ON THE EMBEDDABILITY OF COMPACTA IN N-BOOKS: INTRINSIC AND EXTRINSIC PROPERTIES

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This is to certify that the

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ABSTRACT

ON THE EMBEDDABILITY OF COMPACTA IN N-BOOKS:
INTRINSIC AND EXTRINSIC PROPERTIES

by Gail Adele Atneosen

An n-book B^n is the union of n closed disks in E^3 such that each pair of disks meets precisely on a single arc B on the boundary of each. The disks are called the leaves of B^n, and the arc is called its back. The embeddability of compacta in n-books is investigated from two different viewpoints. In Chapters II and III intrinsic properties are considered and in Chapters IV and V extrinsic properties.

Chapter II is concerned with the embeddability of certain continua in n-books. It is shown that all compact, connected 2-manifolds with non-void boundary embed in a 3-book. Examples are given of a one-dimensional, locally connected, locally plane continuum that embeds in a 3-book but not in any 2-manifold, of a one-dimensional locally connected continuum that does not embed in any n-book, and of a one-dimensional locally connected continuum that embeds in B^n but not in B^m for $2 \leq m < n$.

In Chapter III the concept of a polyhedron tame in B^n is introduced, and those polyhedrons tamely embedded are characterized. Necessary and sufficient conditions are

given for a polygonal simple closed curve in a 3-book to span a 2-manifold in the 3-book. The monotone open union of open n-books is shown to be an open n-book.

In Chapter IV extrinsic properties of subsets of n-books in E^3 are investigated. Necessary and sufficient conditions are given for a topological polyhedron in a tame n-book to be tame in E^3. It is shown that every topological umbrella in a tame n-book is locally tame at its tangent point and that no disk pierced by an arc lies in an arbitrary 3-book in E^3. Next questions of cellularity are considered. The cellular hull of a subset A of E^3 is defined to be a cellular set B containing A such that no proper cellular subset of B contains A. An arc A has a cellular hull that lies in a tame 2-complex in E^3 if and only if there is a space homeomorphism h with the property that the image of A under h lies in a tame 3-book. If A is a cellular arc whose set of wild points is non-empty and does not contain an arc and A lies in an arbitrary n-book in E^3, then A has at most one wild point that is not contained in the back of the n-book.

In the last chapter tamely embedded cones over n-books in E^4 are investigated. It is shown that no wild Cantor set lies in a tame cone over an n-book in E^4, and that every 1- or 2-cell or 1- or 2-sphere in a tamely embedded cone over a 1- or 2-book is tame in E^4. Examples are given of wild 2- and 3-cells and 2-spheres in tamely embedded cones over n-books, $n > 2$, in E^4.

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CHAPTER I

INTRODUCTION

An n-book B^n is the union of n closed disks in E^3 such that each pair of disks meets precisely on a single arc B on the boundary of each. The disks are called the leaves of B^n and are denoted by D_i, $i = 1,\ldots,n$, and B is its back.

In this paper we investigate the embeddability of compacta in n-books from two different viewpoints. In Chapters II and III intrinsic properties are considered and in Chapters IV and V extrinsic properties. This investigation of n-books was initiated by P. H. Doyle in [14] when he extended an earlier result [13], and showed that if each of the leaves of an n-book topologically embedded in E^3 is tame then the n-book is tame. C. A. Persinger continued the investigation of extrinsic properties of subsets of n-books in [38, 39, 40]. The concept of n-books has also arisen in a somewhat different context in the work of A. H. Copeland, Jr. [10,11] where some results concerning intrinsic properties of subsets of n-books are found.

Next we give a few comments on notation and some definitions necessary for the reading of this paper. The following notation will be used.

Z^+ = the set of positive integers.

$$E^n = \{x \mid x = (x_1,\dots,x_n) \text{ an n-tuple of real numbers, } n \in Z^+\}.$$

E^n is assumed to have the topology determined by the Euclidean metric d_n.

$$E^n_+ = \{x \mid x = (x_1,\dots,x_n) \quad x_n \geq 0\}.$$

$$S^{n-1} = \{x \mid d_n(x,0) = 1\}.$$

A homeomorphic image of S^1 is called a <u>simple closed curve</u>.

$$\text{Closed n-cell} = \{x \in E^n \mid d_n(x,0) \leq 1\}.$$

$$\text{Open n-cell} = \{x \in E^n \mid d_n(x,0) < 1\}.$$

By a <u>disk</u> will always be meant a space homeomorphic to a closed 2-cell and by an <u>arc</u> a space homeomorphic to a closed 1-cell. If A and B are topological spaces, a homeomorphism of A into B is called an <u>embedding</u>.

By an <u>n-dimensional manifold</u> M is meant a separable metric space such that each point has a neighborhood whose closure is homeomorphic to a closed n-cell. The <u>interior of M</u>, Int M, consists of those points which have neighborhoods homeomorphic to an open n-cell; the <u>boundary of M</u>, Bd M, is defined to be M - Int M. Thus in discussing a disk D, it is clear what is meant by Bd D and Int D. If boundary or interior is used in the usual topological sense then it will be denoted by $\text{Int}_X A$ and $\text{Bd}_X A$ if A is a subset of X. By the <u>interior of an n-book</u> is meant the set $\bigsqcup_{i=1}^{n} \text{Int } D_i \cup \text{Int } B$.

Next some terminology from combinatorial topology is given; the definitions are essentially those of Zeeman [45].

By an <u>n-simplex</u> Δ_n, $0 \leq n$, is meant the convex hull of n+1 linearly independent points (the vertices) $\{x_j \mid j = 0,\ldots,n\}$ in E^p, $n \leq p$. By a <u>r-face</u> Δ_r of Δ_n, denoted by $\Delta_r < \Delta_n$, is meant the convex hull of r+1 distinct points of $\{x_j \mid j = 0,\ldots,n\}$. A <u>simplicial complex</u>, or complex, K of E^p, $p \geq 1$, is a finite collection of simplexes of E^p such that:

(1) if $\Delta \in K$, then all the faces of Δ are in K, and

(2) if Δ_1, $\Delta_2 \in K$ then $\Delta_1 \cap \Delta_2$ is a common face of Δ_1 and Δ_2.

L is called a <u>subcomplex</u> of K if L is a simplicial complex and $L \subset K$. If Δ_1, Δ_2 are simplexes in E^p such that the union of their vertices forms a linearly independent set of points in E^p, then Δ_1 and Δ_2 are <u>joinable</u>. If Δ_1 and Δ_2 are joinable, then the join of Δ_1 and Δ_2, denoted by $\Delta_1 * \Delta_2$, is defined to be the simplex spanned by the union of their vertices. The subcomplex of K consisting of all q-simplexes of K, where $q \leq m$, is called the <u>m-skeleton</u> of K and is denoted by $K^{(m)}$. For $\Delta \in K$ the set $st(\Delta,K) = \{\sigma \in K \mid \Delta < \sigma\}$ is called the <u>star of Δ in K</u>. The underlying point set $|K|$ of a simplicial complex K is called a <u>Euclidean polyhedron</u> or polyhedron.

Sometimes the phrase finite Euclidean polyhedron is used to emphasize the fact that we are only considering simplicial complexes consisting of finitely many simplexes. By $|st(\Delta,K)|$ is meant the space $\bigsqcup\{|\sigma| \mid \sigma \in st(\Delta,K)\}$. If P is a Euclidean polyhedron then a simplicial complex K such that $|K| = P$ is called a <u>triangulation</u> of P, and P is said to be the <u>carrier</u> of K. If K is a simplicial complex and and h is a homeomorphism of $|K|$ onto Q, then the set $\{h(|\sigma|) \mid \sigma \in K\}$ is said to be a <u>curved triangulation</u> of Q. The <u>m-skeleton</u> of Q is the set $\{h(|\sigma|) \mid \sigma \in K^{(m)}\}$. A complex K' is called a <u>subdivision</u> of a complex K if $|K'| = |K|$ and each simplex of K' is contained in some simplex of K. The <u>dimension</u> of a simplicial complex K is the largest integer n such that K contains an n-simplex. The carrier of a 1-dimensional complex is called a <u>graph</u>, note that a graph is not necessarily connected. By an <u>umbrella</u> is meant the Euclidean polyhedron consisting of a disk D and an arc σ such that $D \cap \sigma = \{x\}$ where x is an interior point of D and an endpoint of σ. x is called the <u>tangent point</u> of the umbrella and σ the <u>handle</u>.

Let I be the interval $[0,1] \subset E^1$. For any space X, the <u>cone C(X) over X</u> is the quotient space $(X \times I)/R$ where R is the equivalence relation $(x,1) \sim (x',1)$ for all $x, x' \in X$. Let J be the interval $[-1,1] \subset E^1$. For any space X, the <u>suspension $\Sigma(X)$ of X</u> is the quotient space $(X \times J)/R$, where R is the equivalence relation

$(x,1) \sim (x',1)$, $(x,-1) \sim (x',-1)$ for all x, $x' \in R$.

Next we give a brief discussion of the literature and state some definitions and theorems that will be used in later chapters. An n-book can be viewed as the double cone over n distinct points, and thus can also be considered as the cone over a 1-dimensional complex. It is from this viewpoint they have arisen in A. H. Copeland's work on the isotopy classes of 2-dimensional cones. The results are of an intrinsic nature. In [11] he divides the isotopy classes of cones of dimension less than or equal to two into disjoint sets α and β. Disregarding the cones over homeomorphic spaces, which have the same isotopy type, there is only one member in each class of α. If β_n $(n = 0,2,3,4,\ldots)$ is the isotopy class of cones containing an n-book (a 0-book is an arc) then β is the set of all these classes. Thus the only isotopy classes that contain more than one distinct member are those containing n-books. In [10] necessary and sufficient conditions are given for cones over finitely triangulable spaces to be embeddable in a book.

n-manifolds have been extensively studied. In considering intrinsic properties of n-books, we are mainly concerned with compact, connect 2-manifolds. 2-manifolds are particularly well-known. For a general discussion of 2-manifolds see [Chapt. 1, 34]; for a short proof that all compact 2-manifolds can be triangulated see [20].

In 1908 Schoenflies [43] proved the following result which will be referred to as the plane Schoenflies theorem.

<u>Theorem 1.1</u> If J is a simple closed curve in E^2 and h is a homeomorphism of J onto the unit circle S^1 in E^2, then h can be extended to a homeomorphism of E^2 onto itself.

A corollary to the Schoenflies result is that an umbrella cannot be embedded in the plane. It also follows that any polyhedron embedded in E^2 can be mapped by a space homeomorphism onto a Euclidean polyhedron. It is this notion which is formalized in the following definitions for higher dimensional Euclidean spaces.

A topological polyhedron P in E^n is <u>tamely embedded</u> in E^n if there is a space homeomorphism that carries P onto a finite Euclidean polyhedron. Otherwise P is <u>wildly embedded</u>. A set X in E^n is <u>locally tame at a point p</u> of X if there is a neighborhood N of p and a homeomorphism h of $\overline{N}$ (the closure of N) onto a polyhedron in E^n such that $h(\overline{N} \cap X)$ is a finite Euclidean polyhedron. A set X is said to be <u>wild at a point p</u> if it is not locally tame at p. The definitions of tame and locally tame are due to Fox and Artin [22] and Bing [3], respectively. A set P in E^n is <u>locally polyhedral at a point x</u> of P if there is a neighborhood of x whose closure meets P in a finite Euclidean polyhedron.

The notion of wild and tame can also be applied to spaces that are not polyhedrons. By a <u>Cantor set</u> is meant any homeomorphic image of the classical Cantor ternary set, that is, any compact, perfect, zero-dimensional, non-empty metric space is a Cantor set. <u>A Cantor set $A \subset E^n$ is called tame</u> if it lies on a tame arc in E^n; otherwise A is said to be wild.

Examples of wild arcs in E^3 were known as early as 1921 when Antoine [2] constructed a wild Cantor set in E^3, an arc through this Cantor set is called an Antoine's necklace and is wild. The Alexander Horned Sphere published by Alexander [1] in 1924 is an example of a wild 2-sphere in E^3. In 1948 Fox and Artin [22] gave a number of examples of wild arcs and spheres in E^3 with one or two wild points. These results revived interest in the area of embeddings and since 1948 this has been an active area of research. As an example of the kind of results that have been obtained, and one that we will use later, we list the following theorem due to Bing [3].

<u>Theorem 1.2</u> Each locally tame closed subset K of a triangulated 3-manifold M with boundary is tame. Furthermore, if C is a closed subset of M such that K is locally polyhedral at each point of $K \cap C$, and α is a

positive continuous function on $M - C$, there is a homeomorphism f of M onto itself such that $x = f(x)$ on C, $\rho(x,f(x)) < \alpha(x)$ on $M - C$, and $f(K)$ is a polyhedron.

In connection with n-books Persinger proved in [40] the following two theorems which we will use.

Theorem 1.3 No wild Cantor set lies in a tame n-book in E^3.

Theorem 1.4 There exist wild arcs and disks in tame n-books in E^3, $n > 2$.

