BIFURCATION AND COMPETITIVE EVOLUTION OF NETWORK MORPHOLOGIES IN THE STRONG FUNCTIONALIZED CAHN-HILLIARD EQUATION

By

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ABSTRACT

BIFURCATION AND COMPETITIVE EVOLUTION OF NETWORK MORPHOLOGIES IN THE STRONG FUNCTIONALIZED CAHN-HILLIARD EQUATION

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The Functionalized Cahn-Hilliard (FCH) energy is a higher-order free energy that has been proposed to describe phase separation in blends of amphiphilic polymers and solvent. It balances interfacial solvation energy of ionic groups and volumetric counter-ion and polymer chain self-interaction energy against elastic energy of the underlying polymer backbone. It is hoped that its gradient flows describe the formation of solvent accessible network structures, such as found in polymer electrolyte membranes, lipid membranes, and amphiphlic diblock copolymers. The FCH gradient flows possess long-lived network morphologies of distinct co-dimension and we characterize their geometric evolution, bifurcation and competition through a formal asymptotic reduction. This reduction encompasses a broad class of coexisting network morphologies with different inner structure, such as bilayers and pores. The stability of the different network morphologies is characterized by the meandering and pearling modes associated to the linearized system. For the H^{-1} gradient flow of the FCH energy, using functional analysis and asymptotic methods, we derive a sharp-interface geometric motion which couples the flow of co-dimension 1 and co-dimension 2 network morphologies, in $\mathbb{R}^3$, through the spatially constant far-field chemical potential. In particular, we rigorously characterize the pearling eigenvalues for a class of admissible co-dimension 1 and co-dimension 2 networks.

לאבי, שתמיד מדריך אותי בדרך הנכונה.
לבעלי, שנותן לי את הכח להגשים את חלומותיי.
לבני, האושר שלי.

To my father, who always guides me along the right path.
To my husband, who gives me the strength to follow my dreams.
To my son, my joy.

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KEY TO ABBREVIATIONS

Symbol	Meaning	Detailed in
CH	Cahn-Hilliard	Chapter 1
FCH	Functionalized Cahn-Hilliard	Chapter 1
$W^{k,p}(A)$	The subset of functions $f \in L_p(A)$ such that the function f and its weak derivatives up to some order k have a finite L_p norm, for given $1 \leq p \leq \infty$	Standard
$\mathbb{R}^n$	n dimensional Euclidean space	Standard
$C^k(\Omega)$	The set of functions having all derivatives of order $\leq k$ continuous in Ω	Standard
$C_0^k(\Omega)$	The set of functions in $C^k(\Omega)$ with compact support in Ω	Standard
$C^{k,\alpha}(\Omega)$	The set $C^{k,\alpha}(\Omega) \subset C^k(\Omega)$ consists of functions whose k-th order partial derivatives are locally hölder continuous with exponent α in Ω	Standard
$C_0^{k,\alpha}(\Omega)$	The set of functions in $C^{k,\alpha}(\Omega)$ with compact support in Ω	Standard
Δ	Cartesian representation of the Laplace operator	Standard
$\mathcal{E}$	Cahn-Hilliard free energy	Equation (1.3)
$\mathcal{F}$	FCH free energy	Equation (1.14)
Ω	$\Omega \subset \mathbb{R}^d$, fixed, bounded domain	Chapter 1
u	The volume fraction of one component of the binary mixture	Chapter 1
$W(u)$	Tilted double well potential	Chapter 1
ε	fixed, ration of amphiphilic molecule to domain size	Chapter 1
η_1	$\eta_1 > 0$, $\eta_1 \in \mathbb{R}$	Chapter 1
η_2	$\eta_2 \in \mathbb{R}$	Chapter 1
η_d	$\eta_d = \eta_1 - \eta_2$	Chapter 1
$\mathcal{N}$	Quasi-minimizer network morphology of $\mathcal{F}$	Chapter 1
$\mathcal{Q}_C$	Quasi-stationary network morphology of $\mathcal{F}$, for a fixed $C > 0$	Chapter 1
λ	The Euler-Lagrange multiplier	Equation (1.18)
$\hat{\lambda}$	$\hat{\lambda} = \lambda/\varepsilon$ the scaled Lagrange multiplier	Equation (1.20)
Γ_b	An admissible, co-dimension one manifold	Chapter 2
$\zeta_b(s,z)$	The mapping to the whiskered coordinates for co-dimension one	Equation (2.1)
$\Gamma_{b,\ell}$	The reach of Γ_b	Equation (1.23)
H	The extended curvatures of co-dimension one manifold	Equation (2.10)

J_b	The Jacobian of the change of variables in co-dimension one manifold	Equation (2.6)
H_0	Mean curvature of Γ_b	Standard
Δ_s	The Laplace-Beltrami operator	Equation (2.13)
Δ_G	The extension of the Laplace-Beltrami operator	Equation (2.12)
U_b	The bilayer profile	Equation (2.37)
u_b	The bilayer dressing of Γ_b with U_b	Chapter 1
$u_{b,1}$	The $O(\varepsilon)$ correction to u_b	Equation (1.32)
$L_{b,0}$	The linearization of (1.27) about U_b	Equation (2.39)
μ_1	The far-field chemical potential	Equation (1.33)
α_-	The far-field well coercivity, $\alpha_- = W''(b_-)$	Equation (1.34)
$\tilde{\Gamma}_{b,\ell}$	$\tilde{\Gamma}_{b,\ell} = \Omega \backslash \Gamma_{b,\ell}$	Chapter 1
σ_b	The bilayer 'surface tension'	Equation (1.46)
m_b	The mass of amphiphilic material per unit length of bilayer	Equation (1.50)
μ_b^*	The optimal value of amphiphilic material in the bulk region	Equation (1.52)
Γ_p	An admissible, co-dimension two manifold	Chapter 2
ζ_p	The mapping to the whiskered coordinates for co-dimension two morphology in $\mathbb{R}^3$	Equation (2.58)
∂_G^2	An extension of the line diffusion operator	Equation (2.67)
U_p	The pore profile	Equation (2.75)
u_p	The pore dressing of Γ_p with U_p	Chapter 1
$u_{p,1}$	The $O(\varepsilon)$ correction to u_p	Chapter 1
b	The far-field constant value	Chapter 1
U_m	The rotational symmetric solution for the co-dimension three ODE	Equation (1.41)
u_c	A critical point of CH free energy	Chapter 1
$\mathbb{L}_b$	The linearization of $\mathcal{F}$ about U_b	Equation (1.54)
$\Psi_{b,0,n}$	The pearling eigenfunctions of $\mathbb{L}_b$	Chapter 2
$\Lambda_{b,0,n}$	The pearling eigenvalues of $\mathbb{L}_b$	Chapter 2
$\Psi_{b,1,n}$	The meander eigenfunctions of $\mathbb{L}_b$	Chapter 2
$\psi_{b,j}$	The eigenfunctions of L_0^b	Chapter 2
$\lambda_{b,j}$	The eigenvalues of L_0^b	Chapter 2
S_b	The shape factor of the bilayer	Equation (5.41)
P_b^*	The bilayer pearling condition	Equation (1.59)

$\Phi_{b,j}$	satisfy $L_{b,0}^{j}\Phi_{b,j} = 1$ for $j = 1, 2$,	Equation (2.40)
$\Phi_{p,j}$	satisfy $L_{p,0}^{j}\Phi_{p,j} = 1$ for $j = 1, 2$,	Equation (2.86)

Chapter 1

Introduction

1.1 Amphiphilic Materials

Traditionally the term 'amphiphilic molecule' denotes a small molecule which finds an energetically favorable interaction at the interface of two disparate fluids, such as soap in an oil-water-soap mixture. Indeed, early studies of amphiphilic materials concerned emulsions formed from two immiscible fluids combined with an amphiphilic surfactant. Lipids, formed of a hydrophilic head group and a hydrophobic tail belong to this class of amphiphilic molecules. More recently, developments in synthetic chemistry, such as atom transfer radical polymerization, have simplified the process of attaching charge groups to polymers, greatly expanding the possible classes of amphiphilic polymers that can readily be synthesized, see [Matyjaszewski, 2012] and [Charleux et al., 2012]. Amphiphilic blends typically phase separate; however the propensity of the amphiphilic molecules to form monolayers leads to an energetic preference for thin interfaces. As a result, amphiphilic mixtures typically form network morphologies which support asymptotically large interfaces of various co-dimension. These include co-dimension one bilayers, or co-dimension two pore structures. To make the idea of a network more precise, we offer the following motivation:

Given a small parameter $0 < \varepsilon_0 \ll 1$, we say that a family of closed subdomains $\{D_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0} \subset \mathbb{R}^d,\ d \geq 2$, is a network morphology if the sets are nested, that is $D_{\varepsilon_1} \subset D_{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2 \leq \varepsilon_0$, and each constituent point of D_ε lies within $O(\varepsilon)$ of its compliment. If D_0 is the intersection of D_ε, then the (local) co-dimension of the network is the difference between the dimension, d, of the ambient space and the (local) Hausdorff dimension of D_0.

Intuitively, D_0 is specified and the sets D_ε can be thought of as the points that lie within ε of D_0, – where ε plays the role of the molecular width. Networks have significant value: they describe the arrangements

of amphiphilic molecules, which self assemble into nano-scale structures with huge densities of solvent-accessible surface area. The resulting network morphologies are typically charge-lined, rendering them efficient charge-selective ionic conductors. Due to these traits, amphiphilic materials have found use in many types of energy conversion devices, forming the ionomer membranes in fuel cells, the photo-active collecting matrix in bulk-heterojunction solar cells, and the separator membrane in Lithium ion batteries, [Anderson, 1975, Wilson and Gottesfeld, 1992, Peet et al., 2009].

The casting of blends of amphiphilic mixtures and solvent presents a rich array of distinct morphologies, however control of the end-state morphology is experimentally challenging due to the delicate roles played by solvent type, salt concentration and counter-ion type, di-block composition and polydispersity, temperature, and pH. It has been shown, [Discher and Eisenberg, 2002], that changing the concentration of water in a water-dioxane solvent blend induces bifurcations in amphiphilic di-blocks yielding micelle, micelle-pore, pore (rod), pore-bilayers, and bilayer network morphologies, see Figure 1.1.

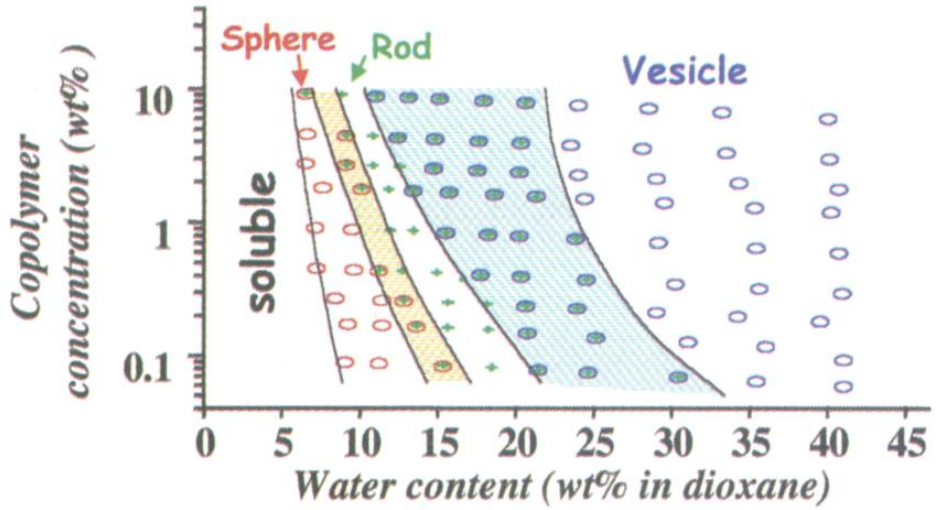


Figure 1.1: Morphological phases and vesicle transformations in dilute solutions. The colored regions between sphere and rod phases and between rod and vesicle phases correspond to coexistence regions, the vertical-axis represent the concentration of polymer by weight and the horizontal-axis is the percent of water. From [Discher and Eisenberg, 2002]. Reprinted with permission from AAAS.

Similar bifurcation were obtained in PEO-PB amphiphilic di-blocks by changing the density of charge groups in the hydrophilic portion, [Jain and Bates, 2004]. Figure 1.2 depicts the morphology diagram of PB-PEO diblock in water as a function of molecular size and composition. The axis, N_{PB} and W_{PEO}, denote the molecular weight of the PB portion of the chain and the weight fraction of the PEO portion, respectively. The four main structures observed are bilayer vesicles (B), cylinders (pores) (C), and spheres (S). As the hydrophilic content (W_{PEO}) is increased, a sequence of structural elements is documented: starting with bilayers, fol-

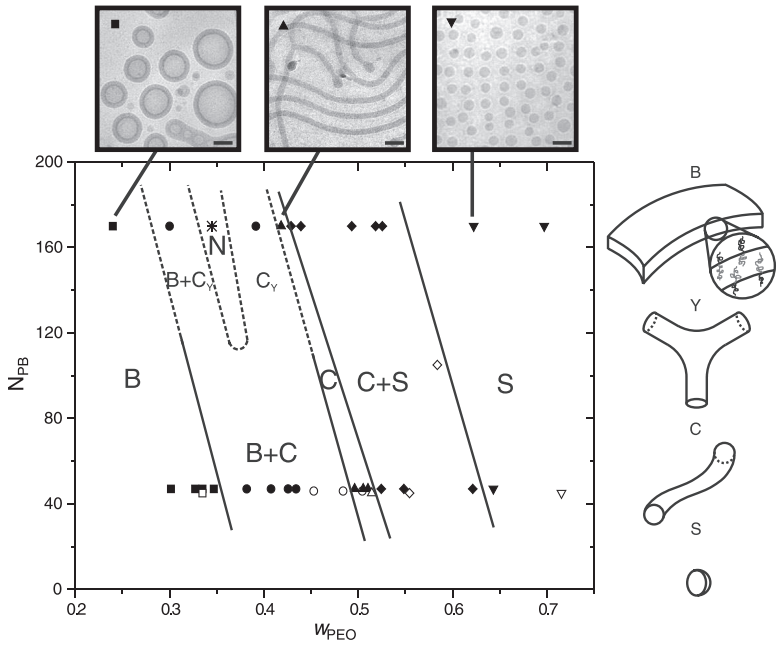


Figure 1.2: Morphology diagram of PB-PEO in water as a function of molecular size and composition describes the different regions of bilayers, pores and micelles. Reprinted (adapted) with permission from [Jain and Bates, 2004]. Copyright 2004 American Chemical Society.

lowed by pores, and then micelles separated by composition windows containing mixed morphologies, such as, bilayers-pores and pores-micelles. Increasing the hydrophilic weight fraction induces greater interfacial surface area, and increases the aspect ratio of the diblock, as PEO is a softer chain than PB, and forms more of a ball-like structure. Morphological reconfigurations can also be achieved through varying temperature, [Zare et al., 2012] and [Gomez et al., 2005], and concentrations of counter-ions [Zhulina and Borisov, 2012]. We pay particular attention to the experimental investigation in [Budin and Szostak, 2011] and [Zhu et al., 2012] addressing the division of primitive lipid membranes. Szostak's group derived a particularly simple method to induce the bilayer to micelle morphological change; they first formed a suspension of spherical vesicles of 10% phospholipid and found that increasing the concentration of free oleo-lipids dispersed in the bulk solvent induced a fingering instability in spherical phospholipid vesicles; this transformation is depicted in the three horizontally arranged panels on the left side of Figure 1.3, the end state of which consists of long, co-dimension two pore morphologies. In a subsequent experiment, the charge density on the surface of cylindrical pores was suddenly increased through a photo-oxidation process; the jump in charge density induces a pearling bifurcation causing the pore structures to break into individual micelles, as depict in the three vertically arranged panels on the right side of Figure 1.3. One aim of this thesis is to present an analysis of related bifurcations within the context of the Functionalized Cahn-Hilliard free energy, which we introduce in the following section.

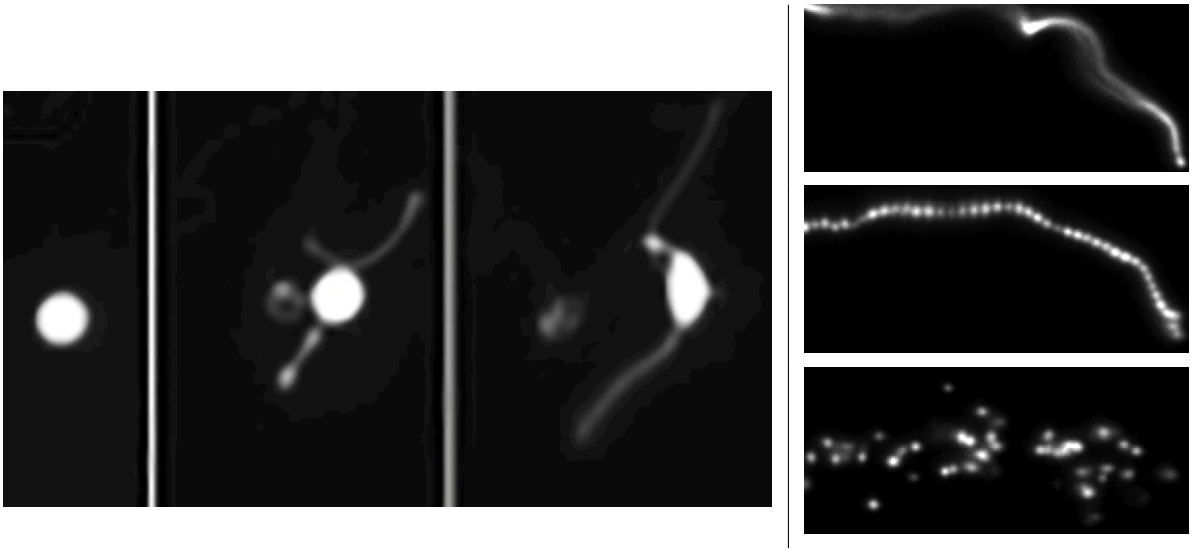


Figure 1.3: Szostak's mechanism for division of primitive cell membrane: (left) raising the background concentration of lipids induces the vesicle to grow worm-like (co-dimension two) protrusions over a 74 nano-second time period [Budin and Szostak, 2011], (right) changing the density of charged groups on the surface via a photochemically induced redox reaction incites the pore to pearl and break into micelles [Zhu et al., 2012]. Reprint permission granted by Proceedings of National Academy of Science.

1.2 The Functionalized Cahn-Hilliard Free Energy

The first step towards the introduction of the FCH is to recall the derivation of the Cahn-Hillard (CH) free energy, [Cahn and Hilliard, 1958], which describes the spinodal decomposition of an immiscible binary mixture. For a fixed domain, $\Omega \subset \mathbb{R}^3$, a phase function $u \in H^1(\Omega)$ describes the volume fraction of one component of the binary mixture, and the free energy is modeled by a function of the density u weakly perturbed by the spatially isotropic gradients

$$\mathcal{E}(u) = \int_\Omega f(u, \varepsilon^2|\nabla u|^2, \varepsilon^2 \Delta u)\, dx. \tag{1.1}$$

Expanding the free energy in orders of ε and keeping terms up to $O(\varepsilon^2)$, yields an expression of the form

$$\mathcal{E}(u) = \int_\Omega \left[f(u,0,0) + \varepsilon^2 A(u)|\nabla u|^2 + \varepsilon^2 B(u)\Delta u \right] dx. \tag{1.2}$$

To obtain a generic normal form for the free energy, Cahn and Hilliard integrated by parts the last term in (1.2), set the resulting coefficient of $|\nabla u|^2$ to $\frac{1}{2}$, and relabeled the potential $f(u,0,0)$ as $W(u)$. The result is the Cahn-Hilliard free energy

$$\mathcal{E}(u) = \int_\Omega \left[\frac{\varepsilon^2}{2}|\nabla u|^2 + W(u) \right] dx. \tag{1.3}$$

The corresponding H^{-1} gradient flow, the Cahn-Hilliard equation, takes the form

$$u_t = \Delta \frac{\delta \mathcal{E}}{\delta u} = \Delta(-\varepsilon^2 \Delta u + W'(u)). \tag{1.4}$$

Subject to zero-flux boundary conditions,

$$\Delta u \cdot \mathbf{n} = 0, \tag{1.5}$$

$$\Delta \mu \cdot \mathbf{n} = 0, \tag{1.6}$$

where $\mathbf{n}$ is the outer normal to $\partial\Omega$, the Cahn-Hilliard equation preserves the total mass

$$\frac{d}{dt} \int_\Omega u(x,t)\, dx = 0, \tag{1.7}$$

and dissipates the Cahn-Hilliard free energy

$$\frac{d}{dt}\mathcal{E}(u) = \left\langle u_t, \frac{\delta \mathcal{E}}{\delta u} \right\rangle_{L^2} = -\left\| \nabla \frac{\delta \mathcal{E}}{\delta u} \right\|_{L^2}^2 \leq 0. \tag{1.8}$$

It is known that the minimizers of the CH free energy over the space $H^1(\Omega)$ subject to a mass constraint are achieved. Moreover, these minimizers satisfy the Euler-Lagrange equation expressed in terms of the variational derivative of $\mathcal{E}$

$$\frac{\delta \mathcal{E}}{\delta u} := -\varepsilon^2 \Delta u + W'(u) = \lambda, \tag{1.9}$$

where λ is a Lagrange multiplier associated to a total mass constraint.

To model amphiphilic mixtures, such as emulsions formed by adding a minority fraction of an oil and soap mixture to water, [Teubner and Strey, 1987] and [Gompper and Schick, 1990] were motivated by small-angle X-ray scattering (SAXS) data to include a higher-order term in the usual Cahn-Hilliard expansion. Inspired by their work, we add the next order term to (1.2),

$$\tilde{\mathcal{F}}(u) := \int_\Omega \left[f(u,0,0) + \varepsilon^2 A(u)|\nabla u|^2 + \varepsilon^2 B(u)\Delta u + \overbrace{C(u)}^{\geq 0} (\varepsilon^2 \Delta u)^2 \right] dx. \tag{1.10}$$

The full form of this system supports too many possible regimes to permit a systematic study. It is important to find the simplest mathematical framework that supports the network morphologies typical of amphiphilic mixtures; we need a new normal form. With this goal we first shift all the differential terms to powers of the Laplacian. Specifically, letting $\overline{A}$ denote the primitive of A, we replace $A(u)\nabla u$ with $\nabla \overline{A}(u)$ and, assuming

appropriate zero-flux boundary conditions, we integrate the term $\nabla\overline{A}\cdot\nabla u$ by parts to obtain

$$\tilde{\mathcal{F}}(u) = \int_\Omega \left[f(u,0,0) + (B(u) - \overline{A}(u))\varepsilon^2\Delta u + C(u)(\varepsilon^2\Delta u)^2 \right] dx. \tag{1.11}$$

The energy density is a quadratic polynomial in $\varepsilon^2\Delta u$, which suggests that we complete the square

$$\tilde{\mathcal{F}}(u) = \int_\Omega C(u)\left[\left(\varepsilon^2\Delta u - \frac{\overline{A}-B}{2C} \right)^2 + f(u,0,0) - \frac{(\overline{A}-B)^2}{4C(u)} \right] dx. \tag{1.12}$$

For simplicity we replace $C(u)$ with $\frac{1}{2}$, and relabel the potential within and outside the squared term by $W'(u)$ and $P(u)$, respectively. The key point is that the first term is the square of the variational derivative of a Cahn-Hilliard type free energy, consequently the case $P \equiv 0$, when the energy is a perfect square, has the special property that its global minimizers are precisely the *critical points* of the corresponding Cahn-Hilliard energy. Indeed, a variant of this case was proposed as a target for $\Gamma-$convergence analysis by De Giorgi, see [Röger and Schätzle, 2006]. Our general form of the network is obtained by perturbing the perfect square with an asymptotically small potential,

$$\tilde{\mathcal{F}}(u) = \int_\Omega \left[\frac{1}{2}\left(\varepsilon^2\Delta u - W'(u) \right)^2 + \delta P(u) \right] dx, \tag{1.13}$$

where $\delta \ll 1$. The function $W(u)$ is assumed to be a double-well potential with two minima at $u = b_\pm$ whose unequal depths are normalized so that $W(b_-) = 0 > W(b_+)$. Typically $b_- = 0$, however it is helpful to give this value a specific name. Thus $u = b_-$ is associated to a bulk solvent phase, while the size of $u - b_- > 0$ is proportional to the density of the amphiphilic phase. The small parameter $\varepsilon \ll 1$, associated to the amphiphilic molecular width, determines the interfacial width and corresponds to the ratio of the typical length of an amphiphilic molecule to the domain size.

The Functionalized Cahn-Hilliard free energy corresponds to a class of two distinguished limits and a particular choice for P,

$$\mathcal{F}(u) := \int_\Omega \left[\frac{1}{2}\left(\varepsilon^2\Delta u - W'(u) \right)^2 - \varepsilon^p \left(\frac{\varepsilon^2\eta_1}{2}|\nabla u|^2 + \eta_2 W(u) \right) \right] dx. \tag{1.14}$$

The functionalization terms, parameterized by $\eta_1 > 0$ and $\eta_2 \in \mathbb{R}$, are analogous to the surface and volume energies typical of models of charged solutes in confined domains, see [Scherlis et al., 2006] and particularly equation (67) of [Andreussi et al., 2012]. The minus sign in front of η_1 is of considerable significance – it incorporates the propensity of the amphiphilic surfactant phase to drive the creation of interface. Indeed,

experimental tuning of solvent quality shows that morphological instability in amphiphilic mixtures is associated to (small) negative values of surface tension, [Zhu et al., 2009] and [Zhu and Hayward, 2012]. In the FCH energy the gradient term, $-\eta_1|\nabla u|^2 < 0$, is localized on interfaces, associated to single-layers of surfactant molecules, whose growth lowers overall system energy – however the effect is *perturbative* and unrestricted growth is arrested by the penalty nature of the square term which keeps u close to the critical points of $\mathcal{E}$. There are two natural distinguished limits corresponding to different choices for the exponent p in the functionalization terms. In the Strong Functionalization, $p = 1$, the functional terms dominate the Willmore corrections from the squared variational term. The Weak Functionalization, corresponding to $p = 2$, is the natural scaling for the Γ-limit as the curvature-type Willmore terms appear at the same asymptotic order as the functional terms.

The well-posedness of the minimization problem for the FCH, including the existence of global minimizers for fixed values of $\varepsilon > 0$ was established in [Promislow and Zhang, 2013] for a more general functional form over various natural function spaces. Depending upon the application, the volume-type η_2 functionalization perturbation incorporates the impact of counter-ion entropy (PEM fuel cells), capillary pressure, or entropic effects from constraint of tail groups (lipid bilayers), [Gavish et al., 2012]. The form $\eta_2 W(u)$ is chosen primarily for convenience, as integrals of $W(u)$ evaluated at critical points of $\mathcal{E}_{\rm CH}$ grow increasingly negative with increasing interfacial co-dimension. We remark that the surface term $\eta_1|\nabla u|^2$ is equivalent to an $\eta_1 u W'(u)$ functional-form since an integration by parts on $-\eta_1|\nabla u|^2$ yields $\eta_1 u\Delta u$ which can be absorbed into the squared variation with a perturbed form of W.

The goal of this study is to present an analysis of the stability and dynamics of classes of quasi-stationary network morphologies $\mathcal{N}$ of $\mathcal{F}$, which we define to be functions $u \in H^2(\Omega)$ which have an asymptotically small minority of amphiphilic phase, satisfy assigned boundary conditions, and render the driving force of the free energy asymptotically small. Specifically for each fixed $C > 0$ we define the set of quasi-stationary network morphologies

$$\mathcal{Q}_C := \left\{ u \in H^2(\Omega) \Big| \int_\Omega |u - b_-|\, dx \le C\varepsilon \text{ and } \left\| \Pi_0 \frac{\delta \mathcal{F}}{\delta u} \right\|_{L^2(\Omega)} \le C\varepsilon^{p+\frac{3}{2}} \right\}, \tag{1.15}$$

where p takes the same value as in equation (1.14) which defines the FCH free energy. The exponent term, $p + \frac{3}{2}$, in the bound on the residual corresponds to temporal dynamics on the ε^{-p} time scale. We also introduce the zero-mass projection

$$\Pi_0 f := f - \frac{1}{\Omega}\int_\Omega f(x)\, dx. \tag{1.16}$$

Our analysis hinges on the construction of quasi-stationary functions whose properly chosen level sets form

locally co-dimension one and two network morphologies in the sense of Definition **??**.

It is important to emphasize the difference between the CH free energy and the FCH free energy. The CH free energy describes the spinodal decomposition of hydrophobic materials. The FCH free energy models network formation in amphiphilic materials. In experimental settings, amphilicity drives the system to phase separate on a molecular length scale. Figure 1.4 (a) resembles an early stage of CH spinodal decomposition.

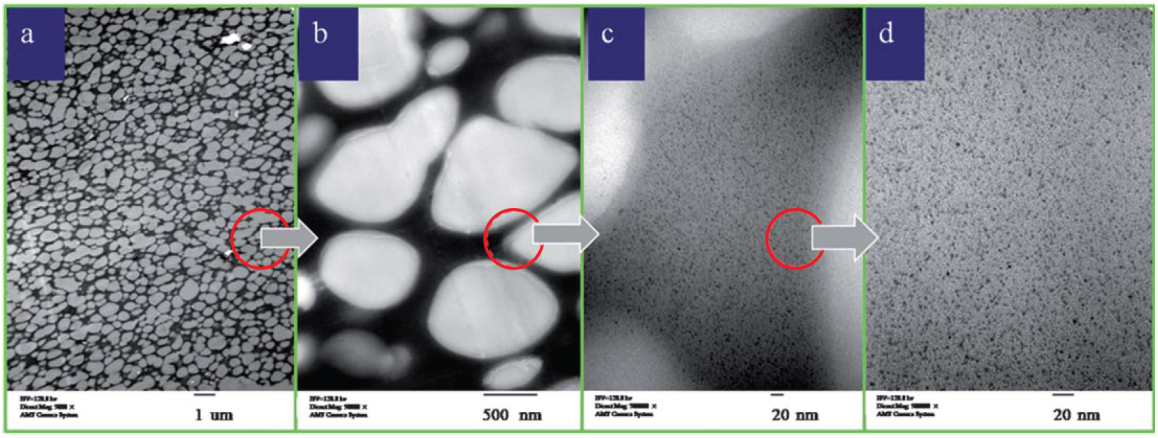


Figure 1.4: A porous membrane assembled from cholormethylated polysufone (CPSF) with pyridine graphed via nucleophilic substitution (ammonium agent). A 500 fold increase in magnification from a 1 micron to a 20 nanometer lengthscale shows a FCH-like nanoscale network morphology embedded within the domain walls of a micron-scale Cahn-Hilliard-like phase separation. The mixture is electroneutral on the micron scale, but has charge separation on the nanometer scale. Reproduced (Adapted) from, [Zhang et al., 2013] with permission of The Royal Society of Chemistry. DOI.

Zooming in where the red circle is, after a 500-fold magnification, the phase separated network morphology is visible within the CH-cell walls, see Figure 1.4 (d). Averaged over a micron length scale the system is electroneutral, and the phase separation is governed by a CH dynamic. On the nanometer length scale the system is not electroneutral, and the phase separation is governed by the FCH with the associated network morphologies.

1.3 Overview of Main Results

The over damped dynamics of amphiphilic polymer suspensions can be received from the Functionalized Cahn-Hilliard free energy via its gradient flows whose evolution preserves the volume fraction of the constituent species and lowers the free energy. Similar to the Cahn-Hilliard gradient flow given in (1.4), the simplest mass preserving gradient flow of the **strong FCH** is generated by the H^{-1} gradient,

$$u_t = \Delta \frac{\delta \mathcal{F}}{\delta u} = \Delta \left[\left(-\varepsilon^2 \Delta + W''(u) - \varepsilon \eta_1 \right) \left(-\varepsilon^2 \Delta u + W'(u) \right) + \varepsilon (\eta_1 - \eta_2) W'(u) \right]. \tag{1.17}$$

This research includes a formal derivative of the geometric evolution of co-dimension one and co-dimension two morphologies under the strong FCH equation, followed by a rigourous analysis of the pearling eigenvalues for morphologies of either codimension.

We start by formal asymptotic reduction: In chapter 3 we derive the geometric motion of a collection of disjoint, far from self-intersecting, closed, co-dimension one morphologies, referred to as bilayer morphologies, in $\mathbb{R}^d$. The key results are that the chemical potential of the pure bilayer system, μ_1, is spatially constant at leading order in the far field, and the H^{-1} gradient flow drives pure bilayer interfaces by a quenched mean-curvature flow, see equation (1.63). Moreover, for a bilayer morphology the far-field chemical potential will converge temporally to a prescribed constant value, μ_b^*. In chapter 4 we investigated the geometric evolution of a family of co-dimension two morphologies, referred to as pore morphologies, in $\mathbb{R}^3$. Away from the interfaces, the chemical potential is spatially constant, and the H^{-1} gradient flow drives pure pore manifolds by a curvature flow, where the vector normal velocity is coupled to the chemical potential, see equation (1.65). Moreover, the far-field chemical potential will decay to a prescribed constant μ_p^*.

Note 1. *The two equilibria points μ_b^* and μ_p^* are determined only by the tilted double-well potential W and the values of the functionalization parameters η_1 and η_2.*

Depending upon the value of the far-field chemical potential the geometric flow can be motion by curvature or motion against curvature; the later induces a strong geometric instability, akin to a backwards heat equation instability for the curvatures, see equation (1.64), which manifests in experiments as a fingering instability, as shown in Figure 1.3. For the bilayer system, if $\mu_1 < \mu_b^*$ then the bilayer will shrink as μ_1 grows, while if $\mu_1 > \mu_b^*$, the bilayer will grow, which may induce fingering of the interface Γ_b. A similar instability mechanism holds for pore morphologies.

To complete the formal analysis we consider the geometric evolution of a co-existing system comprised of a family of disjoint, far from self-intersecting, closed, co-dimension one and two structures, in $\mathbb{R}^3$. In 2014, [Dai and Promislow, 2015] have shown that for the weak FCH the two morphologies can co-exist. However, we conclude that, generically, the **strong** FCH equation does not support co-existence. Morphologies of distinct co-dimension will compete via the common value of the far field chemical potential, and depending upon the initial configuration and the values of the functionalization parameters, η_1 and η_2, the structures will compete for surfactant phase via the common value of the far field chemical potential, μ_1, with various possible outcomes including the extinction of one phase, a pearling bifurcation of one or both phases, or a fingering bifurcation. We also find non-generic values of η_1 and η_2 in which co-dimension one and co-dimension two morphologies can co-exist.

The geometric evolution results are formal, in particular they assume that the underlying bilayer and pore

morphologies are stable. The vulnerability of the matched asymptotics method is that it ignores any possible instabilities. In chapter 2 we review the spectrum of the linearized operators, see Figure 2.3, and show that both the co-dimension one and co-dimension two morphologies have potential instabilities associated to periodic, high-frequency modulations of the interfacial width, called a pearling instabilty.

A rigourous analysis of the eigenvalue problem corresponding to the strong FCH for the bilayer and the pore morphologies is presented in Chapters 5 and 6, respectively. We show that in the strong FCH scaling the leading order behavior of the pearling eigenvalues is independent of the shape of the underlying co-dimension one or two morphology, which allows the definition of associated pearling-stability regions in parameter space. In chapter 7 we analyze the combined bilayer-pore evolution. Under the H^{-1} gradient flow the pearling instability manifests itself on a time scale that is $O(\varepsilon^{-2})$ faster then than the geometric evolution, and hence can be taken to be instantaneous on the geometric time scale. Conversely, the fingering instability occurs on the same time scale as the geometric flow, and may not necessarily immediately manifest itself on the geometric timescales.

1.4 Quasi-Stationary solutions of the strong Functionalized Cahn-Hilliard Free Energy

For simplicity, we focus on the strong FCH, whose critical points, subject to a total mass constraint, are the solutions of the associated Euler-Lagrange equation

$$\frac{\delta \mathcal{F}}{\delta u} := \left(\varepsilon^2 \Delta - W''(u)\right) \boxed{\left(\varepsilon^2 \Delta u - W'(u)\right)} - \varepsilon \left(-\varepsilon^2 \eta_1 \Delta u + \eta_2 W'(u)\right) = \lambda, \tag{1.18}$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier, and the boxed term $(\varepsilon^2 \Delta u - W'(u))$ is the variational derivative of a CH free energy, of the form presented in Equation (1.3). Intuitively, approximate solutions of the CH Euler-Lagrange equation

$$\frac{\delta \mathcal{E}}{\delta u} := -\varepsilon^2 \Delta u + W'(u) = O(\varepsilon). \tag{1.19}$$

are natural starting places for a perturbative construction of solutions of the FCH Euler-Lagrange equation. For such approximate critical points of the CH free energy it is natural that the Lagrange multiplier λ in (1.14) should scale with ϵ, that is $\lambda = \varepsilon \hat{\lambda}$. Within this scaling we may rewrite the FCH Euler-Lagrange

equation,(1.18), as two, coupled second order systems

$$\begin{aligned} \varepsilon^2 \Delta u - W'(u) &= \varepsilon v, \\ \left(\varepsilon^2 \Delta - W''(u)\right) v &= \left(-\varepsilon^2 \eta_1 \Delta u + \eta_2 W'(u)\right) + \hat{\lambda}. \end{aligned} \tag{1.20}$$

The singularly perturbed nature of the FCH Euler-Lagrange system makes it amenable to dimensional reduction, yielding localized solutions build upon immersions in $\mathbb{R}^d$ of different co-dimensions.

1.4.1 Construction of co-dimension 1 Quasi-stationary solutions of FCH

We first review some basic definitions from elementary differential geometry. Let $\Gamma_b \subset \mathbb{R}^d$ be a smooth, co-dim 1 interface, which divides Ω into two disjoint sets $\Omega_+ \cup \Omega_-$, see Figure 1.5. Let $\rho_b(s)$ be its local parametrization, $\rho_b : Q \subset \mathbb{R}^{d-1} \to \mathbb{R}^d$, and $s = (s_1, ..., s_{d-1}) \in Q \subset \mathbb{R}^{d-1}$, and let r be the signed distance (unscaled) from Γ_b. For simplicity we choose the parameterization so the s_i correspond to arc length along

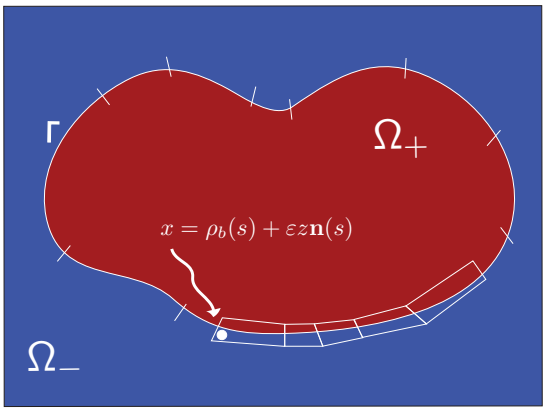


Figure 1.5: The whiskered coordinate system of a generic, admissible, co-dimension one interface.

the i^{th} coordinate curve and the coordinate curves are lines of curvature. In this setting, the vectors $\mathbf{T}^i = (T_1^i, ...T_{d-1}^i)$ defined by

$$\mathbf{T}^i := \frac{\partial \rho}{\partial s_i}, \qquad i = 1, .., d-1, \tag{1.21}$$

form an orthonormal basis for the tangent space to Γ_b at $\rho_b(s,t)$. Denoting the outer normal vector of Γ_b pointing towards Ω_- by $\mathbf{n}(s,t) = (N_1, ..., N_d)$, we have the relations

$$\frac{\partial \mathbf{T}^i}{\partial s_i} = -k_i \mathbf{n}, \qquad \frac{\partial \mathbf{n}}{\partial s_i} = k_i \mathbf{T}^i, \qquad i = 1, .., n-1, \tag{1.22}$$

where k_i are the principle curvatures of Γ_b.

Definition 1.1. *For fixed $K, \ell > 0$ the family, $\mathcal{G}_{K,\ell}$, of "admissible co-dimension one interfaces" is comprised*

of closed (compact and without boundary), oriented 2 dimensional manifolds Γ_b embedded in $\mathbb{R}^d$, which are far from self-intersection and with a smooth second fundamental form. More precisely,
(i) The $W^{4,\infty}(Q)$ norm of the 2nd Fundamental form of Γ_b and its principal curvatures are bounded by K.
(ii) The whiskers of length 3ℓ, in the unscaled distance, defined for each $s_0 \in Q$ by, $w_{s_0} := \{x : s(x) = s_0, |z(x)| < 3\ell/\varepsilon\}$, neither intersect each-other nor $\partial\Omega$ (except when considering periodic boundary conditions).
(iii) The surface area, $|\Gamma_b|$, of Γ_b is bounded by K.

Assume $\Gamma_b \in \mathcal{G}_{K,\ell}$. The set $\Gamma_{b,\ell}$ defined by

$$\Gamma_{b,\ell} := \left\{\zeta_b(s,z) \in \mathbb{R}^d \middle| s \in S, -\ell/\varepsilon \le z \le \ell/\varepsilon\right\} \subset \Omega, \tag{1.23}$$

will be called the **reach** of Γ_b, where we emphasize that ℓ is fixed independent of ε and of $\Gamma_b \in \mathcal{G}_{K,\ell}$. Each $x \in \Gamma_{b,\ell}$ can be uniquely expressed via the **whiskered coordinates** such that

$$x = \zeta_b(s,z) := \rho_b(s) + \varepsilon z \mathbf{n}(s), \tag{1.24}$$

where $z \in [-l/\varepsilon, l/\varepsilon]$ is the scaled signed distance to Γ_b, $z = \frac{r}{\varepsilon}$. The line segments $\{\rho_b(s) \times [-\ell,\ell] \;\big|\; s \in Q\}$ are the **whiskers** of length 2ℓ of Γ_b, and the pair (s,z) form the local whiskered coordinate system. Figure 1.5 presents the whiskered coordinate system. By the Implicit Function Theorem this map is locally and smoothly invertible. In particular, the functions $s = s(x)$ and $z = z(x)$ which relate the whiskered coordinates to the cartesian ones and the associated change of variables, are all C^4 diffeomorphisms on the reach, $\Gamma_{b,l}$, of Γ_b. The white region in Figure 1.6 (right) depicts the reach of the associated immersion Γ_b.

Definition 1.2. *Given $\Gamma_b \in \mathcal{G}_{K,\ell}$ and a function $f : \mathbb{R} \to \mathbb{R}$ which tends to constant values $f_\infty^\pm$ at O(1) exponential rates as $r \to \infty$, we say that we* **dress the interface** *Γ_b with f, obtaining the Γ_b-extension*

$$f_{\Gamma_b}(x) := f(r(x))\chi(|r(x)|/l) + f_\infty^+(1-\chi(|r(x)|/l)) + f_\infty^-(1-\chi(|r(x)|/l)), \tag{1.25}$$

where $r(x)$ is the (unscaled) distance from Γ_b and $\chi : \mathbb{R} \to \mathbb{R}$ is a fixed, smooth cut-off function which is one on $[-\infty, 1]$, while $\chi(s) = 0$ for $s \ge 2$. By abuse of notation we will drop the Γ_b subscript in the Γ_b-extension when doing so creates no confusion.

The first step in the construction of the quasi-stationary solutions is to build the bilayer profile which is a 1-dimension equilibrium of equation (1.20). In the whiskered coordinates the Cartesian Laplacian takes the form

$$\varepsilon^2\Delta = \partial_z^2 + \varepsilon H(s,z)\partial_z + \varepsilon^2\Delta_G, \tag{1.26}$$

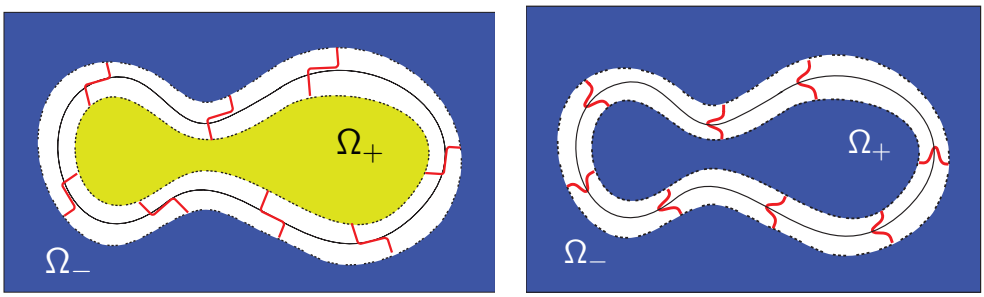


Figure 1.6: Single-layer (left) and Bilayer (right) dressings of the same co-dimension one interface Γ (solid black line). The dressing function is a one-D solution of the CH Euler-Lagrange equation. For the single-layer solution Γ separates regions $u = b_-$ from $u = b_+$, while the bilayer solution corresponds to $u = b_-$ on either side of the bilayer, with a brief excursion $u > b_-$ near Γ.

where H is the extended curvature, defined in terms of the Jacobian, $\mathbf{J}_b$, of the change of variables, defined in Equation (1.24). In particular, at leading order $H(s,z) = H_0(s) + O(\varepsilon z)$ where H_0 is the mean curvature of Γ_b at $\zeta_b(s,0)$ and $\Delta_G = \Delta_s + O(\varepsilon z)$ where Δ_s is the usual Laplace-Beltrami operator on Γ_b, for further details see Section 2.1.

In the whiskered coordinates the first equation of (1.20) reduces, at leading order, to a second-order ODE in z, for the one-dimension profile $\varphi(z)$,

$$\partial_z^2 \varphi(z) = W'(\varphi), \tag{1.27}$$

defined for z in the reach. Since the double-well W is assumed to have unequal depth wells $0 = W(b_-) > W(b_+)$, a simple phase-plane analysis shows that this equation supports a unique solution U_b which is *homoclinic* to b_-, that is $U_b(z) \to b_-$ as $z \to \pm\infty$, see [Homburg and Sandstede, 2010] for a general discussion of homoclinic orbits.

We define the leading-order structure of the bilayer critical point, $u_b = u_b(x;\Gamma_b)$ via the two-term expansion,

$$u_b(x) := U_b(z(x)) + \varepsilon u_{b,1}, \tag{1.28}$$

where U_b is the bilayer dressing of Γ_b within the reach $\Gamma_{b,\ell}$, equal to a constant value on $\Omega \backslash \Gamma_{b,3\ell}$ and smoothly extended to match in the intermediate region, see Figure 1.6 (right). To define the correction term $u_{b,1}$ we first introduce the Sturm-Liouville operator $L_{b,0}$

$$L_{b,0} := \partial_z^2 - W''(U_b), \tag{1.29}$$

which is the linearization of (1.27) about U_b. Evaluating equations (1.20) at u_b and projecting the right-hand

side onto the range of $L_{b,0}$ yields

$$L_{b,0}u_{b,1} = v, \tag{1.30}$$

$$L_{b,0}v = -\eta_1 U_b'' + \eta_2 W'(U_b) + \hat{\lambda}. \tag{1.31}$$

On the reach, $\Gamma_{b,\ell}$, the correction $u_{b,1}$ is chosen to simplify the residual of equation (1.20) when evaluated at u_b, is defined as

$$u_{b,1} \coloneqq L_{b,0}^{-2}\left(-\eta_1 U_b'' + \eta_2 W'(U_b) + \hat{\lambda}\right). \tag{1.32}$$

Remark 1.1. The inverse operator $L_{b,0}^{-1}$ is naturally defined $L_2(\mathbb{R}) \to H^2(\mathbb{R})$, by abuse of notation we apply it to functions on $[-\ell/\varepsilon, \ell/\varepsilon]$ which have a natural extension to $\mathbb{R}$ by applying it to the extension, and then restricting the result.

We further decompose $u_{b,1}$ into a local term $\tilde{u}_{b,1}$ which decays exponentially to zero in z, and is smoothly extended to be zero off of $\Gamma_{b,\ell}$, and a constant term

$$\gamma_1 = \frac{\hat{\lambda}}{\alpha_-^2}, \tag{1.33}$$

where we have introduced the well coercivity

$$\alpha_- \coloneqq W''(b_-) > 0. \tag{1.34}$$

The resulting u_b is our qausi-steady solution

$$u_b(x) = U_b(z) + \varepsilon\left(\gamma_1 + \tilde{u}_{b,1}(z)\right), \tag{1.35}$$

parameterized by $\Gamma_b \in \mathcal{G}_{K,\ell}$ and $\gamma_1 \in \mathbb{R}$. The local term $\tilde{u}_{b,1}$ corrects the structure of U_b within the reach, while the spatial constant γ_1 adjusts the far-field behavior of u_b, which is now $b \coloneqq b_- + \varepsilon\gamma_1$.
In the far-field region u_b takes the spatially constant value

$$u_b(x) = b \coloneqq b_- + \varepsilon\frac{\mu_1}{\alpha_-^2} + O(\varepsilon^2), \qquad x \in \tilde{\Gamma}_{b,\ell}, \tag{1.36}$$

where μ_1, the chemical potential, is the leading order, non-zero term in the first variation of $\mathcal{F}$. By matching

the inner and outer expression of u_b, given in (1.35) and (1.36), respectively, we find that

$$\gamma_1 = \frac{\mu_1}{\alpha_-^2}, \tag{1.37}$$

i.e., the value of μ_1 differs from γ_1 by a factor corresponding to the square of the branch point, α_-^2 of the essential spectrum of $L_{b,0}$.

The quantity μ_1 plays a key role in the evolution and bifurcation of the quasi-steady interfaces. We associate μ_1 with the far-field density of amphiphilic molecule – precisely the quantity that Szostak 'tweaked' by adding oleo-lipids to the bulk solvent phase in his experiment [Budin and Szostak, 2011], see Figure 1.3.

Remark 1.2. There are critical points of $\mathcal{F}$ for which λ is $O(1)$, in particular the *single-layer* solutions, which correspond to heteroclinic orbits of (1.27) that connect two equilibrium values, see Figure 1.6 (left). For the Cahn-Hilliard free energy single-layers form the dominant global minimizers, however they are generically saddle points of the FCH, and are susceptible to meander instabilities in the gradient flow. It is important to emphasize that single-layers and bilayers are distinct morphologies – single-layers separate phase A from phase B while bilayers separate phase A into two regions by a thin layer of phase B, see Figure 1.6. In particular bilayers can rupture, re-uniting the two regions of phase A, as when a lipid bilayer opens a pore, or tears. In addition, the interfacial component is a conserved quantity for bilayers, and when the bilayer is stretched the interface must thin, which naturally increases its free energy as it deforms from its equilibrium profile U_b – bilayers can support non-zero tangential stresses.

1.4.2 Construction of co-dimension 2 and co-dimension 3 Quasi-stationary solutions of the FCH

The FCH Euler-Lagrange equation, (1.20), also possesses co-dimension two and co-dimension three solutions in $\Omega \subset \mathbb{R}^3$. We first consider co-dimension two solutions. These are based upon a foliation of a neighborhood of a smooth, closed, non-self intersecting one-dimensional manifold Γ_p immersed in Ω. The co-dimension two whiskered coordinate system, introduced in Chapter 2, is defined using the mapping $x = \zeta_p(s, z)$, and the ideas of admissibility, reach, and dressing extend naturally from the co-dimension one case.

Within the reach, $\Gamma_{p,\ell}$, of Γ_p the Laplacian admits the local form

$$\varepsilon^2 \Delta = \Delta_z + \varepsilon \frac{\vec{\kappa}}{1 - \varepsilon \vec{z} \cdot \vec{\kappa}} \cdot \nabla_z + \varepsilon^2 \partial_G^2, \tag{1.38}$$

where Δ_z is the Cartesian Laplace operator in the scaled normal distances $\vec{z} = (z_1, z_2)$, the vector $\vec{\kappa} = (\kappa_1, \kappa_2)^T$ is the curvatures vector of Γ_p at $\zeta_p(s, \vec{0})$, and ∂_G^2 reduces to the line diffusion operator, ∂_s^2, on Γ_p

when $\vec{z} = 0$, see Section 2.2 for details. Assuming radial symmetry, the leading order pore profile associated to the Euler-Lagrange equation (1.18) satisfies co-dimension two critical point equation

$$\partial_R^2 U_p + \frac{1}{R}\partial_R U_p = W'(U_p), \tag{1.39}$$

subject to $\partial_R U_p(0) = 0$ and $U_p \to b = b_- + \varepsilon\mu_1 + O(\varepsilon^2)$ as $R \to \infty$. The leading order form for the pore quasi-stationary network arises from the pore profile dressing of a co-dimension two interface Γ_p,

$$u_p(x) := U_p(R(x)) + \varepsilon\left(\mu_1 + u_{p,1}(R)\right), \tag{1.40}$$

where the local term $\tilde{u}_{p,1}$ corrects the structure of U_p within the reach, $\Gamma_{p,\ell}$, while $\mu_1 \in \mathbb{R}$ is a spatial constant that adjusts the far-field chemical potential. It is possible to combine quasi-stationary bilayer and pore morphologies, so long as the associated manifolds have non-intersecting reaches, *and* the far-field constant μ_1 takes a common value. Indeed, the quasi-steady evolution between co-existing co-dimension one and co-dimension two interfaces is driven by the competition between this common far-field value b, which is given by $b = b_- + \varepsilon\mu_1$. If the optimal far-field values, μ_b^* and μ_p^*, associated to distinct co-dimensional morphologies differ, then the morphologies cannot both simultaneously be in equilibrium, see section 4.7 for details.

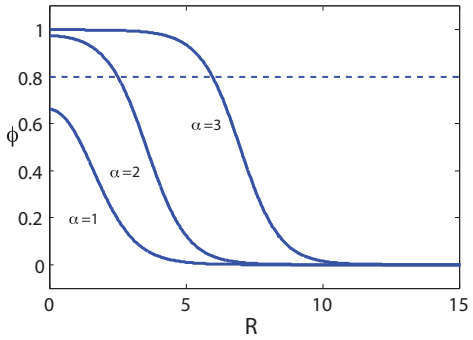
Co-dimension three quasi-stationary solutions, in $\mathbb{R}^3$, are spherically symmetric micelle morphologies. The associated coordinate system reduces to the usual spherical variables and the Laplacian reduces to the associated spherical form. Assuming rotational symmetry, the leading order micelle profile is the unique solution of

$$\partial_R^2 U_m + \frac{2}{R}\partial_R U_m = W'(U_m), \tag{1.41}$$

subject to $\partial_R U_m(0) = 0$ and $U_m \to b$ as $R \to \infty$. An immediate prediction of the FCH free energy is that bilayers must be thinner than pores, which in turn are thinner than micelles. This observation is born out by experimental data, Figure 1.7 (right).

1.4.3 Minimization of the strong FCH free energy over co-dimension 1 quasi-stationary profiles

It is constructive to examine the minimizers of $\mathcal{F}$ over a class of co-dimension 1 quasi-stationary solutions. Momentarily setting aside the mass constraint, there are two classes of free parameters in our construction of u_b, the spatially constant background correction, μ_1, and the interface shape Γ_b. We will show, in



M_n^{core} (g/mol)	2500 ± 40	5850 ± 204
bilayer	8.7 ± 1.2	15.8 ± 2.8
pore	14.3 ± 1.6	25.4 ± 3.3
micelle (nm)	18.4 ± 2.6	38.8 ± 10.2

Figure 1.7: (left) A comparison of co-dimension $\alpha = 1, 2$, and 3 profiles computed from (1.27), (1.39), and (1.41) respectively. The relative widths the profile is most sensitive to the difference in depths of the two wells: $W(b_-) - W(b_+) > 0$. (right) A table of experimental data indicating radii of bilayer, pore, and micelle morphologies obtained by varying the hydrophilic length of polymer in PEO-PB amphiphilic di-blocks with fixed hydrophobic (core) molecule weight, M_n^{core}, as indicated. Reprinted (adapted) with permission from [Jain and Bates, 2004]. Copyright 2004 American Chemical Society.

Equation (1.52), that for the **strong FCH** free energy, the optimal value of amphiphilic material in the bulk region is determine by the double-well potential W and the functionalization terms η_1 and η_2. We first evaluate the free energy, at u_b, which takes the form

$$\mathcal{F}(u_b) = \int_\Omega \frac{1}{2}\left(\varepsilon^2 \Delta u_b - W'(u_b)\right)^2 - \varepsilon\left(\frac{\varepsilon^2\eta_1}{2}|\nabla u_b|^2 + \eta_2 W(u_b)\right) dx, \tag{1.42}$$

and break the integral over the near-field $\Gamma_{b,\ell}$ and far-field $\tilde{\Gamma}_{b,\ell} := \Omega \backslash \Gamma_{b,\ell}$. Denoting the near-field integral by $\mathcal{F}_\ell(u_b)$ we change to local coordinates

$$\begin{aligned}\mathcal{F}_\ell(u_b) &= \int_{\Gamma_{b,\ell}} \left(\varepsilon^2 \Delta u_b - W'(u_b)\right)^2 - \varepsilon\left(\frac{\varepsilon^2\eta_1}{2}|\nabla u_b|^2 + \eta_2 W(u_b)\right) dx, \\ &= \int_{\Gamma_b}\int_{-\ell/\varepsilon}^{\ell/\varepsilon} \frac{1}{2}\left(\partial_z^2 U_b - W'(U_b) + \varepsilon H_0(s) U_b'\right)^2 - \varepsilon\left(\frac{\eta_1}{2}|U_b'|^2 + \eta_2 W(U_b)\right) J_b(s,z)\, dz\, ds,\end{aligned} \tag{1.43}$$

where the Jacobian, defined in (2.6), admits the expansion $J_b = \varepsilon + \varepsilon^2 z H_0(s) + O(\varepsilon^3 z^2)$. Expanding the Jacobian and keeping only leading order terms we find

$$\mathcal{F}_\ell(u_b) = \varepsilon \int_{\Gamma_b}\int_{-\ell/\varepsilon}^{\ell/\varepsilon} \frac{\varepsilon^2}{2}\left(L_{b,0}\left(\frac{\mu_1}{\alpha_-^2} + \tilde{u}_{b,1}\right) + H_0(s) U_b'\right)^2 - \varepsilon\left(\frac{\eta_1}{2}|U_b'|^2 + \eta_2 W(U_b)\right) ds\, dz. \tag{1.44}$$

The localized functions in the squared term will yield $O(\varepsilon^3)$ integrals which are negligible. However, the far-field correction in the squared term $L_{b,0}\frac{\mu_1}{\alpha_-^2} = -W''(U_b)\frac{\mu_1}{\alpha_-^2} \to -\frac{\mu_1}{\alpha_-}$ yields an asymptotically relevant contribution. Moreover integrating (1.27) we see that $(U_b')^2 = 2W(U_b)$. Together these two observations

allow us to rewrite the localized component of the free energy as,

$$\mathcal{F}_\ell(u_b) = \varepsilon^2|\Gamma_b|\left(\ell\frac{\mu_1^2}{\alpha_-^2} - \frac{\eta_1+\eta_2}{2}\sigma_b\right), \tag{1.45}$$

where we introduced the bilayer 'surface tension'

$$\sigma_b := \|U_b'\|^2_{L^2(\mathbb{R})}. \tag{1.46}$$

The value of u_b in the far-field region is given in (1.36), and, by Taylor expansion, we note that $W'(b) = \varepsilon\frac{\mu_1}{\alpha_-} + O(\varepsilon^2)$, and $W(b) = O(\varepsilon^2)$. Denoting the far-field integral by $\tilde{\mathcal{F}}_\ell(u_b)$ we find that its contribution to the energy reduces to the leading order expression,

$$\tilde{\mathcal{F}}_\ell(u_b) = \varepsilon^2(|\Omega| - 2\ell|\Gamma_b|)\frac{1}{2}\frac{\mu_1^2}{\alpha_-^2} + O(\varepsilon^3). \tag{1.47}$$

Combining the near- and far-field expressions, the total energy takes the form

$$\mathcal{F}(u_b) = \varepsilon^2\left(\frac{|\Omega|}{2}\frac{\mu_1^2}{\alpha_-^2} - |\Gamma_b|\frac{\eta_1+\eta_2}{2}\sigma_b\right) + O(\varepsilon^3). \tag{1.48}$$

A similar near-field/ far-field decomposition applied to the the integrals yields the expression for the total mass of amphiphilic material.

$$M := \int_\Omega u_b(x) - b_-\,dx = \int_{\tilde{\Gamma}_{b,\ell}} \varepsilon\frac{\mu_1}{\alpha_-^2}\,dx + \int_{\Gamma_b}\int_{-\ell/\varepsilon}^{\ell/\varepsilon}(U_b + \varepsilon\frac{\mu_1}{\alpha_-^2})J_b\,dz\,ds = \varepsilon|\Omega|\frac{\mu_1}{\alpha_-^2} + \varepsilon|\Gamma_b|m_b, \tag{1.49}$$

where

$$m_b := \int_{\mathbb{R}} U_b(z) - b_-\,dz > 0, \tag{1.50}$$

is the mass of amphiphilic material per unit length of bilayer. Typically the amphiphilic component is scarce within the bulk, so that $M = \varepsilon\hat{M}$ (don't put too much soap in the washing machine!), and since Γ_b is admissible its interfacial area $|\Gamma_b|$ is $O(1)$. These assumptions render u_b a quasi-stationary with respect to $\mathcal{F}$, moreover a prescribed value of $\hat{M}$ and μ_1 determines the area, $|\Gamma_b|$, of the bilayer interface. Consequently, we solve equation (1.49) for $|\Gamma_b|$, and plug the result into equation (1.48) which yields

$$\mathcal{F}(u_b) = \varepsilon^2\left(\frac{|\Omega|}{2}\frac{\mu_1^2}{\alpha_-^2} - \frac{(\hat{M} - |\Omega|\frac{\mu_1}{\alpha_-^2})(\eta_1+\eta_2)\sigma_b}{2m_b}\right). \tag{1.51}$$

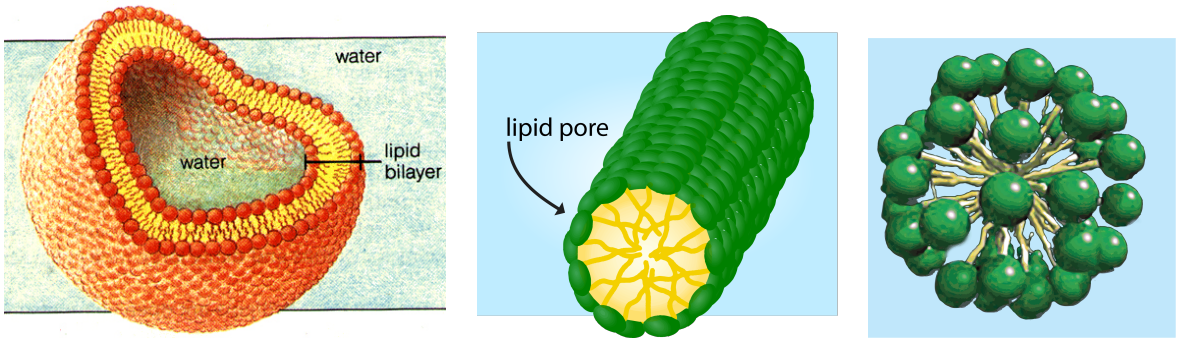


Figure 1.8: Depiction of bilayer (left, source: *academic.brooklyn.cuny.edu*), pore (center), and micelle (right) morphologies of lipids. The co-dimension associated to the morphology is the difference between the space dimension and the number of tangent directions of the minimal manifold whose normal bundle locally foliates the morphology. In $\mathbb{R}^3$ bilayers are co-dimension one, pores are co-dimension two, and micelles are co-dimensionl three.

The minimization of $\mathcal{F}(u_b)$ over Γ_b and μ_1, subject to the mass constraint reduces to the optimization of a quadratic polynomial in μ_1, and the optimal value of amphiphilic material in the bulk region takes the form

$$\mu_b^* = -\frac{\eta_1 + \eta_2}{2}\frac{\sigma_b}{m_b}. \tag{1.52}$$

For the strong functionalization only the area of an admissible co-dimension one interface, and not its curvature, enter into the leading-order determination of the free energy of its bilayer dressing. Moreover bilayers prefer an optimal far-field value of lipid, μ_b^* which is independent of the scaled mass constraint $\hat{M}$ and hence the area of the bilayer – it is a universal property of the system as determined by the shape of the well W through m_b, σ_b, and α_- and through the functionalization parameters η_1 and η_2. For the weak functionalization the Willmore term, the integral of the square of the mean curvature over Γ_b, enters into the free energy at leading order, and the optimization is more subtle.

1.5 Network Bifurcation in the FCH

The quasi-stationary network morphologies developed in Section 1.4 are, at leading order, critical points of the Cahn-Hilliard, however these structures are not perturbations of local minima but rather approximate saddle points of the CH free energy. An essential feature of the functional form of the FCH is its facility to build competitors for its local minima out of the saddle points of the simpler CH free energy. This process is best understood by examining the second variational derivative of the FCH free energy at a smooth critical point, u_c of the Cahn-Hilliard free energy. For traceless boundary conditions, such as periodic boundary conditions, see [Promislow and Zhang, 2013] for a detailed discussion of appropriate boundary conditions,

the second variation takes the form

$$\mathbb{L}_{u_c} := \frac{\delta^2 \mathcal{F}}{\delta u^2}(u_c) = \left(\varepsilon^2 \Delta - W''(u_c)\right)^2 - \varepsilon^p \left(\eta_1 \varepsilon^2 \Delta + \eta_2 W''(u_c)\right). \tag{1.53}$$

For a quasi-steady bilayer, u_b, associated to an admissible, co-dimension one interface Γ_b, the second variational derivative $\mathbb{L}_b := \mathbb{L}_{u_b}$, takes a simplified form when acting on functions $u \in H^4(\Omega)$ whose support lies within the reach, $\Gamma_{b,\ell}$, of Γ_b. On this subspace the operator admits the asymptotic expansion

$$\mathbb{L}_b = \left(L_{b,0} + \varepsilon H \partial_z + \varepsilon^2 \Delta_G\right)^2 - \varepsilon^p \left(\eta_1 \partial_z^2 + \eta_2 W''(U_b)\right) + O(\varepsilon^{p+1}), \tag{1.54}$$

An investigation of the spectrum of the operator $\mathbb{L}_b$ is presented in Chapter 5. Indeed, it was shown by [Hayrapetyan and Promislow, 2014] that there exists $U > 0$, independent of ε such that the eigenfunctions associated to $\mathbb{L}_b$ corresponding to eigenvalues $\Lambda_b < U$ comprise two sets, the *pearling eigenmodes* $\{\Psi_{b,0,n}\}_{n=N_1}^{N_2}$ and the *meander eigenmodes* $\{\Psi_{b,1,n}\}_{n=0}^{N_3}$, see details in Chapter 2. In chapter 5 we characterize the pearling eigenmodes, showing that they are independent of $\Gamma_b \in \mathcal{G}_{K,\ell}$ and consequently determine parametric regions of pearling stability and instability, for the strong FCH. For Γ_b an admissible, generic, co-dimension one interface we consider the eigenvalue problem

$$\mathbb{L}_b \Psi_{b,0,n} = \Lambda_{b,0,n} \Psi_{b,0,n}, \tag{1.55}$$

associated to the second variation of $\mathcal{F}$ about the bilayer dressing u_b. The spectrum of $\mathbb{L}_b$ cannot be localized by a regular perturbation expansion since the eigenvalues are asymptotically close together.

The expression for the pearling stability condition of bilayer interfaces with constant curvatures was established in [Doelman et al., 2014], see Figure 1.9, where the eigenvalues associated to the bilayer dressing of such an interface are uncoupled. In this thesis we extend this result to the linearization about a dressing of generic admissible co-dimension one and co-dimension two manifolds. The main difficulty arises from the coupling among the eigenvalues through the derivatives of the curvatures. The analysis requires bounds on the spectrum that are uniform in $\varepsilon \ll 1$. To this end we introduce the $L^2(\Omega)$ orthogonal projection Π onto the space of the pearling eigenmodes and its complementary projection denoted $\tilde{\Pi} := I - \Pi$. A decomposition of the operator $\mathbb{L}_b$ into a 2×2 block form using the projections takes the form

$$\tilde{\mathbb{L}}_b := \begin{bmatrix} \Pi \mathbb{L}_b \Pi & \Pi \mathbb{L}_b \tilde{\Pi} \\ \tilde{\Pi} \mathbb{L}_b \Pi & \tilde{\Pi} \mathbb{L}_b \tilde{\Pi} \end{bmatrix}. \tag{1.56}$$

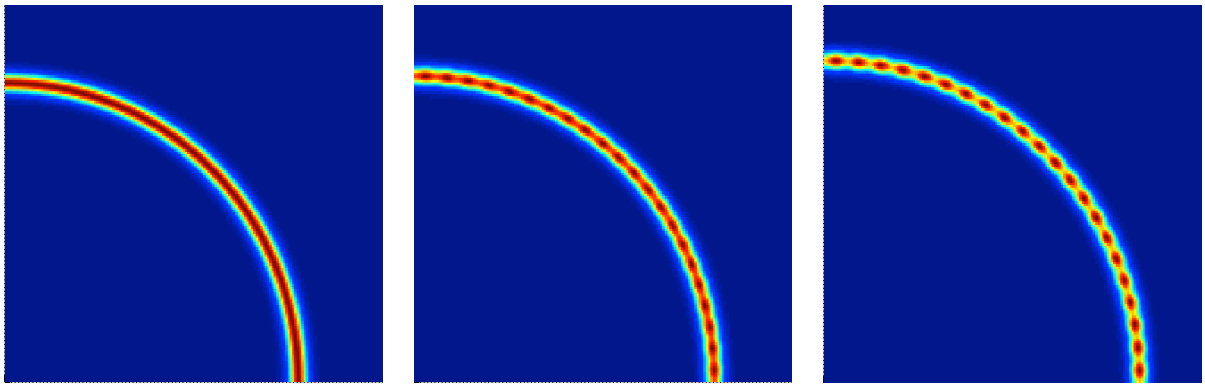


Figure 1.9: Time evolution of a circular, co-dimension one bilayer under the FCH gradient flow (1.17) for vales $\varepsilon = 0.1$ and $\eta_1 = \eta_2 = 2$. The times depicted correspond to $t = 0$, $t = 114$, and $t = 804$ and show the onset of the pearling bifurcation.

