, must hold:

$$\frac{\mu_c(t)}{\mu_c(t+k)} = e^{rk}, \tag{20a}$$

$$\frac{\mu_l(t)}{\mu_l(t+k)} = e^{rk} \frac{F_l(t)}{F_l(t+k)}, \tag{20b}$$

$$\frac{\mu_c(t)}{\mu_l(t+k)} = -e^{rk}\left[\frac{1}{F_l(t+k)}\right]. \tag{20c}$$

These are also standard intertemporal results: marginal rates of substitution must equal price ratios. For example, the present marginal rate of substitution between the rate of consumption at t and the rate of consumption at t + k must equal the ratio of the present price of consumption at t and the present price of consumption at t + k, where prices are expressed in terms of terminal assets.[10,11]

Suppose the nonnegativity constraints are binding somewhere. What is implied? Let the solution c(t) be zero at some t in [τ,T] because of the nonnegativity constraint on c(t); that is, in the absence of the nonnegativity constraint, suppose optimal c(t) would be negative. Then equation (16a) holds with $\eta_c(t) > 0$ at that t. If $a(T) > 0$, then $-\lambda$ equals the present marginal utility of the bequest from (17c). Therefore, equation (16a) states that the present marginal utility of consumption at t is less than the present marginal utility of the bequest multiplied by the price of consumption at t valued in terms of the bequest. Given the market rate of exchange between consumption at t and the bequest and given diminishing marginal utility, the consumer would be willing to reduce his consumption at t in exchange for additional terminal assets. Since c(t) = 0, however, he cannot do this. Under

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[10]For similar necessary conditions in the Pesek-Saving model, see Pesek and Saving [17, pp.336-337]. The only difference in the conditions that appear in the text of this thesis is the integral expression for the marginal utility of the bequest, a consequence of using the bequest as an argument rather than a gift-expenditure function of time as Pesek-Saving use.

[11]It may be pointed out that the ratio of prices is invariant with respect to the choice of a numéraire, the unit in which to express all prices. That is, the same price ratio occurs whether the numéraire is consumption at τ, consumption at τ + k, or assets at T. Consequently, these price ratios can be interpreted a number of ways depending upon the choice of the numéraire.

these conditions, an increase in present assets would lead to an increase in the planned bequest relative to the planned rate of consumption at t.

If $\hat{l}(t) = 0$ at some t in $[\tau,T]$ because of the nonnegativity constraint on $l(t)$ with $a(T) > 0$, then equation (16b) holds with $\eta_{l_0}(t) > 0$ and $\eta_{l_1}(t) = 0$ at that t. The interpretation is that the present marginal utility of leisure at t is less than the present marginal utility of the bequest multiplied by the price of leisure at t in terms of the bequest. The consumer would be willing to trade away some leisure for more terminal assets, but is constrained from doing so. An increase in present assets would lead to an increase in the planned bequest relative to the planned rate of leisure at t.

If $\hat{l}(t) = 1$ or $F[\hat{l}(t),t] = 0$ at some t in $[\tau,T]$ because of the nonnegativity constraint on F with $a(T) > 0$, then equation (16b) holds with $\eta_{l_0}(t) = 0$ and $\eta_{l_1}(t) > 0$ at that t. Here the present marginal utility of leisure at t exceeds the present marginal utility of the bequest multiplied by the price of leisure at t in terms of the bequest. The consumer would be willing to trade away some of the bequest for more leisure at t, but is constrained from doing so. A decrease in present assets would lead to a decrease in the planned bequest relative to the planned rate of leisure at t.

If $\hat{a}(T) = 0$ because of the nonnegativity constraint on $a(T)$, then equation (16c) holds with $\eta_a(T) > 0$. Here the present marginal utility of the bequest, $\int_\tau^T \mu_a(x)dx$, is less than the present marginal utility of present assets multiplied by the price of the bequest in terms of present assets, $e^{r(\tau-T)} \frac{\delta U}{\delta a(\tau)}$. The consumer would be willing to reduce the bequest presumably to increase consumption or leisure somewhere in the remaining life-span. An increase in present assets would lead to an increase in planned consumption or in planned leisure or in both at least somewhere in the remaining life-span relative to the planned bequest.

Thus if any of the nonnegativity constraints becomes operative, the consumer is placed in a "second-best" position. No longer do the regular first-order conditions of equality between price ratios and marginal rates of substitution hold everywhere. Those marginal rates of substitution involving the variable whose value is zero do not equal the proper price ratios. For example, suppose $\hat{c}(t) = 0$ because of the constraint on c(t). Then

$$\mu_c(t) + \lambda e^{r(T-t)} + \eta_c(t) = 0, \tag{21a}$$

$$\mu_l(t) - \lambda e^{r(T-t)} F_l(t) = 0, \tag{21b}$$

$$\int_\tau^T \mu_a(x)dx + \lambda = 0, \quad \text{or} \tag{21c}$$

$$\frac{\mu_c(t) + \eta_c(t)}{\int_\tau^T \mu_a(x)dx} = e^{r(T-t)}, \tag{22a}$$

$$\frac{\mu_c(t) + \eta_c(t)}{\mu_l(t)} = -\frac{1}{F_l(t)}, \tag{22b}$$

$$\frac{\mu_l(t)}{\int_\tau^T \mu_a(x)dx} = -e^{r(T-t)} F_l(t). \tag{22c}$$

For the "constrained" variable, c(t) in this case, equalities hold in (22a) and (22b) only with the presence of the positive terms $\eta_c(t)/\int_\tau^T \mu_a(x)dx$ and $\eta_c(t)/\mu_l(t)$, which indicates that the rate of consumption at t is too large relative to the bequest and to the rate of leisure at t. Necessary conditions for utility maximization involving only variables other than the "constrained" one, however, remain unchanged. For example, utility maximization requires that the marginal rate of substitution between the rate of leisure at t and the bequest equal the relevant price ratio.[12]

Second-Order Utility Maximizing Conditions

Consider briefly the second-order conditions that must hold if utility is being maximized. Assume none of the nonnegativity constraints is binding anywhere; i.e., $\hat{c}(t) > 0$ and $0 < \hat{l}(t) < 1$, for all t in $[\tau,T]$, and $\hat{a}(T) > 0$. Then it is necessary for utility to be maximized for $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$ that $\delta^2 U^{**} \leqq 0$, for $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$, and for all $h_c(t)$, $h_l(t)$, and $\delta a(T)$ such that the wealth constraint is satisfied, $\delta I = 0$.[13] Thus, $\delta^2 U^{**} = \frac{1}{2}\int_\tau^T [\mu_{cc}(t)h_c^2(t)+\mu_{ll}(t)h_l^2(t)$ $+\mu_{aa}(t)\delta^2 a(T)+2\mu_{cl}(t)h_c(t)h_l(t)+2\mu_{ca}(t)h_c(t)\delta a(T)+2\mu_{la}(t)h_l(t)\delta a(T)]dt$ must be nonpositive for $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$, and for all $h_c(t)$, $h_l(t)$, and $\delta a(T)$ such that $\delta a(T) = \int_\tau^T e^{r(T-t)}[F_l(t)h_l(t)-h_c(t)]dt$. Assume that the utility functional is such that the first non-zero even-order $(\delta^2 U^{**}, \delta^4 U^{**}, \delta^6 U^{**}, \ldots)$ variation is the second variation $(\delta^2 U^{**})$, an assumption typically made when considering utility functions. Then the integral $\int_\tau^T [\mu_{cc}(t)h_c^2(t)+\mu_{ll}(t)h_l^2(t)+\mu_{aa}(t)\delta^2 a(T)+2\mu_{cl}(t)h_c(t)h_l(t)$ $+2\mu_{ca}(t)h_c(t)\delta a(T)+2\mu_{la}(t)h_l(t)\delta a(T)]dt < 0$, for $\hat{c}(t)$, $\hat{l}(t)$, and $\hat{a}(T)$, and for all $h_c(t)$, $h_l(t)$, and $\delta a(T)$ such that

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[12]Note that this conclusion contradicts the so-called "General Theorem of the Second-Best" given by Lipsey and Lancaster [15]. The fault lies with the theorem, not the above conclusion. For a refutation of the theorem in general, see Davis and Whinston [4].

[13]Hestenes [13, pp.84-85].

$\delta a(T) = \int_{\tau}^{T} e^{r(T-t)}[F_1(t)h_1(t)-h_c(t)]dt.$

Consider an implication of these conditions. Choose that subset of all admissible variations such that $F_1(t)h_1(t) - h_c(t) = 0$, for all t in $[\tau,T]$; $\delta a(T)$, therefore, is zero. Then, $\int_{\tau}^{T} [\mu_{cc}(t)h_c^2(t)+\mu_{11}(t)h_1^2(t) +2\mu_{c1}(t)h_c(t)h_1(t)]dt < 0$, subject to $F_1(t)h_1(t) - h_c(t) = 0$, for all t in $[\tau,T]$. The integrand $\mu_{cc}(t)h_c^2(t) + \mu_{11}(t)h_1^2(t) + 2\mu_{c1}(t)h_c(t)h_1(t)$, a quadratic form, must be negative semi-definite for every t in $[\tau,T]$ for the chosen subset of $h_c(t)$ and $h_1(t)$, if $\delta^2 U^{**}$ is to be negative for all admissible $h_c(t)$, $h_1(t)$, and $\delta a(T)$. (The integrand may be zero at a finite number of points, but cannot be zero on an interval. If it were zero on an interval $[t_1,t_2]$ contained in $[\tau,T]$, then $\delta^2 U^{**} = 0$, contrary to assumption, for the case where $h_c(t)$ and $h_1(t)$ are nonzero only over $[t_1,t_2]$ and are zero elsewhere. Likewise, the integrand must never be positive.) Except for a possible finite number of points, one can say that the bordered Hessian

$$H = \begin{vmatrix} \mu_{cc}(t) & \mu_{c1}(t) & -1 \\ \mu_{1c}(t) & \mu_{11}(t) & F_1(t) \\ -1 & F_1(t) & 0 \end{vmatrix} \qquad (23)$$

must be positive definite for all t in $[\tau,T]$. This also is a standard condition in utility maximization theory.

Comparative Static Properties

In this section, the effects on the optimal paths of consumption and leisure and on the optimal bequest of a change in initial assets, a change in the rate of interest, and a change in a human wage parameter will be considered. Throughout this section, assume that none of the nonnegativity requirements is anywhere binding.

The Initial Asset Effect

To determine the effects of a change in the value of present assets, take the first variation of the system of equations used to solve for optimal c(t), l(t), and a(T), equations (17a), (17b), (17c), and (8):

$$\mu_{cc}(t)h_c(t) + \mu_{cl}(t)h_l(t) + \mu_{ca}(t)\delta a(T) + e^{r(T-t)}\delta\lambda = 0,$$

$$\mu_{lc}(t)h_c(t) + \mu_{ll}(t)h_l(t) + \mu_{la}(t)\delta a(T) - e^{r(T-t)}F_l(t)\delta\lambda = 0,$$

$$\int_\tau^T \mu_{ac}(x)h_c(x)dx + \int_\tau^T \mu_{al}(x)h_l(x)dx + \int_\tau^T \mu_{aa}(x)dx\delta a(T) + \delta\lambda = 0, \tag{24}$$

$$\int_\tau^T e^{r(T-x)}h_c(x)dx - \int_\tau^T e^{r(T-x)}F_l(x)h_l(x)dx + \delta a(T) + 0 = e^{r(T-\tau)}\delta a(\tau),$$

for all t in $[\tau,T]$. Write $h_c(x) = \phi_c(x)h_c(t_0)$, where $\phi_c(x)$ is a continuous function defined over $[\tau,T]$, and $h_c(t_0)$ is the variation in the optimal consumption path at an arbitrary point t_0 in $[\tau,T]$. $\phi_c(x)$ is that continuous function of time such that $\phi_c(x)h_c(t_0)$ gives the increment in the optimal path of consumption resulting from, in this case, a change in initial assets. Similarly, write $h_l(x) = \phi_l(x)h_l(t_0)$, where $\phi_l(x)$ is a continuous function of time defined over $[\tau,T]$ and $h_l(t_0)$ is the variation in the optimal leisure path at the arbitrary point t_0. The product $\phi_l(x)h_l(t_0)$ gives the increment in the optimal path of leisure. Clearly, an assumption must be made that a point t_0 exists at which consumption and leisure both have nonzero variations. A later assumption regarding normality, i.e., pure wealth or income effects are positive everywhere, for all goods assures such a point. For $t = t_0$, system (24) can be written as

$$\mu_{cc}(t_0)h_c(t_0) + \mu_{cl}(t_0)h_l(t_0) + \mu_{ca}(t_0)\delta a(T) + e^{r(T-t_0)}\delta\lambda = 0,$$

$$\mu_{lc}(t_0)h_c(t_0) + \mu_{ll}(t_0)h_l(t_0) + \mu_{la}(t_0)\delta a(T) - e^{r(T-t_0)}F_l(t_0)\delta\lambda = 0,$$

$$\int_\tau^T \mu_{ac}(x)\phi_c(x)dxh_c(t_0) + \int_\tau^T \mu_{al}(x)\phi_l(x)dxh_l(t_0) + \int_\tau^T \mu_{aa}(x)dx\delta a(T) + \delta\lambda = 0, \qquad (25)$$

$$\int_\tau^T e^{r(T-x)}\phi_c(x)dxh_c(t_0) - \int_\tau^T e^{r(T-x)}F_l(x)\phi_l(x)dxh_l(t_0) + \delta a(T) + 0 = e^{r(T-\tau)}\delta a(\tau).$$

Cramer's Rule can be used to solve for $h_c(t_0)$, $h_l(t_0)$, and $\delta a(T)$. First, consider the determinant of the coefficient matrix,

$$D = \begin{vmatrix} \mu_{cc}(t_0) & \mu_{cl}(t_0) & \mu_{ca}(t_0) & e^{r(T-t_0)} \\ \mu_{lc}(t_0) & \mu_{ll}(t_0) & \mu_{la}(t_0) & -e^{r(T-t_0)}F_l(t_0) \\ \int_\tau^T \mu_{ac}(x)\phi_c(x)dx & \int_\tau^T \mu_{al}(x)\phi_l(x)dx & \int_\tau^T \mu_{aa}(x)dx & 1 \\ \int_\tau^T e^{r(T-x)}\phi_c(x)dx & -\int_\tau^T e^{r(T-x)}F_l(x)\phi_l(x)dx & 1 & 0 \end{vmatrix}. \qquad (26)$$

In order to sign this determinant, assume, for all t in $[\tau,T]$, that (i) $\mu_{ii} < 0$, $\mu_{ij} > 0$, and $|\mu_{ii}| > \mu_{ij}$, where $i \neq j$, and $i,j = c(t)$, $l(t)$, $a(T)$; and (ii) $|\mu_{ii}| > |\mu_{aj}\phi_j|$, where $i = c(t)$, $l(t)$, $a(T)$, and $j = c(t)$, $l(t)$. Assumptions (i) state the acceptance of diminishing present marginal utilities of consumption, leisure, and the bequest everywhere, that consumption at any t, leisure at any t, and the bequest are complements (i and j are defined to be complements if $\mu_{ij} > 0$), and that any direct second-order partial is stronger than any cross-partial. Assumption (ii) is not as unnerving as it might seem; it is an outgrowth of assuming $|\mu_{ii}| > \mu_{ij}$ brought about by the nature of the problem.

Assume that the rates of consumption and leisure are normal at any point in $[\tau,T]$; by normal is meant that pure wealth or income effects operating on consumption or leisure (or the bequest) are positive everywhere on $[\tau,T]$. Then in the absence of any relative price changes, $\phi_c(x)$ and $\phi_l(x)$ are positive for all x in $[\tau,T]$ so that everywhere in $[\tau,T]$, $h_c(t)$ and $h_l(t)$ have the same sign as have $h_c(t_0)$ and $h_l(t_0)$. The size of ϕ_c and of ϕ_l can be made as small as is desired. The closeness of functions that are considered when varying any given $c(t)$, $l(t)$, and $a(T)$ is defined by the norm $\|h\| = \|h_c(t)\| + \|h_l(t)\| + \|\delta a(T)\|$, as mentioned earlier. As $\|h\| \to 0$, $\|h_c(t)\|$, $\|h_l(t)\|$, and $\|\delta a(T)\| \to 0$. As $\|h_c(t)\| \to 0$, $|\phi_c(t)| \to 0$, and as $\|h_l(t)\| \to 0$, $|\phi_l(t)| \to 0$, for all t in $[\tau,T]$. Given the approach of weak variations, therefore, assumption (ii) is patently reasonable. Using this lengthy list of assumptions, D is negative.[14]

It can be shown by Cramer's Rule that $h_c(t_0) = K_c(t_0)\delta a(\tau)$, $h_l(t_0) = K_l(t_0)\delta a(\tau)$, and $\delta a(T) = K_a(t_0)\delta a(\tau)$, where the K's are all positive numbers. Since $h_c(t_0)$ is the variation in the old optimal consumption path that gives the new optimal consumption path and since $\delta a(\tau)$ is the variation in initial assets solely responsible for the shift in the consumption path, the ratio $h_c(t_0)/\delta a(\tau)$ can be interpreted as the partial derivative of the consumption path with respect to initial assets evaluated at t_0. Given the assumption of normality

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[14]The assumption $|\mu_{ij}| > \mu_{ij}$ is necessary because of the indeterminacy of sign of these terms in D:

$$e^{r(T-t_o)}F_l(t_o)\int_\tau^T e^{r(T-x)}F_l(x)\phi_l(x)dx[\mu_{ca}(t_o)\int_\tau^T \mu_{ac}(x)\phi_c(x)dx$$
$$-\mu_{cc}(t_o)\int_\tau^T \mu_{aa}(x)dx],$$

$$e^{r(T-t_o)}\int_\tau^T e^{r(T-x)}\phi_c(x)dx[\mu_{la}(t_o)\int_\tau^T \mu_{al}(x)\phi_l(x)dx-\mu_{ll}(t_o)\int_\tau^T \mu_{aa}(x)dx].$$

everywhere, i.e., positive wealth or income effects everywhere, $\phi_c(x) > 0$, for all points in $[\tau,T]$. Therefore, $\frac{\partial\hat{c}(t)}{\partial a(\tau)} > 0$, for all t in $[\tau,T]$. By similar reasoning, $\frac{\partial\hat{l}(t)}{\partial a(\tau)} > 0$, for all t in $[\tau,T]$, while $\frac{\partial\hat{a}(T)}{\partial a(\tau)} > 0$. Under the above assumptions, an increase in the value of initial assets increases optimal terminal assets and increases the optimal rates of consumption and leisure everywhere in the remaining life-span.

The Rate of Interest Effect

Although definite conclusions result for the initial asset effect by making certain assumptions, those same assumptions are insufficient to arrive at definite results for the rate of interest effect except for some special cases. The difficulty essentially is one of wealth effects and substitution effects both operating when the interest rate changes.