Theorem 1.4 is interesting for it states that wild arcs can lie in very simple subspaces of E^3. In this connection, we note that the arc A of Example 1.1 of Fox and Artin [22] is embeddable in a tame 3-book in E^3 in such a manner that the images of a set of generators of $\pi_1(E^3 - A)$ are also contained in the 3-book. (The fundamental group of the complement of A in E^3, $\pi_1(E^3 - A)$, is non-trivial.) Another simple subspace of E^3 that contains wild arcs is an infinite croquet game. By an <u>infinite croquet game</u> is meant a flat disk D in E^3 union countably many disjoint polygonal arcs $\{\sigma_i \mid i \in Z^+\}$ that intersect D only in their endpoints and such that the $\{\sigma_i\}$ converge to a point p in the interior of D. If A is any arc that lies in a tame 3-book in E^3 and has a single wild point, then it is easy to see that A is equivalent to an arc that lies in an infinite croquet game.

A set C in E^n is said to be <u>cellular</u> if there exists a sequence of topological closed n-cells $\{ C_i \mid i \in Z^+ \}$ such that $C_{i+1} \subset \operatorname{Int} C_i$ and $C = \bigcap_{i=1}^{\infty} C_i$. This notion was defined in 1960 by M. Brown [8]. An arc may be wild and also be cellular as Example 1.2 of [22]. Wild points of cellular subsets of 2-spheres in E^3 are considered by Loveland in [32]. McMillan in [35] has obtained results about cellular subsets of higher dimensional space.

A k-cell in E^n, $k \leq n$, is said to be <u>flatly embedded</u>, or <u>flat</u>, if there is a space homeomorphism of E^n onto itself mapping it onto a k-simplex. A (k-1)-sphere in E^n is said to be <u>flatly embedded</u>, or <u>flat</u>, if there is a space homeomorphism that maps it onto the boundary of a k-simplex in E^n. Thus a polygonal trefoil knot in E^3 is tame but not flat. An <u>n-book is flatly embedded in E^3</u> if each of its leaves is a Euclidean 2-simplex. If M is a k-manifold with boundary contained in an n-manifold N, M is <u>locally flat at a point $p \in \operatorname{Int} M$</u> if there is a neighborhood U of p in N such that $(U, U \cap M)$ is homeomorphic to the pair (E^n, E^k); M is <u>locally flat at a point $p \in \operatorname{Bd} M$</u> if there is a neighborhood U of p in N such that $(U, U \cap M)$ is homeomorphic to (E^n, E^k_+). These notions have been recently studied by Lacher [31] and Kirby [27]. It is known that a locally flat k-cell in E^n is flat in E^n [31].

One other theorem that is used and should be mentioned is the Brouwer Theorem on the invariance of domain.

<u>Theorem 1.5</u> If U and V are homeomorphic subsets of S^n and U is open in S^n, then V is open in S^n.

CHAPTER II

CONTINUA IN N-BOOKS

This chapter is concerned with intrinsic properties of n-books rather than, say, positional properties of n-books in Euclidean space. In particular the embeddability of certain continua, that is compact connected sets, in n-books is considered.

<u>Theorem 2.1</u> All compact, connected 2-manifolds with non-void boundary embed in a 3-book.

<u>Proof.</u> Figure 2.1 illustrates what is meant by a disk with (a) a single bridge, (b) a twisted bridge, (c) a double bridge.

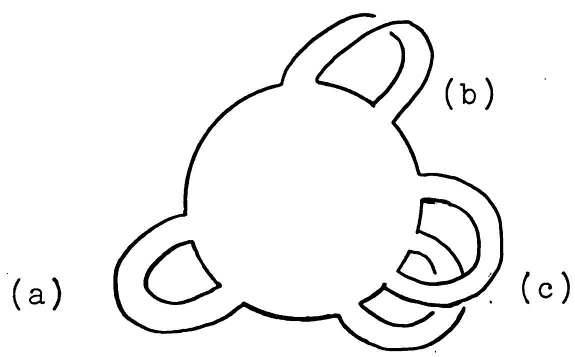


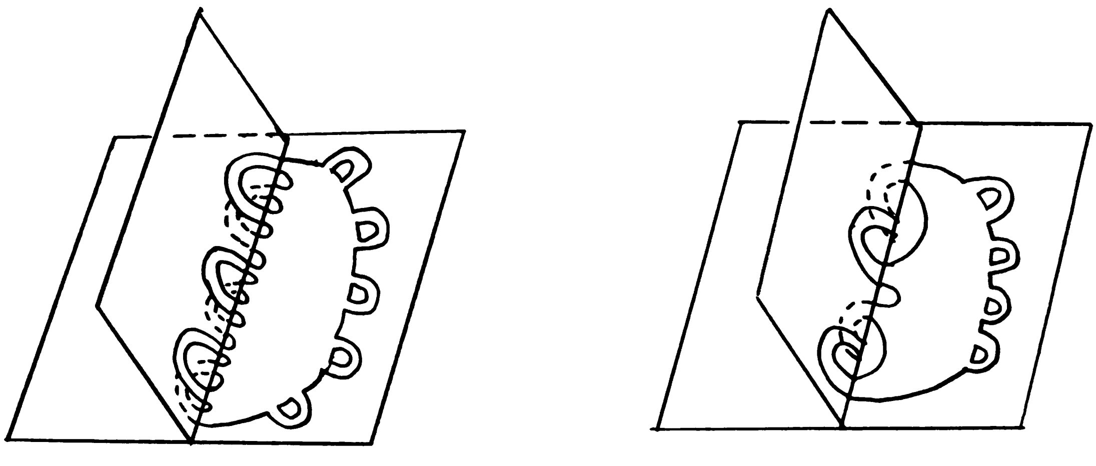
Figure 2.1

Using scissors-and-paste techniques (see Chapter 1 of [34]), it can be shown that all compact, connected 2-manifolds with non-void boundary are homeomorphic to:

(i) a disk with $r \geq 0$ single bridges and $h \geq 0$ double bridges, or

(ii) a disk with $r \geq 0$ single bridges and $q \geq 0$ twisted bridges.

Thus to prove that all 2-manifolds with non-void boundary can be embedded in a 3-book, it suffices to show that 2-manifolds of type (i) or (ii) can be embedded in B^3. Figure 2.2 indicates that this is, in fact, the case. Figure 2.2 (a) consists of a disk with three double bridges and five single bridges and (b) consists of a disk with two twisted bridges and four single bridges.



(a) (b)

Figure 2.2

<u>Corollary 2.2</u> All compact, connected 2-manifolds with non-void boundary can be embedded in a 3-book so as to carry a subcomplex of some triangulation of the 3-book.

Corollary 2.3 Every proper compact subset of a compact connected 2-manifold embeds in a 3-book.

Proof. If C is a proper compact subset of a compact, connected 2-manifold M, then M - C is a non-empty open set. Hence there is a closed disk D contained in M that does not intersect C. M - Int D is a compact, connected 2-manifold with non-void boundary containing C and can be embedded in a 3-book by Theorem 2.1.

Corollary 2.4 All graphs embed in a 3-book.

Proof. Let G be a graph and let n denote the number of vertices of G. Select n distinct points on a 2-sphere and accomodate each arc joining two vertices by a "handle" appropriately attached to the 2-sphere. Thus G can be embedded in a 2-sphere with handles which is a 2-manifold without boundary. Hence by Corollary 2.3 G embeds in a 3-book.

Corollary 2.5 All compact, connected 2-manifolds embed in the triple cone over three points.

Proof. B^3 is the double cone over three points, so it suffices to show that all 2-manifolds embed in $C(B^3)$. Let M be a compact, connected 2-manifold; if Bd M $\neq \emptyset$ then by Theorem 2.1 M embeds in $B^3 \subset C(B^3)$. If Bd M $= \emptyset$, then M - Int D embeds in B^3 where D is a disk in M. But $C(B^3)$ contains the cone over the boundary of D and $(M - \text{Int } D) \cup C(\text{Bd } D)$ is homeomorphic to M. Hence M embeds in $C(B^3)$.

<u>Corollary 2.6</u> All compact, connected 2-manifolds embed in a 2-dimensional continuum in E^4 that fails to be locally polyhedral at only one point.

<u>Proof.</u> There are countably many distinct 2-manifolds with non-void boundary. These can be polyhedrally embedded in B^3, by Corollary 2.2, so as to converge to a point p on the back of B^3. Consider $B^3 \subset E^3 \times 0$, a 3-dimensional hyperplane in E^4, such that these countably many distinct 2-manifolds are embedded in B^3 as described above. In E^4 form the cone over the boundary components of each of these 2-manifolds in such a manner that the cones are disjoint and the vertices of the cones converge to p. Then B^3 union these cones is a 2-dimensional continuum that is locally polyhedral except at p and contains all 2-manifolds with or without boundary.

In Corollary 2.3 it was noted that every proper compact subset of a compact, connected 2-manifold embeds in B^3. Thus the question may be asked if there exists a locally plane, locally connected, one-dimensional continuum that embeds in a 3-book but not in any 2-manifold. By locally plane is meant that each point has a neighborhood that embeds in E^2. An example due to Borsuk [7] is used to answer this question in the affirmative. His example utilizes one of Kuratowski's primitive skew curves [29] which is the union of all the edges of a tetrahedron plus a segment joining two points lying in the interiors of two opposite edges. This graph is not embeddable in the plane.

Example 2.7 [7]   A locally plane, locally connected one-dimensional continuum that is not embeddable in any 2-manifold.

Let a_1, b_1, c_1, and d_1 be the vertices of a tetrahedron in E^3, c_n denote the point dividing the segment $\overline{a_1c_1}$ in the ratio 1: n-1, and d_n the point dividing $\overline{b_1d_1}$ in the same ratio, for $n \in Z^+$. See Figure 2.3 below.

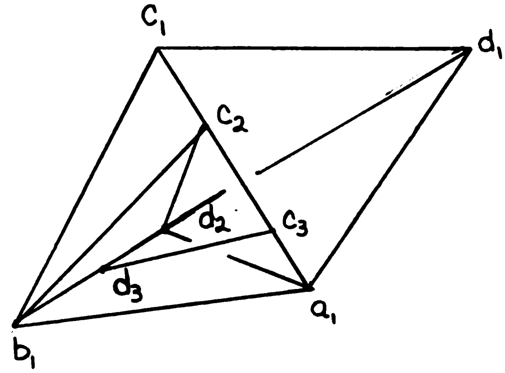


Figure 2.3

Let

$X_n = \overline{a_1b_1} \cup \overline{a_1c_n} \cup \overline{a_1d_n} \cup \overline{b_1c_n} \cup \overline{b_1d_n} \cup \overline{c_nd_n} \cup \overline{c_{n+1}d_{n+1}}$ and

$X = \bigsqcup_{n=1}^{\infty} X_n$.

Note that X is not homeomorphic to any subset of a 2-manifold M. For if g' were such an embedding, then there would exist a disk D in M such that $g'(\overline{a_1b_1}) \subset \operatorname{Int} D$ and for almost all indices n, $g'(X_n) \subset D$. But this is impossible because X_n is not planar.

Next $X - \overline{b_1d_1}$ is mapped into a continuum in E^2 which will enable X to be embedded in a locally connected continuum Y and also show that Y is locally plane. Consider the following points of E^2:

$a' = (0,0)$, $b' = (1,0)$, $c_n' = (0,1/n)$, $b_n' = (1,1/n)$,

$d_n' = (1, 2/(2n+1))$, $d_n'' = (1, -1/n)$ for $n \in Z^+$;

and linear maps

$$
\begin{array}{llll}
f: \overline{a_1b_1} \to \overline{a'b'} & \text{with} & f(a_1) = a', & f(b_1) = b' \ ; \\
g: \overline{a_1c_1} \to \overline{a'c_1'} & \text{with} & g(a_1) = a', & g(c_1) = c_1' \ ; \\
f_n: \overline{c_nd_n} \to \overline{c_n'd_n'} & \text{with} & f_n(c_n) = c_n', & f_n(d_n) = d_n' \ ; \\
f_n': \overline{c_nb_1} \to \overline{c_n'b_n'} & \text{with} & f_n'(c_n) = c_n', & f_n'(b_1) = b_n' \ ; \\
f_n'': \overline{a_1d_n} \to \overline{a'd_n''} & \text{with} & f_n''(a_1) = a', & f_n''(d_n) = d_n'' \ .
\end{array}
$$

Then

$$
h(x) = \begin{cases}
f(x) \text{ for } x \in \overline{a_1b_1} - \{b_1\}, \\
g(x) \text{ for } x \in \overline{a_1c_1}, \\
f_n(x) \text{ for } x \in \overline{c_nd_n} - \{d_n\}, \\
f_n'(x) \text{ for } x \in \overline{c_nb_1} - \{b_1\}, \\
f_n''(x) \text{ for } x \in \overline{a_1d_n} - \{d_n\},
\end{cases}
$$

is a homeomorphism mapping $X - \overline{b_1d_1}$ onto the set

$$Z_o = \overline{a'b'} \cup \overline{a'c_1'} \cup \bigsqcup_{n=1}^{\infty} (\overline{b_n'c_n'} \cup \overline{c_n'd_n'} \cup \overline{a'd_n''}) - \overline{d_1'd_1''} \subset E^2.$$

See Figure 2.4.

Let $L(n,i)$ for $i = 1,\ldots,n$ and $n \in Z^+$ denote the line segment parallel to $\overline{a'c_1'}$ through $a(i,n)$, where $a(i,n)$ is the point dividing $\overline{a'b'}$ in the ratio $i: (n+1-i)$, with one endpoint on $\overline{c_n'd_n'}$ and the other endpoint on $\overline{a'd_n''}$. Then $Y' = Z_o \cup \bigsqcup_{n=1}^{\infty} \bigsqcup_{i=1}^{n} L(n,i)$ is locally connected and the homeomorphism $h^{-1}: Z_o \to X - \overline{b_1d_1}$ can be extended to a homeomorphism g of the set Y' onto a subset of $E^3 - \overline{b_1d_1}$ so that for every $\epsilon > 0$ there exists an $n(\epsilon)$ such that for all $n > n(\epsilon)$ the diameters of the sets $g(L(n,i))$ are less than ϵ. Let $Y = g(Y') \cup \overline{b_1d_1}$, then Y is a locally plane, locally connected continuum that contains X and is hence not embeddable in any surface.

