AUTOMORPHISMS FIXING SUBNORMAL AND NORMAL SUBGROUPS

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

Alphonse H. Baartmans

1966

This is to certify that the

thesis entitled

Automorphisms Fixing Subnormal
and Normal Subgroups

presented by

Alphonse H. Baartmans

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

J. Adney
Major professor

Date May 5, 1967

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ABSTRACT

AUTOMORPHISMS FIXING SUBNORMAL
AND NORMAL SUBGROUPS

by Alphonse H. Baartmans

One of the main objects in the study of finite group theory is the group of automorphisms of a finite group. Our interest here centers around the set of all automorphisms that fix chains of subgroups of a group. In particular, we will consider the set of all automorphisms B_0 that fix every composition series, as well as the set of automorphisms that fix all chief series of a group. If $s : G = G_0 > G_1 > \cdots > G_n = 1$ is a composition series of a solvable group, we define recursively the following:

$$S_0(s) = \{\theta \in A(G)/G_i^\theta = G_i \quad \text{for} \quad i = 0,1,2, \ldots, n\}$$

$$S_k(s) = \{\theta \in A(G)/\theta/G_{k-1}/G_k = 1 \quad \text{for} \quad 1 \leq k \leq n\}.$$

If we let $C(G)$ denote the class of all composition series of a finite solvable group, we may define $B_0 = \bigcap_{s \in C(G)} S_0(s)$ as well as $B_i = \bigcap_{s \in C(G)} S_i(s)$. In a similar manner, if we let $D(G)$ denote the class of all chief series of a group G, we may define $A_0 = \bigcap_{s \in D(G)} S_0(s)$ and $A_i(G) = \bigcap_{s \in D(G)} S_i(s)$.

In the course of study of a set of automorphisms E, we are led to consider two special subgroups of the group G; the group $F(G;E)$ which consists of all $x \in G$ such that $x^\theta = x$ for all $\theta \in E$, and the group $M(G;E) = \langle x^\theta x^{-1} / x \in G; \theta \in E\rangle$. If one can say something about

the groups $F(G;E)$ and $M(G;E)$, then presumably one can say something about how the automorphism acts on the group G. In Chapter I, we will prove some elementary results about the groups $F(G;E)$ and $M(G;E)$.

In Chapter II, we determine what conditions are imposed on the groups B_i if we assume that G is abelian, nilpotent, supersolvable and finally solvable. Some of the results obtained are:

(1) If G is nilpotent π-group, then:

(i) B_0 is abelian

(ii) $B_1 = B_2 = \dots = B_n$ and is a π-group.

Our main interest is to determine what the structure of the groups $M(G;B_0)$ and B_0 must be if G is a solvable group. To this end we prove the following results for $M(G;B_0)$:

(2) Let G be an arbitrary group.

(i) If H is a nilpotent subnormal subgroup of G, then $M(G;B_0)$ normalizes every subgroup of H.

(ii) If H is an abelian normal subgroup of G, then $M(G;B_0)$ centralizes H.

By using result (2) effectively, we obtain the following result for $M(G;B_0)$ for a solvable group G.

(3) If G is a solvable group and F^* denotes the Fitting subgroup of G, then:

(i) $M(G;B_0)'$ is an abelian group

(ii) $M(G;B_0)' \leq Z(F^*)$

(iii) $M(G;B_o)$ is a normal subgroup of G, and is nilpotent of class ≤ 2

(iv) $M(M(G;B_o);B_o) \leq Z(F^*)$.

By using result (3), we are in a position to characterize the group B_o by means of the following results:

(4) If G is a solvable π-group then:

(i) B_o is supersolvable

(ii) B_o' is an abelian π-group

(iii) B_o normalizes every subgroup of B_o'

(iv) B_o has a unique maximal π-subgroup B_o^*, which is the Hall π-subgroup of B_o.

(v) A π'-subgroup of B_o is abelian.

Upon completion of the above, we turn our attention to the groups A_i. Some of the results obtained for A_i are the following:

(5) If G is a p-group, then:

(i) The Sylow-p subgroup, A_o^* of A_o, is normal in $A(G)$

(ii) A_o' is a p-group of class $\leq$ n-1 and $A_o' \leq A_n$

(iii) $A_o^* = A_n$

(iv) A p'-subgroup of A_o is abelian

(v) A_o splits over A_o^*

(vi) If H is a normal nilpotent subgroup of A_o, then H is a p-group

Next, we investigate $A_o(G)$ in case G is nilpotent or supersolvable.

We conclude the Chapter with a theorem relating A_0, B_0 and $I(G)$. We obtain:

(6) If G is a solvable group, then:

(i) Every normal subgroup of $I(G)$ is normal in A_0

(ii) B_0 belongs to the norm of $I(G)$.

In Chapter III we try to determine what conditions are imposed on the group G, if we assume that G admits an automorphism that fixes all subnormal subgroups. In particular, we will investigate how the groups $F(G;E)$ and $M(G;E)$, for E, a subgroup of B_0, are imbedded in G. We obtain:

(7) If $\theta \in B_0(G)$ and $F = F(G;\theta)$ and $M = M(G;\theta)$, then the following are equivalent:

(i) $F \cap M = 1$

(ii) G is generated by F and M

(iii) G is a semi-direct product of F and M

(iv) $M = M(M;\theta) = M(F^*;\theta)$, where F^* is the Fitting subgroup of G.

(8) Let E be a subgroup of $B_0(G)$ and let $F = F(G;E)$ and $M = M(G;E)$. If G is generated by F and M, then:

(i) $F \cap M = 1$

(ii) $M = M(M;E) = M(F^*;E)$, where F^* is the Fitting subgroup of G

(iii) M is a Hall subgroup of F^* and $M \leq Z(F^*)$

(iv) F^* is generated by M and $F(F^*;E)$

(v) Every $\theta \in E$ is a power automorphism on F^*.

In Chapter II, we show that a group may admit an automorphism $\theta \in B_0$ such that $(|\theta|, |G|) = 1$. For these types of automorphisms, as well as a more general class of automorphisms, we obtain:

(9) If $\theta \in B_0$, such that $(|\theta|, |M(G;\theta)|) = 1$, then:

(i) All conclusions of (7) hold.

(10) If E is a subgroup of B_0, such that $(|E|, |M(G;E)|) = 1$, then:

(i) All conclusions of (8) hold.

(11) If $(|B_0|, |G|) = 1$, then:

(i) $G = M(G;E) \times F(G;E)$ and $M(G;E) \leq Z(G)$ and $F(G;E) \leq G'$.

(ii) $F(G;E)$ is a normal Hall subgroup of G.

(iii) $M(G;B_0)$ is abelian and its p-Sylow subgroups are elementary abelian

(iv) $B_0(F(G;B_0)) = 1$.

Next, we investigate the inner automorphisms of G that fix all subnormal subgroups. In particular, we study the group $\overline{N}$, having the property that if $g \in \overline{N}$, then the inner automorphism induced by g fixes all subnormal subgroups. We obtain:

(12) If G is solvable, then:

(i) $\overline{N}$ is supersolvable

(ii) If H is subnormal in $\overline{N}$, then H is normal in $\overline{N}$

(iii) $\overline{N}'$ is abelian and every subgroup of $\overline{N}'$ is normal in $\overline{N}$

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(iv) $\overline{N}' \leq Z(F^*)$.

We conclude the Chapter by investigating the structure of G, if we assume that $I(G) \leq B_0$ or, equivalently, that $\overline{N} = G$.

AUTOMORPHISMS FIXING SUBNORMAL AND NORMAL SUBGROUPS

By

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A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

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PREFACE

The author wishes to express his appreciation for the encouragement and advice given to him by his major professor, Dr. J. E. Adney. This advice and suggestion, given to me during the research and in the preparation of this thesis, was of immeasurable value.

A note of thanks should go to Dr. W. E. Deskins, for his assistance in the completion of the dissertation.

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INTRODUCTION

Our main object of investigation is the study of the automorphisms of a finite group G that leave chains of subgroups invariant. In particular, we will center our attention on the set B_0 of all automorphisms that fix all composition series of a group, as well as on the set A_0 of all automorphisms that fix all chief series of a group G.

In Chapter I, we define the groups B_0 and A_0 and we define recursively the groups A_i and B_i. In the study of a set of automorphisms E of a group G, we are led to consider two special subgroups of the group G; the group $F(G;E)$, which consists of all $x \in G$ such that $x^{\theta} = x$ for all $\theta \in E$, and the group $M(G;E) = \langle x^{\theta}x^{-1} / x \in G, \theta \in E \rangle$. In Chapter I we will determine some elementary properties of the groups $F(G;E)$ and $M(G;E)$.

In Chapter II, we determine what conditions are imposed on the groups B_i if we assume that the group G is abelian, nilpotent, supersolvable and finally solvable. For a solvable group G, we shall determine the structure of the group B_0 and the structure of the group $M(G;B_0)$. We will then focus our attention on the group A_0 for the case that G is a p-group and after that, for the case that G is nilpotent.

In Chapter III, we determine what conditions are imposed on the group G, if we assume that G admits an autom

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We turn our attention next to the inner automorphisms that fix all composition series of the group G. In particular, we will investigate the group $\overline{N}$, having the property that if $x \in \overline{N}$ then the inner automorphism induced by x fixes all subnormal subgroups. We will conclude the chapter by investigating what the structure of a solvable group must be if every inner automorphism fixes all subnormal subgroups.

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CHAPTER I

BASIC PROPERTIES AND DEFINITIONS

If G is a group, we let $A(G)$ denote the set of automorphisms of the group G and $I(G)$ the set of inner automorphisms of G. If α is an automorphism of G, we will define invariance under α as follows:

Definition 1.1: A subgroup H is invariant under α or fixed by α iff for every element h of H, h^α is an element of H. If H is invariant under α, we shall denote this by $H^\alpha = H$. A subgroup H is fixed elementwise by α iff for every h, and element of H, $h^\alpha = h$.

We will be concerned with automorphisms that leave chains of subgroups invariant. For an arbitrary chain of subgroups of a group G, we define the following:

Definition 1.2: Let s be a chain of subgroups of the group G of length n; $s: G = G_o > G_1 > G_2 \cdots > G_n = 1$ which terminates in the identity. As in [4] we define $S_o(s) = \{\theta \in A(G) / G_i^\theta = G_i \text{ for } i = 1, 2, \cdots, n\}$ and $S_i(s) = \{\theta \in S_{i-1}(s) / (G_i x)^\theta = G_i x \text{ for all } x \in G_{i-1}\}$.

Definition 1.3: If s is a chain of subgroups of a group G; $s: G > G_1 \cdots > G_{n-1} > G_n = 1$. We define as in [1] the stability group $A(s)$ of the chain s as follows:

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$A(s) = \{\theta \in A(G) \ / \ (G_i x)^\theta = G_i x$ for all $x \in G_{i-1}$ and for $i = 1, 2, \ldots, n\}$.

We note that the stability group of a chain s of length n is the group $S_n(s)$ of Definition 1.2.

For an arbitrary chain s of length n, terminating in the identity, Philip Hall in [1] has shown that $A(s)$ is nilpotent and of class $\leq 1/2\, n(n-1)$.

As in [4] we will be mainly concerned with chains of normal and subnormal subgroups of G; in particular, chief series and composition series of G. If we let $C(G)$ denote the class of all composition series of G, we may define $B_o(G)$ thus:

<u>Definition 1.4</u>: $B_o(G) = \bigcap_{s \in C(G)} S_o(s)$ and $B_i(G) = \bigcap_{s \in C(G)} S_i(s)$; $i = 1, 2, \ldots, n$.

As in [4] we have that each $B_i = B_i(G)$ is a normal subgroup of the automorphisms group of G and that $B_o = B_o(G) = \{\theta \in A(G) \ / \ H^\theta = H$ for all subnormal subgroups H of $G\}$.

If we let $D(G)$ denote the class of chief series of G, we may similarly define $A_o(G)$ as follows:

<u>Definition 1.5</u>: $A_o(G) = \bigcap_{s \in D(G)} S_o(s)$ and $A_i(G) = \bigcap_{s \in D(G)} S_i(s)$.

Again we have that each $A_i = A_i(G)$ is a normal subgroup of the automorphism group of G and that $A_o = A_o(G) = \{\theta \in A(G) \ / \ H^\theta = H$ for all normal subgroups H of $G\}$.

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In what follows, we will have frequent occasion to refer to the following subgroup of the automorphism group:

<u>Definition 1.6</u>: The dilation group $\Delta(G)$ of a group G is the set of all automorphisms θ of G that leave every subgroup of G invariant. In other words, $\Delta(G) = \{\theta \in A(G)/H^{\theta} = H$ for all subgroups H of $G\}$.

If θ is an automorphism of G that fixes all subgroups of G, then θ will surely fix all subnormal subgroups of G. Furthermore, if θ fixes all subnormal subgroups of G, then θ must fix all normal subgroups of G. Therefore $\Delta(G) \leq B_0(G) \leq A_0(G)$.

We will exhibit the subgroups B_0, A_0, and $\Delta(G)$ by means of two examples.

<u>Example 1</u>: Let $G = \langle a,b/a^3 = b^2 = 1,\ b^{-1}\ ab = a^2\rangle$, then G is the symmetric group on three letters, $A(G) = \langle \alpha,\beta/\alpha^3 = \beta^2 = 1;\ \beta^{-1}\alpha\beta = \alpha^2\rangle$ where $\alpha\colon a \to a\ ;\ b \to ab$

$\beta\colon a \to a^2;\ b \to b.$

Since the alternating group on three letters is the only subnormal subgroup of G, we must have $A(G) = A_0 = B_0$, and $\Delta(G) = 1$.

The next example shows that A_0, B_0, and $\Delta(G)$ may all be distinct.

<u>Example 2:</u> Let $G = A_4$ be the alternating group on four letters. $A_4 = \langle a,b,c/a^2 = b^2 = c^3 = 1;\ c^{-1}ac = ab;\ c^{-1}bc = a\rangle$ then $A_0 = A(G)$. $B_0(G) = \langle \alpha,\beta\rangle$ where:

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$\Delta(G) = 1$, and we have that $\Delta(G) \lneq B_o(G) \lneq A_o(G)$.

In what follows we will have frequent occasion to use the subgroups $M(G;E)$ and $F(G;E)$ which are defined below:

<u>Definition 1.7</u>: Let E be a complex of elements of $A(G)$ of a group G; an element $g \in G$ such that $g^\theta = g$ for all $\theta \in E$ is called an E-fixed element. The E-fixed elements form a group $F(G;E)$, the E-fixed subgroup and $F(G;E) = \{g \in G \,/\, g^\theta = g \text{ for all } \theta \in E\}$.

<u>Definition 1.8</u>: Let E be a complex of elements of $A(G)$ of a group G. Then an element $g^\theta g^{-1}$ is called an E-multiplier element. The group generated by the E-multiplier elements is called the E-multiplier group and will be denoted by $M(G;E) = \langle g^\theta g^{-1} / g \in G \text{ and } \theta \in E \rangle$.

<u>Theorem 1.9</u>: If E is a complex of elements of $A(G)$, then $M(G;E)$ is a normal subgroup of G. Moreover, $M(G;E)$ is invariant under E and is the smallest normal subgroup whose factor group remains fixed, elementwise, by E.

<u>Proof</u>: Let $g \in G$. Consider $(g^\theta g^{-1})^h$ where h is an arbitrary element of G. $(g^\theta g^{-1})^h = h^{-1} g^\theta g^{-1} h$

$$= h^{-1}h^\theta (h^\theta)^{-1} g^\theta g^{-1} h = h^{-1}h^\theta (h^{-1}g)^\theta (h^{-1}g)^{-1}$$
$$= ((h^{-1})^\theta h)^{-1} (h^{-1}g)^\theta (h^{-1}g)^{-1}$$

therefore $(g^\theta g^{-1})^h \in M(G;E)$. Since $M(G;E) = \langle g^\theta g^{-1} / g \in G;\ \theta \in E \rangle$.

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we must have $M(G;E)^h = M(G;E)$; but h was arbitrary. Consequently $M(G;E)$ is normal in G.

Let $g^\theta g^{-1}$ be a generator of $M(G;E)$, and let $\alpha \in E$.
Then:
$$\begin{aligned}(g^\theta g^{-1})^\alpha &= g^{\theta\alpha} (g^\alpha)^{-1}\\ &= g^{\theta\alpha} (g^\theta)^{-1} g^\theta (g^{-1}g) (g^\alpha)^{-1}\\ &= [(g^\theta)^\alpha (g^\theta)^{-1}] (g^\theta g^{-1}) (g^\alpha g^{-1})^{-1}.\end{aligned}$$

Since $(g^\theta)^\alpha(g^\theta)^{-1}$, $g^\theta g^{-1}$ and $g^\alpha g^{-1}$ all belong to $M(G;E)$, we must have that $(g^\theta g^{-1})^\alpha$ is an element of $M(G;E)$. Consequently, $M(G;E)$ is invariant under E.

