STUDIES ON THE BENDING
OF ELASTIC PLATES

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1982

ABSTRACT

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The bending of thin elastic plates with clamped edges is studied under two related theories. The classical theory, resulting in the familiar biharmonic boundary value problems, is well-known. The so-called improved theory of Timoshenko and Reissner differs from the classical theory by taking into account the effect of shear deformations. A set of three linear elliptic partial differential equations of the second order results.

In addition to showing the existence of solutions to the problem formulated in the improved theory, a detailed analysis is made to establish the relationship between the two theories, in terms of a single small parameter ϵ. Standard functional-analytic methods, along with a perturbation technique of Babuska, result in integral estimates comparing solutions for the clamped plate problem under the two respective theories.

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The feasibility of numerical approximations to solutions using the finite element method in the improved theory is also examined. It is shown that accuracy is adversely affected by small values of ϵ.

Finite elements that are customarily used to obtain approximate solutions in the classical theory are limited by the requirement that they have continuous first derivatives. The improved theory may be regarded as one of the so-called penalty function methods, by which the smoothness requirements on the finite elements may be relaxed. The behavior of the finite element solutions in the improved theory for small values of ϵ is studied, as is their usefulness in approximating solutions of the problem in the classical theory.

Additional asymptotic analyses are made, which serve to illustrate the above theoretical results. Numerical examples are presented, using piecewise linear elements for problems for clamped beams and plates.

ACKNOWLEDGEMENT

The author wishes to thank Dr. David Yen, of the Michigan State University Department of Mathematics for the expert guidance and inspiration he provided throughout the process leading to this thesis. His confidence has been essential.

Appreciation goes also to Mrs. Cindy Lou Smith for her superhuman effort, against all odds, in typing this work.

A special thanks goes to the author's parents, Mr. and Mrs. John Assiff for their unwavering support and infinite patience.

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CHAPTER 1 - INTRODUCTION

1-1. Statement of Problems and Background

This thesis concerns the bending of thin elastic plates, studied under two related theories. The classical theory of plate bending is usually attributed to G. Kirchhoff and A.E.H. Love [12,14]. The "improved" theory, developed by E. Reissner and R.D. Mindlin [19,16], is derived from Timoshenko's beam theory [23]. It takes into account the effect of shear deformation, which is neglected in the classical plate theory. The classical plate theory can then be regarded as the limit of the improved plate theory as the shear rigidity becomes infinite.

Throughout this work, a plate with the clamped edge conditions is assumed. The governing boundary value problems are compared, and a study is made to determine how solutions to the problem in the improved theory tend to those in the classical theory, in the limit mentioned above.

The feasibility of using the finite element method to obtain approximate solutions in the improved theory is then studied.

Approximating solutions by finite elements in the classical theory is hampered by differentiability requirements not imposed in the improved theory. The question then addressed is whether finite element approximations constructed using simple elements such as piecewise linear ones, can be used in the improved theory to approximate the solutions to the problem in the classical plate theory. In this way, the improved theory is used as a penalty function method approach to approximate the solution in the classical plate theory, thus avoiding the construction of complicated finite elements needed to satisfy the differentiability requirements.

The two boundary value problems are cast in their strong and weak forms, and functional-analytic methods are applied to show existence of solutions and to derive error estimates in terms of integral norms. Convergence is demonstrated for solutions in the improved plate theory to solutions in the classical theory by estimating the difference, in terms of a small parameter ϵ, which measures the reciprocal of the shear rigidity. The estimates hold for domains with smooth boundaries and their sharpness is studied . The problems of the clamped beam and circular plate are studied as special cases, for which improved estimates are presented, along with asymptotic analysis and explicit solutions.

Estimates based on the approximating properties of finite element spaces are used to guarantee convergence of the finite element solutions to solutions of the boundary value problem in the improved theory for fixed $\epsilon > 0$. The dependence upon negative powers of ϵ observed in these estimates poses a practical computational difficulty. This dependence is studied asymptotically for the clamped beam and plate.

Using piecewise linear finite elements, a numerical study of the clamped beam in the Timoshenko theory is carried out. Solutions are then used to illustrate the adverse effect of small ϵ, and a method is developed for recovering reliable results.

A further numerical study of the clamped square plate is offered as an example where the error estimates may not strictly hold due to the presence of corners. Again, piecewise linear elements are constructed within the improved theory, and the problem is solved for constant, variable, and point loads. These results are analyzed and conjectures are offered toward explaining some of the difficulties that they present.

The well known governing equation for the plate in the classical theory is an inhomogeneous biharmonic equation

$$\nabla^4 w = f$$

and the clamped edge requirement provides homogeneous Dirichlet boundary conditions. The variational form of the biharmonic equation, from which the finite element approximations derive, involves energy integrals of squares of second derivatives. For the energy to be finite, the finite element approximations must be continuously differentiable. This is a cumbersome restriction.

The governing equations for the same clamped plate in the improved theory of Timoshenko and Reissner are expressed as an elliptic system of three coupled second order partial differential equations, along with homogeneous boundary conditions. The boundary value problem will be stated precisely in 1-4. The system depends on the small parameter ε, in such a way that as $\varepsilon \to 0$, the biharmonic governing equation is recovered.

The variational form of the clamped plate problem in the improved theory, as derived in 2-1, imposes the less restrictive condition that a finite element approximation need only be continuous. This reduced restriction on the finite elements is at the cost of accuracy, which is adversely affected when the shear rigidity is large (ε small).

One of the objectives of this thesis is to attempt to use simply-constructed finite elements in the improved theory with small parameter ε, to approximate solutions in the classical theory thereby avoiding the restrictive requirements usually imposed by the latter. This indirect approach for applying "non-conforming" elements to the biharmonic problem can be viewed as an alternative to satisfying the "patch test" (see [21]). The cost, again, is that small ε requires excessively small mesh size h in the finite element construction.

1-2. Organization of the Dissertation

The remainder of this chapter is organized as follows. Section 1-3 includes notations and definitions of function spaces, along with an informal discussion of the finite element method. 1-4 introduces the equations of equilibrium for the clamped plate in the improved theory. Comparison is made to the classical theory, and a physical interpretation is offered for the parameter ε. Problem (C) and problem (I) are formally stated.

In section 2-1, weak forms of the boundary value problems are derived and the existence of solutions is shown for the improved theory, making use of the Lax-Milgram theorem, a standard Hilbert Space result. The variational property that a solution minimizes its potential energy is stated and easily proved, although this is also a standard result of the calculus of variations [15].

Section 2-2 quantifies the relationship between the classical and improved theories, by demonstrating convergence, as ε tends to zero, of solutions to the clamped plate problem in the improved theory to those in the classical theory. Direct integral estimates are derived, verifying the energy estimate claimed by Westbrook [25]. The sharpness and implications of this and related estimates are discussed.

Section 2-3 contains improvements on the estimates of 2-2 for the special cases of the clamped beam and the axisymmetrically loaded clamped circular plate. These estimates are derived by a method introduced by Babuska [3]. These results are contrasted with those of 2-2, and compared to asymptotic behavior discussed in Chapter 3 and 4.

In section 2-4, estimates are provided, following again the procedure of Babuska, to show convergence of finite element approximations to solutions of the clamped plate bending problem in the improved theory. These estimates show the presence of $\varepsilon^{-1/2}$ multiplying the mesh size h (for piecewise linear elements) in the error bounds, implying that accuracy for small ε may require an excessively fine mesh. Finally estimates are also provided to show the convergence of finite element approximations, formed in the improved theory for small ε, to solutions in the classical theory. Again, error bounds contain terms proportional to $\varepsilon^{-1/2}h$ (for piecewise linear elements) as well as terms with positive powers of ε, suggesting that convergence occurs only when ε and h both tend to zero, in a somewhat restricted way.

Chapter 3 is a study of the clamped beam in the Timoshenko improved theory. In section 3-1, a regular perturbation expansion is developed for the solution in powers of ε. For symmetric loads, the expansion actually truncates, showing that the solution is linear in ε.

Section 3-2 studies the discretization error due to the use of piecewise linear finite elements on the Timoshenko beam problem. Consistency between the finite element system and the boundary value problem are shown to depend adversely on the small parameter ε. An observation of the structure of this adverse ε dependence leads to its removal.

Section 3-3 presents the construction of the finite element stiffness matrix associated with piecewise linear elements, in terms of the element stiffness matrix which is constructed explicitly from the potential energy functional.

Section 3-4 contains numerical results for the clamped beam under constant, variable and point loads. Effectiveness in approximating exact solutions in the classical and Timoshenko theories is analyzed.

Chapter 4 is a study of the clamped plate in the improved theory. The difficulty of deriving an asymptotic series in ε for the general case is discussed in section 4-1.

Section 4-2 studies the clamped circular plate. Explicit solutions are constructed using a constant load and a non-axisymmetric load, $\frac{p}{D} = \cos \theta$. The latter serves to demonstrate the sharpness of the main estimates for general plates, exhibitng boundary layer behavior in its twisting moment.

Section 4-3 is analogous to 3-2, containing analysis of discretization error. Again, consistency with the continuous problem is shown to depend on ε, through powers of its reciprocal.

In section 4-4, the element stiffness matrix is constructed as in 3-3. The programming involved in assembling the global stiffness matrix is briefly outlined.

Section 4-5 contains numerical results for the square clamped plate under constant, variable and point loads. The possible effects of domains with corners are discussed. Kondrat'ev's fundamental work [13] predicts singular behavior in the shear forces at the corners of a square clamped plate in the classical theory. Possible implications for the improved theory are considered.

Chapter 5 contains a summary of the results and a discussion of their implications, as well as conjectures and suggestions for further work.

1-3. Notation and Function Spaces

Several different notations are used regarding partial differentiation. Let u_1 and u_2 be sufficiently differentiable functions. The partial derivative of u_1 with respect to x is denoted by any of the following equivalent forms

$$\frac{\partial u_1}{\partial x} \equiv u_{1,x} \equiv D^{(1,0)}u_1 .$$

Similarly,

$$\frac{\partial u_1}{\partial y} \equiv u_{1,y} \equiv D^{(0,1)}u_1 .$$

Likewise, higher derivatives are expressed in the same fashion:

$$\frac{\partial^2 u_1}{\partial x^2} \equiv u_{1,xx} \equiv D^{(2,0)}u_1$$

$$\frac{\partial^3 u_1}{\partial x \partial y^2} \equiv u_{1,xyy} \equiv D^{(1,2)}u_1 .$$

The general form involving the multi-index is

$$D^{\alpha}u_1 = D^{(\alpha_1,\alpha_2)}u_1 = \frac{\partial^{|\alpha|}u_1}{\partial x^{\alpha_1}\partial y^{\alpha_2}} ,$$

where $\alpha = (\alpha_1,\alpha_2)$ and $|\alpha| = \alpha_1 + \alpha_2$.

Vector notation is also used when convenient.

$$\nabla u_1 = \begin{bmatrix} u_{1,x} \\ u_{1,y} \end{bmatrix}$$

$$\nabla \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_{1,x} + u_{2,y}$$

$$\nabla^2 u_1 = u_{1,xx} + u_{1,yy}$$

$$\nabla^4 u_1 = u_{1,xxxx} + 2u_{1,xxyy} + u_{1,yyyy} .$$

Vector functions are generally denoted by capital letters, e.g.

$$U = [u_1,u_2,u_3]^T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} .$$

$\hat{N}$ denotes the outward normal to the boundary where this can be defined and $\partial/\partial N$ denotes the normal derivative at the boundary.

In the discussions of the beam, similar notation will be used although only one independent variable is involved

$$\frac{du}{dx} = u' = u^{(1)}$$

$$\frac{d^k u}{dx^k} = u^{(k)} \quad .$$

In fact vector notation may be used to maintain the analogy to the more general case

$$\nabla u = \frac{du}{dx}$$

$$\nabla^2 u = \frac{d^2 u}{dx^2}$$

and capital letters will denote vector functions of two components

$$U = [u_1, u_2]^T = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} .$$

It will also be understood that whenever sufficient smoothness is lacking, the governing equations will be interpreted in a weak sense.

The standard order of magnitude symbols will be used, and interpreted in the following sense

$$A = O(b^n)$$

if $|A/b^n|$ remains bounded as $b \to 0$, with all other parameters in expression A held constant. Since more than one parameter will be involved at times, the expression $O(b^0)$ will be used instead of the ambiguous $O(1)$.

Consider the bounded open connected region Ω in $\mathbb{R}^2$ and its boundary $\partial\Omega$. $\overline{\Omega}$ represents the closed region occupied by the undeformed plate, hence it is the domain for the dependent variables: transverse displacement and angular displacements. Assume the boundary is such that Ω is Lipschitzian, as defined in [3]. Roughly, this means that, locally, the domain Ω is all on one side of $\partial\Omega$, and that the region is free of cusps.

Let $C^\infty(\overline{\Omega})$ be the space of all real infinitely differentiable functions on Ω where all their derivatives can be extended continuously to the boundary $\partial\Omega$. Let $C_0^\infty(\overline{\Omega})$ be the subspace of $C^\infty(\overline{\Omega})$ consisting of all functions with compact support in Ω. Let $H^0(\Omega)$ be the usual Hilbert space $L_2(\Omega)$, i.e. the space of square integrable functions on Ω, with norm given by

$$\|u\|_0^2 = \iint_\Omega u^2 dA$$

and associated inner product

$$(u,v)_0 = \iint_\Omega uv dA \ .$$

Let $\ell \geq 1$ be an integer.

Define $H^{\ell}(\Omega)$ to be the closure of $C^{\infty}(\overline{\Omega})$ under the norm

$$\|u\|_{\ell}^{2} = \sum_{0\leq|\alpha|\leq\ell} \|D^{\alpha}u\|_{0}^{2}$$

Likewise, define $H_0^{\ell}(\Omega)$ to be the closure of $C_0^{\infty}(\overline{\Omega})$ under the norm $\|\cdot\|_{\ell}^{2}$.

Define the semi-norms

$$|u|_{\ell}^{2} = \sum_{|\alpha|=\ell} \|D^{\alpha}u\|_{0}^{2}, \quad \text{for } \ell \geq 1 \text{ integer}$$

Product spaces and their norms are defined in the usual way:

$$(H^{\ell})^{3} = H^{\ell} \times H^{\ell} \times H^{\ell}$$

with norm defined by

$$\|U\|_{\ell}^{2} = \|u_1\|_{\ell}^{2} + \|u_2\|_{\ell}^{2} + \|u_3\|_{\ell}^{2}, \quad \text{where } U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in (H^{\ell})^{3}.$$

Analogous definitions will hold for

$$(H_0^{\ell})^{3},\ (H^{\ell})^{2},\ (H_0^{\ell})^{2}.$$

A convenient space related to the potential energy for the clamped plate in the improved theory is defined for each fixed $\epsilon > 0$. For

$$U = [u_1, u_2, u_3]^T \in (H_0^1)^3 ,$$

define

$$\|U\|_\varepsilon^2 = |U|_1^2 + \varepsilon^{-1} \iint_\Omega \left| \nabla u_3 + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right|^2 dA .$$

$\|U\|_\varepsilon$ is clearly a norm and the following theorem shows it to be equivalent to $\|U\|_1$ on $(H_0^1)^3$. It follows that H_ε and $(H_0^1)^3$ consist of the same set of functions.

<u>Theorem</u>: The norms $\|\cdot\|_1$ and $\|\cdot\|_\varepsilon$ are equivalent. In fact, for domain Ω with largest dimension unity,

$$\frac{4}{5}\|V\|_1^2 \le \|V\|_\varepsilon^2 \le (1 + 2\varepsilon^{-1})\|V\|_1^2 .$$

<u>Proof</u>: Lemma (c) from section (2.3) provides that for $u \in H_0^1$,

$$\|u\|_0^2 \le \frac{1}{2} \|u_{,x}\|_0^2$$

$$\|u\|_0^2 \le \frac{1}{2} \|u_{,y}\|_0^2$$

so

$$\|u\|_0^2 \le \frac{1}{4}(\|u_{,x}\|_0^2 + \|u_{,y}\|_0^2) .$$

Let $V = [v_1, v_2, v_3]^T \in (H_0^1)^3$.

Then

$$\|v\|_1^2 = \|v\|_0^2 + |v|_1^2$$

$$= \|v_1\|_0^2 + \|v_2\|_0^2 + \|v_3\|_0^2 + |v|_1^2$$

and by Lemma (c),

$$\le \tfrac{1}{4}(\|v_{1,x}\|_0^2 + \|v_{1,y}\|_0^2) + \tfrac{1}{4}(\|v_{2,x}\|_0^2 + \|v_{2,y}\|_0^2)$$

$$+ \tfrac{1}{4}(\|v_{3,x}\|_0^2 + \|v_{3,y}\|_0^2) + |v|_1^2$$

$$\le \tfrac{5}{4}\,|v|_1^2 \le \tfrac{5}{4}\,\|v\|_\varepsilon^2 \,.$$

Also,

$$\|v\|_\varepsilon^2 = |v|_1^2 + \varepsilon^{-1}\left\|\nabla v_3 + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right\|_0^2$$

$$\le |v|_1^2 + \varepsilon^{-1}\left(\|\nabla v_3\|_0^2 + 2\|\nabla v_3\|_0\left\|\begin{matrix} v_1 \\ v_2 \end{matrix}\right\|_0 + \left\|\begin{matrix} v_1 \\ v_2 \end{matrix}\right\|_0^2\right)$$

$$\le |v|_1^2 + \varepsilon^{-1}(|v|_1^2 + 2|v|_1\|v\|_0 + \|v\|_0^2)$$

$$= |v|_1^2 + \varepsilon^{-1}(|v|_1 + \|v\|_0)^2$$

$$\le |v|_1^2 + 2\varepsilon^{-1}(|v|_1^2 + \|v\|_0^2)$$

$$\le (1 + 2\varepsilon^{-1})\|v\|_1^2 \,.$$

In discussing the finite element method, it will be necessary to work with certain finite dimensional subspaces of H^k, which are generated by bases made up of finite element functions. These subspaces are defined in terms of their ability to approximate functions from infinite dimensional spaces. In practice these spaces are generally made up of piecewise polynomials with certain degrees of continuity. The domain in question is subdivided into simple regions (say triangles or rectangles) called finite elements, and a space of functions is defined as all functions which are polynomial of a prescribed degree over each element, and which satisfy certain prescribed continuity conditions across the boundaries between elements. The space is then shown to be generated by a finite basis consisting of functions (also refered to as finite elements) which are themselves piecewise polynomial, and which have the addition property that they are zero except over a certain few adjacent elements of the domain.

Although finite element spaces are occasionally non-polynomial in nature, these are by far the exception, and then are usually an attempt to deal with difficulties (such as singularities) in specific problems. The definition, which will follow, is taken from Chapter 4 of [3]. It and related ones omitted here, provided the common thread which ties all finite element generated subspaces; namely, a formal expression of a finite dimensional space's ability to approximate functions belonging to an infinite dimensional space.

In the following definition, the parameter k represents the degree of continuity (in a mean square sense) present in the functions belonging to the finite dimensional spaces being defined. In the context of piecewise polynomial spaces, t indicates the degree of the polynomials present over each element ($t = 2$ means linear, $t = 3$ means quadratic, etc.)

The parameter h is the mesh size. That is, each element in the subdivided domain has a largest dimension or diameter. h is the maximum of these diameters over all subdivisions. Finite element subspaces are usually constructed so that the mesh size can be varied. Each different mesh size leads to different subspaces. The finer the mesh size (smaller h) the more subdivisions are necessary, and the higher the dimension of the corresponding subspace becomes. As the dimension of the subspace increases and the mesh size decreases, the ability to approximate a particular function improves. As h tends to zero, the system of subspaces should provide a sequence of functions which tend to the desired one. The definition which follows expresses an expectation of this approximation property; an expectation which is met by standard types of finite element spaces, such as piecewise polynomials defined over regular subdivisions such as triangles or rectangles.

As defined in Chapter 4 of [3], let $S_h^{t,k}(\Omega)$ denote any linear system of functions with the following properties: for $t > k \geq 0$,

i) $S_h^{t,k}(\Omega) \subset H^k(\Omega)$

ii) $\forall\ g \in H^\ell(\Omega)$, $\ell \geq 0$ and $0 \leq s \leq \min(\ell,k)$ there exists $\phi \in S_h^{t,k}(\Omega)$ such that

$$\|g - \phi\|_s \leq Ch^\mu \|g\|_\ell ,$$

where $\mu = \min(t - s, \ell - s)$ and

C is independent of g and h .

Such a system is called a (t,k)-system. In the present context, a (t,k)-system will be a subspace of H_0^1 or H^1 spanned by finite element base functions, constructed over an appropriately subdivided domain. Primary importance in this thesis is given to the spaces generated by piecewise linear base functions ("roof" functions) constructed over a domain subdivided by right triangles formed from squares, all bisected by the diagonal with a common direction. For these piecewise linear base functions, $t = 2$ above.

1-4. Equations of Equilibrium for a Clamped Plate in the Improved Theory

Consider a thin homogeneous isotropic plate of arbitrary shape, with its largest lateral dimension unity. The complete list of assumptions which comprise the classical theory, sometimes known as "small deflection, thin plate" theory, can be found in [22], and others. The assumptions used to derive the improved theory are the same, except for the deletion of the assumption:

> "Lines originally perpendicular to the center surface of the plate before deformation remain perpendicular to the deformed surface after".

The deletion of this requirement allows the inclusion of the shear deformations, which characterize the improved theory of plates.

The equilibrium equations for plate bending under the improved plate assumptions are:

$$\frac{D}{2}[(1-\mu)\nabla^2\psi_x + (1+\mu)\frac{\partial\phi}{\partial x}] - G'H(\psi_x + \frac{\partial w}{\partial x}) = 0$$

$$(1\text{-}4.1) \qquad \frac{D}{2}[(1-\mu)\nabla^2\psi_y + (1+\mu)\frac{\partial\phi}{\partial y}] - G'H(\psi_y + \frac{\partial w}{\partial y}) = 0$$

$$G'H(\nabla^2 w + \phi) = -p$$

where

w = vertical displacement

ψ_x, ψ_y = angular displacements (rotations) due to bending

$$\phi = \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$G' = \kappa^2 G$ = modified shear modulus

G = shear modulus

κ^2 = a constant introduced by R.D. Mindlin [16] to bring about agreement between wave speeds of short transverse waves obtained from two- and three- dimensional analysis of infinite plates.*

$D = \frac{GH^3}{6(1-\mu)}$ = bending modulus

μ = Poisson's ratio, $0 \leq \mu \leq 0.5$

H = plate thickness

p = transverse load over surface of plate

---

* κ^2 is a function of μ, varying roughly linearly from $\kappa^2 = .76$ at $\mu = 0$, to $\kappa^2 = .91$ for $\mu = 0.5$.

Defining $\varepsilon = \dfrac{H^2}{6(1-\mu)\varkappa^2}$, equation (1) becomes:

$$\tfrac{1}{2}[(1-\mu)\nabla^2\psi_x + (1+\mu)\frac{\partial\phi}{\partial x}] - \varepsilon^{-1}(\psi_x + \frac{\partial w}{\partial x}) = 0 \tag{1-4.2a}$$

$$\tfrac{1}{2}[(1-\mu)\nabla^2\psi_y + (1+\mu)\frac{\partial\phi}{\partial y}] - \varepsilon^{-1}(\psi_y + \frac{\partial w}{\partial y}) = 0 \tag{1-4.2b}$$

$$\varepsilon^{-1}(\nabla^2 w + \phi) = -\frac{p}{D} \tag{1-4.2c}$$

where

$$\phi = \frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y}\ .$$

Equations (1-4.2) for the "Improved" plate theory may be compared with the equation for classical plate bending. The classical equation is:

$$\nabla^4 w = \frac{p}{D}\ . \tag{1-4.3}$$

In order to compare, let the load p be at least twice differentiable. Differentiating (1-4.2a) with respect to x, and (1-4.2b) with respect to y, and adding, gives

$$\nabla^2\phi - \varepsilon^{-1}(\phi + \nabla^2 w) = 0$$

with (1-4.2c) this becomes

$$\nabla^2\phi = -\frac{p}{D}\ . \tag{1-4.4}$$

Now taking the Laplacian of (1-4.2c), combining with (1-4.4),

$$\nabla^4 w = \frac{p}{D} - \varepsilon\ \nabla^2 \frac{p}{D} \ . \tag{1-4.5}$$

With the angular displacements eliminated, (1-4.5) represents an equation to determine the transverse displacement under the improved plate theory assumptions. The striking feature of (1-4.5) is its similarity to the governing equation for the plate under the classical theory, equation (1-4.3), which neglected shear affects. Equation (1-4.5) merely has the additional term $-\varepsilon\nabla^2 \frac{p}{D}$, due to inclusion of shear effects, which vanishes as ε tends to zero. Since the classical governing equation is recovered as the limit as ε tends to zero, the inference is that ε is a measure of the effect of including shear. The solution w governed by (1-4.5) is still coupled to ψ_x and ψ_y through boundary conditions, but assuming the boundary conditions are chosen to be compatable between classical and improved problems, it is reasonable to expect the displacement w produced by the improved theory to approach the displacement produced by the classical theory, as ε tends to zero. This convergence will be examined for the clamped plate boundary conditions in both an integral sense, and asymptotically in a pointwise sense.

A physical interpretation of this convergence is worth noting. The process of convergence as ϵ tends to zero may be viewed within the context of a fixed material (meaning μ, $\varkappa$, and G are fixed). Then $\epsilon = \dfrac{H^2}{6(1-\mu)\varkappa^2}$ tending to zero means that the plate thickness tends to zero. Consider the transverse load p being scaled also, in such a way that $\frac{p}{D}$ remained constant, independent of plate thickness H. That is, let p be proportional to H^3. It must be remembered that in both models, classical and improved, all in-plane forces are neglected so the limiting process above should not be viewed as leading physically to a description of the "membrane" problem. Rather, as the thickness decreases, intuition suggests that the transverse load is carried less and less by the shear stresses and more and more by the bending moments. In the limit there is no shear, and the improved plate model becomes identical with the classical plate theory of bending.

It will be useful to express equations (1-4.2) in operator form:

$$LU = F \tag{1-4.6}$$

where

$$L = L_B + \varepsilon^{-1} L_s$$

$$L_B = \begin{bmatrix} \frac{1}{2}[(1-\mu)\nabla^2 + (1+\mu)\frac{\partial^2}{\partial x^2}] & \frac{1}{2}(1+\mu)\frac{\partial^2}{\partial x \partial y} & 0 \\ \frac{1}{2}(1+\mu)\frac{\partial^2}{\partial x \partial y} & \frac{1}{2}[(1-\mu)\nabla^2 + (1+\mu)\frac{\partial^2}{\partial y^2}] & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$L_S = \begin{bmatrix} -1 & 0 & -\frac{\partial}{\partial x} \\ 0 & -1 & -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \nabla^2 \end{bmatrix}$$

$$U = \begin{bmatrix} \psi_x \\ \psi_y \\ w \end{bmatrix} , \quad F = \begin{bmatrix} 0 \\ 0 \\ -\frac{p}{D} \end{bmatrix} .$$

Define, also the following related energy expressions: for $U = [u_1, u_2, u_3]^T$ and $W = [w_1, w_2, w_3]^T$,

$$P_S(U,W) = \iint_\Omega [(\frac{\partial u_3}{\partial x} + u_1)(\frac{\partial w_3}{\partial x} + w_1) + (\frac{\partial u_3}{\partial y} + u_2)(\frac{\partial w_3}{\partial y} + w_2)] dA$$

$$P_B(U,W) = \frac{1}{2}\iint_\Omega [(1+\mu)(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y})(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y})$$

$$+ (1-\mu)(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial y})(\frac{\partial w_1}{\partial x} - \frac{\partial w_2}{\partial y})$$

$$+ (1-\mu)(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x})(\frac{\partial w_1}{\partial y} + \frac{\partial w_2}{\partial x})]\,dA$$

$$P_L(F,W) = -\iint_\Omega W^T F \; dA = \iint_\Omega \frac{p}{D} w_3 \; dA$$

$$B_\epsilon(U,V) = P_B(U,V) + \epsilon^{-1} P_S(U,V) .$$

The quadratic forms derived from the bilinear energy expressions have some physical interpretation. $\epsilon^{-1}P_S(U,U)$ represents the potential energy due to shear deformation present in the solution $U = [u_1,u_2,u_3]^T$ where u_3 gives the displacement and u_1 and u_2 give angular rotations due to bending. Note that the quantities $\frac{\partial u_3}{\partial x} + u_1$ and $\frac{\partial u_3}{\partial y} + u_2$ are respectively the angular rotations due to shear in the x- and y-directions. $P_B(U,U)$ represents the potential energy due to bending in the solution U. $P_L(F,U)$ represents the potential energy due to the applied transverse load p. $B_\epsilon(U,U)$ represents the total energy due to both bending moments and shear. If U represents the solution to the classical plate bending problem, then $U = [-\frac{\partial w}{\partial x}, -\frac{\partial w}{\partial y}, w]^T$, since rotations due to shear are zero. Consequently, $P_S(U,U)$ vanishes and

$$P_B(U,U) = \iint w_{,xx}^2 + 2w_{,xy}^2 + w_{,yy}^2 + 2\mu(w_{,xx}w_{,yy} - w_{,xy}^2)dA$$

$$= \iint (w_{,xx} + w_{,yy})^2 - 2(1-\mu)(w_{,xx}w_{,yy} - w_{,xy}^2)dA \ .$$

This is the strain energy (up to a normalizing constant) due to bending for the classical plate [22]. Under the clamped plate assumption, the boundary conditions $w = \frac{\partial w}{\partial N} = 0$ cause the term $\iint w_{,xx}w_{,yy} - w_{,xy}^2 \, dA$ to vanish. Then

$$P_B(U,U) = \iint (w_{,xx} + w_{,yy})^2 dA \ .$$

The boundary value problems for the clamped plate under load $f = p/D$ can now be stated precisely. The problem in the classical theory is

<u>Problem (C)</u>: find $U_0 = [-\frac{\partial w_0}{\partial x}, -\frac{\partial w_0}{\partial y}, w_0]^T$ satisfying

$$\nabla^4 w_0 = f \quad \text{on} \quad \Omega$$

$$w_0 = 0 \quad \text{on} \quad \partial\Omega$$

$$\frac{\partial w_0}{\partial N} = 0 \quad \text{on} \quad \partial\Omega \ .$$

The problem in the improved theory is

<u>Problem (I)</u>: find $U_\varepsilon = [\psi_x, \psi_y, w]^T$

satisfying, on Ω,

$$\frac{1}{2}[(1-\mu)\nabla^2\psi_x + (1+\mu)(\frac{\partial^2\psi_x}{\partial x^2} + \frac{\partial^2\psi_y}{\partial x\partial y})] - \varepsilon^{-1}(\psi_x + \frac{\partial w}{\partial x}) = 0$$

$$\frac{1}{2}[(1-\mu)\nabla^2\psi_y + (1+\mu)(\frac{\partial^2\psi_x}{\partial x\partial y} + \frac{\partial^2\psi_y}{\partial y^2})] - \varepsilon^{-1}(\psi_y + \frac{\partial w}{\partial y}) = 0$$

$$\varepsilon^{-1}(\nabla^2 w + \frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y}) = -f$$

and

$$\psi_x = \psi_y = w = 0 \quad \text{on} \quad \partial\Omega .$$

Problem (C) and Problem (I) will be interpreted in their weak forms when sufficient smoothness of solutions is lacking. The weak forms, derived in 2-1, are as follows. For $F = [0,0,-f]^T$, $f = \frac{p}{D}$.

<u>Problem (C)</u>: find $U_0 = [-\frac{\partial w_0}{\partial x}, -\frac{\partial w_0}{\partial y}, w_0]^T$

where $w_0 \in H_0^2$ and

$$P_B(U_0,V) = P_L(F,V)$$

for all

$$V = [-\frac{\partial v}{\partial x}, -\frac{\partial v}{\partial y}, v]^T$$

where

$$v \in H_0^2 .$$

<u>Problem (I)</u>: find $U_\epsilon = [\psi_x, \psi_y, w]^T \in (H_0^1)^3$ where

$$B_\epsilon(U_\epsilon, V) = P_L(F, V)$$

for all

$$V \in (H_0^1)^3 .$$

CHAPTER 2 - EXISTENCE, CONVERGENCE AND FINITE ELEMENT APPROXIMATIONS FOR SOLUTIONS OF PROBLEM (I)

Chapter 2 studies the existence of solutions to problem (I) and their convergence to solutions of problem (C), as the shear rigidity tends to infinity. The special cases of a clamped beam and a circular plate with axisymmetric loading are studied as examples where improvement in the estimates for convergence can be achieved. Finite element approximations to solutions of problem (I) are developed, and their behavior investigated as the shear rigidity tends to infinity.