In Chapter 5, Section 5.3, we prove that the off-diagonal operators are small, in norm, independent of ε. The spectrum of the fully infinite dimensional piece, $\tilde{\Pi}\mathbb{L}_b\tilde{\Pi}$, is bounded from below by a constant $U > 0$ independent of ε, [Hayrapetyan and Promislow, 2014]. The upper-left element $\Pi\mathbb{L}_b\Pi$ can be reduced to a large matrix $M \in \mathbb{R}^{N\times N}$ where $N \approx \varepsilon^{3/2-d}$. *The spectrum of* $\mathbb{L}_b$ *is controlled by the spectrum of the matrix* M *and the singular scaling is reflected in the growth of* N *as* $\varepsilon \to 0$. Care must be taken to distinguish between the size of the entries of M and the size of M as an operator from $l^2(\mathbb{R}^N)$ to $l^2(\mathbb{R}^N)$, as the latter generically scales like $\sqrt{N}$ times the l^∞ norm of the entries. For simplicity we focus only on the pearling modes $j = 0$, neglecting the meander terms associated to $j = 1$. In Chapter 5 we observe that the matrix M admits an asymptotic decomposition

$$M = M^0_{\text{diag}} + \varepsilon A, \tag{1.57}$$

where M^0_{diag} is a diagonal matrix and A is uniformly bounded as an operator on $l^2(\mathbb{R}^N)$ as long as the curvatures are sufficiently smooth. Therefore, at leading order, the eigenvalues of M are the diagonal entries of M^0_{diag} which take the form

$$\Lambda_{b,0,n} = (\lambda_{b,0} - \varepsilon^2\beta_n)^2 - \varepsilon(\mu_1\alpha_-^2 S_b + \lambda_{b,0}(\eta_1 - \eta_2)\|\psi_{b,0}\|_2^2), \tag{1.58}$$

where $\lambda_{b,0}$ is the ground state eigenvalues of the linear operator $L_{b,0}$ with the corresponding eigenfunction $\psi_{b,0}$, and β_k is an eigenvalue of the Laplace-Beltrami operator Δ_s, corresponding to the eigenfunction Θ_k. The coefficient S_b is the " bilayer shape factor", defined in equation (5.41), whose sign determines if the pearling bifurcation absorbs or releases amphiphilic material from the bulk.

The positive quadratic term in the pearling eigenvalue expression (1.58) is dominant except when the Laplace-

Beltrami eigenvalue β_n is approximately equal to $\lambda_{b,0}\varepsilon^{-2}$. By the Weyl asymptotic formula for the Laplace-Beltrami eigenmodes, the residual of the dominant term is $O(\varepsilon)$ for an asymptotically large value, N of indices n. The nature of the bilayer pearling bifurcation depends sensitively upon the sign of S_b. For $S_b < 0$, which holds for a generic class of double-well potentials W, see section 5 of [Doelman et al., 2014], the spectrum of M will be strictly positive if and only if μ_1 satisfies the *pearling stability condition*

$$P_b^* := -\frac{\lambda_{b,0}(\eta_1 - \eta_2)\|\psi_{b,0}\|_{L^2}^2}{\alpha_-^2 S_b} > \mu_1. \tag{1.59}$$

We also identify a class of wells for which $S_b > 0$, in which case the the direction of the inequality in (1.59) is reversed.

In Chapter 5, Section 5.4, we connect the spectrum of M to that of $\mathbb{L}_b$, showing that the eigenvalues of M are in fact a small perturbation of the small eigenvalues of $\mathbb{L}_b$, and we obtain a perturbation estimate. We also examine the solution of the linear flow generated by $\mathbb{L}_b$. Assuming the eigenvalues of M are stable under pearling, in terms of Equation (1.59), we will show that the semi-groups generated by $\mathbb{L}$ decay exponentially fast and describe the resulting exponential dichotomy.

A similar analysis can be performed for co-dimension two pore structures, parametrized by the one-dimensional immersion Γ_p, see Chapter 6. Assuming a negative value of the "pore shape factor" S_p, defined in (6.33) we show that the pore structure will remain pearling stable if and only if μ_1 satisfies the *pearling stability condition*

$$P_p^* := -\frac{\eta_d\left(\left\|\psi_{p,0}'\right\|_{L_R}^2 + \lambda_{p,0}\left\|\psi_{p,0}\right\|_{L_R}^2\right)}{S_p} > \mu_1, \tag{1.60}$$

where $\lambda_{p,0}$ is the ground state eigenvalue of L_p with the corresponding eigenfunction $\psi_{p,0}$.

This analysis is consistent with Szostak's experiment, see [Zhu et al., 2012], in which a photo-induced increase in charge on the lipid heads induced a pearling bifurcation which drove pores to micelles, see Figure 1.3 (right). The increase in charge corresponds, within the FCH, to an instantaneous increase in η_1; a sufficiently large increase, for a fixed value of μ_1, will trigger the bilayer pearling condition (1.59) as well as the pore pearling condition (1.60). Figure 7.9 depicts the pearling as a result of instantaneously increase in η_1.

In Section 5.5 we relate the small eigenvalues of $\Delta\mathbb{L}$ to those of $\mathbb{L}$ and find that the pearling eigenvalues of $\Delta\mathbb{L}$ which are two orders of ε larger than the pearling eigenvalues of $\mathbb{L}$.

1.6 Competitive Geometric Evolution of Bilayers and Pores

In this thesis we model the over-damped dynamics of amphiphilic polymer suspensions via the mass-preserving H^{-1} gradient flow given in equation (1.17). The quasi-stationary network morphologies constructed in section 1.4 are not stationary solutions of the FCH gradient flow, but generate slow dynamics which may be locally parameterized by the interfacial sub-manifolds of bilayers and pores, respectively Γ_b and Γ_p. Indeed, when the bilayer pearling stability condition holds, then the bilayers meander eigenmodes $\{\Psi_{b,1,n}\}_{n=0}^{N_3}$, depict in Figure 2.3, comprise the potentially negative eigenspace of the associated linearization. The flow of the underlying interfacial structure can be obtained by projecting the residual $\frac{\delta\mathcal{F}}{\delta u}(u_b)$ of the critical point equation (1.18) onto this eigenspace. The method of matched asymptotic expansion provides an accessible, but formal method to derive the interfacial motion. For a bilayer morphology, the ansatz (1.35) for u_b is augmented by taking the signed distance z to the interface Γ_b and the background state μ_1 to be functions of the slow scaled time $\tau = t/\varepsilon$, and the gradient flow is solved by matching fluxes, particularly across the interfacial layers. For single-layer morphologies, under the Cahn-Hilliard gradient flow this results in a Mullins-Sekerka problem for the interface, see [Pego, 1989]. For $\varepsilon \ll 1$ it was shown that the leading order normal velocity of the interface of the spinodal domains is determined by the jump in the normal derivative of the chemical potential defined across the interface, separate the complementary domains. More rigorous derivations of Pego's results quickly followed, particularly [Alikakos et al., 1994] and [De Mottoni and Schatzman, 1995].

For the FCH gradient flow, (1.17) reduces, at leading order, to

$$\varepsilon U_b'(z)\frac{\partial z}{\partial \tau} + \varepsilon\frac{d\mu_1}{d\tau} = \Delta\frac{\delta\mathcal{F}}{\delta u}(u_b) = \varepsilon\Delta H_0(s)U_b'(z) + O(\varepsilon^2). \tag{1.61}$$

The leading order residual arises from the mean-curvature term which was neglected in the construction of the bilayer, u_b. This term now becomes a driving force for the evolution of the interface Γ_b through the time derivative in the signed distance function. Indeed, the quantity

$$V_b(s) := -\frac{\partial z}{\partial \tau}, \tag{1.62}$$

is the normal velocity of the interface Γ_b. The asymptotic reduction does lead to a Mullins-Sekerka problem for the far-field chemical potential, however its driving force is given by the interfacial mean curvature times the jump of the bilayer profile across the interface, $H_0(s)[\![U_b]\!]$. Since the bilayer is a homoclinic orbit its jump $[\![U_b]\!] = 0$, and the Mullins-Sekerka problem is trivial. The outer chemical potential reduces to a spatial constant, and the far-field is characterized by amphiphilic density, $\mu_1(\tau)$, whose value in deter-

mined by conservation of total mass, see [Dai and Promislow, 2013] for details for bilayers under the weak functionalization. For the strong functionalization the resulting system takes the form

$$\begin{aligned} V_b &= \nu_b(\mu_1 - \mu_b^*)H_0, \\ \frac{d\mu_1}{dt_1} &= -\nu_b m_b(\mu_1 - \mu_b^*)\int_{\Gamma_b} H_0^2\, dS, \end{aligned} \tag{1.63}$$

where H_0 is the mean curvature, $\nu_b := \frac{m_b}{\int_{\mathbb{R}}(U_b - b_-)^2\, dz} > 0$ and μ_b^* is the optimal far-field amphiphilic density, the same quantity derived by the optimization process in (1.52). The H^{-1} gradient flow drives pure bilayer interfaces by a quenched mean-curvature flow. While the flow drives μ_1 to its optimal value μ_b^*, the sign of the difference $\mu_1 - \mu_b^*$ is consequential. Indeed, in two space dimension, modulo reparameterization of the evolving interface, the curvature driven flow can be recast as an evolution equation of the single curvature H_0,

$$\frac{\partial H_0}{\partial t_1} = -(\partial_s^2 + H_0^2)V_b = \nu_b(\mu_1 - \mu_b^*)(\partial_s^2 + H_0^2)H_0, \tag{1.64}$$

see section 3.3 of [Gavish et al., 2011] for details. If $\mu_1 > \mu_b^*$, that is if the bulk value of amphiphilic material is in excess then the curvature driven flow is a backwards-heat equation in the curvatures. This is the nature of the fingering instability induced in [Budin and Szostak, 2011] when oleo-lipids were added to the bulk of the spherical bilayer suspension. The fingering instability initiates as a backward heat flow in the curvature. The resulting singularity is associated to the development of the pore type growth emanating from the bilayer surface. Moreover, in [Doelman et al., 2014] the condition $\mu_1 > \mu_b^*$ was identified as the point of bifurcation to linear instability of the meander eigenvalues associated to spherical bilayers. For $\mu_1 < \mu_b^*$ the curvature driven flow is locally well-posed but is subject to finite-time blow-up due to the cubic driving force, H_0^3. This is the familiar finite-time extinction of droplets under curvature driven flow. However, for the quenched flow (1.63) the relaxation of μ_1 to its equilibrium value precludes the blow-up if the initial curvatures are not too large.

A similar reduction can be performed for co-dimension two pore structures, parametrized by the one-dimensional immersion Γ_p. The result is a similar quenched curvature driven flow for the vector valued normal velocity $\vec{V}_p = -(\frac{\partial z_1}{\partial \tau}, \frac{\partial z_2}{\partial \tau})^T$,

$$\begin{aligned} \vec{V}_p &= \nu_p(\mu_1 - \mu_p^*)\vec{\kappa}(s), \\ \frac{d\mu_1}{d\tau} &= -\varepsilon m_p(\mu_1 - \mu_p^*)\int_{\Gamma_p} |\vec{\kappa}|^2\, ds, \end{aligned} \tag{1.65}$$

where $\nu_p := \frac{m_p}{\pi\int_0^\infty (U_p')^2 R\, dR} > 0$, $\vec{\kappa}$ is the vector curvature of Γ_p, $m_p := 2\pi\int_0^\infty (U_p - b_-)R\, dR$ is the mass of

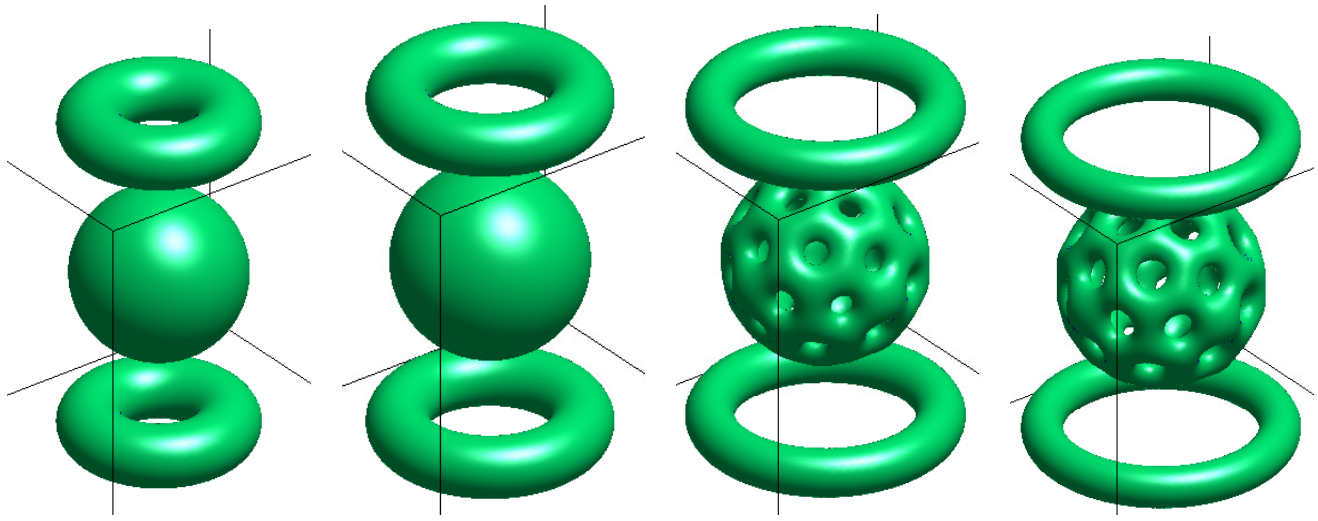


Figure 1.10: Competition for the amphiphilic phase between a spherical bilayer (beach ball) and circular solid pore (hula hoop) as a function of the well tilt $W(b_-) - W(b_+)$. The image shows $t = 100$ end states of the FCH gradient flow (1.17) from identical initial data but with increasing values of the well tilt. Small tilt prefers bilayers, larger tilt prefers pores by increasing μ_b^* and the pearling threshold, P_b^*, which drives bilayers to pearl. Images courtesy of Andrew Christlieb and Jaylan Jones.

amphiphilic material per unit length of pore structure and the equilibrium value

$$\mu_p^* := -\frac{\eta_1}{\alpha_-^2} \frac{\int_0^\infty (U_p')^2 R\, dR}{\int_0^\infty (U_p - b_-)^2 R\, dR}, \tag{1.66}$$

is again independent of Γ_p. Most intriguingly, initial data corresponding to spatially separated pores and bilayers yields a competitive evolution that can be understood as a fight for surfactant, mediated through the common value of the bulk amphiphilic density μ_1, whose evolution is determined to impose the conservation of total mass,

$$\begin{aligned} V_n &= \nu_b(\mu_1 - \mu_b^*)H \\ \vec{V}_p &= \nu_p(\mu_1 - \mu_p^*)\vec{\kappa} \\ \frac{d\mu_1}{dt_1} &= -\nu_b m_b(\mu_1 - \mu_b^*) \int_{\Gamma_b} H_0^2 dS - \varepsilon \nu_p m_p(\mu_1 - \mu_p^*) \int_{\Gamma_p} |\vec{\kappa}|^2 \, ds, \end{aligned} \tag{1.67}$$

The competitive evolution of the bilayers and pores couples through curvature-weighted surface area. However, the two morphologies seek differing equilibria values, which generically satisfy $\mu_b^* \neq \mu_p^*$, making coexistence of bilayers and pores impossible under the strong functionalization, unless one of the structures is flat, since zero curvature interfaces are at equilibrium independent of bulk value of amphiphile. For curved interfaces, the range $\mu_1 \in [\mu_p^*, \mu_b^*]$ is invariant under the flow, and once μ_1 enters this range the bilayers will shrink, while the pore morphologies will grow. Moreover, if the pearling threshold P_b^* lies within the invariant range $[\mu_p^*, \mu_b^*]$ then the value of μ_1 may transiently decrease through the pearling threshold for

bilayers, (1.59), causing the bilayers to pearl as they shrink. Figure 1.9 depicts various $t = 100$ end-states of the FCH gradient flow for a double well potential W with increasing value of well tilt $W(b_+) - W(b_-)$. In all cases the initial data consists of a spherical bilayer and two circular pores placed with antipodal symmetry. Increasing well tilt leads to a pore end state with a larger radius and to pearling of the bilayer. A detailed analysis of the bifurcation structure of the bilayer-pore network morphologies is given in Chapter 7.

Chapter 2

Coordinate System, Definitions and Notation

One main goal of this thesis is to describe the geometric evolution of the functionalized polymer-solvent bilayer and pore morphologies. We introduce the **whiskered coordinate system** which describes the tangential and normal coordinates in a neighborhood of an admissible interface. In Section 2.1 we address the co-dimension one morphology in $\mathbb{R}^d$, $(d \geq 2)$, and establish necessary definitions from elementary differential geometry, and in Section 2.2 we repeat the process for co-dimension two morphology in $\mathbb{R}^3$.

Note 2. *Throughout this thesis we will use subscript b or p to distinguish between quantities associated with the bilayers structures and and those associated with the pore structures, respectively.*

Derivative Notation: Given a function of a single variable, such as $f(x)$, we use $(\cdot)'$ notation to indicate its derivative. e.g., $f' \coloneqq \partial_x f$. If a function involves more then one variable, we specifically write with respect to which variables we differentiate to avoid ambiguity.

2.1 Co-dimension One Morphology in $\mathbb{R}^d$, $d \geq 2$

Admissible co-dimension one interfaces are defined in Definition 1.1. We fix K and ℓ and let $\Gamma_b \in \mathcal{G}_{K,\ell}$ be an admissible, co-dim 1 initial interface, which divides Ω into two disjoint sets, see Figure 1.5. The reach of the interface, $\Gamma_{b,\ell}$ is defined in equation (1.23), and according to equation (1.24), each $x \in \Gamma_{b,\ell}$ can be uniquely expressed using the **whiskered coordinates** such that

$$x = \zeta_b(s, z) \coloneqq \rho_b(s) + \varepsilon z \mathbf{n}(s), \tag{2.1}$$

where ρ_p is a parameterization of Γ_b, $z \in [-l/\varepsilon, l/\varepsilon]$ is the scaled signed distance to Γ_b and $\mathbf{n}$ is the outer normal.

Definition 2.1. *We define* $\vec{k}_b = (k_{b,1}, ...k_{b,d-1})$ *to be the vector of the principle curvatures of* Γ_b.

Let $\mathbf{g}$ the matrix representation of the first fundamental form of Γ_b, whose entries are given by

$$g_{ij} := \left\langle \frac{\partial \rho_b}{\partial s_i}, \frac{\partial \rho_b}{\partial s_j} \right\rangle_{L^2(\mathbb{R}^d)}, \tag{2.2}$$

and the representation of the second fundamental form of Γ_b given by

$$h_{ij} := -\left\langle \frac{\partial \nu}{\partial s_i}, \frac{\partial \rho_b}{\partial s_j} \right\rangle_{L^2(\mathbb{R}^d)}, \tag{2.3}$$

where ν is the Gauss map associated to Γ_b. Then, the Jacobian, $\mathbf{J}_b$, of the transformation $x \to (s, z)$ takes the form

$$\mathbf{J}_b = \left(\frac{\partial \rho_b}{\partial s_1}, ..., \frac{\partial \rho_b}{\partial s_{d-1}}, \mathbf{n} \right) \begin{pmatrix} I_{d-1} - \varepsilon z h_i^j & 0 \\ 0 & \varepsilon \end{pmatrix}, \tag{2.4}$$

where I_{d-1} is the $(d-1) \times (d-1)$ identity matrix, and h_i^j is related to the first and second fundamental forms of Γ_b via the following equation

$$h_i^j = \sum_{m=1}^{d-1} h_{im} g^{mj}. \tag{2.5}$$

The determinant of the Jacobian matrix, $J_b = \det(\mathbf{J}_b)$, satisfies

$$J_b(s, z) = \varepsilon J_0(s) \tilde{J}_b = J_0(\varepsilon + \varepsilon^2 z H_0) + O(\varepsilon^3). \tag{2.6}$$

where J_0 is defined by

$$J_0(s) := \sqrt{\det \mathbf{g}}, \tag{2.7}$$

is related to the matrix representation of the first fundamental form, defined in (2.2), and $\tilde{J}_b$ is given by

$$\tilde{J}_b(s, z) := \prod_{i=1}^{d-1} (1 - \varepsilon z k_{b,i}), \tag{2.8}$$

where $k_{b,i}$ are the principle curvatures of Γ_b. For more details see [Hayrapetyan and Promislow, 2014, Appendix 6].

On the reach $\Gamma_{b,\ell}$, in the whiskered coordinates, the Laplace operator takes the form

$$\varepsilon^2 \Delta_x = \partial_z^2 + \varepsilon H \partial_z + \varepsilon^2 \Delta_G, \tag{2.9}$$

where H is the extended curvature term, given by

$$H := \sum_{j=1}^{d-1} \frac{k_{b,j}}{1 - \varepsilon z k_{b,j}} = \sum_{j=0}^{\infty} H_j(s) \varepsilon^j z^j = H_0(s) + \varepsilon H_1(s) z + O(\varepsilon^2), \tag{2.10}$$

$k_{b,j}$ are the principle curvatures of Γ_b, and H_j are related to the sum of the j^{th} power of the curvatures. To understand the third term on the right hand side of equation (2.9) we first consider the matrix $\mathbf{G} = \mathbf{J}_b^T \mathbf{J}_b$. With this matrix, the generalized Laplace term Δ_G takes the form

$$\Delta_G := J_b^{-1} \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \frac{\partial}{\partial s_i} \mathbf{G}^{ij} J_b \frac{\partial}{\partial s_j}, \tag{2.11}$$

and, according to [Hayrapetyan and Promislow, 2014, Proposition 6.6], the generalized Laplacian term can be written as

$$\Delta_G = \Delta_s + \varepsilon z D_{s,2}. \tag{2.12}$$

Here Δ_s is the **Laplace-Beltrami** operator, define by

$$\Delta_s := J_0^{-1} \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \frac{\partial}{\partial s_i} \mathbf{g}^{ij} J_0 \frac{\partial}{\partial s_j}, \tag{2.13}$$

where $\mathbf{g}$ is the first fundamental form of Γ_b, introduced in equation (2.2), the elements $\mathbf{g}^{ij}$ are the entries of $\mathbf{g}^{-1}$, and $D_{s,2}$ is a 2^{nd} order operator, relatively bounded perturbation of Δ_s, given by

$$D_{s,2} := \sum_{i,j=1}^{d-1} d_{i,j}(s,z) \frac{\partial^2}{\partial s_i \partial s_j} + \sum_{j=1}^{d-1} d_j(s,z) \frac{\partial}{\partial s_j}. \tag{2.14}$$

For admissible Γ_b the coefficients $\{d_{i,j}\}$ and d_j satisfy

$$\max_{i,j} \left(\| \partial_z^m d_{i,j} \|_{L^\infty(\Gamma_l)}, \| \partial_z^m d_j \|_{L^\infty(\Gamma_l)} \right) \leq C \varepsilon^m, \quad \text{for } m = 0, 1, 2 \tag{2.15}$$

for some $C > 0$ independent of ε.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and consider the subspace $H_c^2(\Gamma_{b,\ell})$ where the subscript c denotes compact support within $\Gamma_{b,\ell}$. Then $D_{s,2}$ is a relatively bounded perturbation of Δ_s on $H_c^2(\Gamma_{b,\ell})$.*

The proof of Lemma 2.1 follows from Hölder's Estimates for the second derivatives, given in the following theorem -

Theorem 2.1.1 (Hölder's Estimates for the second derivatives, [Gilbarg and Trudinger, 2001])**.** *Let $u \in C_0^2(\mathbb{R}^n)$, $f \in C_0^\alpha(\mathbb{R}^n)$, satisfy Poisson's equation $\Delta u = f$ in $\mathbb{R}^n$. Then $u \in C_0^{2,\alpha}(\mathbb{R}^n)$ and, if $B = B_R(X_0)$ is any ball containing the support of u, we have*

$$|D^2 u|'_{0,\alpha;B} \le C|f|'_{0,\alpha;B}, \tag{2.16}$$

where $C = C(n,\alpha)$, the Hölder continuous exponent satisfies $0 \le \alpha \le 1$, and the norm is defined by

$$|f|'_{0,\alpha;B} = \sup_{\substack{x,y\in B,\\ x\neq y}} (f(x) - f(y)) + d^\alpha \sup_{\substack{x,y\in B,\\ x\neq y}} \frac{f(x) - f(y)}{|x-y|^\alpha} \tag{2.17}$$

and $d = diam(B)$.

Note 3. *For $\alpha = 1$, the $|\cdot|'_{0,1;B}$ norm is the $W^{1,\infty}(B)$ norm.*

Proof of Lemma 2.1. Fix $f \in C_0^2(\Omega)$ with $supp(f) \subset \Gamma_{b,\ell}$, and $\lambda_* \in \rho(\Delta_s)$. WLOG, take $\lambda_* = 0$. We define the function $u = \Delta_s^{-1} f$. Then, the following calculation shows that we can bound the L^2-norm of $D_{s,2}$ using the L^2-norm of $D^2 u$

$$\begin{aligned}
|D_{s,2}u|'_{0,1;\Omega} &= \left| \left(\sum_{i,j=1}^{d-1} (d_{i,j}(s,z)) \frac{\partial^2}{\partial s_i \partial s_j} + \sum_{j=1}^{d-1} (d_j(s,z)) \frac{\partial}{\partial s_j} \right) u \right|'_{0,1;\Omega} \\
&\le \| d_{i,j}(s,z) \|_{L^\infty(\Gamma_l)} \left| \sum_{i,j=1}^{d-1} \frac{\partial^2}{\partial s_i \partial s_j} u \right|'_{0,1;\Omega} + \| d_j(s,z) \|_{L^\infty(\Gamma_l)} \left| \sum_{j=1}^{d-1} \frac{\partial}{\partial s_j} u \right|'_{0,1;\Omega} \\
&\le \max_{i,j} \left(\| d_{i,j}(s,z) \|_{L^\infty(\Gamma_l)}, \| d_j(s,z) \|_{L^\infty(\Gamma_l)} \right) \left(\left| D^2 u \right|'_{0,1;\Omega} + \left| \sum_{j=1}^{d-1} \frac{\partial}{\partial s_j} u \right|'_{0,1;\Omega} \right) \\
&\le c_1 \left(\left| D^2 u \right|'_{0,1;\Omega} + \left| \sum_{j=1}^{d-1} \frac{\partial}{\partial s_j} u \right|'_{0,1;\Omega} \right) \le c_1 \left(\left| D^2 u \right|'_{0,1;\Omega} + c_2 \left| D^2 u \right|'_{0,1;\Omega} \right) \le C \left| D^2 u \right|'_{0,1;\Omega},
\end{aligned} \tag{2.18}$$

where the third inequality follows from (2.15), the fourth inequality follows from Poincaré inequality and the constants c_1, c_2 and C are independent of ε. Consider the $|\cdot|'_{0,\alpha;B}$ norm of the operator $D^2(\Delta_s)^{-1}$ acting on u and apply (2.16) to u to obtain

$$|D^2 u|'_{0,\alpha;\Omega} \le C|\Delta_s u|'_{0,\alpha;\Omega} = C|f|'_{0,\alpha;\Omega}, \tag{2.19}$$

where the equality follows from replacing $u = \Delta^{-1} f$. Combining equation (2.19) and (2.18) we obtain

$$|D_{s,2}\Delta_s^{-1} f|'_{0,\alpha;\Omega} \leq C|f|'_{0,\alpha;\Omega}. \tag{2.20}$$

In particular, for $\alpha = 1$ we have

$$\left\| D_{s,2}\Delta_s^{-1} f \right\|_{W^{1,\infty}} \leq C \left\| f \right\|_{W^{1,\infty}}. \tag{2.21}$$

We want to show that the following inequality holds

$$\left\| D_{s,2}\Delta_s^{-1} f \right\|_{L^2(\Omega)} \leq C \left\| f \right\|_{L^2(\Omega)}. \tag{2.22}$$

Assuming inequality (2.22) does not holds, i.e.,

$$\left\| D_{s,2}\Delta_s^{-1} f \right\|_{L^2(\Omega)} > C \left\| f \right\|_{L^2(\Omega)}, \tag{2.23}$$

then there exist a sequence $\{f_n\}$ such that

$$\left\| f_n \right\|_{L^2(\Omega)} \to 0, \tag{2.24}$$

$$\left\| D_{s,2}\Delta_s^{-1} f_n \right\|_{L^2(\Omega)} \to 1. \tag{2.25}$$

Since the $W^{1,\infty}$ norm is bounded below by the L^2 norm, equation (2.25) implies that

$$1 \leq \left\| D_{s,2}\Delta_s^{-1} f_n \right\|_{W^{1,\infty}} \leq C \left\| f_n \right\|_{W^{1,\infty}}. \tag{2.26}$$

However, by the Sobolev Embedding Theorem we know that

$$W^{1,\infty} \subset\subset L^p, \tag{2.27}$$

where $\frac{1}{p} = \frac{1}{d}$, and d is the space dimension. The embedding (2.27) implies that there is a subsequence $\{f_{n_k}\}$ such that $\left\| f_{n_k} - f \right\|_{L^p} \to 0$, and $\left\| f \right\|_{L^p} > 1$. However, this contradicts (2.25), and we conclude that

$$\left\| D_{s,2}\Delta_s^{-1} f \right\|_{L^2(\Omega)} \leq C \left\| f \right\|_{L^2(\Omega)}. \tag{2.28}$$

■

To better understand the generalized Laplacian operator, Δ_G, we review some basic facts about the Laplace-

Beltrami operator Δ_s: The eigenvalues, $\{\beta_k\}_{k=0}^{\infty}$, of the $-\Delta_s$, and the corresponding eigenfunctions, $\{\Theta_k\}_{k=0}^{\infty}$, satisfy the following properties :

- $\Delta_s\Theta_k = -\beta_k\Theta_k$,
- $\beta_0 = 0$ and $\beta_k > 0$ for $k > 0$.
- The eigenfunction of Laplace-Beltrami are orthonormal in the Γ_b inner product,

$$(\Theta_k, \Theta_j)_{\Gamma_b} := \int_{\Gamma_b} \Theta_k\Theta_j J_0(s)\, dS = \delta_{k,j}, \tag{2.29}$$

  where J_0 is defined in (2.7).

- According to Weyl's asymptotic formula, [Chavel, 1984], the number of eigenvalues $\leq \lambda$, $N(\beta_n \leq \lambda)$, including multiplicity, satisfy

$$N(\beta_n \leq \lambda) \sim C_1\lambda^{(d-1)/2}. \tag{2.30}$$

  In particular, $\beta_n \sim C_2 n^{2/(d-1)}$, where $C_1, C_2 \in \mathbb{R}$ constants.

Definition 2.2. *Let Γ_b be an admissible interface. We say that $f \in L^1(\Omega)$ is localized on Γ_b if there exist constants $M, \nu > 0$, independent of $\varepsilon > 0$, such that*

$$|f(x(s,z))| \leq Me^{-\nu|z|}, \tag{2.31}$$

for all $x \in \Gamma_{b,\ell}$.

Definition 2.3. *Given a function $f := f(s,z)$ localized on Γ_b we define the jump of f across a given whisker by*

$$[\![f]\!](s) = \lim_{z\to\infty} f(s,z) - \lim_{z\to-\infty} f(s,z). \tag{2.32}$$

Given two functions $f, g \in L^2(\Omega)$ with $supp(f), supp(g) \subset \Gamma_{b,\ell}$ we may change the $L^2(\Omega)$-inner product to the whiskered coordinates

$$(f,g)_{L^2(\Omega)} := \int_\Omega f(x)g(x)\, dx = \int_{\Gamma_b}\int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f(s,z)g(s,z)J_b(s,z)\, dz\, ds, \tag{2.33}$$

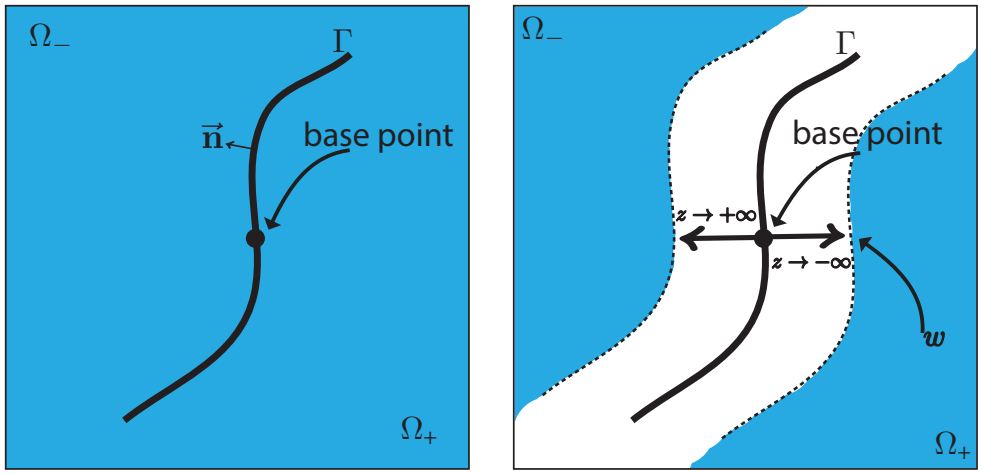
where the Jacobian, J_b, was defined in (2.6). Moreover, integration of a localized function yields

$$\int_\Omega f\, dx = \int_{\Gamma_b}\int_{-\ell/\varepsilon}^{\ell/\varepsilon} f(x(s,z))J_b(s,z)\, dz\, ds + O(\varepsilon^{-\nu l/\varepsilon}). \tag{2.34}$$

We introduce the J_0 inner product, defined as

$$(f,g)_{J_0} := \int_{\Gamma_b} \int_{-\ell/\varepsilon}^{\ell/\varepsilon} f(s,z) g(s,z) J_0 \, dz \, ds. \tag{2.35}$$

Definition 2.4. *For a fixed whisker w, we define the point $(\Gamma_b(s), 0)$ to be its* **base point** *(see Figure 2.1).*



(a) Base Point in Ω **(b)** Base Point in the **whiskered** coordinates

Figure 2.1: Figure (a) is the sharp interface reduction and the base point is a given point $x \in \rho_b(s)$. The white area in Figure (b) is the reach of the interface, $\Gamma_{b,\ell}$, and for the whiskered coordinates, the base point is the intersection point of the whisker with the interface

Lemma 2.2. *The curve length evolves according to*

$$\frac{d|\Gamma_b|}{dt} = \int_{\Gamma_b} VH(s) \, ds. \tag{2.36}$$

We consider the dressing, as defined in Definition 1.2, of an admissible interface, $\Gamma_b \in \mathcal{G}_{K,\ell}$, with the bilayer profile, U_b, which solves

$$\begin{cases} \partial_z^2 U_b = W'(U_b), \\ U_b(\pm\infty) = b_-. \end{cases} \tag{2.37}$$

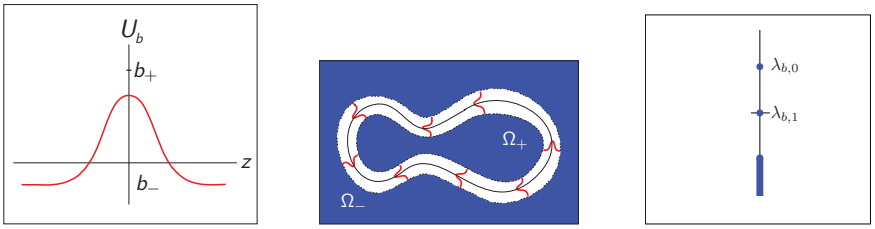
Figure (2.2) depicts the bilayer solution (left) and the dressing of the interface (middle). Observe that U_b is translation invariant, i.e., $U_b(z) \longrightarrow U_b(z+p)$ also solves (2.37). Taking the derivative of (2.37) with respect to z yields

$$L_{b,0} U_b' = 0, \tag{2.38}$$

where $L_{b,0}$ is the linear operator, defined in equation (1.29), given by

$$L_{b,0} \coloneqq \partial_z^2 - W''(U_b). \tag{2.39}$$

From Sturm-Liouville theory, see [Titchmarsh, 1946], we know that the eigenvalue problem $L_{b,0}\hat{\psi}_{b,j} = \lambda_{b,j}\hat{\psi}_{b,j}$



(a) Bilayer Solution U_b **(b)** Dressing Γ_b with U_b **(c)** Spectrum of $L_{b,0}$

Figure 2.2: Subfigure (a) depicts the bilayer profile $U_b(z)$ which converges to b_- as $z \to \infty$. Subfigure (b) describes the dressing of the interface with the bilayer profile U_b marked in red. The blue regions represent the background state and the white region is the neighborhood of the interface $\Gamma_{b,\ell}$. Subfigure (c) depicts the spectrum of $L_{b,0}$, with the vertical axis representing the real line.

has a finite number of simple eigenvalues $\{\lambda_{b,j}\}$, see Figure 2.2 (right). From (2.38) we know that U_b' is an eigenfunction of $L_{b,0}$, and since it has one node, it is the first eigenfunction $\hat{\psi}_{b,1} = U_b'$, i.e., U_b' is the excited-state eigenfunction corresponding to the excited-state eigenvalue $\lambda_{b,1} = 0$. The ground state eigenfunction $\hat{\psi}_{b,0}$ corresponds to the ground state eigenvalue $\lambda_{b,0} > 0$. By *Weyl's essential spectrum theorem*, see [Kato, 1976, Theorem 5.35], the reminder of the spectrum is real, negative and $O(1)$ distance to 0. For further details see Appendix A.2.

We introduce **the co-dimension one, $L^{-j}1$ functions** $\Phi_{b,j} \in L^\infty(\mathbb{R})$ for $j = 1, 2$ which are the solutions of

$$L_{b,0}^j \Phi_{b,j} = 1, \tag{2.40}$$

and are orthogonal to the kernel of $L_{b,0}$. The function $\Phi_{b,1}$ takes the form

$$\Phi_{b,1} = \hat{\Phi}_{b,1} - \frac{1}{\alpha_-}, \tag{2.41}$$

where $\hat{\Phi}_{b,1}$ is the solution of

$$L_{b,0}\hat{\Phi}_{b,1} = \frac{\alpha_- - W''(U_b)}{\alpha_-}, \tag{2.42}$$

and α_- is the well coercivity introduced in equation (1.34). Since $U_b \longrightarrow b_-$ at an exponential rate as $z \longrightarrow \infty$, the right-hand side of (2.42) is in $L^2(\mathbb{R})$, and even about $z = 0$, hence orthogonal to $\ker L_{b,0} = U_b'$. The existence of $\Phi_{b,2}$ follows from a similar argument.

Definition 2.5. *We define the* ***scaled eigenfunctions*** $\psi_{b,k} \coloneqq \chi(z)\tilde{J}^{-1/2}\hat{\psi}_{b,k}$, *where* $\hat{\psi}_{b,k}$ *is the* k^{th} *eigenfunction of* $L_{b,0}$ *and* $\chi(z)$ *is a cut off function,*

$$\chi(z) = \begin{cases} 0 & \text{if } |z| \geq l/\varepsilon, \\ 1 & \text{if } |z| < l/2\varepsilon, \\ \text{monotone in between.} \end{cases} \tag{2.43}$$

The scaled eigenfunctions are orthonormal in the $L^2(\Omega)$-inner product

$$\begin{aligned} \|\psi_{b,k}\|^2_{L^2(\Omega)} &= \int_{\Gamma_b} \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} (\psi_{b,k})^2 J_0(s)\tilde{J}_b \, dz \, ds = \int_{\Gamma_b} \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} (\hat{\psi}_{b,k})^2 \chi^2(z)\tilde{J}_b^{-1} J_0(s)\tilde{J}_b \, dz \, ds \\ &= \int_{\Gamma_b} J_0(s) \, ds \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} (\hat{\psi}_{b,k})^2 \chi^2(z) \, dz \, ds = 1. \end{aligned} \tag{2.44}$$

Definition 2.6. *The full operator is defined by*

$$\mathcal{L}_b \coloneqq L_{b,0} + \varepsilon H \partial_z + \varepsilon^2 \Delta_G, \tag{2.45}$$

where H *and* Δ_G *are given in equations (2.10), (2.11), respectively. The full operator,* $\mathcal{L}_b$*, is self-adjoint in the* $L^2(\Omega)$*-inner product, for more details see appendix (A.3).*

According to [Hayrapetyan and Promislow, 2014], there exists $C > 0$ so that eigenmodes corresponding to eigenvalues from the set $\sigma(\mathcal{L}_b) \cap [-C, C]$ admit the leading order expansion

$$\psi_{b,j,n} = \psi_{b,j}(z)\Theta_n(s) + O(\varepsilon), \tag{2.46}$$

for $j = 0$ or 1. Here the error is in the $L^2(\Omega)$-norm, and we emphasis that the eigenvalues $\psi_{b,j}$ are smoothly extended over the entire domain Ω, see Definition 2.5. Here $\psi_{b,j}$ are the scaled eigenmodes of $L_{b,0}$ introduced in Definition 2.5, the term Θ_n is a Laplace-Beltrami eigenfunction defined in (2.29), and the corresponding eigenvalues take the form

$$\lambda_{b,j,n} = (\lambda_{b,j} + \varepsilon^2 \beta_n) + O(\varepsilon), \tag{2.47}$$

where $\lambda_{b,j}$ are the eigenvalues of $L_{b,0}$ corresponding to the scaled eigenfunctions $\psi_{b,j}$, and β_n are the eigenvalues of $-\Delta_s$, see Figure 2.3 (center) for a depiction of the spectrum of $-\mathcal{L}_b$.

To understand the general structure of the spectrum of the second variation of $\mathcal{F}$ at the bilayer dressing of an admissible interface, we recall that the leading order structure of the second variation of $\mathcal{F}$, $\mathbb{L}_b$, introduced

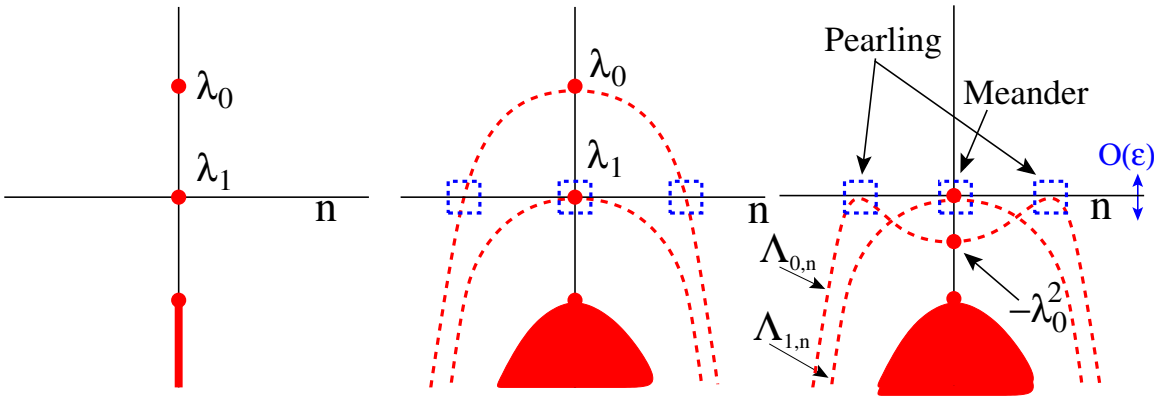


Figure 2.3: The structure of the real spectrum of $-\mathbb{L}_b$, defined in equation (1.54), plotted verses Laplace-Beltrami wavenumber n. (left) The Sturm-Liouville operator $L_{b,0}$, defined in (1.29), has one positive ground state eigenvalue, $\lambda_{b,0} > 0$ and a one dimensional kernel, denoted $\lambda_{b,1}$. (center) The extension of $L_{b,0}$ to $\mathcal{L}_b = L_{b,0} + \varepsilon H \partial_z + \varepsilon^2 \Delta_s$ adds side-bands in n, the Laplace-Beltrami index which bend back negatively at the rate $-(\lambda_{b,0} - \varepsilon^2 \beta_k)^2$. (right) The spectrum of the operator $-\mathbb{L}_b = -\mathcal{L}_b^2 + O(\varepsilon)$, (minus sign chosen to preserve orientation of images) is, to $O(\varepsilon)$, the negative square of the spectrum of $\mathcal{L}_b$. The side-band associated to $\lambda_{b,0}$ has a quadratic tangency at leading order, which may be raised or lowered by the functional terms, η_1 and η_2, the crossing of this spectrum through zero is the mechanism of the pearling instability. Springer and the original publisher [Hayrapetyan and Promislow, 2014], original copyright notice is given to the publication in which the material was originally published, by adding; with kind permission from Springer Science and Business Media

in (1.54), is controlled by $\mathcal{L}_b^2$. The remaining parts of $\mathbb{L}_b$ are relatively bounded and asymptotically small in comparison to $\mathcal{L}_b^2$. The spectral mapping theorem implies that the eigenvalues of $\mathbb{L}_b$ are approximately the square of the eigenvalues of $\mathcal{L}_b$. Figure 2.3 (right) depicts the eigenvalues of the operator $-\mathbb{L}_b$.

The eigenfunctions associated to $\mathbb{L}_b$ corresponding to eigenvalues $\Lambda_{b,j,n} < U$, with $j, n \geq 0$, comprise two sets, the *pearling eigenmodes* $\{\Psi_{b,0,n}\}_{n=N_1}^{N_2}$ and the *meander eigenmodes* $\{\Psi_{b,1,n}\}_{n=0}^{N_3}$, where the index N_3 is the biggest index which satisfies $\Lambda_{b,1,N_3} < U$, and the indices N_1, N_2 are chosen so that N_1 is the first index satisfying $\Lambda_{b,0,N_1} < U$ and N_2 is the biggest index which satisfies $\Lambda_{b,0,N_2} < U$, see Figure 2.4. The indices N_i, $i = 1, 2, 3$ are independent of ε.

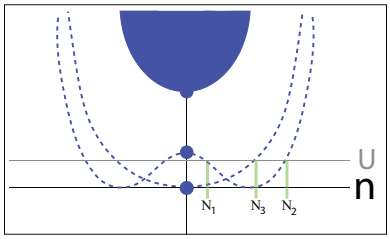


Figure 2.4: The eigenvalues of $\mathbb{L}_b$ with the limit of the meandering eigenmodes, N_3, and the limits of the pearling eigenmodes N_2, N_3.

For $j = 0, 1$ we introduce $\Sigma_{b,j}$, the set of indices n for which $\mathbb{L}_b$ acting on $\psi_j \Theta_n$ is small, i.e.,

$$\Sigma_{b,j} := \{n \mid (\lambda_{b,j} - \varepsilon^2 \beta_n) \sim O(\sqrt{\varepsilon})\}. \tag{2.48}$$

Weyl's asymptotics determine the size of the set $\Sigma_{b,0}$, which satisfies $|\Sigma_{b,0}| \sim O(\varepsilon^{3/2-d}) \gg 1$. We introduce the co-dimension one *meander eigenspace*

$$\mathcal{Y}_{b,me} := span\{\Psi_{b,1,n}\}_{n=0}^{N_3}, \tag{2.49}$$

and the co-dimension two *pearling eigenspace*

$$\mathcal{Y}_{b,pe} := span\{\Psi_{b,0,n}\}_{n=N_1}^{N_2}. \tag{2.50}$$

The co-dimension one morphologies are approximate critical points of the FCH, however they may suffer from both low-frequency (meander or fingering) or high-frequency (pearling) instabilities. We characterize the meander type motion through the bilayer geometric flow in Chapter 3, while the pearling instability of bilayers is characterized in Chapter 5.

2.2 Co-dimension Two Morphology in $\mathbb{R}^3$

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and let $\Gamma_p \subset \mathbb{R}^3$ be a smooth, closed curve, parameterized by ρ_p

$$\Gamma_p = \{\rho_p(s) : [0, L(t)] \to \mathbb{R}^3 \;\Big|\; \rho_p(0) = \rho_p(L(t))\}, \tag{2.51}$$

where s denotes arc-length and L is the total curve length.

At a given point on Γ_p, the unit tangent vector $\mathbf{T}$, the principle normal vector $\mathbf{N}$ and the binormal vector $\mathbf{B}$ defined by

$$\mathbf{T} = \frac{\partial \rho_p}{\partial s}, \tag{2.52}$$

$$\mathbf{N} = \left\| \frac{\partial \mathbf{T}}{\partial s} \right\|^{-1} \frac{\partial \mathbf{T}}{\partial s}, \tag{2.53}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}, \tag{2.54}$$

form the Frenet-Serret frame. we introduce the vectors $\{\mathbf{T}, \mathbf{N}^1, \mathbf{N}^2\}$ which, at each point $\rho_p(s)$ on the

curve Γ_p, form an orthonormal basis for the normal plane and are given by

$$\frac{\partial \mathbf{N}^i}{\partial s} = -\kappa_i \mathbf{T}, \quad i = 1, 2, \tag{2.55}$$

where

$$\vec{\kappa}(s,t) := (\kappa_1, \kappa_2)^t \tag{2.56}$$

is the normal curvature vector with respect to $\{\mathbf{N}^1, \mathbf{N}^2\}$. The local T, N_1, N_2 coordinate system gives a more natural expression for the resulting geometric flow (See [Dai and Promislow, 2015] for further details).

Definition 2.7. *For fixed $K, \ell > 0$ the family $\mathcal{G}^p_{K,\ell}$ of "admissible co-dimension two interfaces" is comprised of smooth, closed curve, 1-dimensional manifolds Γ_p embedded in $\mathbb{R}^3$, which are far from self-intersection and have a smooth second fundamental form.*

The set $\Gamma_{p,\ell}$ defined by

$$\Gamma_{p,\ell} := \left\{ \zeta_p(s,z) \in \mathbb{R}^3 \middle| s \in S, 0 \le z \le \ell/\varepsilon \right\} \subset \Omega, \tag{2.57}$$

will be called the **reach** of Γ_p, where we emphasize that ℓ is fixed independent of ε.
Assume $\Gamma_p \in \mathcal{G}_{p,K,\ell}$ is a co-dimension two admissible interface. Then, by the Implicit Function Theorem each point $x \in \Gamma_{p,\ell}$ is uniquely expressed using the **whiskered coordinates**

$$x = \zeta(s,z) := \rho_p(s,t) + \varepsilon z_1 \mathbf{N}^1(s,t) + \varepsilon z_2 \mathbf{N}^2(s,t), \tag{2.58}$$

where $z = (z_1, z_2)^t$ is the scaled signed distance vector and $t \in [0, \infty]$ represent time, see Figure 2.5.

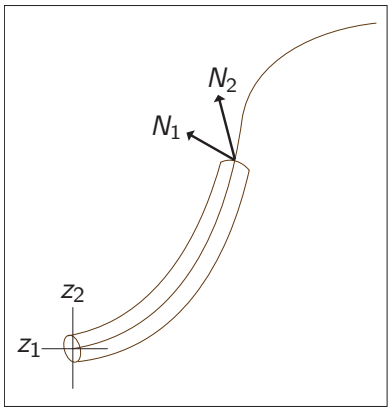


Figure 2.5: Co-dimension 2 whiskered coordinates in $\mathbb{R}^3$

Definition 2.8. *Let $x \in \Gamma_{p,\ell}(t)$ be a point on a given whisker w. We define the point*

$$b_p(x) := \rho_p(s(x,t),t), \tag{2.59}$$

to be the whisker's **base point***, or the base point associated to x.*

Definition 2.9. *For a time-dependent family of admissible surfaces parameterized by $\rho_p(\cdot,t)$, the normal velocity $\mathbf{V} = (V_1, V_2)$ of a point $\rho_p(s,t)$ on Γ_p is defined by*

$$V_i := \mathbf{N}^i \cdot \frac{\partial \rho_p}{\partial t}(s,t) \quad i = 1,2. \tag{2.60}$$

Lemma 2.3 ([Dai and Promislow, 2015])**.** *The $\{\mathbf{T}, \mathbf{N}^1, \mathbf{N}^2\}$ coordinate system satisfies*

$$\frac{\partial \mathbf{T}}{\partial s} = \kappa_1 \mathbf{N}^1 + \kappa_2 \mathbf{N}^2, \tag{2.61}$$

while the curve length evolves according to

$$\frac{d|\Gamma_p|}{dt} = -\int_{\Gamma_p} \mathbf{V} \cdot \vec{\kappa}\, ds. \tag{2.62}$$

Lemma 2.4 ([Dai and Promislow, 2015])**.** *Fix $\Gamma_p \in \mathcal{G}_{p,K,\ell}$ and assume that ℓ is sufficiently small, so that $\|\vec{\kappa}\|_{L^\infty(\Gamma_p)} < 1$. Then, on $\Gamma_{p,\ell}$, in the whiskered coordinates, the Jacobian, J_p, of the transformation $x \to (s,z)$ takes the form*

$$J_p(s,z) = \varepsilon^2 \tilde{J}_p, \tag{2.63}$$

where

$$\tilde{J}_p := (1 - \varepsilon z \cdot \vec{\kappa}), \tag{2.64}$$

and $\vec{\kappa}$ is defined in equation (2.56). Moreover, The Laplace operator takes the form

$$\Delta_x = \varepsilon^{-2}\Delta_z - \varepsilon^{-1} D_z + \partial_G^2 \tag{2.65}$$

where we introduce the operators

$$D_z := \frac{\vec{\kappa}}{\tilde{J}} \cdot \nabla_z, \tag{2.66}$$

$$\partial_G^2 := \partial_s\left(\frac{1}{\tilde{J}^2}\partial_s\right) = \frac{1}{(1-\varepsilon z\cdot\vec{\kappa})^2}\partial_s^2 + \varepsilon\frac{z\cdot\partial_s\vec{\kappa}}{(1-\varepsilon z\cdot\vec{\kappa})^3}\partial_s, \tag{2.67}$$

and the normal velocity $\mathbf{V}$ *takes the form*

$$V_1 = -\varepsilon \frac{\partial z_1}{\partial t} + \varepsilon z_2 \mathbf{N}^2 \cdot \frac{\partial \mathbf{N}^1}{\partial t}, \tag{2.68}$$

$$V_2 = -\varepsilon \frac{\partial z_2}{\partial t} + \varepsilon z_1 \mathbf{N}^1 \cdot \frac{\partial \mathbf{N}^2}{\partial t}. \tag{2.69}$$

For future calculation, we also introduce a more compact version for the Laplacian expansion:

$$\Delta_x = \varepsilon^{-2}\Delta_z - \varepsilon^{-1}\vec{\kappa}\cdot\nabla_z + \partial_s^2 - (z\cdot\vec{\kappa})\vec{\kappa}\cdot\nabla_z + O(\varepsilon). \tag{2.70}$$

We introduce the "co-dimeanion two Laplacian" operator, ∂_s^2, and assume that its eigenvalues, $\{\bar{\beta}_k\}_{k=0}^{\infty}$, and its corresponding eigenfunctions, $\{\bar{\Theta}_k\}_{k=0}^{\infty}$, satisfy the following,

- $\partial_s^2\bar{\Theta}_k = -\bar{\beta}_k\bar{\Theta}_k$,
- $\bar{\beta}_0 = 0$ and $\bar{\beta}_k > 0$ for $k > 0$,
- the eigenfunctions of co-dimeanion two Laplacian are orthonormal in the Γ_p-inner product

$$(\bar{\Theta}_k, \bar{\Theta}_j)_{\Gamma_p} := \int_{\Gamma_p} \bar{\Theta}_k\bar{\Theta}_j \, dS = \delta_{k,j}. \tag{2.71}$$

- Weyl's asymptotic formula, introduced in (2.30), is valid also for the co-dimension two case. For this chapter we fix $d = 3$ and as a result, the number of eigenvalues $\leq \lambda$, $N(\beta_n \leq \lambda)$, including multiplicity, satisfy $N(\beta_n \leq \lambda) \sim C_1\lambda$, and $\beta_n \sim C_2 n$.

Note 4. *By abuse of notation we will drop the bar signs from* $\bar{\beta}_k$ *and* $\bar{\Theta}_k$ *when doing so creates no confusion with* β_k *and* Θ_k *introduced for the co-dimension one interfaces.*

Definition 2.10. *For a radial function* $f : \mathbb{R} \to \mathbb{R}$ *which tends to constant value* f_∞ *at an O(1) exponential rate as* $R \to \infty$*, we say that we* **dress the interface** Γ_p **with** f*, obtaining the* Γ_p*-extended function*

$$f_{\Gamma_p}(x) := f(z(x))\chi(|r(x)|/\ell) + f_\infty(1 - \chi(|r(x)|/\ell)), \tag{2.72}$$

where $\ell > 0$ *is the minimal (unscaled) distance of* Γ_p *to the compliment* $\Gamma_{p,\ell}^c$ *of its neighborhood* $\Gamma_{p,\ell}$ *and* $\chi : \mathbb{R} \to \mathbb{R}$ *is a fixed, smooth cut-off function which is one on* $[0,1]$*, while* $\chi(s) = 0$ *for* $s \geq 2$*. By abuse of notation we will drop the* Γ_p *subscript in the* Γ_p*-extension when doing so creates no confusion.*

Given two functions $f, g \in L^2(\Omega)$ with $supp(f), supp(g) \subset \Gamma_{p,\ell}$ we may change the $L^2(\Omega)$-inner product to

the whiskered coordinates

$$(f,g)_{L^2(\Omega)} \coloneqq \int_\Omega f(x)g(x)\,dx = \int_{\Gamma_p}\int_0^{\ell/\varepsilon} f(s,z)g(s,z)J_p(s,z)\,dz\,ds, \tag{2.73}$$

where the Jacobian, J_p, is defined in (2.63). For two functions $f,g \in L^2(\mathbb{R}^2)$ we may change to polar coordinates and denote the corresponding R-weighted inner product by

$$(f,g)_{L_R} \coloneqq \int_0^{2\pi}\int_0^\infty fgR\,dR\,d\theta. \tag{2.74}$$

We consider the dressing, as defined in Definition 1.2, of an admissible co-dimension two interface, $\Gamma_p \in \mathcal{G}^p_{K,\ell}$, with the pore profile, U_p, which solves

$$\begin{cases} \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R}\right)U_p = W'(U_p), \\ U_p(\infty) = b_-, \\ \frac{\partial U_p}{\partial R}(0) = 0. \end{cases} \tag{2.75}$$

To understand the general structure of the spectrum of $\mathcal{F}$ we consider the second variation of $\mathcal{F}$ at U_p

$$\mathbb{L}_p \coloneqq \mathcal{L}_p^2 + O(\varepsilon), \tag{2.76}$$

where we introduce the full operator

$$\mathcal{L}_p \coloneqq L_p - \varepsilon D_z + \varepsilon^2\partial_G^2. \tag{2.77}$$

To understand the spectrum of $\mathcal{L}_p$ we first investigate the spectrum of the linear operator

$$L_p \coloneqq \partial_R^2 + \frac{1}{R}\partial_R + \frac{1}{R^2}\partial_\theta^2 - W''(U_p), \tag{2.78}$$

We define the spaces $\mathcal{Z}_m$ by

$$\mathcal{Z}_m \coloneqq \{f(R)\cos(m\theta) + g(R)\sin(m\theta) \;\big|\; f,g \in C^\infty(0,\infty), m \in \mathbb{N}\}. \tag{2.79}$$

These spaces are invariant under the operator L_p, and mutually orthogonal in $L^2(\Omega)$. Moreover, on these spaces L_p reduces to

$$L_p(f(R)\cos(m\theta) + g(R)\sin(m\theta)) = \cos(m\theta)L_{p,m}f + \sin(m\theta)L_{p,m}g, \tag{2.80}$$

where

$$L_{p,m} \coloneqq \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{m^2}{R^2} - W''(U_p). \tag{2.81}$$

Each operator $L_{p,m}$ is self-adjoint in the R-weighted inner product, and the operator $L_{p,1}$, introduced in (**??**), has a 1-dimensional kernel spanned by $\partial_R U_p$. For $m > 1$ we observe that $(L_{p,m}f, f)_{L_R} < (L_{p,1}f, f)_{L_R}$ and since $L_{p,1} \leq 0$ we deduce that $L_{p,m} < 0$. In particular the operator $L_{p,m}$ is boundedly invertible for all $m \neq 1$. The operator $L_{p,0}$ satisfied

$$L_{p,0}\psi_{p,0,j} = \lambda_{p,0,j}\psi_{p,0,j}. \tag{2.82}$$

We denote the eigenfunctions and eigenvalues of $L_{p,m}$ by $\{\psi_{p,m,j}\}_{j=0}^{\infty}$ and $\{\lambda_{p,m,j}\}_{j=0}^{\infty}$, respectively.
We differentiate equation (2.75) with respect to R to obtain

$$L_{p,1}U_p' = 0. \tag{2.83}$$

Equation (2.83) implies that the functions $\partial_{z_1}U_p, \partial_{z_2}U_p$ lie in $\ker L_p$.

Assumption 2.2.1. *The operator $L_{p,0}$ has no kernel and it has a 1-dimensional positive eigenspace, i.e., $\lambda_{p,0,0} > 0$ and $\lambda_{p,0,j} < 0$ for every $j \geq 1$.*

It follows from Assumption 2.2.1 that

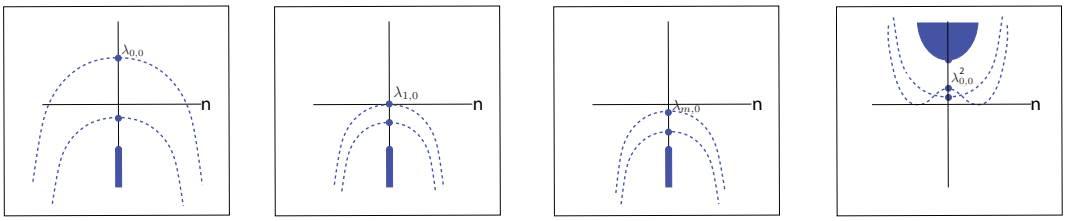
$$\ker(L_p) = span\{\partial_{z_1}U_p, \partial_{z_2}U_p\} = span\{\partial_R U_p \cos\theta, \partial_R U_p \sin\theta\}. \tag{2.84}$$

Under these assumptions, we can write L_p in its block-matrix form

$$L_p = \begin{pmatrix} L_{p,0} & 0 & \\ 0 & L_{p,1} & \\ & & \ddots \end{pmatrix} \tag{2.85}$$

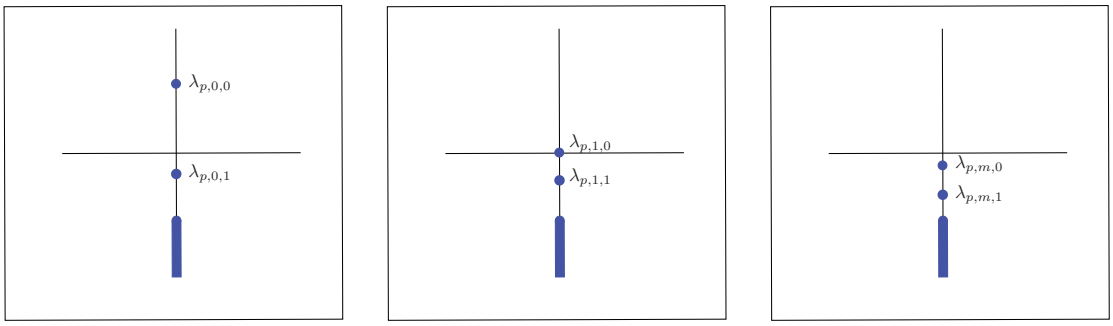
where the spectrum of each operator $L_{p,m}$, $m \geq 0$ is describes in Figure 2.7. The eigenvalues of $\mathcal{L}_p$, defined in (2.77), at leading order, are described in Figure 2.6, where the pearling eigenvalues are the small eigenvalues of the operator $(L_{p,0} + \varepsilon^2\partial_s^2)^2$, see Figure 2.6 (d).
We introduce **the co-dimension two, $L^{-j}1$ functions** $\Phi_{p,j}$, $j = 1, 2$ which solves

$$L_p^j \Phi_{p,j} = 1, \tag{2.86}$$



(a) $\sigma(L_{p,0} + \varepsilon^2\partial_s^2)$ **(b)** $\sigma(L_{p,1} + \varepsilon^2\partial_s^2)$ **(c)** $\sigma(L_{p,m} + \varepsilon^2\partial_s^2)$, $m \geq 2$ **(d)** $\sigma(L_{p,0} + \varepsilon^2\partial_s^2)^2$

Figure 2.6: Full Operators Spectrum



(a) Spectrum of $L_{p,0}$ **(b)** Spectrum of $L_{p,1}$ **(c)** Spectrum of $L_{p,m}$ for $m \geq 2$

Figure 2.7: The spectrum of the sub-operators $L_{p,m}$ for $m = 0, 1, 2$ with the real axis vertical.

and converge exponentially to asymptotic value α_-^{-j} as $R \to \infty$ such that $\Phi_{p,j} - \alpha_-^{-j} \in (\ker L_p)^\perp$, and α_- is the well coercivity, defined in (1.34).

Definition 2.11. *We define the* ***scaled eigenfunctions*** $\psi_{p,k} := \chi(z)J^{-1/2}\psi_{p,0,k}$, *where* $\psi_{p,0,k}$ *is the* k^{th} *eigenfunction of* $L_{p,0}$ *and* $\chi(z)$ *is the cut off function, defined in (2.43)*

The scaled eigenfunctions are orthonormal in the $L^2(\Omega)$-inner product

$$(\psi_{p,k}, \psi_{p,j})_{L^2(\Omega)} = \int_{\Gamma_p} \int_0^{\ell\setminus\varepsilon} \psi_{p,0,k}\psi_{p,0,j}\chi^2(z)\, dz\, ds = 0. \tag{2.87}$$

Note 5. *Any* $f \in L^2(\mathbb{R}^2)$ *admits the Fourier expansion*

$$f = f_0(R) + \sum_{m=1}^{\infty} (f_m(R)\cos(m\theta) + g_m(R)\sin(m\theta)), \tag{2.88}$$

and as long as $\{f_1, g_1\} \perp \ker L_{p,1}$, *we have the inverse formulation*

$$L_p^{-1} f = L_{p,0}^{-1} f_0 + \sum_{m=1}^{\infty} ((L_{p,m}^{-1} f_m(R))\cos(m\theta) + (L_{p,m}^{-1} g_m(R))\sin(m\theta)), \tag{2.89}$$

Assumption 2.2.2. *We assume that the results from [Hayrapetyan and Promislow, 2014] hold for the co-dimension* 2 *morphology.*

Assumption 2.2.2, implies that there exists $C > 0$ such that $\sigma(\mathbb{L}_p) \cap [-C, C]$ have the leading order expansion

$$\Psi_{p,j,n} = (\psi_{p,j}(z)\Theta_n(s))^2 + O(\varepsilon) \tag{2.90}$$

where $\psi_{p,j}$ are the scaled eigenfunctions of L_p, introduced in Definition 2.11, and Θ_n are the eigenfunctions of co-dimeanion two Laplacian, and the corresponding eigenvalues take the form

$$\Lambda_{p,j,n} = (\lambda_{p,j} + \varepsilon^2\beta_n)^2 + O(\varepsilon), \tag{2.91}$$

where $\lambda_{p,j}$ are the eigenvalues of L_p and β_n are the eigenvalues of ∂_s^2.
Similarly to the co-dimension one case, see Figure 2.3 (right), the eigenfunctions associated to $\mathbb{L}_p$ corresponding to eigenvalues $\Lambda_{p,j,n} < U$, with $j, n \geq 0$, comprise two sets, the *pearling eigenmodes* $\{\Psi_{p,0,n}\}_{n=N_1}^{N_2}$ and the *meander eigenmodes* $\{\Psi_{p,1,n}\}_{n=0}^{N_3}$, where the index N_3 is the biggest index which satisfies $\Lambda_{p,1,N_3} < U$, and the indices $N_1,\ N_2$ are chosen so that N_1 is the first index satisfying $\Lambda_{p,0,N_1} < U$ and N_2 is the biggest index which satisfies $\Lambda_{p,0,N_2} < U$, and the indices $N_i,\ i = 1, 2, 3$ are independent of ε. For $j = 0, 1$ we introduce $\Sigma_{p,j}$, the set of indices n for which $\mathbb{L}_p$ acting on $\psi_j\Theta_n$ is small, i.e.,

$$\Sigma_{p,j} := \{n \mid (\lambda_{p,j} - \varepsilon^2\beta_n) \sim O(\sqrt{\varepsilon})\}. \tag{2.92}$$

The size of the set $\Sigma_{p,0}$ follows From Weyl's asymptotic formula, which implies that $|\Sigma_0| \sim O(\varepsilon^{3/2-d}) \gg 1$.

Definition 2.12. *The space, X_Σ, corresponding to the small eigenvalues of $\mathbb{L}$ is defined as*

$$X_\Sigma := \{\psi_0\Theta_k \mid k \in \Sigma\}, \tag{2.93}$$

The L^2-orthogonal projection, Π, onto X_Σ is given by

$$\Pi\ f := \sum_{k\in\Sigma} \frac{(f, \psi_0\Theta_k)_{L^2(\Omega)}}{\|\psi_0\Theta_k\|^2_{L^2(\Omega)}}\psi_0\Theta_k = \sum_{k\in\Sigma}(f, \psi_0\Theta_k)_{L^2(\Omega)}\psi_0\Theta_k, \tag{2.94}$$

and its complementary projection is $\tilde{\Pi} = I - \Pi$.

Assumption 2.2.3. *We assume that the restricted operator $\tilde{\Pi}\mathbb{L}_p\tilde{\Pi}$ is uniformly coercive on $X_\Sigma^\perp$ and its spectrum is bounded from below by $\delta > 0$ which may be chosen independent of sufficiently small $\varepsilon > 0$.*

Assumption 2.2.3 implies that $\sigma(\mathbb{L}_p)/X_\Sigma$ is strictly positive.

We introduce the co-dimension two *meander eigenspace*

$$\mathcal{Y}_{p,me} \coloneqq span\{\Psi_{p,1,n}\}_{n=0}^{N_3}, \tag{2.95}$$

and the co-dimension two *pearling eigenspace*

$$\mathcal{Y}_{p,pe} \coloneqq span\{\Psi_{p,0,n}\}_{n=N_1}^{N_2}. \tag{2.96}$$

Chapter 3

Geometric Evolution of Bilayers in $\mathbb{R}^d$

In this chapter we derive the geometric evolution of admissible co-dimension one interfaces in $\mathbb{R}^d$ under the H^{-1} gradient flow of the **strong FCH**. In contrast to analysis of single-layer interfaces, multi-scale analysis shows that the Stefan and Mullins-Sekerka problemsfor bilayers are trivial, and the sharp interface limit yields a simple, quenched mean curvature-driven normal velocity at leading order. To obtain the flow of the underlying interfacial structure we project the residual $\frac{\delta \mathcal{F}}{\delta u}(u_b)$ of the critical point equation, (1.18), onto the *meandering eigenspace*, defined in (2.49).

Note 6. *By abuse of notation we will drop the b subscript in the u_b critical point when doing so creates no confusion.*

Recall the strong FCH free energy which corresponds to the choice $p = 1$ in (1.14),

$$\mathcal{F} = \int_\Omega \frac{1}{2}(\varepsilon^2 \Delta u - W'(u))^2 - \varepsilon \left(\frac{\varepsilon^2 \eta_1}{2} |\nabla u|^2 + \eta_2 W(u) \right) dx, \tag{3.1}$$

where $\Omega \subset \mathbb{R}^d,\ d \geq 2$ is a bounded domain, $W(u)$ is a tilted double-well potential with two minima at $b_\pm$, $u : \Omega \to \mathbb{R}$ is the density of one of the amphiphilic species, $\varepsilon \ll 1$ controls the width of the boundary layer and η_1 and η_2 are the functionalization constants.