Suppose H is a normal subgroup of G such that G/H is fixed elementwise by E. This implies that if Hg is a coset of H in G, then $(Hg)^\theta = Hg^\theta = Hg$ or $g^\theta g^{-1} \in H$ for arbitrary $g \in G$ and $\theta \in E$. Hence $M(G;E) = \langle g^\theta g^{-1}/g \in G;\ \theta \in E\rangle \leq H$ and the result follows.

<u>Definition 1.10</u>: E is said to be a normal complex of $A(G)$ if $\alpha^{-1}\theta\alpha \in E$ for all $\alpha \in A(G)$ and for all $\theta \in E$.

<u>Theorem 1.11</u>: Let E be a normal complex of $A(G)$.
Then:
(1) $M(G;E)$ is characteristic in G.
(2) $F(G;E)$ is characteristic in G.

<u>Proof</u>: Let $g \in G$, $\theta \in E$ and $\alpha \in A(G)$. Then $(g^\theta g^{-1})^\alpha = (g^\alpha)^{\alpha^{-1}\theta\alpha}(g^\alpha)^{-1}$, but $\alpha^{-1}\theta\alpha \in E$ since E is a normal complex of $A(G)$; consequently $(g^\theta g^{-1})^\alpha \in M(G;E)$ and $M(G;E)$ is characteristic in G.

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Let $g \in F(G;E)$ and $\alpha \in A(G)$. If $\theta \in E$. $\alpha\theta\alpha^{-1} = \overline{\theta}$ is an element of E. Therefore $\alpha\theta = \overline{\theta}\alpha$. Hence $(g^{\alpha})^{\theta} = (g^{\overline{\theta}})^{\alpha}$, but $\overline{\theta} \in E$ whence $g^{\overline{\theta}} = g$. Hence $(g^{\alpha})^{\theta} = (g^{\overline{\theta}})^{\alpha} = g^{\alpha}$ or $g^{\alpha} \in F(G;E)$ and $F(G;E)$ is characteristic in G.

The next theorem shows that we only need to consider subgroups of $A(G)$.

<u>Theorem 1.12</u>: If E is a complex of elements of $A(G)$, then:

(1) $M(G;E) = M(G;\langle E\rangle)$

(2) $F(G;E) = F(G;\langle E\rangle)$.

<u>Proof</u>: If α and β are elements of E, then

$g^{-1}g^{\alpha\beta} = (g^{-1}g^{\alpha})\ [(g^{\alpha})^{-1}(g^{\alpha})^{\beta}] \in M(G;E)$.

Therefore $M(G;\langle E\rangle) \leq M(G;E)$. Since $E \leq \langle E\rangle$, we have $M(G;E) \leq M(G;\langle E\rangle)$ and therefore $M(G;E) = M(G;\langle E\rangle)$.

If $g \in F(G;E)$, then, for α and β elements of E, $g^{\alpha\beta} = (g^{\alpha})^{\beta} = g^{\alpha} = g$ and therefore $F(G;E) \leq F(G;\langle E\rangle)$. Consequently $F(G;E) = F(G;\langle E\rangle)$.

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CHAPTER II

STRUCTURE AND PROPERTIES OF THE GROUPS B_i AND A_i

Introduction:

The aim of this chapter is to investigate the properties of B_i and A_i. We will start with abelian groups, extend our arguments to nilpotent groups and finally determine the structure and certain properties of the B_i and A_i for supersolvable and solvable groups.

I. The Structure and Properties of the Groups B_i, $i = 1,2, \ldots, n$.

Theorem 2.1: If G is a direct product of groups H and K, E a subgroup of B_0, E_H the restriction of E to H, E_K the restriction of E to K, then:

(1) $M(G;E) = M(H;E_H) \times M(K;E_K)$

(2) $F(G;E) = F(H;E_H) \times F(K;E_K)$.

If $(|H|, |K|) = 1$ and $E = B_0$, then:

(3) $A(G) = A(H) \times A(K)$

(4) $B_0(G) \leq B_0(H) \times B_0(K)$.

If G is nilpotent and $(|H|, |K|) = 1$, then:

(5) $B_0(G) = B_0(H) \times B_0(K)$

(6) $A(G)/B_0(G) = A(H)/B_0(H) \times A(K)/B_0(K)$

(7) If $M(H;B_0(H)) \leq Z(H)$ and $M(K;B_0(K)) \leq Z(K)$, then: $M(G;B_0(G)) \leq Z(G)$.

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(8) If $F(H;B_0(H)) \geq H'$ and $F(K;B_0(K)) \geq K'$, then $F(G;B_0(G)) \geq G'$.

Proof: If $G = H \times K$, then every element $g \in G$ has a unique representation of the form $g = hk$ with $h \in H$, $k \in K$. The groups H and K are normal subgroups of G; consequently if $\theta \in E$ then $H^\theta = H$ and $K^\theta = K$. Let $g^\theta g^{-1}$ be an E-multiplier element of G. Then

$$\begin{aligned} g^\theta g^{-1} &= (hk)^\theta (hk)^{-1} \\ &= h^\theta k^\theta k^{-1} h^{-1} \\ &= h^\theta h^{-1} k^\theta k^{-1} \end{aligned}$$

since the elements of H and K permute. The elements $h^\theta h^{-1}$ and $k^\theta k^{-1}$ belong to $M(H;E_H)$ and $M(K;E_K)$. Consequently $M(G;E) = \langle g^\theta g^{-1} / g \in G;\ \theta \in E\rangle \leq M(K;E_K) \times M(H;E_H)$. Conversely $M(H;E_H)$ and $M(K;E_K)$ are subgroups of $M(G;E)$; moreover, they are normal subgroups of G. For if $g \in G$, then $g = hk$ with $h \in H$, $k \in K$. If $u \in H$, then

$$\begin{aligned} g^{-1}(u^{-1}u^\theta)g &= k^{-1}h^{-1}(u^{-1}u^\theta)hk \\ &= h^{-1}(u^{-1}u^\theta)h \\ &= [(uh)^{-1}(uh)^\theta](h^{-1})^\theta h \in M(H;E_H). \end{aligned}$$

Consequently $M(H;E_H) = \langle u^\theta u^{-1}/u \in H;\ \theta \in E\rangle$ is normal in G. By a similar argument we obtain that $M(K;E_K)$ is normal in G. Since $M(H;E_H) \leq H$ and $M(K;E_K) \leq K$, we have that $M(H;E_H) \cap M(K;E_K) \leq H \cap K = 1$. Consequently the union of

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$M(H;E_H)$ and $M(K;E_K)$ is a direct product and $M(H;E_H) \times M(K;E_K) \leq M(G;E)$. Therefore $M(G;E) = M(H;E_H) \times M(K;E_K)$.

If $g \in F(G;E)$, then $g = hk$ with $h \in H$ and $k \in K$. Since

$$\begin{aligned} 1 &= g^{\theta} g^{-1} \\ &= (hk)^{\theta} (hk)^{-1} \\ &= (h^{\theta} h^{-1})(k^{\theta} k^{-1}), \end{aligned}$$

the last equality implies that $h^{\theta}h^{-1} = 1$ and $k^{\theta}k^{-1} = 1$. Hence $h \in F(H;E_H)$ and $k \in F(K;E_K)$, and therefore $F(G;E) \leq F(H;E_H) \times F(K;E_K)$. Conversely $F(H;E_H)$ and $F(K;E_K)$ are normal subgroups of $F(G;E)$; for if $h \in F(G;E_H)$, $g = h_1k_1 \in F(G;E)$, then

$$g^{-1}hg = k_1^{-1}h_1^{-1}hh_1k_1 = h_1^{-1}hh_1.$$

Consequently $F(G;E_H)$ is normal in $F(G;E)$. By a similar argument we obtain that $F(K;E_K)$ is normal in $F(G;E)$. Therefore $F(G;E_H) \times F(K;E_K) \leq F(G;E)$ and (2) follows.

Let $G = H \times K$ with $(|H|,|K|) = 1$, then H and K are characteristic subgroups of G. If θ_H is an automorphism of H and θ_K is an automorphism of K, then $\theta = \theta_H \cdot \theta_K$ is an automorphism of G. The product $\theta = \theta_H \cdot \theta_K$ is to be interpreted as follows: θ acts on H as θ_H does and on K as θ_K does. Then θ_H is an automorphism of H. Similarly θ restricted to K, θ_K, is an automorphism of K. Therefore $A(G) \geq A(H) \times A(K)$. If θ is an automorphism of G then θ restricted to H, θ_H, is an automorphism of H. Similarly θ restricted to K, θ_K, is an automorphism of K. Therefore $A(G) \leq A(H) \times A(K)$ and therefore $A(G) = A(H) \times A(K)$.

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Every automorphism $\theta \in B_0$ may now be written in the form $\theta = \theta_H \cdot \theta_K$ where $\theta_H = \theta/H$ and $\theta_K = \theta/K$. If θ_H does not fix all subnormal subgroups of H, then θ does not fix all subnormal subgroups G, similarly for θ_K. Therefore $B_0(G) \lesssim B_0(H) \times B_0(K)$.

If $\theta_H \in B_0(H)$, $\theta_K \in B_0(K)$, where θ_K induces the identity on H and θ_H induces the identity on K, then $\theta = \theta_H \cdot \theta_K$ is an automorphism of G. If G is nilpotent, by Theorem [T-1], every subgroup of G is subnormal in G; hence if $\theta \in B_0(G)$, then θ must fix all subgroups of G; hence if $y \in G$, then $y^\theta = y^{s(y;\theta)}$, where $(s(y;\theta), |y|) = 1$. If $g = hk$ is an element of G, then

$$g^\theta = (hk)^\theta.$$

If $\theta = \theta_H \cdot \theta_K$ then

$$\begin{aligned} g^\theta &= h^{\theta_H} k^{\theta_K} \\ &= h^{s(h;\theta)} k^{s(k;\theta)}, \quad \text{where} \end{aligned}$$

$(s(h;\theta), |h|) = 1$ and $(s(k;\theta), |k|) = 1$. If t is an integer, then $g^t = (hk)^t = h^t k^t$. Consider the system:

$$\begin{aligned} t &\equiv s(h;\theta) \bmod |h| \\ t &\equiv s(k;\theta) \bmod |k|. \end{aligned}$$

This has a unique solution, $t \equiv t_0 \bmod |g| = |h||k|$. Consequently $g^\theta = (hk)^\theta = h^\theta k^\theta = h^{s(h;\theta)} k^{s(k;\theta)} = h^{t_0} k^{t_0} = g^{t_0}$. Hence θ fixes every cyclic subgroup of G and θ is a dilation of G. Consequently $B_0(G) = B_0(H) \times B_0(K)$, and (5) follows.

From (3) and (5) we obtain (6).

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$Z(G) = Z(H) \times Z(K)$ and $G' = H' \times K'$, together with (1) and (2) give (7) and (8).

<u>Corollary 2.2</u>: If a group G is the direct product of subgroups $H_1, H_2, \cdots, H_n$, E a subgroup of B_0 and E_i the restriction of E to H_i, $i = 1,2,\cdots,n$, then:

(1) $M(G;E) = M(H_1;E_1) \times M(H_2;E_2) \ldots \times M(H_n;E_n)$.

(2) $F(G;E) = F(H_1;E_1) \times F(H_2;E_2) \ldots \times F(H_n;E_n)$.

If the orders of the H_i are relatively prime in pairs and $E = B_0(G)$, then:

(3) $A(G) = A(H_1) \times A(H_2) \ldots \times A(H_n)$.

(4) $B_0(G) \leq B_0(H_1) \times B_0(H_2) \ldots \times B_0(H_n)$.

If the orders of the H_i are relatively prime in pairs and G is nilpotent, then:

(5) $B_0(G) = B_0(H_1) \times B(H_2) \ldots \times B(H_n)$.

(6) $A(G)/B_0(G) = A(H_1)/B_0(H_1) \times A(H_2)/B_0(H_2) \ldots \times A(H_n)/B_0(H_n)$.

(7) If $M(H_i;B_0(H_i)) \leq Z(H_i)$, then $M(G;B_0) \leq Z(G)$.

(8) If $F(H_i;B_0(H_i)) \geq H_i'$, then $F(G;B_0) \geq G'$.

The proof of the Corollary is the same as that of Theorem 2.1. With the aid of Theorem 2.1 and Corollary 2.2 we are now in a position to discuss the B_i for abelian and nilpotent groups.

(A) <u>$B_0, B_1, B_2, \cdots, B_n$ for abelian groups.</u>

If G is an abelian group, then G is a direct product of its Sylow-subgroups. Since the orders of these Sylow-subgroups are relatively prime in pairs, we may apply

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Corollary 2.2, and it suffices to consider B_0 for abelian p-groups. If G is an abelian group, then $B_0(G) = \Delta(G)$, for if U is a subgroup of G, then U is normal in G; consequently $U^\theta = U$, for $\theta \in B_0(G)$ and $\theta \in B_0(G)$ fixes all subgroups of G; therefore $B_0(G) \leq \Delta(G)$. But $\Delta(G) \leq B_0(G)$, therefore $B_0(G) = \Delta(G)$. R. H. Jäschke in [3] has shown:

<u>Theorem 2.3:</u> If G is an abelian group, and $\Delta(G)$ is the set of dilations of G, then

(1) $\Delta(G)$ is an abelian normal subgroup of $A(G)$.

(2) If $\theta \in \Delta(G)$ then θ has the form $g^\theta = g^{s(\theta)}$ where $s(\theta)$ is an integer relatively prime to the exponent of G.

(3) $\Delta(G)$ is isomorphic to the prime residue classes modulo the exponent of G.

<u>Corollary 2.4:</u> If G is an abelian group, then $B_0 = \Delta(G)$, the set of dilations of G, and $B_0(G)$ satisfies all conclusions of Theorem 2.3.

<u>Proof</u>: This follows immediately from the discussion preceding Theorem 2.3.

<u>Definition 2.5:</u> If G is a group, the subgroup of G generated by all the non-generators of G, is called the Frattini subgroup G.

It can be easily shown that the Frattini subgroup of G is the intersection of all maximal subgroups of G. The Frattini subgroup of G will be denoted by $\Phi(G)$.

We will next state two Lemmas that will be needed in what follows.

Lemma 2.6: If $G = H_1 \times H_2 \times H_3 \times \cdots \times H_n$, then $\Phi(G) = \Phi(H_1) \times \Phi(H_2) \times \cdots \times \Phi(H_n)$.

Proof: The proof of this may be found in [6, page 165].

Lemma 2.7: Let G be a group and let U be a subgroup of A(G) such that $|U| = p^k$ where k is an integer and $k \geq 1$. Let M_1 and M_2 be subgroups of G with $M_2 \triangleleft M_1$. If $M_1^\theta = M_1$ and $M_2^\theta = M_2$ for all $\theta \in U$ and $|M_1/M_2| = p$, then every $\theta \in U$ must induce the identity on M_1/M_2.

Proof: Let $\theta \in U$; let M_2x be a coset of M_2 in M_1; then $(M_2x)^\theta = M_2x^\theta$. Since $|M_1/M_2| = p$, we must have $M_2x^\theta = M_2x^k$ where $(k,p) = 1$ and $k < p$. Consider the action of θ^{p^n} on M_1/M_2:

$$(M_2x)^{\theta^{p^n}} = M_2x \quad \text{since} \quad |\theta| = p^n.$$

On the other hand:

$$\begin{aligned}(M_2x)^{\theta^{p^n}} &= (M_2x^\theta)^{\theta^{p^n-1}} \\ &= (M_2x^k)^{\theta^{p^n-1}} \\ &= M_2x^{k^{p^n}}\end{aligned}$$

but by Fermat's Theorem

$$k^{p^n} \equiv k \mod p.$$

Consequently:

$$(M_2x)^{\theta p^n} = (M_2x^{kp^n}) = M_2x^k .$$

Therefore:

$M_2x^k = M_2x$ or $k = 1$ and therefore $(M_2x)^\theta = M_2x$ and θ induces the identity on M_1/M_2 and the lemma is proven.

We are now in a position to apply our results to abelian p-groups and abelian groups in general. We will first focus our attention on abelian p-groups and then generalize it to arbitrary groups.

Theorem 2.8: Let G be an abelian p-group, then:

(1) B_1 is an abelian p-group

(2) $B_1 = B_2 = \cdots = B_n$.