2-1. Existence of Solution to Problem (I)

Let w_0 be the solution to the classical clamped plate problem

$$\begin{aligned} &\nabla^4 w_0 = \frac{p}{D} \quad \text{on} \quad \Omega . \\ (2\text{-}1.1) \qquad & \\ &w_0 = \frac{\partial w_0}{\partial N} = 0 \quad \text{on} \quad \partial\Omega . \end{aligned}$$

The existence of a unique solution $w_0 \in H_0^2$ is guaranteed (see [8]), at least for $\frac{p}{D} \in H^0$ and smooth domain Ω, problem (C) being an elliptic boundary value problem with homogeneous Dirichlet boundary conditions. For the weak form of (2-1.1), let $v \in H_0^2$ then

$$0 = \iint_\Omega (\nabla^4 w_0 - \frac{p}{D}) v \, dA$$

using Green's Identity

$$\iint_\Omega (\alpha\nabla^2\beta - \beta\nabla^2\alpha)\, dA = \int_{\partial\Omega} (\alpha \frac{\partial\beta}{\partial N} - \beta \frac{\partial\alpha}{\partial N})\, ds$$

(2-1.2) $$0 = \iint_\Omega (\nabla^2 w_0 \nabla^2 v - \frac{p}{D} v)\, dA \; .$$

In terms of the energy definitions of Chapter 1, (2-1.2) states that for every $v \in H_0^2$, w_0 satisfies

(2-1.3) $$P_B(U_0, V) = P_L(F, V)$$

$$\text{where} \quad U_0 = [-w_{0,x}, -w_{0,y}, w_0]^T$$

$$V = [-v_{,x}, -v_{,y}, v]^T \, .$$

(2-1.3) can be derived also as the Euler equation for problem of finding the minimum over H_0^2 of the quadratic functional

(2-1.4) $$J_{CLASS} = P_B(U,U) - 2P_L(F,U)$$

Let $U_\epsilon = [\psi_x, \psi_y, w]^T$ satisfy equations (1-2.2) along with the homogeneous boundary conditions. That is,

$$LU_\varepsilon = F \quad \text{on} \quad \Omega$$

$$U_\varepsilon = [0,0,0]^T \quad \text{on} \quad \partial\Omega .$$

Then for any $V = [v_1, v_2, v_3]^T \in (H_0^1)^3$,

$$(2\text{-}1.5) \quad \iint_\Omega (V^T L U_\varepsilon - V^T F)\,dA$$

$$= \iint_\Omega \{\tfrac{1}{2}[(1-\mu)\nabla^2\psi_x + (1+\mu)\tfrac{\partial\phi}{\partial x}] - \varepsilon^{-1}(\psi_x + \tfrac{\partial w}{\partial x})\}v_1$$

$$+ \{\tfrac{1}{2}[(1-\mu)\nabla^2\psi_y + (1+\mu)\tfrac{\partial\phi}{\partial y}] - \varepsilon^{-1}(\psi_y + \tfrac{\partial w}{\partial y})\}v_2$$

$$+ \{\varepsilon^{-1}(\nabla^2 w + \phi) + \tfrac{p}{D}\}v_3 \; dA = 0 .$$

Integration by parts yields

$$\iint_\Omega \Bigg\{ -\tfrac{1}{2}(1-\mu)\nabla\psi_x \cdot \nabla v_1 - \tfrac{1}{2}(1+\mu)\nabla \cdot \begin{pmatrix}\psi_x\\ \psi_y\end{pmatrix} \frac{\partial v_1}{\partial x}$$

$$- \varepsilon^{-1}(\psi_x + \tfrac{\partial w}{\partial x})v_1$$

$$-\tfrac{1}{2}(1-\mu)\nabla\psi_y \cdot \nabla v_2 - \tfrac{1}{2}(1+\mu)\nabla \cdot \begin{pmatrix}\psi_x\\ \psi_y\end{pmatrix} \frac{\partial v_2}{\partial y}$$

$$- \varepsilon^{-1}(\psi_y + \tfrac{\partial w}{\partial y})v_2$$

$$- \varepsilon^{-1}\,\nabla w \cdot \nabla v_3 - \varepsilon^{-1} \begin{pmatrix}\psi_x\\ \psi_y\end{pmatrix} \cdot \nabla v_3 + \tfrac{p}{D}\,v_3 \Bigg\} dA = 0$$

$$\iint_\Omega \Bigg\{ -\tfrac{1}{2}(1-\mu)[\nabla\psi_x \cdot \nabla v_1 + \nabla\psi_y \cdot \nabla v_2] - \tfrac{1}{2}(1+\mu)\nabla \cdot \begin{pmatrix}\psi_x\\ \psi_y\end{pmatrix} \nabla \cdot \begin{pmatrix}v_1\\ v_2\end{pmatrix}$$

$$- \varepsilon^{-1}\left(\left(\begin{pmatrix}\psi_x\\ \psi_y\end{pmatrix} + \nabla w\right)\cdot\left(\begin{pmatrix}v_1\\ v_2\end{pmatrix} + \nabla v_3\right)\right) + \tfrac{p}{D}\,v_3 \Bigg\} dA = 0$$

$$(2\text{-}1.6) \quad \iint_\Omega \{ -\frac{1}{2}(1-\mu)[\frac{\partial\psi_x}{\partial x}\frac{\partial v_1}{\partial x} + \frac{\partial\psi_y}{\partial y}\frac{\partial v_1}{\partial y} + \frac{\partial\psi_y}{\partial x}\frac{\partial v_2}{\partial x} + \frac{\partial\psi_y}{\partial y}\frac{\partial v_2}{\partial y}]$$

$$-\frac{1}{2}(1+\mu)(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y})(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y})\}dA$$

$$-\epsilon^{-1}P_s(U_\epsilon,V) + P_L(F,V) = 0 \ .$$

Now consider

$$(2\text{-}1.7) \quad \iint (-\frac{\partial\psi_x}{\partial x}\frac{\partial v_2}{\partial y} + \frac{\partial\psi_x}{\partial y}\frac{\partial v_2}{\partial x} - \frac{\partial\psi_y}{\partial y}\frac{\partial v_1}{\partial x} + \frac{\partial\psi_y}{\partial x}\frac{\partial v_1}{\partial y})dA$$

by integration by parts

$$= \iint (-\frac{\partial\psi_x}{\partial x}\frac{\partial v_2}{\partial y} - \frac{\partial\psi_x}{\partial y\partial x}v_2 - \frac{\partial\psi_y}{\partial y}\frac{\partial v_1}{\partial x} - \frac{\partial^2\psi_y}{\partial x\partial y}v_1)dA$$

by integration by parts

$$= \iint (-\frac{\partial\psi_x}{\partial x}\frac{\partial v_2}{\partial y} + \frac{\partial\psi_x}{\partial x}\frac{\partial v_2}{\partial y} - \frac{\partial\psi_y}{\partial y}\frac{\partial v_1}{\partial x} + \frac{\partial\psi_y}{\partial y}\frac{\partial v_1}{\partial x})dA$$

$$= 0 \ .$$

Inserting this into (2-1.6), it becomes

$$-P_B(U_\epsilon,V) - \epsilon^{-1}P_S(U_\epsilon,V) + P_L(F,V) = 0$$

$$(2\text{-}1.8) \quad P_B(U_\epsilon,V) + \epsilon^{-1}P_S(U_\epsilon,V) = P_L(F,V) \qquad \forall \ V \in (H_0^1)^3 \ .$$

(2-1.8) can be written

$$B_\varepsilon(U_\varepsilon, V) = P_L(F,V) \quad \forall \quad V \in (H_0^1)^3 . \tag{2-1.9}$$

(2-1.8) or (2-1.9) can be derived as the Euler equation for the problem of finding the minimum over $(H_0^1)^3$ of the quadratic functional

$$J_{\text{IMPROVED}} = P_B(U,U) + \varepsilon^{-1} P_S(U,U) - 2P_L(F,U) \tag{2-1.10}$$

or

$$J_{\text{IMPROVED}} = B_\varepsilon(U,U) - 2P_L(F,U) . \tag{2-1.11}$$

As before, let $U_0 = [-w_{0,x}, -w_{0,y}, w_0]^T$ be the solution of problem (C). That is, w_0 solves

$$\nabla^4 w_0 = \frac{p}{D} \quad \text{on} \quad \Omega$$

$$\left.\begin{aligned} w_0 &= 0 \\ \frac{\partial w_0}{\partial N} &= 0 \end{aligned}\right\} \text{ on } \partial\Omega . \tag{2-1.12}$$

Another weak form will be needed for U_0. Consider, for $V = [v_1, v_2, v_3]^T \in (H_0^1)^3$

$$(2\text{-}1.13)\quad P_B(U_0,V) = \iint \tfrac{1}{2}(1+\mu)(-w_{0,xx} - w_{0,yy})(v_{1,x} + v_{2,y})$$

$$+ \tfrac{1}{2}(1-\mu)(-w_{0,xx} + w_{0,yy})(v_{1,x} - v_{2,y})$$

$$+ \tfrac{1}{2}(1-\mu)(-w_{0,xy} - w_{0,yx})(v_{1.y} + v_{2,x})dA$$

$$= \iint \nabla(\nabla^2 w_0) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dA\ ,$$

by integration by parts .

Now $\nabla^4 w_0 - \frac{p}{D} = 0$, so

$$\iint (\nabla^4 w_0 - \tfrac{p}{D}) v_3 dA = 0\ .$$

Integrating by parts here also,

$$\iint (-\nabla(\nabla^2 w_0) \cdot \nabla v_3 - \tfrac{p}{D}\ v_3)dA = 0$$

and

$$(2\text{-}1.14)\quad \iint \left\{ -\nabla(\nabla^2 w_0) \cdot \left(\nabla v_3 + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) - \frac{p}{D}\ v_3 \right\} dA$$

$$= -\iint \nabla(\nabla^2 w_0) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dA\ .$$

Using (2-1.13), (2-1.14) becomes

$$P_B(U_0,V) = \iint \nabla(\nabla^2 w_0) \cdot \left(\nabla v_3 + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) dA + \iint \frac{p}{D}\ v_3\ dA$$

or

$$(2\text{-}1.15) \quad P_B(U_O,V) = P_L(F,V) + \iint \nabla(\nabla^2 w_O) \cdot \left(\nabla v_3 + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) dA \qquad \forall \; V \in (H_O^1)^3 .$$

The existence of solutions for problem (I) is now considered. The Lax-Milgram Theorem is used below to show that (2-1.9) has a unique solution, U_ϵ, for each $\epsilon > 0$. In fact, since B_ϵ is symmetric, it represents an inner product on H_ϵ. In this context the Lax-Milgram theorem reduces to the Riesz representation theorem.

Theorem 2 (Lax-Milgram): Given a bilinear form $B(U,V)$ defined on $S \times S$, where S is a Hilbert space with norm $\|\cdot\|$, if:

i) There exists $M > 0$ independent of U and V such that $|B(U,V)| \leq M\|U\|\|V\| \;\; \forall \;\; U,V \in S \times S$, and

ii) There exists $\gamma > 0$ independent of V such that $B(V,V) \geq \gamma\|V\|^2, \;\; \forall \; V \in S$,

then the equation

$$B(U,V) = F(V) \, , \quad \forall \; V \in S$$

has a unique solution $U \in S$ for every continuous linear functional F defined on S.

The theorem is proved in Gilbarg and Trudinger [9].

A number of lemmas are required before Theorem 2 can be used to show the existence of a unique solution to problem (I).

Lemma (A):

$$\|u_{1,x} + u_{2,y}\|_0 \le \sqrt{2}\ |U|_1$$

$$\|u_{1,y} + u_{2,x}\|_0 \le \sqrt{2}\ |U|_1$$

$$\|u_{1,x} - u_{2,y}\|_0 \le \sqrt{2}\ |U|_1 \ .$$

Proof:

$$\begin{aligned}\|u_{1,x} + u_{2,y}\|_0^2 &\le (\|u_{1,x}\|_0 + \|u_{2,y}\|_0)^2 \\ &\le 2(\|u_{1,x}\|_0^2 + \|u_{2,y}\|_0^2) \\ &\le 2|U|_1^2 \ .\end{aligned}$$

Similarly,

$$\|u_{1,y} + u_{2,x}\|_0^2 \le 2|U|_1^2$$

$$\|u_{1,x} - u_{2,y}\|_0^2 \le 2|U|_1^2 \ .$$

Lemma (B):

$$\|U\|_\varepsilon \|V\|_\varepsilon \ge |U|_1 |V|_1 + \varepsilon^{-1} \sqrt{P_S(U,U)}\ \sqrt{P_S(V,V)}$$

<u>Proof</u>: For any real numbers a,b,c,d,

$$0 \le (ad - bc)^2 = a^2d^2 + b^2c^2 - 2\,abcd$$

that is,

$$a^2d^2 + b^2c^2 \ge 2\,abcd\ .$$

Adding $a^2c^2 + b^2d^2$ gives

$$a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 \ge a^2c^2 + b^2d^2 + 2\,abcd\ ,$$

that is,

$$(a^2 + b^2)(c^2 + d^2) \ge (ac + bd)^2\ .$$

Setting $a = |U|_1$, $c = |V|_1$, $b = \epsilon^{-1/2}\sqrt{P_S(U,U)}$, $d = \epsilon^{-1/2}\sqrt{P_S(V,V)}$

$$(|U|_1|V|_1 + \epsilon^{-1/2}\sqrt{P_S(U,U)}\sqrt{P_S(V,V)})^2$$
$$\le (|U|_1^2 + \epsilon^{-1}P_S(U,U))(|V|_1^2 + \epsilon^{-1}P_S(V,V)) = \|U\|_\epsilon^2\ \|V\|_\epsilon^2$$

<u>Lemma (C)</u>: For $u \in H_0^1$,

$$\|u\|_0^2 \le \tfrac{1}{2}\|u,_x\|_0^2$$

$$\|u\|_0^2 \le \tfrac{1}{2}\|u,_y\|_0^2$$

Proof: Denote

$$\Omega_x = \{(x,y) \in \Omega, \quad \text{for} \quad x \quad \text{constant}\}$$

$$\Omega_y = \{(x,y) \in \Omega, \quad \text{for} \quad y \quad \text{constant}\} .$$

Let $(x,y) \in \Omega$. Let $(x_0,y) \in \partial\Omega$, where $x_0 < x$ and $\forall \ \xi \ni x_0 < \xi < x$, $(\xi,y) \in \Omega$, then

$$u(x,y) = \int_{x_0}^{x} \frac{\partial u}{\partial \xi} (\xi,y) d\xi$$

$$|u(x,y)| = |\int_{x_0}^{x} \frac{\partial u}{\partial \xi} (\xi,y) d\xi|$$

$$\leq \int_{x_0}^{x} |\frac{\partial u}{\partial \xi} (\xi,y)| d\xi$$

and by Schwartz inequality,

$$\leq (\int_{x_0}^{x} |\frac{\partial u}{\partial \xi} (\xi,y_0)|^2 d\xi)^{1/2} |x - x_0|^{1/2}$$

Now,

$$\|u\|_0^2 = \iint_\Omega |u(x,y)|^2 dy \, dx$$

$$\leq \iint_\Omega \int_{x_0}^{x} |\frac{\partial u(\xi,y)}{\partial \xi}|^2 \partial \xi |x - x_0| dy \, dx$$

$$\leq \iint_\Omega \int_{\Omega_y} |\frac{\partial u}{\partial \xi}|^2 d\xi |x - x_0| dx \, dy$$

$$= \iint_\Omega |\frac{\partial u}{\partial \xi}|^2 \int_{\Omega_y} |x - x_0| dx \, d\xi \, dy$$

$$\leq \frac{1}{2} \iint_\Omega |\frac{\partial u}{\partial \xi}|^2 d\xi \, dy \leq \frac{1}{2} \|u,_x\|_0^2$$

Similarly

$$\|u\|_0^2 \le \tfrac{1}{2}\|u_{,y}\|_0^2 \ .$$

<u>Lemma (D)</u>: For $u_3 \in H_0^1$, and $u_1 \in H^1$, $u_2 \in H^1$ and for all $0 < p < 1$,

$$\iint_\Omega (u_{3,x} + u_1)^2 dA \ge (1-p) \iint_\Omega u_{3,x}^2 \, dA - \frac{2}{p} \iint_\Omega u_{1,x}^2 \, dA$$

$$\iint_\Omega (u_{3,y} + u_2)^2 dA \ge (1-p) \iint_\Omega u_{3,y}^2 \, dA - \frac{2}{p} \iint_\Omega u_{2,y}^2 \, dA$$

that is,

$$P_S(U,U) \ge (1-p)|u_3|_1^2 - \frac{2}{p} \iint (u_{1,x}^2 + u_{2,y}^2) dA \ .$$

<u>Proof</u>:

$$\iint (u_{3,x} + u_1)^2 dA$$

$$\ge \iint (u_{3,x}^2 + 2u_{,}u_{3,x}) dA$$

$$= \iint (u_{3,x}^2 - 2u_3 u_{1,x}) dA \qquad \text{(by integration by parts)}$$

$$= (1-p) \iint u_{3,x}^2 \, dA + p \iint u_{3,x}^2 \, dA - 2 \iint u_3 u_{1,x} \, dA$$

$$\ge (1-p) \iint u_{3,x}^2 \, dA + \frac{p}{2} \iint u_3^2 \, dA - 2 \iint u_3 u_{1,x} \, dA$$

(by Lemma (C))

$$= (1-p) \iint u_{3,x}^2 \, dA + \frac{p}{2} \iint \left(u_3^2 - \frac{4}{p} u_3 u_{1,x} + \frac{4u_{1,x}^2}{p^2}\right) dA$$

$$- \frac{2}{p} \iint u_{1,x}^2 \, dA$$

$$= (1-p) \iint u_{3,x}^2 \, dA + \frac{p}{2} \iint \left(u_3 - \frac{2}{p} u_{1,x}\right)^2 dA$$

$$- \frac{2}{p} \iint u_{1,x}^2 \, dA$$

$$\geq (1-p) \iint u_{3,x}^2 \, dA - \frac{2}{p} \iint u_{1,x}^2 \, dA \, .$$

Similarly

$$\iint (u_{3,y} + u_2)^2 dA \geq (1-p) \iint u_{3,y}^2 \, dA - \frac{2}{p} \iint u_{2,y}^2 \, dA \, .$$

<u>Theorem 3</u>. Equation (2-1.9), which is problem (I), has a unique solution $U_\epsilon \in H_\epsilon$. For each $\epsilon > 0$ and for each $F \in (H^0)^3$.

<u>Proof</u>: Clearly $P_L(F,V)$ is a continuous linear functional on $(H_0^1)^3$. It remains to show that the bilinear form $B_\epsilon(U,V) = P_B(U,V) + \epsilon^{-1} P_S(U,V)$ is continous and coercive (i.e. satisfies conditions i) and ii) of the Lax-Milgram Theorem) on $H_\epsilon \times H_\epsilon$. The result then follows from Theorem 1.

Let $U = [u_1, u_2, u_3]^T$ and $V = [v_1, v_2, v_3]^T \in H_\epsilon$.

i) To show $B_\epsilon(U,V)$ is continuous on $H_\epsilon \times H_\epsilon$,

$$
\begin{aligned}
&|B_\epsilon(U,V)| \\
&= |P_B(U,V) + \epsilon^{-1}\ P_S(U,V)| \\
&\le |P_B(U,V)| + \epsilon^{-1}|P_S(U,V)| \\
&= |\tfrac{1}{2} \iint\ (1+\mu)(u_{1,x} + u_{2,y})(v_{1,x} + v_{2,y}) \\
&\qquad\qquad + (1-\mu)(u_{1,x} - u_{2,y})(v_{1,x} - v_{2,y}) \\
&\qquad\qquad + (1-\mu)(u_{1,y} + u_{2,x})(v_{1,y} + v_{2,x})dA| \\
&\quad + \epsilon^{-1}|\iint\ (u_{3,x} + u_1)(v_{3,x} + v_1) \\
&\qquad\qquad + (u_{3,y} + v_2)(u_{3,y} + v_2)dA|
\end{aligned}
$$

and by Schwartz inequality,

$$
\begin{aligned}
&\le \tfrac{1}{2}(1+\mu)\|u_{1,x} + u_{2,y}\|_0\|v_{1,x} + v_{2,y}\|_0 \\
&\qquad\qquad + \tfrac{1-\mu}{2}\ \|u_{1,x} - u_{2,y}\|_0\|v_{1,x} - v_{2,y}\|_0 \\
&\quad + \tfrac{1}{2}(1-\mu)\|u_{1,y} + u_{2,x}\|_0\|v_{1,y} + v_{2,x}\|_0 \\
&\quad + \epsilon^{-1}\|u_{3,x} + u_1\|_0\|u_{3,x} + v_1\|_0 \\
&\qquad\qquad + \epsilon^{-1}\|u_{3,y} + u_2\|_0\|v_{3,y} + v_2\|\ _0
\end{aligned}
$$

and by lemma (A)

$$
\begin{aligned}
&\le (3-\mu)|U|_1|V|_1 + \epsilon^{-1}\|u_{3,x} + u_1\|_0\|v_{3,x} + v_1\|_0 \\
&\qquad\qquad + \epsilon^{-1}\|u_{3,y} + u_2\|_0\|v_{3,y} + v_2\|_0 \\
&\le (3-\mu)[|U|_1|V|_1 + \epsilon^{-1}\sqrt{P_S(U,U)}\ \sqrt{P_S(V,V)}]
\end{aligned}
$$

.

$$|B_\epsilon(U,V)|$$

$$= |P_B(U,V) + \epsilon^{-1} P_S(U,V)|$$

$$\leq |P_B(U,V)| + \epsilon^{-1}|P_S(U,V)|$$

$$= |\tfrac{1}{2} \iint (1+\mu)(u_{1,x} + u_{2,y})(v_{1,x} + v_{2,y})$$

$$+ (1-\mu)(u_{1,x} - u_{2,y})(v_{1,x} - v_{2,y})$$

$$+ (1-\mu)(u_{1,y} + u_{2,x})(v_{1,y} + v_{2,x})dA|$$

$$+ \epsilon^{-1}|\iint (u_{3,x} + u_1)(v_{3,x} + v_1)$$

$$+ (u_{3,y} + v_2)(u_{3,y} + v_2)dA|$$

and by Schwartz inequality,

$$\leq \tfrac{1}{2}(1+\mu)\|u_{1,x} + u_{2,y}\|_0\|v_{1,x} + v_{2,y}\|_0$$

$$+ \frac{1-\mu}{2} \|u_{1,x} - u_{2,y}\|_0\|v_{1,x} - v_{2,y}\|_0$$

$$+ \tfrac{1}{2}(1-\mu)\|u_{1,y} + u_{2,x}\|_0\|v_{1,y} + v_{2,x}\|_0$$

$$+ \epsilon^{-1}\|u_{3,x} + u_1\|_0\|u_{3,x} + v_1\|_0$$

$$+ \epsilon^{-1}\|u_{3,y} + u_2\|_0\|v_{3,y} + v_2\|_0$$

and by lemma (A)

$$\leq (3-\mu)|U|_1|V|_1 + \epsilon^{-1}\|u_{3,x} + u_1\|_0\|v_{3,x} + v_1\|_0$$

$$+ \epsilon^{-1}\|u_{3,y} + u_2\|_0\|v_{3,y} + v_2\|_0$$

$$\leq (3-\mu)[|U|_1|V|_1 + \epsilon^{-1}\sqrt{P_S(U,U)}\sqrt{P_S(V,V)}]$$

.

and by Lemma (B),

$$\leq (3-\mu)\|U\|_\varepsilon \|V\|_\varepsilon$$

$$\leq 3\|U\|_\varepsilon \|V\|_\varepsilon$$

<u>Note</u>: $|B_\varepsilon(U,V)| \leq M\|U\|_\varepsilon \|V\|_\varepsilon$ where M is independent of ε and μ as well as U and V.

ii) to show $B_\varepsilon(V,V)$ is coercive, that is $B_\varepsilon(V,V) \geq C\|V\|_\varepsilon^2$, for all $V \in (H_0^1)^3$ where C is independent of V and ε.

$$B_\varepsilon(V,V)$$

$$= P_B(V,V) + \varepsilon^{-1} P_S(V,V)$$

$$= \frac{1}{2} \iint (1+\mu)(v_{1,x} + v_{2,y})^2 + (1-\mu)(v_{1,x} - v_{2,y})^2 + (1-\mu)(v_{1,y} + v_{2,x})^2 \, dA$$

$$+ \varepsilon^{-1} \iint (v_{3,x} + v_1)^2 + (v_{3,y} + v_2)^2 \, dA$$

$$\geq \frac{1}{2}(1-\mu) \iint (v_{1,x} - v_{2,y})^2 + (v_{1,y} + v_{2,x})^2 \, dA + \varepsilon^{-1} \iint (v_{3,x} + v_1)^2 + (v_{3,y} + v_2)^2 dA$$

$$= \frac{1}{2}(1-\mu) \iint v_{1,x}^2 + v_{2,y}^2 + v_{1,y}^2 + v_{2,x}^2 - 2v_{1,x}v_{2,y} + 2v_{1,y}v_{2,x} \, dA$$

$$+ \varepsilon^{-1} \iint (v_{3,x} + v_1)^2 + (v_{3,y} + v_2)^2 \, dA$$

and by integration by parts,

$$= \tfrac{1}{2}(1-\mu) \iint v_{1,x}^2 + v_{2,y}^2 + v_{1,y}^2 + v_{2,x}^2 \; dA$$

$$+ \epsilon^{-1} \iint (v_{3,x} + v_1)^2 + (v_{3,y} + v_2)^2 \; dA$$

$$= \tfrac{1}{2}(1-\mu) \iint v_{1,x}^2 + v_{2,y}^2 + v_{1,y}^2 + v_{2,x}^2 \; dA$$

$$+ (\epsilon^{-1} - \tfrac{1}{18}) \iint (v_{3,x} + v_1)^2 + (v_{3,y} + v_2)^2 \; dA$$

$$+ \tfrac{1}{18} \iint (v_{3,x} + v_1)^2 + (v_{3,y} + v_2)^2 \; dA$$

and by Lemma (D), with $p = \frac{1}{2}$

$$\geq \tfrac{1}{2}(1-\mu) \iint v_{1,x}^2 + v_{2,y}^2 + v_{1,y}^2 + v_{2,x}^2 \; dA$$

$$+ (\epsilon^{-1} - \tfrac{1}{18}) \iint (v_{3,x} + v_1)^2 + (v_{3,y} + v_2)^2 \; dA$$

$$+ \tfrac{1}{36} \iint v_{3,x}^2 + v_{3,y}^2 \; dA - \tfrac{2}{9} \iint v_{1,x}^2 + v_{2,y}^2 \; dA$$

$$\geq (\tfrac{1}{2}(1-\mu) - \tfrac{2}{9}) \iint v_{1,x}^2 + v_{2,y}^2 + v_{1,y}^2 + v_{2,x}^2 \; dA$$

$$+ \tfrac{1}{36} \iint v_{3,x}^2 + v_{3,y}^2 \; dA$$

$$+ (\epsilon^{-1} - \tfrac{1}{18}) \iint (v_{3,x} + v_1)^2 + (v_{3,y} + v_2)^2 \; dA$$

and for $0 < \epsilon \leq \frac{35}{2}$, and since $0 < \mu < \frac{1}{2}$,

$$B_\epsilon(V,V) \geq \tfrac{1}{36} \iint v_{1,x}^2 + v_{2,y}^2 + v_{1,y}^2 + v_{2,x}^2 + v_{3,x}^2 + v_{3,y}^2 \; dA$$

$$+ \frac{\epsilon^{-1}}{36} \iint (v_{3,x} + v_1)^2 + (v_{3,y} + v_2)^2 \; dA$$

$$= \tfrac{1}{36} \left(|V|_1^2 + \epsilon^{-1} P_S(V,V) \right) = \tfrac{1}{36} \|V\|_\epsilon^2 .$$

Note that the constant of coercivity $C = \frac{1}{36}$ for the range of $0 < \varepsilon < \frac{35}{2}$, and since ε is considered a small parameter, this is more than sufficient. The constant can be improved slightly by more judicious choice of certain constants in the estimates, as well as allowing C to vary with μ. However these improvements are slight. Any improvement will also be valid for a smaller range of ε. Likewise, if necessary, coercivity may be shown for large ε, at the cost of reducing the constant C.

<u>Theorem 4:</u> U_ε minimizes the functional

$$J(U) = B_\varepsilon(U,U) - 2P_L(F,U) \quad \text{over} \quad H_\varepsilon \ .$$

<u>Proof</u>: Since U_ε satisfies $B_\varepsilon(U_\varepsilon,V) = P_L(F,V)\ \forall\ V \in H_\varepsilon$,

$$\begin{aligned} J(U) &= B_\varepsilon(U,U) - 2B_\varepsilon(U_\varepsilon,U) \\ &= B_\varepsilon(U,U) - 2B_\varepsilon(U_\varepsilon,U) + B_\varepsilon(U_\varepsilon,U_\varepsilon) - B_\varepsilon(U_\varepsilon,U_\varepsilon) \\ &= B_\varepsilon(U-U_\varepsilon,U-U_\varepsilon) - B_\varepsilon(U_\varepsilon,U_\varepsilon) \ . \end{aligned}$$

Clearly $J(U)$ is minimized by $U = U_\varepsilon$, since $B_\varepsilon(V,V)$ is positive definite by virtue of property ii) (coercivity) of Theorem 3.

2-2. Convergence of Solutions of Problem (I) to Solutions of Problem (C)

From (2-1.15) and (2-1.9), the solution vectors U_O and U_ϵ, for problems (C) and (I) respectively, satisfy for all $V \in (H_O^1)^3$,

$$(2\text{-}2.1) \qquad P_B(U_O,V) = P_L(F,V) + \iint_\Omega \nabla(\nabla^2 w_O) \cdot \left(\nabla v_3 + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) dA$$

$$(2\text{-}2.2) \qquad B_\epsilon(U_\epsilon,V) = P_L(F,V)$$

Since $P_S(U_O,V) = 0$ for all $V \in (H_O^1)^3$, (2-2.1) can be rewritten as

$$(2\text{-}2.3) \qquad B_\epsilon(U_O,V) = P_L(F,V) + \iint_\Omega \nabla(\nabla^2 w_O) \cdot \left(\nabla v_3 + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) dA \ .$$

Subtracting gives

$$B_\epsilon(U_\epsilon - U_O,V) = \iint \nabla(\nabla^2 w_O) \cdot \left(\nabla v_3 + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) dA \ , \quad \text{for all } V \in (H_O^1)^3 \ .$$

Then

$$|B_\epsilon(U_\epsilon - U_O,V)| \le \left(\iint |\nabla(\nabla^2 w_O)|^2 dA\right)^{1/2} \left(\iint \left|\nabla v_3 + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right|^2 dA\right)^{1/2}$$

which is

$$(2\text{-}2.4) \qquad |B_\varepsilon(U_\varepsilon - U_0, V)| \le \|\nabla(\nabla^2 w_0)\|_0 (P_S(V,V))^{1/2} .$$

From $B_\varepsilon(V,V) = P_B(V,V) + \varepsilon^{-1} P_S(V,V)$ it follows $P_S(V,V) \le \varepsilon\, B_\varepsilon(V,V)$ so (2-2.4) becomes

$$(2\text{-}2.5) \qquad |B_\varepsilon(U_\varepsilon - U_0, V)| \le \varepsilon^{1/2}\|\nabla(\nabla^2 w_0)\|_0 (B_\varepsilon(V,V))^{1/2} .$$

Setting $V = U_\varepsilon - U_0$ in (2-2.5) results in

$$(2\text{-}2.6) \qquad (B_\varepsilon(U_\varepsilon - U_0, U_\varepsilon - U_0))^{1/2} \le \varepsilon^{1/2}\|\nabla(\nabla^2 w_0)\|_0$$

or

$$(2\text{-}2.7) \qquad B_\varepsilon(U_\varepsilon - U_0, U_\varepsilon - U_0) \le \varepsilon\|\nabla(\nabla^2 w_0)\|_0^2$$

which is Westbrook's energy estimate.