Consider system (27), formed in a manner similar to system (25):

$$\mu_{cc}(t_0)h_c(t_0) + \mu_{cl}(t_0)h_l(t_0) + \mu_{ca}(t_0)\delta a(T) + e^{r(T-t_0)}\delta\lambda = -\lambda(T-t_0)e^{r(T-t_0)}\delta r;$$

$$\mu_{lc}(t_0)h_c(t_0) + \mu_{ll}(t_0)h_l(t_0) + \mu_{la}(t_0)\delta a(T) - e^{r(T-t_0)}F_l(t_0)\delta\lambda = \lambda(T-t_0)F_l(t_0)e^{r(T-t_0)}\delta r;$$

$$\int_\tau^T \mu_{ac}(x)\phi_c(x)dxh_c(t_0) + \int_\tau^T \mu_{al}(x)\phi_l(x)dxh_l(t_0) + \int_\tau^T \mu_{aa}(x)dx\delta a(T) + \delta\lambda = 0; \quad (27)$$

$$\int_\tau^T e^{r(T-x)}\phi_c(x)dxh_c(t_0) - \int_\tau^T e^{r(T-x)}F_l(x)\phi_l(x)dxh_l(t_0) + \delta a(T) =$$

$$\{e^{r(T-\tau)}\partial_r a(\tau) + (T-\tau)e^{r(T-\tau)}a(\tau) - \int_\tau^T (T-x)e^{r(T-x)}\Big[c(x)-F[l(x),x]\Big]dx\}\delta r,$$

where $\partial_r a(\tau)$ is the partial derivative of initial assets with respect to the rate of interest, but which equals zero in this model due to the assumption that nonhuman assets held always earn whatever the rate of

interest might be. By using Cramer's Rule and the assumptions made in the previous section, the signs of $\frac{\partial\hat{c}(t)}{\partial r}$, $\frac{\partial\hat{l}(t)}{\partial r}$, and $\frac{\partial\hat{a}(T)}{\partial r}$ are indeterminate. For the case where initial assets are nonnegative and optimal terminal assets are positive, one can say that an increase in the interest rate will increase the bequest, while a decrease in the interest rate will decrease the bequest, but this is not much.

To gain insight into the general indeterminacy, look carefully at the case involving an increase in the interest rate. First, the increase in r affects the consumer's wealth constraint or terminal asset constraint, a wealth effect. Any positive savings earn a higher rate of return than before the change, while any negative savings force the consumer to pay higher interest charges. To the extent that the consumer is a net saver, the increase in r increases his lifetime wealth; to the extent that he is a net dissaver, it decreases his lifetime wealth. In terms of the change in terminal assets, one has, for no changes in planned leisure or consumption,

$$\delta\hat{a}(T) = \frac{\partial\hat{a}(T)}{\partial r}\,\delta r = (T-\tau)e^{r(T-\tau)}a(\tau)\delta r + \left[\int_{\tau}^{T}(T-t)e^{r(T-t)}\{F[\hat{l}(t),t]-\hat{c}(t)\}dt\right]\delta r \gtreqless 0. \tag{28}$$

Consequently, if the consumer were to begin with a positive asset position and were to plan to save throughout his remaining life-span, then the increase in r would increase his terminal assets given the same consumption and leisure plans. If the consumer were to begin with a negative asset position and were to plan to save over his remaining life-span, then the increase in r might reduce his terminal assets. Whether terminal assets would fall, rise, or remain unchanged would depend upon

how fast the consumer planned to get out of debt.

Second, if his terminal asset position changes as a result of an interest rate change given the same leisure and consumption plans, then the assumption that all goods are normal (are subject to positive wealth effects) throughout the remaining life-span implies that the paths of consumption and leisure will shift over $[\tau,T]$ in the same direction as terminal assets. Consequently, if the consumer were to begin with a positive asset position and were to plan to save throughout his remaining life-span, then the increase in r would lead to an increase in the bequest and upward shifts in consumption and leisure everywhere over $[\tau,T]$.

Third, the change in r affects relative prices and consequently creates pure substitution effects. If r rises, then the price of future economic activity (future consumption, future bequest) falls in terms of present economic activity (e.g., present consumption). Pure substitution effects of this type indicate that additional future activity will be substituted for reduced present activity. The intratemporal and intertemporal first-order conditions, eq. (18) and (20), also give these results. In the set of equations (18), it is clear that, for equilibrium, the marginal rates of substitution between the rate of consumption at any t or the rate of leisure at any t or initial assets and the bequest must increase if r increases. Considering only the substitution effect, holding wealth constant, the increase in r must, therefore, decrease the rate of consumption at t, the rate of leisure at t, and initial assets relative to the bequest. Similarly, in the set of equations (20), it is clear that, for equilibrium, the marginal rates of substitution between the rate of consumption at any t and

the rate of consumption at $t+k$, between the rate of leisure at any t and the rate of leisure at $t+k$, and between the rate of consumption at any t and the rate of leisure at $t+k$ must increase if r increases. That is, the rate of consumption at t or the rate of leisure at t must fall relative to the rate of consumption at $t+k$ or the rate of leisure at $t+k$. The conclusion is reached that a change in r will affect the optimal paths of consumption and leisure and consequently the bequest even if only substitution effects are considered. By combining the substitution effects with the wealth effects, the general indeterminacy regarding shifts in consumption and leisure is easily explained.

The rate of interest effect can be summarized this way. An increase in the rate of interest: (1) may change the bequest even if planned consumption and leisure were to remain unchanged everywhere (a pure wealth effect); (2) if such a wealth effect holds, then given positive wealth or income effects everywhere by assumption, the bequest and the paths of consumption and leisure will all move in the same direction as the wealth effect; and (3) will cause changes in relative prices such that the consumer tends to substitute additional more distant future economic activity for reduced less distant or present economic activity. It is easy to explain why use of Cramer's Rule gives a determinate solution for the bequest in the case of nonnegative initial assets, originally planned positive terminal assets, and an increase in r. The wealth effect here is positive leading to an increase in the bequest, while the substitution effect is in favor of increasing the bequest relative to consumption and leisure. Since both forces are pulling on the bequest in the same direction, a determinate solution exists. For the rates of consumption or leisure at any point in the remaining life-span, however, wealth effects and substitution effects pull against one another, leaving the results to depend upon relative sizes.

The Wage Effect

Here an upward shift in the human wage rate throughout the remaining life-span will be considered. As with the rate of interest effect, the wage effect involves some indeterminacy in its effects upon optimal consumption, leisure, and the bequest. In particular, the effect upon the rate of leisure at any t is, in general, indeterminate because of opposing wealth and substitution effects.

Consider first the effect on human income earned throughout the remaining life-span. An increase in the wage rate everywhere will definitely lead to more human income being earned everywhere given the assumption that human income is normal. (Such an assumption is implicit in the earlier assumption of normality for all goods everywhere.) First, the wage rate increase changes relative prices of leisure and income in favor of income; earning income is less expensive in terms of foregone leisure. The substitution effect, therefore, operates such that the rate of leisure will be reduced and the rate of income earned will be increased at any t in $[\tau,T]$. Second, the wage rate increase provides a positive wealth effect. For the same rate of leisure, a higher rate of income will be earned. Consequently, the positive wealth effect will lead to more leisure and more income given that both goods are normal. At any t in $[\tau,T]$, therefore, a higher rate of income definitely will be taken and a higher, lower, or the same rate of leisure will be enjoyed depending upon the relative sizes of the wealth and substitution effects on the rate of leisure.

There is no change in relative prices regarding the trade-off between the rate of consumption or terminal assets and the rate of income. The prices of consumption and terminal assets in terms of income are unaffected by the wage rate increase. (This is not to say, however, that relative prices of consumption or terminal assets and leisure are unchanged.) The expansion in the rate of income earned at every point in $[\tau,T]$, therefore, definitely leads to a greater rate of consumption at every point in $[\tau,T]$ and to greater terminal assets, the result of a pure positive wealth effect operating on assumed normal goods. Alternatively, considering substitution effects on the rate of consumption or terminal assets and the rate of leisure (instead of on the rate of income and the rate of leisure), relative prices of consumption and terminal assets have fallen in terms of leisure. Consequently, a greater rate of consumption will be taken at every t, more terminal assets will be taken, and a lower rate of leisure will be taken at every t. The inherent positive wealth effect of the wage rate increase increases the rates of consumption and leisure at every t and the bequest.

The conclusions may be stated simply: given normality throughout, an increase in the human wage rate everywhere in the remaining life-span will shift the consumption path upwards throughout the remaining life-span, will increase the bequest, and may increase, decrease, or leave unchanged the rate of leisure at any point in the remaining life-span.

An Example

First-order conditions (17a)-(17c) along with the constraint (8) can be used to solve for the optimal paths of consumption and leisure and for the optimal bequest, assuming an interior solution exists. For

illustrative purposes, an example will be presented here.

To simplify the arithmetic somewhat, choose a consumer with the following utility functional,

$$U = \int_{\tau}^{T} e^{j(\tau-t)}\{[\alpha_1+\alpha_2\left(\frac{t-\tau}{T-\tau}\right)]\ln c(t)+[\beta_1+\beta_2\left(\frac{t-\tau}{T-\tau}\right)]\ln l(t) + [\gamma_1+\gamma_2\left(\frac{t-\tau}{T-\tau}\right)]\ln a(T)\}dt, \tag{29}$$

where α_1, α_2, β_1, β_2, γ_1, γ_2, and j are constants, and where $0 < e^{j(\tau-t)}[\xi_1+\xi_2\left(\frac{t-\tau}{T-\tau}\right)] < 1$, for all t in $[\tau,T]$ and for $\xi = \alpha, \beta, \gamma$. Suppose the consumer's wealth constraint is

$$a(T) = e^{r(T-\tau)}a(\tau) + \int_{\tau}^{T} e^{r(T-t)}\{[\omega_0 e^{k(t-\tau)}][1-l(t)]-c(t)\}dt, \tag{30}$$

where $\omega_0 e^{k(t-\tau)}$ is the human wage function (ω_0 = constant > 0, and $0 \leqq k < 1$). The problem of utility maximization reduces to

$$\max_{c,l,a(T)} U^* = \int_{\tau}^{T}\Bigg(e^{j(\tau-t)}\{[\alpha_1+\alpha_2\left(\frac{t-\tau}{T-\tau}\right)]\ln c(t) + [\beta_1+\beta_2\left(\frac{t-\tau}{T-\tau}\right)]\ln l(t) + [\gamma_1+\gamma_2\left(\frac{t-\tau}{T-\tau}\right)]\ln a(T)\} + \lambda\{e^{r(T-t)}\left(c(t)-[\omega_0 e^{k(t-\tau)}][1-l(t)]\right)+\left(\frac{1}{T-\tau}\right)a(T)\}\Bigg)dt, \tag{31}$$

where λ is a nonzero constant. Following the procedure that led to first-order conditions (17a)-(17c), one obtains

$$e^{j(\tau-t)}[\gamma_1+\gamma_2\left(\frac{t-\tau}{T-\tau}\right)]c(t)^{-1} = -\lambda e^{r(T-t)}, \quad \text{for all t in } [\tau,T]; \tag{32a}$$

$$e^{j(\tau-t)}[\beta_1+\beta_2(\frac{t-\tau}{T-\tau})]l(t)^{-1} = -\lambda e^{r(T-t)}[\omega_0 e^{k(t-\tau)}], \quad \text{for all t in } [\tau,T]; \tag{32b}$$

$$\int_{\tau}^{T} e^{j(\tau-x)}[\gamma_1+\gamma_2\left(\frac{x-\tau}{T-\tau}\right)]a(T)^{-1}dx = -\lambda. \tag{32c}$$

Equations (32a), (32b), (32c), and (30) now can be used to solve for the optimal paths of consumption and leisure and for the optimal bequest. These are[15,16]

$$c(t) = \left(\frac{1}{A}\right) j^2[\alpha_1(T-\tau)+\alpha_2(t-\tau)]e^{j\tau-rT+t(r-j)}a(T), \quad \tau \leqq t \leqq T; \tag{33a}$$

$$l(t) = \left(\frac{1}{\omega_0 A}\right) j^2[\beta_1(T-\tau)+\beta_2(t-\tau)]e^{(j+k)\tau-rT+t(r-j-k)}a(T), \quad \tau \leqq t \leqq T; \tag{33b}$$

$$a(T) = \left(\frac{A}{A+B}\right)\{e^{r(T-\tau)}a(\tau)+\frac{\omega_0}{k-r}[e^{k(T-\tau)}-e^{r(T-\tau)}]\}, \tag{33c}$$

where

$$A = j(T-\tau)[\gamma_1-(\gamma_1+\gamma_2)e^{j(\tau-T)}] + \gamma_2(1-e^{j(\tau-T)}), \text{ and} \tag{34a}$$

$$B = j(T-\tau)[(\alpha_1+\beta_1)-(\alpha_1+\alpha_2+\beta_1+\beta_2)e^{j(\tau-T)}]+(\alpha_2+\beta_2)(1-e^{j(\tau-T)}). \tag{34b}$$

Integrating the instantaneous rates of consumption and leisure, given by equations (33a) and (33b), over a unit interval of time, say, the year, yields the (annual) consumption expenditure and the proportion of the interval enjoyed as leisure:[17]

$$\int_{t_0}^{t_0+1} c(t)dt = \left(\frac{j^2}{r-j}\right)e^{j\tau-rT+t_0(r-j)}\left(\frac{C}{A}\right)a(T); \tag{35a}$$

$$\int_{t_0}^{t_0+1} l(t)dt = \left[\frac{j^2}{\omega_0(r-j-k)}\right]e^{(j+k)\tau-rT+t_0(r-j-k)}\left(\frac{D}{A}\right)a(T); \tag{35b}$$

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[15]For the arithmetic involved in solving this example, see Appendix, Section B.

[16]In solving this example, it is assumed that (i) $j \neq 0$, $r \neq 0$, $k \neq 0$; (ii) $r \neq j$, $r \neq k$, $r \neq j + k$.

[17]See Appendix, Section C.

where t_0 and $t_0 + 1$ are both in $[\tau,T]$, and where

$$C = [\alpha_1(T-\tau)-\alpha_2(\tau-t_0)-\frac{\alpha_2}{r-j}](e^{r-j}-1) + \alpha_2 e^{r-j}, \tag{36a}$$

$$D = [\beta_1(T-\tau)-\beta_2(\tau-t_0)-\frac{\beta_2}{r-j-k}](e^{r-j-k}-1) + \beta_2 e^{r-j-k}. \tag{36b}$$

To summarize, the following set of equations gives the optimal consumption path, the optimal leisure path, the optimal bequest, the optimal one-period consumption expenditure, and the optimal one-period leisure proportion:

$$A = j(T-\tau)[\gamma_1-(\gamma_1+\gamma_2)e^{j(\tau-T)}] + \gamma_2(1-e^{j(\tau-T)}); \tag{37a}$$

$$B = j(T-\tau)[(\alpha_1+\beta_1)-(\alpha_1+\alpha_2+\beta_1+\beta_2)e^{j(\tau-T)}] + (\alpha_2+\beta_2)(1-e^{j(\tau-T)}); \tag{37b}$$

$$C = (e^{r-j}-1)[\alpha_1(T-\tau)-\alpha_2(\tau-t_0)-\frac{\alpha_2}{r-j}] + \alpha_2 e^{r-j}; \tag{37c}$$

$$D = (e^{r-j-k}-1)[\beta_1(T-\tau)-\beta_2(\tau-t_0)-\frac{\beta_2}{r-j-k}] + \beta_2 e^{r-j-k}; \tag{37d}$$

$$c(t) = \left(\frac{1}{A}\right) j^2[\alpha_1(T-\tau)+\alpha_2(t-\tau)]e^{j\tau-rT+t(r-j)}a(T), \quad \tau \leqq t \leqq T; \tag{37e}$$

$$l(t) = \left(\frac{1}{\omega_0 A}\right) j^2[\beta_1(T-\tau)+\beta_2(t-\tau)]e^{(j+k)\tau-rT+t(r-j-k)}a(T), \quad \tau \leqq t \leqq T; \tag{37f}$$

$$a(T) = \left(\frac{A}{A+B}\right)\{e^{r(T-\tau)}a(\tau) + \frac{\omega_0}{k-r}[e^{k(T-\tau)}-e^{r(T-\tau)}]\}; \tag{37g}$$

$$\int_{t_0}^{t_0+1} c(t)dt = \left(\frac{j^2}{r-j}\right) e^{j\tau-rT+t_0(r-j)} \left(\frac{C}{A}\right) a(T); \tag{37h}$$

$$\int_{t_0}^{t_0+1} l(t)dt = \left[\frac{j^2}{\omega_0(r-j-k)}\right] e^{(j+k)\tau-rT+t_0(r-j-k)} \left(\frac{D}{A}\right) a(T). \tag{37i}$$

Suppose that for the consumer the following parameters hold: $r = .05$, $j = .03$, $k = .01$, $a(\tau) = 1000$, $\omega_0 = 30{,}000$, $\tau = 0$, $T = 40$, $\alpha_1 = .300$, $\alpha_2 = -.002$, $\beta_1 = .700$, $\beta_2 = .007$, $\gamma_1 = .005$, $\gamma_2 = .008$. The following table presents for this consumer the optimal bequest and optimal one-period consumption expenditure and leisure for selected time periods.[18]

Table A

Selected Solution Values for the Hypothetical Consumer Planning Example*

t_0	0	1	2	3	4	9	19	29	39
$\int_{t_0}^{t_0+1} c(t)dt$	7720	7875	8032	8193	8357	9229	11,253	13,722	16,732
$\int_{t_0}^{t_0+1} l(t)dt$	0.60	0.60	0.61	0.62	0.62	0.66	0.73	0.80	0.89
$a(T)$	36,042								

*The values for t_0 are the dates of the initial point of each one-year interval, the one-year intervals of consumption are the optimal real (constant) dollar total annual values of consumption spending, the one-year intervals of leisure are the optimal proportions of each year enjoyed as leisure, and the value for $a(T)$ is the optimal bequest in real dollars.

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[18]The above values for the parameters were chosen because (i) they are convenient to work with, (ii) they allow the subjective evaluation of consumption to fall relative to those of leisure and the bequest over time, and (iii) they yield reasonable numerical results. For the calculations here and those later in the thesis, data from Smail [19, pp.552-558] were used.

It might be mentioned briefly why positively sloped consumption and leisure paths occur in this example. The reason essentially is that the rate of interest, r, has been assumed to exceed the subjective rate of discount, j, and the wage rate growth factor, k. Given the assumed values of the other parameters, as long as r exceeds j by at least .0002 (approximately), the rate of consumption increases everywhere with time. If $j \geqq r$, the rate of consumption would decrease everywhere with time. As long as r exceeds .0398 (approximately), the rate of leisure also increases everywhere with time. If $j + k \geqq r + .0003$ (roughly), then the rate of leisure would decrease everywhere with time. The presence of other parameters complicates somewhat the relations that must exist between r, j, and k for positively sloped or negatively sloped paths of consumption and leisure, as equations (38) show, but roughly, if $r > j$, the rate of consumption increases everywhere with time, and if $r > j + k$, the rate of leisure increases everywhere with time.

$$c'(t) = \left(\frac{1}{A}\right) j^2 e^{j\tau - rT + t(r-j)} a(T)\{(r-j)[\alpha_1(T-\tau)+\alpha_2(t-\tau)]+\alpha_2\}, \quad (38a)$$

$$l'(t) = \left(\frac{1}{\omega_0 A}\right) j^2 e^{(j+k)\tau - rT + t(r-j-k)} a(T)\{(r-j-k)[\beta_1(T-\tau) + \beta_2(t-\tau)]+\beta_2\} \quad (38b)$$

where, given the values of the parameters, A and a(T) are positive.