Proof: Let G be an abelian p-group, then $B_1(G)$ is a subgroup of $B_0(G)$. By Theorem 2.3, we have that B_0 is an abelian group, hence B_1 is an abelian group. Let $\theta \in B_1(G)$, then θ induces the identity automorphism on G/M for every maximal normal subgroup M of G. Let Mx be a coset of M in G, where x is an arbitrary element of G. Then $(Mx)^\theta = Mx^\theta = Mx$, hence $x^\theta x^{-1} \in M$. Since M and x are arbitrary, we must have $x^\theta x^{-1} \in \cap\{M/M$ maximal normal$\}$. Since all subgroups of G are normal, we have $\cap\{M/M$ is maximal normal$\} = \Phi(G)$. Hence $x^\theta x^{-1} \in \Phi(G)$ and therefore $M(G;B_1) \leq \Phi(G)$. By Theorem 1.9, B_1 must induce the identity on $G/\Phi(G)$. By Theorem [T-2] we have that B_1 is a p-group. Consider now two subgroups M_1 and M_2 of

G, where M_2 is a maximal normal subgroup of M_1. If $\theta \in B_1$, $M_1^\theta = M_1$ and $M_2^\theta = M_2$. Therefore $\theta \in B_1$ must induce an automorphism on M_1/M_2. Since $|M_1/M_2| = p$ and $|\theta| = p^t$ for some positive integer t, we must have by Lemma 2.7 that θ induces the identity on M_1/M_2. Since M_1 and M_2 were arbitrary, θ must induce the identity on all composition factors of G. Therefore $\theta \in B_n$ and consequently $B_1 \leq B_n$. Since $B_n \leq B_1$ by definition, we must have $B_1 = B_2 \cdots = B_n$, and the Theorem follows.

Although $B_1 \leq B_o$ in Theorem 2.8, we may not conclude that $B_o = B_1$, as may be seen from the following example.

<u>Example 3</u>: Let G be the elementary abelian group of order 9. Then $G = \langle a,b/[a,b] = 1,\ a^3 = b^3 = 1\rangle$. Then $B_o = \langle \alpha/\alpha^2 = 1\rangle$, where $a^\alpha = a^2$; $b^\alpha = b^2$. On the other hand, if $\theta \in B_1(G)$, then θ must induce the identity on $G/\phi(G)$. Since $\phi(G) = 1$, we must have that θ must induce the identity on G. Consequently $\theta = 1$ and therefore $B_1 = 1$. We have therefore that $B_1(G) \lneq B_o(G)$.

<u>Definition 2.9</u>: Let π be a non-empty set of primes. A π-number is a natural number each of whose prime factors is in π: a π-element of a group G is an element whose order is a π-number; a π-group is a group each of whose elements is a π-element.

For any set of primes π, the number 1 is a π-number; the element 1 is a π-element; and the subgroup 1 is a π-subgroup.

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The set of all the primes not in π will be denoted by π'.

<u>Theorem 2.10</u>: Let G be a π-group. If s is a chain $G = G_o > G_1 > \cdots > G_n = 1$ of subgroups of G, then the stability group $A(s)$ is a π-group. Furthermore, if s is a chain of length 2, say $s : G > G_1 > G_2 = 1$, then $A(s)$ is a π-group if G_1 is a π-group.

<u>Proof</u>: Let $|\theta| = p$, $\theta \in A(s)$ and $(p, |G|) = 1$. Apply induction on the length of the chain. Since θ fixes the chains $s' : G_1 > G_2 > G_3 \cdots > G_n = 1$, we have $\theta \in A(s')$. Since $(|\theta|, |G_2|) = 1$, θ restricted to G_1 is the identity.

Let $g \in G$, then $g^{\theta}g^{-1} \in G_1$. Let $g^{\theta} = gx$ with $x \in G_1$: then $g^{\theta^p} = g$; hence $g^{\theta^p} = gx^p$, and $g = gx^p$ and $x^p = 1$. But $(|p|, |G_1|) = 1$, hence $x^p \neq 1$ unless $x = 1$. If $x = 1$, then $g^{\theta} = gx = g$ and θ is the identity. The first part of the theorem follows by induction.

Let s be a chain of length 2, say s: $G > G_1 > G_2 = 1$. Let $\theta \in A(s)$ such that $|\theta| = p$ and $(p, |G_1|) = 1$. If $g \in G$, then $g^{\theta}g^{-1} \in G_1$ and $g^{\theta} = g_1 g$ where $g_1 \in G_1$. Since $|\theta| = p$, $g^{\theta^p} = g$; but $g^{\theta^p} = (g_1 g)^{\theta^{p-1}} = g_1^p g$. Therefore $g = g_1^p g$ and $g_1^p = 1$. Since $(p, |G_1|) = 1$, $g_1^p \neq 1$ unless $g_1 = 1$. If $g_1 = 1$, then $g^{\theta} = g_1 g = g$ and θ is the identity. Therefore if G_1 is a π-group, then $A(s)$ is a π-group.

Theorem 2.11: If G is an abelian π-group, then:

(1) B_0 is an abelian group

(2) B_1 is a π-group

(3) $B_1 = B_2 = \ldots = B_n$.

Proof: The group B_0 is a subgroup of $\Delta(G)$, so by Theorem 2.3 we have that $\Delta(G)$ is an abelian group and consequently B_0 is an abelian group.

By Theorem 2.3 every $\theta \in B_0$ has the form $g^{\theta} = g^{s(\theta)}$ where $s(\theta)$ is an integer relatively prime to the exponent of G. We will show that if $\theta \in B_1$, then $s(\theta) \equiv 1 \mod p_i$ for each prime divisor p_i of the order of G.

Let M be a maximal subgroup of G of index p_i. Let Mx be a coset of M in G. If $\theta \in B_1$, then $(Mx)^{\theta} = Mx^{\theta} = Mx$. Since $B_1 \preceq \Delta(G)$, we must have that $x^{\theta} = x^{s(\theta)}$ where $(s(\theta). \exp G) = 1$. Consequently $(Mx)^{\theta} = Mx^{\theta} = Mx^{s(\theta)}$. Therefore $Mx^{s(\theta)} = Mx$; hence $x^{s(\theta)-1} \in M$. Since the index of M in G is equal to p_i, we must have $s(\theta)-1 \equiv 0 \mod p_i$ and therefore $s(\theta) \equiv 1 \mod p_i$. Since G is an abelian group, then G must have a maximal subgroup of index p_i for each prime divisor p_i of G. Consequently $s(\theta) \equiv 1 \mod p_i$ for each prime divisor p_i of the order of G.

Let G_i/G_{i+1} be a composition factor of G of order p_i. Let $G_{i+1}x$ be a coset of G_{i+1} in G_i and let $\theta \in B_1$. Therefore $(G_{i+1}x)^{\theta} = G_{i+1}x^{s(\theta)}$. From the previous paragraph we have that $s(\theta) \equiv 1 \mod p_i$. Therefore $s(\theta) = 1+kp_i$

where k is an integer. Consequently $(G_{i+1}x)^{\theta} = G_{i+1}x^{\theta}$ $= G_{i+1}x^{1+kp_i} = G_{i+1}x^{kp_i}x^1 = G_{i+1}(x^{p_i})^k x$, but $x^{p_i} \in G_{i+1}$ and therefore $(x^{p_i})^k \in G_{i+1}$; hence $(G_{i+1}x)^{\theta} = G_{i+1}(x^p)^k x$ $= G_{i+1}x$. We have, therefore, that each $\theta \in B_1$ must induce the identity on each composition factor. Hence $B_1 \leq B_n$; but, by definition, $B_n \leq B_1$ and therefore $B_1 = B_n$. Since $B_1 \geq B_2 \geq \ldots \geq B_n$ and $B_1 = B_n$, we must have that $B_1 = B_2 = \ldots = B_n$.

The group B_1 induces the identity on all composition factors of G; hence B_1 must belong to the stability group of every composition series of G. By Theorem 2.10 this implies that B_1 is a π-group.

(B) <u>$B_0, B_1, \ldots, B_n$ in case G is Nilpotent:</u>

As was done for abelian groups, we will first discuss the case for which G is a non-abelian p-group, and then generalize to nilpotent groups.

<u>Lemma 2.12</u>: If G is a nilpotent group, then $B_0(G) = \Delta(G)$, the set of dilations of G.

<u>Proof</u>: If G is nilpotent, we have by Theorem [T-1] that every subgroup of G is subnormal in G. Since B_0 fixes all subnormal subgroups of G, we must have that B_0 fixes all subgroups of G. Hence $\theta \in B$ implies $\theta \in \Delta(G)$, and $B_0(G) \leq \Delta(G)$. If an automorphism fixes all subgroups of a group G, then it surely fixes all subnormal subgroups

of G; therefore for any group G, $\Delta(G) \leq B_0(G)$. Therefore we must have that $B_0(G) = \Delta(G)$, the set of dilations of G.

Next we will state a Theorem by Huppert [2] which will be needed in what follows.

Theorem 2.13: If G is a non abelian p-group, then $B_0(G)$ is a p-group.

Theorem 2.14: If G is a non abelian p-group, then the following hold.

(1) B_0 is a p-group

(2) $B_0 = B_1 \ldots = B_n$.

Proof: By Theorem 2.13 we obtain (1).

Let $\theta_0 \in B_0$. Let M_1 and M_2 be two subgroups of G such that $M_2 \leq M_1$ and $|M_1/M_2| = p$. Now M_2 and M_1 are subnormal in G. Therefore if $\theta \in B_0(G)$, then $M_2^\theta = M_2$, $M_1^\theta = M_1$ and consequently θ must induce an automorphism on M_1/M_2. But $|M_1/M_2| = p$ and $|\theta| = p^t$ for some positive integer t. By Lemma 2.7 this implies that θ must induce the identity on the factor group M_1/M_2. Since θ, M_1, and M_2 are arbitrary, we must have that B_0 induces the identity on all composition factors. Hence $B_0 \leq B_n$, but $B_n \leq B_0$; hence $B_0 = B_n$.

Definition 2.15: If G is a group, the intersection of all the normalizers of all cyclic subgroups of G is called the norm of G and will be denoted by $N(G)$.

We will state some results which will be needed in later development, which have already been shown by H. J. Jaŝchke in [3].

<u>Theorem 2.16</u>: If G is a group then the following hold:

(1) $M(G;\Delta(G)) \leq N(G)$

(2) $G/F(G;\Delta(G))$ is nilpotent of class 2.

(3) $\Delta(G) \cap I(G) \leq Z(I(G))$, hence $\Delta(G) \cap I(G)$ consist of central automorphisms.

(4) $M(N(G):\Delta(G)) \leq Z(G)$.

If G is nilpotent, then $B_0 = \Delta(G)$ and the above Theorem holds if we replace $\Delta(G)$ by B_0. With the aid of the above Theorem we are now in a position to discuss B_0 for nilpotent groups.

<u>Theorem 2.17</u>: If G is a nilpotent π-group, then if none of its Sylow subgroups are abelian:

(1) B_0 is an abelian π-group.

(2) $B_0 = B_1 \ldots = B_n$.

If G has an abelian Sylow subgroup then:

(3) B_0 is an abelian group

(4) $B_1 = B_2 \ldots = B_n$ and is a π-group.

<u>Proof</u>: Let $G = S_{p_1} \times S_{p_2} \cdots \times S_{p_n}$ be a decomposi- of G into a direct product of its Sylow subgroups. By Corollary 2.3, we have that $B_0(G) = B_0(S_{p_1}) \times B_0(S_{p_2}) \cdots \times B_0(S_{p_n})$. Since $B_1(G) \leq B_0(G)$, every $\theta \in B_1(G)$ may be

written in the form $\theta = \theta_1 \cdot \theta_2 \cdots \theta_n$ where $\theta_i \in B_0(S_{p_i})$. Moreover θ_i acts trivially on S_{p_j} for $j \neq i$.

We will show that $\theta_i \in B_1(S_{p_i})$ for all i. Let M_i be a maximal subgroup of S_{p_i}. Then $\overline{M}_i = S_{p_1} \times S_{p_2} \times \cdots \times S_{p_{i-1}} \times M_i \times S_{p_{i+1}} \cdots \times S_{p_n}$ is a maximal subgroup of G. Since $\theta \in B_1(G)$, we must have that for every $x \in S_{p_i}$, $(\overline{M}_i x)^\theta = \overline{M}_i x^\theta = \overline{M}_i x$. Consequently $x^\theta x^{-1} \in \overline{M}_i$ for every $x \in S_{p_i}$, and therefore $M(S_{p_i};\theta) \leq \overline{M}_i$. Since S_{p_i} is a characteristic subgroup of G, we have $M(S_{p_i};\theta) \leq S_{p_i}$. Therefore $M(S_{p_i};\theta) \leq \overline{M}_i \cap S_{p_i} = M_i$. Since M_i was arbitrary, $M(S_{p_i};\theta) \leq M_i$ for all maximal subgroups M_i of S_{p_i}. Since θ restricted to S_{p_i} is equal to θ_i, we must have that $M(S_{p_i};\theta) = M(S_{p_i};\theta_i) \leq M_i$ for all maximal subgroups M_i of S_{p_i}. Consequently θ_i induces the identity on all factor groups S_{p_i}/M_i where M_i is a maximal subgroup of S_{p_i} and therefore $\theta_i \in B_1(S_{p_i})$. By Theorems 2.8 and 2.14, we must have $\theta_i \in B_n(S_{p_i})$.

Let M_1/M_2 be a composition factor of G, of order p_i. Let M_2x be a coset of M_2 in M_1. We may choose $x_i \in S_{p_i}$ as a coset representative of M_2x. For if $x \in G$, then $x = x_1 \cdots x_i \cdot x_{i+1} \cdots x_n$ where each $x_j \in S_{p_j}$ and $|x_j| = p_j^{n_j}$. Let $y = x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n$. Then

$x = x_i y$; but $x^{p_i^{n_i}} = (x_i y)^{p_i^{n_i}} = x_i^{p_i^{n_i}} y^{p_i^{n_i}} = y^{p_i^{n_i}}$. Since $|M_1/M_2| = p_i$, we must have $x^{p_i^{n_i}} \in M_2$ and $y^{p_i^{n_i}} \in M_2$. Since $|y_i| = \pi_{j \neq i} p_j^{n_j}$, we have $(p_i^{n_i}, |y|) = 1$ and $\langle y^{p_i}\rangle = \langle y \rangle$. However $y^{p_i^{n_i}} \in M_2$; consequently $y \in M_2$. Therefore $M_2 x = M_2 y x_i = M_2 x_i$.

Let $\theta \in B_1(G)$; then $(M_2 x)^\theta = (M_2 x_i)^\theta = M_2 x_i^\theta$. Since $\theta = \theta_1 \theta_2 \cdots \theta_i \cdots \theta_n$ and $x_i \in S_{p_i}$, we have $x_i^\theta = x_i^{\theta_i}$, and $(M_2 x)^\theta = M_2 x_i^{\theta_i}$. But $\theta_i \in B_1(S_{p_i})$ and therefore $|\theta_i| = p_i^t$ for some integer t. Furthermore, $|M_1/M_2| = p_i$. Therefore by Lemma 2.7, θ_i must induce the identity of M_1/M_2. Consequently $(M_2 x)^\theta = M_2 x^\theta = M_2 x_i^\theta = M_2 x_i^{\theta_i} = M_2 x_i = M_2 x$, and θ induces the identity on M_1/M_2. The composition factor M_1/M_2 was arbitrary; consequently θ must induce the identity on all composition factors. Therefore $B_1(G) \preceq B_n(G)$, but by definition $B_n(G) \preceq B_1(G)$, hence $B_1(G) = B_n(G)$.

If none of the Sylow subgroups of G are abelian, then $B_0 = B_1$. For if $\theta \in B_0$, then $\theta = \theta_1 \cdot \theta_2 \cdots \theta_n$ where $\theta_i \in B_0(S_{p_i})$. Since each S_{p_i} is a non-abelian p_i-group, we must have by Theorem 2.14 that $\theta_i \in B_n(S_{p_i})$ for all i. If M is a maximal subgroup of G of index p_i, we may choose $x_i \in S_{p_i}$ as a coset representative of M in G.

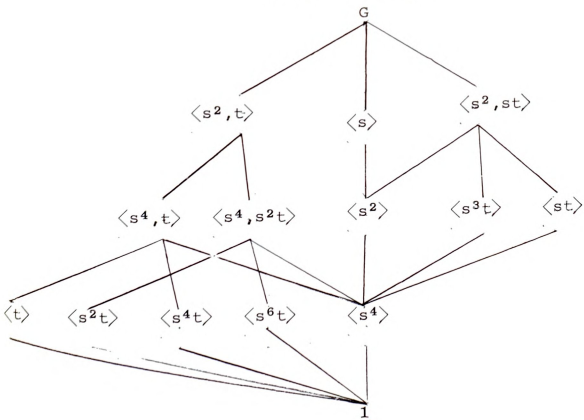
Consequently $(Mx_i)^{\theta} = Mx_i^{\theta} = Mx_i^{\theta_i}$. Since $|\theta_i| = p_i^t$ for some integer t and $|G/M| = p_i$, we must have by Lemma 2.7 that $Mx_i^{\theta} = Mx_i$. Therefore $(Mx_i)^{\theta} = Mx_i^{\theta_i} = Mx_i$ and θ induces the identity on G/M. Since M was an arbitrary maximal subgroup of G, θ must induce the identity on G/M for all maximal subgroups M of G. Consequently $\theta \in B_1(G)$. The element θ was chosen arbitrarily and therefore we must have $B_0(G) = B_1(G)$. This establishes the Theorem.