The chemical potential, μ, is defined as the first variation of $\mathcal{F}$,

$$\mu := \frac{\delta \mathcal{F}}{\delta u}(u) = (\varepsilon^2 \Delta - W''(u) + \varepsilon \eta_1)(\varepsilon^2 \Delta u - W'(u)) + \varepsilon \eta_d W'(u), \tag{3.2}$$

where $\eta_d := \eta_1 - \eta_2$. In this chapter we present a formal reduction of the strong FCH equation,

$$u_t = \Delta \left[(\varepsilon^2 \Delta - W''(u) + \varepsilon \eta_1)(\varepsilon^2 \Delta u - W'(u)) + \varepsilon \eta_d W'(u) \right], \tag{3.3}$$

for functions u that are close to a bilayer dressing of an admissible interface in Ω, subject to periodic or zero-flux boundary conditions. We may rewrite the strong FCH equation using the definition of the chemical potential, given in (3.2),

$$u_t = \Delta\mu. \tag{3.4}$$

3.1 Inner and Outer Expansions

Assuming an admissible initial co-dimension one interface $\Gamma_b(t_0) \in \mathcal{G}_{K,\ell}$. We describe the geometric evolution of the interface as a flow in time t, yielding the curve $\Gamma_b(t)$, see Figure 3.1, by performing a multi-scale analysis

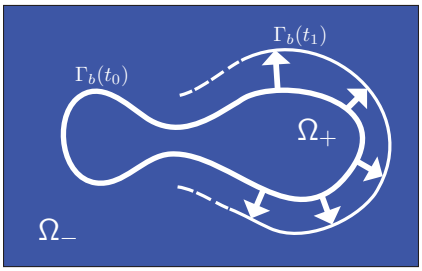


Figure 3.1: The geometric evolution of a generic, admissible, co-dimension one interface, $\Gamma_b(t_0)$ is the initial interface and $\Gamma_b(t_1)$ describes the interface at a later time $t_1 > t_0$.

of the solution u and the chemical potential μ. Away from the interface Γ_b, in the far-field, $\tilde{\Gamma}_{b,\ell}$, the outer solution u and the outer chemical potential μ have the expansions

$$u(x,t) = u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + O(\varepsilon^3), \tag{3.5}$$

$$\mu(x,t) = \mu_0(x,t) + \varepsilon \mu_1(x,t) + \varepsilon^2 \mu_2(x,t) + O(\varepsilon^3). \tag{3.6}$$

In the reach $\Gamma_{b,\ell}$, at a time-scale τ, we have the inner spatial expansions

$$u(x,t) = \tilde{u}(s,z,\tau) = \tilde{u}_0(s,z,\tau) + \varepsilon \tilde{u}_1(s,z,\tau) + \varepsilon^2 \tilde{u}_2(s,z,\tau) + O(\varepsilon^3), \tag{3.7}$$

$$\mu(x,t) = \tilde{\mu}(s,z,\tau) = \tilde{\mu}_0(s,z,\tau) + \varepsilon \tilde{\mu}_1(s,z,\tau) + \varepsilon^2 \tilde{\mu}_2(s,z,\tau) + O(\varepsilon^3). \tag{3.8}$$

The normal velocity V_n of Γ_b at a point $s(t)$ is defined by

$$V_\tau(s) := -\frac{\partial r}{\partial t} = \varepsilon^{-1}\frac{\partial z}{\partial t}. \tag{3.9}$$

where r is the signed distance away from Γ_b and $z = \frac{r}{\varepsilon}$ is the scaled distance. We develop an expression for the time derivative of the outer density function $\tilde{u}$, defined in (3.7), using the whiskered coordinates. To this end, we treat s and z as functions of t and apply the chain rule to obtain

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial s}{\cancel{\partial t}}^{0} \nabla_s \tilde{u} + \frac{\partial \tilde{u}}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial \tilde{u}}{\partial \tau}\frac{\partial \tau}{\partial t}, \tag{3.10}$$

where the first term on the right-hand side of (3.10) is zero since we may reparameterize the evolved curve locally.

Plugging the normal velocity, (3.9), into (3.10) yields

$$\frac{\partial \tilde{u}}{\partial t} = -\varepsilon^{-1} V_\tau(s)\frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{u}}{\partial \tau}\frac{\partial \tau}{\partial t}. \tag{3.11}$$

3.2 Matching Conditions

We connect the inner and outer solutions via matching conditions across the inner-outer boundary. We formally expand the outer solution $u(x,t)$ given in (3.5) and the inner solution $\tilde{u}(s,z,\tau)$ given in (3.7). Fix a whisker, w, and let $x \in \Gamma_b$ be its base point, see Definition 2.4, such that $x + hn \in w$. Then, the matching condition can be written as

$$\lim_{h\to 0} u(x+hn,t) \approx \lim_{z\to\infty} \tilde{u}(s,z,\tau), \tag{3.12}$$

see Figure 2.1. An expansion of the left hand side of equation (3.12) around x, as $h \to 0^+$, is given by

$$u_0^+(x,t) + \varepsilon(u_1^+(x,t) + z\partial_n u_0^+(x,t)) + \varepsilon^2(u_2^+(x,t) + z\partial_n u_1^+(x,t) + z^2\partial_n^2 u_0^+(x,t)) + O(\varepsilon^3), \tag{3.13}$$

where ∂_n is the derivative in the normal direction of Γ_b, and u_i^+ are defined as

$$u_i^+ = \lim_{h\to 0} u_i(x+hn,t), \tag{3.14}$$

for all $i \geq 0$. We can obtain similar expression as $h \to 0^-$. Using (3.7) to expand the right hand side of equation (3.12) and matching it to the left hand side, (3.13), yields the following matching conditions

$$u_0^\pm = \lim_{z\to\pm\infty} \tilde{u}_0, \tag{3.15}$$

$$u_1^\pm + z\partial_{\mathbf{n}} u_0^\pm = \lim_{z\to\pm\infty} \tilde{u}_1. \tag{3.16}$$

Similarly, we can obtain matching conditions for the chemical potential

$$\mu_0^{\pm} = \lim_{z\to\pm\infty} \tilde{\mu}_0, \tag{3.17}$$

$$\mu_1^{\pm} + z\partial_{\mathbf{n}}\mu_0^{\pm} = \lim_{z\to\pm\infty} \tilde{\mu}_1, \tag{3.18}$$

$$\mu_2^{\pm} + z\partial_{\mathbf{n}}\mu_1^{\pm} + \frac{1}{2}z^2\partial_{\mathbf{n}}^2\mu_0^{\pm} = \lim_{z\to\pm\infty} \tilde{\mu}_2, \tag{3.19}$$

$$\mu_3^{\pm} + z\partial_{\mathbf{n}}\mu_2^{\pm} + \frac{1}{2}z^2\partial_{\mathbf{n}}^2\mu_1^{\pm} + \frac{1}{6}z^3\partial_{\mathbf{n}}^2\mu_0^{\pm} = \lim_{z\to\pm\infty} \tilde{\mu}_3. \tag{3.20}$$

3.3 Expansion of the chemical potential

We will also have recourse to the inner and outer expansions of the chemical potential

$$\mu := \left(-\varepsilon^2\Delta + W''(u) - \varepsilon\eta_1\right)\left(-\varepsilon^2\Delta u + W'(u)\right) + \varepsilon\eta_d W'(u). \tag{3.21}$$

3.3.1 Outer Expansion of the Chemical Potential

At a given time scale τ, the outer expansion for the density function $u(x,t)$ is given by equation (3.5). Plugging (3.5) into (3.21) and rewriting the chemical potential μ in orders of ε yields

$$\mu(x,t) = \mu_0(x,\tau) + \varepsilon\mu_1(x,\tau) + \varepsilon^2\mu_2(x,\tau) + ..., \tag{3.22}$$

where

$$\mu_0 = W''(u_0)W'(u_0), \tag{3.23}$$

$$\mu_1 = (W'''(u_0)u_1 - \eta_1)W'(u_0) + (W''(u_0))^2u_1 + \eta_d W'(u_0), \tag{3.24}$$

$$\begin{aligned}\mu_2 = &\left(-\Delta + W'''(u_0)u_2 + \frac{1}{2}W^{(4)}(u_0)u_1\right)W'(u_0) + (W'''(u_0)u_1 - \eta_1)W''(u_0)u_1 \\ &+ W''(u_0)\left(-\Delta u_0 + W''(u_0)u_2 + \frac{1}{2}W'''(u_0)u_1^2\right) + \eta_d W''(u_0)u_1.\end{aligned} \tag{3.25}$$

See Appendix B.1 for further calculation details.

3.3.2 Inner Expansion of the Chemical Potential

At a given time scale τ, the inner expansion for the density function $u(x,t)$ is given by equation (3.7), and in local coordinates, the Laplacian operator, see (2.9), takes the form

$$\varepsilon^2\Delta_x = \partial_z^2 + \varepsilon H_0\partial_z + \varepsilon^2 z H_1\partial_z + \varepsilon^2\Delta_G + O(\varepsilon^3), \tag{3.26}$$

where H_i are defined in (2.10). Plugging (3.7) and (3.26) into (3.21), we can rewrite the chemical potential μ in orders of ε

$$\mu(x,t) = \tilde{\mu}_0(s,z,\tau) + \varepsilon\tilde{\mu}_1(s,z,\tau) + \varepsilon^2\tilde{\mu}_2(s,z,\tau) + \varepsilon^3\tilde{\mu}_3(s,z,\tau) + O(\varepsilon^4), \tag{3.27}$$

where

$$\tilde{\mu}_0 = (-\partial_z^2 + W''(\tilde{u}_0))(-\partial_z^2\tilde{u}_0 + W'(\tilde{u}_0)), \tag{3.28}$$

$$\begin{aligned}\tilde{\mu}_1 =& (-\partial_z^2 + W''(\tilde{u}_0))(-H_0\partial_z\tilde{u}_0 - \partial_z^2\tilde{u}_1 + W''(\tilde{u}_0)\tilde{u}_1) + \\ & (-H_0\partial_z + W'''(\tilde{u}_0)\tilde{u}_1 - \eta_1)(-\partial_z^2\tilde{u}_0 + W'(\tilde{u}_0)) + \eta_d W'(\tilde{u}_0),\end{aligned} \tag{3.29}$$

$$\begin{aligned}\tilde{\mu}_2 =& (-\partial_z^2 + W''(\tilde{u}_0))(-\partial_z^2\tilde{u}_2 - zH_1\partial_z\tilde{u}_0 - H_0\partial_z\tilde{u}_1 - \Delta_s\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W'''(\tilde{u}_0)\tilde{u}_1^2) + \\ & (-H_0\partial_z + W'''(\tilde{u}_0)\tilde{u}_1 - \eta_1)(-\partial_z^2\tilde{u}_1 - H_0\partial_z\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_1) + \\ & (-zH_1\partial_z - \Delta_s + W'''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2)(-\partial_z^2\tilde{u}_0 + W'(\tilde{u}_0)) + \eta_d W''(\tilde{u}_0)\tilde{u}_1,\end{aligned} \tag{3.30}$$

$$\begin{aligned}\tilde{\mu}_3 =& (-\partial_z^2 + W''(\tilde{u}_0))\left(L\tilde{u}_3 - H_0\partial_z\tilde{u}_2 - zH_1\partial_z\tilde{u}_1 - \Delta_s\tilde{u}_1 - \Delta_1\tilde{u}_0 + W'''(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}W^{(4)}(\tilde{u}_0)\tilde{u}_1^3\right) \\ & + (-H_0\partial_z + W'''(\tilde{u}_0)\tilde{u}_1 - \eta_1)\left(L\tilde{u}_2 - H_0\partial_z\tilde{u}_1 - zH_1\partial_z\tilde{u}_0 - \Delta_s\tilde{u}_0 + \frac{1}{2}W'''(\tilde{u}_0)\tilde{u}_1^2\right) \\ & + \left(-zH_1\partial_z - \Delta_s + W'''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2\right)(L\tilde{u}_1 - H_0\partial_z\tilde{u}_0) \\ & + \left(-\Delta_1 + W'''(\tilde{u}_0)\tilde{u}_3 + W^{(4)}(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}W^{(5)}(\tilde{u}_0)\tilde{u}_1^3\right)(-\partial_z^2\tilde{u}_0 + W'(\tilde{u}_0)) \\ & + \eta_d\left(W''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W'''(\tilde{u}_0)\tilde{u}_1^2\right).\end{aligned} \tag{3.31}$$

See Appendix B.3 for further calculation details.

3.4 Time scale $\tau = t$: a Gradient Flow

We start by looking for approximations of the solutions of the strong FCH equation

$$u_t = \Delta_x \overbrace{\left[(\varepsilon^2\Delta - W''(u) + \varepsilon\eta_1)(\varepsilon^2\Delta u - W'(u)) + \varepsilon\eta_d W'(u)\right]}^{\mu} \quad \text{in } \Omega, \tag{3.32}$$

for the time scale $\tau = t$.

3.4.1 Outer expansion

Away from the interface, plugging the outer expansion for the density function $u(x)$ and the outer expansion of the chemical potential μ, given in (3.5) and (3.22), respectively, into the strong FCH equation, (3.32), yields, at leading order, $O(1)$,

$$\frac{\partial u_0}{\partial \tau} = \Delta\left(W''(u_0)W'(u_0)\right) \qquad \text{in} \quad \Omega_- \cup \Omega_+. \tag{3.33}$$

This second order problem has boundary conditions on Ω but to solve it we also need boundary conditions on Γ_b. This leads us to the inner expansion.

3.4.2 Inner expansion

We express each of the terms in (3.32) using the whiskered coordinates. Plugging the inner expansion of u, given in (3.7), into the left-hand side of equation (3.32) yields

$$u_t = -\varepsilon^{-1}V_\tau(s)\frac{\partial \tilde{u}_0}{\partial z} - V_\tau(s)\frac{\partial \tilde{u}_1}{\partial z} + \frac{\partial \tilde{u}_0}{\partial \tau} + O(\varepsilon), \tag{3.34}$$

see Appendix B.2 for calculations details. An expand expression of the Laplacian operator in the whiskered coordinates is given in (3.26) and an expression for the inner expansion of the chemical potential is given in (3.27). Plugging (3.34), (3.26) and (3.27) back into the evolution equation (3.32) and comparing orders of ε yields, at leading order, $O(\varepsilon^{-2})$,

$$0 = \partial_z^2 \tilde{\mu}_0 \quad \text{in } \Gamma_{b,\ell}, \tag{3.35}$$

and at the next order, $O(\varepsilon^{-1})$, we have

$$-V_\tau(s)\partial_z \tilde{u}_0 = \partial_z^2 \tilde{\mu}_1 + H_0 \partial_z \tilde{\mu}_0 \quad \text{in } \Gamma_{b,\ell}. \tag{3.36}$$

Consider the leading order equation, (3.35), and recall that $\tilde{\mu}_0$ is related to $\tilde{u}_0$ through (3.28). Then, equation (3.35) has the solution $\tilde{u}_0 = U_b(z)$ where U_b is the homoclinic profile defined in (2.37). For this choice of $\tilde{u}_0$ it follows that $\tilde{\mu}_0 = 0$ and that $\tilde{\mu}_1$, defined in (3.29), takes the form

$$\tilde{\mu}_1 = L_{b,0}^2 \tilde{u}_1 + \eta_d W'(U_b), \tag{3.37}$$

where the linear operator $L_{b,0}$ is defined in (2.39). Moreover, the next order equation, (3.36), reduces to

$$-V_\tau(s) U_b' = \partial_z^2 \tilde{\mu}_1 \quad \text{in } \Gamma_{b,\ell}, . \tag{3.38}$$

3.4.3 Jump Conditions on the Outer Solution : Gradient Flow

An outer approximation of (3.32) is given in (3.33) which is defined on each domain Ω_- and Ω_+. We would like to solve (3.33) and to connect the two outer solution to obtain a solution over the entire domain Ω. To this end, we use the inner approximation of (3.32) given in equation (3.38) and the matching conditions from Section 3.2 to obtain suitable jump conditions over the interface Γ_b.

Motivated by Definition 2.3 of the interfacial jump, we integrate equation (3.38) with respect to z from $-\infty$ to ∞ obtaining

$$-V_\tau(s) \left(\overbrace{U_b(\infty) - U_b(-\infty)}^{=0} \right) = \partial_z \tilde{\mu}_1 \Big|_{z=\infty} - \partial_z \tilde{\mu}_1 \Big|_{z=-\infty}. \tag{3.39}$$

Since U_b is a homoclinic orbit, equation (3.39) leads to the two key identities

$$\lim_{z\to\infty} \tilde{u}_0(z) - \lim_{z\to-\infty} \tilde{u}_0(z) = \lim_{z\to\infty} U_b(z) - \lim_{z\to-\infty} U_b(z) = 0, \tag{3.40}$$

$$\lim_{z\to\infty} \partial_z \tilde{\mu}_1(z) - \lim_{z\to-\infty} \partial_z \tilde{\mu}_1(z) = 0. \tag{3.41}$$

Differentiating the matching condition (3.18) with respect to z yields

$$\lim_{z\to\pm\infty} \partial_z \tilde{\mu}_1(z) = \partial_{\mathbf{n}} \mu_0^\pm, \tag{3.42}$$

and the combination of equation (3.41) and equation (3.42) implies that the normal derivative of the outer chemical potential is continuous across the interface Γ_b. Similarly, combining the matching condition (3.15) and equation (3.40) we conclude that the outer density function, u_0, is continuous over the interface. we

summarize these results in the **jump conditions** on the outer solution

$$[\![u_0]\!] = 0, \tag{3.43}$$

$$[\![\partial_{\mathbf{n}}\mu_0]\!] = 0. \tag{3.44}$$

Combining the jump condition (3.43) with the outer equation (3.33) implies that u_0 is a solution of (3.33) over the entire domain Ω and that u_0 can be solved independently of Γ_b at this order. The resulting equation for u_0 is

$$\frac{\partial u_0}{\partial t} = \Delta(W''(u_0)W'(u_0)) \qquad \text{in } \Omega, \tag{3.45}$$

subject to the boundary conditions. This evolution equation has is a mass preserving H^{-1} gradient flow on the reduced energy

$$\mathcal{F}_0(u_0) := \int_\Omega \frac{1}{2}(W'(u_0))^2\,dx. \tag{3.46}$$

Consider initial data of the form $u_0 = b_- + v_0$ where $\| v_0 \|_{L^2(\Omega)} \ll 1$, and track the evolution of $v(t) := u(t) - b_-$. Plugging $u = b_- + v$ into equation (3.45) yields the linear evolution equation for v

$$v_t = \alpha_-^2 \Delta\left(v + \frac{W'''(b_-)}{\alpha_-}v^2 + O(v^3)\right), \tag{3.47}$$

where $\alpha_- := W''(b_-)$ is the well-coercivity constant. If $\| v_0 \|_{L^2(\Omega)}$ is sufficiently small, then as long as $\| v \|_{L^2(\Omega)}$ remains small it is plausible that the dynamics of the nonlinear system (3.45) are primarily governed by those of the linear system

$$v_t = \alpha_-^2 \Delta v, \tag{3.48}$$

and it is reasonable to expect that for u_0 close to the equilibria.

For simplicity of presentation, we assume that at leading-order the initial value satisfies

$$u_0(t = 0) = b_-, \tag{3.49}$$

where b_- is the spatial constant, which is an equilibria to equation (3.45).

Return to (3.38) and note that $U_b' = \hat{U}_b'$ where $\hat{U}_b$ is given by

$$\hat{U}_b := U_b - b_-, \tag{3.50}$$

and $\hat{U}_b$ enjoys the property $\hat{U}_b \longrightarrow 0$ as $z \to \pm\infty$. To obtain an expression for the normal velocity, we

integrate (3.38) twice w.r.t z from 0 to z and solve for $\tilde{\mu}_1$ to obtain

$$\tilde{\mu}_1(z) = -V_\tau(s) \int_0^z \hat{U}_b(w)\, dw + z\left(\boxed{V_\tau(s)\hat{U}_b(0)} + \partial_z \tilde{\mu}_1(z)\Big|_{z=0} \right) + \tilde{\mu}_1(0). \tag{3.51}$$

Furthermore, integrating from $z = -\infty$ to $z = 0$ yields the expression

$$-\boxed{V_\tau(s)\hat{U}_b(0)} = \partial_z \tilde{\mu}_1(z)\Big|_{z=0} - \lim_{z\to-\infty} \partial_z \tilde{\mu}_1(z) = \partial_z \tilde{\mu}_1(z)\Big|_{z=0} - \partial_n \mu_0(z), \tag{3.52}$$

for $V_\tau(s)\hat{U}_b(0)$, where the second equality follows from the matching condition (3.18). Since $u_0 \equiv b_-$ we have

$$\mu_0 := W''(b_-)W'(b_-) = 0, \tag{3.53}$$

and equation (3.52) further reduces to

$$\boxed{V_\tau(s)\hat{U}_b(0)} = -\partial_z \tilde{\mu}_1(z)\Big|_{z=0}. \tag{3.54}$$

Using (3.54) to replace $V_\tau(s)\hat{U}_b(0)$ in equation (3.51) yields an expression for the inner chemical potential

$$\tilde{\mu}_1(z) = -V_\tau(s) \int_0^z \hat{U}_b(z)\, dz + \tilde{\mu}_1(0). \tag{3.55}$$

Recall that equation (3.37) relates $\tilde{\mu}_1$ to $\tilde{u}_1$. Plugging (3.37) into (3.55) and solving for $L^2_{b,0}\tilde{u}_1$ yields

$$L^2_{b,0}\tilde{u}_1 = -V_\tau(s) \int_0^z \hat{U}_b(w)\, dw + \tilde{\mu}_1(0) - \eta_d W'(U_b). \tag{3.56}$$

By the Fredholm Alternative, see [Grisvard, 1985], this equation has a solution $\tilde{u}_1 \in L^2(\mathbb{R})$ if and only if the right-hand side is perpendicular to $\ker L_{b,0}$. The solvability condition expressed as

$$\int_{\mathbb{R}} \left(-V_\tau(s) \int_0^z \hat{U}_b(w)\, dw + \tilde{\mu}_1(0) - \eta_d W'(U_b) \right) U_b'\, dz = 0. \tag{3.57}$$

Since U_b' is an odd function it is orthogonal to constants which implies that the integral involving $\tilde{\mu}_1(0)$ is zero. For the η_d term, we evaluate the integral to find

$$\int_{\mathbb{R}} W'(U_b)U_b'\, dz = \int_{\mathbb{R}} (W(U_b))'\, dz = W(U_b)\Big|_{-\infty}^{\infty} = 0, \tag{3.58}$$

where the last inequality follows from the fact that $U_b \to b_-$ as $z \to \pm\infty$. Finally, Integrating the second

integral in the $V_\tau(s)$ term in (3.57) by parts to obtain the equality

$$-V_\tau(s)\int_{\mathbb{R}}\int_0^z U_b(w)\,dw\;\hat{U}_b'(z)\,dz = V_\tau(s)\left\|\hat{U}_b\right\|^2_{L^2(\mathbb{R})}. \tag{3.59}$$

These calculations, combined with the solvability condition (3.57), yield the result

$$V_\tau(s)\left\|\hat{U}_b\right\|^2_{L^2(\mathbb{R})} = 0, \tag{3.60}$$

and since $\left\|\hat{U}_b\right\|^2_{L^2(\mathbb{R})} \neq 0$, it follows that the normal velocity $V_\tau = 0$ at this time scale.

3.5 Time scale $\tau = \varepsilon t$: Mean Curvature Driven Flow

Using the inner equations we obtain jump conditions on the outer solution over the interface and an expression for the normal velocity of the interface. We will see that the reduced system is a trivial Mullins-Sekerka type system and the normal velocity is driven by a curvature-type flow. Finally, we use the mass preserving property of the system to obtain the coupled system for the normal velocity, V_τ, and the external chemical potential, μ_1.

3.5.1 Outer Expansion

Away from the interface, the outer expansion of the density function u is given in (3.5). At this time scale, $\tau = \varepsilon t$, the time derivative ∂_t expands as

$$u_t = \varepsilon u_{0,\tau} + \varepsilon^2 u_{1,\tau} + O(\varepsilon^3). \tag{3.61}$$

Plugging (3.61) and the outer expansion of the chemical potential, μ, given in (3.22), into the strong FCH equation, (3.32), and comparing orders of ε yields, at leading order, $O(1)$,

$$0 = \Delta\left(W''(u_0)W'(u_0)\right) \quad \text{in } \Omega_-\cup\Omega_+, \tag{3.62}$$

and at the next order, $O(\varepsilon)$,

$$u_{0,\tau} = \Delta\left((W'''(u_0)u_1 - \eta_1)W'(u_0) + (W''(u_0))^2 u_1 + \eta_d W'(u_0)\right) \quad \text{in } \Omega_-\cup\Omega_+. \tag{3.63}$$

The first equation, (3.62), is consistent with our assumption that $u_0 = b_-$ and the second equation (3.63) reduces to

$$0 = \alpha_-^2 \Delta u_1 \quad \text{in } \Omega_- \cup \Omega_+, \tag{3.64}$$

and $\alpha_- > 0$ is the well-coercivity defined in (1.34). This second order problem has boundary conditions on Ω, which we supplement with matching conditions on the inner boundary Γ_b that are developed in the next section.

3.5.2 Inner Expansion

We express each of the terms in (3.32) in inner coordinates. Plugging the inner expansion of u, given in (3.7), into the left-hand side of equation (3.32) yields

$$u_t = \frac{\partial r}{\partial \tau} \partial_z \tilde{u}_0 + O(\varepsilon), \tag{3.65}$$

see Appendix B.2 for calculation details. An expand expression of the Laplacian in local coordinates is given in (3.26) and an expression for the inner expansion of the chemical potential is given in (3.27). Plugging (3.65), (3.26) and (3.27) back into the evolution equation (3.32) and comparing orders of ε yields, at leading order, $O(\varepsilon^{-2})$,

$$0 = \partial_z^2 \tilde{\mu}_0 \quad \text{in } \Gamma_{b,\ell}, \tag{3.66}$$

at the next order, $O(\varepsilon^{-1})$,

$$0 = \partial_z^2 \tilde{\mu}_1 + H_0 \partial_z \tilde{\mu}_0 \quad \text{in } \Gamma_{b,\ell}, \tag{3.67}$$

and at $O(1)$ we have

$$-V_\tau(s) \partial_z \tilde{u}_0 = \partial_z^2 \tilde{\mu}_2 + H_0 \partial_z \tilde{\mu}_1 + z H_1 \partial_z \tilde{\mu}_0 + \Delta_G \tilde{\mu}_0 \quad \text{in } \Gamma_{b,\ell}, \tag{3.68}$$

where $\tilde{\mu}_0$, $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are defined in (3.28), (3.29) and (3.30), respectively.

Equation (3.66) is consistent with our choice $\tilde{u}_0 = U_b$, where U_b is the dressing of the interface with the bilayer solution, defined in (2.37). This choice implies that $\tilde{\mu}_0 = 0$. Moreover, for this choice of $\tilde{u}_0$ the next orders equations reduce to

$$0 = \partial_z^2 \tilde{\mu}_1 \quad \text{in } \Gamma_{b,\ell}, \tag{3.69}$$

$$-V_\tau(s) \partial_z \tilde{u}_0 = \partial_z^2 \tilde{\mu}_2 + H_0 \partial_z \tilde{\mu}_1 \quad \text{in } \Gamma_{b,\ell}, \tag{3.70}$$

3.5.2.1 Solving Equation (3.69)

Equation (3.69) is a second order PDE which we want to solve for $\tilde{\mu}_1$. Integrating equation (3.69) twice, w.r.t z, yields

$$\tilde{\mu}_1 = \tilde{A}_1 z + \tilde{B}_1. \tag{3.71}$$

The matching condition (3.18) implies that $\partial_z \tilde{\mu}_1 = \partial_{\mathbf{n}} \mu_0 = 0$ as $z \longrightarrow 0$, which, together with (3.69) implies that $\tilde{A}_1 = 0$ and that $\tilde{\mu}_1$ is independent of z, i.e.,

$$\tilde{\mu}_1 = \tilde{\mu}_1(s,t). \tag{3.72}$$

Since $\tilde{u}_0 = U_b$ we can simplify the inner expression for $\tilde{\mu}_1$ in (3.29) and solve for $L^2_{b,0}\tilde{u}_1$ to obtain

$$L^2_{b,0}\tilde{u}_1 = \tilde{\mu}_1 - \eta_d W'(U_b). \tag{3.73}$$

By the Fredholm Alternative, this equation has a solution $\tilde{u}_1 \in L^2(\mathbb{R})$ if and only if the right-hand side is perpendicular to $\ker L_{b,0}$. Recall that $\ker L_{b,0} = U_b'$ and it is odd about z=0, see the discussion regarding $\sigma(L_{b,0})$ in Chapter 2.1. The fact that the right-hand side of equation (3.73) in perpendicular to U_b' follows from the facts that $\tilde{\mu}_1$ is constant in z, and $W'(U_b)$ is even. Consequently, there exist a solution $\tilde{u}_1$ denoted

$$\tilde{u}_1 = \tilde{\mu}_1 \Phi_{b,2} - \eta_d L^{-2}_{b,0} W'(U_b), \tag{3.74}$$

where $\Phi_{b,2}$ solves $L^2_{b,0}\Phi_{b,2} = 1$, defined in (2.40). Since $W'(U_b) = \partial_z^2 U_b$, using identity (B.43), we can rewrite $\tilde{u}_1$

$$\tilde{u}_1 = \tilde{\mu}_1 \Phi_{b,2} - \eta_d L^{-1}_{b,0}\left(\frac{z}{2}U_b'\right), \tag{3.75}$$

or, alternatively,

$$L_{b,0}\tilde{u}_1 = \tilde{\mu}_1 \Phi_{b,1} - \eta_d \frac{z}{2} U_b'. \tag{3.76}$$

Note that since $\tilde{\mu}_1$ is independent of z, the next order approximation of (3.32), equation (3.70), reduces to

$$-V_\tau(s)\partial_z \tilde{u}_0 = \partial_z^2 \tilde{\mu}_2 \quad \text{in } \Gamma_{b,\ell}. \tag{3.77}$$

3.5.2.2 Jump Conditions

We are looking for a solution of (3.32) in Ω. An outer approximation of (3.32) is given in (3.64) which is defined on each domain, Ω_- and Ω_+. We would like to solve (3.64) and to connect the two outer solution

to obtain a solution over the entire domain Ω. We obtain the jump condition of the external chemical potential μ_1 over the interface from (3.100) and the matching condition (3.18): since $\tilde{\mu}_1$ is independent of z we have

$$\lim_{z\to\pm\infty} \tilde{\mu}_1 = \tilde{\mu}_1(s,t) = \mu_1^{\pm}, \tag{3.78}$$

and the jump condition takes the form

$$[\![\mu_1]\!] = 0. \tag{3.79}$$

To obtain a second jump condition on μ_1, we turn to the Definition 2.3 of the interfacial jump, we integrate equation (3.77) with respect to z from $-\infty$ to ∞ to obtain

$$-V_\tau(s)\left(\overbrace{\hat{U}_b(\infty) - \hat{U}_b(-\infty)}^{=0}\right) = \lim_{z\to\infty} \partial_z\tilde{\mu}_2 - \lim_{z\to-\infty} \partial_z\tilde{\mu}_2 \tag{3.80}$$

Since U_b is a homoclinic orbit, equation (3.80) leads to the key identity

$$\lim_{z\to\infty} \partial_z\tilde{\mu}_2(z) - \lim_{z\to-\infty} \partial_z\tilde{\mu}_2(z) = 0. \tag{3.81}$$

Differentiating the matching condition (3.19) with respect to z yields

$$\lim_{z\to\pm\infty} \partial_z\tilde{\mu}_2(z) = \partial_{\mathbf{n}}\mu_1^{\pm}, \tag{3.82}$$

and the combination of equation (3.81) and equation (3.82) implies that the normal derivative of the outer chemical potential is continuous across the interface Γ_b. we summarize these results in the **jump conditions** on the outer solution

$$[\![\mu_1]\!] = 0, \tag{3.83}$$

$$[\![\partial_{\mathbf{n}}\mu_1]\!] = 0. \tag{3.84}$$

3.5.2.3 The Normal Velocity at $\tau = \varepsilon t$

We would like to determine the evolution of the interface Γ_b. To this end, recall equation (3.77), which involves the normal velocity V_τ, and the inner chemical potential $\tilde{\mu}_2$. The definition of $\tilde{\mu}_2$ is given in (3.30), and since $\tilde{u}_0 = U_b$, (3.30) reduces to

$$\tilde{\mu}_2 = L_{b,0}^2\tilde{u}_2 - L_{b,0}\mathcal{R} + (-H_0\partial_z + W'''(U_b)\tilde{u}_1 - \eta_1)(-L_{b,0}\tilde{u}_1 - H_0U_b') + \eta_d W''(U_b)\tilde{u}_1, \tag{3.85}$$

where

$$\mathcal{R} := -zH_1U_b' - H_0\tilde{u}_1' + \frac{1}{2}W'''(U_b)\tilde{u}_1^2. \tag{3.86}$$

In order to get an expression for the normal velocity we solve (3.77) for $\tilde{\mu}_2$, by integrating (3.77) twice w.r.t z from 0 to z to obtain

$$\tilde{\mu}_2(z) = \tilde{\mu}_2(0) + z(\partial_z\tilde{\mu}_2(0) + \boxed{V_\tau(s)\hat{U}_b(0)}) - V_\tau(s)\int_0^z \hat{U}_b(w)\,dw. \tag{3.87}$$

Furthermore, integrating from $z = -\infty$ to $z = 0$ and recalling that $\hat{U}_b \to 0$ as $z \to \pm\infty$, yields the expression

$$\boxed{V_\tau(s)\hat{U}_b(0)} = \lim_{z\to-\infty} \partial_z\tilde{\mu}_2 - \partial_z\tilde{\mu}_2(0) = \partial_{\mathbf{n}}\mu_1 - \partial_z\tilde{\mu}_2(0). \tag{3.88}$$

for $V_\tau(s)\hat{U}_b(0)$, where the second equality follows from the matching condition (3.19). Using (3.88) to replace $V_\tau(s)\hat{U}_b(0)$ in equation (3.87) yields

$$\tilde{\mu}_2(z) = \tilde{\mu}_2(0) + z\partial_{\mathbf{n}}\mu_1 - V_\tau(s)\int_0^z \hat{U}_b(w)\,dw. \tag{3.89}$$

Replacing $\tilde{\mu}_2$ in (3.89) with its expression from (3.85) and solving for $L_{b,0}^2\tilde{u}_2$ yields

$$L_{b,0}^2\tilde{u}_2 = L_{b,0}\mathcal{R} - (-H_0\partial_z + W'''(U_b)\tilde{u}_1 - \eta_1)(-L_{b,0}\tilde{u}_1 - H_0U_b') - \eta_d W''(U_b)\tilde{u}_1 + \tilde{\mu}_2(0) + z\partial_{\mathbf{n}}\mu_1 - V_\tau(s)\int_0^z \hat{U}_b(w)\,dw \tag{3.90}$$

By the Fredholm Alternative, this equation has a solution $\tilde{u}_2 \in L^2(\mathbb{R})$ if and only if the right-hand side is perpendicular to $\ker L_{b,0}$. Recall that $\ker L_{b,0} = U_b'$ and consider the inner product of U_b' with the right-hand side of equation (3.90). Since $\tilde{u}_1$, defined in (3.74), is even and the operator $L_{b,0}$ preserves symmetry, parity considerations show that the Fredholm condition reduces to

$$\left(-H_0\partial_z L_{b,0}\tilde{u}_1 + H_0W'''(U_b)\tilde{u}_1U_b' - \eta_1H_0U_b' + z\partial_{\mathbf{n}}\mu_1 - V_\tau(s)\int_0^z \hat{U}_b(w)\,dw, U_b'\right)_{L^2(\mathbb{R})} = 0. \tag{3.91}$$

Simplifying the integrals in the inner product, and solving for the normal velocity V_τ yields the expression

$$V_\tau(s) = \frac{(H_0\mu_1 + \partial_{\mathbf{n}}\mu_1)m_b + \frac{1}{2}H_0(\eta_1 + \eta_2)\sigma_b}{B_1}, \tag{3.92}$$

where m_b is the mass amphiphilic material per unit length of bilayer, defined in (1.50), and we introduce the constants

$$B_1 := \left\| \hat{U}_b \right\|_{L^2}^2 , \tag{3.93}$$

$$\sigma_b := \left\| \hat{U}_b' \right\|_{L^2}^2 . \tag{3.94}$$

Detailes of the calculations leading to equation (3.92) can be found in Appendix B.4.

3.5.3 Sharp interface limit : Trivial Mullins-Sekerka and Curvature Driven Flow

The preceding calculation show that, in a neighborhood of the dressed solution, the $\tau = \varepsilon t$ time scale evolution of (3.32) reduces to a Mullins-Sekerka system for the unknown external chemical potential, μ_1.

$$(3.64): \qquad \Delta \mu_1 = 0 \quad \text{in } \Omega_- \cup \Omega_+, \tag{3.95}$$

$$(3.83): \qquad [\![\mu_1]\!] = 0, \tag{3.96}$$

$$(3.84): \quad [\![\partial_n \mu_1]\!] = 0, \tag{3.97}$$

$$(3.92): \qquad V_\tau(s) = \frac{(H_0 \mu_1 + \partial_{\mathbf{n}} \mu_1) m + \frac{1}{2} H_0 (\eta_1 + \eta_2) \sigma_b}{B_1} \quad \text{on } \Gamma. \tag{3.98}$$

The Mullins-Sekerka system (3.95-3.97) is trivial because the jump in the normal derivative of μ_1 balances against the jump of the z derivative of the inner chemical potential $\tilde{\mu}_1$ across the inner structure. This later quantity is zero as the underlying profile is homoclinic. Equations (3.95-3.97) imply that

$$\Delta \mu_1 = 0 \quad \text{in } \Omega, \tag{3.99}$$

and subject to the boundary conditions on $\partial\Omega$ it follows from the maximum principle, [Evans, 2010], that μ_1 is spatially constant, i.e.,

$$\mu_1(x, \tau) = \mu_1(\tau) \quad \forall x \in \Omega. \tag{3.100}$$

Moreover, from equation (3.100), which implies that μ_1 is spatially constant, and the jump condition (3.97), we conclude that μ_1 is continuous over the interface, and

$$\tilde{\mu}_1(\tau) = \mu_1(\tau). \tag{3.101}$$

Combining (3.99) and (3.100), we conclude that $\partial_{\mathbf{n}}\mu_1 = 0$ on Γ_b, and the expression for the normal velocity, (3.98), reduces to a curvature driven expression coupled to the spatially constant chemical potential μ_1

$$V_\tau(s) = \frac{\mu_1 m_b + \frac{1}{2}(\eta_1 + \eta_2)\sigma_b}{B_1} H_0 \quad \text{on } \Gamma_b. \tag{3.102}$$

3.5.4 Equilibria estimate for time scale $\tau = \varepsilon t$

The far-field external chemical potential μ_1 is characterized by the density function, u, whose value is determined by conservation of total mass. For this time scale, we summarize our approximation for the density function in each region. In the outer region $\tilde{\Gamma}_{b,\ell} := \Omega \backslash \Gamma_{b,\ell}$, our assumption that $u_0 = b_-$ combined with equation (3.24) yields

$$u(x,t) = b_- + \varepsilon \frac{\mu_1}{\alpha_-^2} + O(\varepsilon^2) \quad \text{in } \Omega/\Gamma_{b,\ell}, \tag{3.103}$$

where $\alpha_- := W''(b_-) > 0$. In the reach, $\Gamma_{b,\ell}$, our choice $\tilde{u}_0 = U_b$ combined with equation (3.74) yields

$$u(x,t) = U_b + \varepsilon(\mu_1 \Phi_{b,2} - \eta_d L_{b,0}^{-2} W'(U_b)) + O(\varepsilon^2) \quad \text{in } \Gamma_{b,\ell}. \tag{3.104}$$

We use mass balance to determine μ_1 and to obtain the coupled μ_1, V_τ system evolution. The total mass of the system is given by

$$M := \int_\Omega u(x,t) - b_- \, dx = \int_\Omega u(x,0) - b_- \, dx, \tag{3.105}$$

which is fixed by the initial data. Inserting the expressions for the density function, given in (3.103) and (3.104), into the total mass yields

$$M = \varepsilon \int_{\tilde{\Gamma}_{b,\ell}} \frac{\mu_1}{\alpha_-^2} \, dx + \int_{\Gamma_{b,\ell}} \hat{U}_b + \varepsilon(\mu_1 \Phi_{b,2} - \eta_d L_{b,0}^{-2} W'(U_b)) \, dx + O(\varepsilon^2). \tag{3.106}$$

We assume that $|\Gamma| \sim O(1)$, and change to the whiskered coordinates in the localized integral to obtain

$$M = \varepsilon \left(|\Omega| \frac{\mu_1}{\alpha_-^2} + \int_{\Gamma_b} \int_{-\ell/\varepsilon}^{\ell/\varepsilon} \hat{U}_b \, dz \, ds \right) + O(\varepsilon^2). \tag{3.107}$$

We expand $M = \varepsilon \hat{M} + O(\varepsilon^2)$ and the surface area $|\Gamma_b| = \gamma_0 + \varepsilon\gamma_1 + O(\varepsilon^2)$, evaluate the integrals in equation (3.107) and comparing orders of ε yields, at leading order,

$$\hat{M} = |\Omega| \frac{\mu_1}{\alpha_-^2} + \gamma_0 m_b, \tag{3.108}$$

where m_b is defined in (1.50). Moreover, solving for μ_1 yields

$$\mu_1 = \frac{\alpha_-^2}{|\Omega|}\left(\hat{M} - \gamma_0 m\right). \tag{3.109}$$

On the other hand, equation (2.36) implies that, subject to the normal velocity at time scale $\tau = \varepsilon t$, the interfacial surface area growth is given by

$$\frac{\partial|\Gamma_b|}{\partial\tau} = \int_\Gamma V_\tau(s) H_0(s)\, ds, \tag{3.110}$$

so that, subject to (3.98) the interface has the leading order growth

$$\frac{d}{d\tau}\gamma_0 = \frac{\mu_1 m_b + \frac{1}{2}\left(\eta_1+\eta_2\right)\sigma_b}{B_1}\int_\Gamma H_0^2(s)\, ds, \tag{3.111}$$

where B_1 and σ_b are defined in (3.93) and (3.94), respectively. Taking the time derivative, $\frac{d}{d\tau}$, of equation (3.109), solving for $\frac{d}{d\tau}\gamma_0$ and plugging the result expression into (3.111) yields

$$\frac{d}{d\tau}\frac{\hat{M} - |\Omega|\frac{\mu_1}{\alpha_-^2}}{m_b} = \frac{\mu_1 m_b + \frac{1}{2}\left(\eta_1+\eta_2\right)\sigma_b}{B_1}\int_\Gamma H_0^2(s)\, ds, \tag{3.112}$$

and we arrive at the ODE for the chemical potential

$$\frac{d}{d\tau}\mu_1 = -\left(\frac{\mu_1 m_b^2\alpha_-^2}{|\Omega|\, B_1} + \frac{\frac{1}{2}(\eta_1+\eta_2)\sigma_b m_b\alpha_-^2}{|\Omega|\, B_1}\right)\int_\Gamma H_0^2(s)\, ds. \tag{3.113}$$

These results show that the evolution of the interface is governed by the coupled system

$$(3.102)\ :\ V_\tau(s) = \frac{\mu_1 m_b + \frac{1}{2}\left(\eta_1+\eta_2\right)\sigma_b}{B_1} H_0, \tag{3.114}$$

$$(3.113)\ :\ \frac{d}{d\tau}\mu_1 = -\frac{m_b\alpha_-^2}{|\Omega|B_1}\left(\mu_1 m_b + \frac{\eta_1+\eta_2}{2}\sigma_b\right)\int_\Gamma H_0^2(s)\, ds. \tag{3.115}$$

The H^{-1} gradient flow drives pure bilayer interfaces by a quenched mean-curvature flow. While the flow drives the external chemical potential to its equilibria value

$$\mu_1 \longrightarrow -\frac{1}{2}\left(\eta_1+\eta_2\right)\frac{\sigma_b}{m_b}, \tag{3.116}$$

the sign of the right-hand side of (3.114), determined by initial data, is influential. If the right-hand side is positive, motion *against* mean curvature leads the interfacial area to grow uncontrollably, and the reduced

geometric flow is ill-posed. However, the system is a locally well-posed motion by mean-curvature flow for right-hand side is negative. While mean-curvature driven flows can exhibit finite-time singularities, in the quenched flow the singularity can be arrested by the decay of μ_1 to its equilibria value. Since $\mu_1 = \alpha_-^2 u_1$, the density function decays to

$$u_1 \longrightarrow -\frac{1}{2}\left(\eta_1 + \eta_2\right)\frac{\sigma_b}{m\alpha_-^2}, \tag{3.117}$$

and the far-field behaviour of the density, u, takes the form

$$u = b_- - \varepsilon\frac{1}{2}\left(\eta_1 + \eta_2\right)\frac{\sigma_b}{m\alpha_-^2} + O(\varepsilon^2). \tag{3.118}$$

Assuming the system decay to an equilibria with an admissible interface Γ_b, then the analysis can be continued to the next time-scale, however our goal is to investigate the coupled bilayer-pore evolution which occurs at this time scale.

Chapter 4

Geometric Evolution of Pores in $\mathbb{R}^3$

In this chapter we derive the geometric evolution of admissible co-dimension two interfaces in $\mathbb{R}^3$ under the H^{-1} gradient flow of the **strong FCH**. Using multi-scale analysis we derive an expression for the curvature-driven normal velocity at $O(\varepsilon^{-1})$ time scale. We describe the competitive evolution of disjoint collections of bilayers and pores which couple through curvature-weighted surface area, and show that, generically, the two morphologies seek different equilibria values, making coexistence of bilayers and pores impossible under the **strong functionalization**, unless one of the structures is flat, since zero curvature interfaces are at equilibrium independent of bulk value of amphiphile.

Recall the strong FCH free energy which corresponds to the choice $p = 1$ in (1.14),

$$\mathcal{F}(u) = \int_\Omega \frac{1}{2}(\varepsilon^2\Delta u - W'(u))^2 - \varepsilon\left(\frac{\varepsilon^2\eta_1}{2}|\nabla u|^2 + \eta_2 W(u)\right) dx, \tag{4.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $W(u)$ is a tilted double-well potential with two minima at $b_\pm$, $u : \Omega \to \mathbb{R}$ is the density of one of the amphiphilic species, $\varepsilon \ll 1$ controls the width of the boundary layer and η_1 and η_2 are the functionalization constants.

Note 7. *By abuse of notation we will drop the p subscript in the u_p critical point when doing so creates no confusion.*

The chemical potential, μ, is defined as the first variation of $\mathcal{F}$,

$$\mu := \frac{\delta\mathcal{F}}{\delta u}(u) = (\varepsilon^2\Delta - W''(u) + \varepsilon\eta_1)(\varepsilon^2\Delta u - W'(u)) + \varepsilon\eta_d W'(u), \tag{4.2}$$

where $\eta_d := \eta_1 - \eta_2$. In this chapter we present a formal reduction of the strong FCH equation,

$$u_t = \Delta\left[(\varepsilon^2\Delta - W''(u) + \varepsilon\eta_1)(\varepsilon^2\Delta u - W'(u)) + \varepsilon\eta_d W'(u)\right], \tag{4.3}$$

for functions u that are close to a bilayer dressing of an admissible interface in Ω, subject to periodic or zero-flux boundary conditions. We may rewrite the strong FCH equation using the definition of the chemical potential, given in (4.2),

$$u_t = \Delta\mu. \tag{4.4}$$

4.1 The whiskered coordinate system and inner-expansions

Assuming an admissible initial co-dimension two interface $\Gamma_p(t_0)$, we perform a multi-scale analysis of the solution u. Away from the interface, in the far-field region $\tilde{\Gamma}_{p,\ell}$, the outer solution u has the expansion

$$u(x,t) = u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + O(\varepsilon^3). \tag{4.5}$$

On the reach, $\Gamma_{p,\ell}$, at a given time-scale τ, the outer solution's inner expansion takes the form

$$u(x,t) = \tilde{u}(s,z,\tau) = \tilde{u}_0(s,z,\tau) + \varepsilon\tilde{u}_1(s,z,\tau) + \varepsilon^2\tilde{u}_2(s,z,\tau) + O(\varepsilon^3). \tag{4.6}$$

4.2 Matching Conditions

Fix a whisker, w, and let $x \in \Gamma_p$ be its base point. We take two vectors $\mathbf{n}, \mathbf{m} \in \text{span}\{\mathbf{N}^1, \mathbf{N}^2\}$ in the normal plane of Γ_p at x, and specify that

$$\mathbf{n} = \cos(\theta)\mathbf{N}^1 + \sin(\theta)\mathbf{N}^2. \tag{4.7}$$

The usual directional derivative along $\mathbf{n}$ is denoted

$$\partial_{\mathbf{n}} := \mathbf{n}\cdot\nabla_x = \cos(\theta)\mathbf{N}^1\cdot\nabla_x + \sin(\theta)\mathbf{N}^2\cdot\nabla_x, \tag{4.8}$$

and for $f \in C^\infty(\Omega/\Gamma_p)$ we introduce the $\mathbf{n},\mathbf{m}$ limit

$$\partial_{\mathbf{n}}^j f^{\mathbf{m}}(x) := \lim_{h\to 0^+} (\mathbf{n}\cdot\nabla_x)^j f(x + h\mathbf{m}, t) \quad \text{for all } j \geq 0, \tag{4.9}$$

and the limit of the gradient

$$\nabla_x f^{\mathbf{m}}(x) \coloneqq \lim_{h\to 0^+} \nabla_x f(x + h\mathbf{m}, t), \tag{4.10}$$

where the limit exists. If $f \in C^1(\Omega)$ then the normal derivative of f will satisfy

$$\partial_{\mathbf{n}} f^{-\mathbf{m}} = \partial_{\mathbf{n}} f^{\mathbf{m}}. \tag{4.11}$$

This motivates the following definition of the jump condition

Definition 4.1. *Given a radial function $f \coloneqq f(R)$ localized on Γ_p, we define the jump of f across a given whisker by*

$$[\![\partial_{\mathbf{n}} f^{\mathbf{m}}]\!]_{\Gamma_p}(x) \coloneqq \partial_{\mathbf{n}} f^{\mathbf{m}}(x) - \partial_{\mathbf{n}} f^{-\mathbf{m}}(x) \tag{4.12}$$

which is zero when f has a smooth extension through Γ_p.

With this notation we examine the matching condition

$$u(x + h\mathbf{n}, t) \approx \tilde{u}(s, R, \theta, \tau). \tag{4.13}$$

An expansion of the left-hand side of equation (4.13) around x, as $h \to 0^+$, is given by

$$u^{\mathbf{n}}(x,t) + \varepsilon(u_1^{\mathbf{n}}(x,t) + z\partial_{\mathbf{n}} u_0^{\mathbf{n}}(x,t)) + \varepsilon^2(u_2^{\mathbf{n}}(x,t) + z\partial_{\mathbf{n}} u_1^{\mathbf{n}}(x,t) + \frac{1}{2}z^2\partial_{\mathbf{n}}^2 u_0^{\mathbf{n}}(x,t)) + O(\varepsilon^3), \tag{4.14}$$

and equating orders of ε the matching condition (4.13) yields

$$u_0^{\mathbf{n}} = \lim_{R\to\infty} \tilde{u}_0(s, R, \theta, \tau), \tag{4.15}$$

$$u_1^{\mathbf{n}} + R\partial_{\mathbf{n}} u_0^{\mathbf{n}} = \lim_{R\to\infty} \tilde{u}_1(s, R, \theta, \tau). \tag{4.16}$$

Similarly, we can obtain matching conditions for the chemical potential

$$\mu_0^{\mathbf{n}} = \lim_{R\to\infty} \tilde{\mu}_0, \tag{4.17}$$

$$\mu_1^{\mathbf{n}} + R\partial_{\mathbf{n}}\mu_0^{\pm} = \lim_{R\to\infty} \tilde{\mu}_1, \tag{4.18}$$

$$\mu_2^{\mathbf{n}} + R\partial_{\mathbf{n}}\mu_1^{\mathbf{n}} + \frac{1}{2}R^2\partial_{\mathbf{n}}^2\mu_0^{\mathbf{n}} = \lim_{R\to\infty} \tilde{\mu}_2, \tag{4.19}$$

$$\mu_3^{\mathbf{n}} + R\partial_{\mathbf{n}}\mu_2^{\mathbf{n}} + \frac{1}{2}R^2\partial_{\mathbf{n}}^2\mu_1^{\mathbf{n}} + \frac{1}{6}R^3\partial_{\mathbf{n}}^2\mu_0^{\mathbf{n}} = \lim_{R\to\infty} \tilde{\mu}_3. \tag{4.20}$$

4.3 Expansion of the Chemical Potential

We will have recourse to the inner and outer expansions of the chemical potential

$$\mu := \left(-\varepsilon^2\Delta + W''(u) - \varepsilon\eta_1\right)\left(-\varepsilon^2\Delta u + W'(u)\right) + \varepsilon\eta_d W'(u), \tag{4.21}$$

4.3.1 Outer Expansion of the Chemical Potential

At a given time scale τ, the outer expansion for the density function $u(x,t)$ is given by equation (4.5). Plugging (4.5) into (4.21) and rewriting the chemical potential μ in orders of ε yields

$$\mu(x,t) = \mu_0(x,\tau) + \varepsilon\mu_1(x,\tau) + \varepsilon^2\mu_2(x,\tau) + ..., \tag{4.22}$$

where

$$\mu_0 = W''(u_0)W'(u_0), \tag{4.23}$$

$$\mu_1 = (W'''(u_0)u_1 - \eta_1)W'(u_0) + (W''(u_0))^2 u_1 + \eta_d W'(u_0), \tag{4.24}$$

$$\begin{aligned}\mu_2 = &\left(-\Delta + W'''(u_0)u_2 + \frac{1}{2}W^{(4)}(u_0)u_1\right)W'(u_0) + (W'''(u_0)u_1 - \eta_1)W''(u_0)u_1 \\ &+ W''(u_0)\left(-\Delta u_0 + W''(u_0)u_2 + \frac{1}{2}W'''(u_0)u_1^2\right) + \eta_d W''(u_0)u_1.\end{aligned} \tag{4.25}$$

Note that the outer expansion of the chemical potential is identical for both co-dimension one and co-dimension two.

4.3.2 Inner Expansion of the Chemical Potential

At a given time scale τ, the inner expansion for the density function $u(x,t)$ is given by equation (4.6), and in local coordinates, the Laplacian is given in equation (2.70). Plugging (4.6) and (2.70) into (4.21), we can rewrite the chemical potential μ in orders of ε

$$\mu(x,t) = \tilde{\mu}_0(s,z,\tau) + \varepsilon\tilde{\mu}_1(s,z,\tau) + \varepsilon^2\tilde{\mu}_2(s,z,\tau) + O(\varepsilon^3), \tag{4.26}$$

where

$$\tilde{\mu}_0 = (-\Delta_z + W''(\tilde{u}_0))(-\Delta_z\tilde{u}_0 + W'(\tilde{u}_0)), \tag{4.27}$$

$$\tilde{\mu}_1 = (-\Delta_z + W''(\tilde{u}_0))(-\Delta_z\tilde{u}_1 + \vec{\kappa}\cdot\nabla_z\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_1) + \tag{4.28}$$

$$
\begin{aligned}
&(\vec{\kappa}\cdot\nabla_z + W'''(\tilde{u}_0)\tilde{u}_1 - \eta_1)(-\Delta_z\tilde{u}_0 + W'(\tilde{u}_0)) + \eta_d W'(\tilde{u}_0)\\
\tilde{\mu}_2 =&(-\Delta_z + W''(\tilde{u}_0))(-\Delta_z\tilde{u}_2 + \vec{\kappa}\cdot\nabla_z\tilde{u}_1 + \partial_s^2\tilde{u}_0 - (z\cdot\vec{\kappa})\vec{\kappa}\cdot\nabla\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W'''(\tilde{u}_0)\tilde{u}_1^2) +\\
&(\vec{\kappa}\cdot\nabla_z + W'''(\tilde{u}_0)\tilde{u}_1 - \eta_1)(-\Delta_z\tilde{u}_1 + \vec{\kappa}\cdot\nabla_z\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_1) +\\
&(-\partial_s^2 - (z\cdot\vec{\kappa})\vec{\kappa}\cdot\nabla_z + W'''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2)(-\Delta_z\tilde{u}_0 + W'(\tilde{u}_0)) + \eta_d W''(\tilde{u}_0)\tilde{u}_1
\end{aligned}
\tag{4.29}
$$

4.4 Time scale $\tau = \varepsilon^{-2}t$

On the fast time scale, the initial data is expected to relax into an equilibria solution. We start by looking for approximations of the solutions of the strong FCH equation

$$
u_t = \Delta_x \overbrace{\left[(\varepsilon^2\Delta - W''(u) + \varepsilon\eta_1)(\varepsilon^2\Delta u - W'(u)) + \varepsilon\eta_d W'(u)\right]}^{\mu} \quad \text{in } \Omega, \tag{4.30}
$$

for the time scale $\tau = \varepsilon^{-2}t$.

4.4.1 Outer expansion

Away from the pore Γ_p, plugging the outer expansion for the density function $u(x)$ and the outer expansion of the chemical potential μ, given in (4.5) and (4.22), respectively, into the strong FCH equation, (4.30), and equating orders of ε yields

$$
O(\varepsilon^{-2}) \ : u_{0,\tau} = 0, \tag{4.31}
$$

$$
O(\varepsilon^{-1}) \ : u_{1,\tau} = 0, \tag{4.32}
$$

$$
O(1) \ : u_{2,\tau} = \Delta\left(W''(u_0)W'(u_0)\right) \qquad \text{in} \quad \tilde{\Gamma}_{p,\ell}. \tag{4.33}
$$

On the $\tau = \varepsilon^{-2}t$ time scale, the solution u is stationary to first and second order. Equation (4.33) has boundary conditions on Ω but to solve it we also need boundary conditions on Γ_p. This leads us to the inner expansion.

4.4.2 Inner expansion

We express each of the terms in (4.30) in whiskered coordinates. Plugging the inner expansion of u, given in (4.6), into the left-hand side of equation (4.30), the time derivative of u takes the form

$$
u_t = \varepsilon^{-2}\left(\nabla_z\tilde{u}\frac{\partial z}{\partial\tau} + \partial_s\tilde{u}\frac{\partial s}{\partial\tau} + \tilde{u}_\tau\right). \tag{4.34}
$$

In light of the normal velocity relations given in equations (2.68) and (2.69), equation (4.34) reduces to

$$u_t = \varepsilon^{-3}\mathbf{V}\cdot\nabla_z\tilde{u}_0 + \varepsilon^{-2}\left(\left(z_2N^2\cdot\frac{\partial N_1}{\partial\tau}, z_1N^1\cdot\frac{\partial N_2}{\partial\tau}\right)\cdot\nabla_z\tilde{u} + \partial_s\tilde{u}\frac{\partial s}{\partial\tau} + \tilde{u}_\tau\right) + O(\varepsilon^{-1}) \tag{4.35}$$

An expression of the Laplacian in local coordinates is given in (2.70) and an expression for the inner expansion of the chemical potential is given in (4.26). Plugging (4.35), (2.70) and (4.26) back into the evolution equation (4.30) and comparing orders of ε yields,

$$O(\varepsilon^{-3}): \quad \mathbf{V}\cdot\nabla_z\tilde{u}_0 = 0, \tag{4.36}$$

$$O(\varepsilon^{-2}): \quad \mathbf{V}\cdot\nabla_z\tilde{u}_1 + \partial_s\tilde{u}_0\frac{\partial s}{\partial\tau} + \tilde{u}_{0,\tau} + z_2N^2\cdot\frac{\partial N_1}{\partial\tau}\frac{\partial\tilde{u}_0}{\partial z_1} + z_1N^1\cdot\frac{\partial N_2}{\partial\tau}\frac{\partial\tilde{u}_0}{\partial z_2} = \Delta_z\tilde{\mu}_0, \tag{4.37}$$

where $\tilde{\mu}_0$ is given in (4.27). We are interested in non-trivial solutions based upon a quasi-stationary radial profile, consequently we assume that the transient dynamics on the τ time scale have equilibrated, that is $\mathbf{V} = 0$ and all τ partial are zero, so that the system of equations reduces to

$$0 = \Delta_z\tilde{\mu}_0. \tag{4.38}$$

These assumptions are consistent with equilibria which at leading order have radially symmetric profiles that render $\tilde{\mu}_0 = 0$.

The next time scale $\tau = \varepsilon^{-1}t$ yields the same results and we skip the calculations.

4.5 Time Scale $\tau = t$: Sharp Interface Limit

Recall that we are looking for approximations for solutions of the strong FCH equation

$$u_t = \Delta_x\overbrace{\left[(\varepsilon^2\Delta - W''(u) + \varepsilon\eta_1)(\varepsilon^2\Delta u - W'(u)) + \varepsilon\eta_d W'(u)\right]}^{\mu} \quad \text{in } \Omega. \tag{4.39}$$

For the time scale $\tau = t$. We will obtain an evolution equations for the outer and inner regions.

4.5.1 Outer Expansion

Away from the interface, the outer expansion of the density function u is given in (4.5), and the outer expansion of the chemical potential, μ, given in (4.22). Plugging (4.5) and (4.22) into (4.39) yields at leading

order, $O(1)$,

$$u_{0,\tau} = \Delta\mu_0 \quad \text{in } \tilde{\Gamma}_{p,\ell}, \tag{4.40}$$

where μ_0 is given in (4.23).

4.5.2 Inner Expansion

We express each of the terms in (4.39) in inner coordinates. Plugging the inner expansion of u, given in (4.6), into the left-hand side of equation (4.39) yields

$$u_t = \tilde{u}_\tau + \tilde{u}_s \frac{\partial s}{\partial \tau} + \frac{\partial z}{\partial \tau} \cdot \nabla_z \tilde{u} = -\varepsilon^{-1} \mathbf{V} \cdot \nabla_z \tilde{u}_0 + O(1). \tag{4.41}$$

An expand expression of the Laplacian in local coordinates is given in (2.70) and an expression for the inner expansion of the chemical potential is given in (4.26). Plugging (4.41), (2.70) and (4.26) back into the evolution equation (4.39) and comparing orders of ε yields,

$$O(\varepsilon^{-2}) \ : 0 = \Delta_z \tilde{\mu}_0 \quad \text{in } \Gamma_{p,\ell}, \tag{4.42}$$

$$O(\varepsilon^{-1}) \ : \ -\mathbf{V} \cdot \nabla_z \tilde{u}_0 = \Delta_z \tilde{\mu}_1 - \vec{\kappa} \cdot \nabla_z \tilde{\mu}_0, \quad \text{in } \Gamma_{p,\ell}. \tag{4.43}$$

Equation (4.43) has the solution $\tilde{u}_0 = U_p$ where U_p is the pore profile defined in (2.75). For this choice of $\tilde{u}_0$ it follows that $\tilde{\mu}_0 = 0$ and that $\tilde{\mu}_1$, defined in (4.28), takes the form

$$\tilde{\mu}_1 = L_{p,0}^2 \tilde{u}_1 + \eta_d W'(U_b), \tag{4.44}$$

where $L_{p,0}$ is defined in (2.81). Moreover, the next order equation, (4.43), reduces to

$$-\mathbf{V} \cdot \nabla_z \tilde{u}_0 = \Delta_z \tilde{\mu}_1, \quad \text{in } \Gamma_{p,\ell}. \tag{4.45}$$

4.5.3 Jump Conditions

We would like to determine the normal velocity of the interface Γ_p for the time scale $\tau = t$. To this end, we need to determine an explicit solution for $\tilde{\mu}_1$, in equation (4.45), subject to the matching conditions with the outer solution, given in Section 4.2.

Turning to polar coordinates, we use the Fourier mode expansion, (2.88), to obtain an express for $\tilde{\mu}_1$,

$$\tilde{\mu}_1 = A_1(s,R)\cos\theta + B_1(s,R)\sin\theta + C(s,R) + \xi(s,R,\theta), \tag{4.46}$$

where

$$\xi(s,R,\theta) = \sum_{m=2}^{\infty} \left(A_m(s,R)\cos(m\theta) + B_m(s,R)\sin(m\theta)\right). \tag{4.47}$$

From the matching condition (4.18) we see that $\tilde{\mu}_1$ grows at most linearly as $R \to \infty$ and

$$\lim_{R\to\infty} \frac{\partial \tilde{\mu}_1}{\partial R} = \partial_{\mathbf{n}} \mu_0^{\mathbf{n}}. \tag{4.48}$$

Using the definition of the directional derivative along $\mathbf{n}$, given in (4.8), we rewrite equation (4.48) in terms of the sine and cosine functions

$$\lim_{R\to\infty} \frac{\partial \tilde{\mu}_1}{\partial R} = \cos\theta \mathbf{N}^1 \cdot \nabla_z \mu_0^{\mathbf{n}} + \sin\theta \mathbf{N}^2 \cdot \nabla_z \mu_0^{\mathbf{n}}. \tag{4.49}$$

Taking the R derivative of (4.46) yields

$$\frac{\partial \tilde{\mu}_1}{\partial R} = (C_1 - a'(R)V_1)\cos\theta + (C_2 - a'(R)V_2)\sin\theta + \frac{\partial \xi}{\partial R}. \tag{4.50}$$

Comparing (4.50) with (4.49) we conclude that $\xi = \xi(s,\theta)$. Using basic trigonometric identities, (C.35), we note that $\frac{\partial \tilde{\mu}_1}{\partial R}$ also satisfies

$$\frac{\partial \tilde{\mu}_1}{\partial R}(s,R,\theta,\tau) = -\frac{\partial \tilde{\mu}_1}{\partial R}(s,R,\theta+\pi,\tau). \tag{4.51}$$

Combining (4.51) and (4.48) we obtain the jump condition over the interface Γ_p

$$[\![\partial_{\mathbf{n}} \mu_0^{\mathbf{n}}]\!] = 0, \tag{4.52}$$

for any choice of normal vector $\mathbf{n}$.

4.5.4 The Normal Velocity

We would like to determine the evolution of the interface Γ_p. To this end, recall equation (4.45), which involve the normal velocity $\mathbf{V}$, and the inner chemical potential $\tilde{\mu}_1$. Using the polar coordinates extension of the Laplacian, given in (A.3), and the expression for $\tilde{\mu}_1$ given in (4.46), the right-hand side of equation (4.45)

takes the form

$$\begin{aligned}\Delta_z\tilde{\mu}_1 =&(C'' + \frac{1}{R}C') + (A_1'' + \frac{1}{R}A_1' + \frac{1}{R^2}A_1)\cos\theta + (B_1'' + \frac{1}{R}B_1' + \frac{1}{R^2}B_1)\sin\theta \\ &+ \sum_{m=2}^{\infty} \frac{m}{R^2}A_m(s)\cos(m\theta) + \frac{m}{R^2}B_m(s)\sin(m\theta).\end{aligned} \tag{4.53}$$

Using the polar gradient, (A.4), the left-hand side of equation (4.45) becomes

$$\mathbf{V}\cdot\nabla_z U_p = (V_1 U_p'\cos\theta\ ,\ V_2 U_p'\sin\theta) \tag{4.54}$$

Plugging (4.53) and (4.54) into (4.45) and matching coefficients of corresponding trigonometric terms yields the system

$$C'' + \frac{1}{R}C' = 0, \tag{4.55}$$

$$A_1'' + \frac{1}{R}A_1' + \frac{1}{R^2}A_1 = \mathbf{V}_1 U_p', \tag{4.56}$$

$$B_1'' + \frac{1}{R}B_1' + \frac{1}{R^2}B_1 = \mathbf{V}_2 U_p', \tag{4.57}$$

$$\frac{m}{R^2}A_m = 0, \tag{4.58}$$

$$\frac{m}{R^2}B_m = 0. \tag{4.59}$$

From equations (4.58) and (4.59) we deduce that $A_m = B_m = 0$, for $m \geq 2$. Equation (4.55) has the solution

$$C = C_0(s), \tag{4.60}$$

and the non-homogeneous equations, (4.56) and (4.57), have the solutions

$$A(s,R) = C_1(s)R - a(R)V_1(s), \tag{4.61}$$

$$B(s,R) = C_2(s)R - a(R)V_2(s), \tag{4.62}$$

where $a(R)$ is the solution of the non-homogeneous ODE

$$a'' + \frac{1}{R}a' - \frac{1}{R^2}a = U'(R), \tag{4.63}$$

and is given by the explicit formula

$$a(R) = \frac{1}{R}\int_0^R r\hat{U}_p\,dr, \tag{4.64}$$

where we introduce

$$\hat{U}_p := U_p - b_-. \tag{4.65}$$

Note that $\hat{U}_p$ is positive and $\hat{U}_p \to 0$ as $R \to \infty$. Plugging (4.60), (4.61) and (4.62) into (4.46) and taking $A_m = B_m = 0$ we see that $\tilde{\mu}_1$ takes the form

$$\tilde{\mu}_1 = C_0(s) + (C_1(s)R - a(R)V_1(s))\cos\theta + (C_2(s)R - a(R)V_2(s))\sin\theta. \tag{4.66}$$

Recall that $\tilde{\mu}_1$ relates to the density function $\tilde{u}$ through equation (4.28), which for the choice $\tilde{u}_0 = U_p$ takes the form

$$L_p^2\tilde{u}_1 = \tilde{\mu}_1 - \eta_d W'(U_p), \tag{4.67}$$

where the linear operator L_p was introduced in (2.78). By the Fredholm Alternative, this equation has a solution $\tilde{u}_1 \in L^2(\mathbb{R})$ if and only if the right-hand side is perpendicular to $\ker L_p$. Recall that $\ker L_p = (U_p'\cos\theta, U_p'\sin\theta)$ and consider the inner product of $(U_p'\cos\theta, U_p'\sin\theta)$ with the right-hand side of equation (4.67). We know that $\eta_d W'(U_p)$ belong to the space $\mathcal{Z}_0$, defined in (2.79), and hence is perpendicular to $\ker L_p$. From orthogonality in θ, the only non-trivial condition is imposed on the $\sin\theta$ and $\cos\theta$ terms of $\tilde{\mu}_1$, in equation (4.66), and the Fredholm condition (4.67) reduces to

$$\int_0^\infty (C_i(s)R - a(R)V_i(s))U_p' R\,dR = 0, \quad \text{for } i = 1, 2. \tag{4.68}$$

Plugging the definition of $a(R)$, given in (4.64) into (4.68) and integrating by parts yields the relation

$$C_i = V_i \frac{S_2}{2S_1} \tag{4.69}$$

where we introduce the constants

$$S_1 := \int_0^\infty \hat{U}_p\, RdR \tag{4.70}$$

$$S_2 := \int_0^\infty \hat{U}_p^2\, RdR. \tag{4.71}$$

Plugging (4.69) into (4.66) and taking the R derivative of (4.66) yields

$$\frac{\partial\tilde{\mu}_1}{\partial R} = V_1\left(\frac{S_2}{2S_1} - a'(R)\right)\cos\theta + V_2\left(\frac{S_2}{2S_1} - a'(R)\right)\sin\theta. \tag{4.72}$$

Equating coefficients of $\sin\theta$ and $\cos\theta$ in equations (4.72) and (4.49) yields

$$\mathbf{N}^i \cdot \nabla_x \mu_0^{\mathbf{N}^i} = V_i\left(\frac{S_2}{2S_1} - \lim_{R\to\infty} a'(R)\right), \tag{4.73}$$

and since

$$\lim_{R\to\infty} a'(R) = \lim_{R\to\infty}\left(\frac{1}{R^2}\int_0^R r\hat{U}_p'\,dr + \hat{U}_p\right) = 0, \tag{4.74}$$

we find that the normal velocity satisfies

$$V_i = \frac{2S_1}{S_2}\,\mathbf{N}^i \cdot \nabla_x \mu_0^{\mathbf{N}^i}, \quad \text{for } i = 1, 2. \tag{4.75}$$

4.5.5 Sharp Interface Limit

The preceding calculation shows that, in a neighborhood of the dressed solution, the $\tau = t$ time scale evolution of (4.39) reduces to a sharp interface limit problem for the evolution of Γ_p

$$(4.40):\ u_{0,\tau} = \Delta\mu_0 \quad \text{in } \tilde{\Gamma}_{p,\ell}, \tag{4.76}$$

$$\mathbf{n}\cdot\nabla_x\mu_0 = 0 \quad \text{on } \partial\Omega, \tag{4.77}$$

$$\mu_0 = 0 \quad \text{on } \Gamma_p, \tag{4.78}$$

$$(4.52):\ [\![\partial_{\mathbf{n}}\mu_0^{\mathbf{n}}]\!] = 0 \quad \text{on } \Gamma_p, \text{ for all normal vectors } \mathbf{n} \text{ of } \Gamma_p, \tag{4.79}$$

$$(4.75):\ V_i = \frac{2S_1}{S_2}\,\mathbf{N}^i \cdot \nabla_x \mu_0^{\mathbf{N}^i}, \quad \text{for all } x \in \Gamma_p(t),\ i = 1, 2. \tag{4.80}$$

We are following the argument of [Dai and Promislow, 2015] and prove the following Lemma -

Lemma 4.1. *Assume that the co-dimension two interface $\Gamma_p \subset \Omega$ has finite one dimensional Hausdorff measure. Then the only equilibrium solution of (4.76)-(4.80) is the trivial solution $\mu_0 \equiv 0$, however the curve Γ_p can have arbitrary shape.*

Proof. At equilibium we have

$$\Delta\mu_0 = 0, \quad \text{in } \tilde{\Gamma}_{p,\ell}, \tag{4.81}$$

$$\mathbf{n}\cdot\nabla_x\mu_0 = 0 \quad \text{on } \partial\Omega, \tag{4.82}$$

$$\mu_0 = 0 \quad \text{on } \Gamma_p. \tag{4.83}$$

Since μ_0 is analytic off of a set of finite hausdorff measure, then μ_0 has an analytic extension to Ω, $\bar{\mu}_0$, see [**?**]

and [Polking, 1984], and we drop the bar notation. The extended function satisfies

$$\Delta\mu_0 = 0, \quad \text{in } \Omega, \tag{4.84}$$

$$\mathbf{n} \cdot \nabla_x \mu_0 = 0 \quad \text{on } \partial\Omega, \tag{4.85}$$

then, by the *Strong Maximum principle* implies that μ_0 is spatially constant. Finally, since $\mu_0 = 0$ on Γ_p, we conclude that $\mu_0 \equiv 0$. ■

We subsequently assume the system has achieved equilibrium on the $\tau = t$ time-scale.