A Summary

This chapter presented a certainty model of consumer behavior. In the model, the consumer knows or assumes to know his date of death and plans to his date of death. The utility functional is dependent on the paths of leisure and consumption, the bequest, and the present date and date of death. First-order and second-order conditions for utility maximization were derived which intratemporally and intertemporally are the same as those of other consumer models. In particular, marginal rates of substitution must equal the proper price ratios. The problem of intertemporal utility maximization was illustrated by a numerical example. In addition, some comparative static properties were derived. Unfortunately, because of opposing substitution effects and wealth effects, many of the comparative static results are indeterminate in general.

CHAPTER III

THE PLANNING HORIZON AND REMAINING LIFETIME

In the certainty model of Chapter II, it was assumed that the consumer chooses a plan defined over his entire remaining lifetime. The problem to be considered in this chapter involves the conditions which would hold if he were to plan only over an interval less than his known remaining life. In other words, suppose that in maximizing utility, the consumer chooses a date in the future, short of his known date of death, beyond which he presently does not plan *explicit* economic activity.

Presumably, if expanding the planning horizon involves positive costs to him, the consumer might select a terminal point for his planning interval that is short of his date of death. If planning were costless, one would expect the consumer to plan to his date of death, unless the marginal utility of planning became zero at some point short of the date of death. The point at which a consumer stops planning, one would expect, essentially revolves around any "planning costs."

One might also look upon this problem as a prelude to the problem in the next chapter. There the date of death will be taken to be a random variable; i.e., the consumer does not know with certainty or assume to know his date of death. Of special interest will be the conditions which must be satisfied by the terminal point of the planning interval in that model, in general, a point which one would not expect to be the last possible date of life. The principles derived here in this simpler context hopefully will be of benefit in interpreting the principles derived in the more complex context.

A Statement of the Problem and Some Necessary Conditions

In the calculus of variations, this problem is a constrained, variable end-point one. Let the terminal point of the consumer's planning interval be ρ. The utility functional can be written as

$$U[c,l,a(\rho),\rho;\tau] = \int_{\tau}^{\rho} \mu[c(t),l(t),a(\rho),\tau,\rho,t]dt, \tag{1}$$

while the wealth constraint is

$$a(\rho) = e^{r(\rho-\tau)}a(\tau) + \int_{\tau}^{\rho} e^{r(\rho-t)}\{F[l(t),t]-c(t)\}dt. \tag{2}$$ [1]

Since the consumer knows he will live from ρ to T, one would expect assets at ρ to have an influence on current utility; hence the appearance of $a(\rho)$ in U. Any decisions regarding the interval from ρ to T, such as, the allocation of $a(\rho)$ between consumption from ρ to T and the bequest or the earning of human income from ρ to T, however, are not being made at τ.

For purposes of this problem, suppose that any nonnegativity conditions that might be imposed would be nonoperative in equilibrium.

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[1]The functional in (1) uses ρ as the future date in time used by the consumer for purposes of subjectively evaluating the importance of assets and leisure relative to consumption. For example, presumably the nearer a point in any plan is to ρ, the more important leisure is relative to consumption. This accounts for changing marginal rates of substitution as a consumer ages. Rather than using ρ for this purpose, one might use the date of death, T. That is, as a consumer ages towards T, leisure becomes relatively more important than consumption. The utility functional, then, would appear as $\int_{\tau}^{\rho} \mu[c(t),l(t),a(\rho),\tau,T,\rho,t]dt$.

To solve this variable end-point problem, form the functional,

$$U^*[c,l,a(\rho),\rho;\tau,\lambda] = \int_\tau^\rho \Big(\mu[c(t),l(t),a(\rho),\tau,\rho,t] + \lambda\Big[e^{r(\rho-t)}\{c(t)-F[l(t),t]\}\Big]\Big)dt + \lambda[a(\rho)-e^{r(\rho-\tau)}a(\tau)]. \tag{3}$$

Using the "isoperimetric theorem" once more, if $\hat{c}(t)$, $\hat{l}(t)$, $\hat{a}(\rho)$, and $\hat{\rho}$ are extremals of U subject to the constraint, then they are extremals of U*. Therefore, take the first variation of U* and set it equal to zero,[2]

$$\begin{aligned}\delta U^* = \int_\tau^\rho \Big(&[\mu_c(t)h_c(t)+\mu_l(t)h_l(t)+\mu_a(t)\{\delta a(\rho)+a'(\rho)\delta\rho\} + \mu_\rho(t)\delta\rho] \\ &+ \lambda\Big[e^{r(\rho-t)}\{h_c(t)-F_l(t)h_l(t)\}+re^{r(\rho-t)}\{c(t)-F[l(t),t]\}\delta\rho\Big]\Big)dt \\ &+ \lambda[\big(\delta a(\rho)+a'(\rho)\delta\rho\big)-re^{r(\rho-\tau)}a(\tau)\delta\rho] \\ &+ \mu[c(\rho),l(\rho),a(\rho),\tau,\rho,\rho]\delta\rho + \lambda\{c(\rho)-F[l(\rho),\rho]\}\delta\rho = 0,\end{aligned} \tag{4}$$

where $a'(\rho)$ is the time derivative of the path of nonhuman assets evaluated at the point ρ.

One must pause here to consider what is meant by the variation in a function at an end-point in a variable end-point problem. Since the interval over which functions are being defined may now vary, the variation in a function no longer may be considered as the simple vertical shift in the function everywhere. In this problem in particular, the change in terminal assets is not simply the upward or downward shift in assets at the point ρ, where ρ is held constant, $\delta a(\rho)$, but rather is

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[2]Use is made here of the theorem regarding differentiating an integral with respect to a parameter where the limits of integration depend upon that parameter. See I. S. Sokolnikoff [20, p.121] or D. V. Widder [23, p.353].

$\delta a(\rho)$ plus the effect on $a(\rho)$ of varying the planning horizon, $a'(\rho)\delta\rho$. The total variation in assets at ρ is, therefore, $\bar{\delta}a(\rho) \equiv \delta a(\rho) + a'(\rho)\delta\rho$.[3] Similarly, if one were to talk about the variation in consumption at ρ or in leisure at ρ, one would define $\bar{h}_c(\rho) \equiv h_c(\rho) + c'(\rho)\delta\rho$, and $\bar{h}_l(\rho) \equiv h_l(\rho) + l'(\rho)\delta\rho$. Using this notation, (4) can be written as

$$\begin{aligned}
&\int_\tau^\rho [\mu_c(t)+\lambda e^{r(\rho-t)}]h_c(t)dt + \int_\tau^\rho [\mu_l(t)-\lambda e^{r(\rho-t)}F_l(t)]h_l(t)dt \\
&\quad + \int_\tau^\rho \mu_a(t)\bar{\delta}a(\rho)dt + \lambda\bar{\delta}a(\rho) + \int_\tau^\rho [\mu_\rho(t)+\lambda r e^{r(\rho-t)} \\
&\qquad \{c(t)-F[l(t),t]\}]\delta\rho dt + \lambda\{c(\rho)-F[l(\rho),\rho] - re^{r(\rho-\tau)}a(\tau)\}\delta\rho \\
&\quad + \mu[c(\rho),l(\rho),a(\rho),\tau,\rho,\rho]\delta\rho = 0.
\end{aligned} \tag{5}$$

In order for the first variation in utility to equal zero for arbitrary $h_c(t)$, $h_l(t)$, $\bar{\delta}a(\rho)$, and $\delta\rho$ which satisfy the constraint, it is necessary that

$$\int_\tau^\rho [\mu_c(t)+\lambda e^{r(\rho-t)}]h_c(t)dt = 0, \tag{6a}$$

$$\int_\tau^\rho [\mu_l(t)-\lambda e^{r(\rho-t)}F_l(t)]h_l(t)dt = 0, \tag{6b}$$

$$\int_\tau^\rho \mu_a(t)\bar{\delta}a(\rho)dt + \lambda\bar{\delta}a(\rho) = 0, \tag{6c}$$

$$\begin{aligned}
&\int_\tau^\rho \mu_\rho(t)\delta\rho dt + \mu[c(\rho),l(\rho),a(\rho),\tau,\rho,\rho]\delta\rho + \lambda\{c(\rho)-F[l(\rho),\rho] \\
&\qquad - re^{r(\rho-\tau)}a(\tau) + \int_\tau^\rho re^{r(\rho-t)}\left[c(t)-F[l(t),t]\right]dt\}\delta\rho = 0.
\end{aligned} \tag{6d}$$

Therefore, first-order conditions are

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[3]See Gelfand and Fomin [9, pp.54-55].

$$\mu_c(t) + \lambda e^{r(\rho-t)} = 0, \qquad \text{for all } t \text{ in } [\tau,\rho]; \quad (7a)$$

$$\mu_l(t) - \lambda e^{r(\rho-t)} F_l(t) = 0, \qquad \text{for all } t \text{ in } [\tau,\rho]; \quad (7b)$$

$$\int_\tau^\rho \mu_a(x)dx + \lambda = 0; \quad (7c)$$

$$\int_\tau^\rho \mu_\rho(x)dx + \mu[c(\rho),l(\rho),a(\rho),\tau,\rho,\rho] \quad (7d)$$

$$+ \lambda\{c(\rho)-F[l(\rho),\rho]-ra(\rho)\} = 0.$$

Substituting for λ from (7c), the first-order conditions can be written as

$$\mu_c(t) = e^{r(\rho-t)}\int_\tau^\rho \mu_a(x)dx, \qquad \text{for all } t \text{ in } [\tau,\rho]; \quad (8a)$$

$$\mu_l(t) = -e^{r(\rho-t)}F_l(t)\int_\tau^\rho \mu_a(x)dx, \qquad \text{for all } t \text{ in } [\tau,\rho]; \quad (8b)$$

$$\int_\tau^\rho \mu_\rho(x)dx + \mu[c(\rho),l(\rho),a(\rho),\tau,\rho,\rho] \quad (8c)$$

$$= \{c(\rho)-F[l(\rho),\rho]-ra(\rho)\}\int_\tau^\rho \mu_a(x)dx.$$

Conditions (8a) and (8b) are the same as those of the previous chapter's model with the exception that ρ replaces T. This is to be expected. Once the terminal point of a variable end-point problem is selected, the problem is equivalent to a fixed end-point problem. An extremal of the variable end-point problem, therefore, is an extremal of the equivalent fixed end-point problem. Conditions to be met by the paths of consumption and leisure over the planning interval, therefore, should be the same in the two cases. Again, intratemporal and intertemporal marginal rates of substitution between the rate of consumption, the rate of leisure, and assets at ρ, throughout the planning interval, must equal the proper price ratios.

Condition (8c) is a new one; one introduced by the variability of the planning horizon. Written in marginal rate of substitution form,

$$\frac{\int_{\tau}^{\rho} \mu_{\rho}(x)dx + \mu[c(\rho),l(\rho),a(\rho),\tau,\rho,\rho]}{\int_{\tau}^{\rho} \mu_{a}(x)dx} = c(\rho) - F[l(\rho),\rho] - ra(\rho). \quad (8c')$$

Here also the marginal rate of substitution between expanding the planning horizon and assets at ρ must equal the ratio of prices where the price of expanding the planning horizon in the difference between the rate of consumption at ρ and the rate of income at ρ, human plus non-human. The marginal utility of increasing the planning horizon, the numerator of the left-hand side of (8c'), equals the sum of the effect of increasing ρ on the discounting of economic activity over the planning interval, $\int_{\tau}^{\rho} \mu_{\rho}(x)dx$, and the present value of the rate of utility to be enjoyed from economic activity at ρ, $\mu[c(\rho),l(\rho),a(\rho),\tau,\rho,\rho]$. The marginal utility of assets at ρ is given by the denominator. Given positive marginal utilities and positive values for all arguments, (8c') states that, for $\hat{\rho}$ to be the optimal planning horizon, assets at $\hat{\rho}$ must be declining. Under these conditions, the left-hand side of (8c') is positive; the right-hand side, therefore, must also be positive. But the right-hand side is minus one times the time derivative of the path of assets evaluated at ρ,

$$a'(t) = re^{r(t-\tau)}a(\tau) + r\int_{\tau}^{t} e^{r(t-x)}\{F[l(x),x]-c(x)\}dx \quad (9a)$$

$$+ F[l(t),t] - c(t) = ra(t) + F[l(t),t] - c(t),$$

or at $t = \rho$,

$$a'(\rho) = ra(\rho) + F[l(\rho),\rho] - c(\rho). \tag{9b}$$

Since $c(\rho) - F[l(\rho),\rho] - ra(\rho)$ is positive, $a'(\rho)$ must be negative. Assets at $\hat{\rho}$ must be declining if $\hat{\rho}$ is the optimal planning horizon. Further, (8c') implies under these conditions, that if at a point in (τ,T) assets are increasing, then that point cannot be the optimal planning horizon. (If nonhuman assets are allowed to be negative short of the date of death, then it would be possible for (8c') to be met with the sum of the rate of consumption and the rate of interest *payments* exceeding the rate of human income; here also, though, nonhuman assets would be declining.)

The first-order conditions clearly state that expanding the planning horizon involves costs to the consumer even in the absence of any possible current resource costs in planning. The kind of cost involved is an alternative or opportunity cost. The consumer, in expanding his planning horizon, plans for additional future leisure and consumption. But the additional leisure and consumption may come at the expense of reduced assets. That is, if the consumer is planning to dissave at the end of his planning horizon, then extending the planning horizon forces him to face the trade-off between additional leisure and consumption and reduced assets. The marginal alternative cost of planning in this model is the rate at which assets are changing. If this cost is positive somewhere, i.e., if the planned path of assets is falling somewhere, then it is possible that the consumer will not plan out to the end of his life. If he does not plan out to the end of his life, then at the terminal point of his planning interval assets must be declining.

An Example

In an effort to simplify the arithmetic somewhat, consider a consumer who has a fixed present stock of wealth to allocate to consumption over time and to terminal assets; assume he earns no human income (permanently retired). Also suppose that the utility functional involved differs from the one used in the previous chapter. In particular, assume the individual has the following separable or additive utility functional,

$$U = \int_0^{\rho} e^{-jt} a c(t)^{\alpha} dt + e^{-j\rho} b a(\rho)^{\beta}, \qquad (10)$$

where a and b are positive constants, α, β, and j are constants lying strictly between zero and unity, and ρ is the planning horizon. The present point in time is taken to be zero, and ρ falls short of the known date of death. The wealth constraint is

$$e^{-r\rho} a(\rho) + \int_0^{\rho} e^{-rt} c(t) dt = K, \qquad (11)$$

where K is a positive constant.

This example is solved by forming the functional

$$U^* = \int_0^{\rho} [a e^{-jt} c(t)^{\alpha} + \lambda e^{-rt} c(t)] dt + b e^{-j\rho} a(\rho)^{\beta} + \lambda [e^{-r\rho} a(\rho) - K], \qquad (12)$$

setting its first variation equal to zero, solving for first-order conditions, and using the first-order conditions along with the constraint to solve for optimal c(t), optimal a(ρ), optimal ρ, and λ. The first-order conditions are[4]

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[4]See Appendix, Section D.

$$a\alpha e^{(r-j)t}c(t)^{\alpha-1} = -\lambda, \quad \text{for all } t \text{ in } [0,\rho]; \tag{13a}$$

$$b\beta e^{(r-j)\rho}a(\rho)^{\beta-1} = -\lambda; \tag{13b}$$

$$e^{-j\rho}[ac(\rho)^{\alpha}-jba(\rho)^{\beta}] + \lambda e^{-r\rho}[c(\rho)-ra(\rho)] = 0. \tag{13c}$$

Using conditions (13a), (13b), (13c), and (11) leads to the solution:

$$c(t) = \left(\frac{b\beta}{a\alpha}\right)^{\left(\frac{1}{\alpha-1}\right)} e^{(r-j)(\rho-t)\left(\frac{1}{\alpha-1}\right)} \left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{\beta-1}{(\beta-\alpha)(\alpha-1)}}, \tag{14a}$$

$$0 \leq t \leq \rho;$$

$$a(\rho) = \left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{1}{\beta-\alpha}}; \tag{14b}$$

$$K = e^{-r\rho}\{\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\left(\frac{1}{\beta-\alpha}\right)} \tag{14c}$$

$$+ \left(\frac{\alpha-1}{j-\alpha r}\right)\left(\frac{b\beta}{a\alpha}\right)^{\left(\frac{1}{\alpha-1}\right)}\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{\beta-1}{(\beta-\alpha)(\alpha-1)}}\}$$

$$- e^{\left(\frac{r-j}{\alpha-1}\right)\rho}\left(\frac{\alpha-1}{j-\alpha r}\right)\left(\frac{b\beta}{a\alpha}\right)^{\frac{1}{\alpha-1}}\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{\beta-1}{(\beta-\alpha)(\alpha-1)}},$$

where (14c) is an equation containing only ρ as an unknown.[5]

In order to have numbers that are convenient to work with, suppose the parameters have the following values: $a = 12$, $b = 3$, $\alpha = \frac{1}{2}$, $\beta = \frac{2}{3}$, $r = .06$, $j = .30$, and $K = 1{,}000{,}000$. The table below gives some

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[5]The solution assumes $\alpha \neq \beta$, $j \neq r\alpha$, $j \neq r\beta$. Equation (14c) also can be written as $K = \{a(\rho) + \left(\frac{\alpha-1}{j-\alpha r}\right)c(\rho)\}e^{-r\rho}$ $- \{\left(\frac{\alpha-1}{j-\alpha r}\right)c(\rho)\}e^{(r-j/\alpha-1)\rho}$.

selected values for the solution to this variable end-point, planning horizon example.

Table B

Selected Solution Values for the Hypothetical Planning Horizon Example*

ρ	10 (approximately)
$a(\rho)$	12,289
$c(\rho)$	4,793
$\int_0^1 c(t)dt$	462,446
$\int_1^2 c(t)dt$	286,178
$\int_2^3 c(t)dt$	176,996
$\int_9^{10} c(t)dt$	6,187

*The value for ρ is the number of years beyond the present date at which the optimal planning horizon occurs; the value for $a(\rho)$ is the real dollar value of assets at ρ; the value for $c(\rho)$ is the real dollar (annual) rate of consumption spending at ρ; the one-period integrals of consumption are the real dollar (total annual) values of consumption spending.