We shall now give an example to illustrate some of these concepts and properties that were derived above.

<u>Example 4</u>: G a p-group.

Let $G = \langle s, t / \ s^8 = t^2 = 1; \ t^{-1} s \ t = s^3 \rangle$.

The lattice of subgroups (subnormal subgroups) is the following:



$Z(G) = \langle s^4 \rangle$

B_0 is generated by the following automorphism:

$$\alpha : t \longrightarrow t \qquad s \longrightarrow s^3$$

$$\beta : t \longrightarrow t \qquad s \longrightarrow s^5$$

$$\gamma : t \longrightarrow t \qquad s \longrightarrow s^7 .$$

Therefore:

$$B_0 = \langle \alpha, \beta, \gamma / [\alpha,\beta] = [\alpha,\gamma] = [\beta,\gamma] = 1; \quad \alpha^2 = \beta^2 = \gamma^2 = 1 \rangle .$$

Hence B_0 is an elementary abelian group of order 8.

$$M(G;B_0) = \langle s^4 \rangle$$

$$F(G;B_0) = \langle s^4, t \rangle .$$

Example 5: G nilpotent. $G = D_8 \times C_9$ where D_8 is the dihedral group of order 8 and $C_9 = \langle a / a^9 = 1 \rangle$ is a cyclic group of order 9. Then $\Delta(G)$ is generated by:

$$\langle \theta : g^{\theta} = g^5 \text{ for all } g \in G \rangle,$$

i.e., $\Delta(G) = B_0(G) = \langle \theta / \theta^6 = 1 \rangle .$

$$M(G;B_0) = C_3 = \langle a^3 \rangle$$

$$F(G;B_0) = D_8 .$$

All automorphisms of B_0 are power automorphisms.

(C) The group $B_0(G)$ for a solvable group G.

In order to discuss the case B_0 for solvable groups we shall need several Lemmas. We shall present those first, in order that they may be used for reference in the later development.

Lemma 2.18: If G is a solvable group, then G possesses a non-trivial, normal, nilpotent subgroup.

Proof: Since G is solvable, G has a derived series $G = G^0 \geq G' \geq G'' \geq \dots \geq G^n = 1$. Now G^i/G^{i+1} is abelian and since every term in the derived series is characteristic, we have G^{n-1} is normal in G. Hence $G^{n-1} \neq 1$ is a normal nilpotent (abelian) subgroup of G.

Definition 2.19: The maximum, normal nilpotent subgroup of a group G is called the Fitting subgroup of G.

Lemma 2.20: If G is a solvable group, then G has a nontrivial Fitting subgroup.

Proof: By Theorem [T-3] the product of two normal nilpotent groups is a normal nilpotent group of G. Since G is finite, G has only a finite number of normal nilpotent groups. The product of all the normal nilpotent groups is the Fitting subgroup of G.

Lemma 2.21: Let G be an arbitrary group and let $M(G;B_0)$ be the B_0-multiplier group of G. If H is a subnormal nilpotent subgroup of G, then $M(G;B_0)$ normalizes H.

Proof: If H is a subnormal nilpotent subgroup of G, then every subgroup of H is subnormal in H. Since H is subnormal in G, we have by the transitivity of subnormality that every subgroup of H is subnormal in G. Therefore if $\theta \in B_0(G)$, then θ must fix all subgroups of H.

Let $x \in G$ and let $h \in H$, then $x^{-1}hx \in H^x$. Since H^x is isomorphic to H, H^x is nilpotent. Furthermore, since any automorphism maps a composition series onto a composition series, H^x must occur in a composition series for G, and H^x is subnormal. Therefore, H^x is a subnormal nilpotent subgroup of G. Consequently, if $\theta \in B_o(G)$, then θ must fix all subgroups of H^x. Then:

$$(x^{-1}hx)^\theta = (x^{-1}hx)^{s(x^{-1}hx;\theta)} \text{ where } (s(x^{-1}hx,\theta), |x^{-1}hx|) = 1.$$

On the other hand $(x^{-1}hx)^\theta = x^{-\theta}h^\theta x^\theta$

$$= x^{-\theta}h^{s(h;\theta)}x^\theta \text{ where } (s(h;\theta), |h|) = 1.$$

Therefore $(x^{-1}hx)^{s(x^{-1}hx;\theta)} = x^{-\theta}h^{s(h;\theta)}x^\theta$

$$x^{-1}h^{s(x^{-1}hx;\theta)}x = x^{-\theta}h^{s(h;\theta)}x^\theta.$$

Hence $x^\theta x^{-1}h^{s(x^{-1}hx;\theta)}(x^\theta x^{-1})^{-1} = h^{s(h;\theta)}$

Since $(s(x^{-1}hx;\theta), |x^{-1}hx|) = 1$, this implies that $(s(x^{-1}hx;\theta), |h|) = 1$. Consequently $h^{s(h;\theta)}$ and $h^{s(x^{-1}hx;\theta)}$ are generators of $\langle h \rangle$. Therefore $x^\theta x^{-1}$ normalizes every subgroup of H. Since $x^\theta x^{-1}$ was an arbitrary generator of $M(G;B_o)$, we have that $M(G;B_o)$ normalizes every subgroup of H.

Lemma 2.22: Let G be an arbitrary group and let M be the B_o-multiplier subgroup of G. If A is a normal abelian subgroup of G, then M centralizes A.

Proof: If A is a normal abelian subgroup of G, then every subgroup of A is subnormal in G. Therefore every $\theta \in B_o(G)$ must fix all subgroups of A. By Theorem 2.3, $\theta \in B_o(G)$ restricted to A has the form: $a^\theta = a^{s(\theta)}$ for all $a \in A$, where $(s(\theta), \exp A) = 1$. Let $x \in G$, $\theta \in B_o(G)$ and $a \in A$, then $x^{-1}ax \in A$. Therefore

$$(x^{-1}ax)^{\theta} = (x^{-1}ax)^{s(\theta)} \quad \text{where} \quad (s(\theta), \exp A) = 1$$
$$= x^{-1}\, a^{s(\theta)}\, x = x^{-1}\, a^{\theta}\, x$$

On the other hand: $(x^{-1}ax)^{\theta} = (x^{\theta})^{-1}\, a^{\theta}\, x^{\theta}$.

Therefore $(x^{\theta})^{-1}\, a^{\theta}\, x^{\theta} = x^{-1}\, a^{\theta}x$ or $(x^{\theta}x^{-1})^{-1}\, a^{\theta}(x^{\theta}x^{-1}) = a^{\theta}$.

Since θ restricted to A is an automorphism of A, $x^{\theta}x^{-1}$ must centralize all elements of A. Consequently, $M(G;B_0) = \langle x^{-1}x^{\theta} / x \in G, \theta \in B_0(G)\rangle$ must centralize A.

<u>Theorem 2.23</u>: Let G be a solvable group, $M = M(G;B_0)$, and let F^* denote the Fitting subgroup of G, then:

(1) M' is abelian

(2) $M' = M(G;B_0)' \leq Z(F^*)$

(3) M is nilpotent of class ≤ 2

(4) $M(M;B_0) \leq Z(F^*)$.

<u>Proof</u>: By Lemma 2.21, M normalizes every subgroup of the F^*. Every element of M induces an automorphism on F^*, and since it also fixes every subgroup of F^*, it must induce a dilation on F^*. Since the dilations of a group form an abelian group, we must have that the inner automorphisms induced by M' fix all elements of F^*. Therefore M' must centralize F^*. By Theorem T-4, the centralizer of F^* is contained in F^*; hence $M' \leq Z(F^*)$. Since $Z(F^*)$ is an abelian group, we have M' is abelian; hence (1) and (2) follow. The center of F^* is a normal abelian subgroup; hence, by Lemma 2.22, M must centralize

the $Z(F^*)$. Since $M' \leq Z(F^*)$, then M centralizes M'. Hence M' is contained in the center of M. Hence M is nilpotent of class 2 or abelian.

Let $x \in M$, $a \in F^*$, and $\theta \in B_0$, then:

$$(x^{-1}ax)^{\theta} = (x^{\theta})^{-1}a^{\theta}\, x^{\theta}.$$

If $x \in M$, then x induces a dilation on F^*, and since the dilations of a nilpotent group form an abelian group, we must have:

$$(x^{-1}ax)^{\theta} = x^{-1}a^{\theta}x\ .$$

Consequently

$$(x^{\theta})^{-1}a^{\theta}x^{\theta} = x^{-1}a^{\theta}x$$

or $$((x^{\theta})x^{-1})^{-1}a^{\theta}((x^{\theta})x^{-1}) = a^{\theta}$$ or $x^{\theta}x^{-1}$ centralizes F^*. By Theorem [T-4], we must have $x^{\theta}x^{-1} \leq Z(F^*)$. Therefore $M(M;B_0) = \langle x^{\theta}x^{-1} / x \in M,\ \theta \in B_0\rangle \leq Z(F^*)$.

<u>Theorem 2.24</u>: If G is solvable, then the following hold:

(1) B_0 is solvable

(2) B_0' is abelian.

<u>Proof</u>: If G is solvable, then G has a composition series whose factors are of prime order. Let $G = G_0 > G_1 > G_2 > \cdots > G_n = 1$ be such a composition series such that $|G_i/G_{i+1}| = p_i$. By definition $G_i^{\theta} = G_i$ for all $\theta \in B_0$. Hence B_0 must induce an automorphism on each of the factor groups G_i/G_{i+1}. Since these groups are of order p_i, their automorphism group has order p_i-1 and

is cyclic. Hence B_0' must induce the identity on all such factors. Hence B_0' is nilpotent of class n-1 by the Theorem of Philip Hall [T-5]. But this implies that B_0 is solvable. We can, however, obtain a better condition on the nilpotent class of B_0'. Since B_0 fixes $G/M(G;B_0)$, and the restriction of B_0 to $M(G;B_0)$ is abelian, we have that B_0' fixes $G/M(G;B_0)$ and $M(G;B_0)$. Hence B_0' fixes a chain of length 2. By Philip Hall's Theorem [T-5], we must have that B_0' is abelian.

Lemma 2.25: Let H be a normal subgroup of G. Let S be the subgroup of $A(G)$ consisting of all automorphisms of G such that $M(G;S) \leq H$ and such that $H \leq F(G;S)$, then:

(1) S is abelian

(2) $M(G;S) \leq Z(H)$.

Proof: (1) follows immediately from the Theorem by Philip Hall [T-5], since S is contained in the stability subgroup of the chain $G > H > 1$.

Let $h \in H$, $g \in G$ then $[h,g]^\theta = [h,g]$ since $H \triangleleft G$. On the other hand:

$$\begin{aligned}[h,g]^\theta &= [h^{-1}g^{-1}hg]^\theta \\ &= h^{-1}(g^\theta)^{-1}\, h\, g^\theta.\end{aligned}$$

Since $g^\theta g^{-1} \in H$, let $g^\theta = kg$ with $k \in H$, then:

$$\begin{aligned}[h,g]^\theta &= h^{-1}(g^\theta)^{-1}h\, g^\theta \\ &= h^{-1}(kg)^{-1}h\,(kg) \\ &= [h,kg]\end{aligned}$$

but $$[h,kg] = [h,g][h,k]^g ;$$

hence
$$\begin{aligned}[h,g]^\theta &= [h,g][h,k]^g \\ &= [h,g] .\end{aligned}$$

Therefore $[k,h] = 1$ hence $k = g^\theta g^{-1} \in Z(H)$.

Theorem 2.26: If G is solvable then B_0 normalizes every subgroup of B_0'.

Proof: Let $M = M(G;B_0)$, and let F^* denote the Fitting subgroup of G. We have that $M \leq F^*$; consequently B_0 induces the identity on G/F^*. The group B_0 restricted to F^* induces dilation on F^*, and since the dilations of a group form an abelian group, we must have that B_0' centralizes F^*. The group B_0' therefore induces the identity on G/F^* and on F^* and consequently belongs to the stability group of the chain $G > F^* > 1$. By Lemma 2.25, we have that $M(G;B_0') \leq Z(F^*)$, and by Lemma 2.21, we have that $M \leq N(F^*)$. Let $\alpha \in B_0$, $g \in G$, then:

$$g^\alpha = n_\alpha g \qquad \text{with} \quad n_\alpha \in N(F^*) .$$

$$g^{\alpha^{-1}} = n_{\alpha^{-1}} g \qquad \text{with} \quad n_{\alpha^{-1}} \in N(F^*) .$$

Since $(g^\alpha)^{\alpha^{-1}} = g,$ we have that

$$\begin{aligned}(g^\alpha)^{\alpha^{-1}} &= (n_\alpha g)^{\alpha^{-1}} \\ &= n_\alpha^{\alpha^{-1}} n_{\alpha^{-1}} g ;\end{aligned}$$

consequently $n_\alpha^{\alpha^{-1}} n_{\alpha^{-1}} = 1.$

Let $\theta \in B_0'$. If $g \in G$, $g^{\theta} = N_{\theta}g$ with $n_{\theta} \in Z(F^*)$, then:

$$\begin{aligned} g^{\alpha\theta\alpha^{-1}} &= (n_{\alpha}g)^{\theta\alpha^{-1}} \\ &= (n_{\alpha}^{\theta}g^{\theta})^{\alpha^{-1}} \\ &= (n_{\alpha}g^{\theta})^{\alpha^{-1}} \quad \text{since } \theta \text{ centralizes } F^* \\ &= (n_{\alpha}n_{\theta}g)^{\alpha^{-1}} \\ &= n_{\alpha}^{\alpha^{-1}} n_{\theta}^{\alpha^{-1}} n_{\alpha^{-1}} g \end{aligned}$$

but $n_{\theta} \in Z(F^*)$, which is an abelian normal subgroup; hence

$$n_{\theta}^{\alpha^{-1}} = n^{s(\alpha^{-1})} \quad \text{where } (s(\alpha^{-1}), \exp Z(F^*)) = 1.$$

Moreover $n_{\theta}^{\alpha^{-1}} \in Z(F^*)$ and therefore permutes with $n_{\alpha}^{\alpha^{-1}}$ and $n_{\alpha^{-1}}$.

$$\begin{aligned} \text{Therefore } g^{\alpha\theta\alpha^{-1}} &= n_{\alpha}^{\alpha^{-1}} n_{\theta}^{\alpha} n_{\alpha^{-1}} g \\ &= n_{\theta}^{\alpha^{-1}} (n_{\alpha}^{\alpha^{-1}} n_{\alpha^{-1}}) g \\ &= n_{\theta}^{\alpha^{-1}} g \\ &= n_{\theta}^{s(\alpha^{-1})} g \end{aligned}$$

but $g^{\theta^{s(\alpha^{-1})}} = n_{\theta}^{s(\alpha^{-1})}g$. Therefore we have that $\alpha\theta\alpha^{-1} = \theta^{s(\alpha^{-1})}$, and therefore B_0 normalizes every subgroup of B_0'.

Theorem 2.27: If G is solvable, then B_o is supersolvable.

Proof: Since B_o is solvable, B_o' must occur in a chief series of B_o. We may refine this chief series to a composition series for B_o; hence we have
$B_o = H_o > H_1 > \cdots \quad H_i = B_o' \quad H_{i+1} > \cdots > H_n = 1$. Now every subgroup of B_o' is normal in B_o; hence the groups H_i, H_{i+1}, $\cdots$, H_n are normal subgroups of B_o. Since all subgroups of B_o that contain B_o' are normal in B_o, the above composition series is a chief series for B_o. Since the factors of a composition series for a solvable group are of prime order, the above series is a chief series whose chief factors have prime order.

We might be inclined to show that B_o is nilpotent if G is solvable. This is not possible, as may be seen from Example 1. In this example $G = S_3$, the symmetric group on three letters, and consequently G is solvable, but $B_o = A(G) = S_3$ is not nilpotent.

At this point it might be worthwhile to illustrate some of the problems that may occur in what follows by means of an example.

Example 6: Let G be the semi-direct product of a cyclic group of order 7 and a cyclic group of order 2. Then $G = \langle a,b / a^7 = 1, b^2 = 1, bab = a^6 \rangle$. The chief series of G is its composition series: $G > \langle a \rangle > 1$. Therefore,

$A_o = B_o = A(G)$. $A(G) = \langle \alpha, \beta / \alpha^7 = 1;\ \beta^6 = 1$ and $\beta^{-1}\alpha\beta = \alpha^5 \rangle$ where

$$\alpha : a \longrightarrow a \qquad \beta : a \longrightarrow a^5$$
$$: b \longrightarrow a\,b \qquad : b \longrightarrow b.$$

Note, however, that 3 divides $|B_o|$ but 3 does not divide the order of G. In other words; prime divisors of $|B_o|$ are not necessarily prime divisors of the order of G.