4.6 Time scale $\tau = \varepsilon t$: Curvature Driven Flow

We obtain evolution equations for the outer and inner regions. Using the inner equations we obtain a jump conditions of the outer solution over the interface and an expression for the normal velocity of the interface. we will see that the normal velocity is driven by the curvature flow. Finally, we use the mass preserving property of the system to obtain the coupled system for the normal velocity, V, and the external chemical potential, μ_1.

4.6.1 Outer expansion

Away from the interface, the outer expansion of the density function u is given in (4.5), and the outer expansion of the chemical potential, μ, given in (4.22). Plugging (4.5) and (4.22) into (4.39) and equating orders of ε yields

$$O(1) \; : \quad 0 = \Delta_x \mu_0, \quad \text{in } \tilde{\Gamma}_{p,\ell} \tag{4.86}$$

$$O(\varepsilon) \; : \quad u_{0,\tau} = \Delta_x \mu_1, \quad \text{in } \tilde{\Gamma}_{p,\ell}, \tag{4.87}$$

where μ_0 and μ_1 are given in (4.23) and (4.24), respectively. From equations (4.76)-(4.80) we assume that the system has equilibrated to $u_0 = b_-$ in $\tilde{\Gamma}_{p,\ell}$. Under this assumption $\mu_0 = 0$, which satisfies equation (4.86) and equation (4.87) reduces to

$$\Delta_x \mu_1 = 0 \quad \text{in } \tilde{\Gamma}_{p,\ell}. \tag{4.88}$$

4.6.2 Inner Expansion

We express each of the terms in (4.39) in inner coordinates. Plugging the inner expansion of u, given in (4.6), into the left-hand side of equation (4.39) yields

$$u_\tau = -\mathbf{V}\cdot\nabla_z\tilde{u} + O(\varepsilon). \tag{4.89}$$

An expand expression of the Laplacian in local coordinates is given in (2.70) and an expression for the inner expansion of the chemical potential is given in (4.26). Plugging (4.89), (2.70) and (4.26) back into the evolution equation (4.39) and comparing orders of ε yields,

$$O(\varepsilon^{-2}) \ : 0 = \Delta_z\tilde{\mu}_0, \tag{4.90}$$

$$O(\varepsilon^{-1}) \ : 0 = \Delta_z\tilde{\mu}_1 + \vec{\kappa}\cdot\nabla_z\tilde{\mu}_0 \tag{4.91}$$

$$O(1) \ : -V_{t_1}\cdot\nabla_z\tilde{u}_0 = \Delta_z\tilde{\mu}_2 - \vec{\kappa}\cdot\nabla_z\tilde{\mu}_1 + (\partial_s^2 - (z\cdot\vec{\kappa})\vec{\kappa}\cdot\nabla_z)\tilde{\mu}_0, \tag{4.92}$$

where $\tilde{\mu}_0$, $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are defined in equations (4.27), (4.28) and (4.29), respectively. Equation (4.90) is consistent with the assumption that $\tilde{u}_0 = U_p$ which implies that $\tilde{\mu}_0 = 0$. Since $\tilde{\mu}_0 = 0$ equations (4.91) and (4.92) reduces to

$$O(\varepsilon^{-1}) \ : 0 = \Delta_z\tilde{\mu}_1, \tag{4.93}$$

$$O(1) \ : \ -V_{t_1}\cdot\nabla_z\tilde{u}_0 = \Delta_z\tilde{\mu}_2 - \vec{\kappa}\cdot\nabla_z\tilde{\mu}_1. \tag{4.94}$$

4.6.2.1 Solving equation (4.93) for $\tilde{u}_1$

To solve (4.93) we use the explicit expression for $\Delta_z\tilde{\mu}_1$, given in (4.53). Plugging (4.53) into (4.93) and matching cos, sin terms yields

$$C'' + \frac{1}{R}C' = 0, \tag{4.95}$$

$$A_1'' + \frac{1}{R}A_1' + \frac{1}{R^2}A_1 = 0 \tag{4.96}$$

$$B_1'' + \frac{1}{R}B_1' + \frac{1}{R^2}B_1 = 0 \tag{4.97}$$

$$\frac{m}{R^2}A_m = 0, \tag{4.98}$$

$$\frac{m}{R^2}B_m = 0. \tag{4.99}$$

From equations (4.56)-(4.59) we deduce that $A_i = B_i = 0$, for $i \geq 1$. Equation (4.55) has the solution

$$C = C_0(s), \tag{4.100}$$

and we deduce that $\tilde{\mu}_1$ is spatially constant, i.e.,

$$\tilde{\mu}_1 = \tilde{\mu}_1(s,t) = \mu_1. \tag{4.101}$$

Recall that $\tilde{\mu}_1$ is given in (4.28) and since $\tilde{u}_0 = U_p$ it reduces to

$$L_p^2 \tilde{u}_1 = \tilde{\mu}_1 - \eta_d W'(U_p). \tag{4.102}$$

By the Fredholm Alternative, this equation has a solution $\tilde{u}_1$ if and only if the right-hand side is perpendicular to $\ker L_p$. Recall that $\ker L_p \subset \ker L_{p,1}$ and note that $\tilde{\mu}_1, \eta_d W'(U_p) \in \mathcal{Z}_0$. Since the spaces $\mathcal{Z}_m$ are mutually orthogonal, there exists a solution $\tilde{u}_1$ denoted

$$\tilde{u}_1 = \mu_1 \Phi_{p,2} - \eta_d L_p^{-2} W'(U_p) \tag{4.103}$$

where μ_1 is a spacial constant and $\Phi_{p,2}$ satisfied $L_p^2 \Phi_{p,2} = 1$, defined in (2.86).
To determine the interface normal velocity we continue to the equation (4.94). Since equation (4.101) implies that $\tilde{\mu}_1$ is spatially constant and equation (4.94) reduces to

$$O(1) \ : \ -V \cdot \nabla_z \tilde{u}_0 = \Delta_z \tilde{\mu}_2. \tag{4.104}$$

4.6.2.2 Jump Conditions

We would like to determine the normal velocity of the interface Γ_p for the time scale $\tau = \varepsilon t$. To this end, we need to determine an explicit solution for $\tilde{\mu}_2$, in equation (4.104), subject to the matching conditions with the outer solution, given in Section 4.2.
Turning to polar coordinates, we use the Fourier expansion, (2.88), to obtain an express for $\tilde{\mu}_2$,

$$\tilde{\mu}_2 = \bar{A}_1(s,R)\cos\theta + \bar{B}_1(s,R)\sin\theta + \bar{C}(s,R) + \bar{\xi}(s,R,\theta), \tag{4.105}$$

where

$$\bar{\xi}(s,R,\theta) = \sum_{m=2}^{\infty} (\bar{A}_m(s,R)\cos(m\theta) + \bar{B}_m(s,R)\sin(m\theta)). \tag{4.106}$$

From the matching condition (4.19) we see that $\tilde{\mu}_2$ grows at most linearly as $R \to \infty$ and

$$\lim_{R\to\infty} \frac{\partial \tilde{\mu}_2}{\partial R} = \partial_{\mathbf{n}} \mu_1^{\mathbf{n}}. \tag{4.107}$$

Using the definition of the directional derivative along $\mathbf{n}$, given in (4.8), we rewrite equation (4.107) in terms of sin and cos

$$\lim_{R\to\infty} \frac{\partial \tilde{\mu}_2}{\partial R} = \cos\theta \mathbf{N}^1 \cdot \nabla_z \mu_1^{\mathbf{n}} + \sin\theta \mathbf{N}^2 \cdot \nabla_z \mu_1^{\mathbf{n}}. \tag{4.108}$$

Taking the R derivative of (4.105) yields

$$\frac{\partial \tilde{\mu}_2}{\partial R} = (\bar{C}_1 - a'(R)V_1)\cos\theta + (\bar{C}_2 - a'(R)V_2)\sin\theta + \frac{\partial \bar{\xi}}{\partial R}. \tag{4.109}$$

Comparing (4.109) with (4.108) we conclude that $\bar{\xi} = \bar{\xi}(s,\theta)$. Using basic trigonometric identities, (C.35), we note that $\frac{\partial \tilde{\mu}_2}{\partial R}$ also satisfies

$$\frac{\partial \tilde{\mu}_2}{\partial R}(s,R,\theta,\tau) = -\frac{\partial \tilde{\mu}_2}{\partial R}(s,R,\theta+\pi,\tau). \tag{4.110}$$

Combining (4.110) and (4.107) we obtain the jump condition over the interface Γ_p

$$[\![\partial_{\mathbf{n}} \mu_1^{\mathbf{n}}]\!] = 0, \tag{4.111}$$

for any choice of normal vector $\mathbf{n}$. Moreover, plugging (4.109) into (4.108), recalling that $a'(R) \to 0$ as $R \to \infty$, and comparing coefficients of sin and cos yields the relation

$$\bar{C}_i = \mathbf{N}^i \cdot \nabla_z \mu_1^{\mathbf{n}} \tag{4.112}$$

4.6.2.3 The normal velocity

We would like to determine the evolution of the interface Γ_p. To this end, recall equation (4.104), which involves the normal velocity, $\mathbf{V}$, and the inner chemical potential $\tilde{\mu}_2$. The definition of $\tilde{\mu}_2$ is given in (4.29), and since $\tilde{u}_0 = U_p$ (4.29) reduces to

$$\tilde{\mu}_2 = L_p^2 \tilde{u}_2 - L_p(\mathcal{R}) + (\kappa \cdot \nabla_z + W'''(U_p)\tilde{u}_1 - \eta_1)(-L_p\tilde{u}_1 + \vec{\kappa}\cdot\nabla_z U_p) + \eta_d W''(U_p)\tilde{u}_1 \tag{4.113}$$

where

$$\mathcal{R} = \kappa \cdot \nabla \tilde{u}_1 - (z\cdot\vec{\kappa})\vec{\kappa}\cdot\nabla_z U_p + \frac{1}{2}W'''(\tilde{U}_p)\tilde{u}_1. \tag{4.114}$$

In order to get an expression for the normal velocity we want to solve (4.104) for $\tilde{\mu}_2$. Following the same procedure as in Section 4.5.4, specifically equations (4.45)-(4.66), and considering the matching condition (4.19) we deduce that $\tilde{\mu}_2$ takes the form

$$\tilde{\mu}_2 = \bar{C}_0(s) + (\bar{C}_1(s)R - a(R)V_1(s))\cos\theta + (\bar{C}_2(s)R - a(R)V_2(s))\sin\theta, \tag{4.115}$$

where $a(R)$, C_0 and C_1, C_2 are given in (4.64), (4.60), (4.61), and (4.62), respectively.
To solve equation (4.113) for $\tilde{u}_2$ we rewrite it in the following form

$$L_p^2\tilde{u}_2 = \tilde{\mu}_2 - \mathcal{Q} - L_p(R), \tag{4.116}$$

where

$$\mathcal{Q} := (\vec{\kappa}\cdot\nabla_z + W'''(U_p)\tilde{u}_1 - \eta_1)(-L_p\tilde{u}_1 + \vec{\kappa}\cdot\nabla_z U_p) + \eta_d W''(U_p)\tilde{u}_1. \tag{4.117}$$

By the Fredholm Alternative, we can solve equation (4.116) for $\tilde{u}_2$ if and only if the right-hand side is perpendicular to $\ker L_p$. Recall that $\ker L_p = span\{U_p'\cos\theta, U_p'\sin\theta\} \subset \ker L_{p,1}$, and that the $\mathcal{Z}_m$ spaces are mutually orthogonal. Expanding $\mathcal{Q}$, given in (4.117), and decomposing it to its $\mathcal{Z}_m$ components yields

$$\mathcal{Q} = \mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_{0,2}, \tag{4.118}$$

where $\mathcal{Q}_0 \in \mathcal{Z}_0$, $\mathcal{Q}_1 \in \mathcal{Z}_1$, $\mathcal{Q}_{0,2} \in \mathcal{Z}_0 + \mathcal{Z}_2$, and are given by

$$\mathcal{Q}_0 := -W'''(U_p)\tilde{u}_1 L_p\tilde{u}_1 + \eta_1 L_p\tilde{u}_1 + \eta_d W''(U_p)\tilde{u}_1, \tag{4.119}$$

$$\mathcal{Q}_1 := -\vec{\kappa}\cdot\nabla_z L_p\tilde{u}_1 + W'''(U_p)\tilde{u}_1\vec{\kappa}\cdot\nabla_z U_p - \eta_1\vec{\kappa}\cdot\nabla_z U_p, \tag{4.120}$$

$$\mathcal{Q}_{0,2} := (\vec{\kappa}\cdot\nabla_z)^2 U_p. \tag{4.121}$$

By the orthogonality of the $\mathcal{Z}_m$ spaces and since $L_p(\mathcal{R}) \perp \ker L_p$ the Fredholm solvability condition of equation (4.116) reduces to

$$(\tilde{\mu}_2 - \mathcal{Q}_1, \partial_{z_i}U_p)_{L_R([0,\infty))} = 0, \text{ for } i = 1, 2. \tag{4.122}$$

In order to calculate the solvability condition, given in (4.122), we first expand $\mathcal{Q}_1$ using the explicit expression of $\tilde{u}_1$ given in (4.103), such that

$$\mathcal{Q}_1 := -\mu_1\vec{\kappa}\cdot\nabla_z\Phi_{p,1} + \eta_d\vec{\kappa}\cdot\nabla_z L_p^{-1}(W'(U_p)) + \mu_1 W'''(U_p)\Phi_{p,2}\vec{\kappa}\cdot\nabla_z U_p \tag{4.123}$$

$$- \eta_d W'''(U_p) L_p^{-2}(W'(U_p))\vec{\kappa}\cdot\nabla_z U_p - \eta_1 \vec{\kappa}\cdot\nabla_z U_p,$$

and calculating the inner product (see Appendix C.2 for details) yields

$$(\mathcal{Q}_1, \partial_{z_i} U)_{L_R} = -2\pi\mu_1\kappa_i S_1 - \eta_1 \pi \kappa_i S_4, \tag{4.124}$$

where S_1 is defined in (4.70) and

$$S_4 := \int_0^\infty (U_p')^2 R\, dR. \tag{4.125}$$

The inner product of $\partial_{z_i} U_p$ with $\tilde{\mu}_2$, where $\tilde{\mu}_2$ is given in equation (4.115), yields

$$(\tilde{\mu}_2, \partial_{z_i} U) = -2\pi \bar{C}_i S_1 + \pi V_i S_2 = -2\pi \mathbf{N}^i \cdot \nabla_z \mu_1^{\mathbf{N}_i} S_1 + \pi V_i S_2, , \tag{4.126}$$

where S_2 is defined in (4.71), and the second equality follows form the matching conditions, see equation (4.112). Returning to (4.122) and using (4.124) and (4.126), we conclude that the normal velocity is given by

$$V_i = -\frac{2\mu_1 S_1 - \eta_1 S_4}{S_2}\kappa_i + \frac{2S_1}{S_2}\mathbf{N}^i \cdot \nabla_z \mu_1^{\mathbf{N}_i} \quad i = 1, 2. \tag{4.127}$$

4.6.3 Sharp interface limit

On the time scale $\tau = \varepsilon t$, the evolution of the interface, Γ_p is given by the normal velocity

$$(4.127)\ :\ V_i = -\frac{2\mu_1 S_1 + \eta_1 S_4}{S_2}\kappa_i + \frac{2S_1}{S_2}\mathbf{N}^i \cdot \nabla_z \mu_1^{\mathbf{N}_i} \quad i = 1, 2. \tag{4.128}$$

where μ_1 is the solution of the system

$$(4.88)\ :\ \Delta_x \mu_1 = 0 \quad \text{in } \tilde{\Gamma}_{p,\ell}, \tag{4.129}$$

$$\mathbf{n}\cdot\nabla_x \mu_1 = 0 \quad \text{on } \partial\Omega, \tag{4.130}$$

$$(4.111)\ :\ [\![\partial_{\mathbf{n}} \mu_1^{\mathbf{n}}]\!] = 0, \quad \text{on } \Gamma_p, \text{ for all normal vectors } \mathbf{n} \text{ of } \Gamma_p. \tag{4.131}$$

The inner chemical potential satisfies $\mu_1 = \mu_1(s,\tau)$ on Γ_p. Since we assumed that $\mu_1 \in C^2(\Omega/\Gamma_p) \cap C(\bar{\Omega})$ we may use Lemma 4.1 to conclude that $\Delta\mu_1 = 0$ on the entire domain Ω. Applying the Strong Maximum Principle (see [Evans, 2010]) we deduce that μ_1 is spatially constant.

Since μ_1 is spatially constant we have $\nabla_x \mu_1 = 0$ and the normal velocity reduces to

$$\mathbf{V} = -\frac{2\mu_1 S_1 - \eta_1 S_4}{S_2}\vec{\kappa}, \tag{4.132}$$

where S_1 and S_4 are defined in (4.70), (4.125), respectively.

4.6.4 Equilibria estimate for time scale $\tau = \varepsilon t$

The far-field external chemical potential μ_1 is characterized by the density function, u, whose value is determined by conservation of total mass. For this time scale, we summarize our approximation for the density function in each region. In the outer region $\tilde{\Gamma}_{p,\ell}$, our assumption that $u_0 = b_-$ combined with equation (4.24) yields

$$u(x,t) = b_- + \varepsilon\frac{\mu_1}{\alpha_-^2} + O(\varepsilon^2) \quad \text{in } \tilde{\Gamma}_{p,\ell}, \tag{4.133}$$

where α_- is defined in (1.34). In the inner region, $\Gamma_{p,\ell}$, our choice $\tilde{u}_0 = U_p$ combined with equation (4.103) yields

$$u(x,t) = U_p + \varepsilon(\mu_1\Phi_{p,2} - \eta_d L_p^{-2} W'(U_p)) + O(\varepsilon^2) \quad \text{in } \Gamma_{p,\ell}. \tag{4.134}$$

We use mass balance to determine μ_1 and to obtain the coupled μ_1, V system evolution. The total mass of the system is given by

$$M := \int_\Omega u(x,t) - b_-\, dx = \int_\Omega u(x,0) - b_-\, dx = \int_{\Omega\setminus\Gamma_{p,\ell}} (u - b_-)\, dx + \int_{\Gamma_{p,\ell}} (u - b_-)\, dx \tag{4.135}$$

Using (4.133), the outer integral becomes

$$\int_{\Omega\setminus\Gamma_{p,\ell}} (u - b_-)\, dx = \varepsilon\frac{\mu_1}{\alpha_-^2}(|\Omega| - \boxed{|\Gamma_{p,\ell}|}) + O(\varepsilon^2) \tag{4.136}$$

Using (4.134) and the Jacobian, defined in (2.63), the inner integral takes the form

$$\int_{\Gamma_{p,\ell}} (u - b_-)\, dx = \varepsilon^2 \int_{\Gamma_p}\int_{\mathbb{R}^2} \left((U_p - b_-) + \varepsilon(\mu_1\Phi_{p,2} - \eta_d L_p^{-2} W'(U_p)) + O(\varepsilon^2)\right)(1 - \varepsilon z\cdot\vec{\kappa})\, dz\, ds \tag{4.137}$$

$$= \varepsilon^2 2\pi|\Gamma_p|S_1 + \varepsilon^3 2\pi|\Gamma_p|\mu_1\int_0^\infty \Phi_{p,2}\, RdR + \varepsilon^3 2\pi|\Gamma_p|\eta_d\int_0^\infty W'(U_p)\, RdR + O(\varepsilon^4|\Gamma_p|) \tag{4.138}$$

Adding and subtracting the term $\varepsilon^2 2\pi\frac{\mu_1}{\alpha_-^2}|\Gamma_p| = \varepsilon\frac{\mu_1}{\alpha_-^2}|\Gamma_{p,\ell}|$ to (4.137), the inner integral becomes

$$\int_{\Gamma_{p,\ell}} (u - b_-)\, dx = \varepsilon^2 2\pi|\Gamma_p|S_1 + \varepsilon\frac{\mu_1}{\alpha_-^2}|\Gamma_{p,\ell}| + \varepsilon^3 2\pi|\Gamma_p|\mu_1 S_3 + O(\varepsilon^3|\Gamma_p|) \tag{4.139}$$

where we introduce the constant S_3

$$S_3 \coloneqq \int_0^\infty \Phi_{p,2} - \alpha_-^{-2}\, RdR. \tag{4.140}$$

Combining (4.139) and (4.136) into (4.135) and assuming that $|\Gamma_p| \sim O(\varepsilon^{-1})$, which implies that

$$|\Gamma_p| = \varepsilon^{-1}\gamma_{-1} + O(1), \tag{4.141}$$

we rewrite the total mass in orders of ε

$$M = \varepsilon\left(\frac{\mu_1}{\alpha_-^2}|\Omega| + 2\pi S_1\gamma_{-1}\right) + O(\varepsilon^2). \tag{4.142}$$

Taking the $\tau = \varepsilon t$ time derivative of the total mass, (4.142), and solving for $\frac{d\gamma_{-1}}{d\tau}$ yields

$$\frac{d\gamma_{-1}}{d\tau} = -\frac{|\Omega|}{2\pi\alpha_-^2 S_1}\frac{d\mu_1}{d\tau}. \tag{4.143}$$

On the other hand, taking the time derivative of (4.141) yields

$$\frac{d|\Gamma_p|}{d\tau} = \varepsilon^{-1}\frac{d\gamma_{-1}}{d\tau} + O(1). \tag{4.144}$$

Combining equation (2.62), which relates the interfacial surface area growth with the normal velocity, with equation (4.144) yields

$$\varepsilon^{-1}\frac{d\gamma_{-1}}{d\tau} = -\int_{\Gamma_p} \vec{\kappa}\cdot\mathbf{V}\, ds = \frac{2\mu_1 S_1 + \eta_1 S_4}{S_2}\int |\vec{\kappa}|^2\, ds, \tag{4.145}$$

where for the second equality we used the expression of $\mathbf{V}$ obtained in equation (4.132). Plugging (4.143) into equation (4.145) and solving for $\frac{d\mu_1}{d\tau}$ we obtain the leading order evolution equation

$$\frac{d\mu_1}{d\tau} = -\varepsilon\left(\frac{4\pi\alpha_-^2 S_1^2}{|\Omega|S_2}\int_{\Gamma_p}|\vec{\kappa}|^2\, ds\right)\mu_1 + \varepsilon\frac{2\pi\alpha_-^2 S_4 S_1\eta_1}{|\Omega|S_2}\int_{\Gamma_p}|\vec{\kappa}|^2\, ds \tag{4.146}$$

These results show that the evolution of the interface is governed by the coupled system

$$(4.132)\ :\ \mathbf{V} = -\frac{2\mu_1 S_1 - \eta_1 S_4}{S_2}\vec{\kappa}, \tag{4.147}$$

$$(4.146)\ :\ \frac{d\mu_1}{d\tau} = -\varepsilon\frac{2\pi\alpha_-^2 S_1}{|\Omega|S_2}\left(2S_1\mu_1 - S_4\eta_1\right)\int_{\Gamma_p}|\vec{\kappa}|^2\, ds \tag{4.148}$$

The H^{-1} gradient flow drives pure pore interfaces by a mean-curvature flow and the external chemical

potential decays exponentially to its equilibria value

$$\mu_1 \longrightarrow \frac{S_4\eta_1}{2S_1}, \tag{4.149}$$

Since $\mu_1 = \alpha_-^2 u_1$, the density function decays to

$$u_1 \longrightarrow \frac{S_4\eta_1}{2S_1\alpha_-^2} \tag{4.150}$$

and the far-field behaviour of the density, u, takes the form

$$u = b_- + \varepsilon\frac{S_4\eta_1}{2S_1\alpha_-^2} + O(\varepsilon^2). \tag{4.151}$$

4.7 Competitive Geometric Evolution of Bilayers and Pores

After obtaining a leading order expression for the evolution equation of the bilayer morphology and the pore morphology, we would like to consider a combined system in which the two morphologies co-exists. Let $\Omega \subset \mathbb{R}^3$ be a given domain with two admissible manifolds Γ_b and Γ_p, for the co-dimension one and the co-dimension two morphology, respectively, which satisfy $|\Gamma_b| \sim O(1)$ and $|\Gamma_p| \sim O(\varepsilon^{-1})$. Let $\Gamma_{b,l}$ and $\Gamma_{p,l}$ be the reaches of Γ_b and Γ_p, respectively, on which the change of coordinates to the whiskered coordinates is unique.

Away from the interface, at time scale $\tau = \varepsilon t$, the leading order expression for the morphologies solutions takes the form

$$u_p = u_b = b_- - \varepsilon\frac{\mu_1}{\alpha_-^2} + O(\varepsilon^2). \tag{4.152}$$

and the composite solution takes the form

$$u_{b,p} = U_b + U_p + b_- - \varepsilon\frac{\mu_1}{\alpha_-^2} + O(\varepsilon^2), \tag{4.153}$$

and we already see that the two structures will compete each other for surfactant phase through the common, slowly varying, chemical potential μ_1.

For u, a combined bilayer-pore solution of the form (4.153), the total mass constraint of the combined system is given by

$$M = \int_\Omega (u - b_-)\,dx = \int_{\Omega\setminus\Gamma_{b,l}\cup\Gamma_{p,l}} (u - b_-)\,dx + \int_{\Gamma_{b,l}} (u - b_-)\,dx + \int_{\Gamma_{p,l}} (u - b_-)\,dx. \tag{4.154}$$

Calculating the outer integral yields

$$\int_{\Omega\setminus\Gamma_{b,l}\cup\Gamma_{p,l}} (u - b_-)\, dx = \varepsilon \frac{\mu_1}{\alpha_-^2} \left(|\Omega| - |\Gamma_{b,l} \cup \Gamma_{p,l}|\right) + O(\varepsilon^2), \tag{4.155}$$

and plugging (4.155) back into (4.154) yields

$$M = \varepsilon \frac{\mu_1}{\alpha_-^2} |\Omega| - \varepsilon \frac{\mu_1}{\alpha_-^2} \left(|\Gamma_{b,l}| + \Gamma_{p,l}|\right) + \int_{\Gamma_{b,l}} (u - b_-)\, dx + \int_{\Gamma_{p,l}} (u - b_-)\, dx. \tag{4.156}$$

Using (4.139) and equation (1.49), we obtain an expression for the inner integrals

$$\int_{\Gamma_{b,l}} (u - b_-)\, dx = \varepsilon m_b |\Gamma_b| + \varepsilon \frac{\mu_1}{\alpha_-^2} |\Gamma_{b,l}| + O(\varepsilon^2), \tag{4.157}$$

$$\int_{\Gamma_{p,l}} (u - b_-)\, dx = \varepsilon^2 m_p |\Gamma_p| + \varepsilon \frac{\mu_1}{\alpha_-^2} |\Gamma_{p,l}| + O(\varepsilon^3), \tag{4.158}$$

where $m_b \coloneqq \int_{\mathbb{R}} \hat{U}_b\, dz$ and $m_p \coloneqq 2\pi S_1$ and S_1 defined in (4.70). Plugging equations (4.157) and (4.158) in (4.156) the total mass takes the form

$$M = \varepsilon \frac{\mu_1}{\alpha_-^2} |\Omega| + \varepsilon m_b |\Gamma_b| + \varepsilon^2 m_p |\Gamma_p|. \tag{4.159}$$

Expanding $M = \varepsilon \hat{M} + O(\varepsilon^2)$ and using our assumption that $|\Gamma_b| \sim O(1)$ and $|\Gamma_p| \sim O(\varepsilon^{-1})$ yields

$$\hat{M} = \frac{\mu_1}{\alpha_-^2} |\Omega| + m_b |\Gamma_b| + \varepsilon m_p |\Gamma_p|, \tag{4.160}$$

Which, yields the constraint on the chemical potential μ_1,

$$\mu_1 = \frac{\alpha_-^2}{|\Omega|} \left(\hat{M} - m_b |\Gamma_b| - \varepsilon m_p |\Gamma_p|\right). \tag{4.161}$$

Recall that the interfacial surface area growth of the pore is given in equation (2.62) and the equivalent interfacial surface area growth of the bilayer is given in equation (3.110). Plugging into (2.62) the normal velocity of the pore, $\mathbf{V}_p$, given in (4.132), and plugging into (3.110) the normal velocity for the bilayer, V_b, given in (3.102), yields leading order expressions for change in bilayer surface area and pore length

$$\frac{d|\Gamma_b|}{dt} = \frac{m_b}{B_1} \left(\mu_1 + (\eta_1 + \eta_2) \frac{\sigma_b}{2m_b}\right) \int_{\Gamma_b} H_0^2\, ds, \tag{4.162}$$

$$\frac{d|\Gamma_p|}{dt} = \frac{2S_1}{S_2} \left(\mu_1 + \frac{S_4}{2S_1} \eta_1\right) \int_{\Gamma_p} |\vec{\kappa}|^2\, ds, \tag{4.163}$$

where B_1 is defined in (3.93). Taking the time derivative of (4.161) yields

$$\frac{d\mu_1}{d\tau} = -\frac{\alpha_-^2}{|\Omega|}\left(m_b\frac{d|\Gamma_b|}{d\tau} + \varepsilon m_p\frac{d|\Gamma_p|}{d\tau}\right), \tag{4.164}$$

and plugging (4.162) and (4.163) into (4.164) yields

$$\frac{d\mu_1}{d\tau} = -\frac{\alpha_-^2}{|\Omega|}\left[m_b\frac{m_b}{B_1}\left(\mu_1 + (\eta_1+\eta_2)\frac{B_2}{2m_b}\right)\int_{\Gamma_b} H_0^2\,ds + \varepsilon m_p\frac{2S_1}{S_2}\left(\mu_1 - \frac{S_4}{2S_1}\eta_1\right)\int_{\Gamma_p}|\vec{\kappa}|^2\,ds\right]. \tag{4.165}$$

These results show that for initial data corresponding to spatially separated pore and bilayer structures yields a competitive evolution that can be understood as a fight for surfactant, mediated through the common value of the chemical potential μ_1, whose evolution is determined to by the conservation of total mass,

$$(4.132)\ :\ \mathbf{V}_p = \nu_p(\mu_1 + \mu_p^*)\vec{\kappa}, \tag{4.166}$$

$$(3.102)\ :\ V_b = \nu_b(\mu_1 + \mu_b^*)H_0, \tag{4.167}$$

$$(4.165)\ :\ \frac{d\mu_1}{d\tau} = -\frac{\alpha_-^2}{|\Omega|}\left[m_b\nu_b\left(\mu_1+\mu_b^*\right)\int_{\Gamma_b} H_0^2\,ds + \varepsilon m_p\nu_p\left(\mu_1+\mu_p^*\right)\int_{\Gamma_p}|\vec{\kappa}|^2\,ds\right], \tag{4.168}$$

where we introduce the constants

$$\begin{aligned} \nu_b &:= \tfrac{m_b}{B_1}, & \nu_p &:= -\tfrac{2S_1}{S_2}, \\ \mu_b^* &:= \tfrac{1}{2}(\eta_1+\eta_2)\tfrac{\sigma_b}{m_b}, & \mu_p^* &:= -\tfrac{\eta_1 S_4}{2S_1}. \end{aligned} \tag{4.169}$$

The competitive evolution of the bilayers and pores couples through curvature weighted surface area. However, the two morphologies seek differing equilibria values, which typically satisfy $\nu_b^* > \nu_p^*$, making coexistence of bilayers and pores impossible under the strong functionalization, unless one of the structures is flat, since zero curvature interfaces are at equilibrium independent of chemical potential. For curved interfaces, the range $\mu_1 \in [\mu_p^*, \mu_b^*]$ is invariant under the flow, and once μ_1 enters this range the bilayers will shrink, while the pore morphologies will grow. In section 7.2 we will show numerically the equilibria of each system for a specific choice of double-well potential, and the dynamically invariant interval is described in Figure 7.2.

Chapter 5

The Pearling Eigenvalue Problem, Co-Dimension 1

In this chapter we address the linear stability of the bilayer morphology in the strong FCH and obtain an explicit expression for the pearling stability condition. We present a rigourous analysis of the eigenvalue problem corresponding to the strong FCH for the co-dimension one structure. We show that in the **strong FCH** scaling the leading order behavior of the pearling eigenvalues is independent of the shape of the underlying co-dimension one morphology. Under the H^{-1} gradient flow the pearling instability manifests itself on a time scale that is $O(\varepsilon^{-2})$ faster than the geometric evolution, and hence can be taken to be instantaneous on the geometric evolution time scale. Conversely, the fingering instability occurs on the same time scale as the geometric flow, and may not necessarily immediately manifest itself on the geometric evolution time scale.

Recall the strong FCH free energy which corresponds to the choice $p = 1$ in (1.14),

$$\mathcal{F}(u) = \int_\Omega \frac{1}{2}(\varepsilon^2\Delta u - W'(u))^2 - \varepsilon\left(\frac{\varepsilon^2\eta_1}{2}|\nabla u|^2 + \eta_2 W(u)\right) dx, \tag{5.1}$$

where $\Omega \subset \mathbb{R}^d,\ d \geq 2$, is a bounded domain, $W(u)$ is a tilted double-well potential with two minima at $b_\pm$, $u : \Omega \to \mathbb{R}$ is the density of one of the amphiphilic species, $\varepsilon \ll 1$ controls the width of the boundary layer and η_1 and η_2 are the functionalization constants. The first variation of $\mathcal{F}$, introduced in equation (1.18), is given by

$$\frac{\delta\mathcal{F}}{\delta u}(u) = (\varepsilon^2\Delta - W''(u) + \varepsilon\eta_1)(\varepsilon^2\Delta u - W'(u)) + \varepsilon\eta_d W'(u), \tag{5.2}$$

where $\eta_d := \eta_1 - \eta_2$. The second variation of $\mathcal{F}$, evaluated at a critical point of $\mathcal{E}$, takes the form

$$\mathbb{L}_b := \frac{\delta^2 \mathcal{F}}{\delta u^2}(u) = \left(\varepsilon^2\Delta - W''(u) + \varepsilon\eta_1\right)\left(\varepsilon^2\Delta - W''(u)\right) - \left(\varepsilon^2\Delta u - W'(u)\right)W'''(u) + \varepsilon\eta_d W''(u). \tag{5.3}$$

We obtain a pearling stability condition for the co-dimension one morphology which is summarized in the following theorem-

Theorem 5.0.1. *For a given admissible interface, Γ_b, the associated bilayer solution constructed in (2.37), is stable with respect to the pearling bifurcation if and only if the far-field chemical potential μ_1 satisfies the* pearling stability condition

$$P_b^* := -\frac{\lambda_{b,0}\eta_d \|\psi_{b,0}\|^2_{L^2(\Omega)}}{\alpha_-^2 S_b} > \mu_1. \tag{5.4}$$

5.1 Overview

We want to investigate the pearling eigenmodes of the co-dimension one bilayer structure: given an admissible interface $\Gamma_b \in \mathcal{G}_{K,\ell}$, assume that the system is at quasi-equilibrium, as defined in (1.15), with

$$u_b = U_b(z) + \varepsilon u_1, \tag{5.5}$$

where U_b is the homoclinic bilayer solution introduced in (2.37), and u_1, derived in equation (3.74), is given by

$$L_{b,0}u_1 = \mu_1\Phi_{b,1} - \eta_d\left(\frac{z}{2}U_b'\right) \quad \Rightarrow \quad u_1 = \mu_1\Phi_{b,2} - \eta_d L_{b,0}^{-1}\left(\frac{z}{2}U_b'\right), \tag{5.6}$$

where $L_{b,0}$ is the linear operator introduced in (2.39), the chemical potential μ_1 is spatially constant and the functions $\Phi_{b,j}$ solves (2.40) for $j = 1, 2$.

To show that u_b, defined in equation (5.5), is a quasi-equilibrium, as defined in (1.15), we plug (5.5) into the first variation, (5.2), which yields

$$\frac{\delta\mathcal{F}}{\delta u}(u_b) = (\varepsilon^2\Delta - W''(u_b) + \varepsilon\eta_1)(\varepsilon^2\Delta u_b - W'(u_b)) + \varepsilon\eta_d W'(u_b). \tag{5.7}$$

Expanding the Laplacian in local coordinates, (2.9), and Taylor expanding the potential terms $W(u_b)$ yields

$$\begin{aligned}\frac{\delta\mathcal{F}}{\delta u}(u_b) =& \left[L_{b,0} + \varepsilon\left(H\partial_z - W'''(U_b)u_1 + \eta_1\right) + \varepsilon^2\Delta_G\right] \circ \left[\varepsilon(H\partial_z U_b + L_{b,0}u_1) + \varepsilon^2 H\partial_z u_1\right] + \varepsilon\eta_d W'(U_b) \qquad (5.8)\\ &+ O(\varepsilon^2)\\ =& \varepsilon(L_{b,0}^2 u_1 + \eta_d W'(U_b)) \qquad (5.9)\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon^2\Big(H^2U_b'' + H\partial_z L_{b,0}u_1 - H(W''(U_b))'u_1 + \eta_1 HU_b' - W'''(U_b)u_1 L_{b,0}u_1 \\
&\qquad + \eta_1 L_{b,0}u_1 + \eta_d W''(U_b)u_1\Big) + O(\varepsilon^3) \\
&= \varepsilon\mu_1 + O(\varepsilon^2),
\end{aligned} \tag{5.10}
$$

where the second equality follows from the definition of u_1 in (5.6). The leading order term is specially constant, while the $O(\varepsilon^2)$ terms in (5.8) are localized on Γ_b and constant on the reach. Using Π_0 to project away the constant part of (5.8), yields terms that are $O(\varepsilon^2)$ in L^∞ and zero off of the reach, taking the L^2-norm yields

$$
\left\| \Pi_0 \frac{\delta \mathcal{F}}{\delta u}(u_b) \right\|^2_{L^2(\Omega)} = O(\varepsilon^{\frac{5}{2}}). \tag{5.11}
$$

We see that u_b satisfies the definition of quasi-equilibrium, given in (1.15).

We are interested in the pearling eigenmodes of the second variation of $\mathcal{F}$, $\mathbb{L}_b$, defined in (5.3). Consider the eigenvalue problem

$$
\mathbb{L}_b\Psi = \Lambda\Psi. \tag{5.12}
$$

By changing coordinates of the Laplacian, in the operator $\mathbb{L}_b$, to the whiskered coordinates, using (2.9), and plugging-in the expansion of u_b, (5.5), into u, $\mathbb{L}_b$ can be written in orders of ε as

$$
\mathbb{L}_b = \mathcal{L}_b^2 + \varepsilon\mathbb{L}_1 + \varepsilon^2\mathbb{L}_2 + O(\varepsilon^3), \tag{5.13}
$$

where $\mathcal{L}_b$ introduced in (2.45), and the operators $\mathbb{L}_1$, $\mathbb{L}_2$ take the form

$$
\mathbb{L}_1 := -\mathcal{L}_b \circ (W'''(U_b)u_1) - (W'''(U_b)u_1 - \eta_1)\mathcal{L}_b - (HU_b' + \mathcal{L}_b u_1)W'''(U_b) + \eta_d W''(U_b), \tag{5.14}
$$

$$
\begin{aligned}
\mathbb{L}_2 :=& -\mathcal{L}_b \circ \left(W'''(U_b)u_2 + \frac{1}{2}W^{(4)}(U_b)u_1^2\right) - \left(W'''(U_b)u_2 + \frac{1}{2}W^{(4)}(U_b)u_1^2\right)\mathcal{L}_b \\
&+ (W'''(U_b)u_1 + \eta_1)W'''(U_b)u_1 - \left(\mathcal{L}_b u_2 - \frac{1}{2}W'''(U_b)u_1^2\right)W'''(U_b) \\
&- (\mathcal{L}_b u_1 + HU_b')\,W^{(4)}(U_b)u_1 - \eta_d W'''(U_b)u_1,
\end{aligned} \tag{5.15}
$$

see appendix D.1 for details. Note that for $i \geq 1$, the unbounded term in the operators $\mathbb{L}_i$ is $\mathcal{L}_b$, and we can write $\mathbb{L}_b$ in the following form

$$
\mathbb{L}_b = \mathcal{L}_b^2 + \varepsilon\tilde{\mathbb{L}}_b, \tag{5.16}
$$

where $\mathbb{L}_b$ is a relatively bounded perturbation of $\mathcal{L}_b^2$. The eigenvalues of $\mathcal{L}_b^2$ are described in Figure 2.3 (right) where the boxed area contains the pearling eigenvalues.

Recall that $\Sigma_{b,0}$, defined in (2.48), is the set of small eigenvalues associated to $\mathbb{L}_b$, and, according to Weyl's asymptotic formula $|\Sigma_{b,0}| \sim O(\varepsilon^{3/2-d})$. We define

$$P_k := \varepsilon^{-1/2}(\lambda_{b,0} - \varepsilon^2 \beta_k), \tag{5.17}$$

to be the detuning constant depending only on k.

Definition 5.1. *The space, X_Σ, corresponding to the small eigenvalues of $\mathbb{L}_b$ is defined as*

$$X_\Sigma := \{\psi_{b,0}\Theta_k \mid k \in \Sigma_{b,0}\} \cup \{\psi_{b,1}\Theta_k \mid k \in \Sigma_{b,1}\}. \tag{5.18}$$

The meander modes are accounted for in the geometric motion, however, we expand only the pearling modes, for brevity. Looking for solutions of the eigenvalue problem, (5.12), we consider a regular perturbation expansion of the form

$$\Psi_j = \Psi_{0,j} + \varepsilon\Psi_{1,j} + O(\varepsilon^2), \quad \Psi_{0,j} \in X_\Sigma, \quad \Psi_{0,j} = \sum_{k\in\Sigma} \alpha_k \psi_{b,0}\Theta_k, \quad \Psi_{1,j} \in X_\Sigma^\perp, \tag{5.19}$$

$$\Lambda_j = \varepsilon\Lambda_{1,j} + O(\varepsilon^2). \tag{5.20}$$

The L^2-orthogonal projection, Π, onto X_Σ is given by

$$\Pi\, f := \sum_{k\in\Sigma} \frac{(f, \psi_{b,0}\Theta_k)_{L^2(\Omega)}}{\|\psi_{b,0}\Theta_k\|^2_{L^2(\Omega)}} \psi_{b,0}\Theta_k = \sum_{k\in\Sigma} (f, \psi_{b,0}\Theta_k)_{L^2(\Omega)} \psi_{b,0}\Theta_k, \tag{5.21}$$

and its complementary projection is $\tilde{\Pi} = I - \Pi$.

We consider a decomposition of the operator $\mathbb{L}_b$ into a 2×2 block form,

$$\begin{bmatrix} M & B \\ B^T & C \end{bmatrix}, \tag{5.22}$$

where

$$M := \Pi\mathbb{L}_b\Pi, \quad B := \Pi\mathbb{L}_b\tilde{\Pi}, \quad C := \tilde{\Pi}\mathbb{L}_b\tilde{\Pi}. \tag{5.23}$$

By abuse of notation we denote $\mathbb{L}_b$ and its 2×2 decomposition with the same symbol.

[Hayrapetyan and Promislow, 2014] have shown that the restricted operator C is uniformly coercive on $X_\Sigma^\perp$

and its spectrum is bounded from below by $\delta > 0$ which may be chosen independent of sufficiently small $\varepsilon > 0$. In section 5.2 we investigate the operator M, describing its matrix representation and, develop an expression for its pearling eigenvalues, presented in Theorem 5.2.3. In section 5.3, we will show that B, B^T have an ε-bounds as operators from $l^2(\mathbb{R}^{N\times N})$ to $l^2(\mathbb{R}^{N\times N})$. In section 5.4, we conclude, using both semi-group estimates and a perturbation argument, that the spectrum of M determines the pearling eigenmodes of $\mathbb{L}_b$. Finally, in section 5.5, we connect the pearling eigenvalues of $\Delta\mathbb{L}_b$ to those of $\mathbb{L}_b$.

5.2 Eigenvalues of $M := \Pi\mathbb{L}_b\Pi$

Let $v \in X_\Sigma$, i.e., v can be written as

$$v = \sum_{k\in\Sigma} b_k \psi_{b,0}\Theta_k, \tag{5.24}$$

without loss of generality, assume $\|v\|_{L^2} = 1$. The operator $\Pi\mathbb{L}_b\Pi$, acting on v, takes the form

$$\Pi\mathbb{L}_b\Pi \sum_{j\in\Sigma} b_j\psi_{b,0}\Theta_k = \sum_{k\in\Sigma}\left(\mathbb{L}_b \sum_{j\in\Sigma} b_j\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k\right)_{L^2(\Omega)} \psi_{b,0}\Theta_k = \sum_{k\in\Sigma}\sum_{j\in\Sigma} b_j \left(\mathbb{L}_b\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k\right)_{L^2(\Omega)} \psi_{b,0}\Theta_k. \tag{5.25}$$

We define the operator matrix representation $M \in \mathbb{R}^{N_d\times N_d}$, where $N_d \approx \varepsilon^{3/2-d}$, in the following way

$$M_{j,k} := (\mathbb{L}_b\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k)_{L^2(\Omega)}. \tag{5.26}$$

Using the expansion of $\mathbb{L}_b$, (5.13), we can write each entry of M in orders of ε such that

$$(\mathbb{L}_b\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k)_{L^2(\Omega)} = (\mathcal{L}_b^2\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k)_{L^2(\Omega)} + \varepsilon(\mathbb{L}_1\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k)_{L^2(\Omega)} + O(\varepsilon^2), \tag{5.27}$$

and collect the matrix terms into two classes such that

$$M = M^0 + \varepsilon^q\tilde{M}, \tag{5.28}$$

where

$$M^0_{j,k} = (\mathcal{L}_b^2\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k)_{L^2(\Omega)} + \varepsilon(\mathbb{L}_1\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k)_{L^2(\Omega)} + \sum_{i=2}^{q}\varepsilon^i(\mathbb{L}_i\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k)_{L^2(\Omega)}, \tag{5.29}$$

$$\tilde{M}_{j,k} = \sum_{i\geq q}\varepsilon^{(i-q)}(\mathbb{L}_i\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k)_{L^2(\Omega)}. \tag{5.30}$$

We will show that the first term, M^0, can be split into a diagonal and off-diagonal terms, the latter of which can be bounded independently of the matrix size N_d, if the curvatures of the interface Γ_b are sufficiently smooth. The other term, $\tilde{M}$, can be bounded, independent of the dimension, via the L^∞ norm, for q suitably large, depending upon the dimension, d.

5.2.1 Bounding $\tilde{M}$

To establish the bound on $\tilde{M}$, we start with the definition of the l^2-norm of a matrix, followed by a lemma which establish a bound on the l^2-norm using the l^∞ norm:

Definition 5.2. *The induced l^2-norm of a matrix A is given by,*

$$||A||_{l^2} := \sup_{||v||_{l^2}\neq 0} \frac{||Av||_{l^2}}{||v||_{l^2}}. \tag{5.31}$$

Lemma 5.1. *Given a matrix $A \in \mathbb{R}^{N\times N}$, there exist $C > 0$ such that*

$$||A||_{l^2} < C\sqrt{N}||A||_{l^\infty}. \tag{5.32}$$

Proof. Let $v \in \mathbb{R}^N$ with $||v||_{l^2} = 1$. Then,

$$||A||_{l^2} = \sup_{||v||_{l^2}\neq 0} \frac{||Av||_{l^2}}{||v||_{l^2}} = \sqrt{\sum_{j=1}^{N}\left|\sum_{k=1}^{N} A_{j,k}v_k\right|^2} \leq \sqrt{||A||_{l^\infty}^2 \sum_{j=1}^{N}\left|\sum_{j=1}^{N} v_k\right|^2} \leq ||A||_{l^\infty}\sqrt{\sum_{j=1}^{N}||v||_{l^2}} \leq ||A||_{l^\infty}\sqrt{N}. \tag{5.33}$$

■

Corollary 5.2.1. *If $q > \frac{1}{4} + \frac{d}{2}$ and $||\tilde{M}||_{l^\infty} = 1$ then $\varepsilon^p||\tilde{M}||_{l^2} \ll C\varepsilon$, where C is a constant independent of N_d.*

Proof. Since $||\tilde{M}||_{l^\infty} = 1$, applying Lemma (5.1) to $\tilde{M}$ yields $||\tilde{M}||_{l^2} \leq C\sqrt{\varepsilon^{3/2-d}} = C\varepsilon^{3/4-d/2}$, for some constant C. ■

Corollary 5.2.1 implies that for $d = 2, 3$ it suffices to choose $q > \frac{5}{4}, \frac{7}{4}$ respectively, to render the $\varepsilon^q\tilde{M}$ term $O(\varepsilon)$ in the induced l^2-norm.

5.2.2 Bounding M^0

Next, we want to find a bound in $\mathbb{R}^d,\ d \geq 2$, for the matrix M^0. An examination of the first two terms of M^0, given in equation (5.29), shows that they admit the expansions

$$(\mathcal{L}_b^2\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k)_{L^2(\Omega)} = \begin{cases} \varepsilon P_k^2 + O(\varepsilon^2) & \text{if } k = j, \\ \varepsilon^2 \int_\Gamma H_0^2 \Theta_k \Theta_j \, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} ((\psi_{b,0}^0)')^2 \, dz + O(\varepsilon^2\sqrt{\varepsilon}) & \text{if } k \neq j, \end{cases} \tag{5.34}$$

$$(\mathbb{L}_1\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k)_{L^2(\Omega)} = \begin{cases} -\int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} W'''(U_b)(\psi_{b,0}^0)^2 L_{b,0}u_1 \, dz + \eta_d \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} W''(U_b)(\psi_{b,0}^0)^2 \, dz + O(\sqrt{\varepsilon}) & \text{if } k = j, \\ -\varepsilon \int_\Gamma H_1 \Theta_k \Theta_j \, ds \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} W'''(U_b) U_b' (\psi_{b,0}^0)^2 z \, dz + O(\varepsilon^3) & \text{if } k \neq j \end{cases} \tag{5.35}$$

(see Appendix D.2 for calculation details, specifically, equations (D.5) and (D.14)). We may split M^0 into its on/off diagonal matrices

$$M^0 = M^0_{\text{diag}} + M^0_{\text{off-diag}} \tag{5.36}$$

where

$$M^0_{\text{diag}}(j,k) = \begin{cases} M^0_{k,k} + O(\varepsilon\sqrt{\varepsilon}) & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \tag{5.37}$$

and

$$M^0_{\text{off-diag}}(j,k) = \begin{cases} 0 & \text{if } j = k, \\ M^0_{j,k} + O(\varepsilon^2\sqrt{\varepsilon}) & \text{if } j \neq k. \end{cases} \tag{5.38}$$

with entries given by

$$M^0_{k,k} = \varepsilon \left(P_k^2 - \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} \left[W'''(U_b) L_{b,0} u_1 - \eta_d W''(U_b) \right] (\psi_{b,0}^0)^2 \, dz \right) = \varepsilon \left(P_k^2 - \mu_1 S - \eta_d \lambda_{b,0} \|\psi_{b,0}\|_2^2 \right), \tag{5.39}$$

$$\begin{aligned} M^0_{j,k} &= \varepsilon^2 \int_\Gamma H^2 \Theta_k \Theta_j J_0 \, ds \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} ((\psi_{b,0}^0)')^2 z \, dz - \varepsilon^2 \int_\Gamma H_1 \Theta_k \Theta_j J_0 \, ds \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} W'''(U_b) U_b' (\psi_{b,0}^0)^2 z \, dz, \\ &= -\varepsilon^2 \left(S_1 \int_\Gamma H_0^2 \Theta_k \Theta_j J_0 \, ds + S_2 \int_\Gamma H_1 \Theta_k \Theta_j J_0 \, ds \right), \end{aligned} \tag{5.40}$$

with indices $a\varepsilon^{3/2-d} \leq j,k \leq \tilde{a}\varepsilon^{3/2-d}$, $a < \tilde{a},\ a, \tilde{a} \in \mathbb{R}$, S_b is called the "shape factor", and $S_b,\ S_1,\ S_2$ are given by

$$S_b := \int_{-l/\varepsilon}^{l/\varepsilon} \varphi_1 W'''(U) \psi_{b,0}^2 \, dz, \tag{5.41}$$

$$S_1 := \int_{-l/\varepsilon}^{l/\varepsilon} ((\psi_{b,0}^0)')^2 \, dz, \tag{5.42}$$

$$S_2 \coloneqq \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} W'''(U_b)U_b'(\psi_{b,0}^0)^2 z\, dz. \tag{5.43}$$

See appendix D.3 for the derivation of the second equality in equation (5.39).

The entries of M^0_{diag} are $O(\varepsilon)$. If we can bound $M^0_{\text{off-diag}}$ independently of ε then the eigenvalues of M^0 are given, at leading order, by the diagonal entries of M^0_{diag}. To keep $M^0_{\text{off-diag}}$ order of ε^2 we need to bound the two integrals on the right-hand side of equation (5.40). Since S_1 and S_2 are bounded, the main issue is to bound the terms

$$\int_\Gamma H_1\Theta_k\Theta_j J_0\, ds, \text{ and } \int_\Gamma H_0^2\Theta_k\Theta_j J_0\, ds. \tag{5.44}$$

We can write the two terms is (5.44) in a more generic form as

$$\int_\Gamma f(\vec{k}_b)\Theta_k\Theta_j\, J_0 ds \tag{5.45}$$

where $f(\vec{k}_b)$ is a polynomial of the curvatures, $\vec{k}_b$ defined in Definition 2.1.

Lemma 5.2. *Let $\Gamma_b \in \mathbb{R}^d$ be an admissible interface, then, in particular $\vec{k}_b \in W^{2,\infty}$. Let $f : \mathbb{R}^{d-1} \to \mathbb{R}$ be a bounded function, and define the matrix $M \in \mathbb{R}^{N\times N}$, $N \in \mathbb{R}$ with entries*

$$M_{i,j} = \int_\Gamma f(\vec{k}_b)\Theta_i\Theta_j\, J_0 ds, \tag{5.46}$$

where Θ_k are the eigenfunctions of Laplace-Beltrami operator; then, there exists $C > 0$ independent of ε such that

$$\| M \|_{l^2\to l^2} \le C. \tag{5.47}$$

Proof. The operator norm of M from l^2 to l^2 is defined by

$$\| M \|_{l^2\to l^2} \coloneqq \inf\{c > 0 \;\Big|\; |(Mv, w)| \le c \| v \|_{l^2} \| w \|_{l^2}, \text{ for all } v, w \in \mathbb{R}^N\}. \tag{5.48}$$

Let $v, w \in \mathbb{R}^N$, using the definition of M, (5.46), we can write

$$|(Mv, w)| = \left|\sum_{i,j}\int_\Gamma f(s)\Theta_i v_i\Theta_j w_j\, J_0 ds\right| = \left|\int_\Gamma f(s)(\sum_i \Theta_i v_i)(\sum_j \Theta_j w_j)\, J_0 ds\right|, \tag{5.49}$$

and applying Hölder's inequality to this last integral yields

$$|(Mv, w)| \leq \| f \|_{L^\infty(\Gamma)} \left\| \sum_i \Theta_i v_i \right\|_{L_2(\Gamma)} \left\| \sum_j \Theta_j w_j \right\|_{L_2(\Gamma)}. \tag{5.50}$$

Calculating the last norm in (5.50) yields

$$\left\| \sum_j \Theta_j w_j \right\|^2_{L^2(\Gamma)} = \int_\Gamma \left(\sum_j \Theta_j w_j \right)^2 J_0 ds = \sum_j \int_\Gamma \Theta_j^2 w_j^2 \, ds = \sum_j w_j^2 \int_\Gamma \Theta_j^2 \, dS = \| w \|^2_{l^2} \tag{5.51}$$

where the second equality follows form the orthogonality of the Laplace-Beltrami eigenfunctions in the Γ-inner product, see equation (2.71). Similarly, we have

$$\left\| \sum_j \Theta_j v_j \right\|^2_{L^2(\Gamma)} = \| v \|^2_{l^2}. \tag{5.52}$$

Plugging (5.51) and (5.52) back into (5.50) yields

$$|(Mv, w)| \leq \| f \|_{L^\infty(\Gamma)} \| v \|_{l^2} \| w \|_{l^2}, \tag{5.53}$$

and by choosing $C = \| f \|_{L^\infty(\Gamma)}$ and using the operator norm definition, (5.48), we obtain the desired bound,

$$\| M \|_{l^2 \to l^2} \leq C. \tag{5.54}$$

■

Corollary 5.2.2. *The matrix M^0, defined in (5.36), can be written as*

$$M^0 = M^0_{diag} + M^0_{off\text{-}diag}, \tag{5.55}$$

where $M^0_{off\text{-}diag}$ is uniformly bounded as an operator from l^2 to l^2.

Theorem 5.2.3. *The pearling eigenvalues of $\Pi \mathbb{L}_b \Pi$, (5.20), take the leading order form*

$$\Lambda = -\varepsilon \frac{1}{\|\psi_{b,0}\|^2_{L^2(\Omega)}} \left(\mu_1 S_b + \eta_d \lambda_{b,0} \|\psi_{b,0}\|^2_{L^2(\Omega)} \right) + O(\varepsilon^2), \tag{5.56}$$

and, the associated co-dimension one bilayer network is pearling stable if and only if

$$\mu_1 S_b + \eta_d \lambda_{b,0} \|\psi_{b,0}\|^2_{L^2(\Omega)} < 0. \tag{5.57}$$

Proof. Corollary 5.2.2 implies that the eigenvalues of M^0, Λ_k, are, at leading order, the diagonal entries of M^0_{diag}, defined in equation (5.37). From the definition of M, (5.28), and Corollary 5.2.1, we deduce that Λ_k are the eigenvalues of M, at leading order. Since M is the matrix representation of $\Pi\mathbb{L}_b\Pi$, the eigenvalues of $\Pi\mathbb{L}_b\Pi$ are, at leading order, Λ_k, which takes the form

$$\Lambda_k = \varepsilon(P_k^2 - \mu_1 S_b - \eta_d \lambda_{b,0}\|\psi_{b,0}\|^2_{L^2(\Omega)}), \tag{5.58}$$

where S_b is the shape factor defined in (5.41) and P_k is the detuning constant defined in (5.17).

We want to find a lower bound for the eigenvalues of the pearling modes: we have $o(\varepsilon^{3/2-d})$ possible values for $k \in \Sigma_{b,0}$, for which $(\lambda_{b,0} - \varepsilon^2\beta_k) \sim O(\sqrt{\varepsilon})$ and ,for $d = 2$, the Laplace-Beltrami eigenvalues takes the form $\beta_k = \left(\frac{2\pi k}{L}\right)^2$. The distance between two successive scaled eigenvalues is

$$\varepsilon^2\beta_{k+1} - \varepsilon^2\beta_k \sim \varepsilon^2\left(\frac{2\pi}{L}\right)^2(2k+1). \tag{5.59}$$

To determine how close $\varepsilon^2\beta_k$ can get to $\lambda_{b,0}$ we choose k_0 such that $\lambda_{b,0} = \varepsilon^2\left(\frac{2\pi k_0}{L}\right)^2$, so if $k_0 = \frac{\varepsilon^{-1}\sqrt{\lambda_{b,0}}L}{2\pi}$, the closest we can guarantee that $\varepsilon^2\beta_k$ approaches to $\lambda_{b,0}$ is

$$\varepsilon^2\left(\frac{2\pi}{L}\right)^2(2k_0+1) = \varepsilon^2\left(\frac{2\pi}{L}\right)^2\left(\frac{\varepsilon^{-1}\sqrt{\lambda_{b,0}}L}{\pi} + 1\right) \sim O(\varepsilon). \tag{5.60}$$

Recall that $P_k := \varepsilon^{-1/2}(\lambda_{b,0} - \varepsilon^2\beta_k)$ then, the distance between two sequential terms is

$$P_{k+1} - P_k = \varepsilon^{-1/2}(\lambda_{b,0} - \varepsilon^2\beta_{k+1}) - \varepsilon^{-1/2}(\lambda_{b,0} - \varepsilon^2\beta_k) = \varepsilon^{-1/2}(\varepsilon^2\beta_k - \varepsilon^2\beta_{k+1}) = -\varepsilon^{-1/2}\varepsilon^2\left(\frac{2\pi}{L}\right)^2(2k+1) \sim O(\varepsilon^{1/2}). \tag{5.61}$$

Therefore, the detuning parameter P_k satisfies $O(\varepsilon) \le P_k^2 \le O(1)$ for $a\varepsilon^{-1/2} \le k \le \tilde{a}\varepsilon^{-1/2}$. This shows that P_k^2 can be made as small as $O(\varepsilon)$ and therefore it is lower order near the turning point of the pearling spectrum. We conclude that the pearling eigenvalues of $\Pi\mathbb{L}_b\Pi$, (5.20), takes the form

$$\Lambda = -\varepsilon\frac{1}{\|\psi_{b,0}\|^2_{L^2(\Omega)}}\left(\mu_1 S_b + \eta_d\lambda_{b,0}\|\psi_{b,0}\|^2_{L^2(\Omega)}\right) + O(\varepsilon^2), \tag{5.62}$$

■

Note that for a generic interface we recover the same pearling conditions as for interfaces with constant curvature, see [Doelman et al., 2014] for more details.

Recall that our main goal is to find an expression for the pearling eigenvalues of $\mathbb{L}_b$ using our 2×2 represen-

tation of $\mathbb{L}_b$, see (5.22). In this section we found an expression the pearling eigenvalues of the operator M. The next section establish the bounds on the off-diagonal terms B, B^T.

5.3 Bounding the Off-Diagonal Operators

Recall the 2×2 block form representation of $\mathbb{L}_b$, given in (5.22),

$$\begin{bmatrix} M & B \\ B^T & C \end{bmatrix}. \tag{5.63}$$

If the off-diagonal blocks, $B := \Pi \mathbb{L}_b \tilde{\Pi}$ and $B^T := \tilde{\Pi} \mathbb{L}_b \Pi$, are small (same order of the $\Pi \mathbb{L}_b \Pi$ block or less) then we can relate the eigenvalues of $\mathbb{L}_b$ to those of $M := \Pi \mathbb{L}_b \Pi$, see section 5.4. Since both Π and $\tilde{\Pi}$ are self-adjoint operators we have

$$(\Pi \mathbb{L}_b \tilde{\Pi} v, w)_{L^2} = (\mathbb{L}_b \tilde{\Pi} v, \Pi w)_{L^2} = (\tilde{\Pi} v, \mathbb{L}_b \Pi w)_{L^2} = (v, \tilde{\Pi} \mathbb{L}_b \Pi w)_{L^2}. \tag{5.64}$$

So, it is enough to show that one of the off-diagonal blocks is small, i.e., we want to show that there exist a constant C, independent on $N_d \sim O(\varepsilon^{3/2-d})$ such that

$$\|\tilde{\Pi} \mathbb{L}_b \Pi v\|_{L^2(\Omega)} \leq \varepsilon C \|v\|_{L^2(\Omega)}, \qquad \forall v \in X_\Sigma. \tag{5.65}$$

without loss of generality, assume $v \in X_\Sigma$, $v = \sum_{j \in \Sigma} b_j \psi_{b,0} \Theta_j$ and $\|v\|_{L^2(\Omega)} = 1$. Note that

$$\|v\|^2_{L^2(\Omega)} = \int_\Omega \sum_{j,k \in \Sigma} b_j b_k \Theta_j \Theta_k \psi^2_{b,0} \, dx = \sum_{j \in \Sigma} b_j^2 \overbrace{\int_\Omega \Theta_j^2 \psi^2_{b,0} \, dx}^{\|\theta_k \psi_{b,0}\|^2_{L^2(\Omega)} = 1} + \sum_{\substack{j,k \in \Sigma \\ j \neq k}} b_j b_k \overbrace{\int_\Omega \Theta_j \Theta_k \psi^2_{b,0} \, dx}^{=0, \text{by orthogonality of } \Theta_j} = \sum_{j \in \Sigma} b_j^2 = \| b \|^2_{l^2}, \tag{5.66}$$

where $b := (b_1, b_2, ..., b_{N_d})$.

Since $\mathbb{L}_b = \mathcal{L}_b^2 + \varepsilon \tilde{\mathbb{L}}_b$, and $\tilde{\mathbb{L}}_b$ is relatively bounded with respect to $\mathcal{L}_b^2$, we split the proof into three parts: first we show that we appropriately bound the operator $\tilde{\Pi} \mathcal{L}_b \Pi v$, next we appropriately bound the operator $\tilde{\Pi} \mathcal{L}_b^2 \Pi v$ and at finally we appropriately bound $\tilde{\Pi} \mathbb{L}_b \Pi v$.

5.3.1 Bounding $\tilde{\Pi}\mathcal{L}_b\Pi v$

Recall that $v \in X_\Sigma,\ v = \sum_{j\in\Sigma} b_j\psi_{b,0}\Theta_j$ and $\|v\|_{L^2(\Omega)} = 1$. In particular,

$$\Pi v = v, \qquad \tilde{\Pi} v = 0. \tag{5.67}$$

We need to show that there exist C_1 such that

$$\|\tilde{\Pi}\mathcal{L}_b v\|_{L^2(\Omega)} \le \varepsilon C_1 \|v\|_{L^2(\Omega)}. \tag{5.68}$$

Using the expression for $\mathcal{L}_b$, (2.45), yields

$$\mathcal{L}_b v = (L_{b,0}v + \varepsilon H\partial_z v + \varepsilon^2\Delta_G v) = \sum_{j\in\Sigma} b_j \left(\underbrace{\lambda_{b,0}\psi_{b,0}\Theta_j}_{\in X_\Sigma} + \varepsilon H\Theta_j\psi'_{b,0} + \varepsilon^2\Delta_G\psi_{b,0}\Theta_j \right). \tag{5.69}$$

The projection $\tilde{\Pi}$ is the orthogonal projection to X_Σ, there it eliminates the first term and the operator $\tilde{\Pi}\mathcal{L}_b v$ takes the form

$$\tilde{\Pi}\mathcal{L}_b v = \varepsilon\tilde{\Pi}\sum_{j\in\Sigma} b_j \left(H\Theta_j\psi'_{b,0} + \varepsilon\Delta_G\psi_{b,0}\Theta_j \right). \tag{5.70}$$

The L^2-norm of equation (5.70) is given by

$$\left\| \tilde{\Pi}\mathcal{L}_b v \right\|_{L^2(\Omega)} = \left\| \varepsilon\tilde{\Pi}\sum_{j\in\Sigma} b_j \left(H\Theta_j\psi'_{b,0} + \varepsilon\Delta_G\psi_{b,0}\Theta_j \right) \right\|_{L^2(\Omega)} \le \varepsilon \left\| \overbrace{\sum_{j\in\Sigma} b_j H\Theta_j\psi'_{b,0}}^{\mathcal{R}v} \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi}\Delta_G v \right\|_{L^2(\Omega)}, \tag{5.71}$$

where we used triangle inequality and the fact that $\|\tilde{\Pi}u\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)}$ for each u.
We define the operator $\mathcal{R}$ matrix representation $B \in \mathbb{R}^{N_d\times N_d}$, with $N_d \approx \varepsilon^{3/2-d}$ such that

$$B_{j,k} := \left\langle H\Theta_j\psi'_{b,0}\ ,\ H\Theta_j\psi'_{b,0} \right\rangle_{L^2(\Omega)}. \tag{5.72}$$

The entries of B take the form

$$B_{k,j} = \int_\Gamma f\Theta_k\Theta_j J_0\, ds, \tag{5.73}$$

and we can apply Lemma 5.2 to conclude that B is uniformly bounded as an operator from $l^2(\mathbb{R}^N)$ to $l^2(\mathbb{R}^N)$. Since B is the matrix representation of $\mathcal{R}$ we have

$$\| \mathcal{R}v \|^2_{L^2(\Omega)} = \left\| bBb^T \right\|^2_{L^2(\Omega)} \leq \| b \|^2_{l^2} \| B \|^2_{l^2 \to l^2} . \tag{5.74}$$

Theorem 5.2 and (5.74) implies

$$\| \mathcal{R}v \|_{L^2(\Omega)} \leq c \| v \|_{L^2} . \tag{5.75}$$

Going back to (5.71) and plugging in (5.75) yields

$$\left\| \tilde{\Pi} \mathcal{L}_b v \right\|_{L^2(\Omega)} \leq \varepsilon c \| v \|_{L^2} + \varepsilon^2 \left\| \tilde{\Pi} \Delta_G v \right\|_{L^2(\Omega)} \tag{5.76}$$

The following Proposition shows that the $L^2(\Omega)$-norm of $\tilde{\Pi}\Delta_G v$ can be bounded as an operator in the $L^2(\Omega)$ norm:

Proposition 5.3.1. *Let $f(z)$ be a smooth function such that*

$$|f(z)| < c_1 e^{-c_2|z|} \text{ for some } c_i \in \mathbb{R},\ c_i > 0\ , i = 1, 2,\ supp(f) \subset \Gamma_l. \tag{5.77}$$

The operator $\tilde{\Pi}\Delta_G$, where $\tilde{\Pi}$ is the projection off of the space of small eigenvalues X_Σ, defined in (5.18) and Δ_G is defined in (2.12), is bounded on the space

$$Y = \{ f(z)\Theta_k \ \Big|\ k \in \Sigma, \}, \tag{5.78}$$

i.e., there exists $C > 0$ such that

$$\left\| \tilde{\Pi} \Delta_G v \right\|_{L^2(\Omega)} \leq C\varepsilon^{-2} \| v \|_{L^2(\Omega)} , \tag{5.79}$$

for every $v \in Y$. Particularly, for $v \in x_\Sigma$ we obtain the bound

$$\left\| \tilde{\Pi} \Delta_G v \right\|_{L^2(\Omega)} \leq C\varepsilon^{-1} \| v \|_{L^2(\Omega)} . \tag{5.80}$$

Proof. Fix $\lambda_* \in \rho(\Delta_s)$, where $\rho(\Delta_s)$ is the resolvent set of the Laplace-Beltrami operator, then the operator Δ_G can be written as

$$\Delta_G = \Delta_s + \varepsilon z D_{s,2} = (\Delta_s - \lambda_*) + \varepsilon z D_{s,2} (\Delta_s - \lambda_*)^{-1} (\Delta_s - \lambda_*) + \lambda_* . \tag{5.81}$$

The Laplace-Beltrami operator is invariant on Y, i.e, $\Delta_s v \in Y$ for every $v \in Y$ and it satisfies

$$\| \Delta_s v \|_{L^2(\Omega)} \leq \varepsilon^{-2} \| v \|_{L^2(\Omega)} \tag{5.82}$$

and since $D_{s,2}$ is a relatively bounded perturbation of Δ_s, see Lemma 2.1, the operator $D_{s,2}(\Delta_s - \lambda_*)^{-1}$ is bounded, independently on ε, on Y.