As in the example of Chapter II, whether the time path of consumption increases or decreases with time depends essentially on the relation between r and j. Given the particular values assumed for all the other parameters, equation (15) shows that if $j > r$, as in the example, then the rate of consumption falls over time.

$$c'(t) = \left(\frac{b\beta}{a\alpha}\right)^{\left(\frac{1}{\alpha-1}\right)} \left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{\beta-1}{(\beta-\alpha)(\alpha-1)}} \left[-\left(\frac{r-j}{\alpha-1}\right)\right] \times \left[e^{(r-j)(\rho-t)\left(\frac{1}{\alpha-1}\right)}\right] \qquad (15)$$

Some Comparative Static Properties of the Example

Although it would be better to investigate the comparative static properties of the general variable planning horizon model given in this chapter, the arithmetic involved and the *a priori* indeterminateness of some of the relative sizes of some of the terms involved make it impractical to carry out the investigation here. To arrive at even trivial determinate comparative static results in the general model would require going far beyond the kinds of assumptions usually deemed appropriate in economic analysis. Consequently, consider the less ambitious task of determining the comparative static properties of the example. One would follow the same approach if the problem were to consider the properties of the general model. In addition, concentrate the investigation on the terminal point of the planning horizon.

The Initial Asset Effect

To determine the effects of a change in initial wealth on the rate of consumption at ρ, on assets at ρ, and on ρ itself, take the total variation of the system of equations (13a), (13b), (13c), and (11), where in (13a), t is set equal to ρ. After simplifying by use of the first-order conditions, the following system of equations results:

$$(1-\alpha)c(\rho)^{-1}\lambda\bar{h}_c(\rho) + (j-r)\lambda\delta\rho + \delta\lambda = 0,$$

$$(1-\beta)a(\rho)^{-1}\lambda\bar{\delta}a(\rho) + (j-r)\lambda\delta\rho + \delta\lambda = 0,$$

$$(j-r)e^{-r\rho}\lambda\bar{\delta}a(\rho) + (j-r)e^{-r\rho}\lambda[c(\rho)-ra(\rho)]\delta\rho + e^{-r\rho}[c(\rho)-ra(\rho)]\delta\lambda = 0, \quad (16)$$

$$e^{-r\rho}\bar{\delta}a(\rho) + \int_0^\rho e^{-rx}\phi_c(x;\rho)dx\bar{h}_c(\rho) + e^{-r\rho}[c(\rho)-ra(\rho)]\delta\rho = \delta K,$$

where the function $\phi_c(x;\rho)$ is a continuous function of time by which the variation in consumption at ρ is multiplied in order to obtain the variation in the path of consumption over the planning interval. Use of Cramer's rule gives these results: $\bar{h}_c(\rho)/\delta K = 0$, $\bar{\delta}a(\rho)/\delta K = 0$; and $\delta\rho/\delta K > 0$. That is, for this example, a change in initial assets changes the length of the planning interval in the same direction as the change in assets, but the size of terminal assets and the rate of consumption at the new planning horizon are unchanged. These results are not surprising. By inspection of (14a) and (14b), one sees that consumption at the terminal point and assets at the terminal point are independent of the terminal point. That is, economic activity at ρ will be the same regardless of where ρ occurs. A change in assets merely changes the point in time at which planning stops. It also can be shown that over the interval of time for which both consumption plans exist, the new one lies everywhere above (below) the old one given an increase (a decrease) in initial assets.

The Rate of Interest Effect

To determine the effects of a change in the rate of interest on the rate of consumption at ρ, and on ρ itself, again take the total variation of the system of equations (13a), (13b), (13c), and (11), this

time with respect to r, and where in (13a), t = ρ. This leads to

$$(1-\alpha)c(\rho)^{-1}\lambda\bar{h}_c(\rho) + (j-r)\lambda\delta\rho + \delta\lambda = \rho\lambda\delta r;$$

$$(1-\beta)a(\rho)^{-1}\lambda\bar{\delta}a(\rho) + (j-r)\lambda\delta\rho + \delta\lambda = \rho\lambda\delta r;$$

$$\begin{aligned}(j-r)\lambda e^{-r\rho}\bar{\delta}a(\rho) &+ (j-r)\lambda e^{-r\rho}[c(\rho)-ra(\rho)]\delta\rho \\ &+ e^{-r\rho}[c(\rho)-ra(\rho)]\delta\lambda = e^{-r\rho}\lambda[\rho c(\rho)-r\rho a(\rho)+a(\rho)]\delta r;\end{aligned} \tag{17}$$

$$\int_0^\rho e^{-rx}\phi_c(x;\rho)dx\bar{h}_c(\rho) + e^{-r\rho}\bar{\delta}a(\rho) + e^{-r\rho}[c(\rho)-ra(\rho)]\delta\rho$$

$$= [\rho e^{-r\rho}a(\rho) + \int_0^\rho xe^{-rx}c(x)dx]\delta r.$$

Use of Cramer's rule leads to these results:

$$\frac{\bar{h}_c(\rho)}{\delta r} = \left(\frac{1-\beta}{1-\alpha}\right)\frac{b\beta a(\rho)^\beta}{ac(\rho)^\alpha(\beta-\alpha)}; \tag{18a}$$

$$\frac{\bar{\delta}a(\rho)}{\delta r} = \frac{b\beta a(\rho)^{\beta+1}}{ac(\rho)^\alpha(\beta-\alpha)}; \tag{18b}$$

$$\frac{\delta\rho}{\delta r} = \frac{\left[\rho e^{-r\rho}a(\rho) + \int_0^\rho xe^{-rx}c(x)dx\right]}{e^{-r\rho}[c(\rho)-ra(\rho)]} - \frac{b\beta a(\rho)^{\beta+1}}{[c(\rho)-ra(\rho)][ac(\rho)^\alpha(\beta-\alpha)]} \tag{18c}$$

$$- \frac{b\beta(1-\beta)a(\rho)^\beta \int_0^\rho e^{-rx}\phi_c(x;\rho)dx}{(1-\alpha)c(\rho)^{-1}e^{-r\rho}[c(\rho)-ra(\rho)][ac(\rho)^\alpha(\beta-\alpha)]}.$$

If $(\beta-\alpha) > 0$, as in the numerical example, then an increase (decrease) in the rate of interest increases (decreases) assets at ρ and the rate of consumption at ρ. The effect on ρ, however, is indeterminate. If $(\beta-\alpha) < 0$, then an increase (decrease) in the rate of

interest decreases (increases) assets at ρ and the rate of consumption at ρ. The effect on ρ again is indeterminate.

The dependence of the changes in the rate of consumption at ρ and in assets at ρ on the sign of $\beta - \alpha$ merely indicates that the directional changes in $a(\rho)$ and $c(\rho)$ depend upon how the consumer subjectively weights assets at ρ relative to the stream of consumption. If assets are more heavily weighted in the sense that $\beta > \alpha$, then $\alpha(\rho)$ rises with a rise in the rate of interest. From (18a) and (18b), $\bar{h}_c(\rho) = \left(\frac{1-\beta}{1-\alpha}\right) \frac{\bar{\delta} a(\rho)}{a(\rho)}$, or the increase in $a(\rho)$ is accompanied by an increase in $c(\rho)$. Very roughly, the more heavily the consumer weights the distant future relative to the present or near future, the more likely he is to plan to have larger assets and larger consumption at the end of his planning interval if r increases.

The indeterminacy regarding the change in ρ is more complex. Suppose $\beta - \alpha > 0$. Then the first two terms on the right-hand side of (18c) are opposite in sign. The third term's sign is indeterminate in general. It can be shown that the change in the path of consumption is indeterminate in general for $t < \rho$.[6] Therefore, even though $\bar{h}_c(\rho) > 0$ in this case, the sign of $\phi_c(x;\rho)$ may be negative, positive, or zero at any $x < \rho$.[7] The integral $\int_0^\rho e^{-rx}\phi_c(x;\rho)dx$ also, then, is indeterminate in sign. Consequently, one is unable to predict *a priori* in

[6]To show this, take the variation of the system (13a), (13b), (13c), and (11), with respect to a change in the rate of interest, and with $t < \rho$ in (13a). Using Cramer's rule, one finds that the sign of $h_c(t)/\delta r$ is indeterminate. This indeterminacy can be explained by opposing wealth effects and substitution effects as was done in a corresponding section of Chapter II.

[7]Remember $\phi_c(x;\rho)$ is that continuous function of time such that $\phi_c(x;\rho)\bar{h}_c(\rho)$ gives the increment in the optimal path of consumption resulting from, in this case, a change in r.

which direction ρ will respond to a change in r.

An interesting question to ask here seems to be why an increase in r might reduce ρ. Since an increase in r will increase the consumer's stream of nonhuman income, a positive wealth effect, and will cheapen more distant economic activity relative to less distant activity, a substitution effect, one might very well expect that ρ will increase.[8] Suppose, however, that the increase in r makes the consumer expand his planned consumption relative to the increased stream of nonhuman income. Then the point in time at which the rate of consumption begins to exceed the rate of income, a necessary condition for the planning horizon, will occur earlier than was true for the old rate of interest. If so, then it is possible, though not necessary, that the consumer might select an earlier planning horizon.[9]

A Note on Planning Costs

The above model does not include any current resource costs involved in planning. In a certainty world as considered above, such absence is reasonable. In a world of uncertainty, however, one might want to consider such costs. It can be argued that time must be spent and perhaps money, too, in gathering information or estimates about general future economic opportunities, in forming expectations

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[8]This expectation is made more plausible in light of the initial asset effect results. There it was shown that an increase in initial assets lengthens the planning interval.

[9]Perhaps an analogy might help. Suppose there is a consumer who is planning activity over a 10-year interval. Now he is told that at the end of three years he will receive a substantial gift, but he may not borrow currently on that gift. It seems reasonable that he might reduce his planning interval to three years, intending to plan further activity only after he has received his gift.

regarding the consumer's own outlook, and in establishing a detailed plan for consumption, leisure, and assets. If uncertainty increases the more distant the future point in time, then it may very well be the case that to form more reliable estimates of the future would require greater expenditures of current time and money.

In a more general context, then, one might wish to include in the model a current resource cost function for planning. It might take the form $p = p(\rho-\tau)$, where p are resources currently spent in planning (money outlay plus the monetary equivalent of foregone leisure) and $\rho - \tau$ is the length of the interval over which plans are formulated. Presumably, the marginal cost of extending the planning interval, $\frac{dp}{d(\rho-\tau)}$, would be positive and would be an additional factor leading to a terminal planning point possibly falling short of a known date of death.

CHAPTER IV

A RANDOM HORIZON MODEL OF CONSUMER BEHAVIOR

In Chapters II and III, the date of death was taken to be known or assumed to be known by the consumer. In this chapter, the date of death will be taken to be a random variable.[1] That is, instead of knowing the date of death, it is assumed that the consumer knows with certainty the probability density function of the date of death. Issues to be considered will be the effect of the randomness of death on the utility functional, alternative meanings of utility maximization in this uncertain environment, derivation of conditions necessary for expected utility maximization, and the impact of an uncertain date of death on the choice of a planning horizon. A simple consumer planning example will be presented and solved for a number of environments. The model presented here differs from other intertemporal models in that it allows the consumer to choose the length of his planning interval. The choice of a planning horizon becomes an important and interesting question given that the consumer faces an uncertain future.

A Utility Functional for a Random Horizon Model

The introduction of a random horizon forces one to reconsider the utility functional used in the earlier certainty model. New

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[1]For a definition of a random variable, see either Brunk [3, p.33] or Freund [7, p.62].

arguments can be expected to enter the functional and the meaning of utility maximization must be redefined.

Since the consumer does not know when he will die, it is reasonable to expect that the time path of real nonhuman assets will have a bearing on present utility. The assets planned at any point in time serve as a potential bequest, the bequest which the consumer would leave were he to die at that point in time. Consequently, not only is the value of assets at the end-point of the planning interval important, but the path that assets take to reach that planned terminal value is also important. An additional argument of the utility functional, then, is the path of real nonhuman assets.

One might use the mathematical expectation of the date of death as a new argument. In subjectively evaluating economic activity for some future date t, the consumer might discount such activity with his expected date of death playing a role. The closer a person is to death or to his expected death, the more important leisure or a bequest may be to him relative to consumption. That is, as t approaches the expected date of death, marginal rates of substitution can be expected to change in favor of increased leisure and perhaps increased assets relative to consumption.

In addition to the expected date of death and the path of assets, the probability density function of dying defined over some future time interval enters as an argument. If one assumes that the consumer maximizes expected utility, this density function serves as a weighting function attached to consumer plans. In general, this density function will enter the discounting procedure. For example, the greater the probability of being dead at any future date, the greater the rate of

discount to be attached to consumption and leisure at that future date.

More important than the additional variables is the fact that since the horizon is now a random variable, utility also is a random variable.[2] But if utility is a random variable, then some way must be defined to attach a real number to utility if "maximizing utility" is to make any sense.

The problem is this. Arbitrarily pick a particular admissible consumer plan; i.e., given admissible paths of consumption, leisure, and assets defined over a given interval chosen by the consumer. Let

(i) $\Gamma^o \equiv [c^o(t), l^o(t), a^o(t), \rho^o]$

be that plan, where ρ^o is the given terminal point of the planning interval satisfying $\tau < \rho^o \leqq T^*$, where T^* is the last possible date of life, and $c^o(t)$, $l^o(t)$, $a^o(t)$ are given admissible paths (satisfy certain constraints) defined over $[\tau, \rho^o]$. The present utility of this plan depends upon the date of death, T;

(ii) $U^o \equiv U(\Gamma^o) = f^o(T)$.

Assume that f^o is a continuously differentiable, real function of T (for given paths of consumption, leisure, and assets) with $f^{o\prime}(T) > 0$. If T were to occur before ρ^o, then the plan Γ^o would not be completed, and the present utility actually enjoyed from Γ^o would be the present utility of that segment of Γ^o defined over $[\tau, T]$. In general, the further beyond τ the date of death occurs, the greater the proportion of Γ^o that would be fulfilled and the greater the quantity of present

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[2]For a discussion of a function of a random variable being itself a random variable, see Lindgren [14, pp.10-11].

utility (utility at τ) yielded by Γ^{o}. If T were to occur after ρ^{o}, then the plan Γ^{o} would be fully completed and the present utility would be that of the full plan defined over $[\tau,\rho^{o}]$. Consequently, the plan Γ^{o}, or any other plan, yields not one value for present utility, but yields a distribution of values dependent upon T.

Since T is a random variable which takes on values in $[\tau,T^*]$, there is a probability density function defined on $[\tau,T^*]$, call it q(T), such that

$$\text{(iii)}\quad q(T) \geq 0, \qquad \text{for all } T \text{ in } [\tau,T^*];$$

$$\int_{\tau}^{T^*} q(T)dT = 1.$$

The probability that the consumer will die in some interval $[t_0,t_1]$ contained in $[\tau,T^*]$ is given by

$$\text{(iv)}\quad \text{Prob}\{t_0 \leq T \leq t_1\} = \int_{t_0}^{t_1} q(T)dT,$$

while the probability that the consumer will be dead at some point t_0 in $[\tau,T^*]$ is given by

$$\text{(v)}\quad Q(t_0) = \int_{\tau}^{t_0} q(T)dT.$$

The probability that the consumer will be alive at some point t_0 in $[\tau,T^*]$ is one minus the probability that he will be dead at t_0, or

$$\text{(vi)}\quad L(t_0) = 1 - Q(t_0) = \int_{\tau}^{T^*} q(T)dT - \int_{\tau}^{t_0} q(T)dT = \int_{t_0}^{T^*} q(T)dT.$$ [3]

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[3] It is understood that q, Q, and L are conditional probabilities, conditional upon the present date; e.g., $q(T) = q(T;\tau)$.

Since $U^{o} = f^{o}(T)$, U^{o} is a random variable with which is associated a probability density function $\pi(U^{o})$ dependent on $q(T)$. Density function π is defined over $[U^{o\tau}, U^{o\rho}]$, where $U^{o\tau} = f^{o}(T)|_{T=\tau}$, and $U^{o\rho} = f^{o}(T)|_{T=\rho^{o}}$. Let $U^{oo} = f^{o}(T^{o})$, where $\tau < T^{o} < \rho^{o}$. Then

(vii) $\pi(U^{o\tau}) = q(\tau)$,

$\pi(U^{oo}) = q(T^{o})$, for all T^{o} in (τ, ρ^{o}),

$\pi(U^{o\rho}) = L(\rho^{o})$,

where $L(\rho^{o})$ is the probability of being alive at ρ^{o}. If the concept of "maximizing utility" is to be meaningful, a functional now must be defined to ascribe a real number to the random variable--the present utility of any given plan.

In short, the present utility of any given plan, Γ^{o}, depends upon the date of death, T. Since T is a random variable with an assumed known probability density function, q(T), the utility of Γ^{o} is a random variable with a known probability density function, $\pi(U^{o})$, where π is dependent on q. Since maximizing a random variable is meaningless, some means of ascribing a real number to utility is now required. In other words, a functional must be defined which assigns a real number to the utility of any given plan. The consumer then chooses that plan to which this functional assigns the largest real number.

A simple example may be helpful. In a simple intertemporal model, a value must be attached to a function--the time path of real consumption--defined over some interval $[\tau, T]$. A functional such as the integral of some function of the consumption path is defined for this purpose:

(viii) $U[c;\tau,T] = \int_{\tau}^{T} \mu[c(t)]dt.$

Maximizing utility (subject to some constraint) simply means finding that permissible c(t) for which U takes on its highest value.

If the end-point T is a random variable, $U[c;\tau,T]$ is a random variable, and a functional V must be defined to assign a real value to U. Maximizing utility now means maximizing $V^*[c;\tau,T] = V[U(c;\tau,T)]$.

Infinitely many candidates exist to fill the role of the required functional V. Yaari [26, p.139] mentions the minimax principle and the expected utility hypothesis. Under the expected utility hypothesis, V is a functional such that the expectation of the random variable utility is calculated. The consumer then is viewed as maximizing expected utility, the mathematical expectation of utility. Using the minimax principle, the consumer is viewed as a pessimist who chooses that plan for which the minimum utility value for which the density function π is positive is the greatest. The consumer could be viewed as the optimistic counterpart, one who chooses that plan for which the maximum utility value for which π is positive is the greatest. The consumer may also be viewed as maximizing the median or the mode of utility. In general, the consumer may maximize any functional which ascribes real values to alternative density functions respectively associated with alternative consumer plans.

Without entering a discussion of which maximization principle is most appropriate for consumer theory, assume the consumer is an expected utility maximizer. This is the approach taken by Yaari [26] in his model regarding an uncertain death, and by Hadar [10, pp.271-277] in his model regarding an uncertain income. In Yaari's uncertain death

model a planning horizon is forced on the consumer, namely, the last possible date of life. That is, the consumer must plan over $[\tau, T^*]$. The approach in this chapter allows the consumer to choose some ρ, possibly short of T^*, beyond which he does not currently plan any explicit economic activity.