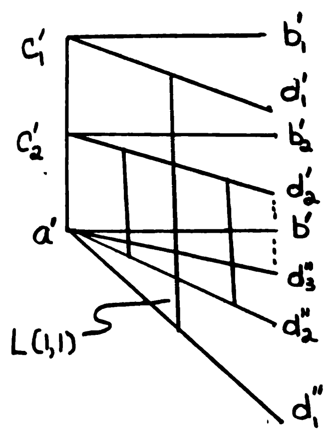


Figure 2.4

<u>Theorem 2.8</u> There exists a locally plane, locally connected one-dimensional continuum that embeds in a 3-book but not in any 2-manifold.

Proof. Example 2.7 is a locally plane, locally connected one-dimensional continuum that embeds in no 2-manifold. To show that it is embeddable in B^3 consider Figure 2.5.

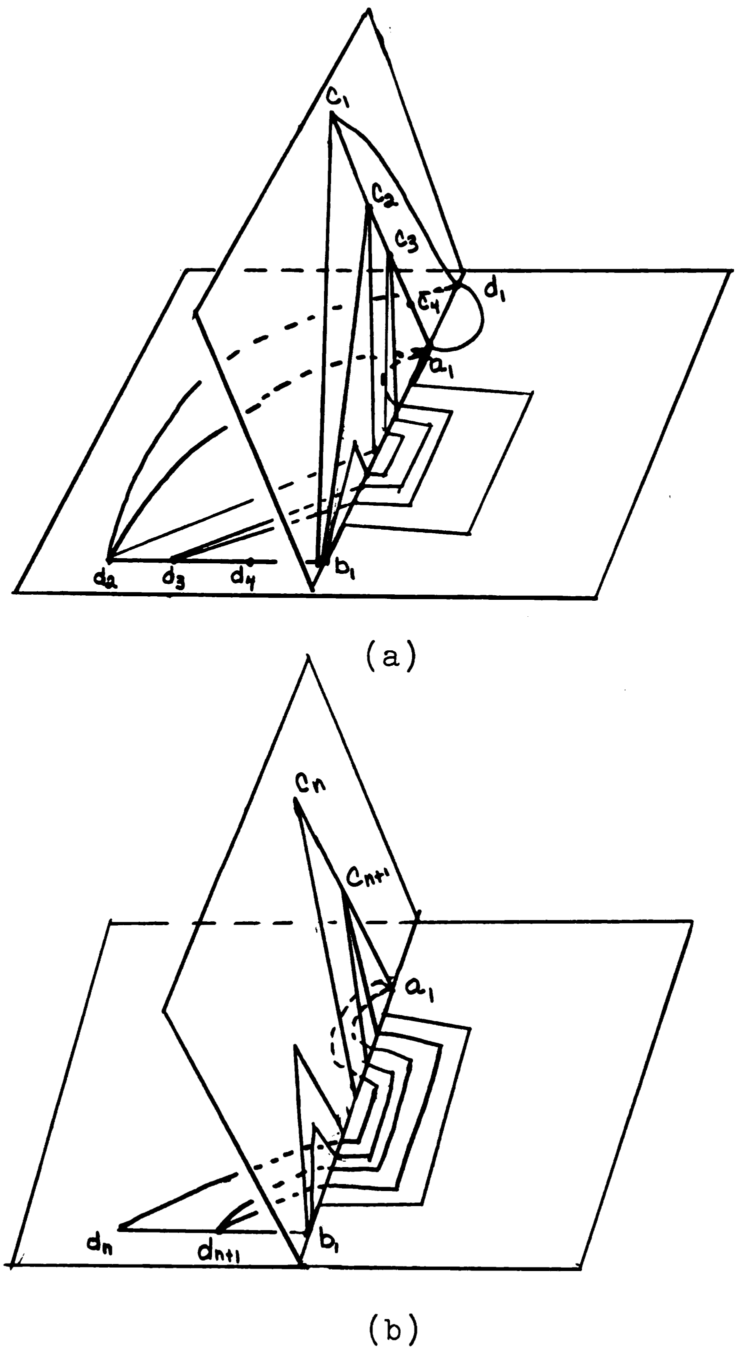


Figure 2.5

Figure 2.5 (a) shows the embedding of $X_1 \cup X_2 \cup X_3 - \overline{c_4 d_4}$. In general, if $\bigsqcup_{i=1}^{n} X_i - \overline{c_{n+1} d_{n+1}}$ has been embedded in B^3, then $X_{n+1} - \overline{c_{n+2} d_{n+2}}$ is embedded in B^3 as in Figure 2.5 (b). Thus continuing in this manner it is clear that X embeds in B^3 and also that Y embeds in B^3.

However, all one-dimensional continua do not necessarily embed in B^n, for some positive integer n, as the argument of the following theorem illustrates.

Theorem 2.9 There exists a one-dimensional, locally connected continuum that cannot be embedded in an n-book for any positive integer n.

Proof. In order to define such a continuum let:

K = the graph which is the union of all the edges of a tetrahedron plus a segment joining two points lying in the interiors of two opposite edges,

$J(1) = \left\{ (x,0,0) \in E^3 \mid 0 \leq x \leq 1 \right\}$,

$K(1,2)$ = a graph in E^3 homeomorphic to K whose diameter is less than 1/2 and such that $K(1,2)$ intersects $J(1)$ only in the point $(1/2,0,0)$, and

$J(2) = J(1) \cup K(1,2)$.

Assume that $J(n)$ has been defined and let:

$$J(n+1) = J(n) \cup \bigsqcup_{m=1}^{n} K(m,n+1).$$

$K(m,n+1)$ is a graph in E^3 homeomorphic to K such that:

(i) $K(m,n+1) \cap J(n) = (m/(n+1),0,0)$ $m = 1,\dots,n$,

(ii) $K(m,n+1) \cap K(m',n+1) = \emptyset$ if $m \neq m'$, and

(iii) the diameter of $K(m,n+1)$ is less than $1/2^{n+1}$ for $m = 1,\dots,n$.

Let $J = \bigcup_{n=1}^{\infty} J(n)$ with the relative topology of E^3. Then J is compact, connected, and locally connected by construction; furthermore, J is one-dimensional since it is the countable union of closed one-dimensional sets.

Next it will be shown that J does not embed in any n-book. Suppose this is not the case, and there is an embedding $h: J \to B^n$, then $h(J(1)) \subset B$, the back of B^n. For if not, then there exists an $x \in J(1)$ such that $d(h(x), B) = \epsilon > 0$. Since J is compact, h is a uniform homeomorphism and there exists a $\delta > 0$ such that if $d(u,v) < \delta$ then $d(h(u),h(v)) < \epsilon/3$. Thus there exists $(m/p,0,0) \in J(1)$ such that $d(K(m,p),x) < \delta$ and hence $h(K(m,p)) \subset D_i$, a leaf of B^n. But this contradicts the fact that K cannot be embedded in the plane so $h(J(1)) \subset B$.

Let z be an interior point of the interval $J(1)$, and N a neighborhood of $h(z)$ in B^n such that $N - h(J(1))$ consists of n components $C_i \subset D_i$, $i = 1,\dots,n$. Then there exists integers m and p such that $h(K(m,p)) \subset N$. Since $h(J(1))$ separates N, $h(K(m,p) - (m/p,0,0)) \subset C_i$ for some i, but this again contradicts the fact that K cannot be embedded in the plane. Hence there does not exist an embedding of J into any n-book.

Next it is shown that n-books can be distinguished by the one-dimensional continua that embed in them. A locally connected, one-dimensional continuum $A(n)$ is constructed with the property that if X is a compact set of dimension less than or equal to one in the interior of an n-book, then $X \subset A'(n)$ where $A'(n)$ is homeomorphic to $A(n)$.

<u>Theorem 2.10</u> There exists a locally connected, one-dimensional continuum that embeds in B^n but not in B^m for $2 \leq m < n$.

<u>Proof.</u> Let $\left\{E_i \mid i \in Z^+\right\}$ be a sequence of mutually disjoint disks in B^n that do not intersect the boundary of any of the leaves of B^n, such that $\bigsqcup_{i=1}^{\infty} E_i$ is dense in B^n, and such that the diameters of the E_i converge to zero. Define $A(n) = B^n - \bigsqcup_{i=1}^{\infty} \operatorname{Int} E_i$, then $A(n)$ is a locally connected, one-dimensional continuum that embeds in B^n. Note that if D_i is a leaf of B^n then $D_i \cap A(n)$ is homeomorphic to Sierpinski's universal plane curve [42, 44].

Assume that there exists an embedding h mapping $A(n)$ into B^m for $2 \leq m < n$ and reach a contradiction. By definition of $A(n)$, the back of B^n is contained in $A(n)$; denote this set by B' in $A(n)$. Then $h(B') \subset B$, the back of B^m. For if not there exists a $z \in B'$ such that $d(z,B) = 3\epsilon > 0$. Since h is a uniform homeomorphism there exists a $\delta > 0$ such that if $d(u,v) < \delta$, then $d(h(u),h(v)) < \epsilon$.

Note that $A(n) \cap D_1$ is arcwise connected. Let $c(i)$, $i = 1,\ldots,n$, be an arc in $A(n) \cap (D_1 - \{z\})$ such that: (1) $d(c(i),z) < \delta$, (2) $c(i)$ intersects B' only in its endpoints a and b, which are the same for all i, and (3) if $\overline{ab}$ denotes the line segment in B' joining a to b then $z \in \overline{ab}$. Let $a(i)$ denote an arc in $A(n)$ joining a point on $c(i) - B'$ to z such that $a(i) \cap B' = \{z\}$, (thus the diameter of $a(i)$ is less than δ). Then $h(\bigsqcup_{i=1}^{n}(c(i) \cup a(i)) \cup \overline{ab})$ is a graph that is entirely contained in a leaf of B^m. This graph contains one of Kuratowski's primitive skew curves which is not embeddable in the plane [29], thus $h(B') \subset B$.

Next, using the fact that $h(B') \subset B$ it will be shown that $A(n)$ cannot be embedded in B^m. Let $z \in B'$ such that $d(h(z), B - h(B'))$ is greater than $3\epsilon > 0$. Choose $c(i)$ and $a(i)$ as before, $i = 1,\ldots,n$. Then since $m < n$ there exists $i \neq j$ such that $h(c(i)) \cup h(c(j))$ is contained in one leaf, say D_k, of B^m and $[h(c(i)) \cup h(c(j))] \cap B = \{h(a)\} \cup \{h(b)\}$. Then $h(c(j)) \cup h(\overline{ab})$, say, bounds a disk containing $h(c(i))$. But then $h(a(j))$ must intersect $h(c(i))$ which contradicts the fact that h is an embedding. Hence $A(n)$ cannot be embedded in B^m for $2 \leq m < n$.

The above argument also provides an entirely different proof for Lemma 2.1 of A. H. Copeland in [11].

This lemma is stated in the following corollary.

<u>Corollary 2.11</u> If B^m is contained in B^n then $m \leq n$. If $m > 2$ then the back of B^m is contained in the back of B^n.

The next corollary states that in some sense $A(n)$ is the "universal curve" for B^n.

<u>Corollary 2.12</u> If X is a compact set in the interior of an n-book B^n such that $\dim X \leq 1$, then $X \subset A'(n)$ where $A'(n)$ is homeomorphic to $A(n)$.

<u>Proof.</u> Let $X_i = (X \cap D_i) \cup B$; then X_i is a closed subset of D_i for each $i = 1, \ldots, n$ and $\dim X_i \leq 1$. So there exists in $\operatorname{Int} D_i - X_i$, $i = 1, \ldots, n$, a sequence $\{E(i,k) \mid k \in Z^+\}$ of mutually disjoint disks such that $\bigsqcup_{k=1}^{\infty} E(i,k)$ is dense in D_i and the diameters of the $E(i,k)$ converge to zero. Let $G_i = D_i - \bigsqcup_{k=1}^{\infty} \operatorname{Int} E(i,k)$, then by construction $G_i \subset X_i$. But G_i is an S-curve [44], that is a plane, locally connected, one-dimensional continuum S such that the boundary of each complementary domain of S is a simple closed curve and no two of these complementary domain boundaries intersect. Hence by Theorem 3 of [44] there exist homeomorphisms h_i, $i = 1, \ldots, n$, which are the identity map on Bd D_i and map G_i onto $A(n) \cap D_i$. If $\bigsqcup_{i=1}^{n} G_i = A'(n)$ then $X \subset A'(n)$.

Furthermore, the map $h\colon A'(n) \to A(n)$ defined by $h(x) = h_i(x)$ for $x \in G_i$ is a homeomorphism of $A'(n)$ onto $A(n)$.

CHAPTER III

SOME PROPERTIES COMMON TO EUCLIDEAN SPACES AND N-BOOKS

In this chapter various properties of Euclidean spaces are generalized to n-books. In particular the notion of tameness, polygonal simple closed curves spanning 2-manifolds, and monotone unions of open n-books are considered.