Let $v \in Y/X_\Sigma$,

$$v = f(z) \sum_{j \in \Sigma} b_j \Theta_j, \tag{5.83}$$

with $\|v\|_{L^2(\Omega)} = 1$.

Taking the $L^2(\Omega)$-norm of $\tilde{\Pi}\Delta_G$ acting on v yields

$$\begin{aligned}
\left\| \tilde{\Pi}\Delta_G v \right\|_{L^2(\Omega)} &= \left\| \tilde{\Pi}(\Delta_s - \lambda_*)v + \varepsilon z \tilde{\Pi} D_{s,2}(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*)v \right\|_{L^2(\Omega)} && (5.84)\\
&\leq \left\| \tilde{\Pi}\Delta_s v \right\|_{L^2(\Omega)} + \left\| \tilde{\Pi}\lambda_* v \right\|_{L^2(\Omega)} + \varepsilon \left\| z \tilde{\Pi} D_{s,2}(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*)v \right\|_{L^2(\Omega)} && (5.85)\\
&\leq \varepsilon^{-2} \| v \|_{L^2(\Omega)} + |\lambda_*| \| v \|_{L^2(\Omega)} + \varepsilon \left\| D_{s,2}(\Delta_s - \lambda_*)^{-1} \right\|_{L^2(\Omega)} \| (\Delta_s - \lambda_*) z v \|_{L^2(\Omega)} && (5.86)\\
&\leq \varepsilon^{-2} \| v \| + |\lambda_*| \| v \| + c\varepsilon^{-1} \| z v \|_{L^2(\Omega)} && (5.87)
\end{aligned}$$

We conclude that

$$\left\| \tilde{\Pi}\Delta_G \right\|_{L^2(\Omega)} \leq C\varepsilon^{-2} \text{ on } Y. \tag{5.88}$$

Similarly, taking $v \in X_\Sigma \subset Y$,

$$v = \psi_{b,0} \sum_{j \in \Sigma} b_j \Theta_j, \tag{5.89}$$

with $\|v\|_{L^2(\Omega)} = 1$, the $L^2(\Omega)$-norm of $\tilde{\Pi}\Delta_G$ acting on v yields

$$\begin{aligned}
\left\| \tilde{\Pi}\Delta_G v \right\|_{L^2(\Omega)} &= \left\| \tilde{\Pi}(\Delta_s - \lambda_*)v + \varepsilon \tilde{\Pi} D_{s,2}(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*) z v \right\|_{L^2(\Omega)} && (5.90)\\
&\leq \cancelto{0}{\left\| \tilde{\Pi}\Delta_s v \right\|_{L^2(\Omega)}} + \cancelto{0}{\left\| \tilde{\Pi}\lambda_* v \right\|_{L^2(\Omega)}} + \varepsilon \left\| \tilde{\Pi} D_{s,2}(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*) z v \right\|_{L^2(\Omega)} && (5.91)\\
&\leq \varepsilon \left\| D_{s,2}(\Delta_s - \lambda_*)^{-1} \right\|_{L^2(\Omega)} \| z \Delta_s v \|_{L^2(\Omega)} && (5.92)\\
&\leq c\varepsilon^{-1} \| v \|_{L^2(\Omega)} && (5.93)
\end{aligned}$$

We conclude that

$$\left\| \tilde{\Pi}\Delta_G \right\|_{L^2(\Omega)} \leq C\varepsilon^{-1} \text{ on } X_\Sigma. \tag{5.94}$$

■

Combining Proposition 5.3.1 with equation (5.76) we obtain the required bound

$$\left\| \tilde{\Pi}\mathcal{L}_b v \right\|_{L^2(\Omega)} \leq \varepsilon c \| v \|_{L^2} . \tag{5.95}$$

5.3.2 Bounding $\tilde{\Pi}\mathcal{L}_b^2\Pi v$

Recall that $v \in X_\Sigma,\ v = \sum_{j\in\Sigma} b_j \psi_{b,0}\Theta_j$ and $\|v\|_{L^2(\Omega)} = 1$. In particular,

$$\Pi v = v, \qquad \tilde{\Pi} v = 0. \tag{5.96}$$

We want to show there exist C_2, independent on ε, such that

$$\left\| \tilde{\Pi}\mathcal{L}_b^2 v \right\|_{L^2(\Omega)} \leq \varepsilon C_2 \| v \|_{L^2} . \tag{5.97}$$

Writing the $\mathcal{L}_b^2$ operator acting on v explicitly we have

$$\mathcal{L}_b^2 v = \mathcal{L}_b(\mathcal{L}_b v) = (L_{b,0} + \varepsilon H\partial_z + \varepsilon^2\Delta_G)(L_{b,0}v + \varepsilon H\partial_z v + \varepsilon^2\Delta_G v) \tag{5.98}$$

$$= (L_{b,0} + \varepsilon H\partial_z + \varepsilon^2\Delta_G) \sum_{j\in\Sigma} b_j \left(\lambda_{b,0}\psi_{b,0}\Theta_j + \varepsilon H\Theta_j\psi'_{b,0} + \varepsilon^2\Delta_G\psi_{b,0}\Theta_j\right) \tag{5.99}$$

$$= \sum_{j\in\Sigma} b_j \Big[\overbrace{\lambda_{b,0}^2\psi_{b,0}\Theta_j}^{\in X_\Sigma} + \varepsilon L_{b,0}(H\Theta_j\psi'_{b,0}) + \varepsilon^2 L_{b,0}(\Delta_G\psi_{b,0}\Theta_j) + \varepsilon\lambda_{b,0}H\psi'_{b,0}\Theta_j + \varepsilon^2 H\Theta_j\partial_z(H\psi'_{b,0}) \tag{5.100}$$

$$+ \varepsilon^3 H\partial_z(\Delta_G\psi_{b,0}\Theta_j) + \varepsilon^2\lambda_{b,0}\psi_{b,0}\Delta_G\Theta_j + \varepsilon^3\Delta_G\psi'_{b,0}(H\Theta_j) + \varepsilon^4\Delta_G^2\psi_{b,0}\Theta_j \Big]$$

Projecting away from X_Σ using $\tilde{\Pi}$ and taking the L^2-norm yields

$$\left\| \tilde{\Pi}\mathcal{L}_b^2 v \right\|_{L^2(\Omega)} = \| \tilde{\Pi} \sum_{j\in\Sigma} b_j \Big[\overbrace{\lambda_{b,0}^2\psi_{b,0}\Theta_j}^{\in X_\Sigma} + \varepsilon L_{b,0}(H\Theta_j\psi'_{b,0}) + \varepsilon^2 L_{b,0}(\Delta_G\psi_{b,0}\Theta_j)\varepsilon\lambda_{b,0}H\psi'_{b,0}\Theta_j + \varepsilon^2 H\Theta_j\partial_z(H\psi'_{b,0}) \tag{5.101}$$

$$+ \varepsilon^3 H\partial_z(\Delta_G\psi_{b,0}\Theta_j) + \varepsilon^2\lambda_{b,0}\psi_{b,0}\Delta_G\Theta_j + \varepsilon^3\Delta_G\psi'_{b,0}(H\Theta_j) + \varepsilon^4\Delta_G^2\psi_{b,0}\Theta_j \Big] \|_{L^2(\Omega)}$$

$$\leq \varepsilon \| \overbrace{\sum_{j\in\Sigma} b_j \Big[L_{b,0}(H\Theta_j\psi'_{b,0}) + \lambda_{b,0}H\psi'_{b,0}\Theta_j + \varepsilon H\Theta_j\partial_z(H\psi'_{b,0}) \Big]}^{\mathcal{R}_1 v} \|_{L^2(\Omega)} + \varepsilon^3 \left\| \tilde{\Pi} H\partial_z(\Delta_G v) \right\|_{L^2(\Omega)} \tag{5.102}$$

$$+ \varepsilon^3 \left\| \tilde{\Pi}\Delta_G H\partial_z v \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi}\lambda_{b,0}\Delta_G v \right\|_{L^2(\Omega)} + \varepsilon^4 \left\| \tilde{\Pi}\Delta_G^2 v \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi} L_{b,0}(\Delta_G v) \right\|_{L^2(\Omega)}$$

where we used the triangle inequality to derive (5.102) from (5.101). We introduce the matrix $\bar{B} \in \mathbb{R}^{N\times N}$, the matrix representation of $\mathcal{R}_1$, given by

$$\bar{B}_{j,k} \coloneqq \left\langle (L_{b,0}(H\psi'_{b,0}) + \lambda_{b,0}H\psi'_{b,0} + \varepsilon H\partial_z(H\psi'_{b,0}))\Theta_j, (L_{b,0}(H\psi'_{b,0}) + \lambda_{b,0}H\psi'_{b,0} + \varepsilon H\partial_z(H\psi'_{b,0}))\Theta_k \right\rangle_{L^2}. \tag{5.103}$$

The entries of $\bar{B}$ take the form

$$\bar{B}_{j,k} = \int_\Gamma \int_{-\ell/\varepsilon}^{\ell/\varepsilon} f(z,s)\Theta_j\Theta_k \, J \, dz \, ds, \tag{5.104}$$

and by Lemma 5.2, we conclude that there exists $c > 0$, independent of ε, such that

$$\left\| \bar{B} \right\|_{l^2\to l^2} \leq c. \tag{5.105}$$

Using (5.105) we obtain a bound on the operator $\mathcal{R}_1$

$$\| \mathcal{R}_1 v \|^2_{L^2(\Omega)} \leq \| b \|^2_{l^2} \left\| \bar{B} \right\|^2_{l^2\to l^2} \leq c \| v \|^2_{l^2}, \tag{5.106}$$

and equation (5.102) reduces to

$$\begin{aligned} \left\| \tilde{\Pi}\mathcal{L}_b^2 v \right\|_{L^2(\Omega)} \leq & c\varepsilon \| v \|_{L^2(\Omega)} + \varepsilon^3 \left\| \tilde{\Pi} H\partial_z(\Delta_G v) \right\|_{L^2(\Omega)} + \varepsilon^3 \left\| \tilde{\Pi}\Delta_G H\partial_z v \right\|_{L^2(\Omega)} \\ & + \varepsilon^2 \left\| \tilde{\Pi}\lambda_{b,0}\Delta_G v \right\|_{L^2(\Omega)} + \varepsilon^4 \left\| \tilde{\Pi}\Delta_G^2 v \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi} L_{b,0}(\Delta_G v) \right\|_{L^2(\Omega)} \end{aligned} \tag{5.107}$$

Consider the second term in the right-hand side of (5.107).

$$\varepsilon^3 \left\| \tilde{\Pi} H\partial_z(\Delta_G v) \right\|_{L^2(\Omega)} \leq \varepsilon^3 \left\| \tilde{\Pi} H\Delta_G\partial_z v \right\|_{L^2(\Omega)} + \varepsilon^3 \left\| \tilde{\Pi} H[\partial_z\Delta_G] v \right\|_{L^2(\Omega)} \tag{5.108}$$

$$\leq \varepsilon c_1 \| v \|_{L^2(\Omega)} + \varepsilon^3 \left\| \tilde{\Pi} H[\partial_z\Delta_G] v \right\|_{L^2(\Omega)}, \tag{5.109}$$

where the second inequality follows applying Proposition 5.3.1 to the first term in (5.108). In order to show that the second term in (5.108) is bounded, we recall the definition of Δ_G, given in (2.12). Taking the z derivative of (2.12) yields

$$[\partial_z\Delta_G] = \varepsilon D_{s,2} + \varepsilon z[\partial_z D_{s,2}], \tag{5.110}$$

where $[\partial_z D_{s,2}]$ denotes the multiplier operator comprised of the z derivative of the coefficients of $D_{s,2}$, which satisfy the bounds (2.15).

Proposition 5.3.2. *Let $f(z)$ be a smooth function such that*

$$|f(z)| < c_1 e^{-c_2|z|} \text{ for some } c_i \in \mathbb{R},\ c_i > 0\ , i = 1, 2, \text{ and } supp(f) \subset \Gamma_l. \tag{5.111}$$

The operator $[\partial_z \Delta_G]$, where Δ_G is defined in (2.12), is bounded on the space

$$Y = span\{ f(z)\Theta_k \ \Big|\ k \in \Sigma, \}, \tag{5.112}$$

i.e., there exists $C > 0$, C independent of ε and f, such that

$$\| [\partial_z \Delta_G] v \|_{L^2(\Omega)} \leq C\varepsilon^{-1} \| v \|_{L^2(\Omega)}, \tag{5.113}$$

for every $v \in Y$.

Proof. Fix $\lambda_* \in \rho(\Delta_s)$, where $\rho(\Delta_s)$ is the resolvent set of the Laplace-Beltrami operator, then the operator Δ_G can be written as

$$[\partial_z \Delta_G] = \partial_z \left(\Delta_s + \varepsilon z D_{s,2} \right) = \varepsilon z [\partial_z D_{s,2}](\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*) + \varepsilon D_{s,2}(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*). \tag{5.114}$$

From Lemma 2.1 we know that $D_{s,2}$ is a relatively bounded perturbation of Δ_s, i.e., there exists $C > 0$, independent of ε, such that

$$\left\| D_{s,2}(\Delta_s - \lambda_*)^{-1} \right\|_{l^2 \to l^2} \leq C. \tag{5.115}$$

Let $v \in Y$,

$$v = f(z) \sum_{j \in \Sigma} b_j \Theta_j, \tag{5.116}$$

with $\|v\|_{L^2(\Omega)} = 1$.

Taking the $L^2(\Omega)$-norm of $[\partial_z \Delta_G]$ acting on v yields

$$\| [\partial_z \Delta_G] v \|_{L^2(\Omega)} = \left\| \left[\varepsilon z [\partial_z D_{s,2}](\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*) + \varepsilon D_{s,2}(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*) \right] v \right\|_{L^2(\Omega)} \tag{5.117}$$

$$\leq \varepsilon \left\| D_{s,2}(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*) v \right\|_{L^2(\Omega)} + \varepsilon \left\| z [\partial_z D_{s,2}](\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*) v \right\|_{L^2(\Omega)} \tag{5.118}$$

$$\leq \varepsilon \left\| D_{s,2}(\Delta_s - \lambda_*)^{-1} \right\|_{l^2 \to l^2} \| (\Delta_s - \lambda_*) v \|_{L^2(\Omega)} \tag{5.119}$$

$$+ \varepsilon \left\| [\partial_z D_{s,2}](\Delta_s - \lambda_*)^{-1} \right\|_{l^2 \to l^2} \| (\Delta_s - \lambda_*) z v \|_{L^2(\Omega)}$$

Note that

$$[\partial_z D_{s,2}] = \sum_{i,j=1}^{d-1} (\partial_z d_{i,j}(s,z)) \frac{\partial^2}{\partial s_i \partial s_j} + \sum_{j=1}^{d-1} (\partial_z d_j(s,z)) \frac{\partial}{\partial s_j} \tag{5.120}$$

$$\leq \| \partial_z d_{i,j}(s,z) \|_{L^\infty(\Gamma_l)} \sum_{i,j=1}^{d-1} d_{i,j} \frac{\partial^2}{\partial s_i \partial s_j} + \| \partial_z d_j(s,z) \|_{L^\infty(\Gamma_l)} \sum_{j=1}^{d-1} d_j(s,z) \frac{\partial}{\partial s_j} \tag{5.121}$$

$$\leq \max \left[\| \partial_z d_{i,j}(s,z) \|_{L^\infty(\Gamma_l)}, \| \partial_z d_j \|_{L^\infty(\Gamma_l)} \right] D_{s,2} \leq c\varepsilon D_{s,2}, \tag{5.122}$$

for some $c > 0$, where for the last inequality we used the bounds on the z-derivatives of the coefficients of $D_{s,2}$, given in equation (2.15). Combining (5.120) with (5.119) yields

$$\| [\partial_z \Delta_G] v \|_{L^2(\Omega)} \leq c\varepsilon \left\| D_{s,2} (\Delta_s - \lambda_*)^{-1} \right\|_{L^2(\Omega)} \| (\Delta_s - \lambda_*) v \|_{L^2(\Omega)}, \tag{5.123}$$

and we conclude that

$$\| [\partial_z \Delta_G] \|_{L^2(\Omega)} \leq C\varepsilon^{-1} \text{ on } Y. \tag{5.124}$$

■

Returning to (5.108) we have

$$\varepsilon^3 \left\| \tilde{\Pi} H \partial_z (\Delta_G v) \right\|_{L^2(\Omega)} \leq \varepsilon c_1 \| v \|_{L^2(\Omega)} + \varepsilon^3 \left\| \tilde{\Pi} H (\partial_z \Delta_G) v \right\|_{L^2(\Omega)} \tag{5.125}$$

$$\leq \varepsilon c_1 \| v \|_{L^2(\Omega)} + \varepsilon^3 \| H (\partial_z \Delta_G) v \|_{L^2(\Omega)} \tag{5.126}$$

$$\leq \varepsilon c_1 \| v \|_{L^2(\Omega)} + \varepsilon^3 \| H \|_{L^\infty(\Gamma_{b,\ell})} \| [\partial_z \Delta_G] v \|_{L^2(\Omega)}. \tag{5.127}$$

where the third inequality follows from the generalized Hölder inequality, see (D.36). Combining Proposition 5.3.2 with equation (5.127) we obtain the bound

$$\varepsilon^3 \left\| \tilde{\Pi} H \partial_z (\Delta_G v) \right\|_{L^2(\Omega)} \leq \varepsilon c_2 \| v \|_{L^2(\Omega)}, \tag{5.128}$$

where c_2 is independent of ε, but it depend upon $\| H \|_{L^\infty(\Gamma_{b,\ell})}$. Plugging (5.128) and into the right-hand side of (5.107) yields

$$\left\| \tilde{\Pi} \mathcal{L}_b^2 v \right\|_{L^2(\Omega)} \leq \varepsilon \tilde{c} \| v \|_{L^2(\Omega)} + \varepsilon^3 \left\| \tilde{\Pi} \Delta_G H \partial_z v \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi} \lambda_{b,0} \Delta_G v \right\|_{L^2(\Omega)} + \varepsilon^4 \left\| \tilde{\Pi} \Delta_G^2 v \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi} L_{b,0} (\Delta_G v) \right\|_{L^2(\Omega)}. \tag{5.129}$$

To show that the second term in (5.129) is bounded as an operator in the $L^2(\Omega)$ norm with

$$\varepsilon^3 \left\| \tilde{\Pi} \Delta_G H \partial_z v \right\|_{L^2(\Omega)} \leq c_1 \varepsilon \| v \|_{L^2(\Omega)} \tag{5.130}$$

we use the following Lemma -

Lemma 5.3. *Fix $v \in X_\Sigma$,*

$$v = \sum_{j \in \Sigma} b_j \Theta_j \psi_{b,0}, \tag{5.131}$$

with $\| v \|_{L^2(\Omega)} = 1$. Then, if Γ_b is admissible, and, in particular, if $\vec{k} \in W^{2,\infty}(\Gamma)$, then there exists $C > 0$ independent of ε such that

$$\| \Delta_s(Hv) \|_{L^2(\Omega)} \leq C \varepsilon^{-1} \| v \|_{L^2(\Omega)} \tag{5.132}$$

Proof. Recall that H, given in (2.10), has the expansion

$$H = H_0(s) + \varepsilon z H_1(s) + ..., \tag{5.133}$$

and the jacobian takes the form

$$J_b(s,z) = J_0(s)\tilde{J}_b = J_0(s)(\varepsilon + \varepsilon^2 H_0 z + ...). \tag{5.134}$$

The term $\Delta_s(Hv)$ has the explicit form

$$\Delta_s(Hv) = (\Delta_s H)v + (\Delta_s v)H + 2 \nabla_s H \nabla_s v. \tag{5.135}$$

Taking the L^2 norm of equation (5.135)

$$\| \Delta_s(Hv) \|_{L^2(\Omega)} \leq \| (\Delta_s H)v \|_{L^2(\Omega)} + \| (\Delta_s v)H \|_{L^2(\Omega)} + 2 \| \nabla_s H \nabla_s v \|_{L^2(\Omega)}. \tag{5.136}$$

Using the expansion of H and J_b we can bound the first term on the right-hand side of equation (5.136)

$$\begin{aligned} \| (\Delta_s H)v \|^2_{L^2(\Omega)} =& \varepsilon \int_\Gamma (\Delta_s H_0)^2 (\sum_{j \in \Sigma} b_j \Theta_j)^2 J_0 ds \int_{-\ell/\varepsilon}^{\ell/\varepsilon} \psi_{b,0}^2 \, dz \\ &+ \varepsilon^2 \Bigg(\int_\Gamma \Delta_s(H_0 H_1)(\sum_{j \in \Sigma} b_j \Theta_j)^2 J_0 ds \int_{-\ell/\varepsilon}^{\ell/\varepsilon} \psi_{b,0}^2 z^2 \, dz \\ &+ \int_\Gamma H_0 \Delta_s(H_0^2)(\sum_{j \in \Sigma} b_j \Theta_j)^2 J_0 ds \int_{-\ell/\varepsilon}^{\ell/\varepsilon} \psi_{b,0}^2 z^2 \, dz \Bigg) \end{aligned} \tag{5.137}$$

$$
\begin{aligned}
&+ O(\varepsilon^3) \\
&\le c\Big(\varepsilon \left\| (\Delta_s H_0)^2 \right\|_{L^\infty(\Gamma)} \| v \|^2_{L^2(\Omega)} \\
&\quad + \varepsilon^2 \Big[\| \Delta_s (H_0 H_1) \|_{L^\infty(\Gamma)} \| v \|^2_{L^2(\Omega)} + \left\| H_0 \Delta_s H_0^2 \right\|_{L^\infty(\Gamma)} \| v \|^2_{L^2(\Omega)} \Big] \Big) + O(\varepsilon^3),
\end{aligned} \tag{5.138}
$$

and as long as the interface is admissible, i.e., $\vec{k} \in W^{2,\infty}(\Gamma)$, we have

$$
\| (\Delta_s H) v \|_{L^2(\Omega)} \le c\varepsilon^{-1} \| v \|_{L^2(\Omega)} . \tag{5.139}
$$

The second term on the right-hand side of equation (5.136) can be bounded

$$
\begin{aligned}
\| H \Delta_s v \|^2_{L^2(\Omega)} =& \varepsilon \int_\Gamma H_0^2 \Big(\sum_{j \in \Sigma} b_j \Delta_s \Theta_j \Big)^2 J_0 ds \int_{-\ell/\varepsilon}^{\ell/\varepsilon} \psi_{b,0}^2 \, dz \qquad (5.140) \\
&+ \varepsilon^2 \left(\int_\Gamma H_0 H_1 \Big(\sum_{j \in \Sigma} b_j \Delta_s \Theta_j \Big)^2 J_0 ds \int_{-\ell/\varepsilon}^{\ell/\varepsilon} \psi_{b,0}^2 z^2 \, dz + \int_\Gamma H_0 H_0^2 \Big(\sum_{j \in \Sigma} b_j \Delta_s \Theta_j \Big)^2 J_0 ds \int_{-\ell/\varepsilon}^{\ell/\varepsilon} \psi_{b,0}^2 z^2 \, dz \right) \\
&+ O(\varepsilon^3) \\
&\le c\varepsilon \left\| f(\vec{k}) \right\|_{L^\infty(\Gamma)} \| \Delta_s v \|^2_{L^2(\Omega)} . \qquad (5.141)
\end{aligned}
$$

Since $\| \Delta_s v \|_{L^2(\Omega)} \le \varepsilon^{-2} \| v \|_{L^2(\Omega)}$, and since the interface is admissible and f is a polynomial of the curvatures of Γ, we know that $\left\| f(\vec{k}) \right\|_{L^2(\Gamma)}$ is bounded, independent of ε. This implies that

$$
\| H \Delta_s v \|_{L^2(\Omega)} \le c\varepsilon^{-1} \| v \|_{L^2(\Omega)} . \tag{5.142}
$$

We apply similar calculations to the third term on the right-hand side of equation (5.136) to obtain

$$
\| \nabla_s H \nabla_s v \|^2_{L^2(\Omega)} \le c\varepsilon \left\| f(\nabla_s \vec{k}) \right\|_{L^\infty(\Gamma)} \| \nabla_s v \|^2_{L^2(\Omega)} , \tag{5.143}
$$

$$
\begin{aligned}
\| \nabla_s v \|^2_{L^2(\Omega)} &= \int_\Gamma \Big(\sum_j b_j \nabla \Theta_j \Big)^2 J_0 ds \int_{-\ell/\varepsilon}^{\ell/\varepsilon} \psi_{b,0}^2 \, dz = \sum_j b_j^2 \int_\Gamma (\Theta_j \Delta_s \Theta_j) \, J_0 ds \int_{-\ell/\varepsilon}^{\ell/\varepsilon} \psi_{b,0}^2 \, dz \qquad (5.144) \\
&= \sum_j b_j^2 \beta_j \int_\Gamma \Theta_j^2 \, J_0 ds \int_{-\ell/\varepsilon}^{\ell/\varepsilon} \psi_{b,0}^2 \, dz \le \varepsilon^{-2} \| v \|^2_{L^2(\Omega)} , \qquad (5.145)
\end{aligned}
$$

and as long as the interface is admissible, i.e., $\vec{k} \in W^{1,\infty}(\Gamma)$, we obtain the bound

$$
\| \nabla_s H \nabla_s v \|_{L^2(\Omega)} \le c\varepsilon^{-1} \| v \|_{L^2(\Omega)} . \tag{5.146}
$$

Plugging (5.139), (5.142) and (5.146) back into (5.136) we conclude that, as long as the interface is admissible, there exists $C > 0$ independent on ε such that

$$\| \Delta_s(Hv) \|_{L^2(\Omega)} \leq C\varepsilon^{-1} \| v \|_{L^2(\Omega)} . \tag{5.147}$$

■

Proposition 5.3.3. *Fix $v \in X_\Sigma$,*

$$v = \sum_{j \in \Sigma} b_j \Theta_j \psi_{b,0}, \tag{5.148}$$

with $\| v \|_{L^2(\Omega)} = 1$. The operator $\tilde{\Pi}\Delta_G H v$, where $\tilde{\Pi}$ is the projection off of the space of small eigenvalues X_Σ, defined in (5.18) and Δ_G is defined in (2.12), is bounded on the space

$$Y = \{ f(z)\Theta_k \;\Big|\; k \in \Sigma, \}, \tag{5.149}$$

i.e., there exists $C > 0$ such that

$$\left\| \tilde{\Pi}\Delta_G H v \right\|_{L^2(\Omega)} \leq C\varepsilon^{-2} \| v \|_{L^2(\Omega)} \quad \textit{for all } v \in Y. \tag{5.150}$$

Proof. We repeat the proof of Proposition 5.3.1 but replacing equation (5.82) with equation (5.132), and taking the $L^2(\Omega)$ norm of $\tilde{\Pi}\Delta_G$ acting on Hv yields the required result. ■

Proposition 5.3.1 shows that the third term in (5.129) is bounded in the $L^2(\Omega)$ operator norm with

$$\varepsilon^2 \left\| \tilde{\Pi}\lambda_{b,0}\Delta_G v \right\|_{L^2(\Omega)} \leq c_2 \varepsilon \| v \|_{L^2(\Omega)} . \tag{5.151}$$

Inserting (5.151) and (5.130) into (5.129) yields

$$\left\| \tilde{\Pi}\mathcal{L}_b^2 v \right\|_{L^2(\Omega)} \leq \varepsilon \bar{c} \| v \|_{L^2(\Omega)} + \varepsilon^4 \left\| \tilde{\Pi}\Delta_G^2 v \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi} L_{b,0}(\Delta_G v) \right\|_{L^2(\Omega)} . \tag{5.152}$$

The bound on the second term in the right-hand side of (5.152) follows from the following lemma

Lemma 5.4. *Let $v \in X_\Sigma$,*

$$v = \sum_{j \in \Sigma} b_j \Theta_j \psi_{b,0}, \tag{5.153}$$

with $\|v\|_{L^2(\Omega)} = 1$. *Then there exists* $C > 0$ *such that*

$$\left\| \tilde{\Pi} \Delta_G^2 v \right\|_{L^2(\Omega)} \leq C\varepsilon^{-3} \| v \|_{L^2(\Omega)}. \tag{5.154}$$

Proof. Fix $\lambda_* \in \rho(\Delta_s)$, where $\rho(\Delta_s)$ is the resolvent set of the Laplace-Beltrami operator, then the operator Δ_G can be written as

$$\Delta_G^2 = (\Delta_s + \varepsilon z D_{s,2})^2 = \Delta_s^2 + 2\varepsilon z D_{s,2}(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*)\Delta_s + \varepsilon^2 z^2 D_{s,2}^2 (\Delta_s - \lambda_*)^{-2}(\Delta_s - \lambda_*)^2. \tag{5.155}$$

The Laplace-Beltrami operator satisfies

$$\left\| \Delta_s^2 v \right\|_{L^2(\Omega)} \leq \varepsilon^{-4} \| v \|_{L^2(\Omega)}, \tag{5.156}$$

and since $D_{s,2}$ is a relatively bounded perturbation of Δ_s, we conclude that the operator $D_{s,2}(\Delta_s - \lambda_*)^{-1}$ is bounded on Y.

Taking the $L^2(\Omega)$-norm of $\tilde{\Pi}\Delta_G^2 v$ and using equation (5.155) to express Δ_G^2 yields

$$\begin{aligned} \left\| \tilde{\Pi}(\Delta_G^2) v \right\|_{L^2(\Omega)} &\leq \cancel{\left\| \tilde{\Pi} \Delta_s^2 v \right\|_{L^2(\Omega)}} + 2\varepsilon \left\| z D_{s,2}(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*)\Delta_s v \right\|_{L^2(\Omega)} \qquad (5.157) \\ &\quad + \varepsilon^2 \left\| D_{s,2}^2 (\Delta_s - \lambda_*)^{-2}(\Delta_s - \lambda_*)^2 z^2 v \right\|_{L^2(\Omega)}, \\ &\leq 2\varepsilon \left\| D_{s,2}(\Delta_s - \lambda_*)^{-1} \right\|_{L^2(\Omega)} \| (\Delta_s - \lambda_*)\Delta_s z v \|_{L^2(\Omega)} \qquad (5.158) \\ &\quad + \varepsilon^2 \left\| D_{s,2}^2 (\Delta_s - \lambda_*)^{-2} \right\|_{L^2(\Omega)} \left\| (\Delta_s - \lambda_*)^2 z^2 v \right\|_{L^2(\Omega)}. \end{aligned}$$

From Lemma 2.1 we know that $D_{s,2}$ is a relatively bounded perturbation of Δ_s, i.e., there exists $C > 0$, independent of ε, such that

$$\left\| D_{s,2}(\Delta_s - \lambda_*)^{-1} \right\|_{L^2(\Omega)} \leq C. \tag{5.159}$$

Proving that $D_{s,2}^2$ is a relatively bounded perturbation of Δ_s^2 is similar and (5.157) reduces to

$$\left\| \tilde{\Pi}(\partial_z^2 \Delta_G) v \right\|_{L^2(\Omega)} \leq C\varepsilon^{-3} \| v \|_{L^2(\Omega)}. \tag{5.160}$$

■

Combining Lemma 5.4 and equation (5.129) yields

$$\left\| \tilde{\Pi} \mathcal{L}_b^2 v \right\|_{L^2(\Omega)} \leq \varepsilon \bar{c} \| v \|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi} L_{b,0}(\Delta_G v) \right\|_{L^2(\Omega)}. \tag{5.161}$$

Consider the second term in (5.161), and note that $W''(U_b)$ commutes with Δ_G. Then

$$\varepsilon^2 \left\| \tilde{\Pi} L_{b,0}(\Delta_G v) \right\|_{L^2(\Omega)} = \varepsilon^2 \left\| \tilde{\Pi}(\partial_z^2 - W''(U_b))(\Delta_G v) \right\|_{L^2(\Omega)} \tag{5.162}$$

$$\leq \varepsilon^2 \left\| \tilde{\Pi}(\partial_z^2 \Delta_G) v \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| 2\tilde{\Pi}(\partial_z \Delta_G)(\partial_z v) \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi} \Delta_G L_{b,0} v \right\|_{L^2(\Omega)} \tag{5.163}$$

$$= \varepsilon^2 \left\| \tilde{\Pi}(\partial_z^2 \Delta_G) v \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| 2\tilde{\Pi}(\partial_z \Delta_G)(\partial_z v) \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \lambda_{b,0} \tilde{\Pi} \Delta_G v \right\|_{L^2(\Omega)}. \tag{5.164}$$

For the first term in (5.164) we have the following lemma

Lemma 5.5. *Let $v \in X_\Sigma$,*

$$v = \sum_{j \in \Sigma} b_j \Theta_j \psi_{b,0}, \tag{5.165}$$

with $\|v\|_{L^2(\Omega)} = 1$. Then there exists $C > 0$ such that

$$\left\| \tilde{\Pi}(\partial_z^2 \Delta_G) v \right\|_{L^2(\Omega)} \leq C \| v \|_{L^2(\Omega)}. \tag{5.166}$$

Proof. Fix $\lambda_* \in \rho(\Delta_s)$, where $\rho(\Delta_s)$ is the resolvent set of the Laplace-Beltrami operator, then the operator Δ_G can be written as

$$\partial_z^2 \Delta_G = \partial_z^2 (\Delta_s + \varepsilon z D_{s,2}) = \varepsilon z \partial_z^2 (D_{s,2})(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*) + \varepsilon 2 \partial_z D_{s,2}(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*). \tag{5.167}$$

From Lemma 2.1 we know that $D_{s,2}$ is a relatively bounded perturbation of Δ_s, i.e., there exists $C > 0$, independent of ε, such that

$$\left\| D_{s,2}(\Delta_s - \lambda_*)^{-1} \right\|_{L^2(\Omega)} \leq C. \tag{5.168}$$

Taking the $L^2(\Omega)$-norm of $\tilde{\Pi}(\partial_z^2 \Delta_G) v$ combined with equation (5.167) yields

$$\left\| \tilde{\Pi}(\partial_z^2 \Delta_G) v \right\|_{L^2(\Omega)} = \left\| \tilde{\Pi} \left[\varepsilon z (\partial_z^2 D_{s,2})(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*) + 2\varepsilon (\partial_z D_{s,2})(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*) \right] v \right\|_{L^2(\Omega)}, \tag{5.169}$$

$$\leq 2\varepsilon \left\| (\partial_z D_{s,2})(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*) v \right\|_{L^2(\Omega)} + \varepsilon \left\| z(\partial_z^2 D_{s,2})(\Delta_s - \lambda_*)^{-1}(\Delta_s - \lambda_*) v \right\|_{L^2(\Omega)},$$

$$\leq 2\varepsilon \left\| (\partial_z D_{s,2})(\Delta_s - \lambda_*)^{-1} \right\|_{L^2(\Omega)} \| (\Delta_s - \lambda_*) v \|_{L^2(\Omega)} \tag{5.170}$$

$$+ \varepsilon \left\| (\partial_z^2 D_{s,2})(\Delta_s - \lambda_*)^{-1} \right\|_{l^2 \to l^2} \| (\Delta_s - \lambda_*) z v \|_{L^2(\Omega)}.$$

From (5.120) we know that

$$\partial_z D_{s,2} \leq c \varepsilon D_{s,2}, \tag{5.171}$$

and similar calculations shows that there exists a constant $\tilde{c} > 0$ such that

$$\partial_z^2 D_{s,2} \leq \tilde{c}\varepsilon^2 D_{s,2}. \tag{5.172}$$

Plugging (5.172) and (5.120) into (5.169) yields

$$\begin{aligned} \left\| \tilde{\Pi}(\partial_z^2 \Delta_G) v \right\|_{L^2(\Omega)} &\leq 2\varepsilon^2 \left\| (D_{s,2})(\Delta_s - \lambda_*)^{-1} \right\|_{l^2 \to l^2} \| (\Delta_s - \lambda_*) v \|_{L^2(\Omega)} \qquad (5.173) \\ &\quad + \varepsilon^3 \left\| (D_{s,2})(\Delta_s - \lambda_*)^{-1} \right\|_{l^2 \to l^2} \| (\Delta_s - \lambda_*) z v \|_{L^2(\Omega)}, \\ &\leq C \| v \|_{L^2(\Omega)}. \qquad (5.174) \end{aligned}$$

■

Returning to (5.164), and using Lemma 5.5 we have

$$\varepsilon^2 \left\| \tilde{\Pi} L_{b,0}(\Delta_G v) \right\|_{L^2(\Omega)} \leq C\varepsilon^2 \| v \|_{L^2(\Omega)} + \varepsilon^2 \left\| 2\tilde{\Pi}(\partial_z \Delta_G)(\partial_z v) \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \lambda_{b,0} \tilde{\Pi} \Delta_G v \right\|_{L^2(\Omega)}. \tag{5.175}$$

Applying Proposition 5.3.2 to the second term and Proposition 5.3.1 to the third term, equation (5.175) reduces to

$$\varepsilon^2 \left\| \tilde{\Pi} L_{b,0}(\Delta_G v) \right\|_{L^2(\Omega)} \leq \tilde{C}\varepsilon \| v \|_{L^2(\Omega)}. \tag{5.176}$$

Plugging equation (5.176) into (5.161) we obtain the required bound

$$\left\| \tilde{\Pi} \mathcal{L}_b^2 v \right\|_{L^2(\Omega)} \leq \varepsilon C \| v \|_{L^2(\Omega)}, \text{ for } v \in X_\Sigma, \tag{5.177}$$

where C is independent of ε, but it depend upon $\left\| \vec{k}_b \right\|_{L^\infty(\Omega)}$.

5.3.3 Bounding $\tilde{\Pi}\mathbb{L}_b\Pi$

Finally, the following Proposition shows that the operator $\tilde{\Pi}\mathbb{L}_b\Pi$ is bounded.

Proposition 5.3.4. *The operator $\tilde{\Pi}\mathbb{L}_b\Pi : l^2(\mathbb{R}^{N\times N}) \mapsto l^2(\mathbb{R}^{N\times N})$ has an $O(\varepsilon)$ operator norm.*

Proof. Let $v \in X_\Sigma$, $v = \sum_{j\in\Sigma} b_j \psi_{b,0} \Theta_j$ with $\|v\|_{L^2(\Omega)} = 1$. In particular,

$$\Pi v = v, \qquad \tilde{\Pi} v = 0, \tag{5.178}$$

where Π is the projection onto X_Σ and $\tilde{\Pi}$ its complementary.

Fix $\lambda_* \in \rho(\mathcal{L}_b^2)$, where $\rho(\mathcal{L}_b^2)$ is the resolvent set of $\mathcal{L}_b^2$, and rewrite $\mathbb{L}_b$ in the following way

$$\mathbb{L}_b = \mathcal{L}_b^2 + \varepsilon\tilde{\mathbb{L}}_b = (\mathcal{L}_b^2 - \lambda_*) + \varepsilon\tilde{\mathbb{L}}_b(\mathcal{L}_b^2 - \lambda_*)^{-1}(\mathcal{L}_b^2 - \lambda_*) + \lambda_*. \tag{5.179}$$

Taking the L^2-norm of $\tilde{\Pi}\mathbb{L}_b\Pi$ acting on v yields

$$\begin{aligned}
\left\| \tilde{\Pi}\mathbb{L}_b\Pi v \right\|_{L^2(\Omega)} &= \left\| \tilde{\Pi}(\mathcal{L}_b^2 - \lambda_*)v + \varepsilon\tilde{\Pi}\tilde{\mathbb{L}}_b(\mathcal{L}_b^2 - \lambda_*)^{-1}(\mathcal{L}_b^2 - \lambda_*)v \right\|_{L^2(\Omega)}, && (5.180)\\
&\le \left\| \tilde{\Pi}\mathcal{L}_b^2 v \right\|_{L^2(\Omega)} + \varepsilon \left\| \tilde{\Pi}\tilde{\mathbb{L}}_b(\mathcal{L}_b^2 - \lambda_*)^{-1}(\Pi + \tilde{\Pi})(\mathcal{L}_b^2 - \lambda_*)v \right\|_{L^2(\Omega)}, && (5.181)\\
&\le \left\| \tilde{\Pi}\mathcal{L}_b^2 v \right\|_{L^2(\Omega)} && (5.182)\\
&\quad + \varepsilon \left(\left\| \tilde{\mathbb{L}}_b(\mathcal{L}_b^2 - \lambda_*)^{-1}\Pi(\mathcal{L}_b^2 - \lambda_*)v \right\|_{L^2(\Omega)} + \left\| \tilde{\mathbb{L}}_b(\mathcal{L}_b^2 - \lambda_*)^{-1}\tilde{\Pi}(\mathcal{L}_b^2 - \lambda_*)v \right\|_{L^2(\Omega)} \right),\\
&\le \left\| \tilde{\Pi}\mathcal{L}_b^2 v \right\|_{L^2(\Omega)} && (5.183)\\
&\quad + \varepsilon \left(\left\| \tilde{\mathbb{L}}_b(\mathcal{L}_b^2 - \lambda_*)^{-1} \right\|_{l^2\to l^2} \left\| \Pi\mathcal{L}_b^2 v \right\|_{L^2(\Omega)} + \left\| \tilde{\mathbb{L}}_b(\mathcal{L}_b^2 - \lambda_*)^{-1} \right\|_{l^2\to l^2} \left\| \tilde{\Pi}\mathcal{L}_b^2 v \right\|_{L^2(\Omega)} \right).
\end{aligned}$$

Since $\tilde{\mathbb{L}}_b$ is relatively bounded with respect to $\mathcal{L}_b^2$, the operator $\tilde{\mathbb{L}}_b(\mathcal{L}_b^2 - \lambda_*)^{-1}$ has an $O(\varepsilon)$ bound as an operator from $l^2(\mathbb{R}^{N\times N})$ to $l^2(\mathbb{R}^{N\times N})$. In section 5.3.2 we have shown that

$$\left\| \tilde{\Pi}\mathcal{L}_b^2\Pi \right\|_{L^2} \le c\varepsilon. \tag{5.184}$$

Therefore, combining bound (5.184) with the boundedness of $\left\| \tilde{\mathbb{L}}_b(\mathcal{L}_b^2 - \lambda_*)^{-1} \right\|_{l^2\to l^2}$, an inspection of equation (5.93) yields

$$\left\| \tilde{\Pi}\mathbb{L}_b\Pi v \right\|_{L^2} \le \tilde{c}\varepsilon \left((1+\varepsilon) \left\| v \right\|_{L^2} + \left\| \Pi\mathcal{L}_b^2 v \right\|_{L^2} \right) \tag{5.185}$$

To complete the bound on $\left\| \tilde{\Pi}\mathbb{L}_b\Pi \right\|_{L^2}$ we need show that $\left\| \Pi\mathcal{L}_b^2\Pi \right\|_{L^2}$ is bounded. Using the definition of Π, (5.21), we can rewrite the operator $\Pi\mathcal{L}_b^2\Pi v$ as

$$\Pi\mathcal{L}_b^2\Pi \sum_{j\in\Sigma} b_j\psi_{b,0}\Theta_k = \sum_{k\in\Sigma}\sum_{j\in\Sigma} b_j(\mathcal{L}_b^2\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k)_{L^2}\psi_{b,0}\Theta_k, \tag{5.186}$$

and define its matrix representation $\bar{M} \in \mathbb{R}^{N_d\times N_d}$, where $N_d \sim O(\varepsilon^{3/2-d})$, to be

$$\bar{M}_{j,k} := \left(\mathcal{L}_b^2\psi_{b,0}\Theta_j, \psi_{b,0}\Theta_k \right)_{L^2}. \tag{5.187}$$

Using equation (5.34) we can write $\bar{M} = \bar{M}_{\text{diag}} + \bar{M}_{\text{off-diag}}$ where

$$\bar{M}_{\text{diag}} = \begin{cases} (\lambda_{b,0} - \varepsilon^2\beta_k)^2 + O(\varepsilon^2) & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases} \tag{5.188}$$

$$\bar{M}_{\text{off-diag}} = \begin{cases} 0 & \text{if } k = j, \\ \varepsilon^2 \int_\Gamma H_0^2 \Theta_k \Theta_j \, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} ((\psi_{b,0}^0)')^2 \, dz + O(\varepsilon^2\sqrt{\varepsilon}) & \text{if } k \neq j, \end{cases} \tag{5.189}$$

By Theorem 5.2 we know that $\bar{M}_{\text{off-diag}}$ has an $O(\varepsilon)$ bound as an operator from $l^2(\mathbb{R}^{N\times N})$ to $l^2(\mathbb{R}^{N\times N})$. Moreover, we consider $k \in \Sigma$ for which $\lambda_{b,0} - \varepsilon^2\beta_k \sim O(\varepsilon)$. Hence, $\bar{M}$ and $\bar{M}_{\text{diag}}$ have a similar bound and

$$\left\| \Pi \mathcal{L}_b^2 v \right\|_{L^2} \leq c\varepsilon \| v \|_{L^2} . \tag{5.190}$$

Plugging this bound back to (5.185) yields

$$\left\| \tilde{\Pi} \mathbb{L}_b \Pi v \right\|_{L^2} \leq \varepsilon c \| v \|_{L^2} , \tag{5.191}$$

which implies that $\tilde{\Pi}\mathbb{L}_b\Pi$ has $O(\varepsilon)$ bound as an operator $l^2(\mathbb{R}^{N\times N}) \to l^2(\mathbb{R}^{N\times N})$. ∎

Recall the 2×2 block form of $\mathbb{L}_b$, (5.22). In this section we have shown that the off diagonal blocks are $O(\varepsilon)$ bounded as operators from $l^2(\mathbb{R}^{N\times N}) \to l^2(\mathbb{R}^{N\times N})$. The following section focuses on the pearling spectrum of $\mathbb{L}_b$ and we use the bounds on B, B^T to show that the pearling eigenvalues of $\mathbb{L}_b$ are, at leading order, the pearling eigenvalues of M.

5.4 Relating the Eigenvalues of $\mathbb{L}_b$ and $\Pi\mathbb{L}_b\Pi$

Recall the 2×2 block form representation of $\mathbb{L}_b$, given in (5.22),

$$\begin{bmatrix} M & B \\ B^T & C \end{bmatrix}, \tag{5.192}$$

where the submatrices are given by

$$M := \Pi\mathbb{L}_b\Pi, \;\; B := \Pi\mathbb{L}_b\tilde{\Pi}, \;\; B^T := \tilde{\Pi}\mathbb{L}_b\Pi \text{ and } C := \tilde{\Pi}\mathbb{L}_b\tilde{\Pi}. \tag{5.193}$$

We use the estimates from the previous section to relate the small eigenvalues of $\mathbb{L}_b$ to those of M. If Π were a spectral projection associated to $\mathbb{L}_b$ then the two operators would commute, and since $\Pi\tilde{\Pi} = 0$, the off-diagonal terms would be zero. However, X_Σ only approximates a spectral subset of $\mathbb{L}_b$, and the estimates $\|B\|_{L^2(\Omega)} = \|B^T\|_{L^2(\Omega)} \leq c\varepsilon$, found in Section 5.3, are sharp. However, the restricted operator C is uniformly coercive on $X_\Sigma^\perp$ with its spectrum is bounded from below by $\delta > 0$ which may be chosen independent of sufficiently small $\varepsilon > 0$, (see [Hayrapetyan and Promislow, 2014] for more details). Consider $v_1 \in X_\Sigma$ and $v_2 \in X_\Sigma^\perp$. Then, for any $\lambda < \delta$ we reduce the 2×2 representation of the infinite dimensional eigenvalue problem

$$\begin{bmatrix} M & B \\ B^T & C \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{5.194}$$

to a finite dimensional system for the component v_1, which solves

$$(M - \lambda)v_1 = B(C - \lambda)^{-1}B^T v_1. \tag{5.195}$$

We will use two methods to show that the pearling conditions established for M in Corollary 5.2.3 do in fact characterize the small spectrum of $\mathbb{L}_b$: First, we show that the eigenvalues of M are in fact a small perturbation of the small eigenvalues of $\mathbb{L}_b$, and we obtain a perturbation estimate. Second, we look at the solution of the linear flow generated by $\mathbb{L}_b$. Assuming the eigenvalues of M are stable under pearling, in terms of Corollary 5.2.3, we will show that the semi-groups generated by $\mathbb{L}_b$ decay exponentially fast and describe the resulting exponential dichotomy.

5.4.1 Perturbation estimate

Consider $\lambda \in \sigma(\mathbb{L}_b) \cap (-\infty, \delta)$, taking the l^2-norm of both sides of (5.195) and estimating the right-hand side yields

$$\|(M - \lambda)v_1\|_{l^2} \leq \|B\|_{L^2(\Omega)} \left\|(C - \lambda)^{-1}\right\|_{L^2(\Omega)} \left\|B^T\right\|_{L^2(\Omega)} \|v_1\|_{l^2}. \tag{5.196}$$

Using the estimates on the norms of B and B^T, given in equation (5.191), we know there exists $c \in \mathbb{R}$ independent of ε such that

$$\|(M - \lambda)v_1\|_{l^2} \leq c\varepsilon^2 \|R(\lambda, C)\|_{L^2(\Omega)} \|v_1\|_{l^2}. \tag{5.197}$$

where $R(\lambda, C)$ is the resolvent operator of C, defined as

$$R(\lambda; C) := (C - \lambda)^{-1}, \tag{5.198}$$

for $\lambda \notin \sigma(C)$. Since C is self-adjoint, standard estimates based upon spectral decomposition of the resolvent allow us to bound the $L^2(\Omega)$-norm of the resolvent operator

$$\| R(\lambda; C) \|_{L^2(\Omega)} \leq (\text{dist}(\lambda, \sigma(C)))^{-1} \leq \frac{1}{|\lambda - \delta|}. \tag{5.199}$$

Plugging the bound (5.199) in (5.197) yields

$$\|(M - \lambda)v_1\|_{l^2} \leq \frac{c\varepsilon^2}{|\lambda - \delta|}\|v_1\|_{l^2}. \tag{5.200}$$

Let $\{w_i\}$ be the set the eigenvectors of M with the corresponding eigenvalues $\{\lambda^i\}$. The spectral decomposition of v is given by

$$v = \sum \alpha_i w_i, \tag{5.201}$$

and the right-hand side of equation (5.200) can be written as

$$\|(M - \lambda)v_1\|_{l^2} = \| \sum \alpha_i (\lambda^i - \lambda) w_i \|_{l^2} \geq dist(\sigma(M), \lambda) \| \sum \alpha_i w_i \|_{l^2} = dist(\sigma(M), \lambda)\|v_1\|_{l^2} \tag{5.202}$$

Combining equations (5.202) and (5.200) yields

$$dist(\sigma(M), \lambda) \leq \frac{c\varepsilon^2}{|\lambda - \delta|}, \quad \text{for } \lambda < \delta. \tag{5.203}$$

Therefore, for $\lambda \in \mathbb{R}$ an order of one distance below δ this estimate implies that $\text{dist}(\lambda, \sigma(M)) = O(\varepsilon^2)$. We infer that the spectrum of $\mathbb{L}_b$ below δ lies within $O(\varepsilon^2)$ of the spectrum of M. In particular, if the spectrum of M is bounded from below by a positive $O(\varepsilon)$ quantity, then so is the spectrum of $\mathbb{L}_b$.

The spectrum of M is to leading order given by its diagonal terms $M_{k,k}$, which are of the form $M_{k,k} = (\mathbb{L}_b \psi_{b,0}\Theta_k, \psi_{b,0}\Theta_k)_{L^2(\Omega)}$, see (5.26). Since the basis elements $\psi_{b,0}\Theta_k$ have norm one, we infer from the Rayleigh Ritz variational characterization of eigenvalues that the smallest eigenvalue of $\mathbb{L}_b$ is smaller than the smallest eigenvalue of M.

We deduce from these calculations that the pearling condition (5.2.3) applies to $\mathbb{L}_b$.

5.4.2 Semi-group estimates

Let $V = [v_1, v_2]^T$ where $v_1 \in X_\Sigma$ and $v_2 \in X_\Sigma^\perp$. We derive decay estimates on the linear evolution equation

$$V_t = -\mathbb{L}_b V. \tag{5.204}$$

Using the 2×2 block form representation for $\mathbb{L}_b$ we obtain

$$v_{1,t} = -Mv_1 - Bv_2, \tag{5.205}$$

$$v_{2,t} = -B^T v_1 - Cv_2. \tag{5.206}$$

Equations (5.191) and (5.64) afford bounds on the off-diagonal matrices

$$\| B \|_{L^2(\Omega)}, \left\| B^T \right\|_{L^2(\Omega)} \leq \varepsilon c. \tag{5.207}$$

We assume the eigenvalues of M are positive, Corollary 5.2.3 implies that they are order of ϵ. By Rayleigh-Ritz formula we know that $\| M \|_{L^2(\Omega)} \leq \varepsilon c$. Since M is a self-adjoint matrix we can apply the Spectral Mapping Theorem to $-M$ and obtain the decay estimate

$$\left\| e^{-Mt} v \right\|_{L^2(\Omega)} \leq c e^{-\varepsilon \sigma t} \| v \|_{L^2(\Omega)}, \tag{5.208}$$

where $\varepsilon\sigma > 0$ is a lower bound on the spectrum of M. From [Hayrapetyan and Promislow, 2014, Thm 2.5] we know that C is uniformly coercive on $X_\Sigma^\perp$, its spectrum is bounded below by some constant δ which may be chosen independent of sufficiently small $\varepsilon > 0$. Since C is self-adjoint it is sectorial and it generates an analytic semi-group for which we have the semi-group estimates

$$\left\| e^{-Ct} v \right\|_{L^2(\Omega)} \leq c e^{-\delta t} \| v \|_{L^2(\Omega)}. \tag{5.209}$$

We fix $\nu \in (0, \sigma)$ and introduce the quantities

$$M_1(t) \coloneqq \sup_{0 \leq s \leq t} \left(e^{\nu \varepsilon s} \| v_1(s) \|_{L^2(\Omega)} \right), \tag{5.210}$$

$$M_2(t) \coloneqq \sup_{0 \leq s \leq t} \left(e^{\nu \varepsilon s} \| v_2(s) \|_{L^2(\Omega)} \right). \tag{5.211}$$

The quantity $M_1(t)$ affords the estimate

$$\| v_1(s) \|_{L^2(\Omega)} \leq e^{-\nu \varepsilon s} M_1(t), \qquad 0 \leq s \leq t, \tag{5.212}$$

so if M_1 is uniformly bounded, then $\|v_1\|_{L^2}$ decays with exponential rate $\varepsilon\nu$ as $t \longrightarrow \infty$, as do $v_2(t)$ and $M_2(t)$.

Applying Variation of Constants formula to equation (5.206) yields

$$v_2(t) = e^{-Ct}v_2(0) + \int_0^t e^{-C(t-s)}B^T v_1(s)\,ds, \tag{5.213}$$

which in light of the bound for B, see (5.207), and the semi-group estimate (5.209) reduces to

$$\| v_2(t) \|_{L^2(\Omega)} \le e^{-\delta t} \| v_2(0) \|_{L^2(\Omega)} + \varepsilon c \int_0^t e^{-\delta(t-s)} \| v_1(s) \|_{L^2(\Omega)}\, ds, \tag{5.214}$$

$$= e^{-\delta t} \| v_2(0) \|_{L^2(\Omega)} + \varepsilon c e^{-\delta t} \int_0^t e^{\delta s} \| v_1(s) \|_{L^2(\Omega)}\, ds, \tag{5.215}$$

$$\le e^{-\delta t} \| v_2(0) \|_{L^2(\Omega)} + \varepsilon c e^{-\delta t} \int_0^t e^{(\delta-\nu\varepsilon)s} M_1(t)\, ds, \tag{5.216}$$

and the second inequality follows from estimate (5.212). Integration of the last line yields

$$\| v_2(t) \|_{L^2(\Omega)} \le e^{-\delta t} \| v_2(0) \|_{L^2(\Omega)} + \varepsilon c e^{-\delta t} M_1(t) \frac{e^{(\delta-\nu\varepsilon)t} - 1}{\delta - \nu\varepsilon}. \tag{5.217}$$

Fixing $\varepsilon\nu \in (0, \delta)$ implies that $e^{-\delta t}\left(e^{(\delta-\nu\varepsilon)t} - 1\right) = e^{-\nu\varepsilon t} - e^{-\delta t} < e^{-\nu\varepsilon t}$ is decaying, the equation above reduces to

$$\| v_2(t) \|_{L^2(\Omega)} \le \tilde{c}\left(e^{-\delta t} \| v_2(0) \|_{L^2(\Omega)} + \varepsilon e^{-\nu\varepsilon t} M_1(t)\right), \tag{5.218}$$

where $\tilde{c} = \tilde{c}(\nu)$. Since $t \in [0, T]$ is arbitrary, we can fix $0 < t' < t$, replace t with t' and multiply by $e^{\nu\varepsilon t'}$, obtaining

$$e^{\nu\varepsilon t'} \| v_2(t') \|_{L^2(\Omega)} \le \tilde{c} e^{\nu\varepsilon t'} \left(e^{-\delta t'} \| v_2(0) \|_{L^2(\Omega)} + \varepsilon e^{-\nu\varepsilon t'} M_1(t')\right), \tag{5.219}$$

since $M_1(t') \le M_1(t)$, taking the supremum over $0 < t' < t$ yields

$$M_2(t) \le \tilde{c}\left(\| v_2(0) \|_{L^2(\Omega)} + \varepsilon M_1(t)\right). \tag{5.220}$$

To obtain a bound on M_1 we apply the variation of constant formula to the ODE of v_1, eq (5.205), which yields

$$v_1(t) = e^{-Mt}v_1(0) + \int_0^t e^{-M(t-s)}Bv_2(s)\, ds \tag{5.221}$$

applying the bound on B, (5.64), and the semi-group estimate on $-M$, (5.208), yields

$$\| v_1(t) \|_{L^2(\Omega)} \le e^{-\sigma\varepsilon t} \| v_1(0) \|_{L^2(\Omega)} + \varepsilon c \int_0^t e^{-\sigma\varepsilon(t-s)} \| v_2(s) \|_{L^2(\Omega)}\, ds, \tag{5.222}$$

$$= e^{-\sigma\varepsilon t} \| v_1(0) \|_{L^2(\Omega)} + \varepsilon c e^{-\sigma\varepsilon t} \int_0^t e^{\varepsilon\sigma s} \| v_2(s) \|_{L^2(\Omega)} \, ds, \tag{5.223}$$

and recall that $\| v_2(s) \|_{L^2(\Omega)} \leq e^{-\varepsilon\nu s} M_2(t)$ which yields

$$\| v_1(t) \|_{L^2(\Omega)} \leq e^{-\sigma\varepsilon t} \| v_1(0) \|_{L^2(\Omega)} + \varepsilon c e^{-\sigma\varepsilon t} \int_0^t e^{\varepsilon(\sigma-\nu)s} M_2(t) \, ds, \tag{5.224}$$

$$\leq e^{-\sigma\varepsilon t} \| v_1(0) \|_{L^2(\Omega)} + \varepsilon c e^{-\sigma\varepsilon t} \frac{e^{\varepsilon(\sigma-\nu)t} - 1}{\varepsilon(\sigma-\nu)} M_2(t), \tag{5.225}$$

$$\leq e^{-\sigma\varepsilon t} \| v_1(0) \|_{L^2(\Omega)} + \tilde{c}(e^{-\varepsilon\nu t} - e^{-\sigma\varepsilon t}) M_2(t). \tag{5.226}$$

where $\tilde{c} = \tilde{c}(\nu)$. Since $\nu \in (0, \sigma)$, we have $(e^{-\varepsilon\nu t} - e^{-\sigma\varepsilon t}) \leq e^{-\varepsilon\nu t}$ and the inequality reduces to

$$\| v_1(t) \|_{L^2(\Omega)} \leq e^{-\sigma\varepsilon t} \| v_1(0) \|_{L^2(\Omega)} + \tilde{c} e^{-\varepsilon\nu t} M_2(t). \tag{5.227}$$

Applying the bound (5.220) to $M_2(t)$ yields

$$\| v_1(t) \|_{L^2(\Omega)} \leq c \Big[e^{-\sigma\varepsilon t} \| v_1(0) \|_{L^2(\Omega)} + e^{-\varepsilon\nu t} \Big(\| v_2(0) \|_{L^2(\Omega)} + \varepsilon M_1(t) \Big) \Big]. \tag{5.228}$$

Since $t \in [0, T]$ arbitrary, we can fix $0 < t' < t$, replace t with t' and multiply by $e^{\nu\varepsilon t'}$, obtaining

$$e^{\nu\varepsilon t'} \| v_1(t') \|_{L^2(\Omega)} \leq c \Big[e^{(\nu-\sigma)\varepsilon t'} \| v_1(0) \|_{L^2(\Omega)} + \Big(\| v_2(0) \|_{L^2(\Omega)} + \varepsilon M_1(t') \Big) \Big]. \tag{5.229}$$

note that $M_1(t') \leq M_1(t)$ and taking the supremum over $0 < t' < t$ yields

$$M_1(t) \leq c \Big[\| v_1(0) \|_{L^2(\Omega)} + \| v_2(0) \|_{L^2(\Omega)} + \varepsilon M_1(t) \Big]. \tag{5.230}$$

For ε sufficiently small we obtain a uniform bound on $M_1(t)$

$$M_1(t) \leq \frac{c}{1-\varepsilon} \Big(\| v_1(0) \|_{L^2(\Omega)} + \| v_2(0) \|_{L^2(\Omega)} \Big), \tag{5.231}$$

valid for all $t > 0$. Combining bound (5.231) on M_1 and estimate (5.212) on $v_1(t)$, yields

$$\| v_1(t) \|_{L^2(\Omega)} \leq \bar{c} e^{-\varepsilon\nu t} \Big(\| v_1(0) \|_{L^2(\Omega)} + \| v_2(0) \|_{L^2(\Omega)} \Big), \tag{5.232}$$

and we see that $\| v_1 \|_{L^2(\Omega)}$ decays with exponential rate $\varepsilon\nu$ as $t \longrightarrow \infty$. To obtain a bound on $\| v_2 \|_{L^2(\Omega)}$, we

combine bound (5.231) on $M_1(t)$ with equation (5.218) and this yields

$$\| v_2(t) \|_{L^2(\Omega)} \leq \hat{c} \Big(\varepsilon e^{-\nu\varepsilon t} \| v_1(0) \|_{L^2(\Omega)} + (e^{-\delta t} + \varepsilon e^{-\nu\varepsilon t}) \| v_2(0) \|_{L^2(\Omega)} \Big), \tag{5.233}$$

$$\leq \hat{c} e^{-\nu\varepsilon t} \Big(\varepsilon \| v_1(0) \|_{L^2(\Omega)} + (1+\varepsilon) \| v_2(0) \|_{L^2(\Omega)} \Big), \tag{5.234}$$

Returning to the original equation (5.204), and recall that $V = [v_1, v_2]^T$ we can bound the norm of V using (5.232) and (5.233) which yields

$$\| V \|_{L^2(\Omega)} \leq \| v_1(t) \|_{L^2(\Omega)} + \| v_2(t) \|_{L^2(\Omega)}, \tag{5.235}$$

$$\leq c e^{-\varepsilon\nu t} \Big[(1+\varepsilon) \| v_1(0) \|_{L^2(\Omega)} + (2+\varepsilon) \| v_2(0) \|_{L^2(\Omega)} \Big]. \tag{5.236}$$

for some constant c depend upon ν and for every $t > 0$. Therefore, the semi-groups generated by $\mathbb{L}_b$ manifest decay with an exponential rate $\varepsilon\nu$.

5.5 Connecting the pearling eigenvalues of $\Delta\mathbb{L}_b$ and those of $\mathbb{L}_b$

In the first part of this section we derived conditions under which the bilayer dressing of an admissible interface is pearling stable. This reduces to an understanding of the the spectrum of $\mathbb{L}_b$, the second variation of $\mathcal{F}$ at the bilayer u_b. However, to understand the dynamic stability of a bilayer under the H^{-1} gradient flow, requires the analysis of the pearling eigenvalues of the linearization, $\Delta\mathbb{L}_b$ of the gradient flow. This analysis is completed in the theorem below.

Proposition 5.5.1. *Fix an admissible interface Γ_b and let u_b denote the associated bilayer solution. Let $\mathbb{L}_b$ be the second variation of $\mathcal{F}$ evaluated at u_b. Then, the spectrum of $\Delta\mathbb{L}_b$ is real, and there exist $U > 0$ such that for each $\Lambda \in \sigma(\Delta\mathbb{L}_b) \cap (-\infty, U)$ there exists $\mu \in \sigma(\mathbb{L}_b) \cap (-\infty, \varepsilon^2 U a)$, such that*

$$\Lambda = \varepsilon^{-2} \frac{\mu}{a} + O(\varepsilon^{3/4 - d/2}), \tag{5.237}$$

where the constant a is defined via

$$a := -\int_{\mathbb{R}} \psi_{b,0} (\partial_z^2 - \lambda_{b,0})^{-1} \psi_{b,0} \, dz > 0. \tag{5.238}$$

In particular, for space dimension $d = 2$ or $d = 3$ the first term gives the leading order form of Λ.

Proof of Theorem 5.5.1. Consider the eigenvalue problem

$$-\Delta \mathbb{L} \Psi = \Lambda \Psi, \tag{5.239}$$

and let $\{v_i\}_{i=1}^N$ be the orthonormal basis of the space X_Σ given in Definition 5.1. We consider the decomposition of the eigenfunction Ψ

$$\Psi = v + v^\perp, \tag{5.240}$$

where $v \in X_\Sigma$, $v = \sum_{i=1}^N \alpha_i v_i$,

$$\| v \|_{L^2(\Omega)} = \| \vec{\alpha} \|_{l^2}, \tag{5.241}$$

and $v^\perp \in V^\perp$. Inserting (5.240) into (5.239) yields

$$\mathbb{L}_b(v + v^\perp) = -\Lambda \Delta^{-1}(v + v^\perp). \tag{5.242}$$

The projection of (5.242) onto $v_j \in X_\Sigma$ yields

$$(\mathbb{L}_b v, v_j)_{L^2(\Omega)} + (\mathbb{L}_b v^\perp, v_j)_{L^2(\Omega)} = -\Lambda(\Delta^{-1} v, v_j)_{L^2(\Omega)} - \Lambda(\Delta^{-1} v^\perp, v_j)_{L^2(\Omega)}. \tag{5.243}$$

We introduce the following matrices

$$\mathbb{D}_{i,j} := (\mathbb{L}_b v_i, v_j)_{L^2(\Omega)}, \tag{5.244}$$

$$\mathbb{E}_{i,j} := -(\Delta^{-1} v_i, v_j)_{L^2(\Omega)}. \tag{5.245}$$

Using the matrices $\mathbb{D}$ and $\mathbb{E}$, defined in (5.244) and (5.245), respectively, we rewrite equation (5.243) as

$$(\mathbb{D} - \Lambda \mathbb{E})\vec{\alpha} = \vec{\gamma}, \tag{5.246}$$

where we introduce the vector $\vec{\gamma}$

$$\vec{\gamma}_j := \Lambda(-\Delta^{-1} v^\perp, v_j)_{L^2(\Omega)} - (\mathbb{L}_b v^\perp, v_j)_{L^2(\Omega)}. \tag{5.247}$$

To bound the right-hand side of (5.247) we need to bound $\| v^\perp \|_{L^2(\Omega)}$. Consider the complementary projection

of (5.242) onto $v^\perp \in V^\perp$ which yields

$$(\mathbb{L}_b v^\perp, v^\perp)_{L^2(\Omega)} = -\Lambda(\Delta^{-1} v, v^\perp)_{L^2(\Omega)} + \Lambda(-\Delta^{-1} v^\perp, v^\perp)_{L^2(\Omega)} - (\mathbb{L}_b v, v^\perp)_{L^2(\Omega)}. \tag{5.248}$$

The operator $\mathbb{L}_b$ is coercive on $X_\Sigma^\perp$ and so, there exists $\nu > 0$ so that

$$\nu \left\| v^\perp \right\|^2_{L^2(\Omega)} \leq (\mathbb{L}_b v^\perp, v^\perp)_{L^2(\Omega)}. \tag{5.249}$$

Proposition (5.3.4) implies that there exists $c > 0$ so that the bilinear form $(\mathbb{L}_b v, v^\perp)_{L^2(\Omega)} = (v, \mathbb{L}_b v^\perp)_{L^2(\Omega)}$ has is bounded

$$(\mathbb{L}_b v, v^\perp)_{L^2(\Omega)} \leq c\varepsilon \left\| v \right\|_{L^2(\Omega)} \left\| v^\perp \right\|_{L^2(\Omega)}. \tag{5.250}$$

Since the term $(-\Delta^{-1} v^\perp, v^\perp)_{L^2(\Omega)}$ is positive, we need to consider the sign of Λ. If $\Lambda < 0$, the last term on the right-hand side of (5.248) is negative, and we can drop it when we are bounding from above. It $\Lambda > 0$, then there exists $c > 0$ so that

$$\Lambda(-\Delta^{-1} v^\perp, v^\perp)_{L^2(\Omega)} \leq c\Lambda \left\| v^\perp \right\|^2_{L^2(\Omega)}. \tag{5.251}$$

Moreover, from equation (5.266) of Lemma 5.6, there exists $c > 0$ so that

$$\Lambda(-\Delta^{-1} v, v^\perp)_{L^2(\Omega)} \leq c_1 |\Lambda| \varepsilon^2 \left\| v \right\|_{L^2(\Omega)} \left\| v^\perp \right\|_{L^2(\Omega)}. \tag{5.252}$$

Using the bounds (5.249), (5.250), (5.251)and (5.252) in equation (5.248) we obtain

$$\nu \left\| v^\perp \right\|^2_{L^2(\Omega)} \leq c_1 |\Lambda| \varepsilon^2 \left\| v \right\|_{L^2(\Omega)} \left\| v^\perp \right\|_{L^2(\Omega)} + \Lambda_+ c_2 \left\| v^\perp \right\|^2_{L^2(\Omega)} + c_3 \varepsilon \left\| v \right\|_{L^2(\Omega)} \left\| v^\perp \right\|_{L^2(\Omega)}, \tag{5.253}$$

where

$$\Lambda_+ = \begin{cases} 0 & \text{if } \Lambda < 0, \\ \Lambda & \text{if } \Lambda > 0. \end{cases} \tag{5.254}$$

Solving (5.253) for $\left\| v^\perp \right\|_{L^2(\Omega)}$, we obtain the upper bound,

$$\left\| v^\perp \right\|_{L^2(\Omega)} \leq c\varepsilon \left\| v \right\|_{L^2(\Omega)}, \tag{5.255}$$

valid so long as $\Lambda_+ < U := \nu / c_2$. Bounding $|\gamma_j|$, defined in (5.247), via (5.250) and (5.252) yields

$$|\gamma_j| \leq (c_1 \varepsilon + c_2 \varepsilon^2 |\Lambda|) \left\| v_j \right\|_{L^2(\Omega)} \left\| v^\perp \right\|_{L^2(\Omega)}. \tag{5.256}$$

Combining (5.256) and (5.255) we obtain a bound on the l^2 norm of $\vec{\gamma}$

$$\| \vec{\gamma} \|_{l^2} \leq c(\varepsilon^2 + \varepsilon^3 |\Lambda|)\sqrt{N} \| \alpha \|_{l^2}, \tag{5.257}$$

where $N \sim O(\varepsilon^{3/2-d})$.