From (2-2.7), using coercivity and norm equivalence,

$$(2\text{-}2.8) \qquad \|U_\varepsilon - U_0\|_\varepsilon \le C\ \varepsilon^{1/2}\|\nabla(\nabla^2 w_0)\|_0$$

$$(2\text{-}2.9) \qquad \|U_\varepsilon - U_0\|_1 \le C\ \varepsilon^{1/2}\|\nabla(\nabla^2 w_0)\|_0 ,$$

where the constant C appearing above will be used in a generic sense, being different in different contexts, but being alway independent of ε and the functions involved.

The estimate (2-2.8) is sharp, but the sharpness of (2-2.9) is an open question. It can certainly have no power greater than $\varepsilon^{3/4}$. As shown in 4-2, the solution to the problem of the circular plate in the improved theory with load $p/D = \cos\theta$ exhibits a term of order $O(\epsilon^{1/2})$ over a boundary layer region. This causes $\|U_\epsilon - U_0\|_1 = O(\epsilon^{3/4})$, ruling out a dependence of U_ϵ on ϵ which is analytic at $\epsilon = 0$. The $\epsilon^{1/2}$ dependence appears in the radial derivative of the rotation component in the θ-direction, that is, $\frac{\partial\psi_\theta}{\partial r}$. $\frac{\partial\psi_\theta}{\partial r}$ corresponds to the twisting moment. The presence of this dependence prohibits improvement of $\epsilon^{1/2}$ to ϵ in (2-2.9), and may be responsible for the lack of an easily constructed asymptotic expansion for U_ϵ for small ϵ, as discussed in Chapter 4. It also may contribute to difficulties in using numerical approximations to U_ϵ in order to approximate U_0.

While (2-2.9) cannot be improved from $\epsilon^{1/2}$ to ϵ in general, this can be done in two cases of special interest. First, for the clamped beam, which may be considered a limiting case of the clamped rectangular plate; and second, for the circular plate with axisymmetric loading.

Estimates (2-2.8) and (2-2.9) are useful whenever the boundary $\partial\Omega$ is smooth. In that case, partial differential equations estimates [21] show that for w_0 satisfying

$$\nabla^4 w_0 = \frac{p}{D} \quad \text{on} \quad \Omega$$

$$w_0 = \frac{\partial w_0}{\partial N} = 0 \quad \text{on} \quad \Omega\,,$$

(2-2.10) $$\|w_0\|_4 \leq C\|\tfrac{p}{D}\|_0 \ .$$

Then

$$\|\nabla(\nabla^2 w_0)\|_0 \leq C\|w_0\|_3 \leq C\|w_0\|_4 \leq C\|\tfrac{p}{D}\|_0 < \infty \ .$$

If the boundary is not smooth,(2-2.10) is not guaranteed, and the presence of singular behavior of solutions at corners is well known. For example, in the important case of a boundary with right angles, it is precisely the square integrability of the third derivatives, needed for (2-2.8) and (2-2.9), which is jeopardized. The corner difficulty is discussed further in Chapter 4, and the possibility of extending (2-2.8) and (2-2.9) in the presence of a domain with corners, is investigated numerically in Chapter 4.

2-3. Convergence of Solutions of Problem (I) to those of Problem (C) for the Beam and Circular Plate with Axisymmetric Loading

The description of the clamped beam can be derived from the equations and energy expressions for the clamped plate by simply deleting all dependence on y, and making the obvious corresponding modifications to the definitions and proofs regarding function spaces, norms, etc. The purpose of this section is to derive estimates analogous to those for the general plate in 2-2, but with some slight but important improvement. This improvement over the general clamped plate estimates is also shared by the axisymmetrically loaded circular clamped plate, and hence it is included here also.

In what follows, the notation used in 2-2 for the general plate will also be used here. For the beam, it will be understood that ∇ and $\nabla\cdot$ mean $\frac{d}{dx}$, etc. Thus (2-1.9) and (2-1.15) are to be interpreted as

$$B_\epsilon(U_\epsilon,V) = P_L(F,V)$$

and

$$P_B(U_O,V) = P_L(F,V) + \int_{-1/2}^{1/2} w_O'''' \,(v_2' + v_1)dx \ ,$$

where

$$U_O = \begin{bmatrix} -w_O' \\ w_O \end{bmatrix}, \quad w_O'''' = \frac{p}{D}, \quad w_O = w' = 0 \quad \text{at} \quad x = \pm 1/2 \ ,$$

$$U_\epsilon = \begin{bmatrix} \psi \\ w \end{bmatrix}, \quad V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in (H_O^1)^2 = H_\epsilon \ .$$

For the circular plate with axisymmetric loading, the notation used in 2-2 is valid in its general form. However, the description of this case would ordinarily be presented in polar coordinates, as is done in Chapter 4. For the sake of the following estimates, this is not done, in order first, to maintain similarity to the beam formulation, and second, to demonstrate why this improvement fails for the general plate, and to comment on its connection with the lack of a satisfactory asymptotic expansion.

Consider the weak forms satisfied by U_ϵ and U_O, respectively, equations (2-1.9) and (2-1.15)

$$(2\text{-}3.1) \qquad B_\epsilon(U_\epsilon, V) = P_L(F,V) , \qquad \forall \quad V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in H_\epsilon ,$$

where

$$B_\epsilon(U_\epsilon, V) = P_B(U_\epsilon, V) + \epsilon^{-1} P_S(U_\epsilon, V)$$

$$(2\text{-}3.2) \qquad P_B(U_O, V)$$

$$= P_L(F,V) + \iint \nabla(\nabla^2 w_O) \cdot \left(\nabla v_3 + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) dA$$

$$\forall \quad V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in H_\epsilon .$$

Following a method similar to that used by Babuska [3] set

$$U_\epsilon = U_O - \epsilon\xi + \eta , \quad \text{where} \quad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \quad \text{and} \quad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} ,$$

$$\xi , \eta \in H_\epsilon .$$

Then (2-3.2) becomes

$$B_\epsilon(U_O - \epsilon\xi + \eta, V) = P_L(F,V)$$

or, since $P_S(U_O, V) = 0$ for all $V \in H_\epsilon$,

$$P_B(U_O, V) - \epsilon \; B_\epsilon(\xi, V) + B_\epsilon(\eta, V) = P_L(F,V) \quad .$$

Subtracting (2-3.2) gives

$$-\epsilon\ B_\epsilon(\xi,V) + B_\epsilon(\eta,V) = -\iint_\Omega \nabla(\nabla^2 w_0)\cdot\left(\nabla v_3 + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) dA$$

$$B_\epsilon(\eta,V) = \epsilon\ P_B(\xi,V) + P_S(\xi,V) - \iint_\Omega \nabla(\nabla^2 w_0)\cdot\left(\nabla v_3 + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) dA$$

(2-3.3) $$B_\epsilon(\eta,V) = \epsilon\ P_B(\xi,V) + \iint_\Omega \left(\left(\nabla\xi_3 + \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}\right) - \nabla(\nabla^2 w_0)\right) \cdot \left(\nabla v_3 + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) dA\ .$$

Now choose $\xi \in H_\epsilon$, so that

(2-3.4) $$\nabla\ \xi_3 + \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \nabla(\nabla^2 w_0)\ .$$

The existence of ξ (or lack of it for the general plate) will be shown below.

(2-3.3) then becomes

$$B_\epsilon(\eta,V) = \epsilon\ P_B(\xi,V), \quad \text{for all} \quad V \in H_\epsilon\ .$$

Now choose $V = \eta$. Then

$$\begin{aligned} B_\epsilon(\eta,\eta) &= \epsilon\ P_B(\xi,\eta) \\ &\leq \epsilon\ \sqrt{P_B(\xi,\xi)}\ \sqrt{P_B(\eta,\eta)} \\ &\leq \epsilon\ \|\xi\|_1\ \sqrt{B_\epsilon(\eta,\eta)} \end{aligned}$$

so dividing by $\sqrt{B_\epsilon(\eta,\eta)}$ gives

(2-3.5) $$\sqrt{B_\varepsilon(\eta,\eta)} \leq \varepsilon \ \|\xi\| \ .$$

It then follows that

(2-3.6) $$\|\eta\|_\varepsilon \leq C \ \varepsilon \ \|\xi\|_1$$

(2-3.7) $$\|\eta\|_1 \leq C \ \varepsilon \ \|\xi\|_1 \quad ,$$ where $\xi \in H_\varepsilon$ satisfying (2-3.4) is independent of ε, depending only upon w_0 .

Now $U_\varepsilon - U_0$ can be estimated using (2-3.6) and (2-3.7), with norm equivalence. Since $U_\varepsilon - U_0 = -\varepsilon\xi + \eta$,

(2-3.8) $$\|U_\varepsilon - U_0\| = \|-\varepsilon\xi + \eta\|_\varepsilon$$
$$\leq \varepsilon\|\xi\|_\varepsilon + \|\eta\|_\varepsilon \leq C \ \varepsilon^{1/2} \ \|\xi\|_1$$

(2-3.9) $$\|U_\varepsilon - U_0\|_1 = \|-\varepsilon\xi + \eta\|_1 \leq \varepsilon\|\xi\|_1 + \|\eta\|_1 \leq C \ \varepsilon \ \|\xi\|_1$$ where ξ depends only on the classical solution w_0.

Although (2-3.8) has the same $\varepsilon^{1/2}$ dependence as does (2-2.8), (2-3.9) shows the improvement over (2-2.9), from $\varepsilon^{1/2}$ to ε.

This improvement was uncovered by splitting the error term and explicitly determinine one part, then using it to estimate the other part more finely. This is analogous to assuming an asymptotic series form for a solution, then explicitly solving for the consecutive terms, using each one to derive the next. The success of this indirect method of estimating $U_\epsilon - U_0$ hinges on the existence of $\xi \in H_\epsilon$, a solution of (2-3.4).

For the beam, the analog of (2-3.4) is

$$\frac{d\xi_2}{dx} + \xi_1 = \frac{d^3w_0}{dx^3} . \tag{2-3.10}$$

Along with the homogeneous boundary conditions $\xi_1 = \xi_2 = 0$ at $x = \pm\frac{1}{2}$ assigned to all members of H_ϵ, a solution can be formed by defining

$$\xi_1 = \frac{d^3w_0}{dx^3} - \frac{d\xi_2}{dx}$$

where ξ_2 is the solution, for any function $g(x)$, of

$$\frac{d^4\xi_2}{dx^4} = g(x)$$

$$\xi_2 = 0 \quad \text{at} \quad x = \pm\frac{1}{2}$$

$$\frac{d\xi_2}{dx} = \frac{d^3w_0}{dx^3} \quad \text{at} \quad x = \pm\frac{1}{2} .$$

By the boundary conditions assigned to ξ_2, it is clear that $\xi_1 = 0$ at $x = \pm\frac{1}{2}$. One possibility is to choose $g(x)$ to be identically zero. However, if $g(x)$ is chosen to be $\frac{p''}{D}$, then ξ agrees with the first order term of the formal asymptotic expansion of U_ε given in Chapter 3.

For the axisymmetrically loaded plate, (2-3.4) can be solved analogously, by defining

$$\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \nabla(\nabla^2 w_0) - \nabla\xi_3 \tag{2-3.11}$$

where ξ_3 satisfies

$$\nabla^4\xi_3 = g$$

$$\xi_3 = 0 \quad \text{on} \quad \partial\Omega$$

$$\frac{\partial\xi_3}{\partial N} = \frac{\partial(\nabla^2 w_0)}{\partial N} \quad \text{on} \quad \partial\Omega .$$

Here it is less clear, but still true that ξ_1 and ξ_2 satisfy the necessary boundary conditions $\xi_1 = \xi_2 = 0$. Considered as a two dimensional vector, $\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$ has components tangential and normal to the boundary, both of which must vanish. By virtue of the boundary conditions assigned to ξ_3, the normal component of (2-3.11) vanishes:

$$\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \cdot \hat{N} = \frac{\partial}{\partial N}(\nabla^2 w_0) - \frac{\partial}{\partial N}\xi_3 = 0 .$$

As for the tangential component, since ξ_3 vanishes on the boundary, its gradient is perpendicular to the boundary. And since w_0 is the classical solution to the axisymmetrically loaded circular plate problem, it is a function of radius only, as is $\nabla^2 w_0$, hence $\nabla(\nabla^2 w_0)$ is also perpendicular to the boundary. Thus the tangential components vanish for each term on the right side of (2-3.11). Since the vector has zero normal and tangential components at the boundary,

$$\xi_1 = \xi_2 = 0 \quad \text{on} \quad \partial\Omega \ .$$

The difficulty in applying this indirect method for the general plate problem is inherent in (2-3.4). Because ξ_3 must vanish on the boundary, its gradient will always be normal to the boundary. Because $\xi_1 = \xi_2 = 0$, the entire left side of (2-3.4) will be normal to $\partial\Omega$, whereas, in general $\nabla(\nabla^2 w_0)$ will have both tangential and normal components. Except for the two special cases above, (2-3.4) is generally inconsistent with the required boundary conditions on ξ. It is of interest that (2-3.4) is exactly the condition which prevents the evolution of even the first order term of an asymptotic expansion of U_ϵ in powers of ϵ. Such inconsistencies are usually an indication of boundary layer phenomena. In one dimension, boundary layers can be handled by matching expansions valid near the boundary to those valid away from the boundary. In two dimensions, even for simple geometries this is not very hopeful, since the boundary layer thickness will tend to vary.

The presence of difficulties near the boundary is not altogether unexpected. Even solutions of problem (C) exhibit shear singularities at corners of the boundary, when they are present. This is discussed in Chapter 4. The boundary phenomena have implications which may affect approximate solutions even well away from the boundary.

2-4. A Finite Element Theory for Problem (I)

Before using a finite element method to approximate solutions of problem (I), it must be determined how the quality of these approximations depends on the shear rigidity, measured by ϵ^{-1}. This is illustrated by error estimates in terms of ϵ and the finite element mesh size h.

In addition, the value of using finite element approximations to solutions of problem (I), in order to approximate solutions of problem (C), is investigated using methods similar to Babuska and Aziz [3]. When successful, this indirect method avoids the use of complicated elements normally required to approximate solutions of problem (C) directly.

Consider the weak form of problem (I). For each $\epsilon > 0$, $U_\epsilon = [\psi_x, \psi_y, w]^T$ satisfies (2-1.9), i.e.,

$$B_\epsilon(U_\epsilon, V) = P_L(F, V), \quad \text{for all} \quad V \in H_\epsilon \ .$$

Let S_h be an N-dimensional subspace of H_ϵ with basis $\{\phi_1, \phi_2, \ldots, \phi_N\}$ as yet unspecified. Let $U_h = \sum_{i=1}^{N} q_i \phi_i$.

In order that U_h best approximate U_ϵ (in the sense of the energy functional $B_\epsilon(\cdot,\cdot)$) the q_i are chosen to satisfy the algebraic system of linear equations

$$B_\epsilon(U_h, \phi_i) = P_L(F, \phi_i) , \quad i = 1, \ldots, N , \tag{2-4.1}$$

that is

$$\sum_{i=1}^{N} q_j B_\epsilon(\phi_j, \phi_i) = P_L(F, \phi_i) , \quad i = 1, \ldots, N . \tag{2-4.2}$$

Equations (2-4.2) can also be derived by minimizing the functional

$$J = B_\epsilon(U,U) - 2P_L(F,U) \tag{2-4.3}$$

over all $U \in S_h$, i.e. by chosing $q_1, q_2, \ldots, q_N$ to minimize

$$\begin{aligned} J &= B_\epsilon\left(\sum_{j=1}^{N} q_j\phi_j, \sum_{i=1}^{N} q_i\phi_i\right) - 2P_L\left(F, \sum_{i=1}^{N} q_i\phi_i\right) \\ &= \sum_{j=1}^{N} \sum_{i=1}^{N} q_j q_i B_\epsilon(\phi_j, \phi_i) - 2 \sum_{i=1}^{N} q_i P_L(F, \phi_i) \quad . \end{aligned} \tag{2-4.4}$$

Setting $\frac{\partial J}{\partial q_i} = 0$ produces (2-4.2), noting that B_ϵ is a symmetric, positive definite bilinear form. The existence of a solution $U_h \in S_h$ to (2-4.1) and its property of minimizing J follow from these corollaries to the theorems in 2-2:

Corollary to Theorem 3.

Let S be any closed subspace of H_ϵ. There exists a unique solution U_s satisfying

$$B_\epsilon(U_s,V) = P_L(F,V) \quad \text{for all} \quad V \in S \ .$$

Corollary to Theorem 4.

U_s minimizes

$$J(U) = B_\epsilon(U,U) - 2P_L(F,U) \quad \text{over} \quad S \ .$$

The proofs are identical to those of the theorems themselves.

The following Theorem, with its elementary proof, define the sense in which $U_h \in S_h$ is best approximation to U_ϵ.

Theorem 5: Let S be a subspace of H_ϵ. If $U_s \in S$ satisfies

$$B_\epsilon(U_s,V) = P_L(F,V) \ , \quad \forall \ V \in S \ ,$$

then

$$B_\varepsilon(U_\varepsilon - U_s, U_\varepsilon - U_s) \le B_\varepsilon(U_\varepsilon - V, U_\varepsilon - V) \quad \forall \ V \in S \ .$$

Proof:

$$B_\varepsilon(U_\varepsilon - V, U_\varepsilon - V) = B_\varepsilon(U_\varepsilon, U_\varepsilon) - 2B_\varepsilon(U_\varepsilon, V) + B_\varepsilon(V,V)$$

$$= B_\varepsilon(U_\varepsilon, U_\varepsilon) - 2P_L(F,V) + B_\varepsilon(V,V) \ .$$

The right hand side is minimized provided $-2P_L(F,V) + B_\varepsilon(V,V)$ is minimized. By the Corollary to Theorem 4, U_s minimizes $J(V) = B_\varepsilon(V,V) - 2P_L(F,V)$.

The system in (2-4.2) is often written in matrix form:

$$KQ = \tilde{F} \tag{2-4.5}$$

where

$$K_{ij} = B_\varepsilon(\phi_j, \phi_i)$$

is refered to as the stiffness matrix. Note that K is symmetric and positive definite.

$$Q = [q_1, q_2, \ldots, q_N]^T$$

$$\tilde{F} = [P_L(F, \phi_i)]^T \ ,$$

the discrete load vector.

Similarly, (2-5.4) can be written

$$J = Q^T K Q - 2\tilde{F}Q \tag{2-4.6}$$

and (2-4.5) derived from it by $\frac{\partial J}{\partial q_i} = 0$.

Equation (2-4.5) is used to determine the finite element approximation to U_ϵ. The construction of K and $\tilde{F}$ will be described in Chapters 3 and 4.

The basic elements used here will be vector functions of the form $\phi_i = [\varphi_i,0,0]^T$ or $\phi_i = [0,\varphi_1,0]^T$ or $\phi_i = [0,0,\varphi_i]^T$. The functions φ_i, used to construct all three types, will be piecewise linear in the numerical examples presented later.

Consider a (t,k)-system $S_h^{t,k}$ as described in Chapter 1. Let $S_h = S_h^{t,k} \times S_h^{t,k} \times S_h^{t,k}$, and assume for each fixed h, that $S_h \subset H_\epsilon$. Equivalently, $S_h \subset (H_0^1)^3$. By the approximation properties of $S_h^{t,k}$, there exists $W \in S_h$ such that

$$\|U_\epsilon - W\|_s \leq C\, h^\mu \|U_\epsilon\|_q\,, \quad \text{where} \quad \mu = \min(t-s, q-s)\ .$$

Taking $s = 1$,

$$\|U_\epsilon - W\|_1 \leq C\, h^\mu \|U_\epsilon\|_q\ . \tag{2-4.7}$$

Now applying Theorem 5, with the coercive and continuity properties of B_ϵ ,

$$
\begin{aligned}
\|U_\varepsilon - U_h\|_\varepsilon^2 &\leq C \; B_\varepsilon(U_\varepsilon - U_h, U_\varepsilon - U_h) \\
&\leq C \; B_\varepsilon(U_\varepsilon - W, U_\varepsilon - W) \\
&\leq C\|U_\varepsilon - W\|_\varepsilon^2 \\
&\leq C \; \varepsilon^{-1}\|U_\varepsilon - W\|_1^2 \\
&\leq C \; \varepsilon^{-1} \; h^{2\mu}\|U_\varepsilon\|_q^2 \; .
\end{aligned}
$$

That is,

$$
(2\text{-}4.8) \qquad \|U_\varepsilon - U_h\|_\varepsilon \leq C \; \varepsilon^{-1/2} \; h^\mu \|U_\varepsilon\|_q
$$

and therefore,

$$
(2\text{-}4.9) \qquad \|U_\varepsilon - U_h\|_1 \leq C \; \varepsilon^{-1/2} \; h^\mu \|U_\varepsilon\|_q \; .
$$

In the important case, $t = 2$ (piecewise linear elements)

$$
(2\text{-}4.10) \qquad \|U_\varepsilon - U_h\|_\varepsilon \leq C \; \varepsilon^{-1/2} \; h\|U_\varepsilon\|_2
$$

$$
(2\text{-}4.11) \qquad \|U_\varepsilon - U_h\|_1 \leq C \; \varepsilon^{-1/2} \; h\|U_\varepsilon\|_2 \; .
$$

The presence of $\varepsilon^{-1/2}$ on the right side of these estimates indicates that difficulty can be expected in the accuracy of finite element approximations to solutions of problem (I), when the shear rigidity is large.

Now consider the approximation of U_0 by U_h. Let $U_0 = [-w_{0,x}, -w_{0,y}, w_0]^T$ solve problem (C), i.e.

$$\nabla^4 w_0 = \frac{p}{D} \quad \text{on} \quad \Omega$$

$$w_0 = \frac{\partial w_0}{\partial N} = 0 \quad \text{on} \quad \partial\Omega .$$

Let S_h be defined as before. By the approximation properties of $S_h^{t,k}$, there exists $W_1 \in S_h$ such that

$$\|U_0 - W_1\|_s \leq C\, h^{\mu} \|U_0\|_q , \tag{2-4.12}$$

$$\text{where} \quad \mu = \min(t-s, q-s) .$$

Using (2-2.8) and norm equivalence,

$$\begin{aligned} \|U_\varepsilon - W_1\|_\varepsilon &= \|U_\varepsilon - U_0 + U_0 - W_1\|_\varepsilon \\ &\leq \|U_\varepsilon - U_0\|_\varepsilon + \|U_0 - W_1\|_\varepsilon \\ &\leq C(\varepsilon^{1/2}\|\nabla(\nabla^2 w_0)\|_0 + \varepsilon^{-1/2}\|U_0 - W_1\|_1) . \end{aligned}$$

Now setting $s = 1$ in (2-4.12), (2-4.13) becomes

$$\|U_\varepsilon - W_1\|_\varepsilon \leq C(\varepsilon^{1/2}\|\nabla(\nabla^2 w_0)\|_0 + \varepsilon^{-1/2}\, h^{\mu}\|U_0\|_q) , \tag{2-4.14}$$

$$\mu = \min(t-1, q-1) .$$

Of primary interest here is the case $t = 2$ (piecewise linear finite elements).

Then (2-4.14) gives

$$(2\text{-}4.15)\qquad \|U_\varepsilon - W_1\|_\varepsilon \le C(\varepsilon^{1/2}\|\nabla(\nabla^2 w_0)\|_0 + \varepsilon^{-1/2}\, h\|U_0\|_2)$$

Since $\|\nabla(\nabla^2 w_0)\|_0 \le C\|U_0\|_2$, (2-4.15) can be written more concisely as

$$(2\text{-}4.16)\qquad \|U_\varepsilon - W_1\|_\varepsilon \le C(\varepsilon^{1/2} + \varepsilon^{-1/2}\, h)\|U_0\|_2 \ .$$

If piecewise linear finite elements are abandoned in favor of piecewise quadratics, the power on h in (2-4.14) can be improved to 2, but only in the presence of enough smoothness in U_0 to make $\|U_0\|_3 < \infty$. The question of smoothness for the solution of problem (C) will be discussed later for the case of domains with non-smooth boundaries. If the boundary is infinitely differentiable, the result mentioned in 2-2 holds, namely

$$\|w_0\|_4 \le C\|f\|_0 \ .$$

In terms of the vector U_0, and with $f = \frac{p}{D} \in H_0$

$$(2\text{-}4.17)\qquad \|U_0\|_3 \le C\|\tfrac{p}{D}\|_0 \ .$$

The load p may be smoother, allowing potentially better approximation, but the main limitation is the degree of piecewise polynomials used in the finite elements.

A compromise can be reached between piecewise linear and piecewise quadratic elements by using the quadratics to generate the third component (displacement) and linear elements for the first two. In this case a power h^2 in (2-4.14) is retained as if quadratics had been used for all three components.

Using (2-4.16), which is based on piecewise linear finite elements, a comparison can be derived between U_ε and U_h, the finite element solution of (2-4.1). By Theorem 5, coercivity and continuity of B_ε, and (2-4.16),

$$\begin{aligned}\|U_\varepsilon - U_h\|_\varepsilon^2 &\le C\ B_\varepsilon(U_\varepsilon - U_h, U_\varepsilon - U_h)\\ &\le C\ B_\varepsilon(U_\varepsilon - W_1, U_\varepsilon - W_1)\\ &\le C\|U_\varepsilon - W_1\|_\varepsilon^2\\ &\le C(\varepsilon^{1/2} + \varepsilon^{-1/2}\ h)^2\|U_0\|_2^2\ .\end{aligned}$$

That is

$$\|U_\varepsilon - U_h\|_\varepsilon \le C(\varepsilon^{1/2} + \varepsilon^{-1/2}\ h)\|U_0\|_2 \tag{2-4.18}$$

and by norm equivalence

$$\|U_\varepsilon - U_h\|_1 \le C(\varepsilon^{1/2} + \varepsilon^{-1/2}\ h)\|U_0\|_2\ . \tag{2-4.19}$$

Finally, by (2-4.18) and (2-2.8),

$$\begin{aligned}\|U_0 - U_h\|_\varepsilon &\le \|U_0 - U_\varepsilon\|_\varepsilon + \|U_\varepsilon - U_h\|_\varepsilon\\ &\le C(\varepsilon^{1/2} + \varepsilon^{-1/2}\ h)\|U_0\|_2\ .\end{aligned} \tag{2-4.20}$$

Similarly, by (2-4.19) and (2-2.9),

$$(2\text{-}4.21) \qquad \|U_0 - U_h\|_1 \leq C(\epsilon^{1/2} + \epsilon^{-1/2}\, h)\|U_0\|_2 \,.$$

Estimates (2-4.20) and (2-4.21) show first that no convergence is expected either for h fixed and ϵ going to zero, or for ϵ fixed with h going to zero. Rather both must go to zero, and it is even important that h go to zero faster than $\epsilon^{1/2}$. Both terms on the right can be made to converge at the same rate by choosing ϵ proportional to h. Then

$$(2\text{-}4.22) \qquad \|U_0 - U_h\|_1 \leq C\, h^{1/2}\|U_0\|_2 \,.$$

While (2-4.22) is appealingly simple, and setting ϵ proportional to h seems a natural way of disposing of the freedom in picking ϵ, in practice, highly unreliable numerical output will result. The mesh size h is, in practice, finite. In fact, it is severly limited from becoming too small, due to the rapidly increasing dimension of the stiffness matrix. Since the coefficient of $h^{1/2}$ in (2-4.22) may be very large, (2-4.22) guarantees little if any accuracy. A method of extrapolation as well as some inferences from the asymptotic and numerical results will prove more reliable.

(2-4.21) also reinforces the expectations expressed in [4 ,18] that letting ϵ become extremely small "spoils" the solution. This is seen through the negative power of ϵ multiplying h. It appears h must be made small enough to overcome this adverse ϵ factor, or a way must be found to make inferences about U_0 from finite element solutions U_h generated by ϵ values which are not "too small". Since h is severly limited the second approach must be followed.

It should be pointed out that the requirement $S_h \subset (H_0^1)^3$ is somewhat restrictive. In practice the finite elements usually have their support over a polygonal domain. Generally, nodes of the subdivision of the original domain Ω are placed on the true boundary. If Ω were a smooth non-convex region, being subdivided by triangles, for example, there would be points outside Ω where a finite element trial solution would be non-zero. Since functions in $(H_0^1)^3$ must satisfy homogeneous boundary conditions, the containment $S_h \subset (H_0^1)^3$ would fail, as would the estimates derived. In this case it is necessary to carefully analyze the so called "skin" region near the boundary. The containment $S_h \subset (H_0^1)^3$ can be guaranteed with polygonal subdivisions of Ω, provided Ω is convex, or a finite union of convex sets. Of course, boundary regularity, as discussed earlier, is a concern in addition to that mentioned here.

Estimates (2-4.18) - (2-4.21) have counterparts showing slight improvement for the special cases of section 2-3, namely the clamped beam and the clamped circular plate with axisymmetric loading.

Recalling the method of 2-3, $U_\varepsilon = U_0 - \varepsilon\xi + \eta$, where ξ is a vector function depending only on U_0. With the same definition of S_h as before, there exists W_1 and W_2 in S_h such that

(2-4.23) $$\|U_0 - W_1\|_1 \le C\ h^\mu \|U_0\|_q \qquad \mu = (t-1, q-1)$$

(2-4.24) $$\|\xi - W_2\|_1 \le C\ h^\mu \|\xi\|_q \qquad \mu = (t-1, q-1)\ .$$

Now using (2-3.7), (2-4.23), (2-4.24)

(2-4.25) $$\begin{aligned} &\|U_\varepsilon - W_1 + \varepsilon\ W_2\|_\varepsilon \\ &= \|U_0 - \varepsilon\xi + \eta - W_1 + \varepsilon\ W_2\|_\varepsilon \\ &\le \|U_0 - W_1\|_\varepsilon + \varepsilon\|\xi - W_2\|_\varepsilon + \|\eta\|_\varepsilon \\ &\le C(\varepsilon^{-1/2}\|U_0 - W_1\|_1 + \varepsilon^{1/2}\|\xi - W_2\|_1 + \varepsilon\|\xi\|_1) \\ &\le C(\varepsilon^{-1/2}\ h^\mu\|U_0\|_q + \varepsilon^{1/2}\ h^\mu\|\xi\|_q + \varepsilon\|\xi\|_1)\ . \end{aligned}$$

Taking $q = 2$, $t = 2$,

(2-4.26) $$\begin{aligned} &\|U_\varepsilon - W_1 + \varepsilon\ W_2\|_\varepsilon \\ &\le C(\varepsilon^{-1/2}\ h\|U_0\|_2 + \varepsilon^{1/2}\ h\|\xi\|_2 + \varepsilon\|\xi\|_1)\ . \end{aligned}$$

Using Theorem 5, and coercivity and continuity of B_ε and (2-4.26)

$$(2\text{-}4.27) \qquad \|U_\varepsilon - U_h\|_\varepsilon^2 \le C\ B_\varepsilon(U_\varepsilon - U_h, U_\varepsilon - U_h)$$
$$\le C\ B_\varepsilon(U_\varepsilon - W_1 + \varepsilon\ W_2, U_\varepsilon - W_1 + \varepsilon\ W_2)$$
$$\le C\|U_\varepsilon - W_1 + \varepsilon\ W_2\|_\varepsilon^2$$
$$\le C(\varepsilon^{-1/2}\ h\|U_0\|_2 + \varepsilon^{1/2}\ h\|\xi\|_2 + \varepsilon\|\xi\|_1)^2\ ,$$

so

$$(2\text{-}4.28) \qquad \|U_\varepsilon - U_h\|_\varepsilon$$
$$\le C(\varepsilon^{-1/2}\ h\|U_0\|_2 + \varepsilon^{1/2}\ h\|\xi\|_2 + \varepsilon\|\xi\|_1)$$

and (2-4.28) with norm equivalence gives

$$(2\text{-}4.29) \qquad \|U_\varepsilon - U_h\|_1$$
$$\le C(\varepsilon^{-1/2}\ h\|U_0\|_2 + \varepsilon^{1/2}\ h\|\xi\|_2 + \varepsilon\|\xi\|_1)\ .$$

Now

$$(2\text{-}4.30) \qquad \|U_0 - U_h\|_\varepsilon \le \|U_0 - U_\varepsilon\|_\varepsilon + \|U_\varepsilon - U_h\|_\varepsilon$$

combined with (2-4.28) and (2-3.8) gives

$$(2\text{-}4.31) \qquad \|U_0 - U_h\|_\varepsilon$$
$$\le C(\varepsilon^{-1/2}\ h\|U_0\|_2 + \varepsilon^{1/2}\ h\|\xi\|_2 + \varepsilon\|\xi\|_1 + \varepsilon^{1/2}\|\xi\|_1)\ .$$

However, using (2-4.29) and (2-3.9),

$$\text{(2-4.32)} \quad \|U_0 - U_h\|_1 \le C(\varepsilon^{-1/2} h\|U_0\|_2 + \varepsilon^{1/2} h\|\xi\|_2 + \varepsilon\|\xi\|_1) .$$

(2-4.32) is improved over (2-4.21) by having ε rather than $\varepsilon^{1/2}$ in the term independent of h. The additional term $\varepsilon^{1/2} h\|\xi\|_2$ is of higher order than the term before it, for small ε and h.