The previous example shows that although G is a π-group, B_o need not be a π-group. If G is a π-group, and B_o possesses π-elements, we let B_o^* denote a maximal π-subgroup of B_o. If B_o has no π-element, we let $B_o^* = 1$.

<u>Definition 2.28</u>: A Hall π-subgroup of a group G is a subgroup whose index is a π'-number.

<u>Theorem 2.29</u>: If G is a solvable π-group, then:

(1) B_o has a unique maximal π-subgroup, B_o^*, which contains every π-element and every π-subgroup of B_o.

(2) B_o^* is the Hall π-subgroup of G.

(3) B_o splits over B_o^*.

(4) B_o' is a π-group and a π'-subgroup of B_o is abelian.

<u>Proof</u>: If B_o has no π-elements, then $B_o^* = 1$ and (1), (2), and (3) hold. Without loss of generality, we may assume that B_o has π-elements. B_o' fixes each subgroup in a composition series of G and induces the identity on every composition factor of G. Hence B_o' belongs to the stability group of every composition chain. Hence, by Theorem 2.10, B_o' is a π-group. Let B_o^* be a

maximal π-subgroup of B_o that contains B_o', then B_o^* is normal in B_o. If x is a π-element of B_o, and $x \notin B_o^*$, let K denote the product of B_o^* and $\langle x \rangle$. Then B_o^* is properly contained in K. On the other hand, $|K| = \dfrac{|B_o^*| \; |\langle x \rangle|}{|B_o^* \cap \langle x \rangle|}$. Since $|B_o^*|$, $|\langle x \rangle|$, $|B_o^* \cap \langle x \rangle|$ are all π-numbers, we must have that $|K|$ is a π-number and therefore K is a π-subgroup of B_o.

Since B_o^* is a maximal π-subgroup of G and B_o^* is properly contained in K, we contradict the maximality of B_o^*; consequently $x \in B_o^*$; or $K = B_o^*$.

If p is a prime divisor of $|B_o/B_o^*|$, then B_o/B_o^* has an element x of order p. By the Homomorphism Theorem B_o must contain a subgroup H such that $B_o^* \leq H \leq B_o$ and $[H:B_o^*] = p$. If p is a π-number, then $|H| = [H:B_o^*] \; |B_o^*|$ is a π-number; consequently H is a π-subgroup which contradicts the maximality of B_o^*. Therefore p is a π'-number and $[B_o:B_o^*]$ is a π'-number. Therefore B_o^* is a Hall π-subgroup of B_o. If H is a Hall π-subgroup of B_o, then H is a π-subgroup of B_o. By the first part of the theorem, $H \leq B_o^*$. Consider the index of H in B_o: $[B_o:H] = [B_o:B_o^*] \; [B_o^*: H]$. Since $[B_o:H]$ is a π'-number $[B_o^*:H]$ is a π'-number. Since B_o^* is a π-group, $[B_o^*:H]$ is a π'-number iff $[B_o^*:H| = 1$ or $H = B_o^*$. Hence B_o^* is the Hall π-subgroup of B_o.

Since B_o^* is a normal Hall π-subgroup of B_o, we may apply the Schur-Zassenhaus Theorem [T-6] from which we obtain that $B_o = B_o^* H$ and $B_o^* \cap H = 1$. Consequently B_o splits over B_o^* .

If H is a π'-subgroup of B_o, then $H' \leq B_o'$, but B_o' is a π-subgroup of B_o. Since H is a π'-subgroup of B_o, we must have $H' = 1$ or H is abelian.

II. The Structure and Properties of the Group A_o.

Introduction

We will discuss properties and the structures of A_o for abelian groups, p-groups and nilpotent groups, and derive a few properties of A_o for solvable groups.

If G is an abelian group, then every subgroup of G is normal in G. Consequently A_o must fix all subgroups of G and therefore $A_o = B_o(G) = \Delta(G)$. We have, therefore, that all the conclusions of Theorem 2.11 hold if we replace B_1 by A_1.

Theorem 2.30: If G is a direct product of groups H and K, and E is a subgroup of A_o; if E_H and E_K are the restrictions of E to H and E to K respectively, then:

(1) $M(G;E) = M(H;E_H) \times M(K;E_K)$.

(2) $F(G;E) = F(H;E_H) \times F(K;E_K)$.

Proof: The proof follows the same pattern as that of Theorem 2.1.

Before we go any further, it might be worthwhile to illustrate the difficulties that will be encountered with an example.

Example 7: $G = \langle a, b, c / a^3 = b^3 = c^3 = 1;$

$b^{-1}ab = c^{-1}ac = a; \; c^{-1}bc = ba \rangle$.

$A_o = \langle \alpha, \beta, \theta \,/\, \alpha^3 = \beta^3 = \theta^2 = 1; \; [\alpha,\beta] = [\alpha,\theta] = [\beta,\theta] = 1 \rangle$

We observe that $2 \mid |A_o|$ although $2 \nmid |G|$.

The previous example shows that although G is a p-group, A_o need not be a p-group. If G is a p-group, and $p \mid |A_o|$, we let A_o^* denote a Sylow p-subgroup of A_o. If $p \nmid |A_o|$, we let $A_o^* = 1$.

Theorem 2.31: If G is a p-group of order p^n, then:

(1) A_o' is a p-group of class $\leq n-1$ and $A_o' \leq A_n$.

(2) The Sylow p-subgroup A_o^* of A_o is normal in $A(G)$.

(3) $A_o^* = A_n$.

(4) A p'-subgroup of A_o is abelian.

(5) A_o splits over A_o^*.

Proof: Since G is a p-group, all chief factors of G have order p. Since $|G| = p^n$, a chief series of G must be of length n. Let s: $G = G_o \geq G_1 \geq G_2 \geq \cdots \geq G_n = 1$ be such a chief series. The group A_o must fix all normal subgroups of G. Therefore, every $\theta \in A_o$ must induce an automorphism on each chief factor G_i / G_{i+1} $i = 0, 1, \ldots, n-1$. Since $A(G_i / G_{i+1})$ is cyclic of order $p-1$, A_o' must induce the identity on G_i / G_{i+1} for $i = 0, 1, \ldots, n-1$. Consequently, $A_o' \leq A(s)$. By Theorem 2.10, $A(s)$ is a p-group and therefore A_o' is a p-group. By Theorem [T-5], A_o' is a p-group of class $\leq n-1$.

Let A_o^* be a Sylow subgroup containing A_o'. Since any subgroup containing the commutator subgroup is normal, we have that A_o^* is normal in A_o. The Sylow subgroups of a group are conjugate; therefore A_o^* is characteristic in A_o. Since A_o is normal in $A(G)$, A_o^* must be normal in $A(G)$.

If s is a chief series for G, then $S_n(s) = A(s)$. By Theorem 2.10, $S_n(s)$ is a p-group. Since $A_n = \bigcap_{s \in D(G)} S_n(s)$, we must have that A_n is a p-group.

Let $s: G = G_o \geq G_1 \geq \cdots \geq G_n = 1$ be a chief series for G. If $\theta \in A_o^*$, then $G_i^\theta = G_i$. Therefore, θ must induce an automorphism on G_i / G_{i+1}. Since A_o^* is a p-Sylow subgroup of G, $|\theta| = p^t$. On the other hand, $A(G_i / G_{i+1})$ is cyclic of order $p-1$. By Lemma 2.7, $\theta / G_i / G_{i+1} = 1$, hence $\theta \in S_n(s)$. Since s was arbitrary, $\theta \in \bigcap_{s \in D(G)} S_n(s) = A_n$; therefore $A_o^* \leq A_n$. Since A_n is a p-group and A_o^* is a p-Sylow subgroup of A_o, we must have $A_o^* = A_n$.

Let H be a p'-subgroup of A_o, then $H' \leq A_o'$. Since $|A_o'| = p^t$ and H' is a p'-subgroup of A_o, we must have $H' = 1$ or H is abelian.

We may now apply a theorem by Schur-Zassenhaus [T-6] from which we obtain $A_o = A_o^* H$ and $A_o^* \cap H = 1$, where H is a p'-subgroup of A_o.

If G is a p-group, A_o need not be a p-group, as may be seen from Example 6. We have, however, the following result:

<u>Theorem 2.32</u>: Let G be a p-group. Then:

(1) If H is a normal nilpotent subgroup of A_o, then H is a p-group.

(2) $\phi(A_o)$ is a p-group.

(3) If A_o is nilpotent, then A_o is a p-group.

Proof: Let Q be a Sylow q-subgroup of H, where $q \neq p$. Since H is nilpotent, Q is a characteristic subgroup of H. Since H is normal in A_o, we must have that Q is normal in A_o. The inner automorphism group $I(G)$ is contained in A_o, and therefore $[Q, I(G)] \leq [Q, A_o] \leq Q$. Since $I(G)$ is a normal subgroup of $A(G)$, we must have $[Q, I(G)] \leq I(G)$. Therefore $[Q, I(G)] \leq Q \cap I(G) = 1$, since Q is a q-group and $I(G)$ is a p-group. Therefore Q and $I(G)$ permute, and by Theorem [T-7] we have that $M(G;Q) \leq Z(G)$. Since $(|Q|, p) = 1$, we have by Theorem T-8 that $G' \geq Z(G)$. The group $M(G;Q) \leq Z(G) \leq G'$ and consequently Q must induce the identity on $G/\phi(G)$. By Theorem [T-2], this implies that $|Q| = p^k$ for some positive integer k. Therefore $Q = 1$ and all q-Sylow subgroups of H are equal to the identity. Consequently H is a p-group.

Result (2) follows from (1), since $\phi(A_o)$ is a normal nilpotent subgroup of A_o. If A_o is nilpotent, then a Sylow q-subgroup Q of A_o is normal in A_o and by the first result $Q = 1$ and therefore A_o is a p-group.

Theorem 2.33: If G is a solvable group, then:

(1) Every normal subgroup of $I(G)$ is normal in A_o

(2) B_0 normalizes every subnormal subgroup of $I(G)$.

<u>Proof</u>: Let $\alpha \in A_0$; $\theta \in I(G)$. Let $H \trianglelefteq I(G)$. By the Homomorphism Theorems, there is a one to one correspondence between the normal subgroups of $I(G)$ and the normal subgroups of G, which contain the center of G. Let H correspond to $\overline{H} \trianglelefteq G$. Let $\theta \in H$ and let $g \in \overline{H}$ such that the inner automorphism induced by g is θ. If $x \in G$, then $x^{\alpha^{-1}\theta\alpha} = (g^{-1}(x^{\alpha^{-1}})g)^{\alpha} = (g^{\alpha})^{-1} x\, g^{\alpha}$, but $g \in \overline{H}$ and $\overline{H} \triangleleft G$; hence $g^{\alpha} \in \overline{H}$. Hence the inner automorphism induced by $g^{\alpha} \in H$. Hence $\alpha^{-1}\theta\alpha \in H$ and $H \triangleleft A_0$ and (1) follows.

Let H be a subnormal subgroup of $I(G)$. Let $\overline{H}$ be the subgroup of G that corresponds to H under the homomorphism of G onto $I(G)$. Since $H \triangleleft \triangleleft\, I(G)$, we have $\overline{H} \triangleleft \triangleleft\, G$. If $\theta \in H$, let θ be induced by $g \in G$. If $x \in G$, then $x^{\alpha^{-1}\theta\alpha} = (x^{\alpha^{-1}})^{\theta\alpha} = (g^{-1} x^{\alpha^{-1}} g)^{\alpha} = (g^{\alpha})^{-1} x\, g^{\alpha}$. Since $g \in \overline{H}$ and $\alpha \in B_0$, then $g^{\alpha} \in \overline{H}$ hence $\alpha^{-1}\theta\,\alpha \in H$ and (2) follows.

We will now generalize Theorem 2.31 for nilpotent groups, the proof of which follows the same pattern as that of Theorem 2.31.

If G is a π-group, and A_0 possesses π-elements, we let A_0^* denote a maximal π-subgroup of A_0. If A_0 is a π'-subgroup, we let $A_0^* = 1$.

<u>Theorem 2.34</u>: If G is a nilpotent π-group, then

(1) A_0 has unique maximal π-subgroup A_0^*, which

contains every π-element and every π-subgroup of A_o.

(2) A_o^* is the Hall π-subgroup of A_o.

(3) A_o splits over A_o^*.

(4) A_o' is a π-group and a π'-subgroup of A_o is abelian.

We shall now proceed to derive a few properties for A_o in case G is supersolvable.

Theorem 2.35: If G is a supersolvable π-group, then

(1) A_o' is a π-group and A_o' is nilpotent

(2) A_o is solvable

(3) A π'-subgroup of A_o is abelian.

Proof: If G is supersolvable, then every chief series of G has chief factors of prime order. Let s be such a chief series, i.e., s: $G = G_o > G_1 > \dots > G_n = 1$. Then $G_i^\theta = G_i$ for all i, for all $\theta \in A_o$. Moreover $\theta \in A_o$ induces an automorphism on G_i/G_{i+1}. Since $|A(G_i/G_{i+1})| = p-1$ and $A(G_i/G_{i+1})$ is cyclic, A_o' must induce the identity on G_i/G_{i+1} for $i = 0,\dots,n-1$. Hence $A_o' \leq A(s)$. By Theorem 2.10, A_o' is a π-group, and by Philip Hall's Theorem T-5, A_o' is nilpotent.

Since A_o' is nilpotent and A_o/A_o' is abelian, the group A_o is solvable.

Let H be a π'-subgroup of A_o, then $H' \leq A_o'$; but A_o' is a π-subgroup of A_o. Consequently $H' = 1$ and H is abelian.

CHAPTER III

In this chapter we will try to answer the following question: If a group G admits an automorphism that fixes all subnormal subgroups, what conditions, if any, does this impose on the structure of G? We will restrict our attention to solvable groups.

In particular we will investigate how the groups $F(G;E)$ and $M(G;E)$, for a subgroup E of B_o, are imbedded in G. Furthermore, we will place conditions on the groups $M(G;E)$ and $F(G;E)$ and see what this must imply about the structure of G. We will begin the chapter with two results which hold for arbitrary automorphisms of the group G.

Next we will focus our attention on automorphisms in B_o for which $F(G;\theta) \cap M(G;\theta) = 1$. We saw in Chapter II that although G is a π-group, B_o need not be a π-group. For the π'-elements of B_o, as well as a more general class of automorphisms, we obtain the condition that $F(G;\theta) \cap M(G;\theta) = 1$. For a group G, it is not only possible that B_o may contain a π'-element, but that a subgroup of B_o, or even the whole group B_o, is a π'-group. The condition that B_o be a π'-group if G is a π-group, places strong conditions on the group G.

We will next turn to the inner automorphisms of G which are elements of B_o, and in particular, we will try to determine some properties of the group $\overline{N}$, which has the property that if $x \in \overline{N}$, then the inner automorphism induced by x fixes all subnormal subgroups.

<u>Lemma 3.1</u>: Let E be a subgroup of the automorphism group of a group G. Let $M = M(G;E) = \langle x^{\theta}x^{-1} / x \in G$ and $\theta \in E\rangle$ and $F = F(G;E) = \{g \in G / g^{\theta} = g\}$.

If W is the subgroup generated by M and F, then $M(W;E) = M(M;E)$.

<u>Proof</u>: By Theorem 1.9, we have that for any subgroup E of $A(G)$, M is normal in G. Hence if W is generated by M and F, then W must be the product of M and F; therefore $W = M \cdot F$. If $w \in W$, then $w = mf$ with $m \in M$ and $f \in F$. Therefore, if $\theta \in E$, $w^{\theta}w^{-1} = (mf)^{\theta}(mf)^{-1} = m^{\theta}f^{\theta}f^{-1}m^{-1} = m^{\theta}m^{-1}$. Hence $w^{\theta}w^{-1} \in M(M;E)$ for all $\theta \in E$ and for all $w \in W$. Consequently, $M(W;E) = \langle w^{\theta}w^{-1}/w \in W$ and $\theta \in E\rangle \leq M(M;E)$; but $M \leq W$ and therefore $M(M;E) \leq M(W;E)$; hence, $M(W;E) = M(M;E)$.

<u>Theorem 3.2</u>: Let θ be an automorphism of G and $M = M(G;\theta)$, $F = F(G;\theta)$. If $F(M;\theta) = 1$, then G is generated by M and F.