A topological polyhedron in Euclidean n-dimensional space is said to be <u>tamely embedded</u> if there is a homeomorphism of E^n onto itself which transforms the embedded polyhedron into a Euclidean polyhedron. It is this notion which is generalized to n-books. For the remainder of this chapter B^n will be considered embedded in E^3 in such a manner that each leaf is a Euclidean 2-simplex Thus B^n inherits a linear structure from E^3 and the notion of tameness in B^n can be introduced. A set $A \subset B^n$ is said to be a <u>polyhedron in B^n</u> if A, when considered embedded in E^3, is a Euclidean polyhedron. A topological polyhedron embedded in the interior of an n-book is said to be <u>tame in B^n</u> iff there is a homeomorphism of B^n onto itself which transforms the embedded polyhedron into a polyhedron in B^n.

<u>Lemma 3.1</u> If a Euclidean polyhedron is embedded in B^n in such a manner that the image of the 1-skeleton is the

carrier of a subcomplex relative to some triangulation of B^n, then the image of the polyhedron is also the carrier of a subcomplex relative to this triangulation.

Proof. Let h be an embedding of a Euclidean polyhedron $|P|$, with triangulation P, into B^n such that $h(|P^{(1)}|)$ is the carrier of a subcomplex of the triangulation K of B^n. Let $J = \{\sigma \in K \mid \sigma < \sigma_1 \ , \ \ \text{Int } |\sigma_1| \cap h(|P|) \neq \emptyset \text{, and } \sigma_1 \in K\}$ then J is a subcomplex of K. It will be shown that $|J| = h(|P|)$. By construction $h(|P|) \subset |J|$. To prove that $|J| \subset h(|P|)$ assume not and reach a contradiction. Suppose there exists $x \in |J| - h(|P|)$, then there are three cases to consider.

Case 1. $x \in |\sigma|$ where σ is a face, not necessarily proper, of a 2-simplex $\Delta \in J$. By definition of J, there is a $y \in \text{Int } |\Delta| \cap h(|P|)$. So there is an arc t joining x to y such that $t - \{x\} \subset \text{Int } |\Delta|$. Let z be the first point of $t \cap h(|P|)$ in the direction from x. Then z must belong to the image of the 1-skeleton of P, since it does not lie in the interior of an open 2-cell in $h(|P|)$. But $z \in \text{Int } |\Delta|$, which contradicts the hypothesis that the image of the 1-skeleton is the carrier of a subcomplex of K.

Case 2. $x \in |\sigma|$ where σ is a 1-simplex in J that is not the face of any 2-simplex in J. But this implies there is a $y \in h(|P|) \cap \text{Int } |\sigma|$ such that y must be the carrier of a 0-simplex of K if $h(|P^{(1)}|)$ is the carrier of a subcomplex of K. Thus this case cannot occur.

Case 3. $x \in |\sigma|$ where σ is a 0-simplex in J and σ is not the face of any other simplex in J. But by the definition of J then $\sigma \notin J$ so this case cannot occur.

The above three cases exhaust all possibilities so $|J| \subset h(|P|)$ and the lemma follows.

The following two lemmas are proved elsewhere but are needed in several arguments so are stated here.

Lemma 3.2 The intersection of two Euclidean polyhedrons is a Euclidean polyhedron.

Proof. Corollary 2 to Lemma 1, Chapter 1 of [45].

Lemma 3.3 If $|K| \supset |L|$ then there exists a subdivision K' of K and L' of L such that L' is a subcomplex of K'.

Proof. Lemma 4, Chapter 1 of [45].

The proof of the following lemma depends upon the plane Schoenflies theorem and is similar to one given by Doyle in [16].

Lemma 3.4 Let T be a finite graph, not necessarily connected, and h an embedding of T into the closed unit square D in E^2 such that $h(T) \cap \text{Bd } D$ is a finite Euclidean polyhedron. Then there is a homeomorphism g of D onto itself such that $g(h(T))$ is the union of finitely many straight line segments, and g restricted to the boundary of K is the identity map.

<u>Proof.</u> Since $h(T) \cap \text{Bd } D$ is a finite Euclidean polyhedron, there are only finitely many points in Bd D that are limit points of $h(T) \cap \text{Int } D$. Let $\{x_i \mid i = 1,\ldots,n\}$ denote these points plus the images of the vertices of T. Let $N(x_i)$ be closed symmetric neighborhoods of x_i, $i = 1,\ldots,n$, in D such that any two are disjoint. Let $\sigma(i,j)$, $i = 1,\ldots,n$ and $j = 1,\ldots,k(i)$, denote the finitely many arcs in $h(T) \cap N(x_i)$ such that: (1) $\sigma(i,j)$ has endpoints x_i and $y_{i,j}$, $y_{i,j} \in \text{Bd } N(x_i)$, and (2) $\sigma(i,j) - \{y_{i,j}\}$ is contained in $\text{Int}_D N(x_i)$. Let $\sigma'(i,j)$ be a straight line segment joining x_i to $y_{i,j}$ for $i = 1,\ldots,n$ and $j = 1,\ldots,k(i)$. Then by the plane Schoenflies theorem there are homeomorphisms g_i, $i = 1,\ldots,n$, such that:

(i) $g_i | D - \text{Int } N(x_i)$ = the identity map, and

(ii) $g_i(\sigma(i,j)) = \sigma'(i,j)$ $j = 1,\ldots,k(i)$.

Then $h(T) - \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{k(i)} \sigma(i,j)$ is the union of finitely many arcs t_i, $i = 1,\ldots,m$. Let U_i be a closed neighborhood of t_i homeomorphic to a closed disk such that $U_i \cap h(T)$ is an arc. Let $g'' = g_1 \ldots g_n$, then $g''(U_i) \cap g''(h(T) - \text{Int } t_i)$ consists of two polygonal arcs, say a_i and b_i, which were obtained by the above application of the plane Schoenflies theorem. Let $t_i' \subset \text{Int } g''(U_i)$ be a polygonal arc joining the endpoints of $g''(t_i)$ such that t_i' intersects $g''(h(T))$ only in its endpoints. Then by the plane Schoenflies theorem there are homeomorphisms g_i', $i = 1,\ldots,m$, such that:

(i) $g_i' \mid (D - \text{Int } g''(U_i)) \cup (a_i \cup b_i)$ is the identity map, and

(ii) $g_i'(g''(t_i)) = t_i'$.

Then $g = g_1' \ldots g_m' g''$ is the desired homeomorphism. That is, $g \mid \text{Bd } D =$ the identity map and $gh(T)$ consists of finitely many straight line segments.

The following theorem is a characterization of those polyhedrons which are tamely embedded in B^n.

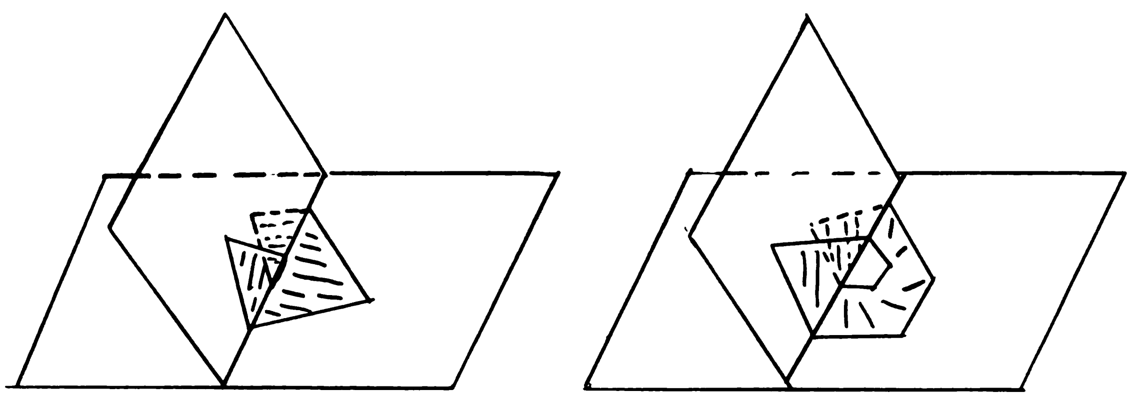
<u>Theorem 3.5</u> An embedding of a polyhedron in Int B^n is tame in B^n iff the polyhedron has a curved triangulation such that the image of the 1-skeleton intersected with the back of B^n is a polyhedron in B^n.

<u>Proof.</u> Let P be a polyhedron and $h: P \to B^n$ an embedding. If P is tamely embedded in B^n, then by definition there is a homeomorphism g of B^n onto itself such that $gh(P)$ is a polyhedron in B^n. Let K be any triangulation of B^n and L any triangulation of $gh(P)$, (then K and L are

simplicial complexes in E^3). By Lemma 3.3 there are subdivisions L' of L and K' of K such that L' is a subcomplex of K'. Then $(gh)^{-1}\colon |L'| \to P$ is a curved triangulation of P. The back of B^n is a subcomplex of every triangulation of B^n, so by Lemma 3.2 $|L'^{(1)}| \cap B$ is a polyhedron in B^n.

Conversely, suppose that P has a curved triangulation such that the image of the 1-skeleton, denoted by R, intersects the back of B^n in a finite polyhedron. Then $R \cap D_i$ is the topological image of a finite graph and $R \cap \mathrm{Bd}\, D_i$ is a finite Euclidean polyhedron, $i = 1,\ldots,n$. Hence by Lemma 3.4 there are homeomorphisms $g_i\colon D_i \to D_i$, $i = 1,\ldots,n$, which are the identity map on Bd D_i and such that $g_i(R)$ consists of finitely many points and straight line segments (when considered in E^3). By Lemma 3.3 there is a triangulation, say K', of B^n such that $g_1\cdots g_n(R)$ is the carrier of a subcomplex of K'. By Lemma 3.1 $g_1\cdots g_n h(P)$ is a Euclidean polyhedron, and h is a tame embedding of P into B^n.

Every polygonal simple closed curve in E^3 spans an orientable surface in E^3, that is, there exists a compact, connected orientable 2-manifold in E^3 such that the manifold boundary is precisely the simple closed curve. One method of obtaining such an orientable surface is given by R. H. Fox in [21]. By a polygonal simple closed curve in B^3 is meant a homeomorphic image of S^1 that is a polyhedron in B^3. Theorem 3.8 characterizes those polygonal simple closed curves in B^3 that span compact, connected 2-manifolds in B^3. All such curves do not necessarily span a surface as is indicated in Figure 3.1 (a). Even if a polygonal simple closed curve does span a 2-manifold, as in Figure 3.1 (b), the 2-manifold need not be orientable. In this case it is a Möbius band.



(a) (b)

Figure 3.1

One approach to the desired characterization is to consider the simplicial homology of B^3 with coefficients in Z_2, the group of integers modulo two. For an exposition of simplicial homology theory and related terminology see [Chapter 6, 25]. By a t-dimensional chain on a simplicial complex K with coefficients in Z_2 is meant a function m on the t-simplexes of K with values in Z_2.

It facilitates notation to let m also denote the subcomplex of K which is the simplicial closure of all t-simplexes of K on which m has non-zero value. The geometric realization of this subcomplex will also be denoted by m rather than $|m|$. No confusion should arise since Z_2 coefficients are particularly well suited to geometric interpretation, and this notation will be used only in relation to chains.

<u>Lemma 3.6</u> If C is a polygonal simple closed curve in B^3 then there exists a mod 2 2-chain, m_c, on B^3 such that $\partial m_c = C$ and m_c has the properties listed below.

(i) $C \subset m_c$ and m_c is compact and connected.

(ii) If $x \in m_c - C$ then x has a neighborhood in m_c homeomorphic to an open disk.

(iii) If $x \in C$ then x has a neighborhood in m_c whose closure is homeomorphic to either:

(1) a closed disk,

(2) B^3,

(3) the union of two closed disks whose intersection is precisely $\{x\}$, x being an interior point of one disk and a boundary point of the other disk, or

(4) the union of two closed disks which intersect along an arc σ with endpoints x and p such that σ is in the boundary

of one disk and $\sigma - \{p\}$ is in the interior of the other disk.

Proof. By Lemma 3.3, B^3 has a triangulation K such that C is contained in the carrier of the 1-skeleton of K. Consider the simplicial homology of K with coefficients in Z_2, and let σ_1 denote the 1-chain that has value 1 on all 1-simplexes of K that are contained in C and 0 on all other 1-simplexes of K. Since B^3 is contractible, $H_1(B^3, Z_2)$, the first simplicial homology group of B^3 with coefficients in Z_2, is trivial. Hence there is a 2-chain m_c on K such that $\partial m_c = \sigma_1$. It will now be shown that m_c has the properties stated in the lemma.

(i) Since $\sigma_1 = C$, $\partial m_c = C$ and $m_c \supset C$. m_c is compact because it is the point set union of finitely many compact 2-simplexes. To prove that m_c is connected assume not and reach a contradiction. Suppose that m_c can be expressed as the disjoint union of two non-empty closed sets, A_1 and A_2. Since C is connected it may be assumed that, say, $C \subset A_1$. Let m_c' be the 2-chain on K that has non-zero value only on those 2-simplexes of K contained in A_2. Since $\partial m_c = C$, each 1-simplex of K in A_2 is the face of an even number of 2-simplexes in A_2. Thus $\partial m_c' = 0$, but $H_2(B^3, Z_2) = 0$, since B^3 is contractible, hence there must be a 3-chain on K whose boundary is m_c'. Since the only 3-chain is the zero 3-chain this implies

A_2 is empty and hence m_c is connected.