CHAPTER 3 - BENDING OF A CLAMPED TIMOSHENKO BEAM

The clamped Timoshenko beam is the one-dimensional analog of problem (I). Its formulation can be derived from that of problem (I) by deleting all dependence on y. Similarly, the clamped beam in the classical theory can be described. Both reduce to two point boundary value problems which can be compared and analyzed in more detail than the two dimensional boundary value problems posed by problems (C) and (I).

An asymptotic analysis of the two beam problems is carried out in 3-1. 3-2 contains an asymptotic pointwise analysis of discretization error generated by approximating the Timoshenko beam solution by piecewise linear finite elements. In 3-3 the element stiffness matrix is constructed, and 3-4 contains numerical results.

3-1. Asymptotic Expansion for the Solution to the Clamped Beam Problem in Timoshenko's Theory

Define the operator L by

$$(3\text{-}1.1) \qquad L = \begin{bmatrix} -\frac{d^2}{dx^2} & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon^{-1} \begin{bmatrix} 1 & \frac{d}{dx} \\ -\frac{d}{dx} & -\frac{d^2}{dx^2} \end{bmatrix}$$

and

(3-1.2) $$U = \begin{bmatrix} \psi \\ w \end{bmatrix}, \qquad F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

The differential system

(3-1.3a) $$LU = F$$

represents the Timoshenko beam equations when $f_1 = 0$ and $f_2 = \frac{p}{D}$. Consider also the clamped boundary condition

(3-1.3b) $$U = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ at } x = \pm\frac{1}{2}.$$

In what follows, f_1 and f_2 will both be allowed to be non-zero, and will be considered infinitely differentiable. This more general right hand side will need to be considered in the discussion of principal error in 3-2.

Due to the homogeneous boundary conditions, a solution U to the system $LU = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ can be written as

(3-1.4) $$U = U^{(1)} + U^{(2)},$$

where

(3-1.5) $$LU^{(1)} = \begin{bmatrix} f_1 \\ 0 \end{bmatrix}$$

(3-1.6) $$LU^{(2)} = \begin{bmatrix} 0 \\ f_2 \end{bmatrix}$$

and

$$(3\text{-}1.7) \qquad U = U^{(1)} = U^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ at } x = \pm\tfrac{1}{2} .$$

Let

$$(3\text{-}1.8) \qquad U^{(1)} = \begin{bmatrix} \alpha \\ u \end{bmatrix}, \qquad U^{(2)} = \begin{bmatrix} \beta \\ w \end{bmatrix}.$$

Expanding each in powers of the small parameter ϵ, and solving for the coefficients produces a uniform asymptotic expansion for each solution $U^{(1)}$ and $U^{(2)}$.

First, let

$$(3\text{-}1.9) \qquad U^{(1)} = \sum_{i=0} \epsilon^{i} U_i^{(1)}$$

$$\text{where } U_i^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ at } x = \pm\tfrac{1}{2} ,$$

that is,

$$(3\text{-}1.10a) \qquad \alpha = \sum_{i=0} \epsilon^{i} \alpha_i , \quad \alpha_i = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2} ,$$

$$(3\text{-}1.10b) \qquad u = \sum_{i=0} \epsilon^{i} u_i , \quad u_i = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2} .$$

Substituting equations (3-1.10) into (3-1.5) and equating powers of ϵ yields

$$(3\text{-}1.11a) \qquad \alpha_0 + u_0' = 0$$

$$(3\text{-}1.11b) \qquad \alpha_0' + u_0'' = 0$$

(3-1.12a) $$\alpha_1 + u_1' = f_1 + \alpha_0''$$

(3-1.12b) $$\alpha_1' + u_1'' = 0$$

(3-1.13a) $$\alpha_2 + u_2' = \alpha_1''$$

(3-1.13b) $$\alpha_2' + u_2'' = 0$$

etc., in general

(3-1.14a) $$\alpha_i + u_i' = \alpha_{i-1}''$$

(3-1.14b) $$\alpha_i' + u_i'' = 0 \quad \text{for} \quad i = 2,3,4,\ldots \ .$$

Differentiating (3-1.11b), substituting into (3-1.12a), differentiating (3-1.12a) and subtracting (3-1.12b) yields

(3-1.15a) $$u_0'''' = f_1' \ .$$

Since $u_0 = 0$ at $x = \pm\frac{1}{2}$ (3-1.11a) implies

(3-1.15b) $$u_0' = -\alpha_0 = 0 \quad \text{at} \quad x = \pm\frac{1}{2} \ ,$$

and u_0 is uniquely determined. (3-1.11a) then gives

(3-1.15c) $$\alpha_0 = -u_0' \ .$$

In a similar manner, u_1 is seen to satisfy

(3-1.15a) $$u_1'''' = 0$$

with

$$u_1 = 0 \text{ and } u_1' = f_1 - u_0''' \text{ at } x = \pm\frac{1}{2}\,. \tag{3-1.15b}$$

Then

$$\alpha_1 = f_1 - u_0'''' - u_1'\,. \tag{3-1.16c}$$

And u_2 satisfies

$$u'''' = 0 \tag{3-1.17a}$$

$$u_2 = 0 \text{ and } u_2' = -u_1''' \text{ at } x = \pm\frac{1}{2}\,. \tag{3-1.17b}$$

Then

$$\alpha_2 = -u_1''' - u_2'\,. \tag{3-1.17c}$$

In general, for $i = 2,3,4,\ldots$

$$u_i'''' = 0 \tag{3-1.18a}$$

$$u_i = 0 \text{ and } u_i' = -u_{i-1}''' \text{ at } x = \pm\frac{1}{2}\,. \tag{3-1.18b}$$

Then

$$\alpha_i = -u_{i-1}''' - u_i'\,. \tag{3-1.18c}$$

In exactly the same manner $U^{(2)}$ is determined.

Let

$$(3\text{-}1.19) \qquad U^{(2)} = \sum_{i=0} \varepsilon^i U_i^{(2)}$$

$$\text{where} \quad U_i^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{at} \quad x = \pm\tfrac{1}{2} ,$$

that is,

$$(3\text{-}1.20a) \qquad \beta = \sum_{i=0} \varepsilon^i \beta_i , \quad \beta_i = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}$$

$$(3\text{-}1.20b) \qquad w = \sum_{i=0} \varepsilon^i w_i , \quad w_i = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2} .$$

Substituting equations (3-1.20) into (3-1.6) and equating powers of ε as before produces the following conditions which determine the q_i and w_i:

$$(3\text{-}1.21a) \qquad w_0'''' = f_2$$

$$(3\text{-}1.21b) \qquad w_0 = w_0' = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}$$

$$(3\text{-}1.21c) \qquad \beta_0 = -w_0'$$

$$(3\text{-}1.22a) \qquad w_1'''' = -f_2''$$

$$(3\text{-}1.22b) \qquad w_1 = 0, \quad w_1' = -w_0''' \quad \text{at} \quad x = \pm\tfrac{1}{2}$$

$$(3\text{-}1.22c) \qquad \beta_1 = -w_0''' - w_1'$$

$$(3\text{-}1.23a) \qquad w_2'''' = 0$$

$$(3\text{-}1.23b) \qquad w_2 = 0, \quad w_2' = -f_2' - w_1''' \quad \text{at} \quad x = \pm\tfrac{1}{2}$$

$$(3\text{-}1.23c) \qquad \beta_2 = -f_2' - w_1''' - w_2'$$

(3-1.24a) $$w_3'''' = 0$$

(3-1.24b) $$w_3 = 0, \quad w_3' = -w_2''' \quad \text{at} \quad x = \pm\frac{1}{2}$$

(3-1.24c) $$\beta_3 = -w_2''' - w_3'$$

and for $i = 3,4,5,\ldots$

(3-1.25a) $$w_i'''' = 0$$

(3-1.25b) $$w_i = 0, \quad w_i' = -w_{i-1}''' \quad \text{at} \quad x = \pm\frac{1}{2}$$

(3-1.25c) $$\beta_i = -w_{i-1}''' - w_i' \ .$$

Some useful observations can be made from these expansions. If f_2 is an even function $(f_2(-x) = f_2(x))$, then the series in (3-1.10) truncate after only two terms for the following reasons: First, if f_2 is even, then each w_i is even. Next, the function defined by $A(x) = -f_2' - w_1'''$ is constant, since $A'(x) = -f_2'' - w_1'''' \equiv 0$ by (3-1.22a). However by (3-1.23b) $w_2'(\frac{1}{2}) = A(\frac{1}{2}) = A(-\frac{1}{2}) = w_2'(-\frac{1}{2})$. Since w_2 is even, w_2' is odd, so that $-w_2'(\frac{1}{2}) = w_2'(-\frac{1}{2})$. So they both vanish. Thus from (3-1.23a) and (3-1.23b), w_2 satisfies

(3-1.26a) $$w_2'''' = 0$$

(3-1.26b) $$w_2 = w_2' = 0 \quad \text{at} \quad x = \pm\frac{1}{2}$$

implying that

(3-1.27) $$w_2 \equiv 0 \ .$$

The argument above also implies that $A(x)$ vanishes, and from (3-1.23c) it follows that

$$\beta_2 \equiv 0 . \tag{3-1.28}$$

It then follows by the recursive nature of relations (3-1.25) that

$$\beta_i \equiv w_i \equiv 0 \quad \text{for} \quad i = 3,4,\ldots . \tag{3-1.29}$$

To summarize, if f_2 is an even function, then the solution $U^{(1)} = \begin{bmatrix} q \\ w \end{bmatrix}$ to the system

$$LU = \begin{bmatrix} 0 \\ f_2 \end{bmatrix}, \qquad \beta = w = 0 \quad \text{at} \quad x = \pm\frac{1}{2}$$

is given simply by

$$w = w_0 + \varepsilon\, w_1 \tag{3-1.30a}$$

$$\beta = -w_0' - \varepsilon\,(w_0''' - w_1') \tag{3-1.30b}$$

where w_0 and w_1 are determined in (3-1.21) and (3-1.22).

In particular, if $f_2 = \frac{p}{D}$, p being the transverse load applied along the beam, then $\begin{bmatrix} \beta \\ w \end{bmatrix}$ represents the solution $\begin{bmatrix} \psi \\ w \end{bmatrix}$ to the clamped beam problem in Timoshenko's theory, and w_0 represents the solution to the clamped beam problem in the classical theory. Similarly each w_i

represents a solution to a particular clamped beam problem in the classical theory with slope at $x = \pm\frac{1}{2}$ determined by the previous term. Now if p is a symmetric load, then w and β differ from w_0 and $-w_0'$, respectively, by a term simply linear in ϵ.

No such conclusion can be reached for p being an antisymmetric loading. However, it is useful to notice that since any load p can be written as a symmetric plus an antisymmetric part, i.e.

$$p = p_s + p_A \tag{3-1.31}$$

and under the homogeneous boundary conditions the solution $\begin{bmatrix} \beta \\ w \end{bmatrix}$ can be likewise decomposed

$$\begin{bmatrix} \beta \\ w \end{bmatrix} = \begin{bmatrix} \beta_s \\ w_s \end{bmatrix} + \begin{bmatrix} \beta_A \\ w_A \end{bmatrix} \tag{3-1.32}$$

where

$$L\begin{bmatrix} \beta_s \\ w_s \end{bmatrix} = \begin{bmatrix} 0 \\ p_s \end{bmatrix} \tag{3-1.33a}$$

$$L\begin{bmatrix} \beta_A \\ w_A \end{bmatrix} = \begin{bmatrix} 0 \\ p_A \end{bmatrix}. \tag{3-1.33b}$$

It can then be seen that all nonlinear dependence on ϵ is due solely to the antisymmetric part of the load. Moreover, if the quantity of interest is the displacement at the beams midpoint $x = 0$, then $w(0) = w_s(0) + w_A(0)$ is simply linear in ϵ since $w_s(0)$ is linear, and $w_A(0)$ vanishes.

A similar truncation takes place for $\begin{bmatrix} \alpha \\ u \end{bmatrix}$ where f_1 is an odd function.

A second observation will prove useful in dealing with principle error in 3-2. If

$$f_2 = -f_1' \, , \tag{3-1.34}$$

then by comparing (3-1.21) and (3-1.15), it can be seen that

$$w_0 = -u_0 \quad \text{and} \quad \beta_0 = -\alpha_0 \; . \tag{3-1.35}$$

This means that the leading term of $U = U^{(1)} + U^{(2)}$ vanishes. Thus the solution to the system

$$LU = \begin{bmatrix} f_1 \\ -f_1' \end{bmatrix} \; , \qquad U = 0 \quad \text{at} \quad x = \pm\frac{1}{2} \tag{3-1.36}$$

is of first order in the small parameter ϵ, i.e.

$$U = O(\epsilon) \; .$$

In 3-2, it will be observed that terms representing error from discretizing the problem by finite elements will solve systems with the special form of (3-1.36). This observation will lead to great improvement to the form of that error.

To summarize, the solution to the clamped beam problem in Timoshenko's theory can be uniformly expanded as powers of ϵ. The leading term represents the solution (slope and displacement) for the clamped plate problem in the classical theory. The difference $\begin{pmatrix}\psi\\ w\end{pmatrix} - \begin{pmatrix}-w_0'\\ w_0\end{pmatrix}$ is of order ϵ, in fact is proportional to ϵ for symmetric loading. Other results useful in controlling discretization error are also uncovered here.

3-2. Principal Error for the Clamped Beam in Timoshenko's Theory

In order to examine the error due to discretizing the boundary value problem by use of finite elements, that is, by replacing

$$\text{(3-2.1)} \qquad LU = F\,, \quad U = 0 \quad \text{at} \quad x = \pm\frac{1}{2}$$

with a system of linear algebraic equations,

$$\text{(3-2.2)} \qquad KQ = \tilde{F}$$

it is useful to keep in mind that equations (3-2.2) are simply a type of finite difference equations which are selected to satisfy a variational principle rather than being chosen in the more conventional way by replacing derivatives in the operator L by certain standard difference quotients.

Viewing (3-2.2) as a system of finite difference equations, an analysis of the local discretization error can be carried out by determining the error as a power series of the discretization parameter, in the case of finite elements, the mesh size h.

To compare solutions to the discrete problem and the continuous one at a node x_i of the finite element grid, it is necessary to examine two consecutive rows of the stiffness matrix, namely, the $(2i-1)$th and $2i$th, where the unknowns q_{2i-1} and q_{2i} represents the approximations to $\psi(x_i)$ and $w(x_i)$ respectively. This can be done either by superimposing element stiffness matrices constructed in 3-3, as they would be assembled in the global stiffness matrix, or by computing the elements K_{ij} directly from the energy functional, i.e.

$$K_{ij} = B_\varepsilon(\phi_i,\phi_j) \, . \tag{3-2.3}$$

The former is simpler here, with rows given in (3-3.20). Note that it is necessary to divide equation (3-2.2) by h in order to observe a "difference" form.

After dividing by h, designate these two rows of equation (3-2.2) by

$$L^h U^h = F^h \tag{3-2.4}$$

where

(3-2.5)

$$L^h = \begin{bmatrix} \left(-\frac{1}{h^2}+\frac{\epsilon^{-1}}{6}\right) & \frac{\epsilon^{-1}}{2h} \\ -\frac{\epsilon^{-1}}{2h} & -\frac{\epsilon^{-1}}{h^2} \\ \left(-\frac{2}{h^2}+\frac{2\epsilon^{-1}}{3}\right) & 0 \\ 0 & \frac{2\epsilon^{-1}}{h^2} \\ \left(-\frac{1}{h^2}+\frac{\epsilon^{-1}}{6}\right) & -\frac{\epsilon^{-1}}{2h} \\ \frac{\epsilon^{-1}}{2h} & -\frac{\epsilon^{-1}}{h^2} \end{bmatrix}^T , \quad U^h = \begin{bmatrix} q_{2i-3} \\ q_{2i-2} \\ q_{2i-1} \\ q_{2i} \\ q_{2i+1} \\ q_{2i+2} \end{bmatrix}$$

(3-2.6)

$$F^h = \begin{bmatrix} 0 \\ f^h \end{bmatrix}$$

(3-2.7)

$$f^h = \frac{2}{h}\int_{-1/2}^{1/2} f\,\varphi_i\,dx \;.$$

The difference form of L^h is made apparent by multiplying L^h by the vector function

$$\tilde{U} = \begin{bmatrix} \psi(x_i - h) \\ w(x_i - h) \\ \psi(x_i) \\ w(x_i) \\ \psi(x_i + h) \\ w(x_i + h) \end{bmatrix} \tag{3-2.8}$$

which corresponds to U^h. In this way L^h can be treated as a difference operator applied to vector functions $U = [\psi, w]^T$.

For convenience, define the following difference operators at a node x_i:

$$D^0[g] = \frac{g(x_i+h) + 4g(x_i) + g(x_i - h)}{6}$$

$$D^1[g] = \frac{g(x_i+h) - g(x_i-h)}{2h}$$

$$D^2[g] = \frac{g(x_i+h) - 2g(x_i) + g(x_i-h)}{h^2} \quad .$$

These can be seen to arise naturally by multiplying $L^h\tilde{U}$ and rearranging

$$L^h\tilde{U} = \begin{bmatrix} -D^2[\psi] + \varepsilon^{-1}(D^0[\psi] + D^1[w]) \\ -\varepsilon^{-1}(D^1[\psi] + D^2[w]) \end{bmatrix} \quad . \tag{3-2.9}$$

To simplify notation, let L^h represent both an operator on a vector function $U = \begin{bmatrix} \psi \\ w \end{bmatrix}$, and the matrix defined in (3-2.7). That is, for $U = \begin{bmatrix} \psi \\ w \end{bmatrix}$ define the operator L^h by

$$(3\text{-}2.10) \qquad L^h U = L^h \tilde{U}$$

where $L^h\tilde{U}$ is the matrix product given explicitly in (3-2.9). Thus (3-2.10) written explicitly is

$$(3\text{-}2.11) \quad L^h U = \begin{bmatrix} -D^2[\psi] + \varepsilon^{-1}(D^0[\psi] + D^1[w]) \\ -\varepsilon^{-1}(D^1[\psi] + D^2[w]) \end{bmatrix} .$$

Since

$$(3\text{-}2.13) \qquad LU = \begin{bmatrix} -\psi'' + \varepsilon^{-1}(\psi + w') \\ -\varepsilon^{-1}(\psi' + w'') \end{bmatrix}$$

the similarity between L^h and L is obvious.

Note that for all local analysis which follows, function evaluation is understood to take place at node x_i unless otherwise indicated.

The variational method, in generating the finite difference equations by use of piecewise linear finite elements, "prefers" to replace all the derivatives in

L by central differences, but elects to replace the function evaluation of ψ by a "Simpson's average", $\frac{\psi(x+h) + 4\psi(x) + \psi(x-h)}{6}$.

Error due to replacing a continuous system by a discrete one is examined by substituting the solution to the continuous problem into the discrete operator, and comparing to the result of applying the discrete operator to the discrete solution. That is, if $LU = F$ is discretized by

$$L^h U^h = F^h ,$$

then the quantity

$$L^h (U^h - U) = F^h - L^h U$$

is a measure of consistency between the two systems. Using Taylor series about a grid point, $(F^h - L^h U)$ is expressed as a series in h (the mesh size). The leading term being, say $O(h)$, indicates consistency. Further refinement can be done by defining this leading term to be hF_1, where F_1 independent of h, and defining e_1 to satisfy the continuous problem

$$Le_1 = F_1$$

(with appropriate boundary conditions) then computing

$L^h(U^h - U - he_1) = F^h - L^hU - hL^he_1$ which in turn is expected to be, say $O(h^2)$, and so on. In this way an error expansion

$$U = U^h - \sum_{i=1} e_i h^i$$

is obtained.

The following theorem gives the computation needed to produce each $L^h e_i$, given Le_i. The notation $f^{(\beta)}$ indicates the βth derivative of f.

Theorem 6. If $LU = F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ and $U = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at $x = \pm\frac{1}{2}$, then for u, f_1 and f_2 infinitely differentiable, the following asymptotic form holds:

(3-2.20)

$$L^hU = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} 2\ f_1^{(2\alpha)} - 2\alpha\ f_2^{(2\alpha-1)} \\ 2\ f_2^{(2\alpha)} \end{bmatrix} + \varepsilon^{-1} \sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} -\frac{4}{3}(\alpha^2-1)(f_1^{(2\alpha-2)} + f_2^{(2\alpha-3)}) \\ 2\alpha\ (f_1^{(2\alpha-1)} + f_2^{(2\alpha-2)}) \end{bmatrix}$$

Proof: Using Taylor series expansions

(3-2.21)

$$u(x+h) = \sum_{\alpha=0} \frac{h^{\alpha}u^{(\alpha)}(x)}{\alpha!}$$

$$u(x-h) = \sum_{\alpha=0} \frac{h^{\alpha}u^{(\alpha)}(x)}{\alpha!}$$

then

(3-2.22)

$$D^{0}[u] = u + \frac{1}{3} \sum_{\alpha=1} \frac{h^{2\alpha}u}{(2\alpha)!}$$

$$D^{1}[u] = \sum_{\alpha=0} \frac{h^{2\alpha}u^{(2\alpha+1)}}{(2\alpha+1)!}$$

$$D^{2}[u] = 2 \sum_{\alpha=1} \frac{h^{(2\alpha-2)}u^{(2\alpha)}}{(2\alpha)!} \quad .$$

Using (3-2.22) in (3-2.11), taking $U = [\psi, w]^T$,

$$L^{h}U = \begin{bmatrix} -2 \sum_{\alpha=1} \frac{h^{(2\alpha-2)}\psi^{(2\alpha)}}{(2\alpha)!} \\ 0 \end{bmatrix}$$

$$+ \epsilon^{-1} \begin{bmatrix} \psi + \frac{1}{3} \sum_{\alpha=1} \frac{h^{2\alpha}\psi^{(2\alpha)}}{(2\alpha)!} + \sum_{\alpha=0} \frac{h^{2\alpha}w^{(2\alpha+1)}}{(2\alpha+1)!} \\ \sum_{\alpha=0} \frac{h^{2\alpha}\psi^{(2\alpha+1)}}{(2\alpha+1)!} + 2 \sum_{\alpha=1} \frac{h^{(2\alpha-2)}w^{(2\alpha)}}{(2\alpha)!} \end{bmatrix}$$

rearranging

(3-2.23)

$$L^h U = \begin{bmatrix} -\psi'' + \epsilon^{-1}(\psi + w') \\ \\ -\epsilon^{-1}(\psi' + w'') \end{bmatrix}$$

$$+ \sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} -2\psi^{(2\alpha+2)} \\ \\ 0 \end{bmatrix}$$

$$+ \epsilon^{-1} \sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} \frac{(2\alpha+2)(2\alpha+1)}{3} \psi^{(2\alpha)} + (2\alpha+2) w^{(2\alpha+1)} \\ \\ (2\alpha+2)\psi^{(2\alpha+1)} + 2w^{(2\alpha+2)} \end{bmatrix} .$$

Now since $L \begin{pmatrix} \psi \\ w \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, the first term on the right of (3-2.23) is $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. Moreover, by differentiating and combining the equations in $LU = F$ the following identities are found:

(3-2.24a) $$\psi''' = -f_1' - f_2$$

(3-2.24b) $$w''' = -\epsilon f_2'' + f_1' + f_2$$

(3-2.24c) $$\psi + w' = \epsilon(\psi'' + f_1)$$

(3-2.24d) $$\psi' + w'' = -\epsilon f_2 \ .$$

Repeated differentiation of these give

$$\psi^{(\beta)} = -f_1^{(\beta-2)} - f_2^{(\beta-3)}, \quad \text{for} \quad \beta \geq 3 \tag{3-2.25a}$$

$$w^{(\beta)} = -\varepsilon\ f_2^{(\beta-2)} + f_1^{(\beta-3)} + f_2^{(\beta-4)}, \quad \text{for} \quad \beta \geq 4 \tag{3-2.25b}$$

$$\begin{aligned} \psi^{(\beta)} &+ w^{(\beta+1)} \\ &= -\varepsilon\ f_2^{(\beta-1)}, \quad \text{for} \quad \beta \geq 1 . \end{aligned} \tag{3-2.25c}$$

Inserting these into (3-2.23), then

(3-2.26)

$$\begin{aligned} L^h U &= \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \frac{h^2}{4!} \begin{bmatrix} -2\psi'''' \\ 0 \end{bmatrix} + \sum_{\alpha=2} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} -2\psi^{(2\alpha+2)} \\ 0 \end{bmatrix} \\ &+ \frac{\varepsilon^{-1} h^2}{4!} \begin{bmatrix} 4\psi'' + 4w''' \\ -4\psi''' - 2w'''' \end{bmatrix} \\ &+ \varepsilon^{-1} \sum_{\alpha=2} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} \frac{(2\alpha+2)(2\alpha+1)}{3} \psi^{(2\alpha)} + (2\alpha+2) w^{(2\alpha+1)} \\ -(2\alpha+2)\psi^{(2\alpha+1)} - 2w^{(2\alpha+2)} \end{bmatrix} \\ &= \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \frac{h^2}{4!} \begin{bmatrix} 2f_1'' - 2f_2' \\ 2f_2'' \end{bmatrix} \\ &+ \sum_{\alpha=2} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} 2f_1^{(2\alpha)} - 2\alpha f_2^{(2\alpha-1)} \\ 2f_2^{(2\alpha)} \end{bmatrix} \end{aligned}$$

$$+ \frac{\varepsilon^{-1}h^2}{4!} \begin{bmatrix} 0 \\ \\ 2f_1' + 2f_2 \end{bmatrix}$$

$$+ \varepsilon^{-1} \sum_{\alpha=2} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} \frac{(2\alpha+2)(2-2\alpha)}{3}(f_1^{(2\alpha-2)} + f_2^{(2\alpha-3)}) \\ \\ 2\alpha(f_1^{(2\alpha-1)} + f_2^{(2\alpha-2)}) \end{bmatrix}.$$

That is,

$$L^h U = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} 2f_1^{(2\alpha)} - 2\alpha f_2^{(2\alpha-1)} \\ \\ 2f_2^{(2\alpha)} \end{bmatrix}$$

$$+ \varepsilon^{-1} \sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} -\frac{4(\alpha^2-1)}{3}(f_1^{(2\alpha-2)} + f_2^{(2\alpha-3)}) \\ \\ 2\alpha(f_1^{(2\alpha-1)} + f_2^{(2\alpha-2)}) \end{bmatrix}$$

which is equation (3-2.20).

To compute the principal error, the expression, $L^h U^h$ must be compared to $L^h U$. $L^h U^h = F^h$ can be expressed as a Taylor series by expanding the integrals defining components of F^h.

The two components of the load term on the right side of (3-2.4) are (see (3-3.3) for definition of φ_i)

(3-2.27a) $$F^h_{2i-1} = 0$$

(3-2.27b) $$F^h_{2i} = \frac{1}{h} \int_{-1/2}^{1/2} f\, \varphi_i \, dx = \frac{1}{h} \int_{x_i-h}^{x_i+h} f\, \varphi_i \, dx \cdot$$

Define

$$f^{(-1)}(x) = \int_{x_i}^{x} f(s)ds \tag{3-2.28}$$

$$f^{(-2)}(x) = \int_{x_i}^{x} f^{(-1)}(s)ds \ . \tag{3-2.29}$$

Integrating by parts

(3-2.30)

$$\int_{x_i-h}^{x_i+h} f(x)\varphi_i(x)dx$$

$$= \int_{x_i-h}^{x_i} f(x)\varphi_i(x)dx + \int_{x_i}^{x_i+h} f(x)\varphi_i(x)dx$$

$$= -\int_{x_i-h}^{x_i} f^{(-1)}(x)\varphi_i^1(x)dx - \int_{x_i}^{x_i+h} f^{(-1)}(x)\varphi_i^1(x)dx$$

$$= -\frac{1}{h}\int_{x_i-h}^{x_i} f^{(-1)}(x)dx + \frac{1}{h}\int_{x_i}^{x_i+h} f^{(-1)}(x)dx$$

$$= \frac{f^{(-2)}(x_i+h) - 2f^{(-2)}(x_i) + f^{(-2)}(x_i-h)}{h} \ .$$

Expanding $f^{(-2)}$ as a Taylor series

$$f^{(-2)}(x_i \pm h) = \sum_{j=0} f^{(j-2)}(x_i)\,\frac{(\pm h)^j}{j!} \ . \tag{3-2.31}$$

Then (3-2.30) becomes

$$(3\text{-}2.32) \qquad \int_{x_i-h}^{x_i+h} f(x)\varphi_i(x)dx$$

$$= 2 \sum_{\substack{j=2 \\ j \text{ even}}} f^{(j-2)}(x_i)\, \frac{h^{j-1}}{j!}$$

$$= 2 \sum_{\alpha=1} f^{(2\alpha-2)}(x_i)\, \frac{h^{2\alpha-1}}{(2\alpha+2)!}$$

$$= 2 \sum_{\alpha=0} f^{(2\alpha)}(x_i)\, \frac{h^{2\alpha+1}}{(2\alpha+2)!}$$

$$= hf(x_i) + 2 \sum_{\alpha=1} f^{(2\alpha)}(x_i)\, \frac{h^{2\alpha+1}}{(2\alpha+2)!}$$

and (3-2.27b) becomes

$$(3\text{-}2.33) \qquad F^h_{2i} = \frac{1}{h} \int_{x_i-h}^{x_i+h} f\, \varphi_i\, dx$$

$$= f(x_i) + 2 \sum_{\alpha=1} f^{(2\alpha)}(x_i)\, \frac{h^{2\alpha}}{(2\alpha+2)!} \;.$$

Inserting these expressions for F^h into (3-2.34)

$$(3\text{-}2.34) \qquad L^h U^h = \begin{bmatrix} 0 \\ \\ f + 2 \displaystyle\sum_{\alpha=1} \frac{h^{2\alpha} f^{(2\alpha)}}{(2\alpha+2)!} \end{bmatrix} .$$

By Theorem 6, if $U = \begin{bmatrix} \psi \\ w \end{bmatrix} \in (H^1_0)^2$ satisfies

$LU = F = \begin{bmatrix} 0 \\ f \end{bmatrix}$, then

(3-2.35)

$$L^h U = \begin{bmatrix} 0 \\ f \end{bmatrix} + \sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} -2\alpha f^{(2\alpha-1)} - \frac{4(\alpha^2-1)}{3} \varepsilon^{-1} f^{(2\alpha-3)} \\ 2f^{(2\alpha)} + 2\alpha \varepsilon^{-1} f^{(2\alpha-2)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ f + 2 \sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!} f^{(2\alpha)} \end{bmatrix} + \sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} -2\alpha f^{(2\alpha-1)} - \frac{4(\alpha^2-1)}{3} \varepsilon^{-1} f^{(2\alpha-3)} \\ 2\alpha \varepsilon^{-1} f^{(2\alpha-2)} \end{bmatrix}.$$

Subtracting (3-2.35) from (3-2.34)

(3-2.36)
$$L^h(U^h - U) = -\sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} -2\alpha f^{(2\alpha-1)} - \frac{4(\alpha^2-1)}{3} \varepsilon^{-1} f^{(2\alpha-3)} \\ 2\alpha \varepsilon^{-1} f^{(2\alpha-2)} \end{bmatrix}.$$

The first thing to be noticed in (3-2.36) is that

$$L^h(U^h - U) = O(h^2) ,$$

meaning that the discrete and continuous problems are consistent, with the error being quadratic in h.

The second is that the factor ε^{-1} is present in the leading and higher order terms. Since ε is a small parameter, this is undesirable. However, by considering an "error" expression

$$e = (1+\frac{\varepsilon^{-1}h^2}{12})U^h - U \tag{3-2.37}$$

in place of $U^h - U$, considerable improvement is attained.