<u>Proof</u>: If $F(M;\theta) = 1$, consider the map $\alpha : x \to x^{\theta}x^{-1}$ for $x \in M$. Then α is a map from M into $M(M;\theta)$. Now α is a one to one map, for if $x^{\alpha} = y^{\alpha}$, then $x^{\theta}x^{-1} = y^{\theta}y^{-1}$ or $(y^{-1}x)^{\theta} = y^{-1}x$. Hence, $y^{-1}x \in F(M;\theta) = 1$ or

$y^{-1}x = 1$ or $x = y$. Since $M(M;\theta) \preceq M(G;\theta)$ and the group G is finite, we must have $M = M(M;\theta)$. In particular, if $x \in M$, then $x = y^{\theta}y^{-1}$ for some $y \in M$. Let Mg be a coset of M in G. If $g^{\theta} = g$, then let g be the coset representative of M. If $g^{\theta} \neq g$, then $g^{\theta}g^{-1} \in M$ and $g^{\theta}g^{-1} \neq 1$. Hence there exists an $x \in M$ such that $x^{\theta}x^{-1} = g^{\theta}g^{-1}$. Therefore, $(x^{-1}g)^{\theta} = (x^{-1}g)$ and $x^{-1}g \in F$; but $x^{-1}g \in Mg$. In this case, let $x^{-1}g$ be the coset representative of Mg. Hence there exists a collection of coset representatives for M in G that is fixed element-wise by θ. If $g \in G$, then $g \in Mx$ with $x \in F$; hence $g = mx$ with $m \in M$ and $x \in F$, or G is the product of M and F.

The hypothesis of the previous Theorem that $F(M;\theta) = 1$ is equivalent to $F \cap M = 1$.

Let us now turn our attention to automorphisms that fix all subnormal subgroups of G. As in Chapter I and II, we will restrict our attention to solvable groups. The next two theorems will show that the action of such an automorphism is to a great extent characterized by its action on the Fitting subgroup $F(G)$ of G. Since the work in this chapter depends upon some of the results of Chapters I and II, we will summarize these results, as in the following two theorems:

<u>Theorem 3.3:</u> Let G be a solvable group and E a subgroup of $B_0(G)$; then:

(1) If H is a subnormal nilpotent subgroup of G, then H and all subgroups of H are fixed by E.

(2) If H is a subnormal nilpotent subgroup of G, and if θ is an element of E, then θ restricted to H is a dilation of H. Hence if $h \in H$, $h^{\theta} = h^{s(h;\theta)}$, where $(s(h;\theta), |h|) = 1$.

(3) If H is an abelian normal subgroup of G and θ is an element of E, then θ restricted to H is a power automorphism of H and $h^{\theta} = h^{s(\theta)}$ for all $h \in H$ where $(s, \exp H) = 1$.

<u>Theorem 3.4</u>: Let G be a solvable group. Let $F^*(G)$ be the Fitting subgroup of G. Let E be a subgroup of $B_0(G)$. Then:

(1) $M(G;E) = \langle x^{\theta}x^{-1}/x \in G;\ \theta \in E \rangle$ is contained in the norm of $F^*(G)$.

(2) If $\theta \in E$ is a power automorphism on $F^*(G)$, then $M(G;\theta) \leq Z(F^*)$.

<u>Proof</u>: (1) follows from Theorem 2.23.

If $\theta \in E$, restricted to $F(G)$, is a power automorphism, let $x \in G$, $f \in F^*(G)$. Then $(x^{-1}fx)^{\theta} = (x^{-1}fx)^{s(\theta)}$, since θ is a power automorphism on F^*, and $x^{-1}fx \in F^*(G)$. Therefore $(x^{-1}fx)^{\theta} = x^{-1}f^{s(\theta)}x = (x^{\theta})^{-1}f^{\theta}x^{\theta} = (x^{\theta})^{-1}f^{s(\theta)}x^{\theta}$ and $(x^{\theta}x^{-1})^{-1}f^{s(\theta)}x^{\theta}x^{-1} = f^{s(\theta)}$. Since $x^{\theta}x^{-1}$ centralizes $f^{s(\theta)}$, it must centralize every power of $f^{s(\theta)}$, but $\langle f^{s} \rangle = \langle f \rangle$; hence $x^{\theta}x^{-1}$ centralizes f. Since f was arbitrary, $x^{\theta}x^{-1} \in C_G(F^*) = Z(F^*)$. But,

x was arbitrary; consequently, $M(G;\theta) = \langle x^{\theta}x^{-1} / x \in g\rangle \leq Z(F^*)$.

The last two theorems show that a careful analysis of the action of the automorphism on the Fitting subgroup of G, should reveal information of the action of the automorphism on the whole group G. In most instances, we will place a condition on an automorphism $\theta \in B_0(G)$, and see what this must imply about the structure of the Fitting subgroup. From the structure of the Fitting subgroup in turn, we try to obtain some information about the structure of the group G.

Lemma 3.5: Let E be a subgroup of B_0 and let $M = M(G;E)$. Then:

(1) $M(M;E)$ is an abelian subnormal subgroup of G.

(2) $M(M;E) \leq Z(F^*)$, where F^* is the Fitting subgroup of G.

Proof: In Chapter II, Theorem 2.23, we have shown that $M(M;E)$ is contained in the center of the Fitting subgroup of G. Since this is an abelian group, this must imply that $M(M;E)$ is abelian. In Chapter I, Theorem 1.9, we have shown that $M(G;E)$ is a normal subgroup of G. Since $M(M;E)$ is a normal subgroup of $M(G;E)$, we must have that $M(M;E)$ is subnormal in G.

Theorem 3.6: Let E be a subgroup of $B_0(G)$, and $M = M(G;E)$, and $F(G;E) = F$; then $M = M(M;E)$ iff $F(M;E) = 1$. In this case, M is an abelian group of odd order.

Proof: Let $M = M(M;E)$ and let us assume that $F(M;E) \neq 1$. We will show that this leads to a contradiction.

If $M = M(M;E)$, then by the previous Lemma, M is an abelian normal subgroup of G. By Theorem 3.3, we have that for $\theta \in E$ and any $g \in M$, $g^\theta = g^{s(\theta)}$, where $(s(\theta), \exp M) = 1$.

If $F(M;E) \neq 1$, then $F \cap M = F(M;E) \neq 1$. Let $x \in F \cap M$, such that $|x| = p$. Then if $\theta \in E$, $x^\theta = x^{s(\theta)}$ since $x \in M$. One the other hand, $x^\theta = x$ since $x \in F$. Consequently $x^{s(\theta)} = x$ and $s(\theta) = 1 \mod p$ for all $\theta \in E$. Since θ is a power automorphism on M, we must have for all $g \in M$, $g^\theta = g^{s(\theta)}$, where $s(\theta) = 1 \mod p$ for all $\theta \in E$.

Let H be a maximal subgroup of M of index p. Then H is normal in M, hence subnormal in G, or $H^\theta = H$ for all $\theta \in E$. Consequently every $\theta \in E$ must induce an automorphism on M/H. Let $g \in M$ and consider the coset Hg. For $\theta \in E$, $(Hg)^\theta = Hg^{s(\theta)}$ but, $s(\theta) = 1 \mod p$ or $s(\theta) = 1 + kp$; therefore $(Hg)^\theta = Hg^{s(\theta)} = Hg^{1+kp} = H(g^p)^k g$. Since the index of H in M is equal to p, $(g^p)^k \in H$; consequently $Hg^\theta = H(g^p)^k g = Hg$. Therefore $Hg^\theta = Hg$ and $g^\theta g^{-1} \in H$ for all $g \in M$ and for all $\theta \in E$. Hence, $M(M;E) = \langle x^\theta x^{-1} / x \in M; \theta \in E \rangle \leq H \lneq M$. Since $M(M;E) = M$, we have a contradiction. Consequently if $M(M;E) = M$, then $F(M;E) = 1$.

Conversely, let $F(M;E) = 1$. By Theorem 2.23, we have that M is nilpotent and of class 2. Therefore, M is a direct product of its Sylow subgroups. Let $M = P_1 \times P_2 \times \ldots \times P_n$ where P_i is a p_i-Sylow subgroup of M. If M is not abelian, then there exists a p_i-Sylow subgroup P_i which is nonabelian. Moreover, P_i is a subnormal subgroup of G. Let $\theta \in E$; then $P_i^\theta = P_i$ and θ is a dilation of P_i. Since the dilations of a nonabelian p-group form a p-group, we must have by Lemma 2.7, that if $x \in P_i$ of order p_i, then $x^\theta = x$. Since θ was arbitrary, we must have $x^\theta = x$ for all $\theta \in E$. Therefore, $x \in F(M;E)$, a contradiction. We may assume then, without loss of generality, that all p_i-Sylow subgroups of M are abelian, and consequently M is abelian.

Since M is abelian, we have for $g \in M$, and $\theta \in E$ that $g^\theta = g^{s(\theta)}$ where $(s(\theta), \exp G) = 1$; therefore $g^\theta g^{-1} = g^{s(\theta)-1}$. Let the greatest common divisor of $s(\theta)-1$ and $|M|$ be $d(\theta)$; i.e., $d(\theta) = (s(\theta)-1, |M|)$. We will show that the greatest common divisor of the $d(\theta)$ for $\theta \in E$ is equal to 1. If the greatest common divisor of the $d(\theta)$ for $\theta \in E$ is not equal to one, then there exists a prime p, such that p divides $d(\theta)$ for all $\theta \in E$. Hence $d(\theta) = p\,t(\theta)$; then $s(\theta)-1 = p\,u(\theta)$ and $s(\theta) = 1 + p\,u(\theta)$. Since p divides the order of M, p must divide the exp M. Therefore $\exp M(G;E) = pw$, where w is a positive integer. If $w = 1$, then M has exponent p and if $x \in M$ of order p, then $x^\theta = x^{s(\theta)} = x^{1+p\,u(\theta)} =$

$= x(x^p)^{u(\theta)} = x$ for all $\theta \in E$. Consequently $x^\theta = x$ and $x \in M \cap F = F(M;E)$. This is a contradiction since $F(M;E) = 1$. Therefore $w \neq 1$; hence $M^w = \langle g^w \mid g \in M(G;E)\rangle \neq 1$. If $g^w \in M(G;E)$ and $g^w \neq 1$, then for $\theta \in E$, $(g^w)^\theta = (g^w)^{1+pu(\theta)}$ $= g^w g^{wpu(\theta)} = g^w(g^{wp})^{u(\theta)} = g^w$. Therefore, $g^w \in F \cap M =$ $F(M;E)$ but $F(M;E) = 1$, a contradiction. Therefore the greatest common divisor of the $d(\theta)$, for $\theta \in E$, is equal to one.

Since the greatest common divisor of the $d(\theta)$ for $\theta \in E$ is equal to one, we must have, for every p_i dividing $|M|$, a $\theta_i \in E$ such that $p_i \not| \, s_i - 1$. Let $M = P_1 \times P_2 \times \cdots \times P_n$ be a direct product of its p_i-Sylow subgroups, P_i, where $|P_i| = p_i^{n_i}$. If $g_i \in P_i$, then $g_i^{\theta_i} g_i^{-1} = g_i^{s_i - 1}$ and $|g_i^{\theta_i} g_i^{-1}| = |g_i^{s_i-1}| = |g_i|$ since $p_i \not| \, s_i - 1$. Consequently $\langle g_i^{\theta_i} g_i^{-1}\rangle = \langle g_i \rangle$. Therefore $M(P_i;\theta_i) = \langle g_i^{\theta_i} g_i^{-1} / \, g_i \in P_i\rangle =$ P_i. Since $P_i = M(P_i;\theta_i) \leq M(P_i;E) \leq P_i$, we must have $M(P_i;E) = P_i$. Therefore $M(M;E) = M(P_1;E) \times M(P_2;E) \times \cdots \times M(P_n;E) = P_1 \times P_2 \times \cdots \times P_n = M(G;E)$. Hence the first part of the Theorem follows.

If we assume that $M(M;E) = M$ and 2 divides $|M|$, let $x \in M(G;E)$ such that $|x| = 2$. Then, since every $\theta \in E$ must fix the subgroup generated by x, we must have $x^\theta = x$ for all $\theta \in E$. This leads to $x \in F(M;E)$, a contradiction.

One would be inclined to prove Theorem 3.2 for an arbitrary subgroup E of $B_0(G)$. That this cannot be done can be seen from the following example:

Example 8: Let $G = S_3$, the symmetric group on three letters. Then G is generated by a and b, subject to defining relations $a^3 = b^2 = 1$ and $b^{-1}ab = a^2$. The only composition series of G is the chain $G \gneq \langle a \rangle > 1$. Since $\langle a \rangle$ is the only subgroup of G that is normal in G, we must have that $\langle a \rangle$ is characteristic. Therefore $B_0 = A_0 = A(G)$. Let $E = A(G) = \langle \alpha, \beta \,/\, \alpha^3 = \beta^2 = 1;\ \beta^{-1}\alpha\beta = \alpha^2 \rangle$; then $F(G;E) = 1$, $M(G;E) = \langle a \rangle$. Hence $F(G;E) \cap M(G;E) = 1$, but $W = F(G;E) \cdot M(G;E) = A_3 \lneq G$.

Theorem 3.7: Let G be a nilpotent group and E be a subgroup of $B_0(G)$. If $F(G;E) = 1$, then $G = M(G;E)$, and G is abelian of odd order.

Proof: In Theorem 3.6, we showed that the condition $F(M;E) = 1$ implies $M(M;E) = M$, where $M = M(G;E)$. To prove this result, consider the following: the only property of $M = M(G;E)$ that was used, was that $M(G;E)$ was nilpotent. Therefore, if we assume that G is nilpotent, then $F(G;E) = 1$ will imply that $G = M(G;E)$, by the same argument as was used in Theorem 3.6.

Definition 3.8: A group G is said to be a semi-direct product of its subgroups H and K iff H is normal in G, $G = HK$ and $H \cap K = 1$.

Lemma 3.9: Let E be a subgroup of $B_0(G)$. Let $M = M(G;E)$ and $F = F(G;E)$. If G is generated by M and F, then:

(1) $F \cap M = 1$ and G is a semi-direct product of M and F.

(2) $M = M(M;E) = M(F^*;E)$, where F^* is the Fitting subgroup of G.

Proof: If G is generated by F and M, by Lemma 3.1, we must have that $M(M;E) = M$ and, by Theorem 3.6, this implies that $F(M;E) = 1$; but, $F(M;E) = F \cap M = 1$. From the normality of M in G, we obtain that G is a semi-direct product of M and F.

The Fitting subgroup F^* is contained in G. Hence $M(F^*;E) \leq M$, but $M < F^*$; therefore, $M(M;E) \leq M(F^*;E)$. Since $M = M(M;E)$, we must have $M(F^*;E) \leq M = M(M;E) \leq M(F^*;E)$ or $M(F^*;E) = M$.

Theorem 3.10: If $\theta \in B_0(G)$ and $F = F(G;\theta)$ and $M = M(G;\theta)$, then the following are equivalent:

(1) G is generated by F and M

(2) G is a semi-direct product of M and F

(3) $M = M(M;\theta) = M(F^*;\theta)$ and is an abelian group of odd order

(4) $F \cap M = 1$.

Proof: The Theorem follows from the previous Theorems and Lemmas: (1) => (2) by Lemma 3.9, (2) => (3) by Lemma 3.9 and Theorem 3.6, (3) => (4) by Theorem 3.6, (4) => (1) by Theorem 3.2.

For a subgroup E of $B_0(G)$, we have the following result.

<u>Theorem 3.11</u>: Let E be a subgroup of $B_0(G)$ and let $F = F(G;E)$, $M = M(G;E)$ and let $F^*(G)$ denote the Fitting subgroup of G. If $F \cap M = 1$, then

(1) $M \leq Z(F^*(G))$.

(2) M is a Hall subgroup of F^*

(3) F^* is generated by M and $F(F^*;E)$.

(4) Every $\theta \in E$ is a power automorphism on F^*.

<u>Proof</u>: By Lemma 3.9 we have that $M = M(F^*;E)$. We will show that $M(F^*;E)$ is a Hall subgroup of F^* and that $M(F^*;E) \leq Z(F^*)$.

Since $M(F^*;E) \cap F(F^*;E) \leq M \cap F = 1$, we have that $M(F^*;E) \cap F(F^*;E) = 1$. Now $M(F^*;E) \leq N(F^*)$, the norm of F^*. Hence if $x \in M(F^*;E)$, then the inner automorphism induced by x is a dilation on F^*. Since the dilations of a group commute, we have for $x \in M(F^*;E)$, $y \in F(F^*;E)$ and $\theta \in E$, that $(y^\theta)^x = (y^x)^\theta$. Since $y \in F(F^*;E)$, we have that $y^\theta = y$; therefore $(y^\theta)^x = y^x$. On the other hand:

$$\begin{aligned}(y^\theta)^x &= (y^x)^\theta \\ &= (x^{-1}yx)^\theta \\ &= (x^\theta)^{-1}y^\theta x^\theta \\ &= (x^\theta)^{-1}yx^\theta.\end{aligned}$$

Therefore $x^{-1}yx = (x^\theta)^{-1}yx^\theta$; hence $x^\theta x^{-1}y(x^\theta x^{-1})^{-1} = y$. Consequently $x^\theta x^{-1} \in C_{F^*}(\langle y\rangle)$. Since x, y and θ are arbitrary, $M(F^*;E) = \langle x^\theta x^{-1} / x \in F^*, \theta \in E\rangle \leq C_{F^*}(F(F^*;E))$. We are now in a position to show that $(|M|, |F(F^*;E)|) = 1$.