(ii) Let $x \in m_c - C$ and show that x has a neighborhood in m_c homeomorphic to an open disk. There are three possible cases to consider.

Case 1. x is contained in the interior of a 2-simplex of m_c. Then x clearly has a neighborhood in m_c homeomorphic to an open disk.

Case 2. x is contained in the interior of a 1-simplex in m_c. Then this 1-simplex does not lie in C and hence must be the face of an even number of 2-simplexes in m_c. By Corollary 2.11 if B^m is embedded in B^n then $m \leq n$, so this 1-simplex is a face of precisely two 2-simplexes in m_c . Hence x has a neighborhood in m_c homeomorphic to an open disk.

Case 3. Lastly, consider the case when x is a 0-simplex in m_c. Then $|st(x,m_c)|$ is the union of n closed disks which contain x in their interior, since every 1-simplex in m_c having x as a face must be the face of precisely two 2-simplexes in m_c. So to show that x has a neighborhood in m_c homeomorphic to an open disk, it suffices to prove that n = 1.

If $x \in B^3 - B$ then n = 1. For if $n \geq 2$ then $|st(x,m_c)|$ contains a topological umbrella which is embeddable in the plane. If $x \in B$ and $n \geq 2$, there are two closed disks, say E_1 and E_2, in $|st(x,m_c)|$ such that $E_1 \cap E_2 = \{x\}$. Lemma 4.1, of the next chapter, implies that x has a neighborhood N_i in E_i, $i = 1,2$, which is an open 2-cell and is contained in precisely two leaves of B^3. But this fact contradicts the disjointness of $E_1 - \{x\}$ and $E_2 - \{x\}$.

(iii) Let $x \in C$, and again consider the various cases.

Case 1. If x is not a vertex of m_c, then x is contained in the interior of a 1-simplex of m_c that lies in C. This 1-simplex is the face of an odd number of 2-simplexes in m_c. By Corollary 2.11 it is the face of one or three 2-simplexes and hence x has a neighborhood in m_c homeomorphic to (1) or (2) of part (iii) of the statement of this lemma.

Case 2. If x is a vertex there are exactly two 1-simplexes, t_1 and t_2, in m_c which lie in C and have x as a face. Thus $st(x,m_c)$ contains two 1-simplexes that are a face of one or three 2-simplexes in $st(x,m_c)$ and all the other 1-simplexes are the face of two 2-simplexes in $st(x,m_c)$. Using the fact that two closed disks which intersect at a single point interior to each cannot be embedded in B^3, as shown above, one obtains the following results. If t_1 and t_2 are both the face of only one 2-simplex in $st(x,m_c)$ then x has a closed neighborhood homeomorphic to (1) or (3). If t_1 is a face of only one 2-simplex in $st(x,m_c)$ and t_2 is a face of three 2-simplexes in $st(x,m_c)$, then x has a closed neighborhood homeomorphic to (4). If both t_1 and t_2 are the face of three 2-simplexes in $st(x,m_c)$, then x has a closed neighborhood homeomorphic to (2). This exhausts all the possibilities and the lemma follows. Figure 3.2 shows the four possible closed neighborhoods.

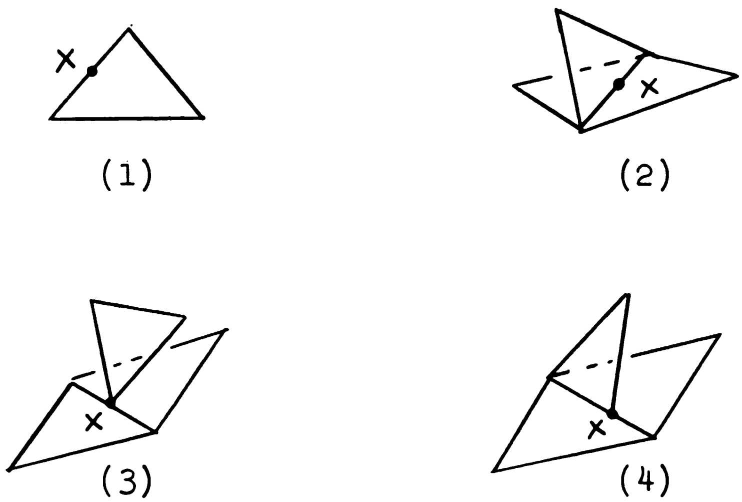


Figure 3.2

The proof of the next lemma is very similar to that of Lemma 3.1 and so is omitted. Since the polyhedron being considered is a 2-manifold weaker conditions can be imposed on its 1-skeleton.

<u>Lemma 3.7</u> Let M be a compact, connected 2-manifold with non-void boundary, C, embedded in B^n. If B^n has a triangulation K such that C is contained in the carrier of the 1-skeleton of K, then M is the carrier of a sub-complex of K.

Theorem 3.8 A polygonal simple closed curve in B^3 bounds a compact, connected 2-manifold in B^3 iff it is a mod 2 cycle which is the boundary of a mod 2 2-chain whose geometric realization contains no umbrellas.

Proof. By Lemma 3.3, if C is a polygonal simple closed curve in B^3 there is a triangulation K of B^3 such that $C \subset |K^{(1)}|$. By Lemma 3.7 the 2-manifold M which C bounds is the carrier of a subcomplex of K. Let z_2 be the mod 2 2-chain which has non-zero value only on those 2-simplexes of K that lie in M. Then $\partial z_2 = C$ and the geometric realization of z_2 is M and hence contains no umbrellas.

Conversely, assume that C is a polygonal simple closed curve in B^3 that is a mod 2 cycle with respect to some triangulation K of B^3, and that there is a mod 2 2-chain z_2 such that $\partial z_2 = C$. By Lemma 3.6 there is a mod 2 2-chain m_c, with the properties stated there, such that $\partial m_c = C$. Hence $\partial(m_c - z_2) = 0$ which implies, since there is only the zero 3-chain on B^3, that $m_c = z_2$. Thus by hypothesis m_c contains no umbrellas, which implies that points of C do not have neighborhoods in m_c of type (2), (3), or (4) of part (iii) of Lemma 3.6. Hence every point of m_c has a closed neighborhood homeomorphic to a closed disk, and C bounds m_c which is a compact, connected 2-manifold in B^3.

Corollary 3.9 A polygonal simple closed curve spans a disk in B^3 iff it is a mod 2 cycle which is the boundary of a mod 2 2-chain whose geometric realization contains no umbrellas and every simple closed curve in the interior of this 2-complex separates it.

Proof. A compact, connected 2-manifold M with non-void boundary is a disk iff every simple closed curve in the interior of M separates M.

A topological space Y is said to be the open monotone union of a space U if $Y = \bigcup_{i=1}^{\infty} U(i)$, and $U(i)$ is open in Y, $U(i)$ is homeomorphic to U for all i, and $U(i) \subset U(i+1)$. In [9] Morton Brown proved that the open monotone union of open n-cells is an open n-cell. Theorem 3.10 states that the same kind of result is true for n-books.

By an open n-book is meant a space homeomorphic to Int B^n which was defined as $\bigcup_{i=1}^{n}$ Int $D_i \cup$ Int B. The space A x $[0,\infty)$ with A x 0 identified to a point v is called the open cone, OC(A), over A. If X is a topological space, a point $x \in X$ is said to have an open cone neighborhood U if there is a homeomorphism f of some OC(A) onto the open set U of X such that $f(v) = x$. Theorem 3 of [30] states that if $U^1 \subset U^2 \subset \ldots$ is a sequence of open cone neighborhoods of x in a locally compact Hausdorff space, then $U = \bigcup_{i=1}^{\infty} U^i$ is also an open cone neighborhood of x homeomorphic to each U^i. We will use this theorem in the proof of the following theorem.

<u>Theorem 3.10</u> If a topological space Y is the open monotone union of open n-books then Y is an open n-book.

<u>Proof.</u> Let $\{U^i \mid i \in Z^+\}$ be a sequence of open n-books such that $U^i \subset U^{i+1}$, and U^i is open in $\bigsqcup_{i=1}^{\infty} U^i = Y$. Next it is established that Y is a locally compact Hausdorff space. Let x and y be two distinct points of Y, then there exists an integer n such that $x, y \in U^n$. U^n is a Hausdorff space so there exist disjoint neighborhoods, V_x and V_y, of x and y, respectively, such that V_x and V_y are open in U^n. Since U^n is open in Y, V_x and V_y are open in Y, and Y is a Hausdorff space. Y is also locally compact for let $x \in Y$, then $x \in U^n$ for some $n \in Z^+$. x has a neighborhood V in U^n such that the closure of V in U^n is a compact set, denoted by $Cl_{U^n} V$. Then U^n open in Y implies V is open in Y; since Y is a Hausdorff space $Cl_{U^n}V$ is also closed in Y. So V is a neighborhood of x in Y whose closure in Y is compact.

Let A be the suspension of n distinct points, then OC(A) is an open n-book. If x is a point on the back of an open n-book U^i, one readily sees that U^i is an open cone neighborhood of x. By an application of Corollary 2.11, since $U^i \subset U^{i+1}$, the open back of U^i must be contained in the open back of U^{i+1}. Thus choose a point x in the back of U^1, by these remarks x is contained in the back of U^i for $i \in Z^+$. So $U^1 \subset U^2 \subset \ldots$ is a sequence of open cone neighborhoods of x in a locally compact Hausdorff space; by Theorem 3 of [30], Y is homeomorphic to an open n-book.

CHAPTER IV

SUBSETS OF N-BOOKS IN E^3

This chapter is concerned with extrinsic properties of compacta in n-books, that is positional properties of subsets of n-books embedded in E^3. Euclidean polyhedrons topologically embedded in tame n-books are investigated, and a characterization is given of those polyhedrons tame in E^3 by considering where they can fail to be locally tame. Next, questions concerning cellularity and n-books are examined.

The first lemma is, however, only concerned with embeddings into n-books; the result will be useful in characterizing wild points of polyhedrons embedded in n-books in E^3.

<u>Lemma 4.1</u> Let h be an embedding of a disk D into B^n, and let x be an interior point of D. Then x has a closed 2-cell neighborhood U in D such that h(U) is contained in the union of two leaves of B^n.

<u>Proof.</u> There are two cases to consider.

<u>Case 1.</u> $h(x) \notin B$, the back of B^n. Then there is a neighborhood V of x such that h(V) is entirely contained in the interior of some leaf of B^n. By invariance of domain, h(V) is open in this leaf, hence there is an $\epsilon > 0$

such that the symmetric neighborhood $\overline{S_\epsilon(x)} \subset h(V)$. Then $U = h^{-1}(\overline{S_\epsilon(x)})$ is the desired neighborhood of x.

Case 2. $h(x) \in B$, the back of B^n. Suppose there does not exist a neighborhood V of x in D such that h(V) is contained in the union of two leaves of B^n. Then there is a sequence of points $\{y_k \mid k \in Z^+\}$ converging to h(x) such that $y_k \in B - h(D)$. Since $h(D) \not\subset B$, there is also a sequence of points $\{x_k \mid k \in Z^+\}$ converging to h(x) such that $x_k \in (B^n - B) \cap h(D)$. Let a_k be an arc in B^n joining x_k to y_k such that $a_k \cap B = \{y_k\}$. Moreover, we may assume that the diameter of $a_k < 1/2^k$. Since $h(D) \cap a_k$ is a compact set there is a first element z_k of a_k, in the direction from y_k to x_k, such that $z_k \in h(D)$. Furthermore, since $y_k \notin h(D)$, $z_k \notin B$. So $z_k \in h(\mathrm{Bd}\ D)$ for otherwise an umbrella could be embedded in the plane. Because of the manner in which the a_k were chosen, the sequence $\{z_k \mid k \in Z^+\}$ converges to h(x). But this implies $\{h^{-1}(z_k) \mid k \in Z^+\}$ is a sequence of points on the boundary of D converging to an interior point x of D which is a contradiction. Thus there is a neighborhood V of x such that h(V) is contained in the union of two leaves of B^n. Then using invariance of domain and preceding as in Case 1, a neighborhood U is obtained with the desired properties.

Lemma 4.1 is not necessarily true for $x \in \mathrm{Bd}\ D$ as is indicated in Figure 4.1.

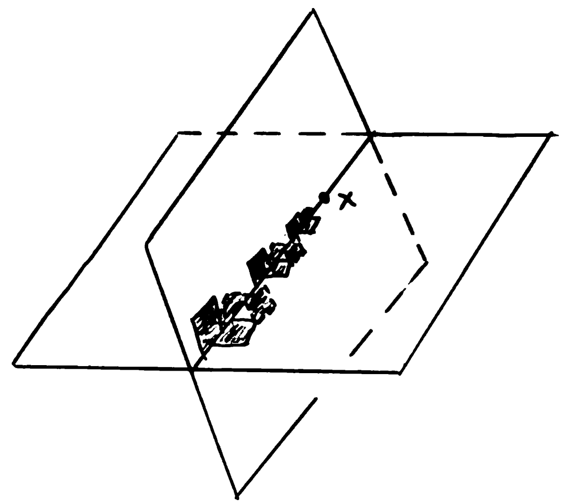


Figure 4.1

Theorem 4.2 Let P be a Euclidean polyhedron embedded in a tame n-book, and let Q be the set of points of P that do not have open 2-cell neighborhoods in P. Then the set of points at which P fails to be locally tame is contained in $Q \cap B$ and is a compact, totally disconnected set.