Using (3-2.35) and (3-2.34),

(3-2.38)

$$L^h((1+\frac{\varepsilon^{-1}h^2}{12})U^h - U)$$

$$= (1+\frac{\varepsilon^{-1}h^2}{12})\begin{bmatrix} 0 \\ f + 2\sum\limits_{\alpha=1} \dfrac{h^{2\alpha}f^{(2\alpha)}}{(2\alpha+2)!} \end{bmatrix} - \begin{bmatrix} 0 \\ f + 2\sum\limits_{\alpha=1} \dfrac{h^{2\alpha}f^{(2\alpha)}}{(2\alpha+2)!} \end{bmatrix}$$

$$- \sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!}\begin{bmatrix} -2\alpha f^{(2\alpha-1)} - \dfrac{4(\alpha^2-1)}{3}\varepsilon^{-1} f^{(2\alpha-3)} \\ 2\alpha\,\varepsilon^{-1} f^{(2\alpha-2)} \end{bmatrix}$$

$$= \frac{\varepsilon^{-1}h^2}{12}\begin{bmatrix} 0 \\ f \end{bmatrix} + \frac{\varepsilon^{-1}}{6}\sum_{\alpha=2}\frac{h^{2\alpha}}{(2\alpha)!}\begin{bmatrix} 0 \\ f^{(2\alpha-2)} \end{bmatrix}$$

$$- \sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} -2\alpha f^{(2\alpha-1)} - \frac{4}{3}(\alpha^2 - 1)\varepsilon^{-1} f^{(2\alpha-3)} \\ \\ 2\alpha\ \varepsilon^{-1}\ f^{(2\alpha-2)} \end{bmatrix}$$

$$= \frac{h^2}{12}\begin{pmatrix} f' \\ 0 \end{pmatrix} + \sum_{\alpha=2} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} 2\alpha f^{(2\alpha-1)} + \frac{4}{3}(\alpha^2-1)\varepsilon^{-1} f^{(2\alpha-3)} \\ \\ \frac{(\alpha-1)(2\alpha-1)}{3}\ \varepsilon^{-1}\ f^{(2\alpha-2)} \end{bmatrix} .$$

The leading term is now free of ε^{-1}. This can now be used to generate the principal error term $\frac{h^2 e_1}{12}$, in e, by letting e_1 satisfy the equation

$$(3\text{-}2.39) \qquad Le_1 = \begin{bmatrix} f' \\ 0 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ at } x = \pm\frac{1}{2} .$$

The quadratic term of the error e having been recovered, the expression

$$(3\text{-}2.40) \qquad (1 + \frac{\varepsilon^{-1}h^2}{12})U^h - U - \frac{h^2 e_1}{12}$$

is expected to be $O(h^4)$. To recover the $O(h^4)$ term, the expression

$$(3\text{-}2.41) \qquad L^h((1 + \frac{\varepsilon^{-1}h^2}{12})U^h - U - \frac{h^2 e_1}{12})$$

is considered. Note first that, from (3-2.39), e_1 has ε-dependence through positive powers only. By equation (3-2.20)

$$(3\text{-}2.42)\qquad L^h e_1 = \begin{bmatrix} f' \\ 0 \end{bmatrix} + \sum_{\alpha=1} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} 2f^{(2\alpha+1)} - \frac{4(\alpha^2-1)}{3}\,\varepsilon^{-1} f^{(2\alpha-1)} \\ 2\alpha\,\varepsilon^{-1}\,f^{(2\alpha)} \end{bmatrix}.$$

So

(3-2.43)

$$\frac{h^2}{12} L^h e_1 = \frac{h^2}{12}\begin{bmatrix} f' \\ 0 \end{bmatrix} + \frac{1}{12}\sum_{\alpha=1} \frac{h^{2\alpha+2}}{(2\alpha+2)!} \begin{bmatrix} 2f^{(2\alpha+1)} - \frac{4(\alpha^2-1)}{3}\,\varepsilon^{-1} f^{(2\alpha-1)} \\ 2\alpha\,\varepsilon^{-1}\,f^{(2\alpha)} \end{bmatrix}$$

$$= \frac{h^2}{12}\begin{matrix} f' \\ 0 \end{matrix} + \sum_{\alpha=2} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} \frac{(\alpha+1)(2\alpha+1)}{3}\,f^{(2\alpha-1)} \\ 0 \end{bmatrix}$$

$$+ \varepsilon^{-1}\sum_{\alpha=2} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} -\frac{2\alpha(\alpha-2)(\alpha+1)(2\alpha+1)}{9}\,f^{(2\alpha-3)} \\ \frac{(\alpha-1)(\alpha+1)(2\alpha+1)}{3}\,f^{(2\alpha-2)} \end{bmatrix}.$$

Now (3-2.43) and (3-2.38) give

(3-2.44)

$$L^h\left(\left(1+\frac{\varepsilon^{-1}h^2}{12}\right)U^h - U - \frac{h^2}{12}\,e_1\right)$$

$$= \frac{h^4}{6!}\begin{bmatrix} -f''' + 4\varepsilon^{-1}\, f' \\ \\ -4\,\varepsilon^{-1}\, f'' \end{bmatrix} + \sum_{\alpha=3} \frac{h^{2\alpha}}{(2\alpha+2)!}\begin{bmatrix} -\frac{(\alpha-1)(2\alpha-1)}{3}\, f^{(2\alpha-1)} \\ \\ 0 \end{bmatrix}$$

$$+\ \varepsilon^{-1} \sum_{\alpha=3} \frac{h^{2\alpha}}{(2\alpha+2)!}\begin{bmatrix} \frac{2(\alpha+1)(2\alpha^3-3\alpha^2+4\alpha-6)}{9}\, f^{(2\alpha-3)} \\ \\ -\frac{(\alpha-1)(2\alpha^2+\alpha+2)}{3}\, f^{(2\alpha-2)} \end{bmatrix}.$$

Let e_2 be the solution of

$$\text{(3-2.45)}\qquad Le_2 = \begin{bmatrix} +f''' - 4\varepsilon^{-1}f' \\ \\ +4\,\varepsilon^{-1}\, f'' \end{bmatrix},\quad e_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ at } x = \pm\tfrac{1}{2};$$

then the expression

$$\text{(3-2.46)}\qquad \left(1+\frac{\varepsilon^{-1}h^2}{12}\right)U^h - U - \frac{h^2 e_1}{12} + \frac{h^4}{6!}\, e_2$$

is expected to be $O(h^6)$. Consider the ε dependence in e_2. Ordinarily the presence of ε^{-1} on the right side of (3-2.45) would indicate that e_2 is $O(\varepsilon^{-1})$. However, ε^{-1} is multiplied by a vector $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ where $f_2 = -f_1'$, a condition which causes the adverse ε

dependence to vanish in the solution e_2. This follows from the asymptotic analysis of the system $LU = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ from 3-1. Hence e_2 has dependence only on positive powers of ε, and again the principal error term remains free of adverse ε dependence.

This special relationship present in (3-2.45) in the ε^{-1}-term provides another advantage, this time in the determination of e_3. To compute Le_3, the expression $\frac{h^4}{6!} L^h e_2$ must be added to equation (3-2.44). Having ε^{-1} in Le_2 would be expected to lead to ε^{-2} dependence in $L^h e_2$, but once again the form of the ε^{-1} dependence in Le_2 and in (3-2.20) combine to leave only ε^{-1} dependence in $L^h e_2$ (coming from the term $\begin{pmatrix} f''' \\ 0 \end{pmatrix}$ in Le_2).

(3-2.47)

$$\frac{h^4 L^h e_2}{6!} = \frac{h^4}{6!} \begin{pmatrix} f''' \\ 0 \end{pmatrix} + \frac{4\varepsilon^{-1} h^4}{6!} \begin{pmatrix} -f' \\ f'' \end{pmatrix} + \sum_{\alpha=3} \frac{h^{2\alpha}}{(2\alpha-2)!6!} \begin{bmatrix} 2f^{(2\alpha-1)} - \frac{4}{3}\varepsilon^{-1}(\alpha+3)(\alpha-1)f^{(2\alpha-3)} \\ 2(\alpha+2)\varepsilon^{-1} f^{(2\alpha-2)} \end{bmatrix}$$

with (3-2.44)

(3-2.48)

$$L^h\left(\left(1+\frac{\varepsilon^{-1}h^2}{12}\right)U^h - U - \frac{h^2}{12}\,e_1 + \frac{h^4}{6!}\,e_2\right)$$

$$= \sum_{\alpha=3} h^{2\alpha} \begin{bmatrix} \left(-\dfrac{(\alpha-1)(2\alpha-1)}{2(2\alpha+2)!} + \dfrac{2}{(2\alpha-2)!6!}\right) f^{(2\alpha-1)} \\ \\ 0 \end{bmatrix}$$

$$+ \ \varepsilon^{-1} \sum_{\alpha=3} \frac{h^{2\alpha}}{(2\alpha+2)!} \begin{bmatrix} \dfrac{2(\alpha+1)(2\alpha^3-3\alpha^2+4\alpha-6)}{9}\ f^{(2\alpha-3)} \\ \\ -\dfrac{(\alpha-1)(2\alpha^2+\alpha+2)}{3}\ f^{(2\alpha-2)} \end{bmatrix}$$

$$+ \ \varepsilon^{-1} \sum_{\alpha=3} \frac{h^{2\alpha}}{(2\alpha-2)!6!} \begin{bmatrix} -\dfrac{4(\alpha+3)(\alpha-1)}{3}\ f^{(2\alpha-3)} \\ \\ 2(\alpha+2)\ f^{(2\alpha-2)} \end{bmatrix}.$$

The leading term which will be used to generate e_3 is

$$\text{(3-2.49)} \qquad \frac{h^6}{8!} \begin{bmatrix} \dfrac{4}{3}\ f^{(s)} - 8\ \varepsilon^{-1}\ f^{(3)} \\ \\ 8\ \varepsilon^{-1}\ f^{(4)} \end{bmatrix}$$

again exhibiting the special form in its ε^{-1} term.

This special property of the ε^{-1} term can be shown to propagate over and over as the error terms are uncovered. Although tedious, an expression can be developed for determining the general principal error function e_k base on extending patterns of coefficients which hold for the first several terms.

Define $e_k \in (H_0^1)^2$ by

$$\text{(3-2.50)} \qquad Le_k = \begin{bmatrix} f^{(2k-1)} \\ 0 \end{bmatrix} - 2k\ \varepsilon^{-1} \begin{bmatrix} f^{(2k-3)} \\ -f^{(2k-2)} \end{bmatrix}$$

for $k = 2,3,4,\ldots$ and

$$\text{(3-2.51)} \qquad Le_1 = \begin{bmatrix} f' \\ 0 \end{bmatrix} .$$

The lending error term is

$$\text{(3-2.52)} \qquad \frac{2\sigma_k h^{2k} Le_k}{(2k+2)!}$$

σ_k satisfies

$$\text{(3-2.53)} \qquad \sigma_k = k - \frac{2}{(2k+4)(2k+3)} \sum_{j=1}^{k-1} \binom{2k+4}{2j+2} \sigma_j \quad .$$

the first few terms of σ_j are :

$$\sigma_1 = 1$$

$$\sigma_2 = -\frac{1}{2}$$

$$\sigma_3 = \frac{2}{3}$$

$$\sigma_4 = -\frac{3}{2}$$

$$\sigma_5 = 5$$

$$\sigma_6 = -\frac{691}{30} \quad .$$

Expressing the principal error itself,

$$(3\text{-}2.54) \qquad U - \left(1+\frac{\varepsilon^{-1}h^2}{12}\right)U^h = \sum_{j=1}^{m} \frac{2\sigma_j h^{2j} e_j}{(2j+2)!} + O(\varepsilon^0 h^{2m+2})$$

where e_j is of order $O(\varepsilon^0)$ for each j.

For $m = 3$,

$$(3\text{-}2.55) \qquad U - \left(1+\frac{\varepsilon^{-1}h^2}{12}\right)U^h$$

$$= \frac{h^2 e_1}{12} - \frac{h^4 e_2}{6!} + \frac{4h^6 e_3}{3\cdot 8!} + O(\varepsilon^0 h^8)\ .$$

To see the significance of the factor $\left(1+\frac{\varepsilon^{-1}h^2}{12}\right)$, rewrite (3-2.55) as

$$(3\text{-}2.56) \qquad \frac{1}{\left(1+\frac{\varepsilon^{-1}h^2}{12}\right)}\, U - U^h$$

$$= \frac{1}{\left(1+\frac{\varepsilon^{-1}h^2}{12}\right)} \left(\frac{h^2 e_1}{12} - \frac{h^4 e_2}{6!} + \frac{4h^6 e_3}{3\cdot 8!} + O(\varepsilon^0 h^8)\right)$$

expanding $\dfrac{1}{1+\frac{\varepsilon^{-1}h^2}{12}}$ for small values of $\dfrac{\varepsilon^{-1}h^2}{12}$,

$$(3\text{-}2.57) \qquad U\left(\sum_{i=0} \left(-\frac{\varepsilon^{-1}h^2}{12}\right)^i\right) - U^h$$

$$= \left(\sum_{i=0} \left(-\frac{\varepsilon^{-1}h^2}{12}\right)^i\right)\left(\frac{h^2 e_1}{12} - \frac{h^4 e_2}{6!} + \frac{4h^6 e_3}{3\cdot 8!} + O(\varepsilon^0 h^8)\right)$$

$$(3\text{-}2.58)\qquad U - U^h = (\varepsilon^{-1}U + e_1)\frac{h^2}{12} - (\varepsilon^{-2}U + \varepsilon^{-1}e_1 + \frac{1}{s}e_2)\frac{h^4}{12!} + O(\varepsilon^{-3}h^6)\ .$$

(3-2.58) is the usual error expression which would have been obtained without involving the factor $(1 + \frac{\varepsilon^{-1}h^2}{12})$. As expected it is "contaminated" by negative powers of ε, in fact to larger and larger negative powers in the higher order terms in h. (3-2.58) even seems to violate the theoretical estimates in Chapter 2, which require $U - U^h$ to be at worst $O(\varepsilon^{-1/2}h)$ (although these are integral estimates and (3-2.58) is pointwise at a node, the ε-dependence should be comparable). However, using

$$\frac{1}{1 + \frac{\varepsilon^{-1}h^2}{12}} = 1 - \frac{\frac{\varepsilon^{-1}h^2}{12}}{1 + \frac{\varepsilon^{-1}h^2}{12}} \quad \text{in (3-2.56)}$$

$$(3\text{-}2.59)\qquad U - U^h = \frac{\frac{\varepsilon^{-1}h^2}{12}}{1 + \frac{\varepsilon^{-1}h^2}{12}}U + \frac{1}{1 + \frac{\varepsilon^{-1}h^2}{12}}\left(\frac{h^2e_1}{12} - \frac{h^4e_2}{6!} + \frac{4h^6e_3}{4\cdot 8!} + O(\varepsilon^0h^8)\right).$$

In this closed form the ε dependence is clearly $O(\varepsilon^0)$ for small ε. However, the leading error term on the right of (3-2.59), though technically $O(h^2)$ for fixed ε, becomes nearly independent of h for very small ε.

This appears numerically by values of U^h being driven to zero as ε becomes small.

It appears that the factor $(1+\frac{\varepsilon^{-1}h^2}{12})$ "collects" the error terms of all orders which contain the adverse dependence on ε^{-1}, allowing the quantity $(1+\frac{\varepsilon^{-1}h^2}{12})U^h$ to approximate U much more reliably than would U^h itself for small ε. Numerical results confirm this effectiveness.

3-3. Construction of the Element Stiffness Matrix for the Clamped Timoshenko Beam

Consider a beam clamped at $x = \pm\frac{1}{2}$ and subdivided by nodes $x_i = ih - \frac{1}{2}$ in $n = \frac{1}{h}$ equal segments.

As in 2-4, the discrete form of the improved plate equations can be derived by minimizing

$$J(U) = B_\varepsilon(U,U) - 2P_L(F,U) \tag{3-3.1}$$

over a finite dimensional space generated by a basis of finite element functions. Taking

$$U = \sum_{i=1}^{2n-2} q_i\,\phi_i\,, \tag{3-3.2}$$

where, for $i = 1,3,5,\ldots,$

$$\phi_i = \begin{bmatrix} \varphi_i \\ 0 \end{bmatrix} \quad \text{and} \quad \phi_i = \begin{bmatrix} 0 \\ \varphi_i \end{bmatrix}.$$

Define

$$\text{(3-3.3)} \qquad \varphi_i = \begin{cases} (x - x_{i-1})/h & \text{for } x_{i-1} < x < x_i \\ (x_{i+1} - x)/h & \text{for } x_i < x < x_{i+1} \end{cases}$$

φ_i are the so-called "roof" functions of Figure 1.

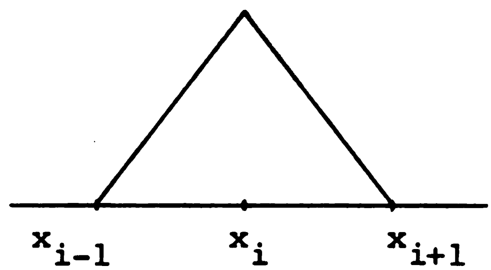


Figure 1. Piecewise linear "roof" function.

With (3-1.2), (3-1.1) becomes

$$\text{(3-3.4)} \qquad J(U) = B_\varepsilon\left(\sum_{i=1}^{2n-2} q_i\phi_i, \sum_{j=1}^{2n-2} q_j\phi_j\right)$$

$$- 2P_L\left(F, \sum_{j=1}^{2n-2} q_j\phi_j\right)$$

$$= \sum_{i=1}^{2n-2} \sum_{j=1}^{2n-2} q_i q_j \, B_\varepsilon(\phi_i, \phi_j)$$

$$- 2 \sum_{j=1}^{2n-2} q_j P_L(F, \phi_j) \; .$$

Or, in matrix form,

$$J(U) = Q^T K Q - 2Q^T \tilde{F} \tag{3-3.5}$$

where

$$K_{ij} = B_\varepsilon(\phi_i, \phi_j)$$

$$Q = [q_1, \ldots, q_{2n-2}]$$

$$\tilde{F}_j = P_L(F, \phi_j)$$

$J(U)$ is minimized over the finite element space by choosing Q to satisfy

$$KQ = \tilde{F} \; . \tag{3-3.6}$$

While K and $\tilde{F}$ are given explicitly in terms of the base functions ϕ_i, it is usually more convenient to construct them indirectly. This is done by constructing energy expression $J(U)$ as a sum of the energies computed over the individual elements of the domain. The fact that U is polynomial when restricted to a single element can be exploited for computing the integrals which make up K and $\tilde{F}$.

$$J(U) = \sum J(U^{(e)}) \; ,$$

the sum over all elements of

$$J(U^{(e)}) = B_{\epsilon}(U^{(e)},U^{(e)}) - 2P_L(F,U^{(e)})$$

where $U^{(e)} = U$ restricted to a particular element.

First, each $U^{(e)}$ is expressed in polynomial form for the purpose of computing $J(U^{(e)})$. Then $U^{(e)}$ is expressed in terms of the finite element base functions having their support over that element of the domain. By transforming the polynomial coefficients into the coefficients of the base functions,

$$J(U^{(e)}) = Q^{(e)T}K^{(e)}Q^{(e)} - 2Q^{(e)T}\tilde{F}^{(e)} \tag{3-3.7}$$

where $Q^{(e)}$ represents those components of Q which multiply the base functions having support over that element.

Once these elemental energies are computed they need only be assembled into the total energy expression $J(U)$. The stiffness matrix K is the resulting assembly of the $K^{(e)}$, and $\tilde{F}$ is formed from $\tilde{F}^{(e)}$.

Consider the finite element space consisting of piecewise linear elements. To compute $K^{(e)}$ and $\tilde{F}^{(e)}$ over a specific element $x_i \leq x \leq x_{i+h}$, let

$$U^{(e)} = \begin{bmatrix} \psi^{(e)} \\ w^{(e)} \end{bmatrix} = \begin{bmatrix} a_1x + a_3 \\ a_2x + a_4 \end{bmatrix} . \tag{3-3.8}$$

Also

$$U^{(e)} = \begin{bmatrix} q_1^{(e)}\varphi_i + q_3^{(e)}\varphi_{i+1} \\ \\ q_2^{(e)}\varphi_i + q_4^{(e)}\varphi_{i+1} \end{bmatrix} . \tag{3-3.9}$$

The relation between coefficients is found as follows.

At $x = x_i$, $\varphi_i = 1$, $\varphi_{i+1} = 0$, so

$$\begin{bmatrix} a_1 x_i + a_3 \\ \\ a_2 x_i + a_4 \end{bmatrix} = \begin{bmatrix} q_1^{(e)} \\ \\ q_2^{(e)} \end{bmatrix}$$

At $x = x_i+h$, $\varphi_i = 0$, $\varphi_{i+1} = 1$, so

$$\begin{bmatrix} a_1(x_i+h) + a_3 \\ \\ a_2(x_i+h) + a_4 \end{bmatrix} = \begin{bmatrix} q_3^{(e)} \\ \\ q_4^{(e)} \end{bmatrix}$$

thus

$$\begin{bmatrix} x_i & 0 & 1 & 0 \\ 0 & x_i & 0 & 1 \\ x_i+h & 0 & 1 & 0 \\ 0 & x_i+h & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} q_1^{(e)} \\ q_2^{(e)} \\ q_3^{(e)} \\ q_4^{(e)} \end{bmatrix} .$$

Inverted, this is

(3-3.10)

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \frac{1}{h} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ x_i+h & 0 & -x_i & 0 \\ 0 & x_i+h & 0 & -x_i \end{bmatrix} \begin{bmatrix} q_1^{(e)} \\ q_2^{(e)} \\ q_3^{(e)} \\ q_4^{(e)} \end{bmatrix}$$

or

(3-3.11) $$A = PQ^{(e)} ,$$

where

(3-3.12)

$$P = \frac{1}{h} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ x_i+h & 0 & -x_i & 0 \\ 0 & x_i+h & 0 & -x_i \end{bmatrix} ,$$

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \qquad Q^{(e)} = \begin{bmatrix} q_1^{(e)} \\ q_2^{(e)} \\ q_3^{(e)} \\ q_4^{(e)} \end{bmatrix} .$$

Next, $J(U^{(e)})$ is computed from the polynomial form (3-3.8),

$$J(U^{(e)}) \tag{3-3.13}$$

$$= P_B(U^{(e)},U^{(e)}) + \epsilon^{-1}P_S(U^{(e)},U^{(e)}) - 2P_L(F,U^{(e)})$$

$$= \int_{x_i}^{x_i+h} a_1^2 dx + \epsilon^{-1} \int_{x_i}^{x_i+h} (a_1x + a_3 + a_2)^2 dx - 2\int_{x_i}^{x_i+h} \frac{p}{D}(a_2x + a_y)dx$$

$$= a_1^2 h + \epsilon^{-1}h\{(x_i^2 + hx_i + \frac{h^2}{3})a_1^2 + a_2^2 + a_3^2 + (a_1a_2 + a_1a_3)(2x_i + h) + 2a_2a_3\} - 2a_2\int_{x_i}^{x_i+h} \frac{p}{D} x\,dx - 2a_4\int_{x_i}^{x_i+h} \frac{p}{D}\,dx \;.$$

In matrix form, (3-3.13) is

$$J(U^{(e)}) = A^T N^{(e)} A - 2A^T \overline{F}^{(e)} \tag{3-3.14}$$

where

$$N^{(e)} = \tag{3-3.15}$$

$$h\begin{bmatrix} 1+\epsilon^{-1}(x_i^2 + hx_i + \frac{h^2}{3}) & \epsilon^{-1}(x_i + \frac{h}{2}) & \epsilon^{-1}(x_i + \frac{h}{2}) & 0 \\ \epsilon^{-1}(x_i + \frac{h}{2}) & \epsilon^{-1} & \epsilon^{-1} & 0 \\ \epsilon^{-1}(x_i + \frac{h}{2}) & \epsilon^{-1} & \epsilon^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{F}^{(e)} = \begin{bmatrix} 0 \\ \int_{x_i}^{x_i+h} \frac{p}{D} x \, dx \\ 0 \\ \int_{x_i}^{x_i+h} \frac{p}{D} \, dx \end{bmatrix} . \tag{3-3.16}$$

Now (3-3.11) can be used to eliminate A from (3-3.14).

$$J(U^{(e)}) = Q^{(e)T}P^TN^{(e)}PQ^{(e)} - 2Q^{(e)T}P^T\bar{F}^{(e)} . \tag{3-3.17}$$

By comparing (3-3.17) to (3-3.7), one has

$$K^{(e)} = P^TN^{(e)}P \tag{3-3.18}$$

$$\tilde{F}^{(e)} = P^T\bar{F}^{(e)} . \tag{3-3.19}$$

In more complicated settings, the matrix multiplication in (3-3.18) and (3-3.19) are left to be done by computer (by choosing local coordinates, P and $N^{(e)}$ each can be expressed independent of the element location). This is done in Chapter 4 for the clamped plate problem. Here $K^{(e)}$ can be computed directly.

$$K^{(e)} = P^T N^{(e)} P \tag{3-3.20}$$

$$= \begin{bmatrix} 1/h & 0 & -1/h & 0 \\ 0 & 0 & 0 & 0 \\ -1/h & 0 & 1/h & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \varepsilon^{-1} \begin{bmatrix} h/3 & -1/2 & h/6 & 1/2 \\ -1/2 & 1/h & -1/2 & -1/h \\ h/6 & -1/2 & h/3 & 1/2 \\ 1/2 & -1/h & 1/2 & 1/h \end{bmatrix} .$$

Once computed, the $K^{(e)}$ can be superimposed to give the global stiffness matrix K. This superposition is guided by the necessity of matching the local coefficients $q_i^{(e)}$ with their proper place in the global Q vector. For the one-dimensional problem, the $K^{(e)}$ are superimposed by placing each successive 4×4 matrix over the previous one by moving it two entries down the diagonal; that is, by adding the $(1,1)$ entry of the second matrix to the $(3,3)$ entry of the first, and so on. This is generally always performed by computer. Note that K is a symmetric band matrix, having a bandwidth of seven, regardless of the dimension of K. The dimension of K, of course, depends on the mesh size. For elements of the domain which border on the boundary, the same formula for $K^{(e)}$ can be used, but with two rows and corresponding columns deleted. For example, the right-most element of the beam would have $q_3^{(e)} = q_4^{(e)} = 0$, by virtue of the clamped (essential) boundary conditions. This corresponds to the fact that there is no base function nonzero at $x = \frac{1}{2}$, hence no corresponding coefficient in the global vector Q. Hence, $K^{(e)}$ for this right-most element contributes only its upper left corner 2×2 submatrix into the global matrix K.

3-4. <u>Numerical Results for the Clamped Beam</u>

Numerical results for the clamped beam verify the expectations expressed in Chapters 2 and 3.

In each of Tables 1 through 7, the first two columns give values of ε and h, respectively, used in the computations. Whenever more than one value is given for ε or h leading to a single computation, a Richardson extrapolation has been performed using the values listed in accordance with error forms given in sections 3-1 and 3-2. The number of values listed indicates the number of extrapolations carried out, e.g. two values of h, say, 1/32 and 1/64, would be used for one extrapolation; five values, 1/4, 1/8, 1/16, 1/32, 1/64 would indicate four extrapolations to produce the final computed results in the right hand columns.

Quantities indicated by $\tilde{\psi}_{\varepsilon,h}$ and $\tilde{w}_{\varepsilon,h}$ are computed from $\psi_{\varepsilon,h}$ and $w_{\varepsilon,h}$ by multiplying by $(1+\varepsilon^{-1}h^2/12)$, as discussed in section 3-2. That is, before extrapolation,

$$\tilde{w}_{\varepsilon,h} = (1+\frac{\varepsilon^{-1}h^2}{12})w_{\varepsilon,h}$$

$$\tilde{\psi}_{\varepsilon,h} = (1+\frac{\varepsilon^{-1}h^2}{12})\psi_{\varepsilon,h} \;.$$

Functions are all evaluated at the beams midpoint.

Tables 1, 2, 4, and 6 give relative errors comparing numerical solutions and exact solutions to problem (I).

As expected, errors relative to ψ_ε for both $\psi_{\varepsilon,h}$ and $\tilde{\psi}_{\varepsilon,h}$ in Table 1 appear to be quadratic in h. After one extrapolation, these errors become quartic, agreeing with the form predicted in 3-2. These same patterns appear in the errors for $w_{\varepsilon,h}$ and $\tilde{w}_{\varepsilon,h}$ for the two symmetric loads $p/D = 1$ and $p/D = \delta(x)$. This holds for $\varepsilon > 2^{-6}$, but fails for $\varepsilon \leq 2^{-15}$, indicating that the small values of ε are causing non-quadratic terms in the principal error to interact with the leading term. $\psi_{\varepsilon,h}$ and $w_{\varepsilon,h}$ then lose all approximating value (in fact they tend to zero as ε tends to zero). However, $\tilde{\psi}_{\varepsilon,h}$ and $\tilde{w}_{\varepsilon,h}$ maintain excellent accuracy for all small values of ε, and remain quadratic in h, except for the most crude mesh sizes (see also Tables 4 and 6).

Tables 2, 4, and 6 show that great improvement in accuracy can be gained by using extrapolations on h. In fact extrapolations using all available mesh sizes extends the range of small ε for which $\psi_{\varepsilon,h}$ and $w_{\varepsilon,h}$ give reliable accuracy. For example, an error of less than one percent can be achieved by $\psi_{\varepsilon,h}$ for ε as small as about 2^{-11} with mesh sizes to $h = 1/64$ with four extrapolations.

Tables 3, 5, and 7 give relative errors comparing numerical solutions to problem (I) with exact solutions to problem (C).

For large values of ε, Table 3 shows that neither $\psi_{\varepsilon,h}$ nor $\tilde{\psi}_{\varepsilon,h}$ bears any resemblance to w_0', even after extrapolating on ε, indicating an interaction of the higher order terms with the linear term in the asymptotic series described in 3-1. For small ε, $\tilde{\psi}_{\varepsilon,h}$ approximates w_0' well, with or without extrapolations. However, $\psi_{\varepsilon,h}$ again loses its ability to approximate w_0' as ε decreases. For $\varepsilon = 2^{-5}$, $\psi_{\varepsilon,h}$ is too large. Then it decreases past the exact solution as ε decreases. Hence there is some ε which could be considered optimal. This is of no use, however, without some method of determining the optimal ε value.

Similar results occur for the solutions represented in Tables 5 and 7, except that for large values of ϵ, $w_{\epsilon,h}$ and $\tilde{w}_{\epsilon,h}$ can be used to approximate w_0 accurately simply by extrapolating once to remove the linear dependence upon ϵ. As shown in 3-1, when p/D is symmetric, the dependence of w_ϵ upon ϵ is linear.

As indicated by the error estimates of Chapter 2 and by the asymptotic formulas of Chapter 3, the results show that $U_{\epsilon,h}$ is a useful approximation to U_ϵ and U_0, but caution must be exercised when ϵ is small. While Richardson-type extrapolations improve accuracy and extend the range of small ϵ for which results remain reliable, limitations are still encountered.

Because the factor $(1+\epsilon^{-1}h^2/12)$ multiplied by the solution $U_{\epsilon,h}$ removes the adverse effect of small ϵ, the quantity $\tilde{U}_{\epsilon,h}$ approximates both U_ϵ and U_0 much more reliably than does $U_{\epsilon,h}$ over the entire range of small ϵ. In the case of the clamped beam, the limitations described above are overcome by this algebraic adjustment to the numerical solutions.

The exact solutions to problems (C) and (I) for the clamped beam under the three different loads considered in 3-4 are given below, along with the quantities (center deflections or slopes) used for comparison in Tables 1-7. In each case, the domain is $-\frac{1}{2} \le x \le \frac{1}{2}$.