To derive the expected utility functional, consider the utility derived from an economic plan should the consumer die at any date $T \leqq \rho$. This utility may be expressed as

$$U[c,l,a(T);\tau,\bar{T},T] = \int_{\tau}^{T} \mu[c(t),l(t),a(T),\tau,\bar{T},T,t]dt, \tag{1}$$

where $\bar{T}$ is the expected date of death,

$$\bar{T} = \int_{\tau}^{T^*} Tq(T)dT. \tag{2}$$

Should the consumer die at some $T > \rho$, the utility derived from the plan would be

$$U[c,l,a(\rho),\rho;\tau,\bar{T}] = \int_{\tau}^{\rho} \mu[c(t),l(t),a(\rho),\tau,\bar{T},\rho,t]dt. \tag{3}$$

Thus, for any given $c(t), l(t)$, and $a(t)$, the utility of the plan can be expressed as a function of T, $U = f(T)$. The utility of any plan ranges from $U[c,l,a(\tau);\tau,\bar{T},\tau] = U^o$ (if $T = \tau$) to $U[c,l,a(\rho),\rho;\tau,\bar{T}] = U^*$ (if $T > \rho$), assuming the utility of any plan increases with the date of death. Defined over this range of possible utility values is a probability density function, $\pi(U)$, such that, for all $U = f(T)$ on $[U^o, U^*)$,

$$\pi(U) = \pi[f(T)] = q(T), \quad \text{for all } T \text{ in } [\tau,\rho], \tag{4a}$$

and, for $U = U^*$,

$$\pi(U^*) = \int_{\rho}^{T^*} q(T)dT = L(\rho). \tag{4b}$$

The expected utility of any plan, then, is

$$V[c,l,a,a(\rho),\rho;\tau,\bar{T},q,L(\rho)] = \int_{\tau}^{\rho} q(T) \int_{\tau}^{T} \mu[c(t),l(t),a(T),\tau,\bar{T},T,t]dtdT \quad (5)$$
$$+ L(\rho) \int_{\tau}^{\rho} \mu[c(t),l(t),a(\rho),\tau,\bar{T},\rho,t]dt.$$

It might be pointed out that if the consumer knew his date of death, i.e., if $q(T^*) = 1$, then $q(T) = 0$, for $\tau \leqq T < T^*$, $L(\rho) = 1$, for $\tau \leqq \rho < T^*$, and the expected date of death $\bar{T}$ would equal T^*. (In this case, let $\bar{T} = T^* = T$.) Then functional (5) reduces to

$$V[c,l,a(\rho),\rho;\tau,T] = \int_{\tau}^{\rho} \mu[c(t),l(t),a(\rho),\tau,T,\rho,t]dt, \quad (6)$$

which is equivalent to a utility functional used for a certainty model.[4]

The expected utility functional given by (5) can be simplified somewhat by assuming an additive integrand μ. Let

$$\mu = \alpha[c(t),l(t),\tau,\bar{T},t] + B[a(T),\tau,\bar{T},T]. \quad (7)$$

Then the first term on the right-hand side of (5) can be written as

$$\int_{\tau}^{\rho} q(T) \int_{\tau}^{T} (\alpha+B)dtdT = \int_{\tau}^{\rho} q(T)[\int_{\tau}^{T} \alpha dt+B\int_{\tau}^{T} dt]dT \quad (8)$$
$$= \int_{\tau}^{\rho} q(T)\int_{\tau}^{T} \alpha dtdT + \int_{\tau}^{\rho} q(T)\beta dT,$$

where $\beta = \beta[a(T),\tau,\bar{T},T] = (T-\tau)B[a(T),\tau,\bar{T},T]$. By reversing the order of integration in the first term on the right-hand side of the last equality in (8),

$$\int_{\tau}^{\rho} q(T)\int_{\tau}^{T} \alpha[c(t),l(t),\tau,\bar{T},t]dtdT = \int_{\tau}^{\rho} \alpha[c(t),l(t),\tau,\bar{T},t]\int_{t}^{\rho} q(T)dTdt. \quad (9)$$

But,

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[4]See footnote 1 of Chapter III.

$$\int_t^{\rho} q(T)dT = \int_t^{T^*} q(T)dT - \int_{\rho}^{T^*} q(T)dT = L(t) - L(\rho). \tag{10}$$

Therefore, substituting from (10) into (9), (9) into (8), and (8) and (7) into (5), one obtains

$$V = \int_{\tau}^{\rho} [L(t)-L(\rho)]\alpha[c(t),l(t),\tau,\bar{T},t]dt + \int_{\tau}^{\rho} q(T)\beta[a(T),\tau,\bar{T},T]dT \tag{11}$$

$$+ L(\rho)\int_{\tau}^{\rho} \alpha[c(t),l(t),\tau,\bar{T},t]dt + L(\rho)\beta[a(\rho),\tau,\bar{T},\rho], \text{ or}$$

$$V[c,l,a,a(\rho),\rho;\tau,\bar{T},q,L(\rho)] = \int_{\tau}^{\rho} L(t)\alpha[c(t),l(t),\tau,\bar{T},t]dt \tag{12}$$

$$+ \int_{\tau}^{\rho} q(T)\beta[a(T),\tau,\bar{T},T]dT + L(\rho)\beta[a(\rho),\tau,\bar{T},\rho],$$

where equation (12) states the expected utility functional for a random horizon model.

A Statement of the Problem and Some Necessary Conditions

The problem facing the consumer is to choose those $c(t)$, $l(t)$, $a(T)$, $a(\rho)$, and ρ which maximize expected utility V subject to the following budget considerations. The terminal value planned for nonhuman assets must satisfy the equation

$$e^{r(\rho-\tau)}a(\tau) + \int_{\tau}^{\rho} e^{r(\rho-t)}F[l(t),t]dt = a(\rho) + \int_{\tau}^{\rho} e^{r(\rho-t)}c(t)dt. \tag{13}$$

This wealth constraint can be looked at as an "isoperimetric constraint"[5] which must be satisfied by the planned terminal value of nonhuman assets for any consumer plan. For any point T along the path of planned human assets, $a(t)$, defined over $[\tau,\rho]$, a similar budget constraint must be satisfied,

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[5]See Gelfand and Fomin [9, pp.42-46].

$$e^{r(T-\tau)}a(\tau) + \int_{\tau}^{T} e^{r(T-t)}F[l(t),t]dt = a(T) + \int_{\tau}^{T} e^{r(T-t)}c(t)dt. \tag{14}$$

This wealth constraint can be looked at as a "finite subsidiary condition"[6] that must be met by the full path of planned nonhuman assets. Consequently, the decision variable a(T) in functional (12) must obey constraint (14), while the decision variable a(ρ) in functional (12) must obey constraint (13). Assuming an interior solution and thereby ignoring any nonnegativity conditions one might wish to impose on the decision functions, those c(t), l(t), a(T), a(ρ), and ρ which maximize V subject to (13) and (14) must maximize V*, where

$$V^* = \int_{\tau}^{\rho} L(t)\alpha[c(t),l(t),\tau,\bar{T},t]dt + \int_{\tau}^{\rho} \{q(T)\beta[a(T),\tau,\bar{T},T] \tag{15}$$
$$+ \lambda(T)[a(T)+\int_{\tau}^{T} e^{r(T-t)}c(t)dt-e^{r(T-\tau)}a(\tau)-\int_{\tau}^{T} e^{r(T-t)}F[l(t),t]dt]\}dT$$
$$+ L(\rho)\beta[a(\rho),\tau,\bar{T},\rho] + \phi\{a(\rho)+\int_{\tau}^{\rho} e^{r(\rho-t)}c(t)dt-e^{r(\rho-\tau)}a(\tau)$$
$$- \int_{\tau}^{\rho} e^{r(\rho-t)}F[l(t),t]dt\},$$

where λ(T) is a continuous function and φ is a Lagrangian multiplier. First-order conditions now are found by setting the first variation of V* equal to zero.[7]

The first variation of V* is

$$\delta V^* = \int_{\tau}^{\rho} L(t)[\alpha_c(t)h_c(t)+\alpha_l(t)h_l(t)]dt + L(\rho)\alpha[c(\rho),l(\rho),\tau,\bar{T},\rho]\delta\rho \tag{16}$$
$$+ \int_{\tau}^{\rho} \{q(T)\beta_a(T)h_a(T)+\lambda(T)[h_a(T)+\int_{\tau}^{T} e^{r(T-t)}h_c(t)dt$$

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[6]See Gelfand and Fomin [9, pp.46-48].

[7]This problem, of course, may also be solved by the direct substitution method; i.e., by substituting for a(T) and a(ρ) in (12) from equations (14) and (13).

$$- \int_{\tau}^{T} e^{r(T-t)} F_1(t)h_1(t)dt]\}dT + \{q(\rho)\beta[a(\rho),\tau,\bar{T},\rho] + \lambda(\rho)[a(\rho)$$

$$+ \int_{\tau}^{\rho} e^{r(\rho-t)} c(t)dt - e^{r(\rho-\tau)} a(\tau) - \int_{\tau}^{\rho} e^{r(\rho-t)} F[l(t),t]dt]\}\delta\rho$$

$$+ L(\rho)\beta_a(\rho)\bar{\delta}a(\rho) + L(\rho)\beta_\rho(\rho)\delta\rho + L'(\rho)\beta[a(\rho),\tau,\bar{T},\rho]\delta\rho$$

$$+ \phi\{\bar{\delta}a(\rho) + \int_{\tau}^{\rho} e^{r(\rho-t)} h_c(t)dt + \int_{\tau}^{\rho} re^{r(\rho-t)} c(t)dt\delta\rho + c(\rho)\delta\rho$$

$$- re^{r(\rho-\tau)} a(\tau)\delta\rho - \int_{\tau}^{\rho} e^{r(\rho-t)} F_1(t)h_1(t)dt - \int_{\tau}^{\rho} re^{r(\rho-t)} F[l(t),t]dt\delta\rho$$

$$- F[l(\rho),\rho]\delta\rho\},$$

where $\bar{\delta}a(\rho) = \delta a(\rho) + a'(\rho)\delta\rho$. This equation can be simplified. By reversing the order of integration of $\int_{\tau}^{\rho} \lambda(T) \int_{\tau}^{T} e^{r(T-t)} h_c(t)dtdT$ and $-\int_{\tau}^{\rho} \lambda(T)[\int_{\tau}^{T} e^{r(T-t)} F_1(t)h_1(t)dt]dT$ and noting that $L'(\rho) = \frac{\partial}{\partial\rho} \int_{\rho}^{T*} q(T)dT = -q(\rho)$, equation (16) can be written as

$$\delta V* = \int_{\tau}^{\rho} L(t)[\alpha_c(t)h_c(t) + \alpha_1(t)h_1(t)]dt + L(\rho)\alpha[c(\rho),l(\rho),\tau,\bar{T},\rho]\delta\rho \quad (17)$$

$$+ \int_{\tau}^{\rho} [q(T)\beta_a(T)h_a(T) + \lambda(T)h_a(T)]dT + \int_{\tau}^{\rho} \int_{t}^{\rho} \lambda(T)e^{r(T-t)} dTh_c(t)dt$$

$$- \int_{\tau}^{\rho} \int_{t}^{\rho} \lambda(T)e^{r(T-t)} F_1(t)dTh_1(t)dt + L(\rho)\beta_a(\rho)\bar{\delta}a(\rho) + L(\rho)\beta_\rho(\rho)\delta\rho$$

$$+ \phi\{\bar{\delta}a(\rho) + \int_{\tau}^{\rho} e^{r(\rho-t)} h_c(t)dt + c(\rho)\delta\rho - F[l(\rho),\rho]\delta\rho - ra(\rho)\delta\rho$$

$$- \int_{\tau}^{\rho} e^{r(\rho-t)} F_1(t)h_1(t)dt\}.$$

Setting (17) equal to zero and treating the variations in the decision functions as independent yields:

$$\int_{\tau}^{\rho} \{L(t)\alpha_c(t) + \int_{t}^{\rho} \lambda(T)e^{r(T-t)} dT + \phi e^{r(\rho-t)}\}h_c(t)dt = 0,$$

$$\int_{\tau}^{\rho} \{L(t)\alpha_1(t) - \int_{t}^{\rho} \lambda(T)e^{r(T-t)} F_1(t)dT - \phi e^{r(\rho-t)} F_1(t)\}h_1(t)dt = 0,$$

$$\int_{\tau}^{\rho} [q(T)\beta_a(T)+\lambda(T)]h_a(T)dT = 0, \tag{18}$$

$$[L(\rho)\beta_a(\rho)+\phi]\bar{\delta}a(\rho) = 0,$$

$$\{L(\rho)\alpha[c(\rho),l(\rho),\tau,\bar{T},\rho] + L(\rho)\beta_{\rho}(\rho) + \phi[c(\rho)-F[l(\rho),\rho]-ra(\rho)]\}\delta\rho = 0.$$

First-order conditions, then, are

$$L(t)\alpha_c(t) + \int_t^{\rho} \lambda(T)e^{r(T-t)}dT + \phi e^{r(\rho-t)} = 0, \text{ for all } t \text{ in } [\tau,\rho]; \tag{19a}$$

$$L(t)\alpha_l(t) - \int_t^{\rho} \lambda(T)e^{r(T-t)}F_l(t)dT - \phi e^{r(\rho-t)}F_l(t) = 0, \quad \text{for all } t \text{ in } [\tau,\rho]; \tag{19b}$$

$$q(T)\beta_a(T) + \lambda(T) = 0, \quad \text{for all } T \text{ in } [\tau,\rho]; \tag{19c}$$

$$L(\rho)\beta_a(\rho) + \phi = 0; \tag{19d}$$

$$L(\rho)\alpha[c(\rho),l(\rho),\tau,\bar{T},\rho] + L(\rho)\beta_{\rho}(\rho) + \phi[c(\rho)-F[l(\rho),\rho]-ra(\rho)] = 0. \tag{19e}$$

Substituting for $\lambda(T)$ and ϕ from (19c) and (19d) into (19a), (19b) and (19e),

$$L(t)\alpha_c(t) - \int_t^{\rho} q(T)\beta_a(T)e^{r(T-t)}dT - L(\rho)\beta_a(\rho)e^{r(\rho-t)} = 0, \quad \text{for all } t \text{ in } [\tau,\rho]; \tag{20a}$$

$$L(t)\alpha_l(t) + \int_t^{\rho} q(T)\beta_a(T)e^{r(T-t)}F_l(t)dT + L(\rho)\beta_a(\rho)e^{r(\rho-t)}F_l(t) = 0, \quad \text{for all } t \text{ in } [\tau,\rho]; \tag{20b}$$

$$\alpha[c(\rho),l(\rho),\tau,\bar{T},\rho] + \beta_{\rho}(\rho) - \beta_a(\rho)[c(\rho)-F[l(\rho),\rho]-ra(\rho)] = 0. \tag{20c}$$

Conditions (20) along with (13) and (14) now may be used to solve for the optimal plan.

Condition (20a) states that the rate of present marginal utility of consumption at any t in the planning interval weighted by the probability of being alive at t must equal a weighted sum of rates of

marginal utilities of assets for the interval $[t,\rho]$. For any t_0 in $[t,\rho]$, the rate of present marginal utility of assets is weighted by the value of the mortality density function at t_0 and multiplied by the rate of exchange between consumption at t and assets at t_0. In addition, the rate of present marginal utility of assets at ρ is weighted by the probability of being alive at ρ and multiplied by the rate of exchange between consumption at t and assets at ρ. In a sense, the weighted rate of marginal utility of consumption at any t must equal the weighted rate of marginal utility of the path of assets over $[t,\rho]$. The weights involved are (i) prices dependent upon r, the rates of exchange between consumption at t and assets over $[t,\rho]$, and (ii) probability density values, either that of dying during the interval $[t,\rho]$ and consequently leaving a bequest or the probability of being alive at the planning horizon ρ and not yet leaving a bequest. Condition (20a) thus points out the twin nature of the path of assets, serving as a potential bequest over $[t,\rho]$, and serving as a stock of wealth at ρ which can be used to finance economic activity beyond ρ should the consumer live beyond ρ.

Condition (20b) states a similar relation which must hold between the rate of present marginal utility of leisure at any t in $[\tau,\rho]$ and that of the stream of assets over $[t,\rho]$. Again, the rate of present marginal utility of leisure at t is weighted by the probability of being alive at t, while the rates of present marginal utilities of assets over $[t,\rho]$ are weighted by the values of the mortality density function over $[t,\rho]$ and the rate of present marginal utility of assets at ρ is weighted by the chance of living at ρ. Again, the rates of exchange between leisure and assets also enter as weights.

Condition (20c) states that the rate of present marginal utility of expanding the planning interval beyond ρ must equal the rate of present marginal utility of assets at ρ multiplied by the rate of change in assets at ρ. In marginal rate of substitution form,

$$\frac{\alpha[c(\rho),l(\rho),\tau,\bar{T},\rho] + \beta_\rho(\rho)}{\beta_a(\rho)} = c(\rho) - F[l(\rho),\rho] - ra(\rho), \tag{21}$$

where the rate of present marginal utility of increasing ρ is given by the rate of present utility from consumption and leisure at ρ, $\alpha[c(\rho),l(\rho),\tau,\bar{T},\rho]$, plus the marginal effect of changing ρ on the present utility from assets at ρ, $\beta_\rho(\rho)$; the rate of present marginal utility of assets at ρ in $\beta_a(\rho)$. The marginal rate of substitution between increasing ρ and assets at ρ, the left-hand side of (21), must equal the rate at which assets at ρ are decreasing, the right-hand side of (21). This is a similar condition to that in the planning horizon model of Chapter III.[8] Given the expected utility functional V of equation (12), the probabilities of being alive or of being dead at ρ do not enter the marginal rate of substitution between increasing ρ and assets at ρ. As in the model of Chapter III, for a point $\hat{\rho} < T^*$ to be the optimal planning horizon, it is necessary that the path of assets be declining at $\hat{\rho}$. Alternatively, if at a point $\rho^o < T^*$ assets are increasing, then ρ^o cannot be the optimal planning horizon.

If the integrand μ of the certainty planning horizon model in Chapter III were additive and if the expected date of death (assumed known to be T) replaced ρ in the discounting procedure, then (20c) in this uncertainty model would be identical with the like condition in

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[8]See equation (8c') of Chapter III.

the certainty model. That is, if

$$U[c,l,a(\rho),\rho;\tau,\bar{T}] = \int_{\tau}^{\rho} \{\alpha[c(t),l(t),\tau,\bar{T},t] + B[a(\rho),\tau,\bar{T},\rho]\}dt \qquad (22)$$

$$= \int_{\tau}^{\rho} \alpha[c(t),l(t),\tau,\bar{T},t]dt + \beta[a(\rho),\tau,\bar{T},\rho],$$

then both the random horizon model of this chapter with utility functional V of equation (12) and the certainty planning horizon model of Chapter III with utility functional U of equation (22) would have equation (20c) as a first-order condition relating the rate of present marginal utility of increasing ρ and the rate of present marginal utility of assets at ρ. Therefore, the introduction of an uncertain date of death need not directly affect the marginal rate of substitution between increasing ρ and assets at ρ. However, since the random horizon does affect the other first-order conditions regarding the rates of marginal utilities of $c(t)$, $l(t)$, and $a(\rho)$ and, consequently, can be expected to influence optimal $c(t)$, $l(t)$, and $a(\rho)$, optimal ρ, itself, can be expected to be influenced by the introduction of a random horizon.