If p is a prime divisor of $|M(F^*;E)|$ and $|F(F^*;E)|$, let $x \in M(F^*;E)$, $y \in F(F^*;E)$, such that $|x| = |y| = p$. If $\theta \in E$, then

$$(xy)^\theta = (xy)^{s(xy;\theta)}, \text{ since } xy \in F^*$$
$$= x^{s(xy,\theta)}y^{s(xy;\theta)}, \text{ since } x \text{ centralizes } y.$$

On the other hand, $(xy)^\theta = x^\theta y^\theta$

$$= xy^{s(y;\theta)}, \text{ since } x \in F(F^*;E).$$

Therefore $x^{s(xy;\theta)}\, y^{s(xy;\theta)} = xy^{s(y;\theta)}$ or

$$x^{-1}x^{s(xy;\theta)} = y^{s(y;\theta)}y^{-s(xy;\theta)}.$$

Since $F(F^*;E) \cap M(F^*;E) = 1$, we must have $x^{-1}x^{s(xy;\theta)} = 1$ and $y^{s(xy;\theta)}\, y^{-s(y;\theta)} = 1$. The former gives $x^{s(xy;\theta)} = x$ or $s(xy;\theta) \equiv 1 \mod p$; the latter gives $y^{s(xy;\theta)} = y^{s(y;\theta)}$ or $s(xy;\theta) \equiv s(y;\theta) \mod p$. Consider the congruence system:

$$s(xy;\theta) \equiv 1 \mod p$$
$$s(xy;\theta) \equiv s(y;\theta) \mod p.$$

Consequently, $s(y;\theta) \equiv 1 \mod p$ or

$$y^\theta = y^{s(y;\theta)}$$
$$= y^{1+kp}$$
$$= (y^p)^k y$$
$$= y, \text{ since } y^p = 1.$$

Hence $y \in F(F^*;E) \cap M = 1$, a contradiction. Hence $(|M|, |F(F^*;E)|) = 1$.

Let P be a p-Sylow subgroup of F^*. If $x \in P$ such that $x^\theta = x$ for all $\theta \in E$, then $x \in F(P;E)$. Consequently $M(P;E) \leq M(G;E)$ must be equal to the identity, since

$(|F(F^*;E)|, |M(F^*;E)|) = 1$. Hence, if p divides $|F(F^*;E)|$, then the p-Sylow subgroup of F^* is contained in $F(G;E)$. Hence $F(F^*:E)$ is a Hall subgroup of F^* and is characteristic in F^*. Hence $F(F^*;E)$ has a complement K in F^*. We claim that $M(G;E)$ is contained in K, for if $g \in F^*$, $g = fk$ with $f \in F(F^*;E)$, $k \in K$; then for $\theta \in E$, $g^\theta g^{-1} = (kf)^\theta (kf)^{-1} = k^\theta k^{-1}$. Hence $M(F^*;E) = M(K;E) \leq K$. Using $F(K;E) = 1$ by Theorem 3.7, and the fact that K is a nilpotent group, we have $M(K;E) = M(G;E) = K$ and $F = F(F^*;E) \times M(G;E)$. (2) and (3) follow.

Since $M(F^*;E)$ is abelian, we must have that every $\theta \in E$ must induce a power automorphism on $M(F^*;E)$. If $x \in M(F^*;E)$ and $y \in F(F^*;E)$, then $x^\theta = x^{s(\theta)}$, where $s(\theta) \equiv 1 \mod \exp M$ and $y^\theta = y$. If $g \in F^*$, then $g = xy$, with $x \in M$, $y \in F(F^*;E)$. Therefore:

$$\begin{aligned} g^\theta &= (xy)^\theta \\ &= x^{s(\theta)} y \\ &= x^t y^t \\ &= (xy)^t \quad \text{and } \theta \text{ is a power automorphism} \end{aligned}$$

on F^*. Here, t is the solution to the congruence system

$$x \equiv s(\theta) \mod \exp M$$

$$x \equiv 1 \mod \exp F(F^*;E).$$

Since $(\exp M, \exp F(F^*;E)) = 1$ and $\exp M \exp F(F^*;E) = \exp F^*$, the above congruence system has a unique solution, modulo the exponent of F^*.

In Chapter II, we saw that a solvable π-group may admit automorphisms $\theta \in B_0$, whose order is a π'-element. For

these automorphisms, as well as a more general class of automorphisms, we are in a position to apply some of the previous results. We will start with a fundamental theorem which will be needed in what follows.

<u>Theorem 3.12</u>: (H. J. Jaschke) Let B be a group of automorphisms of a group G, U a solvable subgroup of G, and let M be a collection of cosets of U in G. If $(|B|, |U|) = 1$ and B leaves U invariant and permutes the cosets of M, then there exists a collection of coset representatives for the cosets of U in G which remains fixed elementwise by B; and the B-fixed subgroup $F(G;B)$ is a supplement for U in G.

<u>Proof</u>: For the cosets of M let $K = (r_1, r_2, \ldots, r_n)$ be a corresponding collection of coset representatives. $\bar{g} \in K$ will denote the coset representative of the coset $Ug \in M$, which contains the element g of the complex UK of G. Since B permutes the cosets of M, we have for $\alpha \in B$, $g \in UK$:

$$(Ug)^{\alpha} = Ug^{\alpha} = U\overline{g^{\alpha}} \quad \text{with} \quad \overline{g^{\alpha}} \in K.$$

Hence there exists $u_{g,\alpha} \in U$ such that $g^{\alpha} = u_{g,\alpha}\overline{g^{\alpha}}$

For $\alpha, \beta \in B$, $r \in K$ we have:

$$r^{\alpha} = u_{r,\alpha}\overline{r^{\alpha}}$$

$$r^{\alpha\beta} = u_{r,\alpha\beta}\overline{r^{\alpha\beta}} = (u_{r,\alpha}\overline{r^{\alpha}})^{\beta} = u^{\beta}_{r,\alpha}u_{\overline{r^{\alpha}},\beta}\overline{\overline{r^{\alpha}}^{\beta}} .$$

By the equality of coset representatives, we have:

$$u_{r,\alpha\beta} = u^{\beta}_{r,\alpha}\, u_{\overline{r^{\alpha}},\beta} .$$

Hence (1) $$u^{\beta^{-1}}_{r,\alpha\beta} = u_{r,\alpha}\, u^{\beta^{-1}}_{\overline{r^{\alpha}},\beta} .$$

Consider (1) for all $\beta \in B$ and fixed $\alpha \in B$ and fixed $r \in K$. As β runs through B, we obtain $|B|$ equations of type 1. If we multiply these $|B|$ equations and let $u^{*}_{g} = \pi_{\beta\in B}\, u^{\beta^{-1}}_{g,\beta}$, we obtain

$$u^{*\alpha}_{r} = \pi_{\beta\in B}\, u^{(\alpha\beta)^{-1}\alpha}_{r,\alpha\beta} = \pi_{\beta\in B}\, u^{\beta^{-1}}_{r,\alpha\beta} = \pi_{\beta\in B}\, u_{r,\alpha}\, u^{\beta^{-1}}_{\overline{r^{\alpha}},\beta} .$$

The elements $u_{g,\gamma}$ with $g \in UK$, $\gamma \in B$, permute modulo U'. Hence, letting $u^{*}_{g} = \pi_{\beta\in B}\, u^{\beta^{-1}}_{g,\beta}$, we obtain

(2) $$u^{*\alpha}_{r} = \pi_{\beta\in B}\, u_{r,\alpha}\, u^{\beta^{-1}}_{\overline{r^{\alpha}},\beta} \equiv u^{|\beta|}_{r,\alpha}\, \pi u^{\beta^{-1}}_{\overline{r^{\alpha}},\beta} \equiv u^{|\beta|}_{r,\alpha}\, u^{*}_{\overline{r^{\alpha}}} \bmod U' .$$

Since $(|B|, |U|) = 1$, we have that $|B|$ has a unique inverse $|B|^{-1}$ modulo the order of $u_{r,\alpha}$, and since $u^{*\,|\beta|^{-1}}_{g} \equiv u_{g} \bmod U'$, we have:

(3) $$(u^{-1}_{r} r)^{\alpha} = u^{-\alpha}_{r}\, r \equiv u^{-1}_{\overline{r^{\alpha}}}\, u^{-1}_{r,\alpha}\, u_{r,\alpha}\, \overline{r^{\alpha}} = u^{-1}_{\overline{r^{\alpha}}}\, \overline{r^{\alpha}} \bmod U' .$$

for $\alpha \in B$, $r \in k$

U' is characteristic in U; hence U' remains invariant under B. Hence B leaves U' invariant and permutes, according to (3), the cosets:

$$U'\, u^{-1}_{r_1} r_1 ,\quad U'\, u^{-1}_{r_2} r_2 \ldots \text{ of } U' \text{ in } G.$$

Apply induction on the solvable length of the subgroup U, assuming the theorem holds for all solvable subgroups whose derived length is smaller than that of U. Then there exists for the cosets $U'u_{r_1}^{-1}r_1$, $U'u_{r_2}^{-1}$, ... of U' in G, a system of coset representatives $K^* = (r_1^*, r_2^*, \ldots)$, which is mapped onto itself by B. Since $U'u_r^{-1}r \subset Ur$ for every $r \in K$, then K^* is also a collection of coset representatives K which is mapped onto itself by B. If B leaves every coset of U in G invariant, then K remains fixed elementwise and hence $K \leq F(G;B)$. Hence, for every $g \in G$, we have $g \in Uk$ or $g = uk$ with $u \in U$, $k \in K \leq F(G;B)$; hence $G = U \cdot F(G;B)$ and the theorem follows.

<u>Theorem 3.13</u>: Let E be a subgroup of B_o, such that $(|E|, |M(G;E)|) = 1$; then:

(1) G is the semi-direct product of $M(G;E)$ and $F(G;E)$.

(2) $M(G;E)$ is a normal complement for $F(G;E)$.

(3) $M(G;E) = M(M(G;E);E) = M(F^*;E)$.

(4) $M(G;E)$ is an abelian group of odd order.

(5) Every $\theta \in E$ induces a power automorphism on F^*.

<u>Proof</u>: By the previous Theorem, we have that G is generated by $F(G;E)$ and $M(G;E)$. By Lemma 3.9 $F(G;E) \cap M(G;E) = 1$ and $M(G;E) = M(MG;E);E) = M(F^*;E)$. By Theorem 3.6, we obtain (4) and by Theorem 3.11, we obtain (5).

At this point it might be worthwhile to give an example.

<u>Example 9</u>: Let $G_1 = \langle a,b,c \,/\, a^3 = b^3 = c^3 = 1;\ [a,b] = [a,c] = 1;\ [b,c] = a\rangle$, $|G_1| = 3^3$, G_1 is nilpotent and $G_1' = \Phi(G_1) = Z(G_1) = \langle a\rangle$. If $\theta \in B_0(G_1)$, then θ must fix all subgroups of G_1; moreover, since G_1 is a non-abelian 3-group, we must have that $B_0(G_1)$ is a 3-group. Hence θ has order a power of 3. Since $|\langle a\rangle| = |\langle b\rangle| = |\langle c\rangle| = 3$, then $a^\theta = a$; $b^\theta = b$; $c^\theta = c$. We must have that θ induces the identity on the subgroups $\langle a\rangle$, $\langle b\rangle$ and $\langle c\rangle$. Hence θ is the identity on G_1 and $B_0(G_1) = 1$. Let $G = \langle G_1 \,,\, d \mid d^2 = 1;\ [a,d] = [b,d] = 1 ;\ [d,c] = c\rangle$; then $B_0(G) = 1$. For if $\theta \in B_0(G)$, then θ restricted to G_1 is a dilation of G_1; hence θ must induce the identity on G_1. Since $|G/G_1| = 2$, θ must induce the identity on G/G_1. Hence θ must belong to the stability group of the chain $G > G_1 > 1$. Consequently $x^\theta x^{-1} \in Z(G_1) = \langle a\rangle$ for all $x \in G$. If $d^\theta = d$, then θ is the identity on G. If $d^\theta d^{-1} \neq 1$, then $d^\theta d^{-1} = a$ or $d^\theta d^{-1} = a^2$; then $d^\theta = ad$ or $d^\theta = a^2 d$; then $|d^\theta| = |a|\ |d|$ or $|d^\theta| = |a^2|\ |d|$. At any rate, $|d^\theta| = 6$; but $|d^\theta| = |d| = 3$, a contradiction; hence $d^\theta = d$ and θ is the identity on G.

We are now in a position to give an example of a supersolvable group, admitting a subgroup E of B_0, such that $(|E| \,,\, |G|) = 1$.

<u>Example 10</u>: Let $\overline{G} = G \times C_{11}$, where G is the group of the previous example and C_{11} is the cyclic group of order 11. Let $C_{11} = \langle e / e^{11} = 1\rangle$. Since $(|G|, |C_{11}|) = 1$, the map,

θ: $e \longrightarrow e^2$ $x \longrightarrow x$ for all $x \in \overline{G}$ is an automorphism of $\overline{G}$. Since θ is a dilation on C_{11} and a dilation on G and $(|G|, |C_{11}|) = 1$, θ is a dilation of G. Now $|\theta| = 10$; consequently $(|\theta|, |\overline{G}|) = 1$. $M(\overline{G};\theta) = C_{11}$, $F(\overline{G};\theta) = G$, $\overline{G} = F(\overline{G};\theta) \times M(\overline{G};\theta)$ and all other properties of the previous theorems can be shown to hold.

The previous examples show that a solvable group may admit automorphisms $\theta \in B_0$, such that $(|\theta|, |G|) = 1$. The next example shows that the whole group B_0 can have order relatively prime to the order of the group G, even though the group is not nilpotent.

<u>Example 11</u>: Let $G_1 = \langle a,b,c / a^7 = b^7 = c^7 = 1;$ $[a,b] = [a,c] = 1;$ $[b,c] = a\rangle$. The same argument as in the previous example, shows that $B_0(G_1) = 1$. Let $G = \langle G_1, d / d^3 = 1, [d,c] = 1, [d,a] = 1, [d,b] = a^2\rangle$; again $B_0(G_1) = 1$. Let $\overline{G} = G x C_{11}$, where C_{11} is the cyclic group of order 11 $C_{11} = \langle x / x^{11} = 1\rangle$. Then $B_0(\overline{G}) = \langle \theta / x^\theta = x^2; g^\theta = g$ for all $g \in G\rangle$. Then $|B_0(\overline{G})| = 10$, $B_0(\overline{G})$ is cyclic and $(|B_0(G)|, |\overline{G}|) = 1$.

For the case that $(|B_0(G)|, |G|) = 1$, we will obtain some special results about the structure of the group G. We first have the following definition:

<u>Definition 3.14</u>: If θ is an automorphism of G such that $M(G;\theta) \leq Z(G)$, then θ is said to be a central automorphism.

<u>Theorem 3.15</u>: If E is a subgroup of B_0 such that E is normal in $A(G)$ and if $(|E|, |M(G;E)|) = 1$, then:

(1) E consists of central automorphisms.

(2) $G = M(G;E) \times F(G;E)$ where $M(G;E) \leq Z(G)$ and $F(G;E) \geq G'$.

(3) $F(G;E)$ is a normal Hall subgroup of G.

<u>Proof</u>: If $(|E|, |M(G;E)|) = 1$, then if $\theta \in E$, θ induces the identity automorphism on $G/M(G;E)$. By Theorem 3.12, G is generated by $M(G;E)$ and $F(G;E)$. By Lemma 3.9, this implies that $M(G;E) = M(M(G;E);E)$, and by Theorem 3.6, we must have that $M(G;E)$ is an abelian group. Since $M(G;E)$ is an abelian normal subgroup of G, an automorphism $\theta \in E$ must induce a power automorphism on $M(G;E)$. Since the power automorphisms of a group are contained in the center of the automorphims group, the group $[E, I(G)]$ must belong to the stability group of the chain $G > M(G;E) > 1$. Therefore, by Theorem 2.10, $[E, I(G)]$ is a π-group if $M(G;E)$ is a π-group. Since E is a normal subgroup of $A(G)$, we must have that $[E, I(G)] \leq E$. Since $[E, I(G)]$ is a π-group and E is a π'-group, we must have $[E, I(G)] = 1$ and therefore E and $I(G)$ permute. By Theorem T-7, this implies that $M(G;E) \leq Z(G)$ and that E consists of central automorphisms.