For $p/D = 4 \sinh 2x$,

$$w_0 = \frac{1}{4} \sinh 2x + \frac{(\cosh 1 - 3 \sinh 1)}{4} x - (\cosh 1 - \sinh 1)x^3$$

$$w_\varepsilon = w_0 - \varepsilon \sinh 2x + \frac{12\varepsilon}{1+12\varepsilon}\left[\frac{(-\cosh 1 + 2(1+4\varepsilon)\sinh 1)}{4} x + \frac{(3 \cosh 1 - 4 \sinh 1)}{3} x^3\right]$$

$$\psi_\varepsilon = -w_0' + \frac{12\varepsilon}{1+12\varepsilon}\left[\frac{(3 \cosh 1 - 4 \sinh 1)}{4} + (-3 \cosh 1 + 4 \sinh 1)x^2\right]$$

$$w_0'(0) = \frac{1}{2} + \frac{1}{4} \cosh 1 - \frac{3}{4} \sinh 1 \approx .43692635 \text{ E-2}$$

$$\psi_\varepsilon(0) = -w_0'(0) + \frac{3\varepsilon}{1+12\varepsilon} (3 \cosh 1 - 4 \sinh 1) .$$

For $p/D = 1$

$$w_0 = \frac{1}{384} (1 - 4x^2)^2$$

$$w_\varepsilon = w_0 + \frac{\varepsilon}{8} (1 - 4x^2)$$

$$\psi_\varepsilon = -w_0'$$

$$w_0(0) = \frac{1}{384} \approx .26041667 \text{ E-2}$$

$$w_\varepsilon(0) = w_0(0) + \frac{\varepsilon}{8} .$$

For $p/D = \delta(x)$,

$$w_0 = \begin{cases} \dfrac{x^3}{12} - \dfrac{x^2}{16} + \dfrac{1}{192} , & -\dfrac{1}{2} \leq x < 0 \\[2ex] -\dfrac{x^3}{12} - \dfrac{x^2}{16} + \dfrac{1}{192} , & 0 \leq x \leq \dfrac{1}{2} \end{cases}$$

$$w_\varepsilon - w_0 = \begin{cases} \varepsilon\left(-\dfrac{x}{2} + \dfrac{1}{4}\right) , & -\dfrac{1}{2} \leq x < 0 \\[2ex] \varepsilon\left(\dfrac{x}{2} + \dfrac{1}{4}\right) , & 0 \leq x \leq \dfrac{1}{2} \end{cases}$$

$$\psi_\varepsilon = -w_0'$$

$$w_0(0) = \frac{1}{192} = .52083333 \text{ E-2}$$

$$w_\varepsilon(0) = w_0(0) + \frac{\varepsilon}{4} .$$

Table 1.

Errors for beam problem (I), $p/D = 4 \sinh 2x$

ϵ	h	$(\psi_{\epsilon,h} - \psi_\epsilon)/\psi_\epsilon$	$(\tilde{\psi}_{\epsilon,h} - \psi_\epsilon)/\psi_\epsilon$
2^5	1/4	-.2531 E0	-.2530 E0
	1/8	-.6342 E-1	-.6344 E-1
	1/16	-.1588 E-1	-.1587 E-1
	1/32	-.3971 E-2	-.3968 E-2
	1/64	-.9928 E-3	-.9922 E-3
2^0	1/4	-.2537 E0	-.2500 E0
	1/8	-.6365 E-1	-.6243 E-1
	1/16	-.1593 E-1	-.1561 E-1
	1/32	-.3983 E-2	-.3902 E-2
	1/64	-.9958 E-3	-.9754 E-3
2^{-5}	1/4	-.2996 E0	-.1829 E0
	1/8	-.7972 E-1	-.4137 E-1
	1/16	-.2028 E-1	-.1007 E-1
	1/32	-.5091 E-2	-.2500 E-2
	1/64	-.1274 E-2	-.6240 E-3
2^{-15}	1/4	-.9945 E0	-.4906 E-1
	1/8	-.9771 E0	.7152 E-3
	1/16	-.9142 E0	.9968 E-3
	1/32	-.7212 E0	.3004 E-3
	1/64	-.4000 E0	.7831 E-4
2^{-60}	1/4	-1.0	-.4868 E-1
	1/8	-1.0	.8349 E-3
	1/16	-1.0	.1028 E-2
	1/32	-1.0	.3084 E-3
	1/64	-1.0	.8031 E-4

Table 2.

Errors for beam problem (I), extrapolated, $p/D = 4 \sinh 2x$

ϵ	h	$(\psi_{\epsilon,h} - \psi_\epsilon)/\psi_\epsilon$	$(\tilde{\psi}_{\epsilon,h} - \psi_\epsilon)/\psi_\epsilon$
2^5	1/64	-.9928 E-3	-.9922 E-3
	1/4, 1/8	-.2641 E-3	-.2538 E-3
	1/32, 1/64	-.6493 E-7	-.6241 E-7
2^0	1/64	-.9958 E-3	-.9754 E-3
	1/4, 1/8	-.3094 E-3	.2051 E-4
	1/32, 1/64	-.7574 E-7	.5288 E-8
2^{-5}	1/64	-.1272 E-2	-.6240 E-3
	1/4, 1/8	-.6430 E-2	.5784 E-2
	1/32, 1/64	-.1885 E-5	.1428 E-5
2^{-15}	1/64	-.4000 E0	.7831 E-4
	1/4, 1/8	-.9713 E0	.1731 E-1
	1/32, 1/64	-.2909 E0	.4271 E-5
2^{-60}	1/64	-1.0	.8031 E-4
	1/4, 1/8	-1.0	.1734 E-1
	1/32, 1/64	-1.0	.4279 E-5
2^{-5}	1/64	-.1274 E-2	-.6240 E-3
	1/4, 1/8, 1/16, 1/32, 1/64	-.1145 E-9	-.8163 E-11
2^{-6}	1/64	-.1744 E-2	-.4439 E-3
	1/4, 1/8, 1/16, 1/32, 1/64	-.2688 E-8	-.7350 E-11
2^{-7}	1/64	-.2863 E-2	-.2665 E-3
	1/4, 1/8, 1/16, 1/32, 1/64	-.6089 E-7	-.1055 E-10
2^{-8}	1/64	-.5307 E-2	-.1265 E-3
	1/4, 1/8, 1/16, 1/32, 1/64	-.1153 E-5	-.1371 E-10

Table 3.

Errors for beam problem (C), $p/D = 4 \sinh 2x$

ϵ	h	$(\psi_{\epsilon,h} - w_0')/w_0'$	$(\psi_{\epsilon,h} - w_0')/w_0'$
2^5	1/64	4.08	4.08
$2^5, 2^4$	1/64	4.06	4.06
2^{-5}	1/64	1.11	1.12
$2^{-5}, 2^{-6}$	1/64	.173	.176
2^{-15}	1/64	-.40	.1577 E-2
$2^{-15}, 2^{-16}$	1/64	-.74	.8058 E-4
	1/32,1/64	-.67	.4554 E-5
2^{-60}	1/64	-1.0	.8031 E-4
$2^{-60}, 2^{-61}$	1/64	-1.0	.8031 E-4
	1/32,1/64	-1.0	.4279 E-5
$2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}$	1/64	-.9934 E-2	.7562 E-3
	1/4,1/8,1/16, 1/32,1/64	.6729 E-3	.6762 E-3

Table 4.

Errors for beam problem (I), p/D = 1

ε	h	$(w_{\varepsilon,h} - w_\varepsilon)/w_\varepsilon$	$(\tilde{w}_{\varepsilon,h} - w_\varepsilon)/w_\varepsilon$
2^5	1/64	-.6358 E-6	-.6341 E-11
	1/4, 1/8	-.6622 E-8	-.1349 E-12
	1/32, 1/64	-.9657 E-11	-.8038 E-11
2^0	1/64	-.2034 E-4	-.5708 E-11
	1/4, 1/8	-.6738 E-5	-.2088 E-12
	1/32, 1/64	-.1663 E-8	-.7169 E-11
2^{-5}	1/64	-.6506 E-3	-.8859 E-11
	1/4, 1/8	-.5714 E-2	-.2387 E-12
	1/32, 1/64	-.1690 E-5	-.1117 E-10
2^{-5}	1/64	-.4000 E0	-.1449 E-8
	1/4, 1/8	-.9714 E0	-.1346 E-11
	1/32, 1/64	-.2909 E0	-.1901 E-8
2^{-60}	1/64	-1.0	-.2107 E-8
	1/4, 1/8	-1.0	-.1513 E-11
	1/32, 1/64	-1.0	-.2761 E-8

Table 5.

Errors for beam problem (C), $p/D = 1$

ε	h	$(w_{\varepsilon,h} - w_0)/w_0$	$(\tilde{w}_{\varepsilon,h} - w_0)/w_0$
2^5	1/64	.1536 E4	.1536 E4
$2^5, 2^4$	1/64	-.9785 E-3	.2627 E-8
	1/4, 1/8	-.3055 E-4	-.1455 E-10
2^{-5}	1/64	.1498 E1	.1500 E1
$2^{-5}, 2^{-6}$	1/64	-.2925 E-2	-.1911 E-10
	1/4, 1/8	-.5302 E-1	-.4370 E-12
2^{-15}	1/64	-.3791 E0	.1465 E-2
$2^{-15}, 2^{-16}$	1/64	-.7431 E0	-.1056 E-8
	1/4, 1/8	-.9997 E0	-.8740 E-12
2^{-60}	1/64	-1.0	-.2107 E-8
$2^{-60}, 2^{-61}$	1/64	-1.0	-.9481 E-9
	1/4, 1/8	-1.0	.2132 E-12

Table 6.

Errors for beam problem (I), $p/D = \delta(x)$

ϵ	h	$(w_{\epsilon,h} - w_\epsilon)/w_\epsilon$	$(\tilde{w}_{\epsilon,h} - w_\epsilon)/w_\epsilon$
2^5	1/64	-.6358 E-6	-.5105 E-11
	1/4, 1/8	-.6621 E-8	-.1065 E-12
	1/32, 1/64	-.8088 E-11	-.6476 E-11
2^0	1/64	-.2034 E-4	-.4608 E-11
	1/4, 1/8	-.6738 E-5	-.1531 E-12
	1/32, 1/64	-.1661 E-8	-.5819 E-11
2^{-5}	1/64	-.6506 E-3	-.7572 E-11
	1/4, 1/8	-.5741 E-2	-.2089 E-12
	1/32, 1/64	-.1690 E-5	-.9533 E-11
2^{-15}	1/64	-.4000 E0	-.1380 E-8
	1/4, 1/8	-.9714 E0	-.1309 E-11
	1/32, 1/64	-.2909 E0	-.1811 E-8
2^{-60}	1/64	-1.0	-.1997 E-8
	1/4, 1/8	-1.0	-.1428 E-11
	1/32, 1/64	-1.0	-.2617 E-8

Table 7.

Errors for beam problem (C), $p/D = \delta(x)$

ϵ	h	$(w_{\epsilon,h} - w_0)/w_0$	$(\tilde{w}_{\epsilon,h} - w_0)/w_0$
2^5	1/64	.1536 E4	.1536 E4
$2^5, 2^4$	1/64	-.9785 E-3	.2179 E-8
2^{-5}	1/64	.1498 E1	.1500 E1
$2^{-5}, 2^{-6}$	1/64	-.2925 E-2	-.1803 E-10
	1/4, 1/8	-.5302 E-1	-.3944 E-12
2^{-15}	1/64	-.3991 E0	.1465 E-2
$2^{-15}, 2^{-16}$	1/64	-.7431 E0	-.1023 E-8
	1/4, 1/8	-.9997 E0	-.7461 E-12
2^{-60}	1/64	-1.0	-.1997 E-8
$2^{-60}, 2^{-61}$	1/64	-1.0	-.8660 E-9
	1/4, 1/8	-1.0	.1918 E-12

CHAPTER 4 - BENDING OF CLAMPED PLATES IN THE IMPROVED THEORY

This chapter studies the problems of the bending of clamped circular and square plates. The solutions U_ϵ of problem (I) for a circular plate under axi-symmetric and non-axisymmetric loading are studied analytically as ϵ tends to zero. Numerical solutions are obtained for U_ϵ for a square plate under several types of loadings. The behaviors of such solutions as ϵ tends to zero are noted and discussed, as they serve to illustrate complexity encountered when dealing with plates with non-smooth boundaries.

4-1. Asymptotic Analysis of the Solution U_ϵ

Because the main estimates of Chapter 2 guarantee that solutions U_ϵ of problem (I) converge to solutions U_0 of problem (C) (in the H^1 sense), it is natural to seek to examine this convergence asymptotically.

A glance at the governing equations reveals the appearance of a singular perturbation problem (small parameter multiplying the highest order derivatives).

Singular perturbation problems usually have solutions exhibiting boundary layer behavior. Such a solution would be quite regular throughout the domain, but would change drastically in a narrow region near the boundary, in order to satisfy the boundary conditions. Schemes are available for handling one-dimensional boundary regions. They usually involve two asymptotic series for the solution, one valid near the boundary, one valid far away from the boundary, and a matching process to connect them.

Such schemes tend to be complicated even for problems in one dimension. Little can then be expected of such approaches for a two dimension problem, where even the very boundary region thickness depends in some unknown way upon the geometry of the boundary as well as on the boundary value problem itself.

On a more hopeful note, the one-dimensional analog of the clamped plate problem, namely the clamped beam problem, exhibits no such boundary layer phenomena at all. In an attempt to imitate the asymptotic analysis of the clamped beam, consider a formal power series expansion of $U_\epsilon = [\psi_x, \psi_y, w]^T$.

$$U_\epsilon = \sum_{i=0} \epsilon^i U_i = \sum_{i=0} \epsilon^i [\psi_{xi}, \psi_{yi}, w_i]^T$$

where $U_i = [\psi_{xi}, \psi_{yi}, w_i]^T \in (H_0^1)^3$.

Substituting these into (1-4.6), assuming sufficient smoothness,

$$LU_\epsilon = F$$

or

$$L_B U_\epsilon + \epsilon^{-1} L_S U_\epsilon = F \; .$$

Collecting terms with like powers of ϵ yields

(4-1.1) $$L_S U_0 = 0$$

(4-1.2) $$L_S U_1 = F - L_B U_0$$

(4-1.3) $$L_S U_2 = -L_B U_1$$

(4-1.4) $$L_S U_3 = -L_B U_2$$

and so on. Now

$$L_S U_0 = \begin{bmatrix} -\psi_{x0} - \dfrac{\partial w_0}{\partial x} \\ -\psi_{y0} - \dfrac{\partial w_0}{\partial y} \\ \dfrac{\partial \psi_{x0}}{\partial x} + \dfrac{\partial \psi_{y0}}{\partial y} + \nabla^2 w_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first two components say

$$U_0 = [\frac{\partial w_0}{\partial x}, \frac{\partial w_0}{\partial y}, w_0]^T \tag{4-1.5}$$

and the third is identically satisfied as a consequence of the first two.

Using (4-1.5), (4-1.2) is

$$\begin{bmatrix} -\psi_{x1} - \frac{\partial w_1}{\partial x} \\ -\psi_{y1} - \frac{\partial w_1}{\partial y} \\ \frac{\partial \psi_{x1}}{\partial x} + \frac{\partial \psi_{y1}}{\partial y} + \nabla^2 w_1 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \nabla^2 w_0 \\ \frac{\partial}{\partial y} \nabla^2 w_0 \\ -f \end{bmatrix}$$

Differentiating the first component by x, the second by y, and adding to the third results in

$$\nabla^4 w_0 = f \ .$$

Since $U_0 = [0,0,0]$ on $\partial\Omega$, U_0 is in fact the solution to problem (C) as expected.

Repeating the same process with (4-1.3),

$$\nabla^2 (\frac{\partial \psi_{x1}}{\partial x} + \frac{\partial \psi_{y1}}{\partial y}) = 0$$

returning now to take the Laplacian of the third component in (4-1.2) yields

$$\nabla^4 w_1 = -\nabla^2 f \ .$$

However U_1 is over-determined on the boundary by (4-1.2). Since U_1 must satisfy $\psi_{x1} = \psi_{y1} = w_1 = 0$ on $\partial\Omega$, the first two components of $L_S U_1$, as a vector, can only be normal to the boundary. This is because $[\psi_{x1}, \psi_{y1}]^T = [0,0]^T$, and $[\frac{\partial w_1}{\partial x}, \frac{\partial w_1}{\partial y}] = \nabla w_1$ has no tangential component at the boundary since $w_1 = 0$ there. On the other hand, $[\partial/\partial x\ \nabla^2 w_0, \partial/\partial y\ \nabla^2 w_0]^T = \nabla(\nabla^2 w_0)$ in general will have both tangential and normal components at the boundary. Thus U_1 cannot meet the boundary conditions, and the regular perturbation procedure breaks down.

As pointed out in Chapter 2, the overly restrictive condition of (4-1.2) is exactly that which could not be met when attempting to apply Babuska's method for improving the energy estimates in the general case. The presence of a boundary layer is even more strongly suggested.

The only two cases where (4-1.2) can be expected to be satisfied are the clamped circular plate with axisymmetric load, and the clamped beam. In these cases the expansion can proceed. This was carried out for the beam in Chapter 3.

The presence of a boundary layer can be verified in one instance, namely that of a circular plate with non-axisymmetric loading. This case with $\frac{p}{D} = \cos\theta$ is carried out explicitly in 4-2.

4-2. The Clamped Circular Plate

The important special case of a clamped circular plate is treated here in the improved theory (problem (I)). Even for most simple geometries exact solutions of problem (I) are rare, as are exact solutions to problem (C). By enabling the construction of exact solutions, the circular plate provides insights into the behavior of solutions to both problems, which otherwise would be missed.

In addition, the clamped circular plate with a non-axisymmetric load allows first hand examination of behavior which prevents improvement in the main energy estimates of 2-2, and which also prevents a simple asymptotic series expansion from being formed.

For solving the circular plate problems it is necessary to transform the boundary value problems into polar coordinates r and θ. Assume plate radius unity.

Problem (C) becomes

$$U_0 = \left[-\frac{\partial w_0}{\partial r}, \frac{-1}{r}\frac{\partial w_0}{\partial \theta}, w_0\right]^T$$

$$\nabla^4 w_0 = \frac{p}{D}$$

$$(4\text{-}2.1) \qquad \left.\begin{aligned} w_0 &= 0 \\ \frac{\partial w_0}{\partial r} &= 0 \end{aligned}\right\} \quad \text{at} \quad r = 1$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

Problem (I) is

$$U_\epsilon = [\psi_r, \psi_\theta, w]^T$$

$$\text{(4-2.2a)} \quad \frac{\partial^2 \psi_r}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_r}{\partial r} + \frac{1}{2r^2}\frac{\partial^2 \psi_r}{\partial \theta^2} - \frac{1}{r^2}\psi_r$$

$$- \frac{3}{2r^2}\frac{\partial \psi_\theta}{\partial \theta} + \frac{1}{2r}\frac{\partial^2 \psi_\theta}{\partial r \partial \theta}$$

$$+ \mu\left(-\frac{1}{2r^2}\frac{\partial^2 \psi_r}{\partial \theta^2} + \frac{1}{2r^2}\frac{\partial \psi_\theta}{\partial \theta} + \frac{1}{2r}\frac{\partial^2 \psi_\theta}{\partial r \partial \theta}\right)$$

$$- \epsilon^{-1}\left(\psi_r + \frac{\partial w}{\partial r}\right) = 0$$

$$\text{(4-2.2b)} \quad \frac{1}{2}\frac{\partial^2 \psi_\theta}{\partial r^2} + \frac{1}{2r}\frac{\partial \psi_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi_\theta}{\partial \theta^2} - \frac{1}{2r^2}\psi_\theta$$

$$+ \frac{3}{2r^2}\frac{\partial \psi_r}{\partial \theta} + \frac{1}{2r}\frac{\partial^2 \psi_r}{\partial r \partial \theta}$$

$$+ \mu\left(-\frac{1}{2}\frac{\partial^2 \psi_\theta}{\partial r^2} - \frac{1}{2r}\frac{\partial \psi_\theta}{\partial r} + \frac{1}{2r^2}\psi_\theta\right.$$

$$\left. - \frac{1}{2r^2}\frac{\partial \psi_r}{\partial \theta} + \frac{1}{2r}\frac{\partial^2 \psi_r}{\partial r \partial \theta}\right)$$

$$- \epsilon^{-1}\left(\psi_\theta + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) = 0$$

$$(4\text{-}2.2c) \qquad \varepsilon^{-1}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial \psi_r}{\partial r} + \frac{1}{r}\psi_r + \frac{1}{r}\frac{\partial \psi_\theta}{\partial \theta}\right) = -\frac{p}{D}$$

with $U^h \equiv [0,0,0]^T$ at $r = 1$.

The first special case is to consider an axisymmetric load. Then $\psi_\theta \equiv 0$, and all derivatives with respect to θ can also be dropped from (4-2.1) and (4-2.2).

In (4-2.1)

$$(4\text{-}2.3) \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \; .$$

In (4-2.2), (4-2.2b) is identically satisfied. The remaining equations are

$$(4\text{-}2.4a) \qquad \frac{\partial^2 \psi_r}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_r}{\partial r} - \frac{1}{r^2}\psi_r - \varepsilon^{-1}\left(\psi_r + \frac{\partial w}{\partial r}\right) = 0$$

$$(4\text{-}2.4b) \qquad \varepsilon^{-1}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial \psi_r}{\partial r} + \frac{1}{r}\psi_r\right) = -\frac{p}{D} \; .$$

Note that the dependence on μ vanishes.

Let $\frac{p}{D} = 1$. The solution to problem (C) is given by

$$(4\text{-}2.5a) \qquad w_0 = \frac{1}{64}\,(1 - r^2)^2$$

$$(4\text{-}2.5b) \qquad \frac{\partial w_0}{\partial r} = \frac{r}{16}\,(1 - r^2) \quad .$$

The solution to problem (I) with $\frac{p}{D} = 1$ is given by

$$w = \frac{1}{64}(1-r^2)^2 + \frac{\varepsilon}{4}(1-r^2) = w_0 + \frac{\varepsilon}{4}(1-r^2) \tag{4-2.6a}$$

$$\psi_r = \frac{r}{16}(1-r^2) = \frac{\partial w_0}{\partial r} \ . \tag{4-2.6b}$$

As in the case of the clamped beam with symmetric loading, the dependence on ε is simply linear. Also the estimates of 2-3 agree with these solutions.

Now consider the non-axisymmetric load $\frac{p}{D} = \cos\,\theta$. The solution can be obtained to problem (C) in a straightforward manner by assuming a form $w_0 = R(r)\cos\,\theta$. This separation of variables leads to solution

$$w_0 = \frac{r}{90}(1-r)^2(2r+1)\cos\,\theta \tag{4-2.7a}$$

$$\frac{\partial w_0}{\partial r} = \frac{-1}{90}(1-r)(8r^2-r-1)\cos\,\theta \tag{4-2.7b}$$

$$\frac{\partial w_0}{\partial \theta} = -\frac{r}{90}(1-r)^2(2r+1)\sin\,\theta \ . \tag{4-2.7c}$$

In order to solve problem (I) for load $\frac{p}{D} = \cos\,\theta$, it is helpful to recall from Chapter 1 that

$$\nabla^4 w = \frac{p}{D} - \varepsilon\,\nabla^2\,\frac{p}{D} \ . \tag{4-2.8}$$

(4-2.8) is valid in polar as well as rectangular coordinates. Although a second boundary condition is needed at $r = 1$, in addition to $w = 0$, a solution can be found in terms of an undetermined constant a:

$$(4\text{-}2.9) \qquad w = r(r-1)[-a(r+1) + \frac{1}{90}(r-1)(2r+1) + \frac{\varepsilon}{3} r]\cos\theta .$$

For convenience define the quantity

$$(4\text{-}2.10) \qquad C(r) = (r-1)[-a(r+1) + \frac{1}{90}(r-1)(2r+1) + \frac{\varepsilon}{3} r] .$$

That is,

$$(4\text{-}2.11) \qquad w = rC(r)\cos\theta .$$

Also define dependent variables ξ, η by

$$(4\text{-}2.12a) \qquad \xi\cos\theta = \psi_r + \frac{\partial w}{\partial r}$$

$$(4\text{-}2.12b) \qquad \eta\sin\theta = \psi_\theta + \frac{1}{r}\frac{\partial w}{\partial r} .$$

Substituting (4-2.11) and (4-2.12) into (4-2.2) results in

$$(4\text{-}2.13a) \qquad r^2\xi'' + r\xi' + (-\frac{3}{2} + \frac{\mu}{2} - \varepsilon^{-1}r^2)\xi$$

$$+ \frac{1+\mu}{2} r\eta' + \frac{\mu-3}{2}\eta$$

$$= r^2(C + rC')'' + r(C + rC')' - 2rC'$$

(4-2.13b) $$\frac{1-\mu}{2} r^2\eta'' + \frac{1-\mu}{2} r\eta' + (\frac{\mu-3}{2} - \epsilon^{-1}r^2)\eta - \frac{1+\mu}{2} r\xi' + \frac{\mu-3}{2}\xi = -r^2C'' - 3rC'$$

(4-2.13c) $$r\xi' + \xi + \eta = -\epsilon r \ .$$

From (4-2.13c)

(4-2.14) $$\eta = -r\xi' - \xi - \epsilon r$$

(4-2.13a) reduces to

(4-2.15) $$r^2\xi'' + 3r\xi' - \frac{2\epsilon^{-1}r^2}{1-\mu}\xi = \frac{2}{1-\mu}[r^2(C + rC')'' + r(C + rC')' - 2rC' - (1-\mu)\epsilon r] .$$

Setting

(4-2.16) $$P = r\xi$$

and

(4-2.17) $$\rho = \alpha r, \quad \text{where} \quad \alpha = \frac{\sqrt{2}}{\sqrt{1-\mu}\,\sqrt{\epsilon}}$$

(4-2.15) becomes the inhomogeneous Bessel Equation

$$(4\text{-}2.18)\qquad \rho^2 P'' + \rho P' - (1+\rho^2)P =$$

$$= \varepsilon\rho^2\left[\frac{1-\mu}{3}\ \varepsilon\ \rho^2 + 8\left(-a - \frac{1}{30} + \frac{\varepsilon}{3}\right)\sqrt{\frac{1-\mu}{2}}\ \sqrt{\varepsilon}\ \rho - (1-\mu)\varepsilon\right] .$$

A general solution to (4-2.18) is given as a particular solution plus a solution to the homogeneous equation.

$$(4\text{-}2.19)\qquad P = P_{HOM} + P_{PART} .$$

The homogeneous solution is of form

$$(4\text{-}2.20)\qquad P_{HOM} = AI_1(\rho) + BK_1(\rho) ,$$

where I_1 and K_1 are modified Bessel functions whose properties are well-documented (see [1]). A particular solution is

$$(4\text{-}2.21)\qquad P_{PART} = -\frac{2}{3}\ \alpha^{-2}\ \varepsilon\ \rho^2 - 8\left(-a - \frac{1}{30} + \frac{\varepsilon}{3}\right)\alpha^{-1}\ \varepsilon\ \rho .$$

From (4-2.16) through (2-2.21) $\xi(r)$ is reconstructed, and boundary conditions can be applied.

$$(4\text{-}2.22)\qquad \xi(r) = \frac{1}{r}[AI_1(\alpha r) + BK_1(\alpha r) + P_{PART}] .$$

Since $\frac{1}{r}\ K_1(\alpha r)$ is singular at the origin, it does not belong to H^1 (or even H^0). Therefore, set $B = 0$. Next, the edge condition at $r = 1$ is

$$\psi_r(1) = 0 .$$

From (4-2.12a), it follows

$$\xi(1) = -2a + \frac{\varepsilon}{3} .$$

From (4-2.22),

$$A = \frac{-8a\varepsilon + \frac{11}{15}\varepsilon + \frac{8}{3}\varepsilon^2 - 2a}{I_1(\alpha)} . \tag{4-2.23}$$

Now $\xi(r)$ can be substituted back into (4-2.14) to determine $\eta(r)$, where the edge condition $\psi_\theta(1) = 0$ determines the constant a. The solutions $\xi(r)$ and $\eta(r)$ can be substituted into (4-2.13b) to verify that all three equations (4-2.13) are satisfied. Edge condition $\psi_\theta(1) = 0$ implies $\eta(1) = 0$

$$0 = \frac{\varepsilon}{3} - (-2a - 8a\varepsilon + \frac{11}{15}\varepsilon + \frac{8}{3}\varepsilon^2)\alpha \frac{I_1'(\alpha)}{I_1(\alpha)} + 8\varepsilon(-a - \frac{1}{30} + \frac{\varepsilon}{3})$$

Using the Bessel function identity

$$I_1'(x) = I_0(x) - \frac{1}{x} I_1(x)$$

and solving for a,

$$a = \frac{-\frac{4}{5}\alpha^{-1}\varepsilon - \frac{16}{3}\alpha^{-1}\varepsilon^2 + \frac{I_0(\alpha)}{I_1(\alpha)}(\frac{11}{15}\varepsilon + \frac{8}{3}\varepsilon^2)}{(2 + 8\varepsilon)\frac{I_0(\alpha)}{I_1(\alpha)} - 2\alpha^{-1} - 16\alpha^{-1}\varepsilon} \tag{4-2.24}$$

where $\alpha = \dfrac{\sqrt{2}}{\sqrt{1-\mu}\,\sqrt{\varepsilon}}$.

Now define $\tilde{a}$ by

$$a = \epsilon\tilde{a} \ . \tag{4-2.25}$$

To summarize,

$$w = \{\frac{r}{90}(1-r)^2(2r+1) + \epsilon r(1-r)(\tilde{a}(r+1) - \frac{r}{3})\}\cos\theta \tag{4-2.26a}$$

$$\psi_r + \frac{\partial w}{\partial r} = \{\epsilon(\frac{11-30\,\tilde{a}}{15\,r}\,\frac{I_1(\alpha r)}{I_1(\alpha)} - \frac{2r}{3} + \frac{4}{15}) + \frac{8\epsilon^2}{3}(\frac{-3\tilde{a}+1}{r}\,\frac{I_1(\alpha r)}{I_1(\alpha)} - 1)\}\cos\theta \tag{4-2.26b}$$

$$\begin{aligned}\psi_\theta + \frac{1}{r}\frac{\partial w}{\partial\theta} = \{&\sqrt{\epsilon}\,\frac{\sqrt{2}}{\sqrt{1-\mu}}\,\frac{30\tilde{a}-11}{15}\,\frac{I_0(\alpha r)}{I_1(\alpha)} \\ &- \epsilon(\frac{30\tilde{a}-11}{15\,r}\,\frac{I_1(\alpha r)}{I_1(\alpha)} - \frac{5r-4}{15}) \\ &- \frac{8}{3}(1-3\tilde{a})\ \epsilon\sqrt{\epsilon}\,\frac{\sqrt{2}}{\sqrt{1-\mu}}\,\frac{I_0(\alpha r)}{I_1(\alpha)} \\ &+ \frac{8}{3}(1-3\tilde{a})\epsilon^2(\frac{I_1(\alpha r)}{I_1(\alpha)} + 1)\}\sin\theta \ .\end{aligned} \tag{4-2.26c}$$

While (4-2.26) gives explicit solutions, the dependence upon ϵ is hidden in the quantities $\tilde{a}$ and α. It is therefore desirable to express (4-2.26) asymptotically. Considering ϵ a small parameter, then α is large. Expansions of Bessel functions for large argument satisfy

$$(4\text{-}2.27) \quad I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \{1 - \frac{(4\nu^2-1)}{8x} + \frac{(4\nu^2-1)(4\nu^2-9)}{2!(8x)^2} \cdots\cdots\}$$

from this

$$(4\text{-}2.28) \qquad \frac{I_0(x)}{I_1(x)} \sim 1 + \frac{1}{2x} + \frac{3}{8x^2} + O(\frac{1}{x^3})$$

using (4-2.28) in (4-2.24) and expanding, (4-2.25) gives

$$(4\text{-}2.29) \quad \tilde{a} = \frac{1}{30}(11 - \sqrt{\varepsilon}\,\frac{\sqrt{1-\mu}}{\sqrt{2}} - \varepsilon(88 + \frac{37(1-\mu)}{8})) + O(\varepsilon^{3/2})$$

Observing that the first term of w in (4-2.26a) is

$$(4\text{-}2.30) \qquad w_0 = \frac{r}{90}(1-r)^2(2r+1)\cos\theta$$

(4-2.26a) can be rewritten asymptotically, with the help of (4-2.29), to express comparison to w_0

$$(4\text{-}2.31a) \qquad w = w_0 + \frac{\varepsilon r}{30}(1-r)(r+11) + O(\varepsilon^{3/2}) \ .$$

Likewise, (4-2.31a) can be used to provide analgous comparisons derived from (4-2.26b) and (4-2.26c)

$$(4\text{-}2.31b) \quad \psi_r = -\frac{\partial w_0}{\partial r} - \frac{\varepsilon}{10}(1-r^2)\cos\theta$$

$$+ \frac{\varepsilon^{3/2}\sqrt{1-\mu}}{15\sqrt{2}}\,\frac{I_1(\frac{\sqrt{2}\ r}{\sqrt{1-\mu}\ \sqrt{\varepsilon}})\cos\theta}{r\ I_1(\frac{\sqrt{2}}{\sqrt{1-\mu}\ \sqrt{\varepsilon}})} + O(\varepsilon^{3/2})$$

$$(4\text{-}2.31c)\qquad \psi_\theta = -\frac{1}{r}\frac{\partial w_0}{\partial \theta} - \frac{\varepsilon}{15}\,\frac{I_0\left(\frac{\sqrt{2}\ r}{\sqrt{1-\mu}\ \sqrt{\varepsilon}}\right)}{I_1\left(\frac{\sqrt{2}}{\sqrt{1-\mu}\ \sqrt{\varepsilon}}\right)}\sin\theta$$

$$+\frac{\varepsilon}{30}(3-r^2)\sin\theta + O(\varepsilon^{3/2}) \ .$$

While the leading error terms in each of (4-2.31) are $O(\varepsilon)$ this is not the case for the radial derivative of (4-2.31c). In fact, using identity $I_0'(x) = I_1(x)$,

$$(4\text{-}2.32)\qquad \frac{\partial}{\partial r}\left(\psi_\theta + \frac{1}{r}\frac{\partial w_0}{\partial \theta}\right)$$

$$= \frac{-\sqrt{\varepsilon}\ \sqrt{2}}{15\sqrt{1-\mu}}\ \frac{I_1\left(\frac{\sqrt{2}\ r}{\sqrt{1-\mu}\ \sqrt{\varepsilon}}\right)}{I_1\left(\frac{\sqrt{2}}{\sqrt{1-\mu}\ \sqrt{\varepsilon}}\right)}\sin\theta + O(\varepsilon)$$

The first term on the right is only $O(\sqrt{\varepsilon})$. This can be seen using (4-2.27) to expand the ratio of Bessel functions, for r bounded away from zero,

$$(4\text{-}2.33)\qquad \frac{I_1(\alpha r)}{I_1(\alpha)} \sim \frac{e^{-\alpha(1-r)}}{\sqrt{r}}\left(1 - \frac{3(1-r)}{8\alpha r} - \frac{3(1-r)(11r+5)}{128\alpha^2 r^2}\ \cdots\right)$$

indicating that

$$\frac{I_1\left(\frac{\sqrt{2}\ r}{\sqrt{1+\mu}\ \sqrt{\varepsilon}}\right)}{I_1\left(\frac{\sqrt{2}}{\sqrt{1-\mu}\ \sqrt{\varepsilon}}\right)} = O(\varepsilon^0) \ .$$

This $\sqrt{\varepsilon}$ dependence, being confined to a boundary layer region, means that

$$\|U_\varepsilon - U_0\|_1 = O(\varepsilon^{3/4}) \tag{4-2.34}$$

so estimate (2-2.9) cannot be improved from $\varepsilon^{1/2}$ to ε. It also indicates the presence of a boundary layer phenomena, since near the boundary

$$\frac{I_1\left(\frac{\sqrt{2}\, r}{\sqrt{1-\mu}\,\sqrt{\varepsilon}}\right)}{I_1\left(\frac{\sqrt{2}}{\sqrt{1-\mu}\,\sqrt{\varepsilon}}\right)} \sim \frac{e^{-\frac{\sqrt{2}\,(1-r)}{\sqrt{1-\mu}\,\sqrt{\varepsilon}}}}{\sqrt{r}} \tag{4-2.35}$$

Exactly at the boundary, this ratio of Bessel functions is one. Away from the boundary, its value diminishes exponentially, as does the $O(\varepsilon^{1/2})$ error term it multiplies. A boundary layer could be defined as that region where the term of $O(\varepsilon^{1/2})$ is significant, say when the ratio in (4-2.25) is larger than 1/10. A boundary layer so defined would have a thickness proportional to $\sqrt{\varepsilon}$, based on (4-2.35).