Proof. Since B^n is tame, we may assume that B^n has planar leaves. If $x \in P - Q$ then x lies in the interior of a topological disk D in P. By Lemma 4.1 we may assume that D lies in the union of two leaves of B^n. Let N be a closed polyhedral neighborhood of x in E^3 such that $N \cap P \subset D$. Then $N \cap P$ is a Euclidean polyhedron and P is locally tame at x.

If $x \in Q - B$ then there is a closed polyhedral neighborhood V of x in B^n such that V is homeomorphic to a closed disk and such that Q intersects the boundary of V in a finite Euclidean polyhedron. Then $Q \cap V$ is the homeomorphic image of a finite graph, so by Lemma 3.4 there is a homeomorphism g mapping V onto V which is fixed on the boundary of V and such that $g(V \cap Q)$ is the union of finitely many straight line segments and points. This homeomorphism can be extended to a closed polyhedral neighborhood N of x in E^3 such that $N \cap P = V$ and $g|$ Bd N is the identity map. By Lemma 3.1 $g(N \cap P)$ is a Euclidean polyhedron . Hence P is locally tame at x.

From the definition of local tameness it follows that the set of points at which P fails to be locally tame is closed and hence compact. Since the back of B^n is tame, the set of wild points of P is also totally disconnected.

Corollary 4.3 If A is an arc in a tame n-book, then $A = E \cup T$ where T is the countable union of tame arcs and E is a compact, totally disconnected set contained in the back of the n-book.

Proof. By Theorem 4.2 the set of points, E, where A fails to be locally tame is compact and totally disconnected and contained in the back of the n-book. A - E is an open subset of A and hence can be expressed as the countable union of open arcs. These open arcs are locally tame and have tame closures.

The set of wild points of an arc in a tame n-book may be uncountable as the next example shows. By an almost tame arc is meant an arc such that every point lies on a tame subarc of the original arc [18].

Example 4.4 An example of a cellular arc with uncountably many wild points that is not almost tame but is embeddable in a tame 3-book in E^3.

Let B^3 be a 3-book in E^3 with planar leaves such that the back B of the 3-book is the unit interval [0,1] on the x-axis of E^3. Let $\{\delta_i \mid i \in Z^+\}$ be the sequence of open intervals deleted from the unit interval to obtain the usual Cantor ternary set, and let $C = B - \bigcup_{i=1}^{\infty} \delta_i$. Replace each closed interval $\overline{\delta_i}$ of B, $i \in Z^+$, with a Wilder arc J_i [23] embedded in B^3 so that the endpoints of J_i concide with the endpoints of $\overline{\delta_i}$, the diameters of the J_i tend to zero, and $J_i \cap J_k = \emptyset$ for $i \neq k$. Then $A = \bigcup_{i=1}^{\infty} J_i \cup C$ is an arc in E^3. If $x \in C$, then every neighborhood of x in E^3 contains a wild arc J_i for sufficiently large i; hence A fails to be locally tame on C plus the set of points where $\bigcup_{i=1}^{\infty} J_i$ fails to be locally tame.

If $x \in C$ but is not an endpoint of $\overline{\delta_i}$ for any i, then x does not lie on a tame subarc of A. Thus, A fails to be locally tame at uncountably many points and is not almost tame.

To see that A is in fact cellular we use a definition and theorem due to Doyle [15]. If A is an arc in S^n, we say that A is <u>p-shrinkable</u> if A has an endpoint q and in each open set U containing q in S^n, there is a closed n-cell $F \subset U$ such that q lies in Int F while Bd F meets A in exactly one point. If A is an arc in S^n such that for each subarc A' of A, A' is p-shrinkable, then every arc in A is cellular [15]. Since Wilder arcs were inserted, the constructed arc satisfies the necessary conditions and is therefore cellular.

Of course, there are wild arcs with uncountably many wild points in tame 3-books that are not cellular. One example could be obtained by inserting Example 1.1 of [22] instead of the Wilder arcs in the above construction.

The next lemma is due to Persinger [40] and is used in the proof of Theorem 4.6.

<u>Lemma 4.5</u> Let D be a closed disk in a tame n-book. Then D is tame iff Bd D is tame.

<u>Theorem 4.6</u> A topological polyhedron P in a tame n-book in E^3 is tame iff it has a triangulation such that the image of the 1-skeleton is locally tame at each point where it meets the back of B^n.

Proof. By Theorem 4.2 and the hypothesis of this theorem the image of the 1-skeleton is locally tame, and so Theorem 1.2 implies the 1-skeleton is tame. By Lemma 4.5 the image of each 2-simplex in P is tame. Theorem 3.1 of Doyle [14] states that a topological polyhedron P in E^3 is tame iff each 2-simplex in P is tame and the 1-skeleton is tame.

Using the notion of tame in B^n, which was defined in Chapter III, one obtains the following theorem.

Theorem 4.7 Let B^n be a tame n-book and P a polyhedron tame in B^n, then P is tame in E^3.

Proof. Let h_1: $E^3 \to E^3$ be a homeomorphism such that the leaves of $h_1(B^n)$ are 2-simplexes. Since P is tame in B^n, there is a homeomorphism h_2: $h_1(B^n) \to h_1(B^n)$ such that $h_2(h_1(P))$ is a Euclidean polyhedron. h_2 can be extended to a homeomorphism of E^3 onto itself, also called h_2. Then $h_2 h_1$: $E^3 \to E^3$ such that $h_2 h_1(P)$ is a Euclidean polyhedron, and hence P is tame in E^3.

Corollary 4.8 Every topological umbrella in a tame n-book is locally tame at its tangent point.

Proof. The tangent point x of the topological umbrella T must lie in the back B of B^n, since an umbrella cannot be embedded in the plane. By Lemma 4.1 there is a closed neighborhood U of x in the 2-cell of T that lies in precisely two leaves of B^n. U may be chosen homeomorphic to a closed 2-cell and such that Bd $U \cap B$ consists of two points. Thus there is a subarc α of the handle of T with endpoint x and such that $\alpha - \{x\}$ lies in the interior of a leaf of B^n. Since $(U \cup \alpha) \cap B$ is homeomorphic to an arc, Theorem 3.5 implies $U \cup \alpha$ is tame in B^n. Corollary 4.7 implies $U \cup \alpha$ is tame in E^3 and hence T is locally tame at its tangent point.

In the next theorem arbitrary 3-books in E^3 are considered. Let D be a disk in Euclidean 3-space. Let e be an arc such that $D \cap e$ is a point p which is an interior point both of D and of e. If for each sufficiently small open neighborhood U of p, U - D is the sum of two disjoint open sets each of which intersects the component of $U \cap e$ that contains p, then <u>e pierces D at p</u>.

<u>Theorem 4.9</u> No disk pierced by an arc lies in an arbitrary 3-book in E^3.

<u>Proof.</u> Suppose there did exist a disk D pierced by an arc e such that $D \cup e \subset B^3$ where B^3 is an arbitrary 3-book in E^3. Then Lemma 4.1 implies there is a closed neighborhood D' of p, $\{p\} = D \cap e$, in D that is homeomorphic to a 2-cell and D' is contained in precisely two leaves of B^3,

say D_1 and D_2. Let the diameter of D' equal $\epsilon > 0$. Without loss of generality, it may be assumed that ϵ is sufficiently small so that if U is a spherical neighborhood of p in E^3 of diameter ϵ, then U - D equals U - D', and U - D is the union of two disjoint open sets, V_1 and V_2, each of which intersects the component e' of $U \cap e$ that contains p. Let $x_1 \in V_1 \cap e'$ and $x_2 \in V_2 \cap e'$. Since $D' \subset D_1 \cup D_2$ it follows that $x_1, x_2 \in D_3$ and there is an arc $A \subset D_3 \cap U$ with endpoints x_1 and x_2 such that $A \cap D = \emptyset$. But this contradicts the fact that x_1 and x_2 are in disjoint components of U - D. Hence a disk pierced by an arc does not lie in an arbitrary 3-book in E^3.

Next questions concerning cellularity and n-books are considered. Recall that a set C in E^3 is said to be <u>cellular</u> if there is a sequence of closed 3-cells $\{C_i \mid i \in Z^+\}$ such that $C_{i+1} \subset \operatorname{Int} C_i$ and $C = \bigcap_{i=1}^{\infty} C_i$. If A is a subset of E^3, then the <u>cellular hull of A</u>, denoted by $\mathcal{H}(A)$, is a cellular set containing A such that no proper cellular set $B \subset \mathcal{H}(A)$ contains A [16]. Thus the cellular hull of a cellular subset of E^3 is the set itself. If A and A_1 are two arcs in E^3, then A is said to be <u>equivalent</u> to A_1 if there is a homeomorphism h mapping E^3 onto itself such that $h(A) = A_1$.

<u>Lemma 4.10</u> Let A be an arc in E^3 and $W \subset A$ be the set

of points at which A fails to be locally tame. If W is 0-dimensional, then A is equivalent to an arc in a flat 3-book iff W lies in a tame set that embeds in E^2.

Proof. This result follows easily from a theorem of Posey [41].

Theorem 4.11 An arc A in E^3 has a cellular hull that lies in a tame 2-complex iff A is equivalent to an arc in a flat 3-book.

Proof. Assume A has a cellular hull that lies in a tame 2-complex. Without loss of generality, it may be assumed that A lies in the carrier of a simplicial complex K of dimension two. If $W \subset A$ is the set of points at which A fails to be locally tame, then by the argument of Theorem 4.2, W is contained in the carrier of the 1-skeleton of K. Furthermore W is a closed, totally disconnected set, so W is contained in a polygonal tree in $|K^{(1)}|$. Hence the conditions of Lemma 4.10 are satisfied and there is a homeomorphism h mapping E^3 onto itself such that h(A) is contained in a flat 3-book.

Conversely, assume A is equivalent to an arc A_1 in a flat 3-book B^3 under a space homeomorphism h. The intersection of a maximal chain (ordered by inclusion) of cellular sets containing A_1 is a cellular hull of A_1. But $B^3 \supset A_1$ and B^3 is a cellular set. Consider a maximal chain of cellular sets (as above) containing B^3 which gives rise to a cellular hull $\mathcal{H}(A_1)$ of A_1, then $h^{-1}(\mathcal{H}(A_1))$ is a cellular hull of A and lies in the tame 2-complex $h^{-1}(B^3)$.

The following corollary follows from the proof of Theorem 4.11.

Corollary 4.12 If A is an arc that lies in a tame 2-complex in E^3, then A is equivalent to an arc in a flat 3-book.

Theorem 4.13 There is an arc A in E^3 with the property that if $\mathcal{H}(A)$ is any cellular hull of A, $\mathcal{H}(A)$ does not lie in a tame 2-complex.

Proof. By Theorem 4.11 it suffices to exhibit an arc that does not embed in a tame 3-book. Let A be an arc through a wild Cantor set in E^3, for example an Antoine's necklace [2]. From Theorem 1.3 it follows that no wild Cantor set lies in a tame 3-book in E^3 and so $\mathcal{H}(A)$ does not lie in a tame 2-complex.

Theorem 4.14 utilizes the following result of McMillan [36]: Suppose that K is a finite complex, L is a subcomplex of K, and that K collapses to L. Let $h: K \to M^n$ be a homeomorphism where M^n is a piecewiselinear n-manifold. If $n \neq 4$ and if $h(K)$ is cellular in M^n, then $h(L)$ is cellular. (For a definition of collapsing see [45].)

<u>Theorem 4.14</u> If B^n is a cellular book in E^3 then each leaf is cellular and the back is cellular, but not conversely.

<u>Proof.</u> Since an n-book collapses to any leaf of the back, it follows immediately from [36] that if B^n is a cellular book in E^3 then each leaf is cellular and the back is cellular.

However, the converse is not true. That is, there are n-books in E^3 such that each leaf plus the back is cellular but the n-book is not cellular. One such example, for $n = 2$, is obtained from the non-cellular arc A of Example 1.1 of [22]. A can be expressed as the union of two arcs A_1 and A_2 such that $A_1 \cap A_2 = \{x\}$, and x is a point in the interior of A. Then A_1 and A_2 are both cellular since they are locally tame except at their endpoints. The arc A can be swollen into a disk D such that $D = D_1 \cup D_2$, D_i, $i = 1,2$, is a cellular disk obtained by swelling A_i, and $D_1 \cap D_2$ is a straight line segment. Thus D is a 2-book with cellular back $D_1 \cap D_2$ and cellular leaves D_1 and D_2 that is not cellular.

In Theorem 4.18, we describe the wild points of cellular arcs in arbitrary n-books in E^3. In the proof of this theorem we use Theorem 10 of C. D. Sikkema ["A duality between certain spheres and arcs in S^3," Trans. Amer. Math. Soc. 22 (1966) 339-415]. P. H. Doyle has recently given an alternate proof of this result which we include. The proofs of Theorems 4.15-4.17 are due to Doyle. Theorem 4.17 is Theorem 10 of Sikkema in the above paper. The space X is either the 3-sphere or Euclidean 3-space. If A is a compact set in X, let Z be the space obtained by identifying A with a point while $n: X \to Z$ is the natural map.

<u>Theorem 4.15</u> Let $A \subset X$ be a wild arc that is locally tame at all points except an endpoint a and let b be the other endpoint. If C is a flat 3-cell in X, $A - \{b\} \subset \text{Int } C$, suppose b lies on Bd C so that $A \cup \text{Bd } C$ is locally tame at b. Then n(Bd C) is wild in Z.