Going back to equation (5.246), we note that the matrix $\mathbb{D}$, defined in (5.244), is precisely the matrix M introduced in (5.26), and from Corollary 5.2.2 we know that, at leading order, $\mathbb{D}$ takes the form

$$\mathbb{D} = \mathbb{D}^0 + \varepsilon^2 \mathbb{D}^1, \tag{5.258}$$

where $\mathbb{D}^0$ is a diagonal matrix whose entries are the eigenvalues of $\mathbb{L}_b$, and $\mathbb{D}^1$ is $O(1)$ in the operator norm. We can express the Laplace inverse operator using the whiskered coordinates system, (2.9),

$$\Delta^{-1}\Big|_{x_\Sigma} = (\varepsilon^{-2}\partial_z^2 + \varepsilon^{-1} H \partial_z + \Delta_G)^{-1} = ((1+T)\mathbb{L}_0)^{-1} = \mathbb{L}_0^{-1} + \mathbb{L}_0^{-1} T + ..., \tag{5.259}$$

where we introduce the operators

$$\mathbb{L}_0 := \varepsilon^{-2}\partial_z^2 - \Delta_s, \tag{5.260}$$

$$T := (\varepsilon^{-1} H \partial_z + \varepsilon z D_{s,2})\mathbb{L}_0^{-1}. \tag{5.261}$$

Plugging (5.259) into (5.245) we see that $\mathbb{E}$ takes the form

$$\mathbb{E} = \tilde{\mathbb{E}}^0 + \tilde{\mathbb{E}}^1, \tag{5.262}$$

where the entries of the matrices $\tilde{\mathbb{E}}^0$ and $\tilde{\mathbb{E}}^1$ take the form

$$\tilde{\mathbb{E}}^0_{i,j} := -(\mathbb{L}_0^{-1} v_i, v_j)_{L^2(\Omega)}, \tag{5.263}$$

$$\tilde{\mathbb{E}}^1_{i,j} := -(\mathbb{L}_0^{-1} T v_i, v_j)_{L^2(\Omega)}. \tag{5.264}$$

To estimate the entries of $\tilde{\mathbb{E}}^0$ we prove the following Lemma

Lemma 5.6. *The inverse operator $\mathbb{L}_0^{-1}$ acting on $v \in X_\Sigma$, $v = \sum_{k \in \Sigma_{b,0}} \alpha_k \psi_{b,0} \Theta_k$, takes the form*

$$\mathbb{L}_0^{-1} \psi_{b,0} \Theta_k = \varepsilon^2 \Theta_k (\partial_z^2 - \varepsilon^2 \beta_k)^{-1} \psi_{b,0}, \quad \forall\ k \in \Sigma_{b,0}. \tag{5.265}$$

In particular, $\mathbb{L}_0^{-1}$ has an $O(\varepsilon^2)$ bound on X_Σ, i.e., there exists $c>0$ so that

$$\left\| \mathbb{L}_0^{-1} v \right\|_{L^2(\Omega)} \leq c\varepsilon^2 \left\| v \right\|_{L^2(\Omega)}, \quad \forall\ v \in X_\Sigma. \tag{5.266}$$

Proof. To obtain an expression for $\mathbb{L}_0^{-1}v$, consider the following equation

$$\mathbb{L}_0 f = \psi_{b,0}\Theta_k, \tag{5.267}$$

and recall that Θ_k is an eigenfunction of Δ_s, which implicates that the function f is of the form $f = g(z)\Theta_k$, and equation (5.267) reduces to

$$(\varepsilon^{-2}\partial_z^2 - \beta_k)g(z)\Theta_k = \psi_{b,0}\Theta_k. \tag{5.268}$$

Factoring ε^{-2} from the left-hand side of (5.268) yields

$$\varepsilon^{-2}(\partial_z^2 - \varepsilon^2\beta_k)g(z)\Theta_k = \psi_{b,0}\Theta_k. \tag{5.269}$$

Inverting the operators in equations (5.269) and (5.267) we conclude that

$$\mathbb{L}_0^{-1}(\psi_{b,0}\Theta_k) = \varepsilon^2\Theta_k(\partial_z^2 - \varepsilon^2\beta_k)^{-1}\psi_{b,0}. \tag{5.270}$$

For a general $v \in X_\Sigma$, $v = \sum_{k\in\Sigma_{b,0}} \alpha_n\Theta_k\psi_{b,0}$, equation (5.270) takes the form

$$\mathbb{L}_0^{-1}v = \varepsilon^2 \sum_{k\in\Sigma_{b,0}} \alpha_k\Theta_k(\partial_z^2 - \varepsilon^2\beta_k)^{-1}\psi_{b,0}. \tag{5.271}$$

Since Θ_j are orthonormal in the Γ_b weighted inner product, see (2.29), the $L^2(\Omega)$-norm of the inverse operator acting on $v \in X_\Sigma$ yields

$$\left\| \mathbb{L}_0^{-1} v \right\|_{L^2(\Omega)} = \varepsilon^2 \left\| A\vec{\alpha} \right\|_{l^2}, \tag{5.272}$$

where A is the diagonal matrix with entries

$$A_i = \|(\partial_z - \varepsilon^2\beta_i)^{-1}\psi_0\|_{L^2(\mathbb{R})}. \tag{5.273}$$

Since $\varepsilon^2\beta_i = \lambda_{b,0} + O(\sqrt{\varepsilon})$ we deduce that A is uniformly bounded and hence

$$\left\| \mathbb{L}_0^{-1} v \right\|_{L^2(\Omega)} = c\varepsilon^2 \left\| v \right\|_{L^2(\Omega)}. \tag{5.274}$$

■

Lemma 5.6 implies that the entries of $\tilde{\mathbb{E}}^0$, defined in (5.263), admit the following expansion

$$\tilde{\mathbb{E}}^0_{i,j} = -\int_{\Gamma_b}\int_{-\ell/\varepsilon}^{\ell/\varepsilon}(\varepsilon^{-2}\partial_z^2+\Delta_s)^{-1}(\psi_{b,0}\Theta_i)\Theta_j\psi_{b,0}\,J_b\,dz\,ds \tag{5.275}$$

$$= -\varepsilon^2\int_{\Gamma_b}\int_{-\ell/\varepsilon}^{\ell/\varepsilon}\Theta_i\Theta_j\psi_{b,0}(\partial_z^2-\varepsilon^2\beta_i)^{-1}\psi_{b,0}\,J_b\,dz\,ds \tag{5.276}$$

$$= \begin{cases} -\varepsilon^2\int_{-\ell/\varepsilon}^{\ell/\varepsilon}\psi_{b,0}(\partial_z^2-\lambda_{b,0})^{-1}\psi_{b,0}\,dz + O(\varepsilon^2\sqrt{\varepsilon}) & \text{if } i=j,\\ -\varepsilon^3\int_\Gamma\int_{-\ell/\varepsilon}^{\ell/\varepsilon}\psi_{b,0}(\partial_z^2-\varepsilon^2\beta_i)^{-1}\psi_{b,0}\Theta_i\Theta_j H_0\,dz\,ds + O(\varepsilon^4) & \text{if } i\neq j, \end{cases} \tag{5.277}$$

where for $i=j$, since $k\in\Sigma_{b,0}$, we obtained (5.277) by expanding $\varepsilon^2\beta_i = \lambda_{b,0}+O(\sqrt{\varepsilon})$. Consequently we may write

$$\tilde{\mathbb{E}}^0 = \varepsilon^2\left(\mathbb{E}^0+\sqrt{\varepsilon}B\right), \tag{5.278}$$

where $\mathbb{E}^0 = aI_d$ is a diagonal matrix, aI_d is the identity matrix multiplies by the constant a defined in (5.238), and the matrix B has entries of the form $\int_{\Gamma_b} f(\vec{k})\Theta_i\Theta_j\,J_0\,ds$. By Lemma 5.2, the matrix B has $O(1)$ bound in the operator norm. The next Lemma shows that the matrix $\tilde{\mathbb{E}}^1$, defined in (5.264), is bounded in the operator norm.

Lemma 5.7. *The matrix $\tilde{\mathbb{E}}^1$, defined in (5.264) has an $O(\varepsilon^3)$ bounded in the operator norm, i.e., there exists $c>0$ so that*

$$\left\|\tilde{\mathbb{E}}^1\right\|_{l^2\to l^2}\le c\varepsilon^3. \tag{5.279}$$

Proof. The operator $\mathbb{L}_0^{-1}$ is self-adjoint in the J_0 inner product, defined as in (2.35), and so the entries of $\tilde{\mathbb{E}}^1$, given in (5.264) can be written as

$$\mathbb{E}^1_{i,j} = -(\mathbb{L}_0^{-1}Tv_i,v_j)_{J_0} = -(Tv_i,\mathbb{L}_0^{-1}v_j)_{J_0}. \tag{5.280}$$

For $v_k\in X_\Sigma$, $v_k=\varphi_{b,0}\Theta_k$, the operator T acting on v_k takes the form

$$Tv_k = \varepsilon H\Theta_k g_k' + \varepsilon^3 z g_k D_{s,2}\Theta_k, \tag{5.281}$$

where

$$g_k := (\partial_z^2-\varepsilon^2\beta_k)^{-1}\psi_{b,0}. \tag{5.282}$$

Using equation (5.281) and Lemma 5.6 we write

$$-(Tv_i, \mathbb{L}_0^{-1} v_j)_{J_0} = -\int_{\Gamma_b} \int_{-\ell/\varepsilon}^{\ell/\varepsilon} (\varepsilon H \Theta_i g_i' + \varepsilon^3 z g_i D_{s,2} \Theta_i) \varepsilon^2 \Theta_j g_j \, J_0 \, dz \, ds \tag{5.283}$$

$$= -\varepsilon^3 \mathbb{E}^1_{i,j} - \varepsilon^5 \mathbb{E}^2_{i,j} \tag{5.284}$$

where the matrices $\mathbb{E}^1$ and $\mathbb{E}^2$ are given by

$$\mathbb{E}^1_{i,j} := \int_{\Gamma_b} \int_{-\ell/\varepsilon}^{\ell/\varepsilon} H g_i' g_j \Theta_i \Theta_j \, J_0 \, dz \, ds, \tag{5.285}$$

$$\mathbb{E}^2_{i,j} := \int_{\Gamma_b} \int_{-\ell/\varepsilon}^{\ell/\varepsilon} z g_i g_j \Theta_j D_{s,2} \Theta_i \, J_0 \, dz \, ds. \tag{5.286}$$

Lemma 5.2 implies that $\mathbb{E}^1$ has an $O(1)$ bound in the operator norm, and according to Corollary 5.2.1, the ε^5 terms are negligible. Going back to (5.280) we conclude that

$$\left\| \tilde{\mathbb{E}}^1 \right\|_{l^2 \to l^2} \leq c\varepsilon^3. \tag{5.287}$$

■

Equation (5.278) and Lemma 5.7 implies that $\mathbb{E}$, defined in (5.262), can be written as

$$\mathbb{E} = \varepsilon^2 a I_d + \varepsilon^{5/2} \mathbb{E}^1, \tag{5.288}$$

where the matrix $\mathbb{E}^1$ has an $O(1)$ bound in the operator norm.

We use the expansions of the matrices $\mathbb{D}$ and $\mathbb{E}$, defined in (5.258) and (5.288), respectively, to expand equation (5.246) so that

$$(\mathbb{D}^0 - \varepsilon^2 a \Lambda) \vec{\alpha} = \vec{\gamma} - \varepsilon^2 (\mathbb{D}^1 - \sqrt{\varepsilon} \mathbb{E}^1) \vec{\alpha}, \tag{5.289}$$

which, dividing by $\varepsilon^2 a$, takes the form

$$\left(\frac{\varepsilon^{-2}}{\alpha} \mathbb{D}^0 - \Lambda \right) \vec{\alpha} = \frac{\varepsilon^{-2}}{\alpha} \left(\gamma - \varepsilon^2 (\mathbb{D}^1 - \sqrt{\varepsilon} \mathbb{E}^1) \vec{\alpha} \right), \tag{5.290}$$

Taking the L^2 norm of equation (5.290), and using inequality (5.257) we deduce that

$$\left\| \left(\frac{\varepsilon^{-}2}{a} \mathbb{D}^0 - \Lambda \right) \vec{\alpha} \right\|_{l^2} \leq C(1 + \varepsilon |\Lambda|) \sqrt{N} \, \| \alpha \|_{l^2}, \tag{5.291}$$

however as $\mathbb{D}^0$ is self-adjoint, it follows that α is close to an eigenvector of $\mathbb{D}^0$ and

$$dist(\Lambda, \frac{\varepsilon^{-2}}{a}\sigma(\mathbb{D}^0)) \leq C(1+\varepsilon|\Lambda|)\varepsilon^{3/4-d/2}. \tag{5.292}$$

Since the spectrum of $\mathbb{D}^0$ constitutes the pearling eigenvalues of $\mathbb{L}_b$, and $\sigma(\mathbb{D}^0) \sim O(\varepsilon)$, we know that $\frac{\varepsilon^{-2}}{a}\sigma(\mathbb{D}^0) \sim O(\varepsilon^{-1})$, and hence $|\varepsilon\Lambda| = O(1)$. As long as the right-hand side of equation (5.292) is $O(\varepsilon^r)$ with $r > -1$, i.e., as long as the dimension size is $d < 3.5$, we may conclude that for $\Lambda \in \sigma(\Delta\mathbb{L}_b) \cap (-\infty, U)$ there exists $\mu \in \sigma(\mathbb{L}_b) \cap (-\infty, \varepsilon^2 U a)$ so that

$$\Lambda = \varepsilon^{-2}\frac{\mu}{a} + O(\varepsilon^{3/4-d/2}). \tag{5.293}$$

■

Chapter 6

The Pearling Eigenvalue Problem, Co-Dimension 2

In this Chapter we address the stability of the bilayer morphology in the strong FCH and obtain an explicit expression for the pearling stability condition. We present a rigourous analysis of the eigenvalue problem corresponding to the strong FCH for the co-dimension two structure. We show that in the **strong FCH** scaling the leading order behavior of the pearling eigenvalues is independent of the shape of the underlying co-dimension two morphology. Under the H^{-1} gradient flow the pearling instability manifests itself on a time scale that is $O(\varepsilon^{-2})$ faster than the geometric evolution, and hence can be taken to be instantaneous on the geometric evolution time scale. Conversely, the fingering instability occurs on the same time scale as the geometric flow, and may not necessarily immediately manifest itself on the geometric evolution time scale.

Recall the strong FCH free energy which corresponds to the choice $p = 1$ in (1.14),

$$\mathcal{F}(u) = \int_\Omega \frac{1}{2}(\varepsilon^2\Delta u - W'(u))^2 - \varepsilon\left(\frac{\varepsilon^2\eta_1}{2}|\nabla u|^2 + \eta_2 W(u)\right) dx, \tag{6.1}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain, $W(u)$ is a tilted double-well potential with two minima at $b_\pm$, $u : \Omega \to \mathbb{R}$ is the density of the amphiphilic species, $\varepsilon \ll 1$ controls the width of the boundary layer and η_1 and η_2 are the functionalization constants. The first variation of $\mathcal{F}$, introduced in equation (1.18), is given by

$$\frac{\delta\mathcal{F}}{\delta u}(u) = (\varepsilon^2\Delta - W''(u) + \varepsilon\eta_1)(\varepsilon^2\Delta u - W'(u)) + \varepsilon\eta_d W'(u), \tag{6.2}$$

where $\eta_d := \eta_1 - \eta_2$. The second variations of $\mathcal{F}$ takes the form

$$\mathbb{L}_p := \frac{\delta^2 \mathcal{F}}{\delta u^2}(u) = \left(\varepsilon^2\Delta - W''(u) + \varepsilon\eta_1\right)\left(\varepsilon^2\Delta - W''(u)\right) - \left(\varepsilon^2\Delta u - W'(u)\right)W'''(u) + \varepsilon\eta_d W''(u). \tag{6.3}$$

We obtain a pearling stability condition for the co-dimension two morphology which is summarized in the following theorem-

Theorem 6.0.2. *For a given admissible interface, Γ_p, the associated pore solution constructed in (2.75), is stable with respect to the pearling bifurcation if and only if the far-field chemical potential μ_1 satisfies the* pearling stability condition

$$P_p^* := -\frac{\eta_d\left(\left\|\psi'_{p,0}\right\|^2_{L_R} + \lambda_{p,0}\left\|\psi_{p,0}\right\|^2_{L_R}\right)}{S_p} > \mu_1, \tag{6.4}$$

6.1 Overview

We want to investigate the pearling eigenmodes of the co-dimension two pore structures: Given an admissible interface Γ, assume that the system is at quasi-equilibrium with

$$u_p = U_p(R) + \varepsilon u_1, \tag{6.5}$$

where U_p is the radial symmetric pore solution of (2.75) and u_1, derived in equation (4.103), is given by

$$u_1 = \mu_1\Phi_{p,2} - \eta_d L_p^{-2}W'(U_p), \tag{6.6}$$

where L_p is the linear operator introduced in (2.78),The chemical potential μ_1 is spatially constant and the functions $\Phi_{p,j}$ solves (2.86) for $j = 1, 2$.

We are interested in the pearling eigenmodes of the second variation of $\mathcal{F}$, $\mathbb{L}_p$, defined in (6.3). Consider the eigenvalue problem

$$\mathbb{L}_p\Psi = \Lambda\Psi. \tag{6.7}$$

By changing coordinates of the Laplacian, in the operator $\mathbb{L}_p$, to the whiskered coordinates, using (2.65), and plugging-in the expansion of u_p, (6.5), into u, we can rewrite $\mathbb{L}_p$ in orders of ε such that

$$\mathbb{L}_p = \mathcal{L}_p^2 + \varepsilon\mathbb{L}_1 + O(\varepsilon^2), \tag{6.8}$$

where

$$\mathbb{L}_1 := -\left(W'''(U_p)u_1 - \eta_1\right) \circ \mathcal{L}_p - \mathcal{L}_p \circ \left(W'''(U_p)u_1\right) - \left(\mathcal{L}_p u_1 - D_z U_p\right) W'''(U_p) + \eta_d W''(U_p), \tag{6.9}$$

and $\mathcal{L}_p$ is defined in (2.77). See appendix (E.2) for detailed calculations of the expansion of $\mathbb{L}_p$.
Recall that $\Sigma_{p,0}$, defined in (2.92), is the set of small eigenvalues associated to $\mathbb{L}_p$, and, according to Weyl's asymptotic formula $|\Sigma_{p,0}| \sim O(\varepsilon^{3/2-d})$. We define

$$P_k := \varepsilon^{-1/2}(\lambda_{p,0} - \varepsilon^2 \beta_k), \tag{6.10}$$

to be the detuning constant depending only on k.

Definition 6.1. *The space, X_Σ, corresponding to the small eigenvalues of $\mathbb{L}_p$ is defined as*

$$X_\Sigma := \{\psi_{p,0}\Theta_k \mid k \in \Sigma\}, \tag{6.11}$$

Looking for solutions of the eigenvalue problem, (6.7), we consider a regular perturbation expansion of the form

$$\Psi_j = \Psi_{0,j} + \varepsilon\Psi_{1,j} + O(\varepsilon^2), \quad \Psi_{0,j} \in X_\Sigma, \quad \Psi_{0,j} = \sum_{k\in\Sigma} \alpha_k \psi_0 \Theta_k, \quad \Psi_{1,j} \in X_\Sigma^\perp, \tag{6.12}$$

$$\Lambda_j = \varepsilon\Lambda_{1,j} + O(\varepsilon^2). \tag{6.13}$$

The L^2-orthogonal projection, Π, onto X_Σ is given by

$$\Pi\, f := \sum_{k\in\Sigma} \frac{(f, \psi_0\Theta_k)_{L^2(\Omega)}}{\|\psi_0\Theta_k\|^2_{L^2(\Omega)}} \psi_0\Theta_k = \sum_{k\in\Sigma} (f, \psi_0\Theta_k)_{L^2(\Omega)} \psi_0\Theta_k, \tag{6.14}$$

and its complementary projection is $\tilde{\Pi} = I - \Pi$.
We consider a decomposition of the operator $\mathbb{L}$ into a 2×2 block form,

$$\begin{bmatrix} M & B \\ B^T & C \end{bmatrix}, \tag{6.15}$$

where

$$M \coloneqq \Pi \mathbb{L}_p \Pi, \quad B \coloneqq \Pi \mathbb{L}_p \tilde{\Pi}, \quad C \coloneqq \tilde{\Pi} \mathbb{L}_p \tilde{\Pi}. \tag{6.16}$$

By abuse of notation we denote $\mathbb{L}_p$ and its 2×2 decomposition with the same symbol.

By Assumption 2.2.3, the restricted operator C is uniformly coercive on $X_\Sigma^\perp$ and its spectrum is bounded from below by $\delta > 0$ which may be chosen independent of sufficiently small $\varepsilon > 0$.

6.2 Eigenvalues of $M \coloneqq \Pi \mathbb{L}_p \Pi$

Let $v \in X_\Sigma$, v can be written as

$$v = \sum_{k \in \Sigma} b_k \psi_0 \Theta_k, \tag{6.17}$$

without loss of generality, assume $\|v\|_{L^2}(\Omega) = 1$. The operator $\Pi \mathbb{L} \Pi$, acting on v, takes the form

$$\Pi \mathbb{L} \Pi \sum_{j \in \Sigma} b_j \psi_0 \Theta_k = \sum_{k \in \Sigma} \left(\mathbb{L}_p \sum_{j \in \Sigma} b_j \psi_0 \Theta_j, \psi_0 \Theta_k \right)_{L^2(\Omega)} \psi_0 \Theta_k = \sum_{k \in \Sigma} \sum_{j \in \Sigma} b_j \left(\mathbb{L}_p \psi_0 \Theta_j, \psi_0 \Theta_k \right)_{L^2(\Omega)} \psi_0 \Theta_k. \tag{6.18}$$

We define the operator matrix representation $M \in \mathbb{R}^{N \times N}$, where $N \approx \varepsilon^{-3/2}$, in the following way

$$M_{j,k} \coloneqq \left(\mathbb{L}_p \psi_0 \Theta_j, \psi_0 \Theta_k \right)_{L^2(\Omega)}. \tag{6.19}$$

Using the expansion of $\mathbb{L}_p$, (6.8), we can write each entry of M in orders of ε such that

$$\left(\mathbb{L}_p \psi_0 \Theta_j, \psi_0 \Theta_k \right)_{L^2(\Omega)} = \left(\mathcal{L}_p^2 \psi_0 \Theta_j, \psi_0 \Theta_k \right)_{L^2(\Omega)} + \varepsilon \left(\mathbb{L}_1 \psi_0 \Theta_j, \psi_0 \Theta_k \right)_{L^2(\Omega)} + O(\varepsilon^2), \tag{6.20}$$

and decompose the matrix into terms of order $\leq \varepsilon^r$ and terms of order $> \varepsilon^r$, such that

$$M = M^0 + \varepsilon^r \tilde{M}, \tag{6.21}$$

where

$$M^0_{j,k} = \left(\mathcal{L}_p^2 \psi_0 \Theta_j, \psi_0 \Theta_k \right)_{L^2(\Omega)} + \varepsilon \left(\mathbb{L}_1 \psi_0 \Theta_j, \psi_0 \Theta_k \right)_{L^2(\Omega)} + \sum_{i=2}^{r} \varepsilon^i \left(\mathbb{L}_i \psi_0 \Theta_j, \psi_0 \Theta_k \right)_{L^2(\Omega)}, \tag{6.22}$$

$$\tilde{M}_{j,k} = \sum_{i \geq r} \varepsilon^{(i-r)} \left(\mathbb{L}_i \psi_0 \Theta_j, \psi_0 \Theta_k \right)_{L^2(\Omega)}. \tag{6.23}$$

We will show that the first term, M^0, can be split into a diagonal and off-diagonal terms, the latter of which can be bounded independently of the matrix size N, assuming that the curvatures of the interface Γ are sufficiently smooth. The other term, $\tilde{M}$, can be bounded, independent of the dimension, via the L^∞ norm. In particular, we have shown in Section 5.2.1, that for a 3-dimensional space, the matrix $\tilde{M}$ is negligible for $r = 2$.

6.2.1 Bounding M^0

Next, we want to find a bound for the matrix M^0 in $\mathbb{R}^3$. An examination of the first two terms of M^0, given in equation (6.22), shows that they admit the expansions

$$(\mathcal{L}_p^2\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} = \begin{cases} \varepsilon P_k^2 + O(\varepsilon^2\sqrt{\varepsilon}) & \text{if } k = j, \\ O(\varepsilon^2\sqrt{\varepsilon}) & \text{if } k \neq j, \end{cases} \tag{6.24}$$

$$(\mathbb{L}_1\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} = -\left(((Lu_1)W'''(U_p) - \eta_d W''(U_p))\psi_0\Theta_j, \psi_0\Theta_k\right)_{L^2(\Omega)} + O(\sqrt{\varepsilon}) \quad \text{if } k = j, \tag{6.25}$$

$$\begin{aligned}(\mathbb{L}_1\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} =& \varepsilon\left(((2W'''(U_p)u_1 - \eta_1)\vec{\kappa}\cdot\nabla_z\psi_0 - \vec{\kappa}\cdot(\nabla_z u_1 - U_p)W'''(U_p)\psi_0)\Theta_j, \psi_0\Theta_k\right)_{L^2(\Omega)} \\ &+ O(\varepsilon^2) \quad \text{if } k \neq j\end{aligned} \tag{6.26}$$

see (E.14) and (E.33) for more details. We may split M^0 into its on/off diagonal matrices

$$M^0 = M^0_{\text{diag}} + M^0_{\text{off-diag}} \tag{6.27}$$

where

$$M^0_{\text{diag}}(j,k) = \begin{cases} M^0_{k,k} + O(\varepsilon\sqrt{\varepsilon}) & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \tag{6.28}$$

and

$$M^0_{\text{off-diag}}(j,k) = \begin{cases} 0 & \text{if } j = k, \\ M^0_{j,k} + O(\varepsilon^2\sqrt{\varepsilon}) & \text{if } j \neq k. \end{cases} \tag{6.29}$$

with entries given by

$$M^0_{k,k} = \varepsilon\left(P_k^2 - \int_0^{l\backslash\varepsilon}\left[W'''(U_p)Lu_1 - \eta_d W''(U_p)\right](\psi_0^0)^2\,dz\right), \tag{6.30}$$

$$= \varepsilon\left[P_k^2 - \mu_1 S_p - \eta_d\left(\left\|(\psi_0^0)'\right\|^2_{L_R} + \lambda_{p,0}\left\|(\psi_0^0)\right\|^2_{L_R}\right)\right], \tag{6.31}$$

$$M^0_{j,k} = \varepsilon^2\left(((2W'''(U_p)u_1 - \eta_1)\vec{\kappa}\cdot\nabla_z\psi_0 - \vec{\kappa}\cdot(\nabla_z u_1 - U_p)W'''(U_p)\psi_0)\Theta_j, \psi_0\Theta_k\right)_{L^2(\Omega)}, \tag{6.32}$$

with indices $a\varepsilon^{-1/2} \leq j,k \leq \tilde{a}\varepsilon^{-1/2}$, $a < \tilde{a},\ a,\tilde{a} \in \mathbb{R}$, and S_p is the shape factor of the pore structure, given by,

$$S_p := 2\pi \int_0^\infty W'''(U_p)\Phi_1(\psi_0^0)^2\, RdR. \tag{6.33}$$

Using Theorem 5.2 we deduce that $M^0_{\text{off-diag}}$ is uniformly bounded as an operator from l^2 to l^2.

Corollary 6.2.1. *The matrix M^0, defined in (6.27), can be written as*

$$M^0 = M^0_{diag} + M^0_{off\text{-}diag}, \tag{6.34}$$

where $M^0_{off\text{-}diag}$ is uniformly bounded as an operator from l^2 to l^2.

At this point we conclude that the eigenvalues of M^0, Λ_k, are, at leading order, the diagonal entries of M^0_{diag}, defined in equation (6.28). By the definition of M, (6.21), we deduce that Λ_k are the eigenvalues of M, at leading order. Since M is the matrix representation of $\Pi\mathbb{L}_p\Pi$, the eigenvalues of $\Pi\mathbb{L}_p\Pi$ are, at leading order, Λ_k, which takes the form

$$\Lambda_k = \varepsilon\left[P_k^2 - \mu_1 S_p - \eta_d\left(\left\|\psi'_{p,0}\right\|^2_{L_R} + \lambda_{p,0}\left\|\psi_{p,0}\right\|^2_{L_R}\right)\right], \tag{6.35}$$

where S_p is the shape factor defined in (6.33) and P_k is the detuning constant defined in (6.10).
Since P_k^2 can be made as small as $O(\varepsilon)$ (see equation (5.61)), it follows that the term involving P_k^2 is lower order near the turning point of the pearling spectrum. This leads us to the following corollary-

Corollary 6.2.2. *The pearling eigenvalues of $\Pi\mathbb{L}_p\Pi$, (6.13), takes the form*

$$\Lambda = -\varepsilon\frac{1}{\|\psi_{p,0}\|^2_{L_R}}\left[\mu_1 S_p + \eta_d\left(\left\|\psi'_{p,0}\right\|^2_{L_R} + \lambda_{p,0}\left\|\psi_{p,0}\right\|^2_{L_R}\right)\right] + O(\varepsilon^2), \tag{6.36}$$

and, in order to have, at leading order, **pearling stability** *we need*

$$\mu_1 S_p + \eta_d\left(\left\|\psi'_{p,0}\right\|^2_{L_R} + \lambda_{p,0}\left\|\psi_{p,0}\right\|^2_{L_R}\right) < 0. \tag{6.37}$$

Recall that our main goal is to find an expression for the pearling eigenvalues of $\mathbb{L}$ using our 2×2 representation of $\mathbb{L}_p$, see (6.15). In this section we found an expression the pearling eigenvalues of the operator M. The next section establish the bounds on the off-diagonal terms B, B^T.

6.3 Bounding the Off-Diagonal Operators

Recall the 2×2 block form representation of $\mathbb{L}_p$, given in (6.15),

$$\begin{bmatrix} M & B \\ B^T & C \end{bmatrix}. \tag{6.38}$$

If the off-diagonal blocks, $B \coloneqq \Pi \mathbb{L}_p \tilde{\Pi}$ and $B^T \coloneqq \tilde{\Pi} \mathbb{L}_p \Pi$, are small (same order of the $M \coloneqq \Pi \mathbb{L}_p \Pi$ block or less) then we can relate the eigenvalues of $\mathbb{L}_p$ to those of $M \coloneqq \Pi \mathbb{L}_p \Pi$, see section 6.2. Since both Π and $\tilde{\Pi}$ are self-adjoint operators we have

$$(\Pi \mathbb{L}_p \tilde{\Pi} v, w)_{L^2} = (\mathbb{L}_p \tilde{\Pi} v, \Pi w)_{L^2} = (\tilde{\Pi} v, \mathbb{L}_p \Pi w)_{L^2} = (v, \tilde{\Pi} \mathbb{L}_p \Pi w)_{L^2}. \tag{6.39}$$

So, it is enough to show that one of the off-diagonal blocks is small, i.e., we want to show that there exist a constant C, independent on $N \sim O(\varepsilon^{-3/2})$ such that

$$\|\tilde{\Pi} \mathbb{L}_p \Pi v\|_{L^2(\Omega)} \leq \varepsilon C \|v\|_{L^2(\Omega)}, \qquad \forall v \in X_\Sigma. \tag{6.40}$$

without loss of generality, assume $v \in X_\Sigma$, $v = \sum_{j \in \Sigma} b_j \psi_0 \Theta_j$ and $\|v\|_{L^2(\Omega)} = 1$. Note that

$$\|v\|^2_{L^2(\Omega)} = \int_\Omega \sum_{j,k \in \Sigma} b_j b_k \Theta_j \Theta_k \psi_0^2 \, dx = \sum_{j \in \Sigma} b_j^2 \overbrace{\int_\Omega \Theta_j^2 \psi_0^2 \, dx}^{\|\theta_k \psi_0\|^2_{L^2(\Omega)} = 1} + \sum_{\substack{j,k \in \Sigma \\ j \neq k}} b_j b_k \overbrace{\int_\Omega \Theta_j \Theta_k \psi_0^2 \, dx}^{=0, \text{by orthogonality of } \Theta_j}, \tag{6.41}$$

$$= \sum_{j \in \Sigma} b_j^2 = \| b \|^2_{l^2}, \tag{6.42}$$

where $b \coloneqq (b_1, b_2, ..., b_{N_d})$.

Note that we can write $\mathbb{L}_p$ in the following form

$$\mathbb{L}_p = \mathcal{L}_p^2 + \varepsilon \tilde{\mathbb{L}}_p, \tag{6.43}$$

where $\mathbb{L}_p$ is a relatively bounded perturbation of $\mathcal{L}_p^2$. We split the proof into three parts: first we show that we can bound the operator $\tilde{\Pi} \mathcal{L}_p \Pi v$, next we bound the operator $\tilde{\Pi} \mathcal{L}_p^2 \Pi v$ and then we bound $\tilde{\Pi} \mathbb{L}_p \Pi v$.

6.3.1 Bounding $\tilde{\Pi}\mathcal{L}_p\Pi v$

Recall that $v \in X_\Sigma,\ v = \sum_{j\in\Sigma} b_j\psi_0\Theta_j$ and $\|v\|_{L^2(\Omega)} = 1$. In particular,

$$\Pi v = v, \qquad \tilde{\Pi} v = 0. \tag{6.44}$$

We need to show that there exist C, independent of ε, such that

$$\|\tilde{\Pi}\mathcal{L}_p v\|_{L^2(\Omega)} \le \varepsilon C\|v\|_{L^2(\Omega)}. \tag{6.45}$$

Using the expression for $\mathcal{L}_p$, (2.77), $\mathcal{L}_p v$ takes the form

$$\mathcal{L}_p v = (Lv - \varepsilon D_z v + \varepsilon^2\partial_G^2 v) = \sum_{j\in\Sigma} b_j \left(\underbrace{\lambda_{p,0}\psi_0\Theta_j}_{\in X_\Sigma} - \varepsilon D_z\psi_0\Theta_j + \varepsilon^2\partial_G^2\psi_0\Theta_j \right). \tag{6.46}$$

Note that $\tilde{\Pi}$ project off of X_Σ, therefore it eliminates the first term and the term $\tilde{\Pi}\mathcal{L}_p v$ becomes

$$\tilde{\Pi}\mathcal{L}_p v = \varepsilon\tilde{\Pi}\sum_{j\in\Sigma} b_j \left(\varepsilon\partial_G^2\Theta_j\psi_0 - D_z\psi_0\Theta_j\right). \tag{6.47}$$

The $L^2(\Omega)$-norm of (6.47) has the bound

$$\left\| \tilde{\Pi}\mathcal{L}_p v \right\|_{L^2(\Omega)} \le \varepsilon \left\| D_z v \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \partial_G^2 v \right\|_{L^2(\Omega)}, \tag{6.48}$$

and we will show that each of the terms on the right-hand side of equation (6.48) is bounded. To show that the first norm on the right-hand side of equation (6.48) is bounded, we introduce the matrix $B \in \mathbb{R}^{N\times N}$, with $N \sim O(\varepsilon^{-3/2})$ such that

$$B_{j,k} := \langle D_z\psi_0\Theta_j, D_z\psi_0\Theta_k\rangle_{L^2(\Omega)}, \tag{6.49}$$

which, using the definition of D_z, given in (2.66), takes the form

$$B_{j,k} = \int_\Gamma f(\vec{\kappa})\Theta_j\Theta_k\, ds, \quad \text{where } f = \int_0^\infty \left(\frac{\vec{\kappa}}{\tilde{J}_p}\cdot\nabla_z\psi_0\right)^2 J\, dz. \tag{6.50}$$

Applying Theorem 5.2 we conclude that B is uniformly bounded operator from $l^2 \to l^2$. Since B is the matrix representation of the operator D_z we obtain the bound

$$\left\| D_z v \right\|_{L^2(\Omega)} \le c \left\| v \right\|_{L^2(\Omega)} \quad \text{for } v \in X_\Sigma, \tag{6.51}$$

where c is independent on ε. Equation (6.48) reduces to

$$\left\| \tilde{\Pi}\mathcal{L}_p v \right\|_{L^2(\Omega)} \leq \varepsilon c \| v \|_{L^2(\Omega)} + \varepsilon^2 \left\| \partial_G^2 v \right\|_{L^2(\Omega)}, \tag{6.52}$$

The following Proposition shows that, over the appropriate space, the operator $\tilde{\Pi}\partial_G^2$ is bounded in the $L^2(\Omega)$ norm:

Proposition 6.3.1. *Let $f(z)$ be a smooth function such that*

$$|f(z)| < c_1 e^{-c_2|z|} \text{ for some } c_i \in \mathbb{R},\ c_i > 0\ , i = 1, 2,\ supp(f) \subset \Gamma_l. \tag{6.53}$$

The operator $\tilde{\Pi}\partial_G^2$, where $\tilde{\Pi}$ is the projection off of the space of small eigenvalues X_Σ, defined in (6.11) and ∂_G^2 is defined in (2.67), is bounded on the space

$$Y = span\{ f(z)\Theta_k \ \Big|\ k \in \Sigma, \}, \tag{6.54}$$

i.e., there exists $C > 0$, independent of ε, such that

$$\left\| \tilde{\Pi}\partial_G^2 v \right\|_{L^2(\Omega)} \leq C\varepsilon^{-2} \| v \|_{L^2(\Omega)}, \tag{6.55}$$

for every $v \in Y$. Particularly, for $v \in X_\Sigma$, i.e., when $f(z) = \psi_0$, we obtain the bound

$$\left\| \tilde{\Pi}\partial_G^2 v \right\|_{L^2(\Omega)} \leq C\varepsilon^{-1} \| v \|_{L^2(\Omega)}. \tag{6.56}$$

Proof. Fix $\lambda_* \in \rho(\partial_s^2)$, where $\rho(\partial_s^2)$ is the resolvent set of the co-dimeanion two Laplacian operator, then the operator ∂_G^2 can be written as

$$\partial_G^2 = \frac{1}{\tilde{J}_p^2}\left[\partial_s^2 + \varepsilon \frac{z \cdot \partial_s \vec{\kappa}}{\tilde{J}_p}\partial_s\right] = \frac{1}{\tilde{J}_p^2}\left[(\partial_s^2 - \lambda_*) + \varepsilon \frac{z \cdot \partial_s \vec{\kappa}}{\tilde{J}_p}\partial_s(\partial_s^2 - \lambda_*)^{-1}(\partial_s^2 - \lambda_*) + \lambda_*\right]. \tag{6.57}$$

where $\tilde{J}_p$ is defined in (2.64) and $\frac{1}{\tilde{J}_p^2}$ have the expansion

$$\frac{1}{\tilde{J}_p^2} = 1 + \varepsilon\mathcal{R}, \quad \text{where } \mathcal{R} := \sum_{i=0}^{\infty}(\tilde{J}_p^2 - 1)^i - 1. \tag{6.58}$$

Without loss of generality, we assume that $\lambda^* = 0$. We note that every $v \in Y$ satisfies the following inequality

$$\left\| \partial_s^2 v \right\|_{L^2(\Omega)} \leq \varepsilon^{-2} \left\| v \right\|_{L^2(\Omega)}, \tag{6.59}$$

and since ∂_s is a relatively bounded perturbation of ∂_s^2, the operator $\partial_s(\partial_s^2 - \lambda_*)^{-1}$ is bounded, independent of ε, on Y.

Considering the case when $f \neq \psi_0$, then v takes the form,

$$v = f(z)\bar{\Theta}(s), \quad \bar{\Theta} := \sum_{j \in \Sigma} b_j \Theta_j, \quad \|v\|_{L^2(\Omega)} = 1 \tag{6.60}$$

Taking the $L^2(\Omega)$-norm of $\tilde{\Pi}\Delta_G$ acting on v yields

$$\left\| \tilde{\Pi} \partial_G^2 v \right\|_{L^2(\Omega)} = \left\| \tilde{\Pi} \frac{1}{\tilde{J}_p^2} \left[\partial_s^2 + \varepsilon \frac{z \cdot \partial_s \vec{\kappa}}{\tilde{J}_p} \partial_s (\partial_s^2)^{-1} \partial_s^2 \right] v \right\|_{L^2(\Omega)}, \tag{6.61}$$

$$\leq \left\| \tilde{\Pi} \frac{1}{\tilde{J}_p^2} \partial_s^2 v \right\|_{L^2(\Omega)} + \varepsilon \left\| \tilde{\Pi} \frac{1}{\tilde{J}_p^2} \frac{z \cdot \partial_s \vec{\kappa}}{\tilde{J}_p} \partial_s (\partial_s^2)^{-1} \partial_s^2 v \right\|_{L^2(\Omega)}. \tag{6.62}$$

Using the expansion of $\tilde{J}_p$, given in (6.58), we can bound the first term on the right-hand side of equation (6.61)

$$\left\| \tilde{\Pi} \frac{1}{\tilde{J}_p^2} \partial_s^2 v \right\|_{L^2(\Omega)} \leq \left\| \tilde{\Pi} \partial_s^2 v \right\|_{L^2(\Omega)} + \varepsilon \left\| \tilde{\Pi} \mathcal{R} \partial_s^2 v \right\|_{L^2(\Omega)} \leq \left\| \partial_s^2 v \right\|_{L^2(\Omega)} + \varepsilon \left\| \mathcal{R} \partial_s^2 v \right\|_{L^2(\Omega)}, \tag{6.63}$$

$$\leq \varepsilon^{-2} \left\| v \right\|_{L^2(\Omega)} + \varepsilon \left\| \mathcal{R} f(z) \right\|_{L^\infty(\Omega)} \left\| \partial_s^2 \bar{\Theta} \right\|_{L^2(\Omega)}, \leq \left(\varepsilon^{-2} + \varepsilon^{-1} \right) c \left\| v \right\|_{L^2(\Omega)}, \tag{6.64}$$

where c is independent of ε and the third inequality follows from the fact that $v = f(z)\bar{\Theta}(s)$, and f decays at $O(1)$ rate in z. As for the second term on the right-hand side of equation (6.61)

$$\varepsilon^2 \left\| \tilde{\Pi} \frac{1}{\tilde{J}_p^2} \frac{z \cdot \partial_s \vec{\kappa}}{\tilde{J}_p} \partial_s (\partial_s^2)^{-1} \partial_s^2 v \right\|_{L^2(\Omega)}^2 \leq \varepsilon^2 \left\| \frac{1}{\tilde{J}_p^2} \frac{z \cdot \partial_s \vec{\kappa}}{\tilde{J}_p} \partial_s (\partial_s^2)^{-1} \partial_s^2 v \right\|_{L^2(\Omega)}^2, \tag{6.65}$$

$$\leq \varepsilon^2 \left\| \frac{1}{\tilde{J}_p^3} \partial_s (\partial_s^2)^{-1} (\partial_s \vec{\kappa}) \cdot \partial_s^2 z v \right\|_{L^2(\Omega)}^2, \tag{6.66}$$

$$\leq \varepsilon^2 \left\| \frac{1}{\tilde{J}_p^3} \right\|_{L^\infty(\Omega)}^2 \left\| \partial_s (\partial_s^2)^{-1} \right\|_{l^2 \to l^2}^2 \left\| (\partial_s \vec{\kappa}) \cdot \partial_s^2 z v \right\|_{L^2(\Omega)}^2, \tag{6.67}$$

$$\leq \varepsilon^2 c_1 \left\| (\partial_s \vec{\kappa}) z f(z) \right\|_{L^\infty(\Omega)}^2 \left\| \partial_s^2 \bar{\Theta} \right\|_{L^2(\Omega)}^2 \leq \varepsilon^{-2} c_2 \left\| v \right\|_{L^2(\Omega)}^2, \tag{6.68}$$

where for the third inequality we used the fact that $f(z)$ decay at $O(1)$ in z. Plugging (6.70) and (6.68)

into (6.61) yields

$$\left\| \tilde{\Pi} \partial_G^2 v \right\|_{L^2(\Omega)} \leq \varepsilon^{-2} c_3 \| v \|_{L^2(\Omega)} . \tag{6.69}$$

For $v \in X_\Sigma$, the only difference in is equation (6.70) which the becomes

$$\left\| \tilde{\Pi} \frac{1}{\tilde{J}_p^2} \partial_s^2 v \right\|_{L^2(\Omega)} \leq \cancel{\left\| \tilde{\Pi} \partial_s^2 v \right\|_{L^2(\Omega)}} + \varepsilon \left\| \tilde{\Pi} \mathcal{R} \partial_s^2 v \right\|_{L^2(\Omega)} \leq \varepsilon \left\| \mathcal{R} \partial_s^2 v \right\|_{L^2(\Omega)} , \tag{6.70}$$

$$\leq \varepsilon \| \mathcal{R} f(z) \|_{L^\infty(\Omega)} \left\| \partial_s^2 \bar{\Theta} \right\|_{L^2(\Omega)} \leq \varepsilon^{-1} c_4 \| v \|_{L^2(\Omega)} , \tag{6.71}$$

and for this case we have

$$\left\| \tilde{\Pi} \partial_G^2 v \right\|_{L^2(\Omega)} \leq \varepsilon^{-1} c \| v \|_{L^2(\Omega)} . \tag{6.72}$$

■

Combining Proposition 6.3.1 with equation (6.52) we obtain the required bound

$$\left\| \tilde{\Pi} \mathcal{L}_p v \right\|_{L^2} \leq \varepsilon C \| v \|_{l^2} . \tag{6.73}$$

6.3.2 Bounding $\tilde{\Pi} \mathcal{L}_p^2 \Pi v$

Recall that $v \in X_\Sigma$, $v = \sum_{j \in \Sigma} b_j \psi_0 \Theta_j$ and $\|v\|_{L^2(\Omega)} = 1$, in particular,

$$\Pi v = v, \qquad \tilde{\Pi} v = 0. \tag{6.74}$$

We want to show there exist C_2, independent on ε, such that

$$\left\| \tilde{\Pi} \mathcal{L}_p^2 v \right\|_{L^2(\Omega)} \leq \varepsilon C_2 \| v \|_{L^2} . \tag{6.75}$$

Writing the $\mathcal{L}_p^2$ operator acting on v explicitly we have

$$\begin{aligned} \mathcal{L}_p^2 v =& \mathcal{L}_p(\mathcal{L}_p v) = (L - \varepsilon D_z + \varepsilon^2 \partial_G^2)(Lv - \varepsilon D_z v + \varepsilon^2 \partial_G^2 v), \\ =& (L_0 - \varepsilon D_z + \varepsilon^2 \partial_G^2) \sum_{j \in \Sigma} b_j \left(\lambda_{p,0} \psi_0 \Theta_j - \varepsilon D_z \psi_0 \Theta_j + \varepsilon^2 \partial_G \psi_0 \Theta_j \right), \\ =& \sum_{j \in \Sigma} b_j \Big[\overbrace{\lambda_{p,0}^2 \psi_0 \Theta_j}^{\in X_\Sigma} - \varepsilon L(D_z \psi_0 \Theta_j) + \varepsilon^2 L(\partial_G^2 \psi_0 \Theta_j) - \varepsilon \lambda_{p,0} D_z \psi_0 \Theta_j + \varepsilon^2 D_z (D_z \psi_0 \Theta_j) \end{aligned} \tag{6.76}$$

$$- \varepsilon^3 D_z(\partial_G^2\psi_0\Theta_j) + \varepsilon^2\lambda_{p,0}\partial_G^2\psi_0\Theta_j - \varepsilon^3\partial_G(D_z\psi_0\Theta_j) + \varepsilon^4\partial_G^2(\partial_G^2\psi_0\Theta_j)\Big].$$

Projecting away from X_Σ using $\tilde{\Pi}$ and taking the L^2-norm yields

$$\begin{aligned}\left\|\tilde{\Pi}\mathcal{L}_p^2 v\right\|_{L^2(\Omega)} = \|\tilde{\Pi}\sum_{j\in\Sigma} b_j\Big[\overbrace{\lambda_{p,0}^2\psi_0\Theta_j}^{\in X_\Sigma} - \varepsilon L(D_z\psi_0\Theta_j) + \varepsilon^2 L(\partial_G^2\psi_0\Theta_j) - \varepsilon\lambda_{p,0}D_z\psi_0\Theta_j + \varepsilon^2 D_z(D_z\psi_0\Theta_j) \quad (6.77)\\ - \varepsilon^3 D_z(\partial_G^2\psi_0\Theta_j) + \varepsilon^2\lambda_{p,0}\partial_G^2\psi_0\Theta_j - \varepsilon^3\partial_G(D_z\psi_0\Theta_j) + \varepsilon^4\partial_G^2(\partial_G^2\psi_0\Theta_j)\Big]\|_{L^2(\Omega)},\\ \leq \varepsilon\|\overbrace{\sum_{j\in\Sigma} b_j\Big[- L(D_z\psi_0\Theta_j) - \lambda_{p,0}D_z\psi_0\Theta_j + \varepsilon D_z(D_z\psi_0\Theta_j)\Big)}^{\mathcal{R}_1 v}\|_{L^2(\Omega)} + \varepsilon^6\left\|\tilde{\Pi}D_z(\partial_G^2 v)\right\|_{L^2(\Omega)} \quad (6.78)\\ + \varepsilon^6\left\|\tilde{\Pi}\partial_G^2 D_z v\right\|_{L^2(\Omega)} + \varepsilon^4\left\|\tilde{\Pi}\lambda_{p,0}\partial_G^2 v\right\|_{L^2(\Omega)} + \varepsilon^8\left\|\tilde{\Pi}\partial_G^2(\partial_G^2 v)\right\|_{L^2(\Omega)} + \varepsilon^4\left\|\tilde{\Pi}L(\partial_G^2 v)\right\|_{L^2(\Omega)},\end{aligned}$$

where we used the triangle inequality. The matrix representation $\bar{B}\in\mathbb{R}^{N\times N}$ of $\mathcal{R}_1$ is given by

$$\bar{B}_{j,k} := \langle(-L(D_z\psi_0) - \lambda_{p,0}D_z\psi_0 + \varepsilon D_z(D_z\psi_0))\Theta_j, -L(D_z\psi_0) - \lambda_{p,0}D_z\psi_0 + \varepsilon D_z(D_z\psi_0))\Theta_k\rangle_{L^2}. \quad (6.79)$$

The entries of $\bar{B}$ take the form

$$\bar{B}_{j,k} = \int_\Gamma\int_0^{l/\varepsilon} f(z,s)\Theta_j\Theta_k\, J\, dz\, ds, \quad (6.80)$$

and, for a smooth function f, Theorem 5.2 implies that there exists $c>0$, independent of ε, such that

$$\left\|\bar{B}\right\|_{l^2\to l^2}\leq c. \quad (6.81)$$

Using (6.81) we obtain a bound on the operator $\mathcal{R}_1$

$$\|\mathcal{R}_1 v\|_{L^2(\Omega)}^2 \leq \|b\|_{l^2}^2\left\|\bar{B}\right\|_{l^2\to l^2}^2 \leq c\|v\|_{l^2}^2, \quad (6.82)$$

and equation (6.78) reduces to

$$\begin{aligned}\left\|\tilde{\Pi}\mathcal{L}_p^2 v\right\|_{L^2(\Omega)} \leq & c\varepsilon\|v\|_{L^2(\Omega)} + \varepsilon^3\left\|\tilde{\Pi}D_z(\partial_G^2 v)\right\|_{L^2(\Omega)} + \varepsilon^3\left\|\tilde{\Pi}\partial_G^2 D_z v\right\|_{L^2(\Omega)} \quad (6.83)\\ & + \varepsilon^2\left\|\tilde{\Pi}\lambda_{p,0}\partial_G^2 v\right\|_{L^2(\Omega)} + \varepsilon^4\left\|\tilde{\Pi}\partial_G^2(\partial_G^2 v)\right\|_{L^2(\Omega)} + \varepsilon^2\left\|\tilde{\Pi}L(\partial_G^2 v)\right\|_{L^2(\Omega)}.\end{aligned}$$

Consider the second term in the right-hand side of (6.83).

$$\varepsilon^3\left\|\tilde{\Pi}D_z(\partial_G^2 v)\right\|_{L^2(\Omega)} \leq \varepsilon^3\left\|\frac{\kappa}{\tilde{J}_p}\cdot\nabla_z\left(\partial_G^2 v\right)\right\|_{L^2(\Omega)}, \quad (6.84)$$

$$\leq \varepsilon^3 \left\| \frac{\kappa}{\tilde{J}_p} \cdot \left[\nabla_z \partial_G^2\right] v \right\|_{L^2(\Omega)} + \varepsilon^3 \left\| \frac{\kappa}{\tilde{J}_p} \cdot \partial_G^2 \left(\nabla_z v\right) \right\|_{L^2(\Omega)}, \tag{6.85}$$

$$\leq \varepsilon^3 \left\| \frac{\kappa}{\tilde{J}_p} \right\|_{L^\infty(\Omega)} \left\| \left[\nabla_z \partial_G^2\right] v \right\|_{L^2(\Omega)} + \varepsilon^3 \left\| \frac{\kappa}{\tilde{J}_p} \right\|_{L^\infty(\Omega)} \left\| \partial_G^2 \left(\nabla_z v\right) \right\|_{L^2(\Omega)}, \tag{6.86}$$

$$\leq \varepsilon^3 c_1 \left\| \left[\nabla_z \partial_G^2\right] v \right\|_{L^2(\Omega)} + c_2 \varepsilon \left\| v \right\|_{L^2(\Omega)}, \tag{6.87}$$

where the last inequality follows applying Proposition 6.3.1. In order to show that the first term in (6.87) is bounded, we note that

$$[\nabla_z \partial_G^2] = \left(\nabla_z \frac{1}{\tilde{J}_p^2}\right) \partial_s^2 + \varepsilon \left(\nabla_z \frac{z \cdot \partial_s \vec{\kappa}}{\tilde{J}_p^3}\right) \partial_s, \tag{6.88}$$

and we consider the following Proposition.

Proposition 6.3.2. *Let $f(z)$ be a smooth function such that*

$$|f(z)| < c_1 e^{-c_2|z|} \text{ for some } c_i \in \mathbb{R},\ c_i > 0\ , i = 1, 2, \text{ and } supp(f) \subset \Gamma_l. \tag{6.89}$$

The operator $[\nabla_z \partial_G^2]$, where ∂_G^2 is defined in (2.67), is bounded on the space

$$Y = span\{ f(z)\Theta_k \ \Big|\ k \in \Sigma, \}, \tag{6.90}$$

i.e., there exists $C > 0$, C independent of ε and f, such that

$$\left\| [\nabla_z \partial_G^2] v \right\|_{L^2(\Omega)} \leq C\varepsilon^{-1} \left\| v \right\|_{L^2(\Omega)}, \tag{6.91}$$

for every $v \in Y$.

Proof. Fix $\lambda_* \in \rho(\partial_s^2)$, where $\rho(\partial_s^2)$ is the resolvent set of the co-dimeanion two Laplacian operator, and, without loss of generality, assume that $\lambda^* = 0$. The operator ∂_G^2 can be written as

$$\partial_G^2 = \frac{1}{\tilde{J}_p^2} \partial_s^2 + \varepsilon \frac{z \cdot \partial_s \vec{\kappa}}{\tilde{J}_p^3} \partial_s = \frac{1}{\tilde{J}_p^2} \partial_s^2 + \varepsilon \frac{z \cdot \partial_s \vec{\kappa}}{\tilde{J}_p^3} \partial_s (\partial_s^2)^{-1} (\partial_s^2). \tag{6.92}$$

Let $v \in Y$ such that $f \neq \psi_0$, then v takes the form,

$$v = f(z)\bar{\Theta}(s), \quad \bar{\Theta} := \sum_{j \in \Sigma} b_j \Theta_j, \tag{6.93}$$

and, without loss of generality, $\|v\|_{L^2(\Omega)} = 1$.

Taking the $L^2(\Omega)$-norm of $\partial_z \Delta_G$ acting on v yields

$$\left\| (\nabla_z \partial_G^2) v \right\|_{L^2(\Omega)} = \left\| (\nabla_z \frac{1}{\tilde{J}_p^2}) \partial_s^2 v + \varepsilon \nabla_z (\frac{z \cdot \partial_s \vec{\kappa}}{\tilde{J}_p^3}) \partial_s (\partial_s^2)^{-1} (\partial_s^2) v \right\|_{L^2(\Omega)} \tag{6.94}$$

$$\leq \left\| (\nabla_z \frac{1}{\tilde{J}_p^2}) f(z) \right\|_{L^\infty(\Omega)} \left\| \partial_s^2 \bar{\Theta} \right\|_{L^2(\Omega)} \tag{6.95}$$

$$+ \varepsilon \left\| \nabla_z (\frac{z \cdot \partial_s \vec{\kappa}}{\tilde{J}_p^3}) f(z) \right\|_{L^\infty(\Omega)} \left\| \partial_s (\partial_s^2)^{-1} \right\|_{l^2 \to l^2} \left\| (\partial_s^2) \bar{\Theta} \right\|_{L^2(\Omega)}$$

$$\leq \varepsilon^{-2} c_1 \left\| (\nabla_z \frac{1}{\tilde{J}_p^2}) f(z) \right\|_{L^\infty(\Omega)} \left\| \bar{\Theta} \right\|_{L^2(\Omega)} + \varepsilon^{-1} c_2 \left\| v \right\|_{L^2(\Omega)}, \tag{6.96}$$

where the first inequality follows from the triangle inequality combined with the generalized Hölder inequality. Note that

$$\nabla_z \frac{1}{\tilde{J}_p^2} = \varepsilon \frac{(1,1) \cdot \vec{\kappa}}{2 \tilde{J}_p^3}, \tag{6.97}$$

which implies that there exists c independent of ε such that

$$\left\| (\nabla_z \frac{1}{\tilde{J}_p^2}) f(z) \right\|_{L^\infty(\Omega)} \leq \varepsilon c_3. \tag{6.98}$$

Plugging (6.98) into (6.96) yields

$$\left\| (\nabla_z \partial_G^2) v \right\|_{L^2(\Omega)} \leq \varepsilon^{-1} c_4 \left\| v \right\|_{L^2(\Omega)}, \quad \text{for } v \in X_\Sigma, \tag{6.99}$$

and we conclude that

$$\left\| \nabla_z \partial_G^2 \right\|_{L^2(\Omega)} \leq C \varepsilon^{-1} \text{ on } Y. \tag{6.100}$$

■

Returning to (6.87), and using Proposition 6.3.2, we have

$$\varepsilon^3 \left\| \tilde{\Pi} D_z (\partial_G^2 v) \right\|_{L^2(\Omega)} \leq \varepsilon c_1 \left\| v \right\|_{L^2(\Omega)}. \tag{6.101}$$

Plugging (6.101) and into the right-hand side of (6.83) yields

$$\left\| \tilde{\Pi} \mathcal{L}_p^2 v \right\|_{L^2(\Omega)} \leq c\varepsilon \left\| v \right\|_{L^2(\Omega)} + \varepsilon^3 \left\| \tilde{\Pi} \partial_G^2 D_z v \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi} \lambda_{p,0} \partial_G^2 v \right\|_{L^2(\Omega)} \tag{6.102}$$

$$+ \varepsilon^4 \left\| \tilde{\Pi} \partial_G^2 (\partial_G^2 v) \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi} L (\partial_G^2 v) \right\|_{L^2(\Omega)}.$$

The second term in (6.102) involves the operator $\partial_G^2 D_z$. Using the definitions of ∂_G^2 and D_z, given in (2.67) and (2.66), respectively, we can write

$$\partial_G^2 D_z v = \partial_G^2 \left(\frac{1}{\tilde{J}_p} \nabla_z v \right) = \frac{\vec{\kappa}}{\tilde{J}_p} \cdot \left(\partial_G^2 (\nabla_z v) \right) + \left(\partial_G^2 \frac{\vec{\kappa}}{\tilde{J}_p} \right) \cdot \nabla_z v + 2 \partial_s \left(\frac{\vec{\kappa}}{\tilde{J}_p} \right) \cdot \partial_s \nabla_z v. \tag{6.103}$$

Taking the $L^2(\Omega)$-norm yields

$$\begin{aligned}
\left\| \tilde{\Pi} \partial_G^2 D_z v \right\|_{L^2(\Omega)} &= \left\| \frac{\vec{\kappa}}{\tilde{J}_p} \left(\partial_G^2 (\nabla_z v) \right) + \left(\partial_G^2 \frac{\vec{\kappa}}{\tilde{J}_p} \right) \nabla_z v + 2 \partial_s \left(\frac{\vec{\kappa}}{\tilde{J}_p} \right) \partial_s \nabla_z v \right\|_{L^2(\Omega)}, && (6.104) \\
&\leq \left\| \frac{\vec{\kappa}}{\tilde{J}_p} \left(\partial_G^2 (\nabla_z v) \right) \right\|_{L^2(\Omega)} + \left\| \left(\partial_G^2 \frac{\vec{\kappa}}{\tilde{J}_p} \right) \nabla_z v \right\|_{L^2(\Omega)} + 2 \left\| \partial_s \left(\frac{\vec{\kappa}}{\tilde{J}_p} \right) \partial_s \nabla_z v \right\|_{L^2(\Omega)}, && (6.105) \\
&\leq \left\| \frac{\vec{\kappa}}{\tilde{J}_p} \right\|_{L^\infty} \left\| \left(\partial_G^2 (\nabla_z v) \right) \right\|_{L^2(\Omega)} + \left\| \left(\partial_G^2 \frac{\vec{\kappa}}{\tilde{J}_p} \right) \nabla_z (f(z)) \right\|_{L^\infty(\Omega)} \left\| \bar{\Theta} \right\|_{L^2(\Gamma)} && (6.106) \\
&\quad + 2 \left\| \partial_s \left(\frac{\vec{\kappa}}{\tilde{J}_p} \right) \right\|_{L^\infty(\Omega)} \left\| \partial_s \nabla_z v \right\|_{L^2(\Omega)}, \\
&\leq \varepsilon^{-2} c_1 \left\| v \right\|_{L^2(\Omega)} + c_2 \left\| v \right\|_{L^2(\Omega)} && (6.107) \\
&\quad + 2 \left\| \partial_s \left(\frac{\vec{\kappa}}{\tilde{J}_p} \right) \right\|_{L^\infty(\Omega)} \left\| \partial_s (\partial_s^{-2}) \right\|_{l^2 \to l^2} \left\| \partial_s^2 \nabla_z v \right\|_{L^2(\Omega)}, \\
&\leq \varepsilon^{-2} C \left\| v \right\|_{L^2(\Omega)}, && (6.108)
\end{aligned}$$

where the first inequality is the triangle inequality, for the second inequality we use Hölder and the third inequality follows from Lemma 6.3.1, combined with the assumption that $\kappa \in W^{2,\infty}$.Plugging (6.108) into (6.102) yields

$$\left\| \tilde{\Pi} \mathcal{L}_p^2 v \right\|_{L^2(\Omega)} \leq c\varepsilon \left\| v \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi} \lambda_{p,0} \partial_G^2 v \right\|_{L^2(\Omega)} + \varepsilon^4 \left\| \tilde{\Pi} \partial_G^2 (\partial_G^2 v) \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi} L (\partial_G^2 v) \right\|_{L^2(\Omega)} \tag{6.109}$$

Proposition 6.3.1 shows that the second term in (6.109) is bounded as an operator in the $L^2(\Omega)$ norm with

$$\varepsilon^2 \left\| \tilde{\Pi} \lambda_{p,0} \partial_G^2 v \right\|_{L^2(\Omega)} \leq c_2 \varepsilon \left\| v \right\|_{L^2(\Omega)}, \tag{6.110}$$

and (6.109) reduces to

$$\left\| \tilde{\Pi} \mathcal{L}_p^2 v \right\|_{L^2(\Omega)} \leq c\varepsilon \left\| v \right\|_{L^2(\Omega)} + \varepsilon^4 \left\| \tilde{\Pi} \partial_G^2 (\partial_G^2 v) \right\|_{L^2(\Omega)} + \varepsilon^2 \left\| \tilde{\Pi} L (\partial_G^2 v) \right\|_{L^2(\Omega)}. \tag{6.111}$$

The bound on the second term in the right-hand side of (6.111) follows from the following lemma

Lemma 6.1. *Let $v \in X_\Sigma$,*

$$v = \sum_{j\in\Sigma} b_j \Theta_j \psi_0, \tag{6.112}$$

with $\|v\|_{L^2(\Omega)} = 1$. Then there exists $C > 0$ such that

$$\left\| \tilde{\Pi} \partial_G^2 (\partial_G^2 v) \right\|_{L^2(\Omega)} \le C\varepsilon^{-3} \|v\|_{L^2(\Omega)}. \tag{6.113}$$

Proof. There exists c, independent of ε, such that

$$\left\| \partial_s^4 v \right\|_{L^2(\Omega)} \le c\varepsilon^{-4} \|v\|_{L^2(\Omega)}. \tag{6.114}$$

Fix $\lambda_* \in \rho(\partial_s^2)$, where $\rho(\partial_s^2)$ is the resolvent set of the second order operator. without loss of generality, assume $\lambda^* = 0$. Then, first we write the operator $\partial_G^2(\partial_G^2 v)$ explicitly -

$$\begin{aligned}
\partial_G^2(\partial_G^2 v) = {} & \frac{1}{\tilde{J}_p^2}\left[(\partial_s^2 \frac{1}{\tilde{J}_p^2})\partial_s^2 + 2(\partial_s \frac{1}{\tilde{J}_p^2})\partial_s^3 + \frac{1}{\tilde{J}_p^2}\partial_s^4 + \varepsilon\left(\partial_s^2 \frac{\kappa\cdot z}{\tilde{J}_p^3}\partial_s + \frac{\kappa\cdot z}{\tilde{J}_p^3}\partial_s^3 + (\partial_s \frac{\kappa\cdot z}{\tilde{J}_p^3})\partial_s^2 \right)\right] v \qquad (6.115)\\
& + \varepsilon \frac{\vec{\kappa}\cdot z}{\tilde{J}_p^3}\left(\left(\partial_s \frac{1}{\tilde{J}_p^2}\right)\partial_s^2 + \varepsilon \frac{1}{\tilde{J}_p^2}\partial_s^3 \right) v + \varepsilon^2 \left((\frac{\kappa\cdot z}{\tilde{J}_p^3})^2 \partial_s^2 + \frac{\kappa\cdot z}{\tilde{J}_p^3}(\partial_s \frac{\kappa\cdot z}{\tilde{J}_p^3})\partial_s \right) v,\\
= {} & \frac{1}{\tilde{J}_p^4}\partial_s^4 + \left(2\frac{1}{\tilde{J}_p^2}(\partial_s \frac{1}{\tilde{J}_p^2}) + \varepsilon \frac{1}{\tilde{J}_p^2}\frac{\vec{\kappa}\cdot z}{\tilde{J}_p^3} + \varepsilon^2 \frac{\vec{\kappa}\cdot z}{\tilde{J}_p^3}\frac{1}{\tilde{J}_p^2}\right)\partial_s^3 \qquad (6.116)\\
& + \left(\frac{1}{\tilde{J}_p^2}(\partial_s^2 \frac{1}{\tilde{J}_p^2}) + \varepsilon \frac{1}{\tilde{J}_p^2}(\partial_s \frac{\vec{\kappa}\cdot z}{\tilde{J}_p^3}) + \varepsilon \frac{\vec{\kappa}\cdot z}{\tilde{J}_p^3}(\partial_s \frac{1}{\tilde{J}_p^2}) + \varepsilon^2 (\frac{\vec{\kappa}\cdot z}{\tilde{J}_p^3})^2 \right)\partial_s^2\\
& + \left(\varepsilon \frac{1}{\tilde{J}_p^2}(\partial_s^2 \frac{\vec{\kappa}\cdot z}{\tilde{J}_p^3}) + \varepsilon^2 \frac{\vec{\kappa}\cdot z}{\tilde{J}_p^3}(\partial_s \frac{\vec{\kappa}\cdot z}{\tilde{J}_p^3})\right)\partial_s.
\end{aligned}$$

To bound the first term on the right-hand side of equation (6.116), note that the Taylor expansion $\frac{1}{\tilde{J}_p^4}$ can be written as

$$\frac{1}{\tilde{J}_p^4} = 1 + \varepsilon\mathcal{R}, \quad \mathcal{R} = \mathcal{R}(z,\vec{\kappa}) = \sum_{i=0}^{\infty} (\tilde{J}_p^4 - 1)^i - 1. \tag{6.117}$$

and we obtain the bound

$$\left\| \tilde{\Pi} \frac{1}{\tilde{J}_p^4} \partial_s^4 v \right\|_{L^2(\Omega)} \le \cancelto{0}{\left\| \tilde{\Pi}\partial_s^4 v \right\|_{L^2(\Omega)}} + \varepsilon \|\mathcal{R} f(z)\|_{L^\infty(\Omega)} \left\| \partial_s^4 \bar{\Theta} \right\|_{L^2(\Omega)} \le c_1 \varepsilon^{-3} \|v\|_{L^2(\Omega)}. \tag{6.118}$$

Similarly, as long as $\vec{\kappa} \in W^{1,\infty}$, then we have

$$\left\| \tilde{\Pi} \left(2\frac{1}{\tilde{J}_p^2}(\partial_s \frac{1}{\tilde{J}_p^2}) + \varepsilon \frac{1}{\tilde{J}_p^2}\frac{\vec{\kappa}\cdot z}{\tilde{J}_p^3} + \varepsilon^2 \frac{\vec{\kappa}\cdot z}{\tilde{J}_p^3}\frac{1}{\tilde{J}_p^2}\right)\partial_s^3 v \right\|_{L^2(\Omega)} \le c_1 \varepsilon^{-3} \|v\|_{L^2(\Omega)}. \tag{6.119}$$

To bound the second operator on the right-hand side of equation (6.116) we require that $\vec{\kappa} \in W^{2,\infty}$, then

$$\left\| \tilde{\Pi} \left(\frac{1}{\tilde{J}_p^2} (\partial_s^2 \frac{1}{\tilde{J}_p^2}) + \varepsilon \frac{1}{\tilde{J}_p^2} (\partial_s \frac{\vec{\kappa} \cdot z}{\tilde{J}_p^3}) + \varepsilon \frac{\vec{\kappa} \cdot z}{\tilde{J}_p^3} (\partial_s \frac{1}{\tilde{J}_p^2}) + \varepsilon^2 (\frac{\vec{\kappa} \cdot z}{\tilde{J}_p^3})^2 \right) \partial_s^2 v \right\|_{L^2(\Omega)} \leq c_3 \varepsilon^{-1} \| v \|_{L^2(\Omega)}, \tag{6.120}$$

and similarly,

$$\left\| \left(\varepsilon \frac{1}{\tilde{J}_p^2} (\partial_s^2 \frac{\vec{\kappa} \cdot z}{\tilde{J}_p^3}) + \varepsilon^2 \frac{\vec{\kappa} \cdot z}{\tilde{J}_p^3} (\partial_s \frac{\vec{\kappa} \cdot z}{\tilde{J}_p^3}) \right) \partial_s v \right\|_{L^2(\Omega)} \leq c_4 \varepsilon^{-1} \| v \|_{L^2(\Omega)}. \tag{6.121}$$

Plugging (6.118), (6.119), (6.120) and (6.121) back into (6.116) we obtain the bound

$$\| \tilde{\Pi} \partial_G^2 (\partial_G^2 v) \|_{L^2(\Omega)} \leq C \varepsilon^{-3} \| v \|_{L^2(\Omega)}. \tag{6.122}$$

■

Combining Lemma 6.1 and equation (6.111) yields

$$\| \tilde{\Pi} \mathcal{L}_p^2 v \|_{L^2(\Omega)} \leq \varepsilon \bar{c} \| v \|_{L^2(\Omega)} + \varepsilon^2 \| \tilde{\Pi} L (\Delta_G v) \|_{L^2(\Omega)}. \tag{6.123}$$

Consider the second term in (6.123), and note that $W''(U_p)$ commutes with ∂_G^2. Then

$$\begin{aligned}
\varepsilon^2 \| \tilde{\Pi} L (\partial_G^2 v) \|_{L^2(\Omega)} &= \varepsilon^2 \| \tilde{\Pi} (\Delta_z - W''(U_p)) (\partial_G^2 v) \|_{L^2(\Omega)}, && (6.124) \\
&\leq \varepsilon^2 \| \tilde{\Pi} [\Delta_z \partial_G^2] v \|_{L^2(\Omega)} + \varepsilon^2 \| 2 \tilde{\Pi} [\nabla_z \partial_G^2] (\nabla_z v) \|_{L^2(\Omega)} + \varepsilon^2 \| \tilde{\Pi} \partial_G^2 L v \|_{L^2(\Omega)}, && (6.125) \\
&\leq \varepsilon^2 \| \tilde{\Pi} [\Delta_z \partial_G^2] v \|_{L^2(\Omega)} + \varepsilon^4 \| 2 \tilde{\Pi} [\nabla_z \partial_G^2] (\nabla_z v) \|_{L^2(\Omega)} + \varepsilon^2 \| \lambda_{p,0} \tilde{\Pi} \partial_G^2 v \|_{L^2(\Omega)}, && (6.126) \\
&\leq \varepsilon^2 \| \tilde{\Pi} [\Delta_z \partial_G^2] v \|_{L^2(\Omega)} + c_2 \varepsilon^3 \| v \|_{L^2(\Omega)} + c_3 \varepsilon \| v \|_{L^2(\Omega)}, && (6.127)
\end{aligned}$$

where the last inequality follows from Propositions 6.3.1 and 6.3.2. The first term in (6.127) has the following bound

$$\begin{aligned}
\varepsilon^2 \| \tilde{\Pi} [\Delta_z \partial_G^2] v \|_{L^2(\Omega)} &\leq \varepsilon^2 \left\| \varepsilon \left[\frac{((1,1) \cdot \vec{\kappa})^2}{6 \tilde{J}_p^4} \partial_s^2 + \left(\frac{((1,1) \cdot \vec{\kappa})^2}{3 \tilde{J}_p^4} + \varepsilon \frac{((1,1) \cdot \vec{\kappa})^2 z \cdot \vec{\kappa}}{12 \tilde{J}_p^5} \right) \partial_s \right] v \right\|_{L^2(\Omega)}, && (6.128) \\
&\leq c_1 \varepsilon \| v \|_{L^2(\Omega)}, && (6.129)
\end{aligned}$$

for some c_1 independent of ε (The explicit calculation is similar to that of Propositions 6.3.1 and 6.3.2). Plugging (6.128) into (6.127) yields

$$\varepsilon^2 \| \tilde{\Pi} L (\partial_G^2 v) \|_{L^2(\Omega)} \leq c_5 \varepsilon \| v \|_{L^2(\Omega)}. \tag{6.130}$$

Plugging equation (6.130) into (6.123) we obtain the required bound

$$\left\| \tilde{\Pi}\mathcal{L}_p^2 v \right\|_{L^2(\Omega)} \leq \varepsilon C \left\| v \right\|_{L^2(\Omega)}, \quad \text{for } v \in X_\Sigma \tag{6.131}$$

where C is independent of ε, but it depend upon $\left\| \kappa \right\|_{L^\infty(\Omega)}$, which is uniformly bounded for Γ_p admissible.