On the other hand, if ρ were to replace $\bar{T}$ in the consumer's discounting procedure for the expected utility functional V so that $\mu = \alpha[c(t),l(t),\tau,\rho,t] + B[a(T),\tau,T]$ in equation (7), then the first-order condition regarding the rates of marginal utility of increasing ρ and of assets at ρ would be

$$\int_{\tau}^{\rho} L(x)\alpha_{\rho}(x)dx + L(\rho)\alpha[c(\rho),l(\rho),\tau,\rho,\rho] + L(\rho)\beta_{\rho}(\rho) \qquad (23)$$

$$- L(\rho)\beta_{a}(\rho)[c(\rho)-F[l(\rho),\rho]-ra(\rho)] = 0,$$

and the probability of being alive at any t in $[\tau,\rho]$ would affect the

marginal rate of substitution between increasing ρ and assets at ρ. Because of this direct impact on the rate of marginal utility of increasing ρ as well as any indirect impact because of different optimal paths of consumption, leisure, and assets, $\hat{\rho}$ can be expected to differ from its value in a similar certainty model.

The main conclusion here is that the probability density function of the date of death and the related probability of being alive at any future date will affect the marginal rates of substitution between consumption, leisure, and assets, and may affect the marginal rate of substitution between increasing ρ and $a(\rho)$. In any case, the introduction of a random horizon will influence the choice of an optimal planning horizon.

Some Examples

In order to reduce greatly the arithmetic, consider a simple intertemporal consumer planning problem. Suppose the only decision to make is the choice of an optimal consumption path where initially it is assumed that the consumer plans to the last possible date of life and where the consumer has a fixed stock of wealth to allocate to consumption. Although the consumer does not know when he will die, it is assumed that he knows the probability density function of death.

Let the expected utility functional for this problem be

$$V = \int_0^{T^*} q(T) \int_0^{T} \mu[c(t),t]dtdT, \tag{24}$$

where the present date is taken to be zero, and T^* is the last possible date of life. By reversing the order of integration in (24), one obtains

$$V = \int_0^{T^*} \mu[c(t),t] \int_t^{T^*} q(T)dTdt = \int_0^{T^*} L(t)\mu[c(t),t]dt, \tag{25}$$

where L(t) is the probability density function of being alive at any t. Suppose

$$\mu[c(t),t] = \ln[e^{-jt}c(t)] = -jt + \ln c(t), \tag{26}$$

where $0 < j < 1$. Suppose also that the mortality density function is rectangular; that is, let $q(T) = 1/T^*$, for all T in $[0,T^*]$. Then

$$L(t) = \int_t^{T^*} q(T)dT = \int_t^{T^*} 1/T^*\, dT = \frac{T^* - t}{T^*}, \quad \text{for all } t \text{ in } [0,T^*]. \tag{27}$$

Constraining the consumer to allocate all present wealth to consumption, the wealth constraint may be written as

$$K = \int_0^{T^*} e^{-rt}c(t)dt, \tag{28}$$

where K is a positive constant, and $0 < r < 1$.

To solve this example, form the functional

$$V^* = \int_0^{T^*} \{\left(\frac{T^*-t}{T^*}\right)[-jt+\ln c(t)] + \lambda e^{-rt}c(t)\}dt - \lambda K, \tag{29}$$

where λ is a Lagrangian multiplier. Setting the first variation of V^* equal to zero yields the following first-order condition:

$$\left(\frac{T^*-t}{T^*}\right)c(t)^{-1} + \lambda e^{-rt} = 0, \quad \text{for all } t \text{ in } [0,T^*], \tag{30}$$

which can be solved easily for $\hat{c}(t)$ in terms of λ:

$$\hat{c}(t) = \left(\frac{t-T^*}{T^*}\right)\left(\frac{1}{\lambda}\right)e^{rt}, \quad \text{for all } t \text{ in } [0,T^*]. \tag{31}$$

Substituting for $\hat{c}(t)$ in the constraint,

$$K = \int_0^{T^*} \left(\frac{1}{\lambda}\right)\left(\frac{t-T^*}{T^*}\right)dt = -\left(\frac{T^*}{2\lambda}\right). \tag{32}$$

Therefore,

$$\frac{1}{\lambda} = -\frac{2K}{T^*}, \text{ and} \tag{33}$$

$$\hat{c}(t) = e^{rt}\left(\frac{2K}{T^*}\right)\left(\frac{T^*-t}{T^*}\right), \text{ for all } t \text{ in } [0,T^*]. \tag{34}$$

Substituting for $\hat{c}(t)$ in the utility functional (25),

$$V = \int_0^{T^*}\left(\frac{T^*-t}{T^*}\right)\left[-jt+\ln e^{rt}\left(\frac{2K}{T^*}\right)\left(\frac{T^*-t}{T^*}\right)\right]dt \tag{35}$$

$$= (\ln 2K-2\ln T^*)\int_0^{T^*}\left(\frac{T^*-t}{T^*}\right)dt + \int_0^{T^*}\left(\frac{T^*-t}{T^*}\right)[(r-j)t+\ln(T^*-t)]dt$$

$$= \frac{1}{6}(r-j)T^{*2} + \frac{1}{2}T^*\ln 2K - \frac{1}{4}T^* - \frac{1}{2}T^*\ln T^*.$$

Suppose the parameters have the following values: $r = .03$, $j = .33$, $T^* = 70$, and $K = 110{,}000$. Then the value of present expected utility for the optimal consumer plan defined over $[0,T^*]$ is 19.35.

Now suppose the planning horizon is another decision variable for the consumer. The expected utility functional for this case can be written as

$$V = \int_0^{\rho} q(T)\int_0^{T}\mu[c(t),t]dtdT + \int_{\rho}^{T^*} q(T)\int_0^{\rho}\mu[c(t),t]dtdT. \tag{36}$$

Upon changing the order of integration in the first term on the right-hand side of (36) and simplifying, one obtains

$$V = \int_0^{\rho} L(t)\mu[c(t),t]dt. \tag{37}$$

Substituting for $L(t)$ and μ in (37) from (27) and (26) and constraining the consumer to allocate all present wealth to consumption over the planning interval,

$$K = \int_0^{\rho} e^{-rt}c(t)dt, \tag{38}$$

then this planning horizon problem is solved by forming

$$V^* = \int_0^{\rho} \{\left(\frac{T^*-t}{T^*}\right)[-jt+\ln c(t)] + \lambda e^{-rt}c(t)\}dt - \lambda K, \tag{39}$$

and setting its first variation equal to zero. This yields the first-order conditions

$$\left(\frac{T^*-t}{T^*}\right)c(t)^{-1} + \lambda e^{-rt} = 0, \quad \text{for all } t \text{ in } [0,\rho]; \tag{40a}$$

$$\left(\frac{T^*-\rho}{T^*}\right)[-j\rho+\ln c(\rho)] + \lambda e^{-r\rho}c(\rho) = 0, \tag{40b}$$

which along with constraint (38) can be used to solve for $\hat{c}(t)$, $\hat{\rho}$, and λ.

By setting $t = \rho$ in (40a), (40a) and (40b) can be used to solve for $\hat{c}(\rho)$ and λ in terms of ρ:

$$\hat{c}(\rho) = e^{1+j\rho}, \tag{41a}$$

$$\lambda = \left(\frac{\rho-T^*}{T^*}\right)e^{\rho(r-j)-1} \tag{41b}$$

Substituting for λ, then, in (40a), $\hat{c}(t)$ can be solved for in terms of ρ:

$$\hat{c}(t) = \left(\frac{T^*-t}{T^*-\rho}\right)e^{\rho(j-r)+1+rt}, \quad \text{for all } t \text{ in } [0,\rho]. \tag{42}$$

Finally, substituting for $\hat{c}(t)$ in the constraint,

$$\int_0^{\rho} \left(\frac{T^*-t}{T^*-\rho}\right)e^{\rho(j-r)+1}dt = K, \tag{43a}$$

which yields

$$\frac{\hat{\rho}(2T^*-\hat{\rho})}{2(T^*-\hat{\rho})}\, e^{\hat{\rho}(j-r)+1} = K. \tag{43b}$$

Given the values $T^* = 70$, $r = .03$, $j = .33$, and $K = 110{,}000$, and given

that $\hat{\rho}$ must lie between 0 and 70, optimal ρ is very near 24.[9]

Substituting into utility functional (37), one obtains

$$V = \int_0^{24} \left(\frac{70-t}{70}\right)\{-.33t+\ln\left[\left(\frac{70-t}{46}\right)e^{24(.3)+1+.03t}\right]\}dt. \qquad (44)$$

From (44), one may obtain the value of expected utility from the optimal consumption path given by (42) and for $\hat{\rho} = 24$: $\hat{V} = 101.2$. Therefore, by planning currently only out to date $\rho = 24$, the value of expected utility from the optimal consumption path is 101.2, while, if planning is forced to take place to the last possible date of life, $T^* = 70$, the value of expected utility at best is 19.35.

As a special case, consider the certainty case where the consumer knows the date of death to be $T = 35$, the expected date of death for the above rectangular mortality density function and for the assumed value of $T^*(=70)$. Then, for any $\rho < 35$, utility functional (36) reduces to

$$V = \int_0^{\rho} \mu[c(t),t]dt = \int_0^{\rho} [-jt+\ln c(t)]dt. \qquad (45)$$

Constraining the consumer to allocate all present wealth to consumption over the planning interval, form the functional

$$V^* = \int_0^{\rho} \{-jt+\ln c(t)+\lambda e^{-rt}c(t)\}dt - \lambda K, \qquad (46)$$

set its first variation equal to zero, and obtain the first-order conditions

$$c(t)^{-1} + \lambda e^{-rt} = 0, \quad \text{for all } t \text{ in } [0,\rho]; \qquad (47a)$$

$$-j\rho + \ln c(\rho) + \lambda e^{-r\rho}c(\rho) = 0. \qquad (47b)$$

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[9] For $\hat{\rho} = 24$, (43b) is 110,173 ≈ 110,000.

From (47a),

$$c(t) = -\left(\frac{1}{\lambda}\right)e^{rt}, \quad \text{for all t in } [0,\rho]. \tag{48a}$$

Substituting (48a) into the constraint,

$$K = \int_0^\rho e^{-rt}[-\left(\frac{1}{\lambda}\right)e^{rt}]dt = -\frac{\rho}{\lambda}, \quad \text{or} \tag{48b}$$

$$-\frac{1}{\lambda} = \frac{K}{\rho}. \tag{48c}$$

Therefore,

$$c(t) = e^{rt}\left(\frac{K}{\rho}\right), \quad \text{for all t in } [0,\rho], \text{ and} \tag{48d}$$

$$c(\rho) = e^{r\rho}\left(\frac{K}{\rho}\right). \tag{48e}$$

Substituting for λ and $c(\rho)$ in (47b), and using the previously assumed values for K, r, and j,

$$-.33\rho + \ln e^{.03\rho}\left(\frac{110,000}{\rho}\right) - 1 = 0, \text{ or} \tag{49a}$$

$$-.33\rho + .03\rho + \ln 110,000 - \ln\rho - 1 = 0. \tag{49b}$$

The value for ρ that satisfies (49b) is very near 24.7. Therefore,

$$\hat{\rho} \approx 24.7, \tag{50a}$$

$$\hat{c}(t) = 4453\ e^{.03t}, \quad \text{for all t in } [0,24.7]. \tag{50b}$$

Substituting the solution into the utility functional, one obtains the value of utility yielded by the optimal plan: $\hat{V} = 116.0$, as expected, a value greater than those yielded by the uncertainty environments.

It should be pointed out for these examples that although the consumer is constrained to allocate all present wealth to consumption over

the planning interval, no presumption is made that the consumer, over time, will always follow these optimal paths. Indeed, one would expect the optimal path to change as the present point in time advances. That is, as the consumer actually grows older and reevaluates any previously selected optimal path of consumption, one would expect the consumer to choose a new optimal path and a new optimal planning horizon given his new wealth position. In the Strotz consistency sense, the consumer would be inconsistent.[10]

Concluding Remarks

As mentioned at the end of the preceding chapter, one might wish to include any current resource costs involved in planning as a formal aspect of an uncertainty model. These costs would involve any time and money spent on forecasting future economic variables such as human income parameters and rates of interest as well as any resources spent on gathering information regarding the mortality density function. The introduction of such explicit planning costs would present the consumer with a direct cost of planning in addition to the alternative planning cost already embodied in the planning model, namely, the rate of change in the path of assets. To the extent that any such marginal planning costs are positive, one would expect the planning horizon to be affected. As the marginal resource cost of planning increases, one would expect the optimal planning horizon to decrease.

For the case of maximizing expected utility, one finds that the probabilities of living or dying serve as weights attached to the present marginal utilities of various kinds of economic variables. Consequently,

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[10]See Strotz [21].

marginal rates of substitution are affected in general and, therefore, optimal consumer plans are affected. The marginal rate of substitution between expanding the planning horizon and assets at the planning horizon, however, need not be affected by the introduction of a random horizon. Whether or not this marginal rate of substitution is affected depends upon the particular utility functional used and especially upon the kind of discounting mechanism used in subjectively evaluating future economic activity for current utility purposes.

As the simple consumer planning example used in this chapter shows, the choice of an optimal planning horizon is affected by the introduction of an uncertain date of death. In particular, allowing the consumer to choose an optimal planning interval rather than forcing on him an interval involving a known date of death or the last possible date of death allows the consumer to increase present utility to be derived from future economic activity.

CHAPTER V

SUMMARY AND FINAL REMARKS

Including the planning horizon as a decision variable for the consumer has been shown in this thesis to be a relatively easy step. In addition, although introducing uncertainty into the environment complicates the problem of intertemporal utility maximization, uncertainty does not make the handling of the problem unmanageable. In particular, considering the planning horizon as a choice variable is not seriously affected by the introduction of uncertainty. Traditional necessary conditions hold intratemporally and intertemporally in the variable planning horizon case, and interesting but not startling conditions must hold at the optimal planning horizon. The main contributions of this thesis are the introduction of the planning horizon as a consumer decision variable, the derivation of necessary conditions for utility maximization and for expected utility maximization, and the way in which uncertainty affects the choice of an optimal planning horizon.

Summary

An intertemporal consumer model allowing a variable planning interval (with or without uncertainty) provides useful generality which the traditional model lacks. The additional generality is obtained with relatively little additional cost. Instead of constraining the consumer to plan over a fixed, predetermined interval such as his expected remaining life-span, the approach here is to allow the consumer to choose an optimal interval of time over which to select optimal economic activity. Essentially, this approach allows the consumer to choose from a larger set of admissible plans, and therein lies its increased generality.

Restricting comments here to those concerning a continuous-time, uncertain death, variable planning horizon model, three major points should be made. First, taking the date of death to be a random variable requires one to take utility to be a random variable. Consequently, the utility of any consumer plan is properly described by a probability density function, not by a single real number. As a result, "maximizing utility" must be defined in a way that makes sense. For this reason, a functional must be defined which assigns a unique real number to the probability density function associated with any plan. The functional defining expected utility comes most readily to mind and is the one used in the uncertainty work in this thesis.

Second, most of the necessary conditions for utility or expected utility maximization are the same in the variable end-point context as is the usual fixed end-point context since once the optimal end-point is selected, the problem is equivalent to a fixed end-point problem. Therefore, the same first-order and second-order conditions must hold at each point in the optimal variable planning interval as must hold at each respective point in the fixed planning interval. Among these conditions are the traditional intratemporal and intertemporal ones of equality between marginal rates of substitution and price ratios, and the traditional uncertainty ones of equality between expected marginal rates of substitution and price ratios.

Third, the only additional condition introduced by a variable planning interval is the "transversality" one, a condition which must be

met by the variable end-point (the planning horizon). The transversality condition also involves a marginal rate of substitution and a price ratio. In particular, in a certainty world the marginal rate of substitution between expanding the planning horizon, ρ, and assets at ρ must equal the rate at which the path of assets at ρ is declining. The interpretation of this result is that at ρ, the rate of marginal utility of consumption (leisure) must equal the rate of marginal utility of assets multiplied by the price of consumption (leisure) in terms of assets. Briefly, extending ρ involves an alternative cost to the consumer of reduced planned terminal assets. Of some interest here is that the transversality condition may be unaffected by the introduction of a random date of death. If the subjective discount function depends only on the difference between any future date and the present date, if the utility function of consumption and leisure is independent of ρ, and if assets yield the same utility at ρ as a bequest or as a stock of wealth from which to finance continued existence,[1] then the end-point condition involving ρ is unaffected by the introduction of a random horizon. Regardless of whether the transversality condition is affected, the remaining necessary conditions are affected by uncertainty, and consequently one would expect the optimal plan to be different in a certainty world from that in an uncertainty world. The smaller the chance of being alive at some future date t_o (identically, the greater the chance of dying in $[\tau,t_o]$), the smaller the expected rates of marginal utility of consumption and leisure at t_o, and the greater the expected rate of marginal utility of assets serving as a potential bequest over $[\tau,t_o]$.

[1] $U = \int_{\tau}^{\rho} j(t-\tau)u[c(t),l(t),t]dt + b[a(\rho),\rho,\tau]$, where j is a subjective discount function, u is a utility function, and b is a utility function.

Consequently, given diminishing marginal utilities one would expect planned consumption and leisure at t_o to be less and planned assets over $[\tau,t_o]$ to be greater, the more likely the consumer is to die before t_o. Although this does not follow directly from the first-order conditions, one might expect that the optimal planning interval will be shorter, the greater the chance of dying within some interval containing the present point in time.

The use of the calculus of variations or optimal control theory in consumer theory seems to be promising. Since there exists a body of received theory in these areas dealing with finite variable end-point problems, one should be able to deal with the planning horizon as a decision variable for a consumer in much the same way as the planning horizon is considered a decision variable for a firm in making investment decisions. This thesis was meant as an initial step in this direction.

Some Possible Extensions of the Basic Model

No life insurance or annuity streams were incorporated in the above models but could easily be included. That is, the wealth constraint could be reformulated to take into account the payoff value of a life insurance policy as well as the insurance premium stream paid over some interval. Yaari's model [26] demonstrates a way in which this might be done.

Other kinds of uncertainty might be introduced. The stream of future human income might be taken to be a random variable. At any point in time, the human wage rate could be looked at as taking on any of a distribution of possible values each with an associated probability. The consumer then would be viewed as choosing consumption and leisure

paths with certainty over the optimal planning interval, but because of the uncertainty of the human income function, would be faced with terminal assets being a random variable. Amounts to be borrowed or lent over any subinterval of the planning interval also, of course, would be random. Along these same lines, the rate of interest could be considered random leading to an uncertain stream of nonhuman income.

Instead of or in addition to the choice of an optimal planning horizon, one might recast the model to consider the choice of an optimal retirement date. That is, find conditions which must be satisfied at a future date in order for that date to be the one for which and beyond which planned labor is zero.