<u>Proof.</u> Let D be a 3-cell that is locally tame except at a, $\text{Bd } D \cap \text{Bd } C$ is a disk E on the boundary of each cell while $A - (\{a\} \cup \{b\}) \subset \text{Int } D$ and b lies in Int E. D is obtained by "swelling A".

By construction $\overline{C - D}$ is not a 3-cell, but it has a wild 2-sphere boundary R. Note that $n | \overline{C - D}$ is a homeomorphism and so n(R) is wild in Z. If n(Bd C) were tame in Z, then n(R) would have the point n(b) accessible by a tame arc from the side having the 3-cell closure. But by [26] this is impossible.

Theorem 4.16 Let A be as in Theorem 4.15. If B is an arbitrary arc in X, $B \cap A = \{b\}$, then n(B) is wild in Z.

Proof. There is a disk D that lies on a tame 2-sphere S^2 such that S^2 bounds a 3-cell C, $A - \{b\} \subset \operatorname{Int} C$, b lies on S^2 and $A \cup S^2$ is locally tame at b; one may obtain D by swelling A near b and S^2 is obtained by the tameness of D and an application of [5]. By Theorem 4.15 $n(S^2)$ is wild in Z. Then by the same argument n(B) contains no tame arc in X with n(b) as endpoint. So n(B) is wild in Z.

Theorem 4.17 Let A_1 and A_2 be disjoint arcs in X that are each wild and fail to be locally tame at just one endpoint each, a_1 and a_2, respectively. If A_3 is any arc in X containing $A_1 \cup A_2$ and having a_1 and a_2 as endpoints, then A_3 is not cellular.

Proof. Suppose A_3 were cellular. Then each subarc of A_3 must be cellular [35]. In Int A_3 select a subarc Q such that $\overline{A_3 - Q}$ is locally tame except at a_1 and a_2. Note that X modulo Q is topologically X again. So for A_3 one may select an arc A that fails to be locally tame at its endpoints and by [33] exactly one interior point. But by [5] and Theorem 4.16 this is impossible.

The proof of the following theorem could have also been obtaining by using the techniques of Sikkema in the paper mentioned on page 53.

Theorem 4.18 Let A be a cellular arc in the interior of an arbitrary n-book in E^3. If the set of wild points of A is non-empty and does not contain an arc, then A has at most one wild point that is not contained in the back of the n-book.

Proof. Let B^n be an arbitrary n-book that contains a cellular arc A. Assume that A has two wild points a_1 and a_2 that are not contained in the back of B^n and reach a contradiction. There are three cases to consider.

Case 1. a_1 and a_2 are both interior points of A. Let A_1 and A_2 be disjoint subarcs of A such that $a_i \in \operatorname{Int} A_i$ and A_i is contained in the union of two leaves of B^n, $i = 1,2$. Then both A_1 and A_2 are cellular arcs by [35]. The argument of Theorem 5 of [5] establishes the existence of a subdisk D' of the two leaves of B^n containing A_1 such that: (1) A_1 is contained in the interior of D', and (2) D' lies on a 2-sphere S_1 in E^3. Theorem 1 of [32] states that a cellular arc on a 2-sphere in E^3 has a set of wild points that is empty, contains an arc, or consists of a single point. Hence by the hypothesis of this theorem, a_1 is the only wild point of A_1. Let T_1 and T_2 be two subarcs of A_1 whose union is A_1 and whose intersection is a_1. We next show that both T_1 and T_2 fail to be locally tame at a_1.

By the Bing approximation theorem [4], it may be assumed that S_1 is locally polyhedral except on A_1. By the above argument, A_1 is locally tame everywhere except at a_1 ; Theorem 1 of [17] implies that S_1 is locally tame except at a_1. It then follows from Theorem 9 of [3] that there is a space homeomorphism h mapping S_1 onto a 2-sphere that is locally polyhedral except at $h(a_1)$. Then Theorem 5 of [17] implies that $h(T_1)$ and $h(T_2)$, and hence T_1 and T_2, are equivalently embedded in E^3. If T_1 and T_2 are both locally tame at a_1, then by an application of Theorem 1 of [17], A_1 would be a tame arc. Hence T_1 and T_2 both fail to be locally tame at a_1. The same argument establishes subarcs U_1 and U_2 of A_2 such that $U_1 \cup U_2 = A_2$ and $U_1 \cap U_2 = \{a_2\}$, and such that U_1 and U_2 both fail to be locally tame at a_2. Let A_3 be a subarc of A with endpoints a_1 and a_2, then these are isolated wild points of A_3 which is a cellular arc by [35]. But this contradicts Theorem 4.17 and hence this case cannot occur.

Case 2. a_1 is an endpoint of A and a_2 is an interior point of A. Let A_3 be a subarc of A with endpoints a_1 and a_2. Then by the same argument as in Case 1, a_2 is an isolated wild point of A_3. Let $a_1 \in T \subset A_3$, where T is an arc contained in the union of two leaves of B^n. By an argument as in Case 1, T may be assumed to lie on a 2-sphere in E^3. So by [32], a_1 is an isolated wild point of T. Thus A_3 is a cellular arc by [35] such that its endpoints are isolated wild points of the arc. This contradicts Theorem 4.17 and so this case cannot occur.

<u>Case 3.</u> a_1 and a_2 are the endpoints of A. As in Case 2, it follows that a_1 and a_2 are isolated wild points of A. So by Theorem 4.17 this case cannot occur.

Since the above three cases cannot occur, it follows that A has at most one wild point that does not lie on the back of B^n.

Next we give an example to show that this is the best possible result for an n-book, $n > 2$. Example 1.2 of Fox and Artin [22] can be swollen into a 3-cell that contains Example 1.2 on its boundary. Let D be a 2-cell contained in the 2-sphere boundary of the 3-cell, such that D contains Example 1.2 in its interior and is locally polyhedral except at the wild point of Example 1.2. Let B^3 be a 3-book in E^3 such that two of the leaves are 2-simplexes and the other leaf is D, and let A be an arc in D that is equivalent to Example 1.2 of [22] such that A intersects the back of B^3 only in its endpoint z. Then Example 4.4 of this chapter can be embedded in B^3 in such a manner that it has z as one of its endpoints, and it intersects A only in this point. The union of Example 4.4 and A is a cellular arc, by the p-shrinkable criterion, that has uncountably many wild points on the back of B^3 and precisely one wild point contained in the interior of a leaf of B^3.

CHAPTER V

SUBSETS OF TAMELY EMBEDDED CONES OVER N-BOOKS IN E^4

In [40] Persinger considered wild and tame subsets of tamely embedded n-books in E^3. As was remarked in Chapter II, an n-book may be considered as the double cone over n points. In this chapter wild and tame subsets of tamely embedded triple cones over n points in E^4 are considered, that is, subsets of tamely embedded cones over n-books in E^4. A cone over an n-book will be denoted by $C(B^n)$.

The argument of the first theorem establishes that there exist no wild Cantor sets in tame cones over n-books in E^4, just as in [40] it is established that there exist no wild Cantor sets in tame n-books in E^3.

<u>Theorem 5.1</u> No wild Cantor set lies in a tame cone over an n-book in E^4.

<u>Proof.</u> Let $C(B^n)$ be a tamely embedded cone over an n-book in E^4. Let h be a homeomorphism of E^4 onto itself such that $h(C(B^n))$ is a Euclidean polyhedron with triangulation K. Let $s_1, \ldots, s_r$ denote the 3-simplexes of K. Suppose C is a Cantor set embedded in $C(B^n)$. Then for each i, $1 \leq i \leq r$, $C \cap h^{-1}(|s_i|)$ is contained in a Cantor set C_i such that $C_i \subset h^{-1}(|s_i|)$. $h(C_i) \subset |s_i|$, and $|s_i|$ is a

Euclidean 3-simplex in E^4 and so is contained in a 3-dimensional hyperplane of E^4. From Klee [28] it follows that $h(C_i)$, hence C_i, is a tame Cantor set in E^4. Theorem 8 of Osborne [37] states that the countable union of tame Cantor sets in E^n is a tame Cantor set. Hence $C \subset \bigcup_{i=1}^{r} C_i$ and $\bigcup_{i=1}^{r} C_i$ is a tame Cantor set in E^4, and so C is a tame Cantor set.

The above proof is valid for any tamely embedded (n-1)-complex in Euclidean n-space, hence the following corollary.

<u>Corollary 5.2</u> A tamely embedded (n-1)-complex in E^n contains no wild Cantor sets.

Next 1-cells and 1-spheres in tamely embedded $C(B^n)$ are considered. The fact that all such 1-cells and 1-spheres are tame in E^4 follows from Theorem 2 of Dancis in [12]. This theorem states: A necessary and sufficient condition that a k-complex K, which is a closed subset of a combinatorial n-manifold (without boundary) $n \geq 2k + 2$, be tame in M is that K lie in the union of a countable number of locally tame (n-k)-simplexes in M.

<u>Theorem 5.3</u> There exist no wild arcs or wild simple closed curves in tame cones over n-books in E^4.

In the case n = 1 or 2, this result can be obtained in another manner which is indicated in the proof of Theorem 5.6. All arcs, simple closed curves, and disks in a tame 1- or 2-book in E^3 are tame in E^3 [40]. The next two theorems are an analogous kind of result for cones over 1- or 2-books tamely embedded in E^4. These theorems depend strongly on some recent results of Kirby [27], and results on embeddings of subsets in 3-dimensional hyperplanes in E^4 of Klee [28], Bing and Klee [6], and Gillman [24].

A finite sequence of distinct 3-simplexes in E^n, $s_1,\ldots,s_r$, is called a <u>circuit</u> if:

(1) $\bigsqcup_{j=1}^{i} |s_j|$ is homeomorphic to a closed 3-cell for $1 \leq i \leq r$, and

(2) $(\bigsqcup_{j=1}^{i} |s_j|) \cap |s_{i+1}|$ for $i < r$ is a 2-cell on the boundary of $\bigsqcup_{j=1}^{i} |s_j|$ and on the boundary of $|s_{i+1}|$.

A circuit which is a sequence with r members is said to have length r.

<u>Lemma 5.4</u> Let β be a Euclidean polyhedron in E^4 homeomorphic to a closed 3-cell, K a triangulation of β, and σ a 1-simplex of K. Then the collection of 3-simplexes of K that have σ as a face can be ordered in such a manner, say $s_1,\ldots,s_r$, so that this sequence is a circuit.

Proof. The collection of all circuits, such that the members of the sequence are 3-simplexes of K having σ as a face, is non-empty and contains a finite number of elements. Thus there is a circuit of maximal length, $s_1,\ldots,s_k$. To prove the lemma it is necessary to show that $k = r$, where r is the number of 3-simplexes of K having σ as a face. Suppose $k \neq r$ and reach a contradiction. Let $G = |s_1| \cup \ldots \cup |s_k|$, then G is homeomorphic to a closed 3-cell. There are two cases to consider.

Case 1. There exists $x \in |\sigma| \cap \text{Int } G$. Then x has an open 3-cell neighborhood N in G which, by invariance of domain, is also a neighborhood of x in β. If there exists $s' \in K$, where s' is a 3-simplex with σ as a face and $s' \neq s_i, i = 1,\ldots,k$, then Int $|s'|$ intersects every neighborhood of x in β. But Int $|s'| \cap N = \emptyset$, hence there does not exist such an s'.

Case 2. There does not exist $x \in |\sigma| \cap \text{Int } G$. Then $|\sigma| \subset \text{Bd } G$ and there are two 2-simplexes, say $\sigma * \langle x_1 \rangle$ and $\sigma * \langle x_2 \rangle$, which have σ as a face and lie in Bd G. (Denote the j-simplex with vertices $q_0,\ldots,q_j$ by $\langle q_0,\ldots,q_j \rangle$.)

Next it will be shown that both $\sigma * \langle x_1 \rangle$ and $\sigma * \langle x_2 \rangle$ lie in the boundary of β. Suppose not, and that Int $|\sigma * \langle x_1 \rangle| \subset \text{Int } \beta$. Then $\sigma * \langle x_1 \rangle$ is the face of two 3-simplexes in K, one of which lies in G and the other $s' = \sigma * \langle x_1, b \rangle$ which is not a member of the circuit determining G. Then

$$|s'| \cap G = \begin{cases} |\sigma^*\langle x_1\rangle| & \text{or} \\ |\sigma^*\langle x_1\rangle| \cup |\sigma^*\langle b\rangle|, \end{cases}$$

for all the 3-simplexes in G have σ as a face. Hence $|s'|$ intersects G in a 2-cell on the boundary of each, and $|s'| \cup G$ is homeomorphic to a closed 3-cell. But then $s_1, \ldots, s_k, s'$ is a circuit which contradicts the maximal length of $s_1, \ldots, s_k$. Hence both $\sigma^*\langle x_1\rangle$ and $\sigma^*\langle x_2\rangle$ lie in the boundary of β.

Since both $\sigma^*\langle x_1\rangle$ and $\sigma^*\langle x_2\rangle$ lie in the boundary of β, if $x \in \operatorname{Int} |\sigma|$ then x has a neighborhood N in G, which by invariance of domain, is also a neighborhood of x in β. So by the same reasoning as in Case 1, there does not exist a 3-simplex s' of k with $\sigma < s'$ and $s' \neq s_i$ for $i = 1, \ldots, k$. The lemma follows from the above argument.

Theorem 5.5 A tame 3-cell β in E^4 is flat.