6.3.3 Bounding $\tilde{\Pi}\mathbb{L}_p\Pi$

Finally, the following Proposition shows that the operator $\tilde{\Pi}\mathbb{L}_p\Pi$ is bounded.

Proposition 6.3.3. *The operator* $\tilde{\Pi}\mathbb{L}_p\Pi : l^2(\mathbb{R}^{N\times N}) \mapsto l^2(\mathbb{R}^{N\times N})$ *has an* $O(\varepsilon)$ *operator norm.*

Proof. Let $v \in X_\Sigma,\ v = \sum_{j\in\Sigma} b_j \psi_0 \Theta_j$ with $\|v\|_{L^2(\Omega)} = 1$. In particular,

$$\Pi v = v, \qquad \tilde{\Pi} v = 0, \tag{6.132}$$

where Π is the projection onto X_Σ and $\tilde{\Pi}$ is the complementary projection.
Fix $\lambda_* \in \rho(\mathcal{L}_p^2)$, where $\rho(\mathcal{L}_p^2)$ is the resolvent set of $\mathcal{L}_p^2$, and rewrite $\mathbb{L}_p$ in the following way

$$\mathbb{L}_p = \mathcal{L}_p^2 + \varepsilon\tilde{\mathbb{L}}_p = (\mathcal{L}_p^2 - \lambda_*) + \varepsilon\tilde{\mathbb{L}}_p(\mathcal{L}_p^2 - \lambda_*)^{-1}(\mathcal{L}_p^2 - \lambda_*) + \lambda_*. \tag{6.133}$$

Taking the L^2-norm of $\tilde{\Pi}\mathbb{L}\Pi$ acting on v yields

$$\begin{aligned}
\left\| \tilde{\Pi}\mathbb{L}_p\Pi v \right\|_{L^2(\Omega)} &= \left\| \tilde{\Pi}(\mathcal{L}_p^2 - \lambda_*)v + \varepsilon\tilde{\Pi}\tilde{\mathbb{L}}(\mathcal{L}_p^2 - \lambda_*)^{-1}(\mathcal{L}_p^2 - \lambda_*)v \right\|_{L^2(\Omega)}, && (6.134)\\
&\leq \left\| \tilde{\Pi}\mathcal{L}_p^2 v \right\|_{L^2(\Omega)} + \varepsilon \left\| \tilde{\Pi}\tilde{\mathbb{L}}(\mathcal{L}_p^2 - \lambda_*)^{-1}(\Pi + \tilde{\Pi})(\mathcal{L}_p^2 - \lambda_*)v \right\|_{L^2(\Omega)}, && (6.135)\\
&\leq \left\| \tilde{\Pi}\mathcal{L}_p^2 v \right\|_{L^2(\Omega)} && (6.136)\\
&\quad + \varepsilon \left(\left\| \tilde{\mathbb{L}}(\mathcal{L}_p^2 - \lambda_*)^{-1}\Pi(\mathcal{L}_p^2 - \lambda_*)v \right\|_{L^2(\Omega)} + \left\| \tilde{\mathbb{L}}(\mathcal{L}_p^2 - \lambda_*)^{-1}\tilde{\Pi}(\mathcal{L}_p^2 - \lambda_*)v \right\|_{L^2(\Omega)} \right),\\
&\leq \left\| \tilde{\Pi}\mathcal{L}_p^2 v \right\|_{L^2(\Omega)} && (6.137)\\
&\quad + \varepsilon \left(\left\| \tilde{\mathbb{L}}(\mathcal{L}_p^2 - \lambda_*)^{-1} \right\|_{L^2(\Omega)} \left\| \Pi\mathcal{L}_p^2 v \right\|_{L^2(\Omega)} + \left\| \tilde{\mathbb{L}}(\mathcal{L}_p^2 - \lambda_*)^{-1} \right\|_{L^2(\Omega)} \left\| \tilde{\Pi}\mathcal{L}_p^2 v \right\|_{L^2(\Omega)} \right).
\end{aligned}$$

Since $\tilde{\mathbb{L}}$ is relatively bounded with respect to $\mathcal{L}_p^2$, the operator $\tilde{\mathbb{L}}(\mathcal{L}_p^2 - \lambda_*)^{-1}$ has an $O(\varepsilon)$ bound as an operator from $l^2(\mathbb{R}^{N\times N})$ to $l^2(\mathbb{R}^{N\times N})$. In section 6.3.2 we have shown that

$$\left\| \tilde{\Pi}\mathcal{L}_p^2\Pi \right\|_{L^2} \leq c\varepsilon. \tag{6.138}$$

Therefore, combining bound (6.138) with the boundedness of $\left\| \tilde{\mathbb{L}}(\mathcal{L}_p^2 - \lambda_*)^{-1} \right\|_{L^2}$, an inspection of equa-

tion (6.137) yields

$$\left\| \tilde{\Pi}\mathbb{L}_p\Pi v \right\|_{L^2} \le \tilde{c}\varepsilon \left((1+\varepsilon) \| v \|_{L^2} + \left\| \Pi\mathcal{L}_p^2 v \right\|_{L^2} \right) \tag{6.139}$$

To complete the bound on $\left\| \tilde{\Pi}\mathbb{L}_p\Pi \right\|_{L^2}$ we need show that $\left\| \Pi\mathcal{L}_p^2\Pi \right\|_{L^2}$ is bounded. Using the definition of Π, (6.14), we can rewrite the operator $\Pi\mathcal{L}_p^2\Pi v$ as

$$\Pi\mathcal{L}_p^2\Pi \sum_{j\in\Sigma} b_j \psi_0\Theta_k = \sum_{k\in\Sigma}\sum_{j\in\Sigma} b_j (\mathcal{L}_p^2\psi_0\Theta_j, \psi_0\Theta_k)_{L^2}\psi_0\Theta_k, \tag{6.140}$$

and define its matrix representation $\bar{M} \in \mathbb{R}^{N\times N}$, where $N \sim O(\varepsilon^{-3/2})$, with entries

$$\bar{M}_{j,k} \coloneqq \left(\mathcal{L}_p^2\psi_0\Theta_j, \psi_0\Theta_k \right)_{L^2}. \tag{6.141}$$

Calculating $\mathcal{L}_p^2 v$ explicitly, we can write $\bar{M} = \bar{M}_{\text{diag}} + \bar{M}_{\text{off-diag}}$ where

$$\bar{M}_{\text{diag}} = \begin{cases} (\lambda_{p,0} - \varepsilon^2\beta_k)^2 + O(\varepsilon^2) & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases} \tag{6.142}$$

$$\bar{M}_{\text{off-diag}} = \begin{cases} 0 & \text{if } k = j, \\ \varepsilon^2\sqrt{\varepsilon}(P_k + P_j) \int_\Gamma \int_0^{l/\varepsilon} (z\cdot\vec{\kappa})\vec{\kappa}\cdot(\nabla\psi_0)\psi_0\Theta_k\Theta_j \, dz\, ds + O(\varepsilon^3) & \text{if } k \neq j, \end{cases} \tag{6.143}$$

see appendix (E.3) for detailed calculations. By Theorem 5.2 we know that $\bar{M}_{\text{off-diag}}$ has an $O(\varepsilon)$ bound as an operator from $l^2(\mathbb{R}^{N\times N})$ to $l^2(\mathbb{R}^{N\times N})$. Moreover, we consider $k \in \Sigma$ for which $(\lambda_{p,0} - \varepsilon^2\beta_k)^2 \sim O(\varepsilon)$. Hence, $\bar{M}$ and $\bar{M}_{\text{diag}}$ has a similar bound and

$$\left\| \Pi\mathcal{L}_p^2 v \right\|_{L^2} \le c\varepsilon \| v \|_{L^2}. \tag{6.144}$$

Inserting this bound back to (6.139) yields

$$\left\| \tilde{\Pi}\mathbb{L}\Pi v \right\|_{L^2} \le \varepsilon c \| v \|_{L^2}, \tag{6.145}$$

which implies that $\tilde{\Pi}\mathbb{L}_p\Pi$ has an $O(\varepsilon)$ bound as an operator $l^2(\mathbb{R}^{N\times N}) \to l^2(\mathbb{R}^{N\times N})$. ■

Recall the 2×2 block form of $\mathbb{L}$, (6.15). In this section we have shown that the off diagonal blocks has $O(\varepsilon)$ bound as operators from $l^2(\mathbb{R}^{N\times N}) \to l^2(\mathbb{R}^{N\times N})$. The next step in the analysis will be to show that, at

leading order, the spectrum of $\mathbb{L}$ is determined by the spectrum of M. However, we already proved this for the Bilayer, and since the analysis does not depend upon the interface co-dimension, the same result holds here. We conclude that the pearling stability condition of $\mathbb{L}$, for the co-dimension 2 structure is, at leading order, the pearling condition of M, given in Corollary 6.2.2. That is, we have shown that for a given admissible interface, Γ_p, the associated pore solution constructed in (2.75), is stable with respect to the pearling bifurcation if and only if the far-field chemical potential μ_1 satisfies the *pearling stability condition* stated in Theorem 6.0.2.

Chapter 7

Analysis of Network Bifurcations

7.1 Introduction

In the chapters 3 and 4 we develop asymptotic expressions for the geometric evolution of admissible bilayer and pore morphologies. These are quenched curvature driven flows, which yield a stable motion by mean curvature for values of the spatially constant chemical potential, μ, that are less than μ_b^* for bilayers and less than μ_p^* for pores. If the chemical potential exceeds either of these critical values, then the evolution becomes motion against mean curvature, which is unstable to fingering growths. In section 4.7, the combined evolution of well separated bilayers and pores is given by equations (4.166-4.169), which couple the evolution of the two morphologies through the spatially constant value of the far-field chemical potential. The stability of bilayers and pores to the pearling bifurcation is characterized in chapters 5 and 6, respectively. Again the stability condition can be expressed in terms of the magnitude of the far field chemical potential with respect to critical values that depend only upon the functionalization parameters, η_2, η_2 and the potential well W. As the stability of the underlying pore and bilayer morphologies is independent of their shape, for a fixed potential well W, we may analyze the stability regions of bilayers and pores with respect to μ_1, η_1, and η_2. For simplicity we fix $\eta_1 = 1$ and present the stability regions in terms of μ_1 and $\eta_d := \eta_1 - \eta_2$. Under the H^{-1} gradient flow the chemical potential μ_1 is dynamic on the $\tau = O(\varepsilon^{-1})$ time-scale. This is the same time scale as the geometric flow, and hence of the fingering instability. However the time scale of the pearling instability is governed by the pearling eigenvalues of $\Delta\mathbb{L}$ which are two orders of ε larger than the pearling eigenvalues of $\mathbb{L}$, that is they scale with $O(\varepsilon^{-1})$ and the time-scale of the of the onset of the pearling instability is $t = O(\varepsilon)$. Thus the pearling instability manifests itself on a time scale that is instantaneous with respect to the underlying geometric evolution.

7.2 Meandering Equilibria

We investigated the geometric evolution of co-dimensions one and two morphologies in Chapters 3 and 4. It was shown that, in the combined system, so long as the underlying network morphologies remain admissible and have non-zero curvature, then the leading order chemical potential μ_1 will decay exponentially to a constant, see equation (4.165). The explicit expression for the equilibria points of the chemical potential for a system with bilayers, μ_b^*, and for a system with pores, μ_p^*, takes the form

$$(3.116) \ : \ \mu_b^*(\eta_d) := -\frac{1}{2}\left(2\eta_1 - \eta_d\right)\frac{\left\|\hat{U}_b'\right\|_{L^2}^2}{\int_{\mathbb{R}} \hat{U}_b \, dz}, \tag{7.1}$$

$$(4.149) \ : \ \mu_p^*(\eta_d) := \eta_1 \frac{\int_0^\infty (\hat{U}_p')^2 R \, dR}{2\int_0^\infty \hat{U}_p \, R dR}. \tag{7.2}$$

According to the results in Section 4.7, for non-flat interfaces, the range $\mu_1 \in [\mu_p^*, \mu_b^*]$ is invariant under the flow, and once μ_1 enters this range one structure will shrink, while the other morphology will grow.
Once a double-well potential W_ξ has been chosen, see equation (7.10), we can calculate the bilayer and pore profiles, U_b and U_p from equations 2.37 and 2.75 respectively, see Figure 7.1.

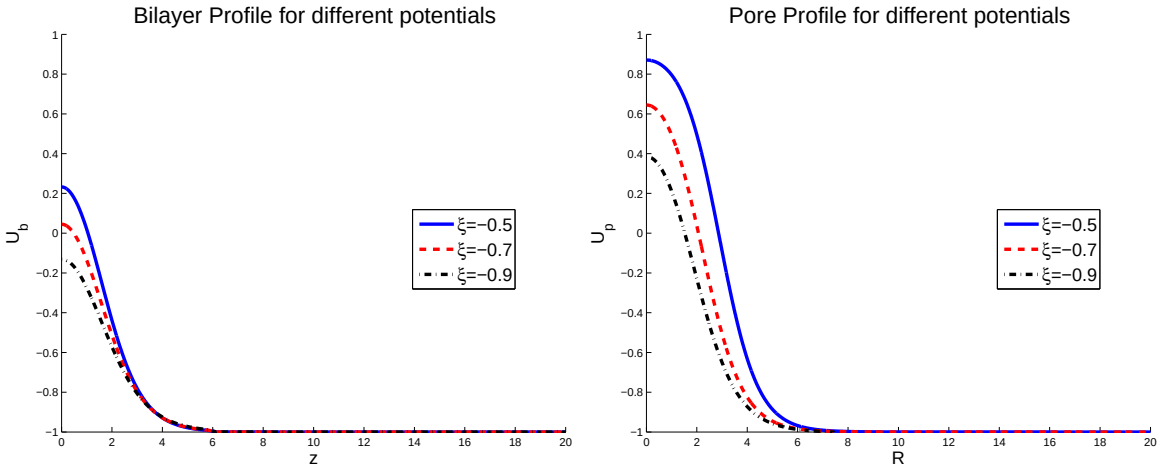


Figure 7.1: The bilayer profile (left) and the pore profile (right) corresponding to the fixed tilted double-well potential W

The functionalization parameter η_2 plays an important rule: since it can be either positive or negative it will determine the relative size of μ_b^* and μ_p^*, and so we consider η_2 as a free parameter through η_d. Once the well, W, and η_1 are fixed, the only two varying parameters in the system are μ_1 and η_2. Figure 7.2 depicts the equilibria points, of the two structures, as a function of η_d. The two equilibria lines intersect and divide the plane into four regions: above the two lines - both structures grow, and the chemical potential decays,

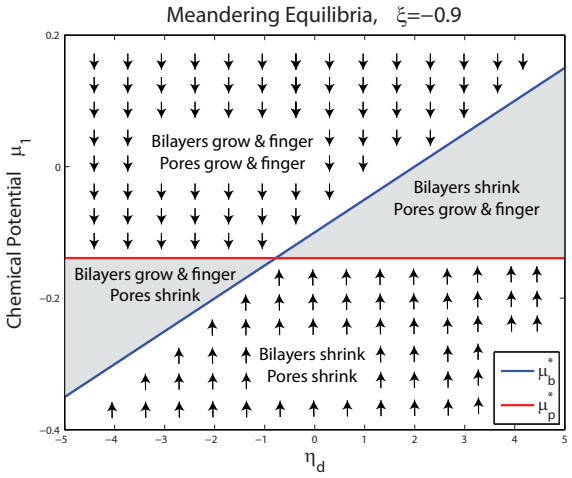


Figure 7.2: The meandering equilibria lines: The blue (solid) line is for the bilayer system, μ_b^*, and the red (dashed) line is for the pore system μ_p^*, as a function of η_d, where $\eta_1 = 1$ fixed and a fixed double-well potential.

below the two lines - both structures shrink, and the chemical potential grows, and two regions between the two lines, where one structure grows and the second decays. The intersection point of the two equilibria lines is given by

$$\eta_d^* := \eta_1 \left(1 + \left(\frac{\| Up \|_{L_R}}{\int_0^\infty \hat{U}_p \, RdR} + \frac{\| U_b \|_{L^2}^2}{2 \int_{\mathbb{R}} \hat{U}_b \, dz} \right) \frac{\int_{\mathbb{R}} \hat{U}_b \, dz}{\| U_b \|_{L^2}^2} \right), \tag{7.3}$$

and, after choosing W_ξ and η_1, the intersection point, η_d^*, is fixed. For this specific value of η_d the strong FCH may, at leading order, support a coexistence of the two morphologies. However, for any value of $\eta_d > \eta_d^*$ the system gives priority to pores, and, similarly, when $\eta_d < \eta_d^*$ the system prefers bilayers.

7.3 Pearling Stability

In Chapters 5 and 6 we derived an explicit leading order expression for the pearling eigenvalues. The pearling stability condition is the condition on μ_1 for which the pearling eigenvalues remain positive. The pearling stability conditions, for the bilayers and the pores, respectively, take the form

$$(5.2.3) \ : \ \mu_1 S_b + \eta_d \lambda_{b,0} \| \psi_{b,0} \|_2^2 < 0, \quad \text{for bilayers} \tag{7.4}$$

$$(6.2.2) \ : \ \mu_1 S_p + \eta_d \left(\left\| \psi'_{p,0} \right\|_{L_R}^2 + \lambda_{p,0} \| \psi_{p,0} \|_{L_R}^2 \right) < 0, \quad \text{for pores,} \tag{7.5}$$

where $\lambda_{b,0}$ is the ground-state eigenvalue of the linear operator $L_{b,0}$, defined in (1.29), with the corresponding eigenfunction $\psi_{b,0}$, $\lambda_{p,0}$ is the ground state eigenvalue of the linear operator $L_{p,0}$, defined in (2.81), with the corresponding eigenfunction $\psi_{p,0}$, and S_b, S_p are the **shape factors** of the bilayers and the pores, respectively, defined by

$$(5.41)\ S_b := \int_{\mathbb{R}} \Phi_{b,1} W'''(U_b)\psi_{b,0}^2\, dz, \tag{7.6}$$

$$(6.33)\ S_p := 2\pi \int_0^\infty \Phi_{p,1} W'''(U_p)\psi_{p,0}^2\, RdR. \tag{7.7}$$

Within the $\mu_1 - \eta_d$ plane the pearling bifurcation occurs along the two "pearling bifurcation lines"

$$(5.2.3)\ :\ P_b^* = \left\{ \mu_1 = -\frac{\eta_d \lambda_{b,0} \| \psi_{b,0} \|_2^2}{S_b} \;\middle|\; \eta_d \in \mathbb{R} \right\}, \tag{7.8}$$

$$(6.2.2)\ :\ P_p^* = \left\{ \mu_1 = -\frac{\eta_d \left(\left\| \psi'_{p,0} \right\|_{L_R}^2 + \lambda_{p,0} \| \psi_{p,0} \|_{L_R}^2 \right)}{S_p} \;\middle|\; \eta_d \in \mathbb{R} \right\}. \tag{7.9}$$

The sign of **shape factors** S_b, S_p determines if the morphology is pearling stable when μ_1 is above the pearling bifurcation lines P_b^*, P_p^*, or if the morphology pearls.

7.4 Numerical Evaluation of Bifurcation Regions

In this section we numerically determine the pearling lines and the meander stability/meander fingering lines and present their partitioning of the $\mu_1 - \eta_d$ plane. We fix the background state, b_-, to be -1, and choose a tilted double-well potential of the form

$$W_\xi(u) = \frac{(u^2 - b_-^2)^2}{4} - \frac{\xi}{3}(u - 2b_-), \tag{7.10}$$

where the parameter ξ determines the depth of the right well. We consider 3 different well tilts, corresponding to $\xi = -0.9, -0.7, -0.5$, see Figure 7.3. We start by fixing $\xi = -0.9$.

To calculate each of the stability lines we must evaluate the ground state eigenvalue, the ground state eigenfunctions, and the value of the shape factor. We use the Evans functions to calculate the ground state eigenvalues, and for the potential defined in (7.10) we find that $\lambda_{b,0} = 0.7421$ and $\lambda_{p,0} = 0.4648$. We normalize the associated eigenfunctions so that $\| \psi_{b,0} \|_2^2 = \| \psi_{p,0} \|_{L_R}^2 = 1$, see Figure 7.4. Next, to evaluate the shape factors S_b and S_p we need an expression for the "$L^{-1}1$ **functions**" $\Phi_{b,1} := L_{b,0}^{-1}1$and $\Phi_{p,1} := L_{p,0}^{-1}1$, defined in (2.40) and (2.86), respectively, and depicted in Figure 7.5, and the resulting values for the shape factors

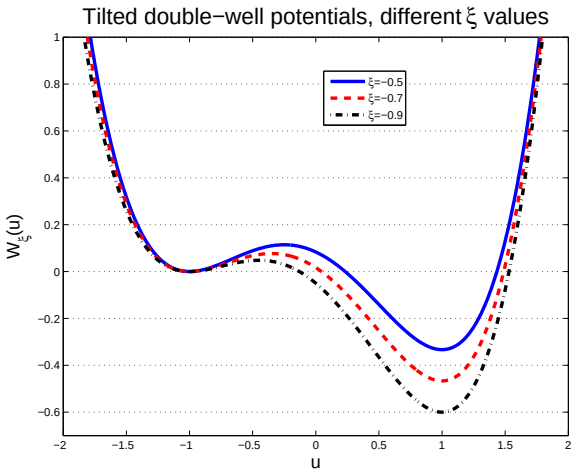


Figure 7.3: The tilted double-well potential $W(u)$.

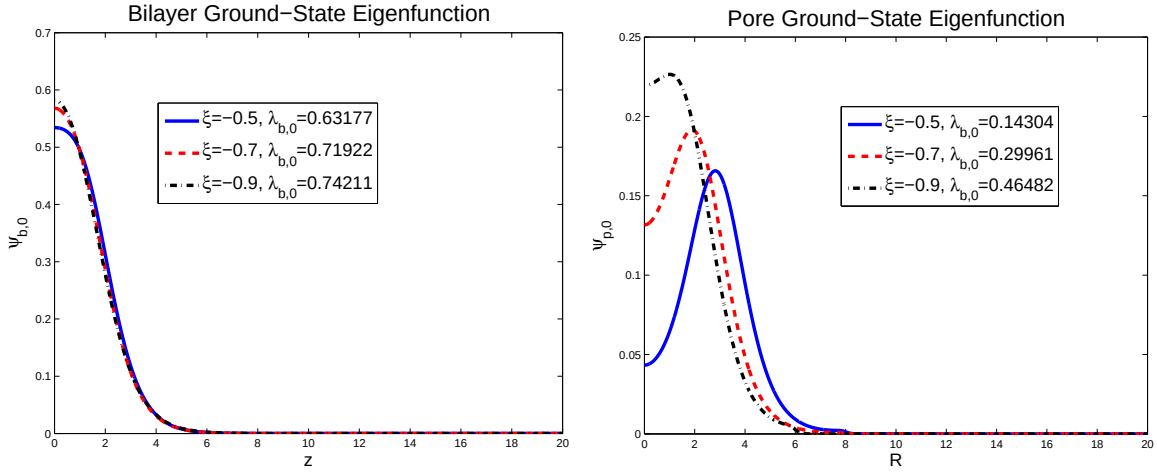


Figure 7.4: The ground state eigenfunctions: $\psi_{b,0}$ (left) and $\psi_{p,0}$ (right).

are given in Table 7.1. The pearling bifurcation lines divide the $\mu_1 - \eta_d$ plane into four regions, when both shape factors are negative. The first region - above the two lines, both morphologies are pearling stable, second region - below the two lines, both morphologies are pearling unstable, the third region where the bilayer is pearling unstable while the pore is stable and the fourth region it is vise versa, see Figure 7.6 (left). The equilibria lines, $\mu_b^*(\eta_d)$ and $\mu_p^*(\eta_d)$, defined in equations (7.1) and (7.2) are also functions of η_d, and figure 7.6 (right) shows the partitioning of the plane from all four lines. The invariant interval for μ_1 is between the two horizontal lines, where the blue (solid) line represents the bilayer equilibria and the red (solid) line represents the pore equilibria line.

Recalling that the parameter μ_1 is dynamic on the slow $\tau = \varepsilon t$ time scale, we consider initial data for which the chemical potential μ_1 lies within the region where both morphologies are pearling stable and are growing,

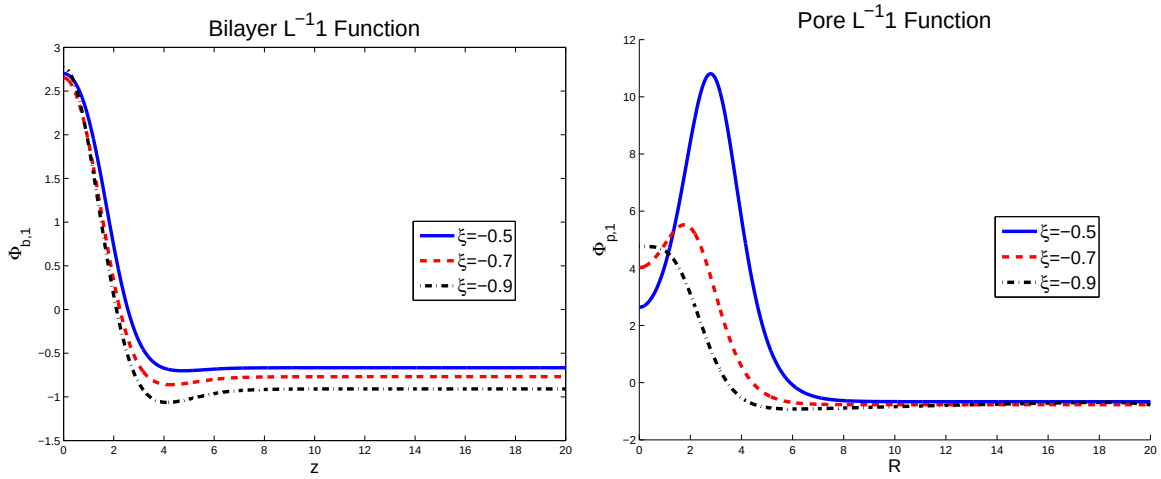


Figure 7.5: The $L^{-1}1$ functions: $\Phi_{b,1}$ (left) and $\Phi_{p,1}$ (right).

ξ	S_b	S_p
-0.9	-2.1958	-0.0431
-0.7	-0.616	-0.033
-0.5	0.8736	-2.1756

Table 7.1: Numerical evaluations of the shape factor of the bilayers S_b and the shape factor of the pores S_p as a function of the tilt of the double well potential $W_\xi(u)$.

see arrow on Figure 7.6 (right). Then for $\eta_d < 0$ the chemical potential will shrink while both the bilayer and pore morphologies, Γ_b and Γ_p, will grow, while they may finger, this is a slow process which can be dominated by the evolution of μ_1 if the curvature weighted integrals in (4.168) are sufficiently large. Assuming that both morphologies remain admissible, then at some time $t \sim O(\varepsilon^{-1})$ the chemical potential will cross the P_b^* line, and the bilayers will pearl on a fast $O(\varepsilon)$ time-scale.

Generically the coupled bilayer and pore evolution is competitive, with the two morphologies seeking incompatible values of the far-field chemical potential at equilibrium. However, by tuning the value of η_d, the equilibrium values can balance, $\mu_b^* = \mu_p^*$, and the codimension one and two morphologies can potentially co-exist. The green circle in Figure 7.6 marked the location of a common equilibria, and for the specific double-well potential, with $\xi = -0.9$, we see that the equilibria point is located in the pearling stable region. For this value of η_d, and an initial value of μ_1 below the equilibria point, the two morphologies with shrink, until reaching the equilibria, without suffering from pearling instability or meandering instability.

For a double tilted well potential W_ξ with $\xi = -0.7$ the results are qualitatively same as the case $\xi = -0.9$. The values of the shape factors are indicated in Table 7.1 and Figure 7.7 (above) depicts the division of the $\mu_1 - \eta_d$ plane by the four meandering equilibria/pearling bifurcation lines.

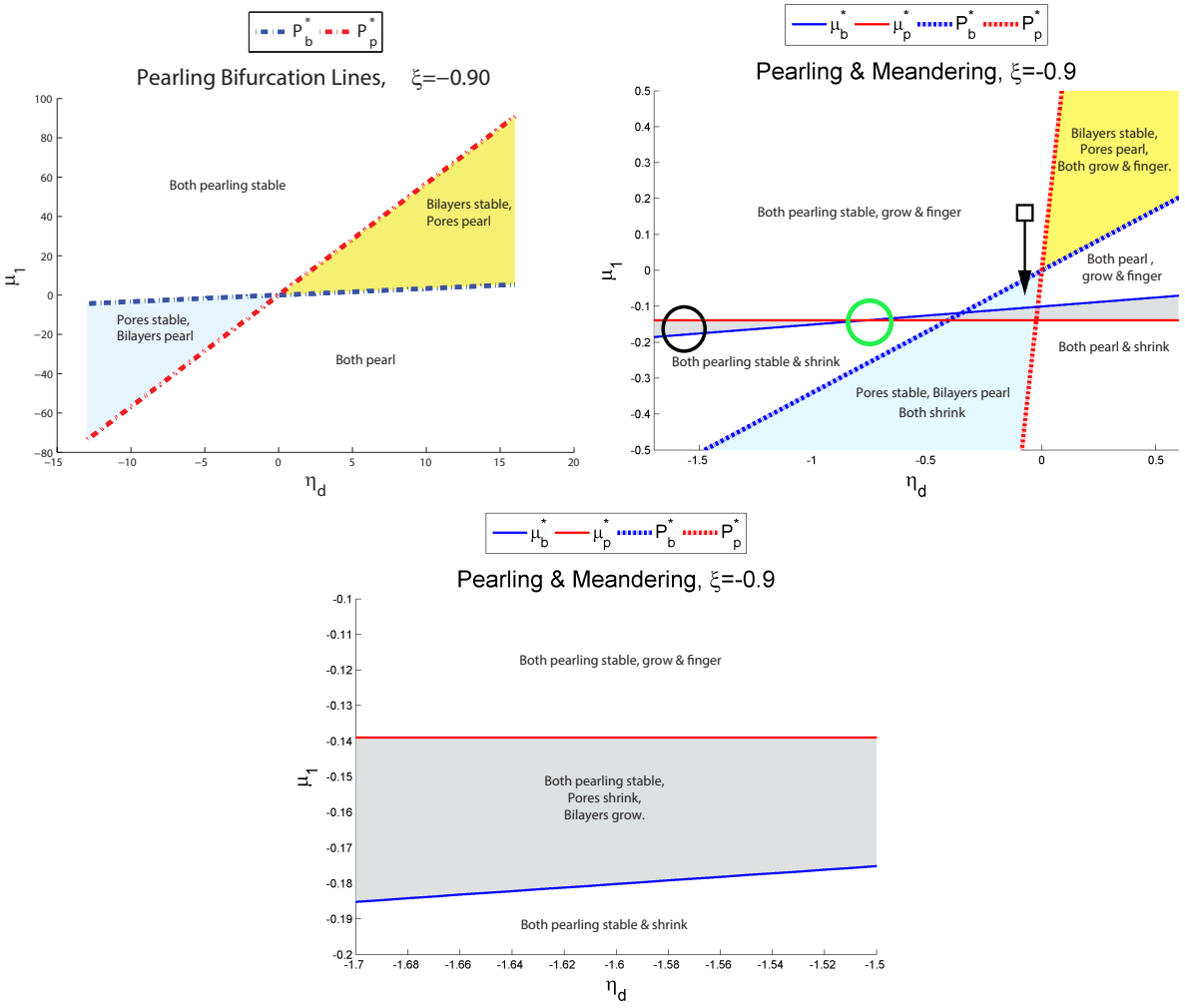


Figure 7.6: Pearling bifurcation lines as a function of η_d (top left), Pearling and Equilibria lines (top right), the co-existence equilibria is marked by a green circle. Zooming onto the black circle in the figure on the right (bottom center).

We consider an even more flattened tilted double well potential W_ξ, with $\xi = -0.5$. In this case, the value the shape factor of the bilayer is positive, see Table 7.1. The change in the sign of S_b implies that the bilayer morphology is stable as long as μ_1 lies below P_b^*. Furthermore, the shape factor of the pore remains negative and the pore structure is stable for μ_1 above P_p^*. For η_d close to zero, the area where both structures are pearling stable is located inside the dynamically invariant interval. The chemical potential will decay to its equilibria, and will stay inside the interval, while the pores will shrink, and the bilayers will grow. As μ_1 decays, it will cross the pore bifurcation line, P_p^*, which will cause the pore structure to pearl before the system reached its equilibria, see Figure 7.7 (bottom).

Note 8. *The choice of the potential W_ξ influence the sign of S_b. By changing the potential tilt, we are able to change the sign of S_b. If the shape factor is identically zero, the corresponding pearling bifurcation line*

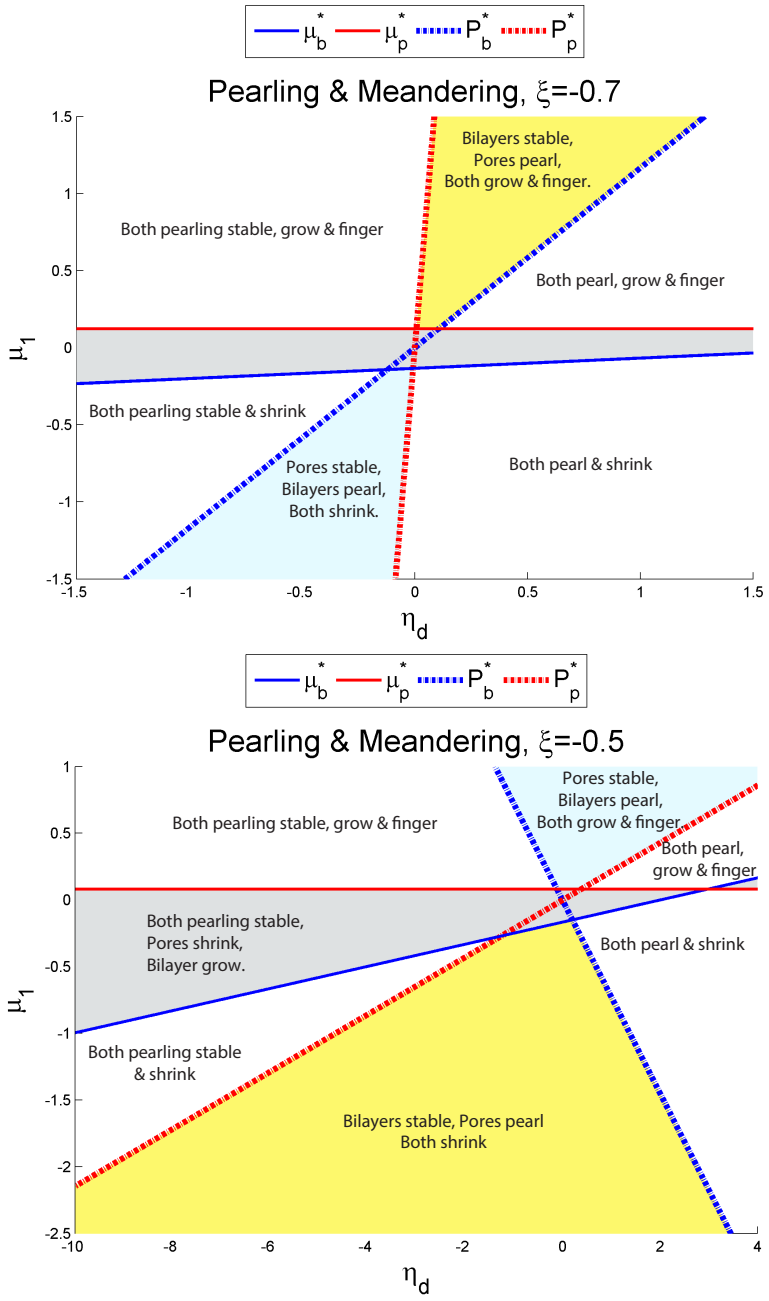


Figure 7.7: The two equilibria lines and the two bifurcation lines for a double tilted well with $\xi = -0.7$ (top) and with $\xi = -0.5$ (bottom).

will be vertical in the $\mu_1 - \eta_d$ plane.

7.5 Comparison to Experiment and Full FCH simulations

The Functionalized Cahn-Hilliard free energy provides a compact description of the energy landscape driving morphological selection in amphiphilic mixtures, such as lipid bilayers. We have shown that the strength of the interactions of the hydrophilic units with the solvent phase, parameterized by $\eta_1 > 0$, the packing entropy of the hydrophobic tails, parameterized by η_2, and the pressure jump between amphiphilic and hydrophobic phases, characterized by the difference in self energies, $W(b_\pm)$ of the amphiphilic and bulk phases, can trigger a range of bifurcations. Specifically the fingering and pearling instabilities observed experimentally in [Budin and Szostak, 2011] and [Zhu et al., 2012] by adjusting the bulk values of lipids and the charge density of the lipids, respectively, can be induced in the FCH framework by varying the corresponding control parameters. The fingerling instability is triggered by a jump in the value of the chemical potential μ_1. Assuming we start with the combined system at its equilibria point, and instantaneously increase μ_1. Then, at least one morphology will start growing, as μ_1 decays back to its equilibria, and this morphology may start fingering, see Figure 7.8 (left). On the other hand, the pearling instability can be triggered by an

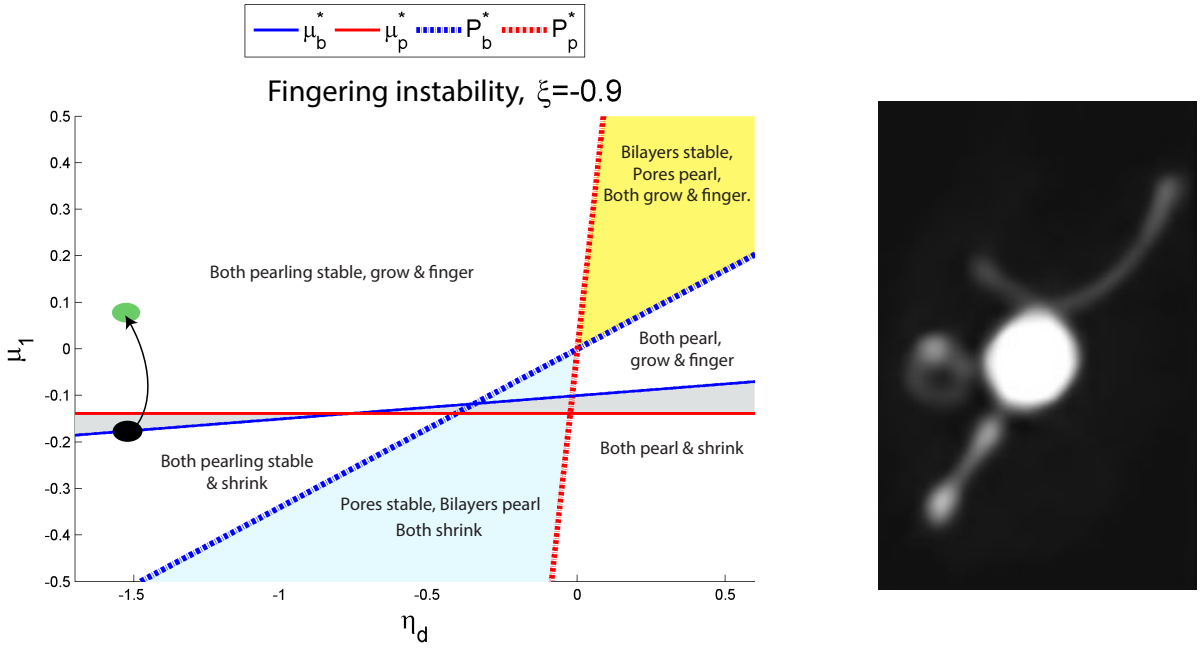


Figure 7.8: (left) Increasing the background state μ_1 moves the black point from its equilibria, which results in growth of both morphologies. (right) Szostak's experiment: raising the background concentration of lipids induces the vesicle to grow worm-like (co-dimension two) protrusions over a 74 nano-second time period [Budin and Szostak, 2011]

increase in η_d which moves the system from its equilibria in a pearling stable region into a pearling unstable region, see figure 7.9 (left).

Another way to change the stability of a system is by changing the tilt of the double-well potential. Fig-

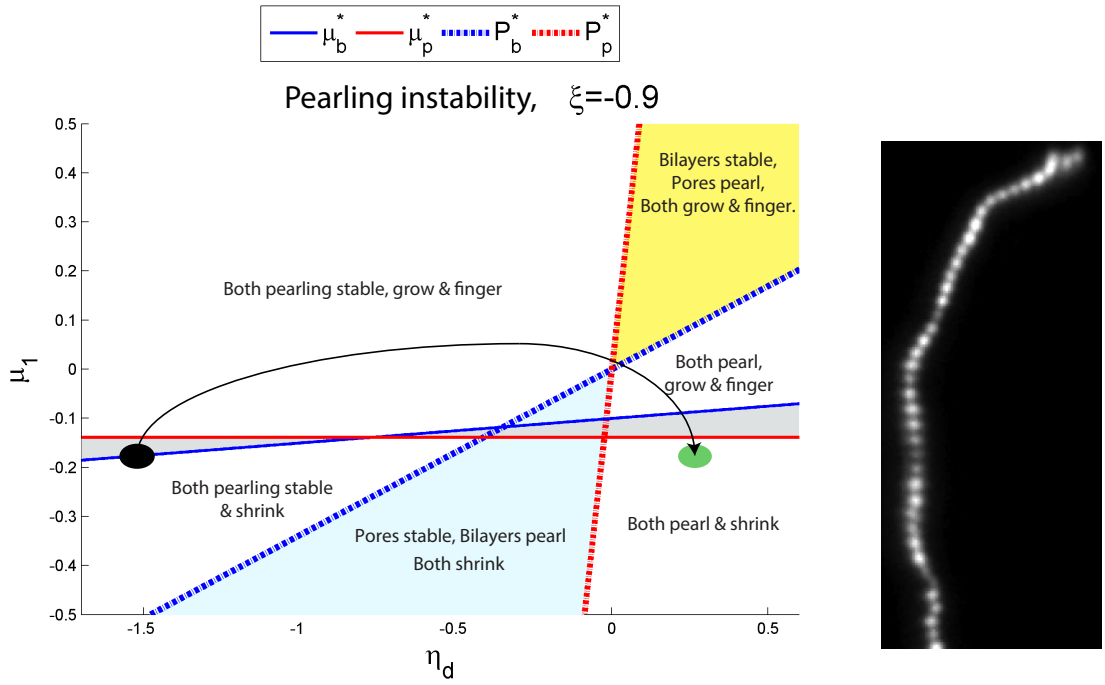


Figure 7.9: (left) Increasing η_d moves the black point to a different pearling stability area, which quickly leads to pearling in the bilayers. (right) Changing the density of charged groups on the surface via a photochemically induced redox reaction incites the pore to pearl and break into micelles [Zhu et al., 2012].

ure 7.10 describes the numerical results for $\xi = -0.7$ (left) for $\xi = -0.9$ (right), where the green dot marks the initial data for which the functionalization terms satisfy $\eta_d = -1$ and the chemical potential, at the initial state, is $\mu_1(0) = -0.4$. For $\xi = -0.7$ the green dot is located at a region which is both pearling stable and meandering stable. According to the analysis, starting with a combined system, with initial data corresponding to the green dot, both morphologies will shrink while μ_1 grows until $\mu_1 = \mu_b^*$, and both structures should remain pearling stable. However, for $\xi = -0.9$, the same green dot would be located in a region corresponding to meandering stability, i.e., the two structures will shrink, however, the region is pearling unstable for the bilayer. These results are in concord with Figure 1.10 which describes the competition for the amphiphilic phase between a bilayer and pores as a function of the tilt.

There are however many avenues to explore, for example the pearling bifurcation induces a periodic dimpling of a bilayer surface which can lead to perforation. Within the biological context of cell membranes, it is of particular interest to understand the energy required to open a single hole. Can a local adjustment of parameter values, such as a spatial variation in η_1, induce the opening of isolated holes in the membrane?

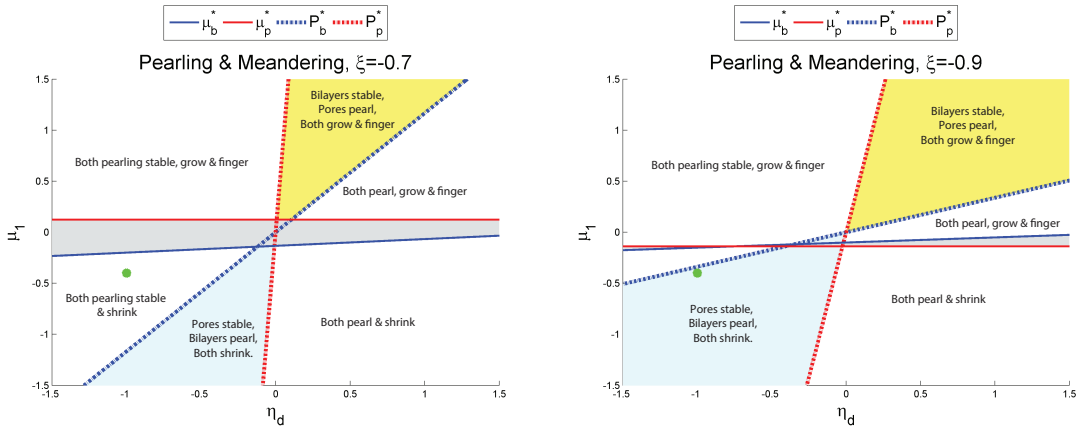


Figure 7.10: Pearling and meandering stability regions for different tilts of the potential W_ξ. For the flat tilt $\xi = -0.7$, the green dot is both meandering and prealing stable (left). For $\xi = -0.9$, the green dot is located in the bilayer unstable pearling region.

7.6 Verifying the Numerical Results

To verify the numerical results presented here, we compare the numerical value of the shape factor, S_b, as well as the numerical values of other key parameters, to the value of their algebraic expressions, given in [Doelman et al., 2014]. It was shown by [Doelman et al., 2014] that the shape factor S_b, of the bilayer is negative for a family of tilted double-well potentials of the form

$$W_p(u) = \tilde{W}_p(u+1) + 20(u - \tilde{m}_p + 1)^{p+1} H(u - \tilde{m}_p + 1), \tag{7.11}$$

where

$$\tilde{W}_p(u) := \frac{1}{p-2}(pu^2 - 2u^p), \quad m_p := \frac{p}{2}^{\frac{1}{p-2}} > 1, \tag{7.12}$$

H is the Heaviside function and $p > 2$. Figure 7.11 depicts W_3. Moreover, in [Doelman et al., 2014] algebraic expressions are derived for key quantities, which we repeat in the following Lemma.

Lemma 7.1 ([Doelman et al., 2014]). *Fix $p > 2$ and let U_b be the homoclinic solution of $\partial_z^2 U_b = W'(U_b)$ for $W = W_p$. Then the ground state eigenvalue of the linearized operator $L_{b,0}$, defined in (1.29), satisfies*

$$\lambda_{b,0} = \frac{1}{2}p(p+2) > 0. \tag{7.13}$$

Moreover, the following equalities hold

$$\| U_b' \|_{L^2(\Omega)} = \| \psi_{b,0} \|_{L^2(\Omega)} = \frac{2}{\sqrt{p-2}} \tilde{m}_p^{\frac{p+2}{2}} I\left(\frac{2}{p-2}\right), \tag{7.14}$$

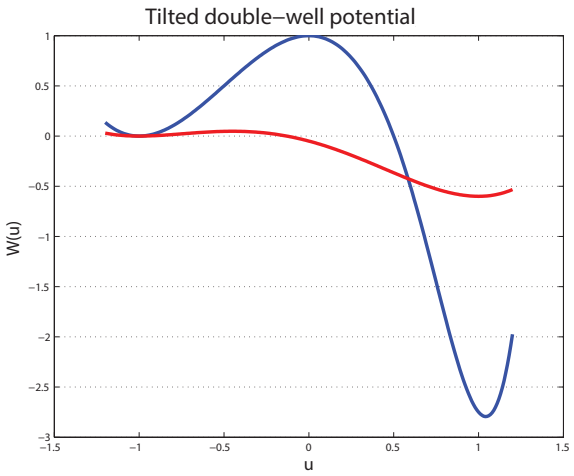


Figure 7.11: A tilted double-well potential of the form (7.11) for $p = 3$ (blue) and of the form (7.10) for $\xi = -0.9$ (red).

where $I(q) = 4^q \int_0^1 z^q (1-z)^q \, dz$, *while the shape factor* S_b *satisfy*

$$S_b = -\frac{2(p-1)}{\sqrt{p-2}} \tilde{m}_p^{\frac{1}{2}(3p-4)} I\left(\frac{1}{p-2}\right). \tag{7.15}$$

For $p = 3$, we use W_3 to obtain the values of $\lambda_{b,0}$, $\|U_b'\|_{L^2(\Omega)}$ and S_b numerically and compare them to the algebraic values, given in Lemma 7.1. The results are shown in table 7.2.

Parameter	Numerical Value	Algebraic value
$\lambda_{b,0}$	7.4985	7.5
$\|U_b'\|_{L^2(\Omega)}$	2.9394	2.9394
S_b	-7.3242	-7.3485

Table 7.2: Numerical evaluations of the key parameters compared to their algebraic values, for the double well potential $W_3(u)$.

Note that the tilted double-well potential W_ξ, defined in (7.10), does not belong to this class of potentials, defined in (7.11). The double well potentials given by (refNV-eq:ArjenW) have strongly unequal depths of the two local minima and a larger value for the local maxima between them. The value of S_b turns positive for W_ξ as the value of the local minima become proximal and the height of the local maxima decreases.

APPENDICES

Appendix A

Coordinates System

A.1 Polar Coordinates

The co-dimension 2 morphologies can be formed from cylindrically, symmetric critical points of the Cahn-Hilliard free energy. In polar coordinates, the scaled destance vector z is given by

$$z = (R\cos\theta \ , \ R\sin\theta), \tag{A.1}$$

where R is the ε-scaled radial distance to Γ_p. We rewrite the z-gradient and the z-Laplace operators in polar coordinates

$$\nabla_z = (\cos\theta\partial_R - \frac{1}{R}\sin\theta\partial_\theta \ , \ \sin\theta\partial_R + \frac{1}{R}\cos\theta\partial_\theta), \tag{A.2}$$

$$\Delta_z = \partial_R^2 + \frac{1}{R}\partial_R + \frac{1}{R^2}\partial_\theta^2. \tag{A.3}$$

For radial symmetric functions the gradient (A.2) and the Laplacian (A.3) reduce to

$$\nabla_z = (\cos\theta\partial_R \ , \ \sin\theta\partial_R), \tag{A.4}$$

$$\Delta_z = \partial_R^2 + \frac{1}{R}\partial_R. \tag{A.5}$$

Plugging (A.5) into (2.70), we obtain a radially symmetric representation of the Laplacian in the inner coordinates

$$\Delta_x = \varepsilon^{-2}\left(\partial_R^2 + \frac{1}{R}\partial_R\right) - \varepsilon^{-1}\vec{\kappa}\cdot(\cos\theta\partial_R \ , \ \sin\theta\partial_R) + \partial_s^2 - (z\cdot\vec{\kappa})\vec{\kappa}\cdot\nabla_z + O(\varepsilon), \tag{A.6}$$

A.2 Detailed investigation of the spectrum of $L_{b,0}$.

Consider the linear, closed, limit operator

$$L_{b,\infty} := \partial_z^2 - W''(b_-). \tag{A.7}$$

According to Strum-Liouville Theory for operators on the real line, the point spectrum of $L_{b,\infty}$ consists of finite number of simple eigenvalues which can be enumerated in a strictly descending order

$$\lambda_0 > \lambda_1 > ... > \lambda_N > W''(b_-). \tag{A.8}$$

We define the matrix

$$A := \begin{pmatrix} 0 & 1 \\ \lambda - W''(b_-) & 0 \end{pmatrix} \tag{A.9}$$

The matrix eigenvalues are given by

$$\mu_\pm = \pm\sqrt{W''(b_-) - \lambda}. \tag{A.10}$$

The essential spectrum of $L_{b,\infty}$ satisfies

$$\sigma_{ess}(L_{b,\infty}) = \{\lambda \in \mathbb{R} \ : \ dim\mathbb{E}^c(A(\lambda)) \neq 0\} = W''(b_-). \tag{A.11}$$

Note that $L_{b,\infty}$ is the limit operator of $L_{b,0}$, defined in (2.39), and $L_{b,0}$ is a close, linear operator. Moreover, since the operator

$$(L_{b,0} - L_{b,\infty})(L_{b,0} - \lambda_*)^{-1} = (W''(U_b) - W''(b_-))(L_{b,0} - \lambda_*)^{-1} \tag{A.12}$$

is a compact for every $\lambda_* \in \sigma(L_{b,0}))$, we know that $L_{b,0}$ is a relatively compact perturbation of $L_{b,\infty}$. We apply Weyl's Theorem to conclude

$$\sigma_{ess}(L_{b,\infty}) = \sigma_{ess}(L_{b,0}). \tag{A.13}$$

A.3 Self-adjoint operators

Consider the $L^2(\Omega)$ inner product defined in (2.73) and the two operators $\tilde{\mathcal{L}}_b := \partial_z^2 - W''(u) + \varepsilon^2\Delta_s$ and the full operator $\mathcal{L}_b := \partial_z^2 + \varepsilon\kappa\partial_z - W''(u) + \varepsilon^2\Delta_G$. The Laplace-beltrami operator is **not** self-adjoint in this inner

product,

$$(\Delta_s f, g)_{L^2(\Omega)} = \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} g J_0^{-1} \nabla_s \cdot (\boldsymbol{g}^{-1} J_0 \nabla_s) f \, J_0 \tilde{J} \, dz \, ds = -\int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} \boldsymbol{g}^{-1} J_0 \nabla_s (g\tilde{J}) \cdot \nabla_s f \, dz \, ds \tag{A.14}$$

$$= \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} \nabla_s (\boldsymbol{g}^{-1} J_0 \nabla_s (g\tilde{J})) f \, dz \, ds \neq (f, \Delta_s g)_{L^2(\Omega)} \tag{A.15}$$

although it is self-adjoint in the Γ_b-inner product. However, Δ_G is self-adjoint in the $L^2(\Omega)$ inner product.

$$(\Delta_G f, g)_{L^2(\Omega)} = \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} g J^{-1} \nabla_s \cdot (\boldsymbol{g}^{-1} J \nabla_s) f \, J \, dz \, ds = -\int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} \boldsymbol{g}^{-1} J \nabla_s (g) \cdot \nabla_s f \, dz \, ds \tag{A.16}$$

$$= \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} \nabla_s (\boldsymbol{g}^{-1} J \nabla_s (g)) f \, dz \, ds = (f, \Delta_G g)_{L^2(\Omega)}. \tag{A.17}$$

Calculating the $L^2(\Omega)$-inner product to the rest of the terms in the operator $\mathcal{L}_b$ yields,

$$(\partial_z^2 f, g)_{L^2(\Omega)} = \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f'' g J \, dz ds = \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f (gJ)'' \, dz ds = \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f g'' J + 2 f g' J' + f g J'' \, dz ds \tag{A.18}$$

$$= (f, \partial_z^2 g)_{L^2(\Omega)} + 2\varepsilon \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f g' \kappa J \, dz ds + \varepsilon \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f g \kappa' J \, dz ds + \varepsilon^2 \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f g \kappa^2 J \, dz ds, \tag{A.19}$$

$$\varepsilon(\kappa \partial_z f, g)_{L^2(\Omega)} = \varepsilon \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} \kappa f' g J \, dz ds = \varepsilon \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f (gJ\kappa)' \, dz ds = -\int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f \varepsilon \kappa g' J + \varepsilon f g J \kappa' + f g \varepsilon \kappa J' \, dz ds \tag{A.20}$$

$$= -\varepsilon (f, \kappa \partial_z g)_{L^2(\Omega)} - \varepsilon \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f g \kappa' J \, dz ds - \varepsilon^2 \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f g \kappa^2 J \, dz ds. \tag{A.21}$$

Each of the terms separately is not self-adjoint in the $L^2(\Omega)$-inner product, however their sum satisfies

$$(\partial_z^2 f + \varepsilon \kappa \partial_z f, g)_{L^2(\Omega)} = (f, \partial_z^2 g + \varepsilon \kappa \partial_z g)_{L^2(\Omega)}.$$

Therefore, our full operator $\mathcal{L}_b$ is self-adjoint in the $L^2(\Omega)$-inner product where the $\tilde{\mathcal{L}}_b$ operator is not.

Appendix B

Geometric Evolution Co-dimension 1

B.1 Outer expansion of the $1^{\underline{\text{st}}}$ variation of $\mathcal{F}$

Recall that the 1^{st} variation of $\mathcal{F}$ is given by

$$\frac{\delta \mathcal{F}}{\delta u}(u) = \left[\left(-\varepsilon^2\Delta + W''(u) - \varepsilon\eta_1\right)\left(-\varepsilon^2\Delta u + W'(u)\right) + \varepsilon\eta_d W'(u)\right]. \tag{B.1}$$

Plugging a formal expansion of $u(x) = u_0(x) + \varepsilon u_1(x) + ...$ and expanding yields

$$\begin{aligned}
&\left(-\varepsilon^2\Delta + W''(u) - \varepsilon\eta_1\right)\left(-\varepsilon^2\Delta u + W'(u)\right) + \varepsilon\eta_d W'(u) = \\
&\left(-\varepsilon^2\Delta + W''(u_0) + \varepsilon(W'''(u_0)u_1 - \eta_1) + \varepsilon^2(W'''(u_0)u_2 + \frac{1}{2}W''''(u_0)u_1^2) + ..\right)\cdot \\
&\left(-\varepsilon^2(\Delta u_0 + \varepsilon\Delta u_1 + \varepsilon^2\Delta u_2) + W'(u_0) + \varepsilon W''(u_0)u_1 + \varepsilon^2(W''(u_0)u_2 + \frac{1}{2}W'''(u_0)u_1^2) + ...\right) \\
&+ \varepsilon\eta_d\left(W'(u_0) + \varepsilon W''(u_0)u_1 + \varepsilon^2(W''(u_0)u_2 + \frac{1}{2}W'''(u_0)u_1^2) + ...\right).
\end{aligned} \tag{B.2}$$

Since $\mu(x,t) := \frac{\delta \mathcal{F}}{\delta u}(u)$ we can rewrite it in order of ε such that

$$\frac{\delta \mathcal{F}}{\delta u}(u) = \mu_0(x,t) + \varepsilon\mu_1(x,t) + \varepsilon^2\mu_2(x,t) + O(\varepsilon^3), \tag{B.3}$$

where

$$\mu_0 = W''(u_0)W'(u_0), \tag{B.4}$$

$$\mu_1 = (W'''(u_0)u_1 - \eta_1)W'(u_0) + (W''(u_0))^2 u_1 + \eta_d W'(u_0), \tag{B.5}$$

$$\mu_2 = \left(-\Delta + W'''(u_0)u_2 + \frac{1}{2}W^{(4)}(u_0)u_1\right)W'(u_0) + (W'''(u_0)u_1 - \eta_1)W''(u_0)u_1 \tag{B.6}$$
$$+ W''(u_0)\left(-\Delta u_0 + W''(u_0)u_2 + \frac{1}{2}W'''(u_0)u_1^2\right) + \eta_d W''(u_0)u_1$$

B.2 Inner expansion of the t-derivative of u

To get an expression for the left-hand side using the whiskered coordinates we take the t-derivative of u and consider $u = \tilde{u}(s, z, \tau)$. Treating both s and z as functions of t, the use of the chain rule is required, and results in

$$u_t = \nabla S \cdot \nabla_s \tilde{u} + \frac{\partial \tilde{u}}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial \tilde{u}}{\partial \tau}\frac{\partial \tau}{\partial t}. \tag{B.7}$$

Assuming that u do not change when s varies normal to Γ_b with z held fixed, see equation (**??**), and using the normal velocity,defined in (3.9), equation (B.7) reduces to

$$\frac{\partial u}{\partial t} = -\varepsilon^{-1}V_\tau(s)\frac{\partial \tilde{u}}{\partial z} + \frac{\partial \tilde{u}}{\partial \tau}\frac{\partial \tau}{\partial t}. \tag{B.8}$$

Expanding $\tilde{u}$ in orders of ε such that $\tilde{u}(s, z, \tau) = \tilde{u}_0(s, z, \tau) + \varepsilon\tilde{u}_1(s, z, \tau) + O(\varepsilon^2)$ and plugging it back into equation (B.8) yields

$$\frac{\partial u}{\partial t} = -\varepsilon^{-1}V_\tau(s)\frac{\partial \tilde{u}_0}{\partial z} - V_\tau(s)\frac{\partial \tilde{u}_1}{\partial z} + \frac{\partial \tilde{u}_0}{\partial \tau}\frac{\partial \tau}{\partial t} - V_\tau(s)\frac{\partial \tilde{u}_2}{\partial z} + \frac{\partial \tilde{u}_1}{\partial \tau}\frac{\partial \tau}{\partial t} + \dots . \tag{B.9}$$

B.3 Inner expansion of the $1^{\underline{\text{st}}}$ variation of $\mathcal{F}$

Recall that the 1^{st} variation of $\mathcal{F}$ is given by

$$\frac{\delta \mathcal{F}}{\delta u}(u) = \left(-\varepsilon^2\Delta + W''(u) - \varepsilon\eta_1\right)\left(-\varepsilon^2\Delta u + W'(u)\right) + \varepsilon\eta_d W'(u), \tag{B.10}$$

and the chemical potential μ is defined by

$$\mu := \frac{\delta \mathcal{F}}{\delta u}(u). \tag{B.11}$$

At a given time scale τ, the inner spatial expansion for the density function $u(t, x)$ is given by

$$u(x, t) = \tilde{u}_0(s, z, \tau) + \varepsilon\tilde{u}_1(s, z, \tau) + \varepsilon^2\tilde{u}_2(s, z, \tau) + \dots, \tag{B.12}$$

and in local coordinates, recall that the Laplacian operator, see (2.9), takes the form

$$\varepsilon^2\Delta_x = \partial_z^2 + \varepsilon\kappa\partial_z + \varepsilon^2\Delta_G. \tag{B.13}$$

First, consider an expansion of each of the terms on the right-hand side of (B.10) :

$$-\varepsilon^2\Delta = -\partial_z^2 - \varepsilon H_0\partial_z - \varepsilon^2(zH_1\partial_z + \Delta_s) - \varepsilon^3\Delta_1 + \dots\,, \tag{B.14}$$

$$\begin{aligned}-\varepsilon^2\Delta\tilde{u} =& -\partial_z^2\tilde{u}_0 - \varepsilon(\partial_z^2\tilde{u}_1 + H_0\partial_z\tilde{u}_0) - \varepsilon^2(\partial_z^2\tilde{u}_2 + H_0\partial_z\tilde{u}_1 + zH_1\partial_z\tilde{u}_0 + \Delta_s\tilde{u}_0)\\ &- \varepsilon^3(\partial_z^2\tilde{u}_3 + H_0\partial_z\tilde{u}_2 + zH_1\partial_z\tilde{u}_1 + \Delta_s\tilde{u}_1 + \Delta_1\tilde{u}_0) + \dots\,,\end{aligned} \tag{B.15}$$

$$\begin{aligned}W''(\tilde{u}) =& W''(\tilde{u}_0) + \varepsilon W'''(\tilde{u}_0)\tilde{u}_1 + \varepsilon^2\Big[W'''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2\Big]\\ &+ \varepsilon^3\Big[W'''(\tilde{u}_0)u_3 + W^{(4)}(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}W^{(5)}(\tilde{u}_0)\tilde{u}_1^3\Big] + O(\varepsilon^4)\,,\end{aligned} \tag{B.16}$$

$$\begin{aligned}W'(\tilde{u}) =& W'(\tilde{u}_0) + \varepsilon W''(\tilde{u}_0)\tilde{u}_1 + \varepsilon^2\Big[W''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W'''(\tilde{u}_0)\tilde{u}_1^2\Big]\\ &+ \varepsilon^3\Big[W''(\tilde{u}_0)\tilde{u}_3 + W'''(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}W^{(4)}(\tilde{u}_0)\tilde{u}_1^3\Big] + O(\varepsilon^4).\end{aligned} \tag{B.17}$$

Next, we collecting the terms and write them in order of ε

$$\begin{aligned}(-\varepsilon^2\Delta + W''(u) - \varepsilon\eta_1) =& (-\partial_z^2 + W''(\tilde{u}_0))\\ &+ \varepsilon\left(-H_0\partial_z + W'''(\tilde{u}_0)\tilde{u}_1 - \eta_1\right)\\ &+ \varepsilon^2\left(-zH_1\partial_z - \Delta_s + W'''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2\right)\\ &+ \varepsilon^3\left(-\Delta_1 + W'''(\tilde{u}_0)\tilde{u}_3 + W^{(4)}(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}W^{(5)}(\tilde{u}_0)\tilde{u}_1^3\right) + O(\varepsilon^4),\end{aligned} \tag{B.18}$$

$$\begin{aligned}(-\varepsilon^2\Delta u + W'(u)) =& (-\partial_z^2\tilde{u}_0 + W'(\tilde{u}_0))\\ &+ \varepsilon\left(-\partial_z^2\tilde{u}_1 - H_0\partial_z\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_1\right)\\ &+ \varepsilon^2\left(-\partial_z^2\tilde{u}_2 - H_0\partial_z\tilde{u}_1 - H_1z\partial_z\tilde{u}_0 - \Delta_s\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W'''(\tilde{u}_0)\tilde{u}_1^2\right)\\ &+ \varepsilon^3\Bigg(-\partial_z^2\tilde{u}_3 - H_0\partial_z\tilde{u}_2 - H_1z\partial_z\tilde{u}_1 - \Delta_s\tilde{u}_1 - \Delta_1\tilde{u}_0\\ &\qquad + W''(\tilde{u}_0)\tilde{u}_3 + W'''(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}W^{(4)}(\tilde{u}_0)\tilde{u}_1^3\Bigg)\\ &+ O(\varepsilon^4).\end{aligned} \tag{B.19}$$

Using expansion (B.18) and (B.19) we can rewrite the $1^{\underline{\text{st}}}$ variation of $\mathcal{F}$ in the following form

$$\frac{\delta \mathcal{F}}{\delta u}(u) = \tilde{\mu}_0(s,z,\tau) + \varepsilon\tilde{\mu}_1(s,z,\tau) + \varepsilon^2\tilde{\mu}_2(s,z,\tau) + O(\varepsilon^3), \tag{B.20}$$

where $\tilde{\mu}_i$ for $i \geq 0$ are the inner expansion of the chemical potential

$$\mu(x,t) = \tilde{\mu}_0(s,z,\tau) + \varepsilon\tilde{\mu}_1(s,z,\tau) + \varepsilon^2\tilde{\mu}_2(s,z,\tau) + ..., \tag{B.21}$$

given by

$$\tilde{\mu}_0 = (-\partial_z^2 + W''(\tilde{u}_0))(-\partial_z^2\tilde{u}_0 + W'(\tilde{u}_0)), \tag{B.22}$$

$$\tilde{\mu}_1 = (-\partial_z^2 + W''(\tilde{u}_0))(-H_0\partial_z\tilde{u}_0 - \partial_z^2\tilde{u}_1 + W''(\tilde{u}_0)\tilde{u}_1) + \tag{B.23}$$
$$(-H_0\partial_z + W'''(\tilde{u}_0)\tilde{u}_1 - \eta_1)(-\partial_z^2\tilde{u}_0 + W'(\tilde{u}_0)) + \eta_d W'(\tilde{u}_0),$$

$$\tilde{\mu}_2 = (-\partial_z^2 + W''(\tilde{u}_0))(-\partial_z^2\tilde{u}_2 - zH_1\partial_z\tilde{u}_0 - H_0\partial_z\tilde{u}_1 - \Delta_s\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W'''(\tilde{u}_0)\tilde{u}_1^2) + \tag{B.24}$$
$$(-H_0\partial_z + W'''(\tilde{u}_0)\tilde{u}_1 - \eta_1)(-\partial_z^2\tilde{u}_1 - H_0\partial_z\tilde{u}_0 + W''(\tilde{u}_0)\tilde{u}_1) +$$
$$(-zH_1\partial_z - \Delta_s + W'''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2)(-\partial_z^2\tilde{u}_0 + W'(\tilde{u}_0)) + \eta_d W''(\tilde{u}_0)\tilde{u}_1,$$

and, merely from pedantic reasons (I don't know if I use this term later on) we also have

$$\begin{aligned}
\tilde{\mu}_3 =& (-\partial_z^2 + W''(\tilde{u}_0))\left(L\tilde{u}_3 - H_0\partial_z\tilde{u}_2 - zH_1\partial_z\tilde{u}_1 - \Delta_s\tilde{u}_1 - \Delta_1\tilde{u}_0 + W'''(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}W^{(4)}(\tilde{u}_0)\tilde{u}_1^3\right) \\
&+ (-H_0\partial_z + W'''(\tilde{u}_0)\tilde{u}_1 - \eta_1)\left(L\tilde{u}_2 - H_0\partial_z\tilde{u}_1 - zH_1\partial_z\tilde{u}_0 - \Delta_s\tilde{u}_0 + \frac{1}{2}W'''(\tilde{u}_0)\tilde{u}_1^2\right) \\
&+ \left(-zH_1\partial_z - \Delta_s + W'''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2\right)(L\tilde{u}_1 - H_0\partial_z\tilde{u}_0) \\
&+ \left(-\Delta_1 + W'''(\tilde{u}_0)\tilde{u}_3 + W^{(4)}(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}W^{(5)}(\tilde{u}_0)\tilde{u}_1^3\right)(-\partial_z^2\tilde{u}_0 + W'(\tilde{u}_0)) \\
&+ \eta_d\left(W''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W'''(\tilde{u}_0)\tilde{u}_1^2\right).
\end{aligned} \tag{B.25}$$

B.4 Normal Velocity Calculations for $\tau = \varepsilon t$

Recall equation (3.91) given by

$$\left(-H_0\partial_z L_{b,0}\tilde{u}_1 + H_0 W'''(U_b)\tilde{u}_1 U_b' - \eta_1 H_0 U_b' + z\partial_{\mathbf{n}}\mu_1 - V_\tau(s)\int_0^z \hat{U}_b(w)\,dw, U_b'\right)_{L^2(\mathbb{R})}. \tag{B.26}$$

To calculate each of the inner products we first note that the inner product of $L_{b,0}\tilde{u}_1$, given in (3.76), with U_b'' yields

$$(L_{b,0}\tilde{u}_1, U_b'')_{L^2(\mathbb{R})} = (\tilde{\mu}_1\varphi_1 - \eta_d\frac{z}{2}U_b', U_b'')_{L^2(\mathbb{R})} = \mu_1\int_{\mathbb{R}}\varphi_1 U_b''\,dz - \eta_d\int_{\mathbb{R}}\frac{z}{4}((U_b')^2)'\,dz \tag{B.27}$$

$$= \mu_1\int_{\mathbb{R}}\varphi_1 L_{b,0}(\frac{z}{2}U_b')\,dz - \eta_d\int_{\mathbb{R}}\frac{z}{4}((U_b')^2)'\,dz \tag{B.28}$$

$$= \mu_1\int_{\mathbb{R}}\frac{z}{2}U_b'\,dz - \eta_d\int_{\mathbb{R}}\frac{z}{4}((U_b')^2)'\,dz \tag{B.29}$$

$$= -\frac{\mu_1}{2}\int_{\mathbb{R}}U_b\,dz + \frac{\eta_d}{4}\int_{\mathbb{R}}(U_b')^2\,dz \tag{B.30}$$

$$= -\frac{\mu_1}{2}m + \frac{\eta_d}{4}\sigma_b \tag{B.31}$$

$$\tag{B.32}$$

where we use identity (B.43) to get (B.28), (B.29) follows from the fact that $L_{b,0}$ is self-adjoint and (B.30) follows from integrating by parts each of the integrals. The last equality we recall m_b and σ_b are defined in (1.50) and (3.94), respectively.