The quantity $\frac{\partial \psi_\theta}{\partial r}$ represents the twisting moment. For all geometries of clamped plates, and for all loads, some twisting moment is expected, hence some boundary layer phenomena may be present in the improved theory. The only clamped plate which is an exception is the circular plate with axisymmetric load. In 2-3, it was shown that such phenomena cannot occur. The absence of a twisting moment appears to be the reason.

4-3. Principal Error

The following is the analog of section 3-2 for the clamped plate with piecewise linear finite elements in the improved theory. The expansions for principal error are considered only as far as the first term, i.e. $L^h U^h - L^h U$ is considered. Consistency is shown, depending on ε^{-1}. The form of the leading term leaves little hope that a factor such as $(1 + \frac{\varepsilon^{-1} h^2}{12})$ will emerge. In fact, because this expression reduces to that for the beam when all dependence upon the y-coordinate is removed, the factor $(1 + \frac{\varepsilon^{-1} h^2}{12})$ is the only conceivable one which could be present. Numerical results suggest that computing $(1 + \frac{\varepsilon^{-1} h^2}{12}) U^h - U$ as was done for the beam does in fact seem to reduce a large share of the error, but no claim can be made, as was done in the beam case, that all adverse dependence upon ε is removed.

The differential operator, L, from (1-4.6) is replaced by the finite difference operator L^h, generated variationally using linear finite elements. A Taylor series expansion is then carried out to check consistency and observe the form of the discretization error, as was done in section 3-2.

The finite difference operator is defined locally in terms of the values of u_1, u_2, u_3 at a central node (subscripted O) and its neighboring nodes (subscripted n, ne, e, s, sw, w, respectively. See Figure 2 below.) .

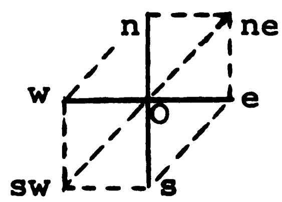


Figure 2. Nodes for local difference operators.

By observing the three consecutive rows of the global stiffness matrix K associated with a particular node (the central node described above), a difference form can be found analogous to that determined for the beam in 3-2. As with the beam, the difference form, L^h, can be seen as a replacement for the differential operator, L. The following difference quotients arise naturally in L^h as replacements for their differential counterparts in L: define

$$D^{(0,0)}[u] = \frac{1}{12}[u_{sw} + u_w + u_s + 6u_0 + u_n + u_e + u_{ne}]$$

$$D^{(1,0)}[u] = \frac{1}{6h}[-u_{sw} - 2u_w + u_s - u_n + 2u_e + u_{ne}]$$

(4-3.1)

$$D^{(0,1)}[u] = \frac{1}{6h}[-u_{sw} + u_w - 2u_s + 2u_n - u_e + u_{ne}]$$

$$D^{(2,0)}[u] = \frac{1}{h^2}[u_w - 2u_0 + u_e]$$

$$D^{(0,2)}[u] = \frac{1}{h^2}[u_s - 2u_0 + u_n]$$

$$D^{(1,1)}[u] = \frac{1}{2h^2}[u_{sw} - u_w - u_s + 2u_0 - u_n - u_e + u_{ne}] \ .$$

The operator $L^h U = [L_1^h U, L_2^h U, L_3^h U]^T$ is defined by the following replacements:

$$L_1 U = -u_{1,xx} - (\frac{1-\mu}{2})u_{1,yy} - (\frac{1+\mu}{2})u_{2,yx} + \varepsilon^{-1}(u_1 + u_{3,x})$$

is replaced by

(4-3.2a) $$L_1^h U = -D^{(2,0)}[u_1] - (\frac{1-\mu}{2})D^{(0,2)}[u_1] - (\frac{1+\mu}{2})D^{(1,1)}[u_2] + \varepsilon^{-1}(D^{(0,0)}[u_1] + D^{(1,0)}[u_3])$$

$$L_2 U = -(\frac{1-\mu}{2})u_{2,xx} - u_{2,yy} - (\frac{1+\mu}{2})u_{1,yx} + \mu^{-1}(u_2 + u_{3,y})$$

is replaced by

$$(4\text{-}3.2b)\qquad L_2^hU = -\left(\frac{1-\mu}{2}\right)D^{(2,0)}[u_2] - D^{(0,2)}[u_2] - \left(\frac{1+\mu}{2}\right)D^{(1,1)}[u_1] + \varepsilon^{-1}(D^{(0,0)}[u_2] + D^{(0,1)}[u_3])$$

$$L_3U = \varepsilon^{-1}(-\nabla^2 u_3 - u_{1,x} - u_{2,y})$$

is replaced by

$$(4\text{-}3.2c)\qquad L_3^hU = \varepsilon^{-1}(-D^{(2,0)}[u_3] - D^{(0,2)}[u_3] - D^{(1,0)}[u_1] - D^{(0,1)}[u_2]) \ .$$

To analyze the discretization error, it is first necessary to expand the difference quotients of L^h as formal Taylor series about the central node, in powers of h. The replacements are (function evaluation is at the central node)

(4-3.3)

$$D^{(0,0)}[u] = u + \frac{1}{6}\sum_{\substack{k=2\\ k\ \text{even}}} \frac{h^k}{k!}[(D^{(1,0)} + D^{(0,1)})^k + D^{(k,0)} + D^{(0,k)}]u$$

(4-3.3) (continued)

$D^{(1,0)}[u]$

$$= D^{(1,0)}u + \frac{1}{3} \sum_{\substack{k=2 \\ k \text{ even}}} \frac{h^k}{(k+1)!}[(D^{(1,0)} + D^{(0,1)})^{k+1} + 2D^{(k+1,0)} - D^{(0,k+1)}]u$$

$D^{(0,1)}[u]$

$$= D^{(0,1)}u + \frac{1}{3} \sum_{\substack{k=2 \\ k \text{ even}}} \frac{h^k}{(k+1)!}[(D^{(1,0)} + D^{(0,1)})^{k+1} + 2D^{(0,k+1)} - D^{(k+1,0)}]u$$

$D^{(2,0)}[u]$

$$= D^{(2,0)}u + 2 \sum_{\substack{k=2 \\ k \text{ even}}} \frac{h^k}{(k+2)!} D^{(k+2,0)}u$$

$D^{(0,2)}[u]$

$$= D^{(0,2)}u + 2 \sum_{\substack{k=2 \\ k \text{ even}}} \frac{h^k}{(k+2)!} D^{(0,k+2)}u$$

$D^{(1,1)}[u]$

$$= D^{(1,1)}u + 2 \sum_{\substack{k=2 \\ k \text{ even}}} \frac{h^k}{(k+2)!}[(D^{(1,0)} + D^{(0,1)})^{k+2} - D^{(k+2,0)} - D^{(0,k+2)}]u .$$

For the purpose of generating principal error terms as was done for the beam problem, the system

(4-3.4) $$LU = F ,$$

where

$$F = [f_1, f_2, f_3]^T$$

is considered, instead of setting $f_1 = f_2 = 0$, which occurs in the original problem (I). The more general right hand side is needed for computing terms of the error beyond the first, although they will not be considered here.

Certain differentiation identities are useful in simplifying the expression L^hU. They are

(4-3.5)

$$\nabla^2(u_{1,x} + u_{2,y}) = -f_3 - f_{1,x} - f_{2,y}$$

$$\nabla^4 u_3 = f_3 - \epsilon\nabla^2 f_3 + f_{1,x} + f_{2,y}$$

$$u_1 + u_{3,x} = \epsilon f_1 + \frac{\epsilon}{2}[(1-\mu)\nabla^2 u_1 + (1+\mu)(u_{1,x} + u_{2,y})_{,x}]$$

$$u_2 + u_{3,y} = \epsilon f_2 + \frac{\epsilon}{2}[(1-\mu)\nabla^2 u_2 + (1+\mu)(u_{1,x} + u_{2,y})_{,y}]$$

$$\nabla^2 u_3 + u_{1,x} + u_{2,y} = -\epsilon f_3 .$$

The following formal series result from using (4-3.5) and definitions (4-3.3) in the expressions for L^hU given by (4-3.2)

(4-3.6a)

$$
\begin{aligned}
& L_1^hU - L_1U \\
&= \sum_{\substack{k=2\\ \text{even}}} \frac{h^k}{k!} \left\{ \frac{-[(1-\mu)(D^{(k+2,0)} + D^{(0,k+2)}) + (1+\mu)D^{(k+2,0)}]u_1}{(k+2)(k+1)} \right. \\
&\quad + \frac{(1+\mu)[-(D^{(1,0)} + D^{(0,1)})^{k+2}u_2 + (D^{(k+2,0)} + D^{(0,k+2)})u_2]}{2(k+2)(k+1)} \\
&\quad + \frac{[(D^{1,0)} + D^{(0,1)})^k + D^{(k,0)} + D^{(0,k)}]f_1}{6} \\
&\quad + \frac{(1-\mu)(D^{1,0)} + D^{(0,1)})^k(D^{(2,0)} + D^{(0,2)})u_1}{12} \\
&\quad + \frac{(1-\mu)(D^{(k,0)} + D^{(0,k)})(D^{(2,0)} + D^{(0,2)})u_1}{12} \\
&\quad + \frac{(1+\mu)(D^{(1,0)} + D^{(0,1)})^k(D^{(2,0)}u_1 + D^{(1,1)}u_2)}{12} \\
&\quad + \frac{(1+\mu)(D^{(k,0)} + D^{(0,k)})(D^{(2,0)}u_1 + D^{(1,1)}u_2)}{12} \\
&\quad + \frac{\epsilon^{-1}(D^{(1,0)} + D^{(0,1)})^k[-(k-1)D^{(1,0)} + 2D^{(0,1)}]u_3}{6(k+1)} \\
&\quad \left. + \frac{\epsilon^{-1}[-(k-3)D^{(k+1,0)} - (k+1)D^{(1,k)} - 2D^{(0,k+1)}]u_3}{6(k+1)} \right\}
\end{aligned}
$$

(4-3.6b)

$$
\begin{aligned}
& L_2^h U - L_2 U \\
&= \sum_{\substack{k=2 \\ \text{even}}} \frac{h^k}{h!} \left\{ \frac{-[(1-\mu)(D^{(k+2,0)} + D^{(0,k+2)}) + (1+\mu)D^{(0,k+2)}]u_2}{(k+2)(k+1)} \right. \\
&+ \frac{(1+\mu)[-(D^{(1,0)} + D^{(0,1)})^{k+2}u_1 + (D^{(k+2,0)} + D^{(0,k+2)})u_1]}{2(k+2)(k+1)} \\
&+ \frac{[(D^{(1,0)} + D^{(0,1)})^k + D^{(k,0)} + D^{(0,k)}]f_2}{6} \\
&+ \frac{(1-\mu)(D^{1,0)} + D^{(0,1)})^k(D^{(2,0)} + D^{(0,2)})u_2}{12} \\
&+ \frac{(1-\mu)(D^{(k,0)} + D^{(0,k)})(D^{(2,0)} + D^{(0,2)})u_2}{12} \\
&+ \frac{(1+\mu)(D^{(1,0)} + D^{(0,1)})^k(D^{(1,1)}u_1 + D^{(0,2)}u_2)}{12} \\
&+ \frac{(1+\mu)(D^{(k,0)} + D^{(0,k)})(D^{(1,1)}u_1 + D^{(0,2)}u_2)}{12} \\
&+ \frac{\epsilon^{-1}(D^{(1,0)} + D^{(0,1)})^k(2D^{(1,0)} - (k-1)D^{(0,1)})u_3}{6(k+1)} \\
&+ \left. \frac{\epsilon^{-1}[-(k-3)D^{(0,k+1)} - (k+1)D^{(k,1)} - 2d^{(k+1,0)}]u_3}{6(k+1)} \right\}
\end{aligned}
$$

(4-3.6c)

$$L_3^h U - L_3 U$$

$$= - \sum_{\substack{k=2 \\ \text{even}}} \Big\{ \frac{h^k}{3\cdot(k+1)!} \quad (D^{(1,0)} + D^{(0,1)})^{k+1}(f_1 + f_2)$$

$$+ \frac{1-\mu}{2}(D^{(1,0)} + D^{(0,1)})^{k+1}(D^{(2,0)} + D^{(0,2)})(u_1 + u_2)$$

$$+ \frac{1+\mu}{2}(D^{(1,0)} + D^{(0,1)})^{k+1}(D^{(2,0)} + D^{(1,1)})u_1$$

$$+ \frac{1+\mu}{2}(D^{(1,0)} + D^{(0,1)})^{k+1}(D^{(1,1)} + D^{(0,2)})u_2$$

$$+ 2(D^{(k+1,0)} + D^{(0,k+1)})(f_1 + f_2) - 3(D^{(0,k+1)}f_1 + D^{(0,k+1)}f_2)$$

$$+ (1-\mu)(D^{(k+1,0)} + D^{(0,k+1)})(D^{(2,0)} + D^{(0,2)})(u_1 + u_2)$$

$$- \frac{3}{2}(1-\mu)(D^{(2,0)} + D^{(0,2)})(D^{(0,k+1)}u_1 + D^{(k+1,0)}u_2)$$

$$+ (1+\mu)(D^{(k+1,0)} + D^{(0,k+1)})(D^{(2,0)} + D^{(1,1)})u_1$$

$$+ (1+\mu)(D^{(k+1,0)} + D^{(0,k+1)})(D^{(1,1)} + D^{(0,2)})u_2$$

$$- \frac{3}{2}(1+\mu)D^{(0,k+1)}(D^{(2,0)}u_1 + D^{(1,1)}u_2)$$

$$- \frac{3}{2}(1+\mu)D^{(k+1,0)}(D^{(1,1)}u_1 + D^{(0,2)}u_2)$$

$$- \epsilon^{-1}(D^{(1,0)} + D^{(0,1)})^{k+2}u_3$$

(4-3.6c) (continued)

$$- 2\varepsilon^{-1} \frac{(k-1)}{k+2} (D^{(k+2,0)} + D^{(0,k+2)})u_3$$

$$\left. + \varepsilon^{-1}(D^{(1,k+1)} + D^{(k+1,1)})u_3 \right\} .$$

(4-3.6) is analogous to (3-2.20), the result of Theorem 6. It can be observed that (4-3.6) shares with (3-2.20) the property that $L_1^h U - L_1 U$ and $L_2^h U - L_2 U$ have leading terms which are free of ε^{-1}, i.e. they are $O(\varepsilon^0 h^2)$. The ε^{-1} dependence in the leading term is limited to the third component $L_3^h U - L_3 U$.

An important difference is that while the right side of (3-2.20) is expressed solely in terms of f_1 and f_2, the right side of (4-3.6) contains direct dependence on $U = [u_1,u_2,u_3]^T$, the solution of $LU = F$, in addition to f_1, f_2, and f_3. u_1, u_2 and u_3 cannot be removed entirely by identities, as was done in deriving (3-2.20). (4-3.6) is expressed in a way which separates the terms of order $O(\varepsilon^{-1})$ from those of order $O(\varepsilon^0)$.

In order to apply (4-3.6) to compute $L^h U^h - L^h U = \tilde{F} - L^h U$, it is necessary to derive a formal expansion for the components of $\tilde{F}$. By tedious integration, it can be shown that

$$\tilde{F} = [0,0,\tilde{f}]^T$$

where

$$(4\text{-}3.7) \qquad \tilde{f} = \frac{1}{h^2} \iint f\varphi \, dA$$

$$= f + 2 \sum_{\substack{k=2 \\ k \text{ even}}} \frac{h^k}{(k+3)!} \sum_{j=0}^{k} \left[\binom{k+2}{j+1} + (-1)^j\right] D^{(k-j,j)} f$$

where φ is the finite element "tent" function, with value 1 at the central node, 0 at all others and linear in-between.

An expression for $L^h U^h - L^h U$ can be written from (4-3.6) and (4-3.7), in order to define the leading principal error term, which is the $O(h^2)$ term of

$$(4\text{-}3.8) \qquad L^h U^h - L^h U = \tilde{F} - L^h U$$

$$= [-L_1^h U,\ -L_2^h U,\ \tilde{F} - L_3^h U]^T \ .$$

As observed, the leading $(O(h^2))$ term of each of the first two components is free of ϵ^{-1}. Since this dependence is of concern, consider only this dependence from the third component for the term proportional to h^2:

(4-3.9)

$$-\frac{h^2\epsilon^{-1}}{6}\left(\frac{1}{2}\,u_{3,xxxx} + u_{3,xxxy} + 2u_{3,xxyy} + u_{3,xyyy} + \frac{1}{2}\,u_{3,yyyy}\right)$$

$$= \frac{-h^2\epsilon^{-1}}{6}\left(\frac{1}{2}\,\nabla^4 u_3 + u_{3,xxxy} + u_{3,xxyy} + u_{3,xyyy}\right)$$

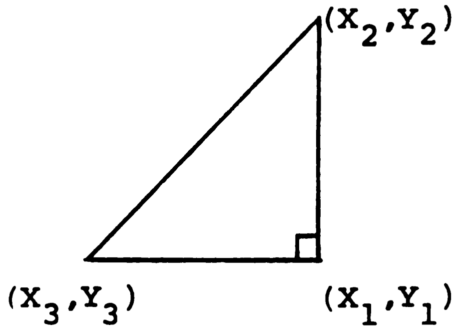
$$= \frac{-h^2\epsilon^{-1}}{6}\left(\frac{1}{2}\,f + u_{3,xxxy} + u_{3,xxyy} + u_{3,xyyy}\right) + O(\epsilon^0) \ .$$

At this point in the one dimensional problem the factor $(1+\epsilon^{-1}h^2/12)$ could be employed to remove the ϵ^{-1} term $-h^2\epsilon^{-1}f/12$. In this case the additional mixed derivatives remain. From this analysis, it is unclear whether the use of this special factor has any advantage. It appears that the adverse effect of small ϵ on finite element solutions is unavoidable. The apparent message here is that very small ϵ values should not be used in numerical computation, as their presence may cause error which cannot be controlled. Information about solutions of problem (I) for small ϵ values will have to come by way of inferences from solving the problem for more moderate values of the parameter.

4-4. Construction of the Element Stiffness Matrix for the Clamped Plate in the Improved Theory

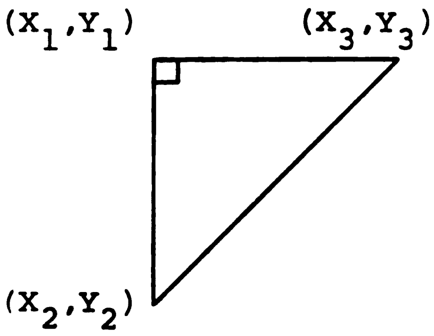
The construction of the stiffness matrix for the clamped plate is similar to that for the clamped beam of Chapter 3, except for being much more complex, both in the computation of the element stiffness matrix and in the assembly of these into the global matrix.

The domain is subdivided into isosceles right triangles of two types, based upon their orientation. A local coordinate system will be assigned to each triangular element of the domain having its origin at the triangle's centroid. The legs of each triangle will be h, the mesh size. The coordinates of the vertices are indexed (locally) (X_1,Y_1), (X_2,Y_2), (X_3,Y_3) starting at the right angle, and proceding counterclockwise. The two types of elements are given by Figures 3 and 4, respectively.



where
$$(X_1,Y_1) = \left(\frac{h}{3}, -\frac{h}{3}\right)$$
$$(X_2,Y_2) = \left(\frac{h}{3}, \frac{2h}{3}\right)$$
$$(X_3,Y_3) = \left(-\frac{2h}{3}, -\frac{h}{3}\right)$$

Figure 3. Type 1 triangular element.



where $(X_1,Y_1) = (-\frac{h}{3}, \frac{h}{3})$

$(X_2,Y_2) = (-\frac{h}{3}, -\frac{2h}{3})$

$(X_3,Y_3) = (\frac{2h}{3}, \frac{h}{3})$

Figure 4. Type 2 triangular element.

Note that by replacing h by $-h$ in the coordinates for a type 1 element, the coordinates for a type 2 element are produced. In this way formulas developed for type 1 elements become valid for type 2 elements by simply negating h.

The approximating finite element solution U to the solution of problem (I) is expressed as a sum of piecewise linear finite elements

$$U = \sum_{i=1}^{N} q_i \, \phi_i \ ,$$

where N is the number of parameters needed to represent the solution over the entire domain, and

$$\phi_i = \begin{bmatrix} \varphi_i \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ \varphi_i \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ 0 \\ \varphi_i \end{bmatrix},$$

to be determined by the indexing scheme. The function φ_i is the finite element "tent" function generating the piecewise linear solutions. It equals 1 at a certain node (depending on the indexing), 0 at all other nodes, and is otherwise linear. Its support is the six elements adjacent to its central node (see Figure 5).

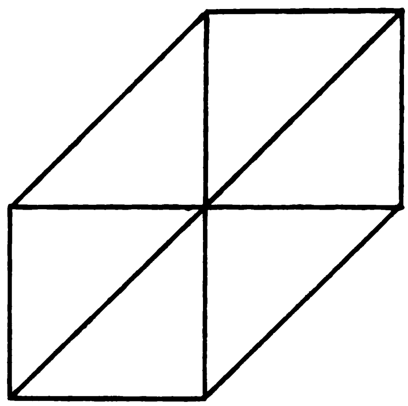


Figure 5. Support for "tent" function.

As with the beam problem of 3-3, the finite element solution is determined by finding the q_i which minimize the energy functional

$$(4\text{-}4.1) \qquad J(U) = B_\epsilon(U,U) - 2P_L(F,U)$$

$$= Q^T K Q - 2Q^T\tilde{F}$$

giving

$$(4\text{-}4.2) \qquad KQ = \tilde{F}$$

(in the same notation as in 3-3). K and $\tilde{F}$ are again determined by summing the energy over the individual elements

$$(4\text{-}4.3) \qquad J(U) = \sum J(U^{(e)})$$
$$= \sum Q^{(e)T}K^{(e)}Q^{e} - 2Q^{(e)T}\tilde{F}^{(e)} \quad .$$

$K^{(e)}$ and $\tilde{F}^{(e)}$ are most conveniently found by expressing $U^{(e)}$ as a linear polynomial in the local coordinates over a single element, then computing the energy in terms of the coefficients, and finally transforming to an expression in terms of the q_i.

Let

$$(4\text{-}4.4) \quad U^{(e)} = \begin{bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \end{bmatrix} = \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} .$$

Let $\varphi_1, \varphi_2, \varphi_3$ be the "tent" functions which are non-zero at (X_1,Y_1), (X_2,Y_2), (X_3,Y_3), respectively. Let $q_1,q_2,\ldots,q_9$ be coefficients, indexed locally, described as follows

$$(4\text{-}4.5)\qquad U^{(e)} = \begin{bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \end{bmatrix} = \begin{bmatrix} q_1 & q_4 & q_7 \\ q_2 & q_5 & q_8 \\ q_3 & q_6 & q_9 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} .$$

Equating (4-4.4) and (4-4.5) at the three nodes of the element,

(4-4.6)

$$\begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix} \begin{bmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} q_1 & q_4 & q_7 \\ q_2 & q_5 & q_8 \\ q_3 & q_6 & q_9 \end{bmatrix} .$$

For a type 1 element (for type 2, negate h)

$$\begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix} \begin{bmatrix} \frac{h}{3} & \frac{h}{3} & -\frac{2h}{3} \\ -\frac{h}{3} & \frac{2h}{3} & -\frac{h}{3} \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} q_1 & q_4 & q_7 \\ q_2 & q_5 & q_8 \\ q_3 & q_6 & q_9 \end{bmatrix} .$$

Inverting gives

(4-4.7)

$$\begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix} = \begin{bmatrix} q_1 & q_4 & q_7 \\ q_2 & q_5 & q_8 \\ q_3 & q_6 & q_9 \end{bmatrix} \begin{bmatrix} \frac{1}{h} & -\frac{1}{h} & \frac{1}{3} \\ 0 & \frac{1}{h} & \frac{1}{3} \\ -\frac{1}{h} & 0 & \frac{1}{3} \end{bmatrix}.$$

(4-4.7) can be expressed as

$$A = PQ^{(e)} \tag{4-4.8}$$

where

$$A = [a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9]^T$$

$$Q^{(e)} = [q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9]^T$$

$$P = \left[\begin{array}{c|c|c} \frac{1}{h}\ I & 0 & -\frac{1}{h}\ I \\ \hline \frac{1}{h}\ I & \frac{1}{h}\ I & 0 \\ \hline \frac{1}{3}\ I & \frac{1}{3}\ I & \frac{1}{3}\ I \end{array}\right]$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To determine $K^{(e)}$ from

$$B_\varepsilon(U^{(e)}, U^{(e)}) = Q^{(e)T} K^{(e)} Q^{(e)} \tag{4-4.9}$$

first define a matrix N by computing

$$B_\epsilon(U^{(e)},U^{(e)}) = A^T NA \quad . \tag{4-4.10}$$

Then

$$A^T NA = Q^{(e)T}K^{(e)}Q^{(e)}$$

and

$$A = PQ^{(e)}$$

imply

$$Q^{(e)T}K^{(e)}Q^{(e)} = Q^{(e)T}P^T NPQ^{(e)} \quad ,$$

that is

$$K^{(e)} = P^T NP \quad . \tag{4-4.11}$$

This matrix multiplication of two nine by nine matrices is best left to the computer. Importantly though, it need be computed only once, and then used repeatedly for the different elements of type 1 (for type 2, negate h).

Computing $B_\epsilon(U^{(e)},U^{(e)})$ involves integrals, over the single elemental triangle, of form

$$P_{rs} = \iint X^r Y^s dXdY \quad .$$

Formulas for P_{rs} are available for triangular regions with local coordinates (see Holland and Bell [10]) saving some tedious work here as well as in computing $\tilde{F}$.

Only the following are required for $B_\varepsilon(U^{(e)},U^{(e)})$:

(4-4.12)
$$P_{00} = \frac{h^2}{2}, \quad P_{01} = P_{10} = 0$$
$$P_{20} = P_{02} = \frac{h^4}{36}, \quad P_{11} = \frac{h^4}{72} \ .$$

Inserting the polynomial forms for u_1, u_2, u_3 from (4-4.4),

(4-4.13)

$$B_\varepsilon(U^{(e)},U^{(e)})$$

$$= \frac{1+\mu}{2} \iint_{(e)} (u_{1,x} + u_{2,y})^2 dA + \frac{1-\mu}{2} \iint_{(e)} (u_{1,x} - u_{2,y})^2 dA$$

$$+ \frac{1-\mu}{2} \iint_{(e)} (u_{1,y} + u_{2,x})^2 dA + \varepsilon^{-1} \iint_{(e)} (u_{3,x}+u_1)^2 + (u_{3,y}+u_2)^2 dA$$

$$= \frac{h^2}{2}\{a_1^2 + a_5^2 + 2\mu a_1 a_5 + \frac{1-\mu}{2} a_4^2 + \frac{1-\mu}{2} a_2^2 + (1-\mu) a_2 a_4\}$$

$$+ \frac{\varepsilon^{-1} h^2}{2}\{a_3^2 + a_7^2 + 2a_3 a_7 + a_6^2 + a_8^2 + 2a_6 a_8\}$$

$$+ \frac{\varepsilon^{-1} h^4}{36}\{a_1^2 + a_4^2 + a_1 a_4 + a_2^2 + a_5^2 + a_2 a_5\} \ .$$

(4-4.13) can be arranged in matrix form A^TNA where N is symmetric. The result is

$$(4\text{-}4.14) \quad B_\epsilon(U^{(e)},U^{(e)}) = A^TNA$$

$$= A^T\left(\frac{h^2}{2} N^\alpha + \frac{\epsilon^{-1}h^2}{2} N^\beta + \frac{\epsilon^{-1}h^4}{36} N^\gamma\right)A$$

where

$$N^\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & \frac{1-\mu}{2} & 0 & \frac{1-\mu}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1-\mu}{2} & 0 & \frac{1-\mu}{2} & 0 & 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N^\beta = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N^{\gamma} = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with N defined in (4-4.14) and P defined in (4-4.8), $K^{(e)}$ can be computed by

$$K^{(e)} = P^T N P \ .$$

The element load vector $\tilde{F}^{(e)}$ can be handled in a similar manner although it will generally depend on the position of the element in the domain

$$Q^{(e)T}\tilde{F}^{(e)} = P_L(F, U^{(e)}) = \iint_{(e)} \frac{p}{D} u_3^{(e)} \, dA$$

$$= [a_3 \ a_6 \ a_9] \iint_{(e)} \frac{p}{D} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} dA$$

$$= [q_3 \ q_6 \ q_9] \begin{bmatrix} \frac{1}{h} & -\frac{1}{h} & \frac{1}{3} \\ 0 & \frac{1}{h} & \frac{1}{3} \\ \frac{1}{h} & 0 & \frac{1}{3} \end{bmatrix} \iint_{(e)} \frac{p}{D} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} dA \ .$$

If the centroid of the elemental triangle is (X_c, Y_c) relative to the global coordinates (x, y), then $x = X + X_c$ and $y = Y + Y_c$ so p/D can be expressed in local coordinates for the sake of integration.

$$\frac{p}{D} = \frac{p}{D}(x,y) = \frac{p}{D}(X + X_c,\ Y + Y_c)$$

so

$$\tilde{F}^{(e)} = [0,0,f_1,0,0,f_2,0,0,f_3]^T$$

where

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{h} & -\frac{1}{h} & \frac{1}{3} \\ 0 & \frac{1}{h} & \frac{1}{3} \\ -\frac{1}{h} & 0 & \frac{1}{3} \end{bmatrix} \iint_{(e)} \frac{p}{D} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} dA$$

is a function of (X_c, Y_c).