Proof. Since β is tame it may be assumed that β is a Euclidean polyhedron in E^4 with triangulation K. In the appendix of [31] Lacher gives a proof that locally flat cells in E^n are flat. So it suffices to prove that β is locally flat. In [27] Kirby also proves that if β_1 and β_2 are two locally flat (n-1)-cells in E^n with $\beta_1 \cap \beta_2 = \operatorname{Bd} \beta_1 \cap \operatorname{Bd} \beta_2 = \beta^{n-2}$ where β^{n-2} is an (n-2)-cell which is locally flat in Bd β_1 and Bd β_2, then $\beta_1 \cup \beta_2$

is a flat (n-1)-cell in E^n. This theorem along with Lemma 5.4 implies that β is locally flat except possibly at its vertices and hence by the previous remarks β is flat.

Theorem 5.6 There exist no wild 1- or 2-cells or 1- or 2-spheres in a tamely embedded cone over a 1- or 2-book in E^4.

Proof. The cone over a 1- or 2-book is homeomorphic to a closed 3-cell. Hence by Theorem 5.5, we may assume that $C(B^n)$ $n = 1,2$ is contained in the hyperplane $E^3 \times 0$ in E^4. By a theorem of Klee [28], any 1-cell embedded in a 3-dimensional hyperplane of E^4 is tame in E^4. In [6] Bing and Klee prove that every simple closed curve in E^3 is unknotted in E^4. By Theorem 3 of Gillman in [24] every 2-sphere or 2-cell in a 3-dimensional hyperplane of E^4 is tame in E^4. Thus every 1- or 2-cell or 1- or 2-sphere in a tamely embedded cone over a 1- or 2-book in E^4 is tame in E^4.

The question of whether or not there exist wild 3-cells in tame $C(B^n)$, $n = 1,2$, has not been answered. However, there do exist wild 2- and 3-cells in tamely embedded cones over n-books, $n > 2$, in E^4. To show that this is the case it is necessary to introduce some definitions from [5]. Let D be a disk. We say that _a map of Bd D into a set Y can be shrunk to a constant in Y_ if the map can be extended to take D into Y. $Y \subset X$ is <u>locally simply connected</u> at a point p of $\overline{Y}$ if for each neighborhood U of p in X there is a neighborhood V of p in X such that each map of Bd D into $V \cap Y$ can be shrunk to a constant in $U \cap Y$.

<u>Theorem 5.7</u> If A is an arc in E^n whose complement fails to be locally simply connected at an endpoint p of A, then $A \times [0,1] \subset E^n \times E^1 = E^{n+1}$ is a wild disk in E^{n+1}.

<u>Proof.</u> Assume $A \times [0,1]$ is tame in E^{n+1} and reach a contradiction. Since $A \times [0,1]$ is tame there exists $t \in (0,1)$ such that if $p' = p \times t$ then $A \times [0,1]$ is locally flat at p'. Hence $E^{n+1} - (A \times [0,1])$ is locally simply connected at p'. $A \times t$ is embedded in $E^n \times t$ as A is embedded in E^n. A contradiction will be reached by proving that $E^n \times t - A \times t$ is locally simply connected at p', and hence that $E^n - A$ is locally simply connected at p.

Let U be any neighborhood of p' in $E^n \times t$, then $U' = U \times (0,1)$ is a neighborhood of p' in E^{n+1}. Since $E^{n+1} - (A \times [0,1])$ is locally simply connected at p', there exists a neighborhood V' of p' in E^{n+1} such that each map of Bd D into $V' \cap (E^{n+1} - (A \times [0,1]))$ can be shrunk to a constant in $U' \cap (E^{n+1} - (A \times [0,1]))$. Let V be a neighborhood of p' in $E^n \times t$ such that $V \subset V'$, and prove that each map of Bd D into $V \cap (E^n \times t - A \times t)$ can be shrunk to a constant in $U \cap (E^n \times t - A \times t)$. Let f be any mapping of Bd D into $V \cap (E^n \times t - A \times t)$, then f(Bd D) is contained in $V' \cap (E^{n+1} - (A \times [0,1]))$. Hence f can be extended to a map, denoted by f', where f' maps D into $U' \cap (E^{n+1} - (A \times [0,1]))$. Let α be the projection of E^{n+1} onto $E^n \times t$; α is a continuous map. Then $\alpha f'$ maps D into $U \cap (E^n \times t - A \times t)$ and is an extension of f. Thus $E^n \times t - A \times t$ is locally simply connected at p'. Hence $E^n - A$ is locally simply connected at p which is a contradiction to the hypothesis of the theorem. So $A \times [0,1]$ is a wild disk in E^{n+1}.

If A is the arc of example 1.1 of Fox and Artin [22], then $E^3 - A$ is not locally simply connected at an endpoint of A. The proof of this fact may be readily obtained by considering the presentation of the fundamental group of the complement of the arc given in [22].

Theorem 5.8 There exists a wild disk in a tamely embedded cone over an n-book, $n > 2$, in E^4.

Proof. Example 1.1 of [22] can be embedded in a polyhedral 3-book, $B^3 \subset E^3$. Then $B^3 \times [-2,2] \subset E^4$ is homeomorphic to the cone over B^3 and is tamely embedded in E^4. By the remarks preceding this theorem and Theorem 5.7, it follows that $A \times [0,1]$ is a wild disk contained in a tamely embedded cone over a 3-book.

Next it will be established that there exist wild 3-cells in tamely embedded cones over n-books, $n > 2$.

Theorem 5.9   If D is a 2-cell in E^3 whose complement fails to be locally simply connected at a point p of Bd D, then D x [0,1] is a wild 3-cell in E^4.

Proof.   Assume D x [0,1] is tamely embedded and reach a contradiction. Let h be a homeomorphism of E^4 onto itself such that h(D x [0,1]) is a Euclidean polyhedron with triangulation K.

If there exists $t \in (0,1)$ such that h(p x t) lies in the interior of a 2-simplex of K, then E^4 - (D x [0,1]) is locally simply connected at p x t. If, however, h(p x [0,1]) is contained in the 1-skeleton of K then there exists $t \in (0,1)$ such that h(p x t) is contained in the interior of a 1-simplex of K. The point-set realization of all those 3-simplexes of K that have this 1-simplex as a face is a closed 3-cell by Lemma 5.4. Furthermore, by [27] and Lemma 5.4, it is a flat 3-cell. Hence E^4 - (D x [0,1]) is locally simply connected at p x t. Using the same argument as in the proof of Theorem 5.7, one obtains that D is locally simply connected at p. Thus a contradiction is reached so D x [0,1] is wildly embedded in E^4.

Theorem 5.10 There exists a wild 3-cell in a tamely embedded cone over an n-book in E^4, $n > 2$.

Proof. Example 1.1 of [22] can be swollen in E^3 into a 2-cell D whose complement in E^3 fails to be locally simply connected at an endpoint p of Example 1.1, which by the "swelling construction" lies on the boundary of the 2-cell. Furthermore, D can be embedded in a polyhedral 3-book in E^3. Then D x [0,1] lies in a tamely embedded cone over an n-book (as in the proof of Theorem 5.8). By Theorem 5.9 D x [0,1] is a wild 3-cell in E^4.

To obtain an example of a wild 2-sphere in a tamely embedded cone over an n-book, $n > 2$, that is constructed in a somewhat different manner then the wild 2- and 3-cells above, we utilize an example of Doyle and Hocking [19].

Let B^3 be a flat 3-book in E^3. Let B_i^3, $i \in Z^+$, be a sequence of disjoint 3-books embedded in B^3 such that: (1) the leaves of B_i^3 are Euclidean 2-simplexes when considered embedded in E^3, (2) the books B_i^3 converge to a point p on the back of B^3, and (3) the diameter of B_i^3 is less than $1/2^i$. Denote the back of B_i^3 by B_i, $i \in Z^+$. By [40] a trefoil knot can be polyhedrally embedded in a 3-book. Let T_i, $i \in Z^+$, be a polygonal trefoil knot embedded in B_i^3 in such a manner that T_i is contained in the interior of B_i^3 except for two straight line segments t_i and s_i on the boundary of a leaf of B_i^3 and in such a manner that $t_i \cup s_i \subset D_1$, a leaf of B^3, for all $i \in Z^+$. Let E_i, $i \in Z^+$, be a polygonal disk in B^3 such that: (1) $E_i \subset D_1$ for all $i \in Z^+$, (2) $E_i \cap T_i = t_i$ and $E_i \cap T_{i+1} = s_{i+1}$, (3) $E_i \cap \bigsqcup_{j=1}^{\infty} B_j^3 = t_i \cup s_{i+1}$, (4) $E_i \cap E_j = \emptyset$ for $i \neq j$, and (5) the diameters of the E_i tend to zero. See Figure 5.1.

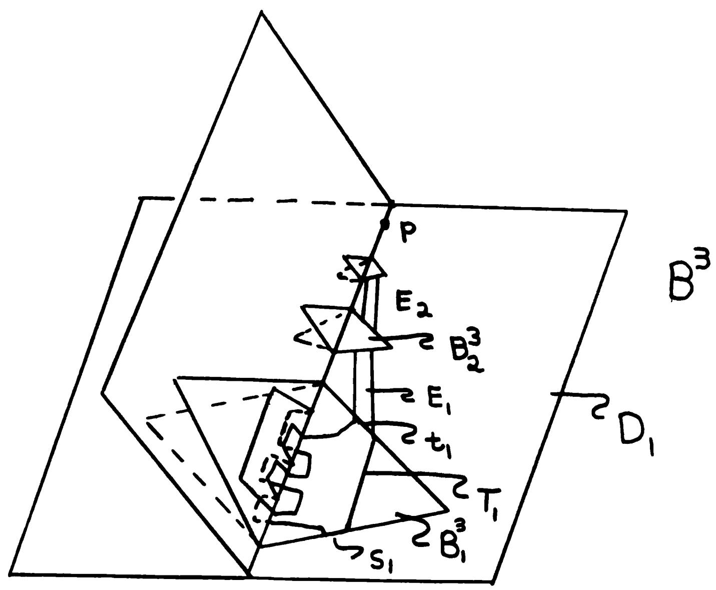


Figure 5.1

Let B^3 be embedded in the 3-dimensional hyperplane $E^3 \times 0 \subset E^4$. Let $\Sigma'(B^3)$ denote the suspension of B^3 in E^4 with suspension points u and v, and let $C_u(B_i)$ denote the cone over B_i which is obtained by joining points in B_i to u, and $C_v(B_i)$ denote the cone over B_i which is obtained by joining points on B_i to v. Let v_i be a point in $C_v(B_i) - (B_i \cup \{v\})$ and let u_i denote a point in $C_u(B_i) - (B_i \cup \{u\})$. Choose the points v_i and u_i, $i \in Z^+$, so that the sequences $\{u_i \mid i \in Z^+\}$ and $\{v_i \mid i \in Z^+\}$ converge to p. Let $\Sigma'(B_i^3)$ denote the suspension of B_i^3 in E^4 with the same suspension points as B^3, then $\Sigma'(B_i^3) \subset \Sigma'(B^3)$. If $\Sigma(T_i)$ denotes the suspension of T_i in E^4 with suspension points u_i and v_i, then $\Sigma(T_i) \subset \Sigma'(B_i^3)$. This fact can be verified using the convexity of the cones over the leaves of B_i^3 in $\Sigma'(B_i^3)$ and the fact that u_i and v_i were chosen in the suspension of the back of B_i^3. Hence $\{\Sigma(T_i) \mid i \in Z^+\}$ consists of a sequence of disjoint 2-spheres converging to the point p. These 2-spheres are now joined together in such a manner that a wild 2-sphere is obtained. The E_i can be swollen into a polyhedral 2-spheres E_i' in E^4 containing E_i such that: (1) $E_i' \subset \Sigma'(D_1)$, (2) $E_i' \cap E_j' = \emptyset$ for $i \neq j$, (3) $E_i' \cap \Sigma(T_i)$ is a polyhedral 2-cell containing t_i and $E_i' \cap \Sigma(T_{i+1})$ is a polyhedral 2-cell containing s_i, (4) the diameters of the E_i' converge to zero, and (5) $S = \bigsqcup_{i=1}^{\infty} (\Sigma(T_i) \cup E_i')$ is locally polyhedral except at p.

Then S is homeomorphic to a 2-sphere and is the example of Doyle and Hocking [19]. Furthermore by construction $S \subset \Sigma'(B^3)$, and $\Sigma'(B^3)$ is homeomorphic to the cone over B^3. $\Sigma'(B^3)$ is tamely embedded in E^4. A tamely embedded 2-sphere in E^4 can fail to be locally flat at only finitely many points, the vertices of a triangulation of the 2-sphere. S fails to be locally flat at the sequences $\{v_i \mid i \in Z^+\}$ and $\{u_i \mid i \in Z^+\}$ which converge to p. Hence S is wildly embedded in E^4. The above discussion yields the following theorem.

Theorem 5.12 There exists a wild 2-sphere in a tamely embedded cone over an n-book, $n > 2$, in E^4.

It is interesting to note that the wild 2-sphere S constructed above is locally tame everywhere except at p and that S fails to be locally flat on a sequence of points converging to p. Furthermore, every arc embedded in S is tamely embedded in E^4. Also S can be expressed as the union of either two wild 2-cells or two tame 2-cells. The wild 2- and 3-cells in tame cones over n-books, $n > 2$, constructed in this chapter were both products of cells; therefore, they are both cellular.

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