The first term in the inner product in equation (B.26) can be written as

$$(-H_0\partial_z L_{b,0}\tilde{u}_1, U_b') = H_0(L_{b,0}\tilde{u}_1, U_b''), \tag{B.33}$$

and the second term in the inner product in equation (B.26) can be written as

$$(H_0W'''(U_b)\tilde{u}_1U_b', U_b') = H_0(\tilde{u}_1, W'''(U_b)(U_b')^2) = H_0(\tilde{u}_1, L_{b,0}U_b'') = H_0(L_{b,0}\tilde{u}_1, U_b'') \tag{B.34}$$

Summing these two inner product together yields

$$(-H_0\partial_z L_{b,0}\tilde{u}_1 + H_0W'''(U_b)\tilde{u}_1U_b', U_b') = 2H_0(L_{b,0}\tilde{u}_1, U_b'') = -H_0\mu_1 m_b + H_0\frac{\eta_d}{2}\sigma_b, \tag{B.35}$$

where the last equality we used equation (B.31). Calculating the integrals in the next two term in equation (B.26) yields

$$(-\eta_1 H_0U_b', U_b') = -\eta_1 H_0\int_{\mathbb{R}}(\hat{U}_b')^2\,dz = -\eta_1 H_0\sigma_b, \tag{B.36}$$

$$(z\partial_{\mathbf{n}}\mu_1, U_b') = \partial_{\mathbf{n}}\mu_1\int_{\mathbb{R}}z\hat{U}_b'\,dz = -\partial_{\mathbf{n}}\mu_1 m_b, \tag{B.37}$$

where for the last equality we integrated by parts. The last term in equation (B.26) involve the normal

velocity $V_\tau(s)$

$$\left(-V_\tau(s)\int_0^z \hat{U}_b(w)\,dw, U_b'\right) = -V_\tau(s)\int_{\mathbb{R}}\int_0^z \hat{U}_b(w)\,dw U_b'\,dz = V_\tau(s)\int_{\mathbb{R}}\hat{U}_b^2\,dz = V_\tau(s)B_1 \tag{B.38}$$

Setting equation (B.26) equal to zero, summarizing the calculation of each inner product and solving for $V_\tau(s)$ yields

$$V_\tau(s) = \frac{(H_0\mu_1 + \partial_{\mathbf{n}}\mu_1)m_b + \frac{1}{2}H_0\left(\eta_1+\eta_2\right)\sigma_b}{B_1}. \tag{B.39}$$

B.5 Useful Identities

Recall that U_b solves

$$W'(U_b) = \partial_z^2 U_b. \tag{B.40}$$

Taking the z derivative of (B.40) yields

$$L_{b,0}U_b' = 0, \tag{B.41}$$

and taking the z derivative again yields

$$L_{b,0}U_b'' = W'''(U_b)(U_b')^2. \tag{B.42}$$

In addition, direct calculation yields

$$L_{b,0}(\frac{z}{2}U_b') = \partial_z^2 U_b. \tag{B.43}$$

Appendix C

Geometric Evolution Co-dimension 2

C.1 Appendix : Detailed expansion of the FCH equation using inner variables

We start by expanding each term on the right-hand side of (4.3) to obtain

$$-\varepsilon^2\Delta = -\Delta_z + \varepsilon\vec{\kappa}\cdot\nabla_z + \varepsilon^2(\partial_s^2 - (z\cdot\vec{\kappa})\vec{\kappa}\cdot\nabla_z) + O(\varepsilon^3), \tag{C.1}$$

$$-\varepsilon^2\Delta\tilde{u} = -\Delta_z\tilde{u}_0 - \varepsilon(\Delta_z\tilde{u}_1 - \vec{\kappa}\cdot\nabla_z\tilde{u}_0) - \varepsilon^2(\Delta_z\tilde{u}_2 - \vec{\kappa}\cdot\nabla_z\tilde{u}_1 - \partial_s^2\tilde{u}_0 + (z\cdot\vec{\kappa})\vec{\kappa}\nabla\tilde{u}_0) + O(\varepsilon^3), \tag{C.2}$$

$$W''(\tilde{u}) = W''(\tilde{u}_0) + \varepsilon W'''(\tilde{u}_0)\tilde{u}_1 + \varepsilon^2\Big[W'''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2\Big] \tag{C.3}$$

$$+ \varepsilon^3\Big[W'''(\tilde{u}_0)u_3 + W^{(4)}(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}W^{(5)}(\tilde{u}_0)\tilde{u}_1^3\Big] + O(\varepsilon^4) \tag{C.4}$$

$$W'(\tilde{u}) = W'(\tilde{u}_0) + \varepsilon W''(\tilde{u}_0)\tilde{u}_1 + \varepsilon^2\Big[W''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W'''(\tilde{u}_0)\tilde{u}_1^2\Big] \tag{C.5}$$

$$+ \varepsilon^3\Big[W''(\tilde{u}_0)\tilde{u}_3 + W'''(\tilde{u}_0)\tilde{u}_1\tilde{u}_2 + \frac{1}{6}W^{(4)}(\tilde{u}_0)\tilde{u}_1^3\Big] + O(\varepsilon^4). \tag{C.6}$$

Next, collecting the terms by orders of ε yields

$$(-\varepsilon^2\Delta + W''(u) - \varepsilon\eta_1) = (-\Delta_z + W''(\tilde{u}_0)) \tag{C.7}$$

$$+ \varepsilon\,(\vec{\kappa}\cdot\nabla_z + W'''(\tilde{u}_0)\tilde{u}_1 - \eta_1) \tag{C.8}$$

$$+ \varepsilon^2\left(\partial_s^2 - (z\cdot\vec{\kappa})\vec{\kappa}\cdot\nabla_z + W'''(\tilde{u}_0)\tilde{u}_2 + \frac{1}{2}W^{(4)}(\tilde{u}_0)\tilde{u}_1^2\right) \tag{C.9}$$

$$+ O(\varepsilon^3), \tag{C.10}$$

$$(-\varepsilon^2\Delta u + W'(u)) = (-\Delta_z\tilde{u}_0 + W'(\tilde{u}_0)) \tag{C.11}$$

$$+\varepsilon\left(-\Delta_z\tilde{u}_1+\vec{\kappa}\cdot\nabla_z\tilde{u}_0+W''(\tilde{u}_0)\tilde{u}_1\right) \tag{C.12}$$

$$+\varepsilon^2\left(-\Delta_z\tilde{u}_2+\vec{\kappa}\cdot\nabla_z\tilde{u}_1+\partial_s^2\tilde{u}_0-(z\cdot\vec{\kappa})\vec{\kappa}\cdot\nabla\tilde{u}_0+W''(\tilde{u}_0)\tilde{u}_2+\frac{1}{2}W'''(\tilde{u}_0)\tilde{u}_1^2\right) \tag{C.13}$$

$$+O(\varepsilon^3). \tag{C.14}$$

C.2 Appendix : Calculation of the solvability condition, $\tau=\varepsilon t$

Recall that the operators $L_{p,m}$ given in (2.81) are self-adjoint in the R-weighted inner product, introduced in (2.74). We calculate each term of the R-weighted inner product of $(\mathcal{Q}_1,U_p)_{L_R}$, where

$$\begin{aligned}\mathcal{Q}_1 :=& -\mu_1\vec{\kappa}\cdot\nabla_z\Phi_{p,1}+\eta_d\vec{\kappa}\cdot\nabla_z L_p^{-1}(W'(U_p))+\mu_1W'''(U_p)\Phi_{p,2}\vec{\kappa}\cdot\nabla_z U_p\\ &-\eta_dW'''(U_p)L_p^{-2}(W'(U_p))\vec{\kappa}\cdot\nabla_z U_p-\eta_1\vec{\kappa}\cdot\nabla_z U_p.\end{aligned} \tag{C.15}$$

Calculation of the R-inner product of the first term in (C.15) yields

$$\begin{aligned}(\mu_1\vec{\kappa}\cdot\nabla_z\Phi_{p,1},\partial_{z_1}U_p)_{L_R}&=\int_0^{2\pi}\int_0^{\infty}\mu_1(\kappa_1\cos\theta+\kappa_2\sin\theta)\Phi_{p,1}'(R)U_p'(R)\cos\theta R\,dR\,d\theta\\ &=\mu_1\kappa_1\int_0^{2\pi}\cos^2\theta\,d\theta\int_0^{\infty}\Phi_{p,1}'U_p'R\,dR=\pi\mu_1\kappa_1\int_0^{\infty}\Phi_{p,1}'U_p'R\,dR,\end{aligned} \tag{C.16}$$

using integration by parts and identity (C.39), equation (C.16) reduces to

$$\begin{aligned}\pi\mu_1\kappa_1\int_0^{\infty}\Phi_{p,1}'U_p'R\,dR&=-\pi\mu_1\kappa_1\int\Phi_{p,1}(U_p'R)'\,dR=-\pi\mu_1\kappa_1\int\Phi_{p,1}L_p\left(\frac{1}{2}RU_p'\right)R\,dR\\ &=-\frac{\pi\mu_1\kappa_1}{2}\int U_p'R^2\,dR=\pi\mu_1\kappa_1\int\hat{U}R\,dR=\pi\mu_1\kappa_1S_1,\end{aligned} \tag{C.17}$$

where S_1 is the total mass, defined in (4.70). Similarly,

$$(\mu_1\vec{\kappa}\cdot\nabla_z\Phi_{p,1},\partial_{z_2}U)_{L_R}=\int_0^{2\pi}\int_0^{\infty}\left(\mu_1(\kappa_1\cos\theta+\kappa_2\sin\theta)\Phi_{p,1}'(R)\right)U'(R)\sin\theta R\,dR\,d\theta=\pi\mu_1\kappa_2S_1. \tag{C.18}$$

Next we calculate the inner product of the 2^{nd} term of $\mathcal{Q}_1$, (C.15), with $\partial_{z_i}U_p$. Recall that $W'(U_p)=\Delta_zU_p$ and, using identity (C.42), we have $L_p^{-1}(W'(U_p))=L_p^{-1}(\Delta_zU_p)=\frac{1}{2}RU_p'$. Therefore, the second term takes the form $\eta_d\vec{\kappa}\cdot\nabla_z(\frac{1}{2}RU')$ and the inner product reduces to

$$(\eta_d\vec{\kappa}\cdot\nabla_z(\frac{1}{2}RU'),\partial_{z_1}U)_{L_R}=\int_0^{2\pi}\int_0^{\infty}\frac{\eta_d}{2}(\kappa_1\cos\theta+\kappa_2\sin\theta)(U'+RU'')U'\cos\theta R\,dR\,d\theta \tag{C.19}$$

$$= \frac{\eta_d\kappa_1}{2}\int_0^{2\pi}\int_0^{\infty}(U' + RU'')U'\cos^2\theta R\,dR\,d\theta = \pi\frac{\eta_d\kappa_1}{2}\int_0^{\infty}(U' + RU'')U'R\,dR \tag{C.20}$$

$$= \pi\frac{\eta_d\kappa_1}{2}\int_0^{\infty}(U')^2R\,dR + \pi\frac{\eta_d\kappa_1}{2}\int_0^{\infty}\frac{1}{2}((U')^2)'R^2\,dR \tag{C.21}$$

$$= \pi\frac{\eta_d\kappa_1}{2}S_4 - \pi\frac{\eta_d\kappa_1}{2}\int_0^{\infty}(U')^2R\,dR = \pi\frac{\eta_d\kappa_1}{2}S_4 - \pi\frac{\eta_d\kappa_1}{2}S_4 = 0 \tag{C.22}$$

where S_4.

Next we calculate the inner product of the 3^{nd} term of $\mathcal{Q}_1$, (C.15), with $\partial_{z_i}U_p$.

$$(\mu_1 W'''(U_p)\Phi_{p,2}\vec{\kappa}\cdot\nabla_z U_p, \partial_{z_1}U_p)_{L_R} = \pi\kappa_1\mu_1\int_0^{\infty}W'''(U_p)(U_p')^2\Phi_{p,2}R\,dR = \pi\kappa_1\mu_1\int_0^{\infty}\mathcal{L}^2(\frac{1}{2}RU_p')\Phi_{p,2}R\,dR \tag{C.23}$$

$$= \pi\kappa_1\mu_1\int_0^{\infty}\frac{1}{2}U_p'R^2\,dR = -\pi\kappa_1\mu_1\int_0^{\infty}\hat{U}_pR\,dR = -\pi\kappa_1\mu_1 S_1. \tag{C.24}$$

The 4^{th} term -

$$(\eta_d W'''(U_p)L_p^{-2}(W'(U_p))\vec{\kappa}\cdot\nabla_z U_p, \partial_{z_1}U_p)_{L_R} = \pi\eta_d\kappa_1\int_0^{\infty}W'''(U_p)(U_p')^2L_p^{-2}(W'(U_p))R\,dR \tag{C.25}$$

$$= \pi\eta_d\kappa_1\int_0^{\infty}L_p^2(\frac{1}{2}RU_p')L_p^{-2}(W'(U_p))R\,dR \tag{C.26}$$

$$= \pi\eta_d\kappa_1\int_0^{\infty}\frac{1}{2}U_p'W'(U_p)R^2\,dR \tag{C.27}$$

$$= \pi\eta_d\kappa_1\int_0^{\infty}\frac{1}{2}U_p'(U_p'' + \frac{1}{R}U_p')R^2\,dR \tag{C.28}$$

$$= \pi\eta_d\kappa_1\int_0^{\infty}\frac{1}{2}U_p'U_p''R^2\,dR + \pi\eta_d\kappa_1\int_0^{\infty}\frac{1}{2}(U_p')^2R\,dR \tag{C.29}$$

$$= \pi\eta_d\kappa_1\int_0^{\infty}\frac{1}{2}(\frac{1}{2}(U_p')^2)'R^2\,dR + \frac{\pi\eta_d\kappa_1}{2}S_2 \tag{C.30}$$

$$= -\pi\eta_d\kappa_1\int_0^{\infty}\frac{1}{2}(U_p')^2R\,dR + \frac{\pi\eta_d\kappa_1}{2}S_2 \tag{C.31}$$

$$= -\frac{\pi\eta_d\kappa_1}{2}S_4 + \frac{\pi\eta_d\kappa_1}{2}S_4 = 0 \tag{C.32}$$

Using the fact that $W'(U_p) = U_p'' + \frac{1}{R}U_p'$. And the inner product of the last term yields

$$(\eta_1\vec{\kappa}\cdot\nabla_z U_p, \partial_{z_1}U_p)_{L_R} = \pi\eta_1\kappa_1\int_0^{\infty}(U_p')^2R\,dR = \pi\eta_1\kappa_1 S_4 \tag{C.33}$$

we can summarize it

$$(\mathcal{Q}_1, \partial_{z_i}U_p)_{L_R} = -2\pi\tilde{B}_1\kappa_i S_1 - \eta_1\pi\kappa_i S_4. \tag{C.34}$$

C.3 Appendix : Useful Identities

The following basic trigonometric identities may come useful

$$\cos(\theta + \pi) = -\cos(\theta), \tag{C.35}$$

$$\sin(\theta + \pi) = -\sin(\theta). \tag{C.36}$$

The following are useful operator identities for $L_{p,m}$. Recall that the spaces $\mathcal{Z}_m$ are orthogonal. We first calculate $L_p(\frac{1}{2}RU_p')$

$$L_p(\frac{1}{2}RU_p') = \frac{1}{2}L_{p,0}(RU_p') = \frac{1}{2}\left(2U_p'' + RU_p''' + \frac{U_p'}{R} + U_p'' - RW''(U_p)U_p'\right) \tag{C.37}$$

$$= \frac{1}{2}\left(2U_p'' + 2\frac{U_p'}{R} + \overbrace{RU_p''' + U_p'' - \frac{U_p'}{R} - RW''(U_p)U_p'}^{=RL_{p,1}U_p'=0}\right) \tag{C.38}$$

This yields the first identity

$$L_p(\frac{1}{2}RU_p') = \frac{1}{2}L_{p,0}(RU_p') = \Delta_z U = (U_p'' + \frac{U_p'}{R}) \tag{C.39}$$

To obtain the next identity, we differential equation (2.75) twice w.r.t R

$$U_p^{(4)} + \frac{1}{R}U_p''' - 2\frac{U_p''}{R^2} + \frac{2U_p'}{R^3} - W''(U_p)U_p'' = W'''(U_p)(U_p)^2 \tag{C.40}$$

Next we calculate $L_p(\Delta_z U_p)$

$$L_p(U_p'' + \frac{U_p'}{R}) = L_{p,0}(U_p'' + \frac{U_p'}{R}) = \boxed{U_p^{(4)} + \frac{1}{R}U_p''' - 2\frac{U_p''}{R^2} + \frac{2U_p'}{R^3} - W''(U_p)U_p''} + \boxed{\frac{1}{R}\left(U_p''' + \frac{U_p''}{R} - \frac{U_p'}{R^2} - W''(U_p)U_p'\right)} \tag{C.41}$$

using identity (C.40) we see that first boxed terms sum up to $W'''(U_p)(U_p)^2$, and the second boxed terms sum up to $\frac{1}{R}L_{p,1}U_p' = 0$. This yields the second identity

$$L_p(\Delta_z U_p) = L_{p,0}(U_p'' + \frac{U_p'}{R}) = W'''(U_p)|\nabla U_p|^2 = W'''(U_p)(U_p')^2. \tag{C.42}$$

Appendix D

Pearling Co-dimension 1

D.1 Calculations of the expansion of $\mathbb{L}_b$

Consider the 2^{nd} variation of $\mathcal{F}$

$$\mathbb{L}_b := \frac{\delta^2 \mathcal{F}}{\delta u^2} = \left(\varepsilon^2\Delta - W''(u) + \varepsilon\eta_1\right)\left(\varepsilon^2\Delta - W''(u)\right) - \left(\varepsilon^2\Delta u - W'(u)\right)W'''(u) + \varepsilon\eta_d W''(u). \tag{D.1}$$

Using the expansion of u_b, given in (5.5), and considering the Taylor expansion of $W(u_b)$ and its derivatives, $\mathbb{L}_b$ takes the form

$$\begin{aligned}
\mathbb{L}_b = &\Big[\varepsilon^2\Delta - W''(U_b) - \varepsilon W'''(U_b)u_1 - \varepsilon^2\left(W'''(U_b)u_2 + \frac{1}{2}W^{(4)}(U_b)u_1^2\right) + \varepsilon\eta_1\Big]\cdot \\
&\Big[\varepsilon^2\Delta - W''(U_b) - \varepsilon W'''(U_b)u_1 - \varepsilon^2\left(W'''(U_b)u_2 + \frac{1}{2}W^{(4)}(U_b)u_1^2\right)\Big] - \\
&\Big[\varepsilon^2\Delta U_b + \varepsilon^3\Delta u_1 - W'(U_b) - \varepsilon W''(U_b)u_1 - \varepsilon^2\left(W''(U_b)u_2 + \frac{1}{2}W'''(U_b)u_1^2\right)\Big]\cdot \\
&\Big[W'''(U_b) + \varepsilon W^{(4)}(U_b)u_1\Big] + \\
&\varepsilon\eta_d\Big[W''(U_b) + \varepsilon W'''(U_b)u_1 + \varepsilon^2\left(W'''(U_b)u_2 + \frac{1}{2}W^{(4)}(U_b)u_1^2\right)\Big] + \text{higher order terms.}
\end{aligned} \tag{D.2}$$

Note that $\varepsilon^2\Delta U_b - W'(U_b) = \varepsilon H U_b' + \varepsilon^2 \cancelto{0}{\Delta_G U_b}$ since U_b is the homoclinic solution and it is independent of s. Using the definition of the full operator $\mathcal{L}_b$, defined in (2.45), equation (D.2) reduces to

$$\begin{aligned}
\mathbb{L}_b = &\Big[\mathcal{L}_b - \varepsilon W'''(U_b)u_1 - \varepsilon^2\left(W'''(U_b)u_2 + \frac{1}{2}W^{(4)}(U_b)u_1^2\right) + \varepsilon\eta_1\Big]\cdot \\
&\Big[\mathcal{L}_b - \varepsilon W'''(U_b)u_1 - \varepsilon^2\left(W'''(U_b)u_2 + \frac{1}{2}W^{(4)}(U_b)u_1^2\right)\Big] -
\end{aligned} \tag{D.3}$$

$$\varepsilon\Big[HU_b' + \varepsilon^2\Delta u_1 - W''(U_b)u_1 - \varepsilon\left(W''(U_b)u_2 + \frac{1}{2}W'''(U_b)u_1^2\right)\Big]\cdot\Big[W'''(U_b) + \varepsilon W^{(4)}(U_b)u_1\Big] +$$
$$\varepsilon\eta_d\Big[W''(U_b) + \varepsilon W'''(U_b)u_1 + \varepsilon^2\left(W'''(U_b)u_2 + \frac{1}{2}W^{(4)}(U_b)u_1^2\right)\Big] + \text{higher order terms.}$$

Collecting $\mathbb{L}_b$ in orders of ε yields

$$\begin{aligned}\mathbb{L}_b =& \mathcal{L}_b^2 - \\ &\varepsilon\Big[\mathcal{L}_b\circ(W'''(U_b)u_1) + (W'''(U_b)u_1 + \eta_1)\mathcal{L}_b + (\mathcal{L}_b u_1 + HU_b')\,W'''(U_b) - \eta_d W''(U_b)\Big] - \\ &\varepsilon^2\Big[\mathcal{L}_b\circ\left(W'''(U_b)u_1 + \frac{1}{2}W^{(4)}(U_b)u_1^2\right) + \left(W'''(U_b)u_1 + \frac{1}{2}W^{(4)}(U_b)u_1^2\right)\mathcal{L}_b \\ &\quad - (W'''(U_b)u_1 + \eta_1)W'''(U_b)u_1 + \left(\mathcal{L}_b u_2 - \frac{1}{2}W'''(U_b)u_1^2\right)W'''(U_b) \\ &\quad + (\mathcal{L}_b u_1 + HU_b')W^{(4)}(U_b)u_1 + \eta_d W'''(U_b)u_1\Big] \\ &+ O(\varepsilon^3).\end{aligned} \tag{D.4}$$

D.2 Calculating M

$$(\mathcal{L}_b^2\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} = (\mathcal{L}_b\psi_0\Theta_j, \mathcal{L}_b\psi_0\Theta_k)_{L^2(\Omega} \tag{D.5}$$

$$= \int_\Gamma\int_{-l/\varepsilon}^{l/\varepsilon}(L_{b,0} + \varepsilon H\partial_z + \varepsilon^2\Delta_G)\psi_0\Theta_j(L_{b,0} + \varepsilon H\partial_z + \varepsilon^2\Delta_G)\psi_0\Theta_k\, J\, dzds \tag{D.6}$$

$$= \int_\Gamma\int_{-l/\varepsilon}^{l/\varepsilon} L_{b,0}\psi_0\Theta_j L_{b,0}\psi_0\Theta_k\, J\, dzds \tag{D.7}$$

$$\begin{aligned}&+ \varepsilon\left(\int_\Gamma\int_{-l/\varepsilon}^{l/\varepsilon} L_{b,0}\psi_0\Theta_j H\partial_z(\psi_0\Theta_k)\, J\, dzds + \int_\Gamma\int_{-l/\varepsilon}^{l/\varepsilon} H\partial_z(\psi_0\Theta_j)L_{b,0}(\psi_0\Theta_k)\, J\, dzds\right) + \\ &\varepsilon^2\Bigg(\int_\Gamma\int_{-l/\varepsilon}^{l/\varepsilon} L_{b,0}\psi_0\Theta_j\Delta_G(\psi_0\Theta_k)\, J\, dzds + \int_\Gamma\int_{-l/\varepsilon}^{l/\varepsilon}\Delta_G(\psi_0\Theta_j)L_{b,0}(\psi_0\Theta_k)\, J\, dzds \\ &\quad + \int_\Gamma\int_{-l/\varepsilon}^{l/\varepsilon} H\partial_z(\psi_0\Theta_j)H\partial_z(\psi_0\Theta_k)\, J\, dzds\Bigg) \\ &+ O(\varepsilon^3)\end{aligned}$$

$$= \varepsilon P_k P_j\int_\Gamma\Theta_k\Theta_j\, J_0\, ds\int_{-l/\varepsilon}^{l/\varepsilon}(\psi_0^0)^2 dz + \varepsilon\sqrt{\varepsilon}(P_j + P_k)\int_\Gamma\int_{-l/\varepsilon}^{l/\varepsilon}\Theta_j\Theta_k H\psi_0'\psi_0\, J\, dzds + \tag{D.8}$$

$$\begin{aligned}&- \varepsilon^2\sqrt{\varepsilon}\varepsilon^2(P_j\beta_k + P_k\beta_j)\int_\Gamma\int_{-l/\varepsilon}^{l/\varepsilon}\Theta_j\Theta_k\psi_0^2\, J\, dzds \\ &+ \varepsilon^2\int_\Gamma\int_{-l/\varepsilon}^{l/\varepsilon} H^2\Theta_j\Theta_k(\psi_0')^2\, J\, dzds + O(\varepsilon^3)\end{aligned}$$

since $\psi_0 = \psi_0^0 \tilde{J}^{-1/2}$, we have

$$\psi_0' = (\psi_0^0)' \tilde{J}^{-1/2} + \psi_0^0 (\tilde{J}^{-1/2})' = (\psi_0^0)' \tilde{J}^{-1/2} - \frac{1}{2} \psi_0^0 (\tilde{J}^{-3/2}) \tilde{J}' \tag{D.9}$$

and using the identity $\tilde{J}' = \varepsilon H \tilde{J}$ we get

$$\psi_0' = (\psi_0^0)' \tilde{J}^{-1/2} - \frac{1}{2} \psi_0^0 (\tilde{J}^{-3/2}) \varepsilon H \tilde{J} = (\psi_0^0)' \tilde{J}^{-1/2} - \frac{1}{2} \varepsilon H \psi_0^0 (\tilde{J}^{-1/2}) = (\psi_0^0)' \tilde{J}^{-1/2} - \frac{1}{2} \varepsilon H \psi_0 \tag{D.10}$$

and

$$(\psi_0')^2 = ((\psi_0^0)')^2 \tilde{J}^{-1} - \varepsilon H (\psi_0^0)' \psi_0^0 \tilde{J}^{-1} + \frac{1}{4} \varepsilon^2 H^2 (\psi_0^0)^2 (\tilde{J}^{-1}) \tag{D.11}$$

$$(\mathcal{L}_b^2 \psi_0 \Theta_j, \psi_0 \Theta_k)_{L^2(\Omega)} \tag{D.12}$$

$$= \varepsilon P_k P_j \int_\Gamma \Theta_k \Theta_j \, J_0 \, ds \int_{-l/\varepsilon}^{l/\varepsilon} (\psi_0^0)^2 dz + \tag{D.13}$$

$$\varepsilon\sqrt{\varepsilon} P_j \int_\Gamma H \Theta_k \Theta_j j_0 \, ds \cancelto{0}{\int_{-l/\varepsilon}^{l/\varepsilon} (\psi_0^0)' \psi_0^0 \, dz} - \frac{1}{2} \varepsilon^2 \sqrt{\varepsilon} P_j \int_\Gamma H^2 \Theta_k \Theta_j \, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} (\psi_0^0)^2 \, dz +$$

$$\varepsilon\sqrt{\varepsilon} P_k \int_\Gamma H \Theta_k \Theta_j j_0 \, ds \cancelto{0}{\int_{-l/\varepsilon}^{l/\varepsilon} (\psi_0^0)' \psi_0^0 \, dz} - \frac{1}{2} \varepsilon^2 \sqrt{\varepsilon} P_k \int_\Gamma H^2 \Theta_k \Theta_j \, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} (\psi_0^0)^2 \, dz +$$

$$- \varepsilon^2 \sqrt{\varepsilon} (P_j \varepsilon^2 \beta_k + P_k \varepsilon^2 \beta_j) \int_\Gamma \Theta_j \Theta_k \, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} (\psi_0^0)^2 \, dz + \varepsilon^2 \int_\Gamma H^2 \Theta_j \Theta_k \, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} ((\psi_0^0)')^2 \, dz$$

$$- \varepsilon^3 \int_\Gamma H^3 \Theta_j \Theta_k \, j_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} (\psi_0^0)' \psi_0^0 \, dz + \frac{1}{4} \varepsilon^4 \int_\Gamma H^4 \Theta_j \Theta_k \, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} (\psi_0^0)^2 \, dz + O(\varepsilon^3)$$

$$= \begin{cases} \varepsilon P_k^2 + \varepsilon^2 \int_\Gamma (H\theta_k)^2 \, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} ((\psi_0^0)')^2 \, dz + O(\varepsilon^2\sqrt{\varepsilon}) & \text{if } k = j, \\ \varepsilon^2 \int_\Gamma H^2 \theta_k \Theta_j \, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} ((\psi_0^0)')^2 \, dz + O(\varepsilon^2\sqrt{\varepsilon}) & \text{if } k \neq j \end{cases}$$

Calculating M_1 yields

$$\begin{aligned} (\mathbb{L}_1 \psi_0 \Theta_j, \psi_0 \Theta_k)_{L^2(\Omega)} = & - \int_\Gamma \int_{-l/\varepsilon}^{l/\varepsilon} (W'''(U_b) u_1) \psi_0 \Theta_j \mathcal{L}_b \psi_0 \Theta_k \, J dz ds \\ & - \int_\Gamma \int_{-l/\varepsilon}^{l/\varepsilon} (W'''(U_b) u_1 - \eta_1) \psi_0 \Theta_k \mathcal{L}_b (\psi_0 \Theta_j) \, J dz ds \\ & - \int_\Gamma \int_{-l/\varepsilon}^{l/\varepsilon} (H U_b' + \mathcal{L}_b u_1) W'''(U_b) \psi_0 \Theta_j \psi_0 \Theta_k \, J dz ds \\ & + \eta_d \int_\Gamma \int_{-l/\varepsilon}^{l/\varepsilon} W''(U_b) \psi_0 \Theta_j \psi_0 \Theta_k \, J dz ds \end{aligned} \tag{D.14}$$

$$\begin{aligned} = & - \sqrt{\varepsilon} P_k \int_\Gamma \int_{-l/\varepsilon}^{l/\varepsilon} W'''(U_b) u_1 \Theta_j \Theta_k \psi_0^2 \, J dz ds \\ & - \sqrt{\varepsilon} P_j \int_\Gamma \int_{-l/\varepsilon}^{l/\varepsilon} (W'''(U_b) u_1 - \eta_1) \Theta_k \Theta_j \psi_0^2 \, J dz ds \end{aligned} \tag{D.15}$$

$$
\begin{aligned}
&- \int_\Gamma \int_{-l/\varepsilon}^{l/\varepsilon} H U_b' W'''(U_b)\Theta_j\Theta_k\psi_0^2\, J dz ds \\
&- \int_\Gamma \int_{-l/\varepsilon}^{l/\varepsilon} \mathcal{L}_b(u_1) W'''(U_b)\Theta_j\Theta_k\psi_0^2\, J dz ds \\
&+ \eta_d \int_\Gamma \int_{-l/\varepsilon}^{l/\varepsilon} W''(U_b)\Theta_j\Theta_k\psi_0^2\, J dz ds
\end{aligned}
$$

using $\psi_0 = \tilde{J}^{-1/2}\psi_0^0$ we get

$$
\begin{aligned}
(\mathbb{L}_1\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} = &- \sqrt{\varepsilon} P_k \int_\Gamma \Theta_k\Theta_j\, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} W'''(U_b) u_1 (\psi_0^0)^2\, dz && \text{(D.16)} \\
&- \sqrt{\varepsilon} P_j \int_\Gamma \Theta_k\Theta_j\, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} (W'''(U_b)u_1 - \eta_1)(\psi_0^0)^2\, dz \\
&- \int_\Gamma H_0\Theta_k\Theta_j\, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} U_b' W'''(U_b)(\psi_0^0)^2\, dz \to 0 && \text{(D.17)} \\
&- \varepsilon \int_\Gamma H_1\Theta_k\Theta_j\, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} U_b' W'''(U_b)(\psi_0^0)^2 z\, dz \\
&- \int_\Gamma \Theta_j\Theta_k\, j_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} \mathcal{L}_b(u_1) W'''(U_b)(\psi_0^0)^2\, dz && \text{(D.18)} \\
&+ \eta_d \int_\Gamma \Theta_j\Theta_k\, J_0 ds \int_{-l/\varepsilon}^{l/\varepsilon} W''(U_b)(\psi_0^0)^2\, dz + O(\varepsilon^3)
\end{aligned}
$$

$$
= \begin{cases}
- \int_{-l\setminus\varepsilon}^{l\setminus\varepsilon} W'''(U_b)(\psi_0^0)^2 L_{b,0} u_1\, dz + \eta_d \int_{-l\setminus\varepsilon}^{l\setminus\varepsilon} W''(U_b)(\psi_0^0)^2\, dz & \\
+ \sqrt{\varepsilon} P_k(\eta_1 - 2) - \varepsilon \int_\Gamma H_1\Theta_k^2\, ds \int_{-l\setminus\varepsilon}^{l\setminus\varepsilon} W'''(U_b) U_b'(\psi_0^0)^2 z\, dz + O(\varepsilon^2) & \text{if } k = j, \\
- \varepsilon \int_\Gamma H_1\Theta_k\Theta_j\, ds \int_{-l\setminus\varepsilon}^{l\setminus\varepsilon} W'''(U_b) U_b'(\psi_0^0)^2 z\, dz + O(\varepsilon^3) & \text{if } k \neq j
\end{cases} \qquad \text{(D.19)}
$$

D.3 Simplifying the expression for $M_{k,k}$

Recall that, $M_{k,k}$ is given by

$$
M_{k,k}^0 = P_k^2 - \int_{-l\setminus\varepsilon}^{l\setminus\varepsilon} \Big[W'''(U_b) L_{b,0} u_1 - \eta_d W''(U_b) \Big] (\psi_0^0)^2\, dz. \qquad \text{(D.20)}
$$

Using the following identities

$$
L_{b,0} u_1 = \mu_1 \Phi_1 - \eta_d \left(\frac{z}{2} U' \right), \qquad \text{(D.21)}
$$

$$
W''(U)\psi_0 = \psi_0'' - L_{b,0}\psi_0 = \psi_0'' - \lambda_0\psi_0, \qquad \text{(D.22)}
$$

we get

$$\eta_d \int W''(U)\psi_0^2 - \int (\mu_1\varphi_1 - \eta_d\frac{z}{2}U')W'''(U)\psi_0^2 = \eta_d \int W''(U)\psi_0^2 \tag{D.23}$$

$$- \overbrace{\int \mu_1\varphi_1 W'''(U)\psi_0^2}^{\mu_1 S_b} - \int \eta_d\frac{z}{2}U'W'''(U)\psi_0^2$$

$$= -\mu_1 S_b + \eta_d \int W''(U)\psi_0^2 - \int \eta_d\frac{z}{2}(W''(U))'\psi_0^2. \tag{D.24}$$

Integrating by parts we have

$$\eta_d \int W''(U)\psi_0^2 - \int (\mu_1\varphi_1 - \eta_d\frac{z}{2}U')W'''(U)\psi_0^2 = -\mu_1 S_b \tag{D.25}$$

$$+ \eta_d \int \left(W''(U)\psi_0^2 - \frac{1}{2}W''(U)\psi_0^2 - W''(U)z\psi\psi'\right)$$

$$= -\mu_1 S_b + \eta_d \int \left(\frac{1}{2}W''(U)\psi_0^2 - W''(U)z\psi_0\psi_0'\right) \tag{D.26}$$

$$= -\mu_1 S_b + \eta_d \int \left(\frac{1}{2}\psi_0 - z\psi_0')\right)W''(U)\psi_0 \tag{D.27}$$

Using identity (D.22) yields

$$\eta_d \int W''(U)\psi_0^2 - \int (\mu_1\varphi_1 - \eta_d\frac{z}{2}U')W'''(U)\psi_0^2 = -\mu_1 S_b + \eta_d \int (\frac{1}{2}\psi_0 - z\psi_0')(\psi_0'' - \lambda_0\psi_0) \tag{D.28}$$

$$= -\mu_1 S_b + \eta_d \int \left(\frac{1}{2}\psi_0\psi_0'' - \frac{1}{2}\lambda_0\psi_0^2 - z\psi_0'\psi_0'' + \lambda_0 z\psi_0\psi_0'\right) \tag{D.29}$$

$$= -\mu_1 S_b + \eta_d \int \frac{1}{2}\psi_0\psi_0'' - \frac{1}{2}\lambda_0\eta_d \int \psi_0^2 \tag{D.30}$$

$$- \eta_d \int z\psi_0'\psi_0'' + \lambda_0\eta_d \int z\psi_0\psi_0'$$

Using integration by parts we can show that

$$\int z\psi_0'\psi_0'' = -\frac{1}{2}\|\psi_0'\|_2^2, \qquad \int z\psi_0\psi_0' = -\frac{1}{2}\|\psi_0\|_2^2 \tag{D.31}$$

therefore,

$$\eta_d \int W''(U)\psi_0^2 - \int (\mu_1\varphi_1 - \eta_d\frac{z}{2}U')W'''(U)\psi_0^2 = -\mu_1 S_b + \eta_d \int \frac{1}{2}\psi_0\psi_0'' - \frac{1}{2}\lambda_0\eta_d \int \psi_0^2 \tag{D.32}$$

$$- \eta_d \int z\psi_0'\psi_0'' + \lambda_0\eta_d \int z\psi_0\psi_0'$$

$$= -\mu_1 S_b - \frac{1}{2}\eta_d \|\psi_0'\|_2^2 - \frac{1}{2}\lambda_0\eta_d\|\psi_0\|_2^2 + \frac{1}{2}\eta_d\|\psi_0'\|_2^2 - \frac{1}{2}\lambda_0\eta_d\|\psi_0\|_2^2 \tag{D.33}$$

$$= -\mu_1 S_b - \lambda_0\|\psi_0\|_2^2\eta_d. \tag{D.34}$$

Hence, the diagonal terms $M_{k,k}$ take the form

$$M_{k,k} = P_k^2 - \mu_1 S_b - \eta_d\lambda_0\|\psi_0\|_2^2. \tag{D.35}$$

D.4 Useful identities and inequalities

Theorem D.4.1 (Hölder's Inequality)**.** *Suppose* $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $1 \le p, q, r \le \infty$. *Then*

$$\|fg\|_1 \le \|f\|_p\|g\|_q. \tag{D.36}$$

Theorem D.4.2 (Generalized Hölder's Inequality)**.** *Suppose* $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ *with* $1 \le p, q, r \le \infty$. *Then*

$$\|fg\|_r \le \|f\|_p\|g\|_q. \tag{D.37}$$

1. An expression for the derivative of ψ_0

$$\psi_0' = (\psi_0^0)'\tilde{J}^{-1/2} - \varepsilon H\psi_0 \tag{D.38}$$

2.

$$L_{b,0}z\psi_0 = z\lambda_0\psi_0 + 2\psi_0' \tag{D.39}$$

3.

$$L_{b,0}\psi_0' = \lambda_0\psi_0' - \psi_0''' + W'''(U)U'\psi_0 \tag{D.40}$$

4.

$$L_{b,0}(z\psi_0') = zL_{b,0}\psi_0' + 2\psi_0'' = z(\lambda_0 - \partial_z^2)\psi_0' + W'''(U)U'\psi_0 + 2\psi_0''. \tag{D.41}$$

5.

$$L_{b,0}(\psi_0 D_{s,2}\Theta_k) = \lambda_0\psi_0 D_{s,2}\Theta_k + 2\psi_0'(D_{s,2})_z\Theta_k + \psi_0(D_{s,2})_{zz}\Theta_k \tag{D.42}$$

Appendix E

Pearling Co-dimension 2

E.1 Appendix : Self-adjoint operators

Consider the $L^2(\Omega)$ inner product defined in (2.73) and the two operators $\tilde{\mathcal{L}}_p := \partial_z^2 - W''(u) + \varepsilon^2 \Delta_s$ and $\mathcal{L}_p := \partial_z^2 + \varepsilon\kappa\partial_z - W''(u) + \varepsilon^2\Delta_G$. The co-dimeanion two Laplacian operator is **not** self-adjoint in this inner product,

$$(\Delta_s f, g)_{L^2(\Omega)} = \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} g J_0^{-1} \nabla_s \cdot (\boldsymbol{g}^{-1} J_0 \nabla_s) f \, J_0 \tilde{J} \, dz \, ds = - \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} \boldsymbol{g}^{-1} J_0 \nabla_s (g\tilde{J}) \cdot \nabla_s f \, dz \, ds \tag{E.1}$$

$$= \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} \nabla_s (\boldsymbol{g}^{-1} J_0 \nabla_s (g\tilde{J})) f \, dz \, ds \neq (f, \Delta_s g)_{L^2(\Omega)} \tag{E.2}$$

although it is self-adjoint in the Γ-inner product. However, Δ_G is self-adjoint in the $L^2(\Omega)$ inner product.

$$(\Delta_G f, g)_{L^2(\Omega)} = \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} g J^{-1} \nabla_s \cdot (\boldsymbol{g}^{-1} J \nabla_s) f \, J \, dz \, ds = - \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} \boldsymbol{g}^{-1} J \nabla_s (g) \cdot \nabla_s f \, dz \, ds \tag{E.3}$$

$$= \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} \nabla_s (\boldsymbol{g}^{-1} J \nabla_s (g)) f \, dz \, ds = (f, \Delta_G g)_{L^2(\Omega)}. \tag{E.4}$$

As for the rest of the terms in the operator,

$$(\partial_z^2 f, g)_{L^2(\Omega)} = \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f'' g J \, dz ds = \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f (gJ)'' \, dz ds = \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f g'' J + 2 f g' J' + f g J'' \, dz ds \tag{E.5}$$

$$= (f, \partial_z^2 g)_{L^2(\Omega)} + 2\varepsilon \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f g' \kappa J \, dz ds + \varepsilon \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f g \kappa' J \, dz ds + \varepsilon^2 \int_\Gamma \int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f g \kappa^2 J \, dz ds, \tag{E.6}$$

$$\varepsilon(\kappa\partial_z f, g)_{L^2(\Omega)} = \varepsilon\int_\Gamma\int_{-l\backslash\varepsilon}^{l\backslash\varepsilon}\kappa f' g J\,dzds = \varepsilon\int_\Gamma\int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f(gJ\kappa)'\,dzds = -\int_\Gamma\int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} f\varepsilon\kappa g' J + \varepsilon f g J\kappa' + fg\varepsilon\kappa J'\,dzds \tag{E.7}$$

$$= -\varepsilon(f, \kappa\partial_z g)_{L^2(\Omega)} - \varepsilon\int_\Gamma\int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} fg\kappa' J\,dzds - \varepsilon^2\int_\Gamma\int_{-l\backslash\varepsilon}^{l\backslash\varepsilon} fg\kappa^2 J\,dzds, \tag{E.8}$$

we see that each of the terms separately is not self-adjoint in the $L^2(\Omega)$-inner product, but the sum of them is

$$(\partial_z^2 f + \varepsilon\kappa\partial_z f, g)_{L^2(\Omega)} = (f, \partial_z^2 g + \varepsilon\kappa\partial_z g)_{L^2(\Omega)}.$$

Therefore, our full operator $\mathcal{L}_p$ is self-adjoint in the $L^2(\Omega)$-inner product where the $\tilde{\mathcal{L}}_p$ operator is not.

E.2 Calculations of the expansion of $\mathbb{L}$

We consider the 2^{nd} variation of $\mathcal{F}$

$$\mathbb{L}_p := \frac{\delta^2\mathcal{F}}{\delta u^2} = \left(\varepsilon^2\Delta - W''(u) + \varepsilon\eta_1\right)\left(\varepsilon^2\Delta - W''(u)\right) - \left(\varepsilon^2\Delta u - W'(u)\right)W'''(u) + \varepsilon\eta_d W''(u). \tag{E.9}$$

Using the expansion of the Laplacian in local coordinates, given in (2.65), writing u_p using (6.5) and Taylor expand $W(u_p)$ and its derivatives.

$$\begin{aligned}
\mathbb{L}_p =& \Big[\Delta_z - W''(U_p) - \varepsilon D_z + \varepsilon^2\partial_G^2 - \varepsilon\left(W'''(U_p)u_1 - \eta_1\right) - \varepsilon^2\left(W'''(U_p)u_2 + \frac{1}{2}W^{(4)}(U_p)u_1^2\right)\Big]\circ \\
&\Big[\Delta_z - W''(U_p) - \varepsilon D_z + \varepsilon^2\partial_G^2 - \varepsilon\left(\vec{\kappa}\nabla_z + W'''(U_p)u_1\right) - \varepsilon^2\left(W'''(U_p)u_2 + \frac{1}{2}W^{(4)}(U_p)u_1^2\right)\Big] - \\
&\Big[\Delta_z U_p - W'(U_p) - \varepsilon D_z U_p + \varepsilon\Delta_z u_1 - \varepsilon^2 D_z u_1 + \varepsilon^2\partial_G^2 U_p - \varepsilon W''(U_p)u_1 - \varepsilon^2\left(W''(U_p)u_2 + \frac{1}{2}W'''(U_p)u_1^2\right)\Big]\circ \\
&\Big[W'''(U_p) + \varepsilon W^{(4)}(U_p)u_1\Big] + \\
&\varepsilon\eta_d\Big[W''(U_p) + \varepsilon W'''(U_p)u_1 + \varepsilon^2\left(W'''(U_p)u_2 + \frac{1}{2}W^{(4)}(U_p)u_1^2\right)\Big] + \text{higher order terms},
\end{aligned} \tag{E.10}$$

Recall that $\mathcal{L}_p := L_p - \varepsilon D_z + \varepsilon^2\partial_G^2$, L_p is defined in (2.78), and U_p is the radial symmetric solution of equation (2.75). Then,

$$\begin{aligned}
\mathbb{L}_p =& \Big[\mathcal{L}_p - \varepsilon\left(W'''(U_p)u_1 - \eta_1\right) - \varepsilon^2\left(W'''(U_p)u_2 + \frac{1}{2}W^{(4)}(U_p)u_1^2\right)\Big]\circ \\
&\Big[\mathcal{L}_p - \varepsilon\left(W'''(U_p)u_1\right) - \varepsilon^2\left(W'''(U_p)u_2 + \frac{1}{2}W^{(4)}(U_p)u_1^2\right)\Big] - \\
&\Big[-\varepsilon D_z U_p + \varepsilon\mathcal{L}_p u_1 - \varepsilon^2\left(W''(U_p)u_2 + \frac{1}{2}W'''(U_p)u_1^2\right)\Big]\circ\Big[W'''(U_p) + \varepsilon W^{(4)}(U_p)u_1\Big] +
\end{aligned} \tag{E.11}$$

$$\varepsilon\eta_d\Big[W''(U_p)+\varepsilon W'''(U_p)u_1+\varepsilon^2\left(W'''(U_p)u_2+\frac{1}{2}W^{(4)}(U_p)u_1^2\right)\Big]+\text{higher order terms},$$

Rewriting $\mathbb{L}_p$ in orders of ε we have

$$\mathbb{L}_p = \mathcal{L}_p^2 + \varepsilon\mathbb{L}_1 + O(\varepsilon^2), \tag{E.12}$$

where

$$\mathbb{L}_1 := -\left(W'''(U_p)u_1-\eta_1\right)\circ\mathcal{L}_p - \mathcal{L}_p\circ\left(W'''(U_p)u_1\right) - \left(\mathcal{L}_p u_1 - D_z U_p\right)W'''(U_p) + \eta_d W''(U_p). \tag{E.13}$$

E.3 Calculating M^0

To obtain an explicit expression for M^0 we calculate the inner products given in (6.22). We start with the inner product involving $\mathcal{L}_p^2$: Recall that $\mathcal{L}_p$, given in (2.77), is self-adjoint in the $L^2(\Omega)$ inner product. Then

$$(\mathcal{L}_p^2\psi_0\Theta_j,\psi_0\Theta_k)_{L^2(\Omega)} = (\mathcal{L}_p\psi_0\Theta_j,\mathcal{L}_p\psi_0\Theta_k)_{L^2(\Omega)} \tag{E.14}$$

$$= (((L+\varepsilon^2\partial_s^2)-\varepsilon\vec{\kappa}\cdot\nabla_z-\varepsilon^2(z\cdot\vec{\kappa})\vec{\kappa}\cdot\nabla_z)\psi_0\Theta_j,((L+\varepsilon^2\partial_s^2)-\varepsilon\vec{\kappa}\cdot\nabla_z-\varepsilon^2(z\cdot\vec{\kappa})\vec{\kappa}\cdot\nabla_z)\psi_0\Theta_k)_{L^2(\Omega)} \tag{E.15}$$

$$+O(\varepsilon^3)$$

$$= ((L+\varepsilon^2\partial_s^2)\psi_0\Theta_j,(L+\varepsilon^2\partial_s^2)\psi_0\Theta_k)_{L^2(\Omega)} \tag{E.16}$$

$$-\varepsilon\left(((L+\varepsilon^2\partial_s^2)\psi_0\Theta_j,\vec{\kappa}\cdot\nabla_z\psi_0\Theta_k)_{L^2(\Omega)}+(\vec{\kappa}\cdot\nabla_z\psi_0\Theta_j,(L+\varepsilon^2\partial_s^2)\psi_0\Theta_k)_{L^2(\Omega)}\right)+O(\varepsilon^2)$$

$$= \varepsilon P_k P_j(\psi_0\Theta_j,\psi_0\Theta_k)_{L^2(\Omega)} - \varepsilon\sqrt{\varepsilon}(P_j+P_k)(\Theta_j\Theta_k\psi_0,\vec{\kappa}\cdot\nabla_z\psi_0)_{L^2(\Omega)}+O(\varepsilon^2) \tag{E.17}$$

where for the second equality we used the expansion of the Laplacian, given in (**??**). Since psi_0 is a radial function we have

$$(\Theta_j\Theta_k\psi_0,\vec{\kappa}\cdot\nabla_z\psi_0)_{L^2(\Omega)} = \int_\Gamma \kappa\Theta_k\Theta_j\,ds\cdot\int_0^{l/\varepsilon}\psi_0\nabla_z\psi_0\,J\,dz \tag{E.18}$$

changing to polar coordinates, equation (E.18) takes the form

$$(\Theta_j\Theta_k\psi_0,\vec{\kappa}\cdot\nabla_z\psi_0)_{L^2(\Omega)} = \int_\Gamma \kappa\Theta_k\Theta_j\,ds\cdot\overbrace{\int_0^{2\pi}(\cos\theta,\sin\theta)\,d\theta}^{=(0,0)}\int_0^\infty\psi_0\psi_0'\,dz+O(\varepsilon) \tag{E.19}$$

That is, the term $\varepsilon\sqrt{\varepsilon}(P_j+P_k)(\Theta_j\Theta_k\psi_0,\vec{\kappa}\cdot\nabla_z\psi_0)_{L^2(\Omega)}$ is actually $O(\varepsilon^2\sqrt{\varepsilon})$ and negligible. Using the orthonormality of the co-dimeanion two Laplacianeigenmodes we can rewrite equation (E.17) in the following

way

$$(\mathcal{L}_p^2\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} = \begin{cases} \varepsilon P_k^2 + O(\varepsilon^2\sqrt{\varepsilon}) & \text{if } k = j, \\ O(\varepsilon^2\sqrt{\varepsilon}) & \text{if } k \neq j \end{cases} \tag{E.20}$$

The next term in (6.22) involves the $\mathbb{L}_1$ operator, given in (6.9).

$$(\mathbb{L}_1\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} = -\overbrace{((W'''(U_p)u_1 - \eta_1) \circ \mathcal{L}_p\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)}}^{(A)} - \overbrace{(\mathcal{L}_p \circ (W'''(U_p)u_1)\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)}}^{(B)} \tag{E.21}$$

$$- \overbrace{((\mathcal{L}_p u_1 - D_z U_p)\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)}}^{(C)} + \overbrace{(\eta_d W''(U_p)\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)}}^{(D)} \tag{E.22}$$

we calculate (E.21) term by term:

(A)

$$\begin{aligned} ((W'''(U_p)u_1 - \eta_1) \circ \mathcal{L}_p\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} =& \sqrt{\varepsilon}P_j((W'''(U_p)u_1 - \eta_1)\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} \\ & - \varepsilon((W'''(U_p)u_1 - \eta_1)\vec{\kappa} \cdot \nabla_z\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} + O(\varepsilon^2) \end{aligned} \tag{E.23}$$

$$= \begin{cases} \sqrt{\varepsilon}P_j((W'''(U_p)u_1 - \eta_1)\psi_0\Theta_k, \psi_0\Theta_k)_{L^2(\Omega)} + O(\varepsilon) & \text{if } k = j, \\ -\varepsilon((W'''(U_p)u_1 - \eta_1)\vec{\kappa} \cdot \nabla_z\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} + O(\varepsilon^2) & \text{if } k \neq j \end{cases} \tag{E.24}$$

(B)

$$\begin{aligned} (\mathcal{L}_p \circ (W'''(U_p)u_1)\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} =& ((W'''(U_p)u_1)\psi_0\Theta_j, \mathcal{L}_p\psi_0\Theta_k)_{L^2(\Omega)} \end{aligned} \tag{E.25}$$

$$= \begin{cases} \sqrt{\varepsilon}P_j(W'''(U_p)u_1\psi_0\Theta_k, \psi_0\Theta_k)_{L^2(\Omega)} + O(\varepsilon) & \text{if } k = j, \\ -\varepsilon(\psi_0\Theta_j, W'''(U_p)u_1\vec{\kappa} \cdot \nabla_z\psi_0\Theta_k)_{L^2(\Omega)} + O(\varepsilon^2) & \text{if } k \neq j \end{cases} \tag{E.26}$$

(C)

$$((\mathcal{L}_p u_1 - D_z U_p)\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} = \begin{cases} ((Lu_1 - (\vec{\kappa}\cdot\nabla_z U_p))W'''(U_p)\psi_0\Theta_k, \psi_0\Theta_k)_{L^2(\Omega)} + O(\varepsilon) & \text{if } k = j, \\ \boxed{-((\vec{\kappa}\cdot\nabla_z U_p)W'''(U_p)\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)}} & \text{if } k \neq j \\ -\varepsilon(\vec{\kappa}\cdot(\nabla_z(u_1 - U_p))W'''(U_p)\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} + O(\varepsilon^2) & \end{cases} \tag{E.27}$$

Recall that U_p, ψ_0^0 and $W'''(U_p)$ are all radial functions, and using (A.2), the calculation of the inner-product $((\vec{\kappa}\cdot\nabla_z U_p)W'''(U_p)\psi_0\Theta_k, \psi_0\Theta_k)_{L^2(\Omega)}$ (boxed) becomes

$$((\vec{\kappa}\cdot\nabla_z U_p)W'''(U_p)\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} = \int_\Gamma \int_{-\ell/\varepsilon}^{\ell/\varepsilon} (\vec{\kappa}\cdot\nabla_z U_p)W'''(U_p)(\psi_0^0)^2\Theta_j\Theta_k\, dz\, ds \tag{E.28}$$

$$= \int_\Gamma \int_0^{2\pi} \int_0^\infty (\vec{\kappa}\cdot(\cos\theta, \sin\theta)\partial_R U_p)W'''(U_p)(\psi_0^0)^2\Theta_j\Theta_k\, R dR\, d\theta\, ds \tag{E.29}$$

$$= \int_\Gamma \vec{\kappa}\Theta_j\Theta_k\, ds \cdot \overbrace{\int_0^{2\pi} (\cos\theta, \sin\theta)\, d\theta}^{=(0,0)} \int_0^\infty (\partial_R U_p)W'''(U_p)(\psi_0^0)^2\, R dR \tag{E.30}$$

$$= 0. \tag{E.31}$$

It follows from (E.28) that equation (E.27) reduces to

$$((\mathcal{L}_p u_1 - D_z U_p)\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} = \begin{cases} ((Lu_1)W'''(U_p)\psi_0\Theta_k, \psi_0\Theta_k)_{L^2(\Omega)} + O(\varepsilon) & \text{if } k = j, \\ -\varepsilon(\vec{\kappa}\cdot(\nabla_z(u_1 - U_p))W'''(U_p)\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} + O(\varepsilon^2) & \text{if } k \neq j \end{cases} \tag{E.32}$$

summarizing the calculation of each of the terms, and returning to (E.21) we see that

$$(\mathbb{L}_1\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} = \tag{E.33}$$

$$= \begin{cases} -(((Lu_1)W'''(U_p) - \eta_d W''(U_p))\psi_0\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} + O(\sqrt{\varepsilon}) & \text{if } k = j, \\ \varepsilon(((2W'''(U_p)u_1 - \eta_1)\vec{\kappa}\cdot\nabla_z\psi_0 - \vec{\kappa}\cdot(\nabla_z u_1 - U_p)W'''(U_p)\psi_0)\Theta_j, \psi_0\Theta_k)_{L^2(\Omega)} + O(\varepsilon^2) & \text{if } k \neq j \end{cases}$$

E.4 Simplifying the expression for $M_{k,k}$

Recall that, $M_{k,k}$ is given by

$$M_{k,k}^0 = P_k^2 - \int_0^{l\setminus\varepsilon} \Big[W'''(U_b)Lu_1 - \eta_d W''(U_b)\Big](\psi_0^0)^2\, dz, \tag{E.34}$$

where u_1 is defined in (6.6) and in polar coordinates, using the identity $L^{-1}(W'(U_p)) = L^{-1}(\Delta_z U_p) = \frac{1}{2} R U_p'$ takes the form

$$Lu_1 = \mu_1 \varphi_1 - \eta_d L^{-1}(W'(U_p)) = \mu_1 \varphi_1 - \eta_d \frac{1}{2} R U_p'. \tag{E.35}$$

For future calculations we have the following identity

$$W''(U_p)\psi_0^0 = (\psi_0^0)'' - L_0\psi_0^0 = (\psi_0^0)'' - \lambda_{p,0}\psi_0^0. \tag{E.36}$$

Since all the function in (E.34) are radial functions, we change to polar coordinates

$$M_{k,k}^0 = P_k^2 - 2\pi \int_0^\infty \left[W'''(U_b) L u_1 - \eta_d W''(U_b) \right] (\psi_0^0)^2 \, R dR, \tag{E.37}$$

and plugging (E.35) into (E.37) yields

$$M_{k,k}^0 = P_k^2 - 2\pi\mu_1 S + 2\pi\eta_d \int_0^\infty \left[W'''(U_b) \frac{1}{2} U_p' R + W''(U_b) \right] (\psi_0^0)^2 \, R dR, \tag{E.38}$$

where S is the shape factor defined in (6.33). Consider the η_d term in (E.37)

$$\int_0^\infty \left[W'''(U_b) \frac{1}{2} U_p' R + W''(U_b) \right] (\psi_0^0)^2 \, R dR = \int_0^\infty [W''(U_b)]' \frac{1}{2} (\psi_0^0)^2 \, R^2 dR + \int_0^\infty W''(U_b)(\psi_0^0)^2 \, R dR \tag{E.39}$$

$$= -\int_0^\infty W''(U_b)(\psi_0^0)(\psi_0^0)' \, R^2 dR - \cancel{\int_0^\infty W''(U_b)(\psi_0^0)^2 \, R dR} \tag{E.40}$$

$$+ \cancel{\int_0^\infty W''(U_b)(\psi_0^0)^2 \, R dR}$$

$$= -\int_0^\infty (\psi_0^0)''(\psi_0^0)' \, R^2 dR - \int_0^\infty \lambda_{p,0}(\psi_0^0)(\psi_0^0)' \, R^2 dR \tag{E.41}$$

$$= -\int_0^\infty \frac{1}{2} (((\psi_0^0)')^2)' \, R^2 dR - \lambda_{p,0} \int_0^\infty \frac{1}{2} ((\psi_0^0)^2)' \, R^2 dR \tag{E.42}$$

$$= -\int_0^\infty (((\psi_0^0)')^2) \, R dR - \lambda_{p,0} \int_0^\infty ((\psi_0^0)^2) \, R dR. \tag{E.43}$$

The second and the last equalities follow from integration by parts, and the third equality follows from identity (E.36). Plugging (E.43) into (E.37) yields

$$M_{k,k}^0 = P_k^2 - 2\pi\mu_1 S - \eta_d \left(\left\| (\psi_0^0)' \right\|_{L_R}^2 + \lambda_{p,0} \left\| (\psi_0^0) \right\|_{L_R}^2 \right). \tag{E.44}$$

E.5 Useful identities and inequalities

Theorem E.5.1 (Young's Inequality). *Suppose* $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ *with* $1 \le p, q, r \le \infty$. *Then*

$$\|f * g\|_r \le \|f\|_p \|g\|_q. \tag{E.45}$$

Theorem E.5.2 (Hölder's Inequality). *Suppose* $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $1 \le p, q, r \le \infty$. *Then*

$$\|fg\|_1 \le \|f\|_p \|g\|_q. \tag{E.46}$$

E.6 Appendix : Detailed calculations of operator bounds

First we consider the following operator $\partial_G^2 D_z$. Using the definitions of each of the operators, given in (2.67), and (2.66), we can write

$$\partial_G^2 D_z v = \partial_G^2 \left(\frac{1}{\tilde{J}_p} \nabla_z v \right) = \frac{\vec{\kappa}}{\tilde{J}_p} \left(\partial_G^2 (\nabla_z v) \right) + \left(\partial_G^2 \frac{\vec{\kappa}}{\tilde{J}_p} \right) \nabla_z v + 2 \partial_s \left(\frac{\vec{\kappa}}{\tilde{J}_p} \right) \partial_s \nabla_z v \tag{E.47}$$

Taking the $L^2(\Omega)$-norm yields

$$\begin{aligned}
\left\| \tilde{\Pi} \partial_G^2 D_z v \right\|_{L^2(\Omega)}^2 &= \left\| \frac{\vec{\kappa}}{\tilde{J}_p} \left(\partial_G^2 (\nabla_z v) \right) + \left(\partial_G^2 \frac{\vec{\kappa}}{\tilde{J}_p} \right) \nabla_z v + 2 \partial_s \left(\frac{\vec{\kappa}}{\tilde{J}_p} \right) \partial_s \nabla_z v \right\|_{L^2(\Omega)} && \text{(E.48)} \\
&\le \left\| \frac{\vec{\kappa}}{\tilde{J}_p} \left(\partial_G^2 (\nabla_z v) \right) \right\|_{L^2(\Omega)}^2 + \left\| \left(\partial_G^2 \frac{\vec{\kappa}}{\tilde{J}_p} \right) \nabla_z v \right\|_{L^2(\Omega)}^2 + 4 \left\| \partial_s \left(\frac{\vec{\kappa}}{\tilde{J}_p} \right) \partial_s \nabla_z v \right\|_{L^2(\Omega)}^2 && \text{(E.49)} \\
&\le \left\| \frac{\vec{\kappa}}{\tilde{J}_p} \right\|_{L^\infty} \left\| \left(\partial_G^2 (\nabla_z v) \right) \right\|_{L^2(\Omega)}^2 + \left\| \left(\partial_G^2 \frac{\vec{\kappa}}{\tilde{J}_p} \right) \right\|_{L^\infty} \| \nabla_z v \|_{L^2(\Omega)}^2 && \text{(E.50)} \\
&\quad + 4 \left\| \partial_s \left(\frac{\vec{\kappa}}{\tilde{J}_p} \right) \right\|_{L^\infty} \| \partial_s \nabla_z v \|_{L^2(\Omega)}^2 \\
&\le \varepsilon^{-4} c_1 \| v \|_{L^2(\Omega)} + c_2 \| v \|_{L^2(\Omega)} + 4 \left\| \partial_s \left(\frac{\vec{\kappa}}{\tilde{J}_p} \right) \right\|_{L^\infty} \left\| \partial_s (\partial_s^{-2}) \right\|_{l^2 \to l^2} \left\| \partial_s^2 \nabla_z v \right\|_{L^2(\Omega)}^2 && \text{(E.51)} \\
&\le \varepsilon^{-4} C \| v \|_{L^2(\Omega)}, && \text{(E.52)}
\end{aligned}$$

where the first inequality is the triangle inequality, for the second inequality we use Hölder, the third inequality follows from Lemma 6.3.1, combined with the assumption that $\kappa \in W^{2,\infty}$.

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