If $\frac{p}{D}$ happens to be polynomial, then the formulas of Holland and Bell can be applied to the integration. If numerical integration is called for, account should be made of any error introduced.

The numerical studies of section 4-5 are carried out for a square clamped plate under constant load, point load, and a particular fourth degree polynomial load. The plate covers the unit square with center at the origin. Because all three loads considered are symmetric in x and y, the problem is solved over the first quadrant only.

The clamped edge condition imposes essential boundary conditions $u_1 = u_2 = u_3 = 0$ at $x = \frac{1}{2}$ and at $y = \frac{1}{2}$. Therefore no unknowns q_i are associated with nodes at the boundary. At each interior node, three unknowns are present, namely q_i, q_{i+1}, q_{i+2} associated with u_1, u_2, u_3 respectively. However, the use of symmetry to restrict the problem to one quadrant introduce constraints along the "edges" $x = 0$ and $y = 0$. Because u_1 must be an odd function of x and an even function of y, an essential "boundary condition" is imposed: $u_1 = 0$ at $x = 0$. Similarly $u_2 = 0$ at $y = 0$. Thus along these two edges, each node will possess only two unknowns instead of the customary three. Special care must be taken to properly index the variables (q_i for $i = 0 \bmod 3$ no longer represents just displacements u_3, for example).

An algorithm can be outlined for the indexing procedure to be used.

1. Place nodes in a rectangular grid over the quarter plate positioning three consecutive indices at each node including those on the boundaries of the quarter square. The elements

will be right triangles with diagonals parallel to that of the full square. Work horizontally to the right along each row to reduce bandwidth in the stiffness matrix. This original indexing information will be retained so that each q_i for $i = 0 \bmod 3$, for example, will be readily associated with a local index which is $0 \bmod 3$ for the purpose of assembling the element stiffness matrices and element load vectors.

2. Systematically delete indices associated with homogeneous essential boundary conditions, storing the original index corresponding to each new one. Call these "new" global indices.

3. For each element, associate the nine local indices to the corresponding original global ones. Indices which have been deleted in step 2 indicate rows and columns of the element stiffness matrix which will not be entered into the global matrix. Indices which have survived the deletions of step 2 indicate entries of the element stiffness matrix to be entered into the global array. The positions in the global matrix to receive the entries

from the local matrix are indicated by the "new" global indices associated with the surviving original indices. In this way, elements adjacent to the boundaries of the quarter square (and there are many) need not be handled as special cases (of which there also would be many).

As the stiffness matrix is being assembled, the load vector may be done simultaneously, in a similar manner. For each element the centroid (X_c, Y_c) is located and the local loads computed.

The number of indices grows rapidly as the mesh size shrinks. If $h = \frac{1}{2n}$ (n being the number of elements along a side of the quarter square) the number of original indices is $3(n+1)^2$.

Even after deleting the boundary indices, the number of "new" indices is still $O(n^2)$ and so the stiffness matrix has $O(n^4)$ entries, many of which are zero due to the bandwidth being $O(n)$. Also the matrix is symmetric, and while the fact alone can be used to reduce the number of operations in assembling the matrix it does not cut down on the size of the array required.

To conserve on operations in the matrix solving routine (a Cholesky, or "square root" routine was used for the symmetric positive definite banded stiffness matrix) it is possible to make use of apriori knowledge that certain entries are zero. This has been done in the computations of 4-5.

More economical storage is also gained by arranging the upper diagonals of the stiffness matrix "rectangularly" in an array. In this way, storage requirements are $O(n^3)$ a substantial savings, but at the cost of computer time and programming complexity. This has also been done for the work in 4-5. The array size required is $(3n^2 - 2n)$ by $(3n+2)$. For $h = 1/18$, this is 225 by 29. Naturally a step 4 in the process is required to associate a pair of "new" indices with a position in the rectangular array.

4-5. Numerical Results

Numerical solutions to problem (I), the clamped plate problem, are computed below for a square plate under three different load distributions. The computations are carried out using piecewise linear finite elements, as described in section 4-4. The plate has sides of (dimensionless) length one. Mesh sizes of $h = 1/4$, $1/8$, $1/16$, and $1/32$ have been used for computing solutions for each ϵ.

Deflections at the plate's center have been displayed in Tables 8, 9, and 10, for ε values ranging from 2^{-1} to 2^{-8}. A Richardson extrapolation has been performed using consecutive mesh sizes in accordance with section 4-3. These appear at the right in the tables.

It should be noted that exact solutions of problem (I) are not generally available for test problems generated with simple load functions. Thus inferences as to the reliability of solutions must come from the numerical results themselves, rather than by computing errors relative to exact solutions. As in the case of the clamped beam of section 3-4, it is expected that numerical solutions will fail to maintain accuracy when ε becomes very small (while the use of the correcting factor $(1+\varepsilon^{-1}h^2/12)$ seems to increase the range of ε for which the results are reliable, there is no solid theoretical basis for its use here, in contrast with the case of the beam).

The estimates of section 2-4 can be interpreted as saying that in order to maintain a given accuracy in the numerical approximation of solutions to problem (I), as ε becomes smaller, a finer and finer mesh size, h, must be used. h is restricted from becoming too small by the rapid growth of the system matrix. $h = 1/32$ produces 736 system variables distributed over the quarter-square, and a stiffness matrix stored "rectangularly" in an array, 736 by 50. In general, for $h = 1/2n$, an array, $(3n^2 - 2n)$ by $(3n+2)$ is required. Thus, there exists a practical limitation on the size of ϵ for which accurate results can be obtained.

The following rather arbitrary criterion will be applied to the center deflections in Tables 8, 9, and 10, to determine the reliability of the numerical outputs:

> The computed values for the center deflection will be considered reliable for all values of ϵ for which there is less than one percent difference between the two values computed using a given mesh sizes and a mesh size half as large, respectively.

In order to investigate the behavior of solutions for smaller ϵ, a less restrictive criterion may be allowed. Consider the center deflections computed with three consecutive mesh sizes, say, $h = 1/8$, $1/16$, and $1/32$. As in the above criterion, a one percent difference will be acceptable between the result of a Richardson extrapolation using $h = 1/8$ and $1/16$, and one computed using $h = 1/16$ and $1/32$.

In all the computations in Tables 8, 9, and 10, the value $\mu = 0.3$ is taken for Poisson's ratio. The problem of the clamped plate is solved for a plate covering the region $-1/2 \leq x \leq 1/2$ and $-1/2 \leq y \leq 1/2$. For the three loads tested, values can be given for center deflections in the classical theory.

For the polynomial load

$$p/D = 24(x^4 + 12x^2y^2 + y^4) - 36(x^2 + y^2) + 5$$

the solution to problem (C) is

$$w_0 = 2^{-8}(4x^2 - 1)^2(4y^2 - 1)^2 \ .$$

The center deflection is then given by

$$w_0(0,0) = 2^{-8} = .390625 \text{ E-2} \ .$$

For a uniform load, $p/D = 1000$, a value for the center deflection has been computed to three significant digits (see [24]) to be

$$w_0(0,0) = 1.26 \ .$$

For a point load, $p/D = 1000\,\delta(x)$ a center deflection to three significant digits (see [24]) is

$$w_0(0,0) = 5.60 \ .$$

Table 8.

Square plate deflections, polynomial load

ε	h	$w_\varepsilon(0,0)$	h	$w_\varepsilon(0,0)$
2^{-1}	1/4	.113116 EO		
	1/8	.113371 EO		
	1/16	.112286 EO	1/8, 1/16	.111925 EO
	1/32	.111707 EO	1/16, 1/32	.111514 EO
2^{-2}	1/4	.578558 E-1		
	1/8	.585893 E-1		
	1/16	.581983 E-1	1/8, 1/16	.580680 E-1
	1/32	.579396 E-1	1/16, 1/32	.578534 E-1
2^{-3}	1/4	.301841 E-1		
	1/8	.311751 E-1		
	1/16	.311419 E-1	1/8, 1/16	.311308 E-1
	1/32	.310471 E-1	1/16, 1/32	.310155 E-1
2^{-4}	1/4	.162721 E-1		
	1/8	.174236 E-1		
	1/16	.175911 E-1	1/8, 1/16	.176469 E-1
	1/32	.175849 E-1	1/16, 1/32	.175829 E-1
2^{-5}	1/4	.918538 E-2		
	1/8	.104783 E-1		
	1/16	.107770 E-1	1/8, 1/16	.108799 E-1
	1/32	.108283 E-1	1/16, 1/32	.108454 E-1
2^{-6}	1/4	.544297 E-2		
	1/8	.685689 E-2		
	1/16	.730850 E-2	1/8, 1/16	.745903 E-2
	1/32	.741368 E-2	1/16, 1/32	.744874 E-2
2^{-7}	1/4	.331845 E-2		
	1/8	.484342 E-2		
	1/16	.548202 E-2	1/8, 1/16	.569489 E-2
	1/32	.566086 E-2	1/16, 1/32	.572047 E-2
2^{-8}	1/4	.200441 E-2		
	1/8	.354541 E-2		
	1/16	.443066 E-2	1/8, 1/16	.472574 E-2
	1/32	.472872 E-2	1/16, 1/32	.482807 E-2

Table 9.

Square plate deflections, uniform load

ϵ	h	$w_\epsilon(0,0)$	h	$w_\epsilon(0,0)$
2^{-1}	1/4	.400761 E2		
	1/8	.389808 E2		
	1/16	.384852 E2	1/8, 1/16	.383200 E2
	1/32	.383036 E2	1/16, 1/32	.382431 E2
2^{-2}	1/4	.205338 E2		
	1/8	.201426 E2		
	1/16	.199252 E2	1/8, 1/16	.198527 E2
	1/32	.198396 E2	1/16, 1/32	.198111 E2
2^{-3}	1/4	.107468 E2		
	1/8	.107151 E2		
	1/16	.106406 E2	1/8, 1/16	.106157 E2
	1/32	.106042 E2	1/16, 1/32	.105920 E2
2^{-4}	1/4	.582425 E1		
	1/8	.598567 E1		
	1/16	.599000 E1	1/8, 1/16	.599145 E1
	1/32	.598029 E1	1/16, 1/32	.597706 E1
2^{-5}	1/4	.331305 E1		
	1/8	.359290 E1		
	1/16	.365063 E1	1/8, 1/16	.366597 E1
	1/32	.365854 E1	1/16, 1/32	.365727 E1
2^{-6}	1/4	.198141 E1		
	1/8	.234978 E1		
	1/16	.245902 E1	1/8, 1/16	.249543 E1
	1/32	.248390 E1	1/16, 1/32	.249220 E1
2^{-7}	1/4	.121882 E1		
	1/8	.165642 E1		
	1/16	.183124 E1	1/8, 1/16	.188951 E1
	1/32	.187941 E1	1/16, 1/32	.189600 E1
2^{-8}	1/4	.741327 E0		
	1/8	.121003 E1		
	1/16	.147093 E1	1/8, 1/16	.155791 E1
	1/32	.155821 E1	1/16, 1/32	.158731 E1

Table 10.

Square plate deflections, point load

ε	h	$w_\varepsilon(0,0)$	h	$w_\varepsilon(0,0)$
2^{-1}	1/4	.191177 E3		
	1/8	.249895 E3		
	1/16	.305900 E3	1/8, 1/16	.324568 E3
	1/32	.361238 E3	1/16, 1/32	.379683 E3
2^{-2}	1/4	.973868 E2		
	1/8	.127631 E3		
	1/16	.155866 E3	1/8, 1/16	.165278 E3
	1/32	.183578 E3	1/16, 1/32	.192816 E3
2^{-3}	1/4	.504340 E2		
	1/8	.664676 E2		
	1/16	.808343 E2	1/8, 1/16	.856232 E2
	1/32	.947386 E2	1/16, 1/32	.993734 E2
2^{-4}	1/4	.268515 E2		
	1/8	.358268 E2		
	1/16	.432908 E2	1/8, 1/16	.457788 E2
	1/32	.503007 E2	1/16, 1/32	.526373 E2
2^{-5}	1/4	.148785 E2		
	1/8	.203995 E2		
	1/16	.244711 E2	1/8, 1/16	.258283 E2
	1/32	.280522 E2	1/16, 1/32	.292459 E2
2^{-6}	1/4	.861552 E1		
	1/8	.125035 E2		
	1/16	.149833 E2	1/8, 1/16	.158099 E2
	1/32	.168855 E2	1/16, 1/32	.175196 E2
2^{-7}	1/4	.513293 E1		
	1/8	.826778 E1		
	1/16	.101184 E2	1/8, 1/16	.107353 E2
	1/32	.112471 E2	1/16, 1/32	.116233 E2
2^{-8}	1/4	.304324 E1		
	1/8	.574003 E1		
	1/16	.749797 E1	1/8, 1/16	.808394 E1
	1/32	.835674 E1	1/16, 1/32	.864300 E1

Tables 8 and 9 indicate that for both a uniform load and the polynomial load, changes in the center deflections for problem (I) occurring between mesh sizes $h = 1/16$ and $h = 1/32$ are less than one percent for $\varepsilon \geq 2^{-5}$ while changes of 1.5% and 1.01% respectively occur at $\varepsilon = 2^{-6}$, indicating a need for finer mesh sizes in order to regain accuracy. Under the first criterion, $\varepsilon = 2^{-5}$ and above would be considered a range of ε for which numerical results are expected to be reliable for the two loads mentioned.

While one percent may be considered a rough estimate of the accuracy of the numerical results relative to the exact solutions of problem (I), improved accuracy can be expected when these "reliable" values are subjected to a Richardson extrapolation.

By using the second criterion to compare previously extrapolated values, the range of ε for which reliable results can be expected is extended to $\varepsilon \geq 2^{-7}$.

Table 10 gives the center deflections under a point load for problem (I). No such indication of convergence is seen as the mesh is refined, for any range of ε. The center deflections steadily increase as the mesh size shrinks. This, however, is not due to the adverse effect of small ε. It indicates the failure of the finite element method to approximate the singular nature of the exact solution of problem (I) under a point load.

As shown in Chapter 1, the displacement component, w, of the solution to problem (I) can be partially decoupled from the slope components ψ_x and ψ_y, leading to the differential equation

$$\nabla^4 w = p/D - \epsilon\nabla^2 p/D \ . \tag{4-5.1}$$

Viewed as a distribution, with p/D proportional to $\delta(x,y)$, w is expected to have a term proportional to ϵ, with a logarithmic singularity at $(0,0)$.

Physically, an infinite shear occurs under a point load, and an infinite displacement results from this infinite shear. Table 10 reflects the failure of the finite element method with polynomial base functions to accurately describe this singular behavior under the point load.

It should also be mentioned that, in contrast, the governing equation for problem (C),

$$\nabla^4 w_0 = p/D \tag{4-5.2}$$

leads to a singularity in w_0 of form $r^2 \ln r$, when p/D represents a point load. The classical theory results in continuous displacements in response to a point load.

The question of approximating solutions of problem (C) by numerical solutions of problem (I) poses several obstacles.

While the estimates of Chapter 2 indicate that solutions of problem (I) converge to solutions of problem (C) as ε tends to zero, this convergence cannot be directly exploited numerically because of the limitations just discussed. The numerical solutions of problem (I) needed for this convergence are for small values of ε. But this is exactly the range of ε for which these numerical solutions are unreliable.

The question then becomes: can approximations to solutions of problem (C) be inferred from numerical solutions of problem (I), using moderate sizes of ε? To address this question, it is necessary to postulate the form through which solutions of problem (I) depend on ε, for small ε. Correctly postulated, this form can be used to extrapolate the numerical data produced for moderate values of ε to produce a value corresponding to $\varepsilon = 0$.

Without the help of an asymptotic expansion, different assumptions may be made leading to quite different conclusions. The only guideline is that any assumed form must satisfy the estimates of section 2-2, namely, that the dependence of $U_\varepsilon - U_0$ on ε cannot be "more singular" than $\varepsilon^{1/2}$.

In fact, even this may be violated on a domain with corners. While estimate (2-2.9) holds for domains with non-smooth boundaries, the right hand side contains the factor $\|\nabla(\nabla^2 w_0)\|_0$ which may be infinite. In the case of the particular polynomial load being consider numerically in this section, w_0 is also a polynomial, so that estimate (2-2.9) is binding on the solution, U_ϵ, to problem (I).

However solutions to problem (C) in general exhibit singular behavior at the corners of the square, even for smooth loads (including uniform loads).

Kondrat'ev [13] studied the effect of corners on solutions of elliptic boundary value problems, finding that, while solutions are analytic up to regions of the boundary which are smooth, singularities are introduced into solutions at angular points of the boundary, the severity of singularity depending upon the angle.

For the biharmonic equation with homogeneous boundary conditions (such as for problem (C)) Kondrat'ev develops an equation to determine the singularity at a corner, as a power of r, the radial distance from the corner. The severity of the singular depends only on the angle of the corner, not on the forcing term of the differential equation. For a right angle corner, the singularity has the form

$$r^{z+1}$$

where z is a complex solution to the equation

$$\sin \frac{2\pi z}{2} = z^2 .$$

Besides the integer solutions $z = 0$ and $z = \pm 1$ it can be shown that solutions lie between

$$\frac{3}{4} < \mathrm{Re}(z) < 1$$

and

$$-1 < \mathrm{Re}(z) < -\frac{3}{4} .$$

Since the integers represent analytic solutions the solutions which may be singular have exponents either

$$\frac{7}{4} < \mathrm{Re}(z+1) < 2$$

or

$$0 < \mathrm{Re}(z+1) < \frac{1}{4} .$$

The latter is rejected because such a singularity will prevent the solution from being in H_0^2. The former however provides a singularity of form

$$w \sim r^{7/4+\delta}, \quad \text{where} \quad 0 < \delta < \frac{1}{4} .$$

Roughly speaking, then

$$\frac{\partial w}{\partial r} \sim r^{3/4 + \delta}$$

$$\frac{\partial^2 w}{\partial r^2} \sim r^{-1/4 + \delta}$$

$$\frac{\partial^3 w}{\partial r^3} \sim r^{-5/4 + \delta}$$

while such a solution has square integrable second derivatives (required for problem (C)), the third derivative has infinite energy. The third derivative corresponds to the shear force.

It may be conjectured that while this shear singularity is confined to the corner in the restrictive setting of problem (C), this same shear effect is released in the less restrictive problem (I). No longer a singularity confined to a corner, its effect spreads throughout the boundary as a boundary layer phenomena.

Being no longer confined to isolated points, its effect is also more readily propagated to the region of the domain away from the boundary. This may account somewhat for the difficulty in using numerical results for problem (I) in order to approximate solutions to problem (C).

Even in the cases where $\|\nabla(\nabla^2 w_0)\|_0$ is finite, some irregular dependence upon ϵ can be expected. In the case of the circular plate with $p/D = \cos\theta$, from section 4-2, it was illustrated that a boundary layer behavior occurs, though this would not show up in the center deflection. Without the symmetry of a circular domain, it is unclear whether or not such behavior would be confined to the boundary region.

The simplest approach to the question of approximating solutions of problem (C) is to assume a simple power series in ε:

$$U_\varepsilon - U_O = \sum_{i=1} a_i \varepsilon^i .$$

While this cannot be expected to describe the solutions over the entire domain it may be valid at least at the center of the square, or perhaps only for the displacement component.

Based on this simple form (see $w_{\varepsilon A}$ on Figure 6), Richardson extrapolations yield the following results: For the polynomial load, extrapolation using $\varepsilon = 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}$ produce a center deflection

$$w_A = .4087008 \text{ E-2} .$$

Since the exact solution to problem (C) is

$$w_O = .390625 \text{ E-2} ,$$

w_A is in error by 4.6%.

The values of ε above were chosen to satisfy the first criterion for reliability. A slightly more accurate result is obtained by using smaller ε values (satisfying the second criterion), $\varepsilon = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$. This produces a center deflection

$$w_A = .396349\ \text{E-2}\ ,$$

which is in error by 1.5%. Clearly, caution must be exercised in interpreting results based on such ad hoc assumptions.

A second approach is to make inferences about the ε dependence from the numerical data itself.

First, it may be observed that, while problem (I) does not in general admit a regular series expansion for small ε, it does so easily for large ε. The form is

$$U_\varepsilon = \varepsilon V_{-1} + V_0 + \sum_{i=1} \varepsilon^{-i} V_i\ .$$

The functions V_i are independent of ε, and have no direct relationship to the terms (if they exist) in an expansion of U_ε for small ε. However this form may serve as a framework to help clarify the behavior of U_ε for the moderate values of ε being used to approximate U_0. Let w_ε^0 be the third component of $\varepsilon V_{-1} + V_0$.

This "outer" expansion indicates an almost linear relationship between U_ε and ε for ε sufficiently large. This is observed numerically in the center deflections for ε as small as about 2^{-3} (see Figure 6).

Since numerical values for ε large are quite reliable, the linear portion of the graph of Figure 6 can be determined accurately. The non-linear region near $\varepsilon = 0$ can be viewed as a departure from the linear form.

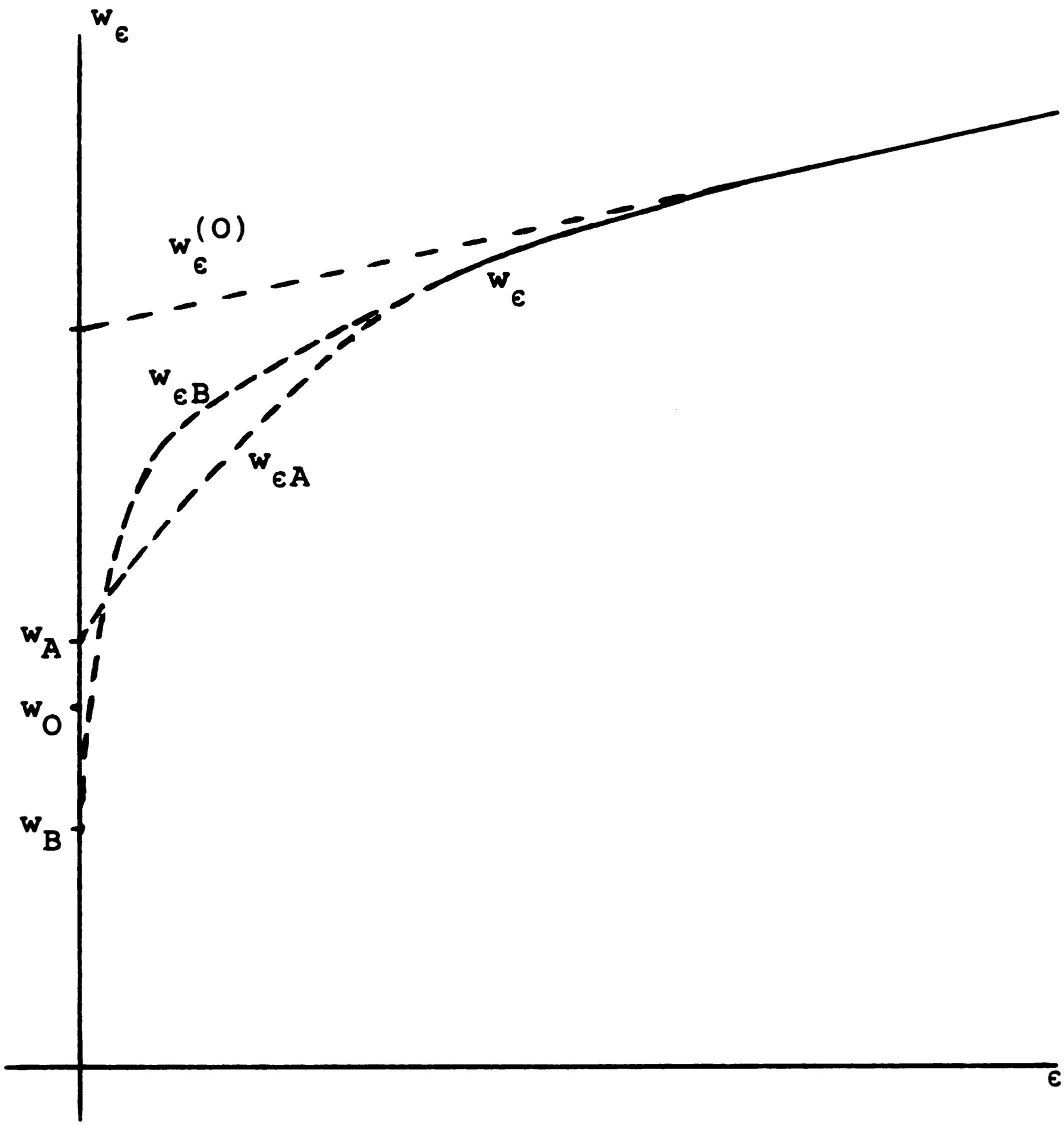
It is noted that in the nonlinear region for which ε is still large enough to produce reliable results ($2^{-7} \leq \varepsilon \leq 2^{-3}$, roughly), there is a very good correlation between the values $w_\varepsilon^{(0)}(0,0) - w_\varepsilon(0,0)$ and an exponential of the form $C_1\, e^{-C_2\sqrt{\varepsilon}}$. These constants can be determined by taking logarithms and applying a linear regression. The value of C_1 will be the extrapolated value w_B (see Figure 6), which approximates w_0.

It is observed that the constants C_1 and C_2 are very sensitive to the data used to produce them. Moreover, while the numerical values used to construct the linear portion of the graph are generally very accurate relative to their size, their precision may not be adequate to produce values, $w_\varepsilon^{(0)}$, which are equally accurate for small ε.

This sensitivity makes it difficult to attain consistently reliable results, and again caution must be exercised in interpreting the results. Generally, values for center deflections extrapolated to $\varepsilon = 0$ are within 2-3% of the center deflection for the classical theory (generally being on the low side).

Qualitatively, the sensitivity of this method may be due to the large (in fact infinite) slope at $\epsilon = 0$ for the curve, $w_{\epsilon B}$, following the exponential form. Reflecting an $O(\sqrt{\epsilon})$ dependence, the curve will tend to be displaced disproportionately at $\epsilon = 0$, when slight changes are introduced at ϵ values in the moderate range.

Whether or not this assumed exponential form is correct, such sensitive behavior is allowed by the main estimate (2-2.9).



<u>Figure 6</u>. Center deflections extrapolated to $\epsilon = 0$.

CHAPTER 5 - CONCLUSIONS AND FURTHER REMARKS

The relationship between the classical theory of plates and the improved theory of Timoshenko and Reissner has been investigated for the clamped plate, in terms of the small parameter ε, a measure of the shear effects included in the improved theory. As ε tends to zero, the solutions in the improved theory tend to solutions in the classical theory.

While this continuity is demonstrated by the estimate (2-2.9), strong evidence is offered to show that the ε dependence is not analytic in ε, nor is it uniform over the surface of the plate. Rather, a boundary layer occurs, possessing different ε dependence than is present throughout the plate. There is reason to believe that this is related to the shear distribution about the boundary in the solutions in the classical theory. For plates with corners it may also be tied to the shear concentrations which are known to occur at the corners in the classical theory. An explanation of these relationships might shed light on the role of shear forces in both theories.

While solutions to problems in the improved theory are rarely attainable in closed form for even very simple loads and for domains with very simple geometry, solutions developed by other means, such as Fourier series, may be used to clarify the possible variations in the ε dependence over the plate surface, as well as to determine the sharpness of (2-2.9).

It would also be useful to develop an estimate analogous to (2-2.9), but estimating the expression $\|U_\varepsilon - U_0\|_0$. The example of section 4-2 shows that $\|U_\varepsilon - U_0\|_1 = O(\varepsilon^{3/4})$ because of a term in a derivative of U_ε. If $\|U_\varepsilon - U_0\|_0 = O(\varepsilon)$, then the numerical difficulties are lessened, in approximating center deflections for the classical theory by those for moderate values of ε in the improved theory. It still must be remembered that of primary interest to engineers are the bending moments, which are estimated in (2-2.9).

By formulating the clamped plate problem in the improved theory, it is possible to approximate solutions using piecewise linear finite elements instead of the more complicated elements generally required to solve plate bending problems in the classical theory. However, the accuracy of the results are adversely affected by the small values usually chosen for the shear parameter ε. While this "spoiling effect" of small ε cannot be avoided, in general, it may be possible to lessen its effect.

Through the term $\varepsilon^{-1}P_s(U,U)$ in the energy expression which is minimized by the solution U_ε to the clamped plate problem in the improved theory, it is clear that each candidate function U is "penalized" according to the value of $P_s(U,U) = \iint \left|\nabla u_3 + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right|^2 dA$. The numerical approximation $U_{\varepsilon,h}$ to U_ε minimizes the same energy expression over a finite subspace, hence is similarly "penalized". Since all three components of $U_{\varepsilon,h}$ are constructed with piecewise linear elements, ∇u_3 is constant within any single element. For a fixed mesh size, h, as ε tends to zero, $\varepsilon^{-1} \iint \left|\nabla u_3 + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right|^2 dA$ over each element must become unbounded unless $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ tends to a constant distribution over the element. Then continuity requirements at the boundaries of the elements, tend to force $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ to the same values for all elements. With homogeneous boundary conditions at the plates edge, $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is forced to zero, and likewise, u_3 must follow suit.

It is tempting to blame this "mismatch" of polynomial degrees in the expression $\nabla u_3 + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ for the behavior of the numerical results as ε tends to zero. The values are, in fact, "driven to zero" as $\varepsilon \to 0$.

Unfortunately, this phenomenon is inherent in the penalty function method form, as can be seen from a simple algebraic example.

Consider the problem: Minimize

$$J = (x-1)^2 + \varepsilon^{-1} y^2$$

over $\mathbb{R}^2$. Clearly the minimum occurs at the point $(1,0)$ for all ε values. Consider the same minimization problem restricted to the subspace $y = hx$. Of course if $h = 0$, the minimum over the entire space $\mathbb{R}^2$ would belong to the subspace, $y = 0$. Otherwise, a "best" approximation to $(1,0)$ would be found by minimizing

$$(x-1)^2 + \varepsilon^{-1}h^2x^2 .$$

The minimum along $y = hx$ is $(1/(1+\varepsilon^{-1}h^2),\ h/(1+\varepsilon^{-1}h^2))$. For a fixed h, as $\varepsilon \to 0$ this point is forced toward the origin, not to a location close to $(1,0)$. The measurement of closeness "penalizes" the y coordinate much more than it does the distance from $(1,0)$.

The similarity of the minimizing point above and the factor $(1 + \frac{\varepsilon^{-1}h^2}{12})$, which proved useful in controlling error for the clamped beam problem cannot be overlooked. If the coordinates of the minimizing point are multiplied by $(1+\epsilon^{-1}h^2)$ the resulting point is $(1,h)$, a very good approximation to $(1,0)$ for small h, and this approximating point is not adversely affected by small ϵ.

Two questions thus arise related to the clamped plate problem. First, can reliable results be obtained for smaller values of ϵ by approximating the third component of U_ϵ by piecewise quadratic elements instead of linear ones? The estimates of section 2-4 suggest that improvement will take place. Correcting the "mismatch" mentioned above may improve accuracy even more so. The second question is whether by approximating by quadratic or higher degree finite elements, a factor such as $(1+\epsilon^{-1}h^2/12)$ can be found to alleviate the adverse ϵ dependence altogether as was done in the case of the beam.

Since the improved plate theory has been chosen as a penalty function method approach to solving the clamped plate problem in the classical theory, in order to both avoid restrictive smoothness requirements and bypass the "patch test" usually required of solutions failing these requirements, a more general question is: can the patch test itself be relaxed by virtue of a penalty function method? If such an approach can be made rigorous, it would open the door to the use of many types of finite elements applied to higher order equations.

The problem of extending this work to the cases of simply-supported plates and plates with free edges is hindered by the failure of the bilinear energy form $B_{\epsilon}(U,U)$ to be coercive. The problem can perhaps be approached by using a device of Cornwell and Yen [5], namely, by attaching torsional springs to the plate surface. As the spring constant is reduced to zero, the simply-supported (or free-edged) plate is recovered. In this limit, it must be shown that estimates similar to (2-2.8) and (2-2.9) are recovered.

In order to successfully apply the penalty function method to solve the problem of bending of a clamped plate in the classical theory by using numerical solutions in the improved theory, either a better understanding must be obtained for the behavior of solutions for small ϵ, or a better numerical process must be developed which is not so adversely affected by small values of ϵ.

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