Of perhaps greater analytic interest would be the extension of the analysis to cover dynamic aspects of consumer planning especially with regard to the choice of an optimal planning horizon. The question to be answered here would be how the planning horizon would be affected as the consumer ages. That is, as time goes on and the consumer grows older, if he reevaluates a previously selected optimal plan, what will happen to the planning interval? Will the terminal planning date remain the same, will it move further into the future, and especially will ρ approach a known date of death or the last possible date of life as the consumer grows older? In a sense, this is another aspect of consumer consistency in the Strotz sense. An interesting question to ask in this regard would be: As a consumer grows older, does the length of his optimal planning interval grow shorter (or longer) with the optimal planning horizon approaching a known date of death or the last possible date of life?

Much work remains to be done in this area of consumer theory, especially work leading up to testable hypotheses or statements of some empirical relevance. One would like to derive meaningful or testable comparative static properties for the uncertain death case and dynamic properties of the optimal consumer plan. One would especially like to indicate how the planning interval changes in response to changes such as an increased expected life-span, an increased human wage function, or the simple aging of a consumer. Answers to these questions are not found easily, and this thesis provides very few such answers. From what has been accomplished in this thesis, one might conclude that the marginal propensity to consume would be greater for a consumer with a short expected remaining life-span and little motive to leave a bequest than for a consumer with a long expected remaining life-span and great motive to leave a bequest. More statements such as this, especially much less trivial ones and ones on an aggregate level, would be useful. How shifts in the age distribution of the population affect the aggregate marginal propensity to consume would be helpful from a macroeconomic policy standpoint.

As a concluding statement, it is hoped that the introductory work done in this thesis indicates that the choice of an optimal planning interval for the consumer is an important and interesting aspect of consumer planning, an aspect that is easily incorporated in an intertemporal model, and that further work in consumer theory might use the planning horizon as an additional choice variable.

APPENDIX

APPENDIX

A. On Interpreting the Lagrangian Multiplier λ

In the certainty model of Chapter II, first-order conditions were derived which involve a Lagrangian multiplier λ. To interpret this multiplier, consider the effects on utility and present wealth of a change in present nonhuman assets. Such a change may occur through the present receipt of a gift, through calculating the present value of a future gift such as an inheritance, etc. By the use of Taylor expansions around $\mu[\hat{c}(t),\hat{l}(t),\hat{a}(T),\tau,T,t]$ and $F[\hat{l}(t),t] - \hat{c}(t)$, the changes in utility and in the present wealth constraint can be written as

$$\Delta U = \int_{\tau}^{T} [\left(\mu_c(t)+\varepsilon_c(t)\right)h_c(t) + \left(\mu_l(t)+\varepsilon_l(t)\right)h_l(t) + \left(\mu_a(t)+\varepsilon_a(t)\right)\Delta a(T)]dt, \tag{1}$$

$$\Delta a(T) - \int_{\tau}^{T} e^{r(T-t)}[F_l(t)h_l(t)-h_c(t)]dt = e^{r(T-\tau)}\Delta a(\tau), \tag{2}$$

where $\varepsilon_c(t) \to 0$ as $h_c(t) \to 0$, $\varepsilon_l(t) \to 0$ as $h_l(t) \to 0$, and $\varepsilon_a(t) \to 0$ as $\Delta a(T) \to 0$. Substituting for $\Delta a(T)$ from (2) into (1),

$$\Delta U = \int_{\tau}^{T} \{[\mu_c(t)+\varepsilon_c(t)]h_c(t) + [\mu_l(t)+\varepsilon_l(t)]h_l(t) + [\mu_a(t)+\varepsilon_a(t)] [e^{r(T-\tau)}\Delta a(\tau)+\int_{\tau}^{T} e^{r(T-x)}[F_l(x)h_l(x)-h_c(x)]dx\}dt. \tag{3}$$

Dividing (3) through by $\Delta a(\tau)$,

$$\frac{\Delta U}{\Delta a(\tau)} = \int_{\tau}^{T} \{[\mu_c(t)+\varepsilon_c(t)]\, \frac{h_c(t)}{\Delta a(\tau)} + [\mu_1(t)+\varepsilon_1(t)]\, \frac{h_c(t)}{\Delta a(\tau)} \tag{4}$$

$$+ [\mu_a(t)+\varepsilon_a(t)][e^{r(T-\tau)} + \int_{\tau}^{T} e^{r(T-x)}[F_1(x)\ h_1(x)/\Delta a(\tau)$$

$$- h_c(x)/\Delta a(\tau)]dx\}dt,$$

which can be written as

$$\frac{\Delta U}{\Delta a(\tau)} = \frac{1}{\Delta a(\tau)}\ \{\int_{\tau}^{T} \Big([\mu_c(t)+\varepsilon_c(t)]h_c(t) - [\mu_a(t)+\varepsilon_a(t)] \int_{\tau}^{T} e^{r(T-x)} h_c(x)dx \tag{5}$$

$$+ [\mu_1(t)+\varepsilon_1(t)]h_1(t) + [\mu_a(t)+\varepsilon_a(t)] \int_{\tau}^{T} e^{r(T-x)} F_1(x)h_1(x)dx\Big)dt\}$$

$$+ \int_{\tau}^{T} [\mu_a(t)+\varepsilon_a(t)]e^{r(T-\tau)}dt.$$

Writing (5) completely as a sum of integrals,

$$\frac{\Delta U}{\Delta a(\tau)} = \frac{1}{\Delta a(\tau)}\ \{\int_{\tau}^{T} [\mu_c(t)+\varepsilon_c(t)]h_c(t)dt + \int_{\tau}^{T} [\mu_1(t)+\varepsilon_1(t)]h_1(t)dt \tag{6}$$

$$- \int_{\tau}^{T} [\mu_a(t)+\varepsilon_a(t)] \int_{\tau}^{T} e^{r(T-x)} h_c(x)dxdt$$

$$+ \int_{\tau}^{T} [\mu_a(t)+\varepsilon_a(t)] \int_{\tau}^{T} e^{r(T-x)} F_1(x)h_1(x)dxdt\} + \int_{\tau}^{T} [\mu_a(t)$$

$$+\varepsilon_a(t)]e^{r(T-\tau)}dt.$$

Upon changing the variables of integration of the double integrals in (6),

$$\frac{\Delta U}{\Delta a(\tau)} = \frac{1}{\Delta a(\tau)}\ \{\int_{\tau}^{T} [\mu_c(t)+\varepsilon_c(t)]h_c(t)dt \tag{7}$$

$$- \int_{\tau}^{T} \left[e^{r(T-t)} \int_{\tau}^{T} [\mu_a(x)+\varepsilon_a(x)]dx\right] h_c(t)dt + \int_{\tau}^{T} [\mu_1(t)+\varepsilon_1(t)]h_1(t)dt$$

$$+ \int_{\tau}^{T} \left[e^{r(T-t)} F_1(t) \int_{\tau}^{T} [\mu_a(x)+\varepsilon_a(x)]dx\right] h_1(t)dt\} +$$

$$+\ e^{r(T-\tau)}\int_{\tau}^{T}[\mu_a(t)+\varepsilon_a(t)]dt.$$

Taking the limit as $\Delta a(\tau) \rightarrow 0$,

$$\lim_{\Delta a(\tau)\rightarrow 0}\frac{\Delta U}{\Delta a(\tau)} = \lim_{\Delta a(\tau)\rightarrow 0}\frac{1}{\Delta a(\tau)}\int_{\tau}^{T}[\mu_c(t)-e^{r(T-t)}\int_{\tau}^{T}\mu_a(x)dx+\varepsilon_1(t)] \tag{8}$$

$$h_c(t)dt + \lim_{\Delta a(\tau)\rightarrow 0}\frac{1}{\Delta a(\tau)}\int_{\tau}^{T}[\mu_1(t)+e^{r(T-t)}F_1(t)\int_{\tau}^{T}\mu_a(x)dx+\varepsilon_2(t)]$$

$$h_1(t)dt + \lim_{\Delta a(\tau)\rightarrow 0} e^{r(T-\tau)}\int_{\tau}^{T}[\mu_a(t)+\varepsilon_a(t)]dt.$$

As $\Delta a(\tau) \rightarrow 0$, $h_c(t) \rightarrow 0$, $h_1(t) \rightarrow 0$, and $\varepsilon_a(t) \rightarrow 0$. As $h_c(t) \rightarrow 0$, $\varepsilon_1(t) \rightarrow 0$, and as $h_1(t) \rightarrow 0$, $\varepsilon_2(t) \rightarrow 0$. Assuming nonnegativity constraints are nowhere binding, the following first-order conditions hold (equations (17) in the second chapter):

$$\mu_c(t) - e^{r(T-t)}\int_{\tau}^{T}\mu_a(x)dx = 0, \qquad \text{for all } t \text{ in } [\tau,T];$$

$$\mu_1(t) + e^{r(T-t)}F_1(t)\int_{\tau}^{T}\mu_a(x)dx = 0, \qquad \text{for all } t \text{ in } [\tau,T]; \tag{9}$$

$$\int_{\tau}^{T}\mu_a(x)dx = -\lambda.$$

Therefore,

$$\lim_{\Delta a(\tau)\rightarrow 0}\frac{\Delta U}{\Delta a(\tau)} \equiv \frac{\delta U}{\delta a(\tau)} = -e^{r(T-\tau)}\lambda, \text{ or} \tag{10}$$

$$-\lambda = e^{r(\tau-T)}\frac{\delta U}{\delta a(\tau)}. \tag{11}$$

Here, $e^{r(\tau-T)}$ is the price of the bequest in terms of present assets, and $\frac{\delta U}{\delta a(\tau)}$ is the present marginal utility of present assets.

B. Solving the Certainty Model Example

$$\max_{c,1,a(T)} \quad U^{*} = \int_{\tau}^{T}\left(e^{j(\tau-t)}\{[\alpha_1+\alpha_2\left(\frac{t-\tau}{T-\tau}\right)]\ln c(t) + [\beta_1+\beta_2\left(\frac{t-\tau}{T-\tau}\right)]\ln l(t)\right.$$

$$+ [\gamma_1+\gamma_2\left(\frac{t-\tau}{T-\tau}\right)]\ln a(T)\} + \lambda\{e^{r(T-t)}\Big(c(t)-[\omega_0 e^{k(t-\tau)}]$$

$$[1-l(t)]\Big) + \left(\frac{1}{T-\tau}\right)a(T)\}\Bigg)dt.$$

$$\delta U^* = \int_\tau^T \Bigg(e^{j(\tau-t)}\{[\alpha_1+\alpha_2\left(\frac{t-\tau}{T-\tau}\right)]c(t)^{-1}h_c(t) + [\beta_1+\beta_2\left(\frac{t-\tau}{T-\tau}\right)]l(t)^{-1}h_l(t)$$

$$+ [\gamma_1+\gamma_2\left(\frac{t-\tau}{T-\tau}\right)]a(T)^{-1}\delta a(T)\} + \lambda\{e^{r(T-t)}\Big(h_c(t)-[\omega_0 e^{k(t-\tau)}]$$

$$[-h_l(t)]\Big) + \left(\frac{1}{T-\tau}\right)\delta a(T)\}\Bigg)dt = 0.$$

Since $\delta U^* = 0$, for any arbitrary $h_c(t)$, $h_l(t)$, and $\delta a(T)$, not all zero, one must have

$$\text{(a)} \quad \int_\tau^T \{e^{j(\tau-t)}[\alpha_1+\alpha_2\left(\frac{t-\tau}{T-\tau}\right)]c(t)^{-1} + \lambda e^{r(T-t)}\}h_c(t)dt = 0,$$

$$\text{(b)} \quad \int_\tau^T \{e^{j(\tau-t)}[\beta_1+\beta_2\left(\frac{t-\tau}{T-\tau}\right)]l(t)^{-1} + \lambda e^{r(T-t)}[\omega_0 e^{k(t-\tau)}]\}h_l(t)dt = 0,$$

$$\text{(c)} \quad \int_\tau^T \{e^{j(\tau-t)}[\gamma_1+\gamma_2\left(\frac{t-\tau}{T-\tau}\right)]a(T)^{-1} + \lambda\left(\frac{1}{T-\tau}\right)\}\delta a(T)dt = 0.$$

For (a) and (b) to hold for any $h_c(t)$ and $h_l(t)$ respectively, one must have (d) and (e) below, while (c) can be written as (f) below:

$$\text{(d)} \quad e^{j(\tau-t)}[\alpha_1+\alpha_2\left(\frac{t-\tau}{T-\tau}\right)]c(t)^{-1} = -\lambda e^{r(T-t)} \qquad \text{, for all } t \text{ in } [\tau,T];$$

$$\text{(e)} \quad e^{j(\tau-t)}[\beta_1+\beta_2\,\frac{t-\tau}{T-\tau}\,]l(t)^{-1} = -\lambda e^{r(T-t)}[\omega_0 e^{k(t-\tau)}],$$

for all t in $[\tau,T]$;

$$\text{(f)} \quad \int_\tau^T e^{j(\tau-x)}[\gamma_1+\gamma_2\left(\frac{x-\tau}{T-\tau}\right)]a(T)^{-1}dx = -\lambda.$$

Working with (f), one has

$$\text{(g)} \quad \int_\tau^T [e^{j(\tau-x)}\left(\gamma_1 - \frac{\gamma_2\tau}{T-\tau}\right)+e^{j(\tau-x)}\frac{\gamma_2 x}{T-\tau}]dx = \left(\gamma_1 - \frac{\gamma_2\tau}{T-\tau}\right)\int_\tau^T e^{j(\tau-x)}dx$$

$$+ \frac{\gamma_2}{T-\tau}\int_\tau^T e^{j(\tau-x)}x\,dx = -\lambda a(T).$$

Performing the integrations in (g),

$$\text{(h)}\quad \left(\gamma_1 - \frac{\gamma_2\tau}{T-\tau}\right)\left[\frac{1}{j}\,(1-e^{j(\tau-T)})\right] + \frac{\gamma_2}{T-\tau}\left(\frac{1}{j}\right)\left[\tau-(T+1/j)e^{j(\tau-T)}+\frac{1}{j}\right] = -\lambda a(T).$$

Simplifying,

$$\text{(i)}\quad \left[\frac{1}{j^2(T-\tau)}\right]\{j(T-\tau)[\gamma_1-(\gamma_1+\gamma_2)e^{j(\tau-T)}] + \gamma_2(1-e^{j(\tau-T)})\} = -\lambda a(T).$$

Let

$$\text{(j)}\quad A = j(T-\tau)[\gamma_1-(\gamma_1+\gamma_2)e^{j(\tau-T)}] + \gamma_2(1-e^{j(\tau-T)}).$$

Then

$$\text{(k)}\quad \left[\frac{1}{j^2(T-\tau)}\right]Aa(T)^{-1} = -\lambda.$$

Substituting for $-\lambda$ in equations (d) and (e), and solving for $c(t)$ and $l(t)$,

$$\text{(l)}\quad \hat{c}(t) = \frac{j^2[\alpha_1(T-\tau)+\alpha_2(t-\tau)]e^{j\tau-rT+t(r-j)}}{A}\,\hat{a}(T), \qquad \tau \leqq t \leqq T;$$

$$\text{(m)}\quad \hat{l}(t) = \frac{j^2[\beta_1(T-\tau)+\beta_2(t-\tau)]e^{(j+k)\tau-rT+t(r-j-k)}}{\omega_0 A}\,\hat{a}(T), \qquad \tau \leqq t \leqq T.$$

Substituting for $\hat{c}(t)$ and $\hat{l}(t)$ in the constraint

$$\hat{a}(T) = e^{r(T-\tau)}a(\tau) + \int_\tau^T e^{r(T-t)}\{[\omega_0 e^{k(t-\tau)}][1-\hat{l}(t)]-\hat{c}(t)\}dt,$$

one obtains

$$\text{(n)}\quad \hat{a}(T) = e^{r(T-\tau)}a(\tau) + \int_\tau^T \omega_0 e^{rT-k\tau+t(k-r)}dt$$

$$- \frac{\hat{a}(T)}{A}\int_\tau^T j^2[\beta_1(T-\tau)+\beta_2(t-\tau)]e^{j(\tau-t)}dt$$

$$- \frac{\hat{a}(T)}{A} \int_{\tau}^{T} j^2[\alpha_1(T-\tau)+\alpha_2(t-\tau)]e^{j(\tau-t)}dt.$$

This leads to

$$\text{(o)} \quad \hat{a}(T) + \frac{j^2\hat{a}(T)}{A} \{[\beta_1(T-\tau)-\beta_2\tau+\alpha_1(T-\tau)-\alpha_2\tau] \int_{\tau}^{T} e^{j(\tau-t)}dt$$

$$+ (\beta_2+\alpha_2) \int_{\tau}^{T} e^{j(\tau-t)}tdt\} = e^{r(T-\tau)}a(\tau) + \omega_0 \int_{\tau}^{T} e^{rT-k\tau+t(k-r)}dt.$$

Integrating,

$$\text{(p)} \quad \hat{a}(T) + \frac{j^2\hat{a}(T)}{A} \{[(\beta_1+\alpha_1)(T-\tau)-(\beta_2+\alpha_2)\tau][\frac{1}{j} - \frac{1}{j} e^{j(\tau-T)}] + (\beta_2+\alpha_2)$$

$$[\left(\frac{1}{j^2}\right)(-e^{j(\tau-T)}+1+j\tau-jTe^{j(\tau-T)})]\} = e^{r(T-\tau)}a(\tau)$$

$$+ \frac{\omega_0}{k-r}(e^{k(T-\tau)}-e^{r(T-\tau)}).$$

Simplifying,

$$\text{(q)} \quad \hat{a}(T)\{1 + \frac{1}{A}\Big(j(T-\tau)[(\alpha_1+\beta_1)-(\alpha_1+\alpha_2+\beta_1+\beta_2)e^{j(\tau-T)}] + (\alpha_2+\beta_2)$$

$$(1-e^{j(\tau-T)})\Big)\} = e^{r(T-\tau)}a(\tau) + \frac{\omega_0}{k-r}(e^{k(T-\tau)}-e^{r(T-\tau)}).$$

Terminal assets, therefore, are

$$\text{(r)} \quad \hat{a}(T) =$$

$$\frac{e^{r(T-\tau)}a(\tau) + \frac{\omega_0}{k-r}(e^{k(T-\tau)}-e^{r(T-\tau)})}{1 + \frac{1}{A}\{j(T-\tau)[(\alpha_1+\beta_1)-(\alpha_1+\alpha_2+\beta_1+\beta_2)e^{j(\tau-T)}] + (\alpha_2+\beta_2)(1-e^{j(\tau-T)})\}}$$

Equations (l), (m), and (r) give the optimal path of consumption, the optimal path of leisure, and the optimal bequest.

Note: In the calculations above, it is assumed that (i) $j \neq 0$, $r \neq 0$, $k \neq 0$; (ii) $r \neq j$, $r \neq k$, $r \neq j + k$.

C. Determining (Annual) Consumption and Leisure in the Certainty Model

Example

$$\int_{t_0}^{t_0+1} c(t)dt = \int_{t_0}^{t_0+1} \left(\frac{1}{A}\right) j^2[\alpha_1(T-\tau)+\alpha_2(t-\tau)]e^{j\tau-rT+t(r-j)}a(T)dt$$

$$= \frac{1}{A} j^2[\alpha_1(T-\tau)-\alpha_2\tau]a(T) \int_{t_0}^{t_0+1} e^{j\tau-rT+t(r-j)}dt$$

$$+ \left(\frac{1}{A}\right) j^2\alpha_2 a(T) \int_{t_0}^{t_0+1} te^{j\tau-rT+t(r-j)}dt.$$

Upon carrying out the integration, one obtains

$$\int_{t_0}^{t_0+1} c(t)dt = \left(\frac{1}{A}\right) j^2[\alpha_1(T-\tau)-\alpha_2\tau]a(T)[\left(\frac{1}{r-j}\right)(e^{r-j}-1)e^{j\tau-rT+t_0(r-j)}]$$

$$+ \left(\frac{1}{A}\right) j^2\alpha_2 a(T)\{\left(\frac{1}{r-j}\right)[e^{j\tau-rT+t_0(r-j)}(e^{r-j}-1)(t_0 - \frac{1}{r-j})$$

$$+ e^{j\tau-rT+(t_0+1)(r-j)}]\}$$

$$= \left(\frac{1}{A}\right)\left(\frac{1}{r-j}\right)^2 j^2 e^{j\tau-rT+t_0(r-j)}a(T)\{(r-j)(e^{r-j}-1)$$

$$[\alpha_1(T-\tau)-\alpha_2\tau] + \alpha_2(e^{r-j}-1)[t_0(r-j)-1] + \alpha_2(r-j)e^{r-j}\}$$

$$= \left(\frac{1}{A}\right)\left(\frac{j}{r-j}\right)^{2\tau j\tau-rT+t_0(r-j)} e \quad a(T)\{(r-j)(e^{r-j}-1)$$

$$[\alpha_1(T-\tau)-\alpha_2\tau+\alpha_2 t_0] + \alpha_2[(r-j)e^{r-j}-(e^{r-j}-1)]\}$$

$$= \left(\frac{1}{A}\right)\left(\frac{j}{r-j}\right)^2 e^{j\tau - rT + t_0(r-j)} a(T)\{(r-j)(e^{r-j}-1)$$

$$[\alpha_1(T-\tau)-\alpha_2(\tau-t_0) - \frac{\alpha_2}{r-j}] + \alpha_2(r-j)e^{r-j}\}.$$

Therefore,

$$\int_{t_0}^{t_0+1} c(t)dt = \left(\frac{j^2}{r-j}\right)e^{j\tau - rT + t_0(r-j)}\left(\frac{C}{A}\right)a(T),$$

where $C = (e^{r-j}-1)[\alpha_1(T-\tau)-\alpha_2(\tau-t_0) - \frac{\alpha_2}{r-j}] + \alpha_2 e^{r-j}$.

$$\int_{t_0}^{t_0+1} l(t)dt = \int_{t_0}^{t_0+1} \left(\frac{1}{\omega_0 A}\right)j^2[\beta_1(T-\tau)+$$

$$+\beta_2(t-\tau)]e^{(j+k)\tau - rT + t(r-j-k)}a(T)dt$$

$$= \left(\frac{1}{\omega_0 A}\right)j^2[\beta_1(T-\tau)-\beta_2\tau]e^{(j+k)\tau - rT}a(T) \int_{t_0}^{t_0+1} e^{t(r-j-k)}dt$$

$$+ \left(\frac{1}{\omega_0 A}\right)j^2\beta_2 e^{(j+k)\tau - rT}a(T) \int_{t_0}^{t_0+1} te^{t(r-j-k)}dt$$

$$= \left(\frac{1}{\omega_0 A}\right)j^2[\beta_1(T-\tau)-\beta_2\tau]e^{(j+k)\tau - rT}a(T)$$

$$\{\left(\frac{1}{r-j-k}\right)e^{t_0(r-j-k)}(e^{r-j-k}-1)\} + \left(\frac{1}{\omega_0 A}\right)j^2[\beta_2 e^{(j+k)\tau - rT}]a(T)$$

$$\{\left(\frac{1}{r-j-k}\right)\left[e^{t_0(r-j-k)}[e^{r-j-k}(t_0+1 - \frac{1}{r-j-k})\right.$$

$$\left. - (t_0 - \frac{1}{r-j-k})]\right]\}$$

$$= \left(\frac{1}{\omega_0 A}\right)\left(\frac{1}{r-j-k}\right) j^2 e^{(j+k)\tau - rT + t_0(r-j-k)} a(T)$$

$$\{(e^{(r-j-k)}-1)[\beta_1(T-\tau)-\beta_2\tau]$$

$$+ \beta_2(t_0 - \frac{1}{r-j-k})(e^{r-j-k}-1) + \beta_2 e^{r-j-k}\}$$

$$= \left(\frac{1}{\omega_0 A}\right)\left(\frac{1}{r-j-k}\right) j^2 e^{(j+k)\tau - rT + t_0(r-j-k)} a(T)\{(e^{r-j-k}-1)[\beta_1(T-\tau)$$

$$- \beta_2\tau + \beta_2(t_0 - \frac{1}{r-j-k})] + \beta_2 e^{r-j-k}\}.$$

Therefore,

$$\int_{t_0}^{t_0+1} l(t)dt = \left[\frac{j^2}{\omega_0(r-j-k)}\right] e^{(j+k)\tau - rT + t_0(r-j-k)} \left(\frac{D}{A}\right) a(T),$$

where

$$D = (e^{r-j-k}-1)[\beta_1(T-\tau)-\beta_2(\tau-t_0) - \frac{\beta_2}{r-j-k}] + \beta_2 e^{r-j-k}.$$

D. Solution of the Planning Horizon Example

$$U^* = \int_0^{\rho} [ae^{-jt}c(t)^{\alpha}+\lambda e^{-rt}c(t)]dt + be^{-j\rho}a(\rho)^{\beta}+\lambda[e^{-r\rho}a(\rho)-K].$$

$$\delta U^* = \int_0^{\rho} [ae^{-jt}\alpha c(t)^{\alpha-1}+\lambda e^{-rt}]h_c(t)dt + be^{-j\rho}\beta a(\rho)^{\beta-1}$$

$$[\delta a(\rho)+a'(\rho)\delta\rho] + \lambda e^{-r\rho}[\delta a(\rho)+a'(\rho)\delta\rho]$$

$$+ [ae^{-j\rho}c(\rho)^{\alpha}+\lambda e^{-r\rho}c(\rho)]\delta\rho - jbe^{-j\rho}a(\rho)^{\beta}\delta\rho$$

$$- r\lambda e^{-r\rho}a(\rho)\delta\rho = 0.$$

$$\delta U^* = \int_0^\rho [ae^{-jt}\alpha c(t)^{\alpha-1}+\lambda e^{-rt}]h_c(t)dt + [b\beta e^{-j\rho}a(\rho)^{\beta-1}$$

$$+ \lambda e^{-r\rho}]\bar{\delta}a(\rho) + [ae^{-j\rho}c(\rho)^{\alpha}-jbe^{-j\rho}a(\rho)^{\beta}+\lambda e^{-r\rho}\left(c(\rho)-ra(\rho)\right)]\delta\rho$$

$$= 0.$$

For arbitrary $h_c(t)$, $\bar{\delta}a(\rho)$, and $\delta\rho$,

$$\int_0^\rho [ae^{-jt}\alpha c(t)^{\alpha-1}+\lambda e^{-rt}]h_c(t)dt = 0,$$

$$[b\beta e^{-j\rho}a(\rho)^{\beta-1}+\lambda e^{-r\rho}]\bar{\delta}a(\rho) = 0,$$

$$[e^{-j\rho}\left(ac(\rho)^{\alpha}-jba(\rho)^{\beta}\right)+\lambda e^{-r\rho}\left(c(\rho)-ra(\rho)\right)]\delta\rho = 0.$$

Thus, these first-order conditions result:

(a) $a\alpha e^{(r-j)t}c(t)^{\alpha-1} = -\lambda$, for all t in $[0,\rho]$;

(b) $b\beta e^{(r-j)\rho}a(\rho)^{\beta-1} = -\lambda$;

(c) $e^{-j\rho}[ac(\rho)^{\alpha}-jba(\rho)^{\beta}]+\lambda e^{-r\rho}[c(\rho)-ra(\rho)] = 0.$

Setting $t = \rho$ in (a), (a), (b), and (c) can be used to solve for $c(\rho)$ in terms of $a(\rho)$:

$$a\alpha e^{(r-j)\rho}c(\rho)^{\alpha-1} = b\beta e^{(r-j)\rho}a(\rho)^{\beta-1},$$

$$ac(\rho)^{\alpha} = b\left(\frac{\beta}{\alpha}\right)a(\rho)^{\beta-1}c(\rho);$$

$$e^{-j\rho}[b\left(\frac{\beta}{\alpha}\right)a(\rho)^{\beta-1}c(\rho)-jba(\rho)^{\beta}] - b\beta e^{(r-j)\rho}a(\rho)^{\beta-1}e^{-r\rho}$$

$$[c(\rho)-ra(\rho)] = 0,$$

$$be^{-j\rho}a(\rho)^{\beta-1}\left[\left(\frac{\beta}{\alpha}\right)c(\rho)-ja(\rho)\right] - be^{-j\rho}a(\rho)^{\beta-1}[\beta c(\rho)-r\beta a(\rho)] = 0,$$

$$\left(\frac{\beta}{\alpha}\right)c(\rho) - \beta c(\rho) - ja(\rho) + r\beta a(\rho) = 0,$$

(d) $$c(\rho) = \left(\frac{\alpha}{\beta}\right)\left(\frac{j-r\beta}{1-\alpha}\right)a(\rho).$$

Using (a) and (b) again with $t = \rho$ in (a), and substituting for $c(\rho)$ from (d),

$$a\alpha\left(\frac{\alpha}{\beta}\right)^{\alpha-1}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}a(\rho)^{\alpha-1} = b\beta a(\rho)^{\beta-1},$$

$$\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1} = a(\rho)^{\beta-\alpha}, \quad \text{and}$$

(e) $$\hat{a}(\hat{\rho}) = \left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{1}{\beta-\alpha}}.$$

One can now solve for $\hat{c}(t)$ by using (a) and (b) and substituting for $\hat{a}(\hat{\rho})$:

$$a\alpha e^{(r-j)t}\hat{c}(t)^{\alpha-1} = b\beta e^{(r-j)\hat{\rho}}\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{\beta-1}{\beta-\alpha}},$$

$$\hat{c}(t)^{\alpha-1} = \left(\frac{b\beta}{a\alpha}\right)e^{(r-j)(\hat{\rho}-t)}\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{\beta-1}{\beta-\alpha}},$$

(f) $$\hat{c}(t) = \left(\frac{b\beta}{a\alpha}\right)^{\frac{1}{\alpha-1}} e^{(r-j)(\hat{\rho}-t)\left(\frac{1}{\alpha-1}\right)}\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{\beta-1}{(\beta-\alpha)(\alpha-1)}},$$

for $0 \leq t \leq \hat{\rho}$.

Substituting from (e) and (f) into the constraint, one has

$$K = e^{-r\hat{\rho}}\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{1}{\beta-\alpha}} + \int_0^{\hat{\rho}} e^{-rt}\left(\frac{b\beta}{a\alpha}\right)^{\frac{1}{\alpha-1}} e^{(r-j)(\hat{\rho}-t)\left(\frac{1}{\alpha-1}\right)}$$

$$\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{\beta-1}{(\beta-\alpha)(\alpha-1)}} dt.$$

The above integral equals

$$\left(\frac{b\beta}{a\alpha}\right)^{\frac{1}{\alpha-1}}\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{\beta-1}{(\beta-\alpha)(\alpha-1)}} e^{\left(\frac{r-j}{\alpha-1}\right)\hat{\rho}} \int_0^{\hat{\rho}} e^{\left(\frac{j-\alpha r}{\alpha-1}\right)t} dt =$$

$$\left(\frac{b\beta}{a\alpha}\right)^{\frac{1}{\alpha-1}}\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{\beta-1}{(\beta-\alpha)(\alpha-1)}} e^{\left(\frac{r-j}{\alpha-1}\right)\hat{\rho}}\left(\frac{\alpha-1}{j-\alpha r}\right)\left[e^{\left(\frac{j-\alpha r}{\alpha-1}\right)\hat{\rho}}-1\right].$$

Therefore,

$$\text{(g)}\quad K = e^{-r\hat{\rho}}\left\{\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\left(\frac{1}{\beta-\alpha}\right)} + \left(\frac{\alpha-1}{j-\alpha r}\right)\left(\frac{b\beta}{a\alpha}\right)^{\left(\frac{1}{\alpha-1}\right)}\right.$$

$$\left.\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{\beta-1}{(\beta-\alpha)(\alpha-1)}}\right\} - e^{\left(\frac{r-j}{\alpha-1}\right)\hat{\rho}}\left(\frac{\alpha-1}{j-\alpha r}\right)\left(\frac{b\beta}{a\alpha}\right)^{\left(\frac{1}{\alpha-1}\right)}$$

$$\left[\left(\frac{a}{b}\right)\left(\frac{\alpha}{\beta}\right)^{\alpha}\left(\frac{j-r\beta}{1-\alpha}\right)^{\alpha-1}\right]^{\frac{\beta-1}{(\beta-\alpha)(\alpha-1)}}.^{*}$$

For $a = 12$, $b = 3$, $\alpha = \frac{1}{2}$, $\beta = \frac{2}{3}$, $r = .06$, $j = .30$, and $K =$ 1,000,000:

---

*From equations (e) and (f), it can be shown that $K =$ $e^{-r\rho}\{a(\rho) + \left(\frac{\alpha-1}{j-\alpha r}\right)c(\rho)\} - e^{\left(\frac{r-j}{\alpha-1}\right)\rho}\{\left(\frac{\alpha-1}{j-\alpha r}\right)c(\rho)\}$.

$$1{,}000{,}000 = e^{-.06\hat{\rho}}\{\left[(4)\left(\frac{3}{4}\right)^{1/2}(.52)^{-1/2}\right]^6$$

$$+ \left(-\frac{1}{.54}\right)\left(\frac{1}{3}\right)^{-2}\left[(4)\left(\frac{3}{4}\right)^{1/2}(.52)^{-1/2}\right]^4\}$$

$$- e^{.48\hat{\rho}}\left(-\frac{1}{.54}\right)\left(\frac{1}{3}\right)^{-2}\left[(4)\left(\frac{3}{4}\right)^{1/2}(.52)^{-1/2}\right]^4.$$

$$1{,}000{,}000 = e^{-.06\hat{\rho}}\{\left[(16)\left(\frac{3}{4}\right)\left(\frac{100}{52}\right)\right]^3 - \left(\frac{100}{54}\right)(9)\left[(16)\left(\frac{3}{4}\right)\left(\frac{100}{52}\right)\right]^2\}$$

$$+ e^{-48\hat{\rho}}\left(\frac{100}{54}\right)(9)\left[(16)\left(\frac{3}{4}\right)\left(\frac{100}{52}\right)\right]^2.$$

$$1{,}000{,}000 = e^{-.06\hat{\rho}}\{12{,}289-\left(\frac{100}{54}\right)4793\} + e^{.48\hat{\rho}}\left(\frac{100}{54}\right)(4793).$$

$$1{,}000{,}000 = e^{-.06\hat{\rho}}\{12{,}289-8876\} + e^{.48\hat{\rho}}(8876).$$

$\hat{\rho} \approx 10$. (For $\hat{\rho} = 10$, one obtains $1{,}000{,}000 \approx 1{,}080{,}396$.)

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Allen, R. G. D. *Mathematical Analysis for Economists.* New York, N.Y.: St. Martin's Press, 1962.

2. Arrow, K. J., and Kurz, M. *Public Investment, The Rate of Return, and Optimal Fiscal Policy.* Baltimore, Md.: The Johns Hopkins Press, 1970.

3. Brunk, H. D. *An Introduction to Mathematical Statistics.* Boston, Mass.: Ginn and Company, 1960.

4. Davis, O. A., and Whinston, A. B. "Welfare Economics and the Theory of Second Best," *Review of Economic Studies,* XXXII (January, 1965), pp. 1-14.

5. Duesenberry, James S. *Income, Saving, and the Theory of Consumer Behavior.* Cambridge, Mass.: Harvard University Press, 1949.

6. Fisher, Irving. *The Theory of Interest.* New York, N.Y.: Macmillan Co., 1930.

7. Freund, John E. *Mathematical Statistics.* Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1962.

8. Friedman, Milton. *A Theory of the Consumption Function.* Princeton, N.J.: Princeton University Press, 1957.

9. Gelfand, I. M., and Fomin, S. V. *Calculus of Variations.* Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1963.

10. Hadar, Josef. *Mathematical Theory of Economic Behavior.* Reading, Mass.: Addison-Wesley Publishing Company, 1971.

11. Hakansson, Nils H. "Optimal Investment and Consumption Strategies Under Risk for a Class of Utility Functions," *Econometrica,* Vol. 38 (September, 1970), pp. 587-607.

12. Henderson, James M., and Quandt, Richard E. *Microeconomic Theory: A Mathematical Approach.* New York, N.Y.: McGraw-Hill Book Company, 1971.

13. Hestenes, Magnus R. *Calculus of Variations and Optimal Control Theory.* New York, N.Y.: John Wiley and Sons, Inc., 1966.

14. Lindgren, B. W. *Statistical Theory.* New York, N.Y.: Macmillan Company, 1962.

15. Lipsey, R. G., and Lancaster, Kelvin. "The General Theory of Second Best," *Review of Economic Studies,* XXIV (January, 1957), pp. 11-32.

16. Modigliani, Franco, and Brumberg, Richard. "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data," *Post-Keynesian Economics.* Edited by K. K. Kurihara. New Brunswick, N.J.: Rutgers University Press, 1954.

17. Pesek, Boris P., and Saving, Thomas R. *Money, Wealth, and Economic Theory.* New York, N.Y.: Macmillan Company, 1967.

18. Ramsey, F. P. "A Mathematical Theory of Saving," *Economic Journal,* XXXVIII (December, 1928), pp. 543-559.

19. Smail, Lloyd L. *Calculus.* New York, N.Y.: Appleton-Century-Crofts, Inc., 1949.

20. Sokolnikoff, Ivan S. *Advanced Calculus.* New York, N.Y.: McGraw-Hill Book Company, 1939.

21. Strotz, R. H. "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies,* XXIII (October, 1956), pp. 165-180.

22. Tintner, Gerhard. "The Maximization of Utility Over Time," *Econometrica,* VI (April, 1938), pp. 154-158.

23. Widder, D. V. *Advanced Calculus.* Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1961.

24. Yaari, Menahem E. "On the Consumer's Lifetime Allocation Process," *International Economic Review,* Vol. 5 (September, 1964), pp. 304-317.

25. _______. "On the Existence of an Optimal Plan in a Continuous-Time Allocation Process," *Econometrica,* Vol. 32 (October, 1964), pp. 576-590.

26. _______. "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer," *Review of Economic Studies,* XXXII (April, 1965), pp. 137-150.