If $M(G;E) \leq Z(G)$, then $F(G;E)$ is a normal subgroup of G. By Theorem 3.13, we have that G is the semi-direct product of $F(G;E)$ and $M(G;E)$. Since $F(G;E)$ is normal in G, we must have that G is the direct product of $M(G;E)$ and $F(G;E)$.

If $F(G;E) = 1$, then $F(G;E)$ is a Hall subgroup of G. If $F(G;E) \neq 1$, let $F^* = G_0 < G_1 < \cdots < G_n = G$ be a composition series joining the Fitting subgroup and G. If we consider E restricted to G_1, then E consists of central automorphisms on G_1; moreover, $G_1' \leq F^*$; therefore $G_1' \leq F(F^*;E)$. By Theorem 3.11, we have that $F(F^*;E)$ is a Hall subgroup of F^*. Since $M(F^*;E) = M(G;E)$, we must have that $M(G_1;E) = M(G;E)$. If p divides $(|F(G_1;E)|, |M(G_1;E)|)$, then p does not divide $|G_1'|$ since $F(F^*;E) \geq G_1'$ and $(|F(F^*;E)|, |M(G;E)|) = 1$. Let $y \in F(G_1;E)$ and $x \in M(G_1;E)$ such that $|x| = |y| = p$. We may choose x and y as coset representatives of G_1/G_1'. If $\theta \in E$, then θ induces a power automorphism on G_1/G_1'. In other words:

$(G_1'z)^\theta = G_1'z^{s(\theta)}$ where $(s(\theta), \exp G/G_1) = 1$. Hence $(G_1'x)^\theta = G_1'x^\theta = G_1'x^{s(\theta)}$ and $(G_1'y)^\theta = G_1'y = G_1y$, but $y \in F(G_1;E)$; hence $y^\theta = y^{s(\theta)} = y$. Consequently $s(\theta) \equiv 1$ mod p; hence $(G_1'x)^\theta = G_1'x^\theta = G_1'x^{1+kp} = G_1'x$ or $x^\theta x^{-1} \in G_1' \cap M(G_1;E) = 1$. Therefore $x^\theta = x$, hence $x \in M(G_1;E) \cap F(G_1;E) = 1$; consequently $x = 1$, and $(|M(G_1;E)|, |F(G_1;E)|) = 1$. We may assume $(|F(G_i;E)|, |M(G_i;E)|) = 1$ for $i = 1, 2, \cdots, n-1$. Then $G' \leq F(G;E)$ and since $G' \leq F(G_{n-1};E)$, we must have $(|G'|, |M(G_{n-1};E)|) = 1$;

but $M(G_{n-1};E) = M(G_n;E)$. Consequently $(|G'|, |M(G;E)|) = 1$. If $y \in F(G;E)$ and $x \in M(G;E)$ such that $|x| = |y| = p$, then the same argument as for G_1 shows that $x \in F(G;E) \cap M(G;E) = 1$. Hence $|F(G;E)|, |M(G;E)|) = 1$, and the Theorem is proven.

Lemma 3.16: Let H and K be subgroups of a group G. Let $G = H \times K$ with $(|H|, |K|) = 1$. Then:

(1) Any dilation on K or H may be extended to a dilation for G.

(2) Any power automorphism on K or H may be extended to a power automorphism for G.

Proof: Let θ be a dilation of K; let $\overline{\theta}$ be the extension of θ to G such that θ restricted to H is the identity. Then $\overline{\theta}$ is an automorphism of G. Let $g \in G$; then $g = hk$, with $h \in H$, $k \in K$.

$$g^{\overline{\theta}} = (hk)^{\overline{\theta}} = h\,k^{\theta} = hk^{s(k,\theta)} \text{ where } (s(k;\theta), |k|) = 1.$$

Now if t is an integer,

$$g^t = (hk)^t = h^t k^t, \quad \text{since } H \text{ and } K \text{ permute.}$$

The congruence system

$$t \equiv s(k;\theta) \mod |k|$$

$$t \equiv 1 \mod |h|$$ has a unique solution, modulo the order of $|hk| = |h|\;|k|$. Hence θ maps every $g \in G$ onto a power $g^{s(g;\theta)}$; hence θ is a dilation of G.

If θ is a power automorphism of K, then $k^{\theta} = k$, where $(s(\theta), \exp K) = 1$ for all $k \in K$.

The congruence system

$$x \equiv s(\theta) \bmod \exp K$$
$$x \equiv 1 \quad \bmod \exp H$$

has a unique solution modulo, the exp G, since (exp K, exp H) = 1; exp K $\cdot$ exp H = exp G. Hence $g^{\overline{\theta}} = (hk)^{\overline{\theta}} = hk^{s(\theta)} = h^x k^x = (hk)^x$ and $\overline{\theta}$ is a power automorphism on G.

In case $(|B_0|, |G|) = 1$, the group G must have a special structure, as will be shown from the following theorem.

<u>Theorem 3.17</u>: If $(|B_0|, |G|) = 1$, then

(1) $G = M(G;B_0) \times F(G;B_0)$, where $M(G;B_0) \preceq Z(G)$; $F(G;B_0) \succeq G'$.

(2) $F(G;B_0)$ is a normal Hall subgroup of G.

(3) $M(G;B_0)$ is abelian.

(4) If P_i is a p_i-Sylow subgroup of $M(G;B_0)$, then P_i is an elementary abelian p_i-group. Moreover, $(p_i-1, |G|)$.

(5) B_0 consists of power automorphisms.

(6) B_0 is abelian.

(7) $B_0(F(G;B_0)) = 1$.

<u>Proof:</u> Theorem 3.15 implies (1), (2) and (3).

The group $M(G;B_0)$ is a Hall subgroup of G. Hence, by the previous Lemma, any dilation or power automorphism of $M(G;B_0)$ may be extended to a dilation or power automorphism of G. If $\theta \in B_0(G)$, then θ must fix $F(G;B_0)$

elementwise. Moreover θ must induce a power automorphism on $M(G;B_0)$, since $M(G;B_0)$ is abelian: therefore all of its dilations are power automorphisms. By Lemma 3.16, θ is a power automorphism on G. Consequently B_0 consists of power automorphisms. Since the power automorphisms of a group are contained in the center of the automorphism group, we have that B_0 is abelian.

Since $M(G;B_0)$ is abelian, we have that $M(G;B_0) = P_1 \times P_2 \times \ldots \times P_n$, where P_i is an abelian p_i-Sylow subgroup of $M(G;B_0)$. Consequently, $B_0(M(G;B_0)) = B_0(P_1) \times B_0(P_2) \times \ldots \times B_0(P_n)$. Since every dilation of $B_0(P_i)$ can be extended to a dilation for $M(G;B_0)$ and consequently to a dilation for G, we can consider $B_0(P_i)$. By a previous Theorem, we have that $B_0(P_i)$ is isomorphic to the prime residue classes module the exp P_i. Therefore if exp $P = p_i^{n_i}$, then $|B_0(P_i)| = p_i^{n_i-1}(p_i-1)$. Consequently if $n_i > 1$, P_i has a dilation of order p_i. Therefore G has a dilation of order p_i, but p_i divides $|G|$. This is contrary to the fact that $(|B_0(G)|, |G|) = 1$. Therefore $n_i = 1$, and P_i is an elementary abelian p_i-group. By the same reasoning, we obtain that $(p_i-1, |G|) = 1$; therefore (4) follows.

Let $F = F(G;B_0)$. Let $B_0(F)$ denote the group of all automorphisms of F that fix all subnormal subgroups of G. Let $\theta_0 \in B_0(F)$ extend θ to an automorphism $\overline{\theta}$ of G by letting $\overline{\theta}$ be the identity of $M(G;B_0)$. Let H be a subnormal subgroup of G, then $H^\alpha = H$ for all $\alpha \in B(G)$.

Since $(|B_0(G)|, |M(H;B_0)|) = 1$, we must have that $H = F(H;B_0(G)) \times M(H;B_0(G))$. Now $F(H;B_0(G)) = H \cap F(G;B_0)$ and since H and $F(G;B_0)$ are subnormal in G, $F(H;B_0(G))$ is a subnormal subgroup of $F = F(G;B_0)$. Hence $H^{\overline{\theta}} = F(H;B_0(G))^{\overline{\theta}} \times M(H;B_0(G))^{\overline{\theta}} = F(H;B_0(G))^{\theta} \times M(H;B_0(G))^{\overline{\theta}}$. The automorphism $\overline{\theta}$ fixes all subnormal subgroups of H and is the identity on $M(G;B_0)$. Therefore:

$$\begin{aligned} H^{\overline{\theta}} &= F(H;B_0(G))^{\theta} \times M(H;B_0(G))^{\overline{\theta}} \\ &= F(H;B_0(G)) \times M(H;B_0(G)) \\ &= H. \end{aligned}$$

Therefore $\overline{\theta}$ fixes all subnormal subgroups of G; consequently $\overline{\theta} \in B_0(G)$. If $\overline{\theta} \in B_0(G)$, then θ must induce the identity of $F(G;B_0)$. Hence $\overline{\theta}$ is the identity and $B_0(F(G;B_0)) = 1$.

Let us now turn our attention to the inner automorphisms of G that fix every subnormal subgroup of G. Every inner automorphism α_g of G is induced by an element $g \in G$. We would like to investigate the subgroup $\overline{N}$ of G such that $g \in \overline{N}$ if α_g, the inner automorphism induced by g, fixes all subnormal subgroups of G.

<u>Lemma 3.18</u>: $\overline{N} = \bigcap_{H \lhd\lhd G} N_G(H)$

<u>Proof</u>: If $x \in \overline{N}$, then $H^x = H$ for all $H \lhd\lhd G$; therefore $x \in N_G(H)$. Since this holds for all $H \lhd\lhd G$, we must have $x \in \bigcap_{H \lhd\lhd G} N_G(H)$ and $\overline{N} \leq \bigcap_{H \lhd\lhd G} N_G(H)$. Conversely, if $x \in \bigcap_{H \lhd\lhd G} N_G(H)$, then $H^x = H$ for all $H \lhd\lhd G$. Consequently

$x \in \overline{N}$ and therefore $\overline{N} \geq \bigcap_{H \lhd\lhd G} N_G(H)$.

Theorem 3.19: If G is solvable, then $\overline{N} = \bigcap_{H \lhd\lhd G} N_G(H)$ has the following properties:

(1) $\overline{N}$ is a characteristic subgroup of G.

(2) If H is subnormal in $\overline{N}$, H is normal in $\overline{N}$.

(3) $\overline{N}$ is supersolvable.

(4) $\overline{N}'$ is abelian and $\overline{N}' \leq Z(F^*)$, where F^* is the Fitting subgroup of G.

Proof: If $\alpha \in A(G)$, and $H \lhd\lhd G$, then $H^\alpha \lhd\lhd G$ and $N_G(H)^\alpha = N_G(H^\alpha)$. Hence $\overline{N}^\alpha = \bigcap_{H^\alpha \lhd\lhd G} N_G(H^\alpha) = \bigcap_{H \lhd\lhd G} N_G(H)$ and $\overline{N}$ is characteristic.

If H is a subnormal subgroup of N, then, since $\overline{N}$ is normal in G, we must have $H \lhd\lhd G$. Consequently $\overline{N} \leq N_G(H)$. Since $H \leq \overline{N}$, this implies that H is normal in $\overline{N}$.

Since $\overline{N} \leq G$ and G is solvable, we must have that $\overline{N}$ is solvable. Hence $\overline{N}$ has a composition series with composition factors of prime order and, since every composition subgroup of $\overline{N}$ is a normal subgroup of $\overline{N}$, this implies that N has a chief series with chief factors of prime order. Hence $\overline{N}$ is a supersolvable group.

If $x \in N$, and H is a subgroup of the Fitting subgroup $F^*(G)$ of G, then $H^x = H$ since $H \lhd\lhd G$; therefore $x \in \overline{N}$ must induce a dilation on F^*. Since the dilations of F^* form an abelian group, then $[x,y] = x^{-1}y^{-1}xy$ must centralize F^*. Hence $[x,y] \in C_G(F^*) = Z(F^*)$.

Since x and y were arbitrary, this means that $\overline{N}' \leq Z(F^*)$.

Since the subgroup $\overline{N}$ possesses such nice properties, one might conjecture that $\overline{N}$ is nilpotent. This is not the case, as may be seen from Example 1. In this example $G = S_3$, the symmetric group on three letters. Since A_3, the alternating group on three letters is the only subnormal subgroup of G, we must have $\overline{N} = G$. Consequently $\overline{N}$ is not nilpotent.

If every inner automorphism fixes every subnormal subgroup of G, then this imposes strong conditions on the structure of the group G, as may be seen from the following Theorem.

<u>Theorem 3.20</u>: Let G be a solvable group. If $2 \nmid G$ and $\overline{N} = G$, then:

(1) Every subnormal subgroup of G is normal in G.

(2) G is supersolvable and G' is abelian.

(3) All Sylow subgroups of G are abelian; i.e., G is an A-group.

(4) $G = G'K$, $G' \cap K = 1$ and $K = N_G(K) = C_G(K)$.

(5) $F^*(G) = G'Z(G)$.

<u>Proof</u>: (1) and (2) follow from the previous Theorem.

If G is supersolvable, then G has a Sylow Tower for the natural ordering of the primes. In other words, G has a normal chain:

$1 \leq S_{p_1} \leq S_{p_1} S_{p_2} \leq \cdots \leq S_{p_1} S_{p_2} \cdots S_{p_n} = G$, where S_{p_i}

is a p_i-Sylow subgroup of G and $p_1 \geq p_2 \geq \cdots \geq p_n$. If H is a subgroup of S_{p_1}, then H is subnormal in S_{p_1}. Since S_{p_1} is normal in G, H must be subnormal in G. By (1), H is normal in G. Hence, all subgroups of S_{p_1} are normal in S_{p_1}, and S_{p_1} is a Hamiltonian group. Since p is odd, we must have by Theorem T-9 that S_{p_1} is abelian. The same argument as above shows that $\prod_{j=1}^{i} S_{p_j} / \prod_{j=1}^{i-1} S_{p_j}$ is an abelian subgroup of $G/\prod_{j=1}^{i-1} S_{p_j}$. Since $\prod_{j=1}^{i} S_{p_j} / \prod_{j=1}^{i-1} S_{p_j}$ is isomorphic to S_{p_i}, we have that S_{p_i} is abelian for $i = 1,2, \cdots, n$. Therefore G is an A-group.

By a Theorem of Taunt [6], we have that G' can be complemented. Therefore $G = G'K$ with $G' \cap K = 1$. Let $x \in N_G(K) \cap G'$, then for all $y \in K$, $[x,y] \in G' \cap K = 1$. This implies that x permutes with all elements of K. Hence x permutes with G' and with K; consequently $x \in Z(G) \cap G'$. By another Theorem of Taunt [6], we have that for an A-group, $G' \cap Z(G) = 1$; consequently, $N_G(K) = K$. Since $K \cong G/G'$, we have that K is abelian; therefore $N_G(K) = C_G(K) = K$.

Since $G' \leq F^*$ and F^* is normal in G, we must have $F^* = G'(F^* \cap K)$. Let $x \in F^* \cap K$. Since K is abelian, x permutes with K. Moreover x permutes with all elements of F^*, since F^* is abelian. Therefore x permutes with G' and with K. Consequently $x \in Z(G)$. Therefore

$F^* \cap K \leq Z(G)$; Hence $G'(F^* \cap K) \leq G'Z(G)$. Since G' and $Z(G)$ are both abelian normal subgroups of G, we must have $G'Z(G) \leq F^*$. Therefore, $F^* = G'(F \cap K) \leq G'Z(G) \leq F^*$. Consequently, $G'Z(G) = F^*$.

INDEX OF NOTATION

I. Relations:

$\subseteq$	Is a subset of
$\subsetneqq$	Is a proper subset of
$\leq$	Is a subgroup of
$\lneqq$	Is a proper subgroup of
$\triangleleft$	Is a normal subgroup of
$\triangleleft\triangleleft$	Is a subnormal subgroup of
$\cong$	Is isomorphic to
$\in$	Is an element of
$\equiv$	Is congruent to

II. Operations:

G^θ	The image of G under the mapping θ
S^x	$x^{-1}Sx$
f/S	Automorphism of S induced by f
G/H	Factor group
$[x,y]$	The commutator of x and y
G^n	The n^{th} derived group of G
x	Direct product of groups
$G:H$	Index of H in G
$[H,K]$	Subgroup generated by all $[h,k]$, $h \in H$, $k \in K$
$\langle \ \rangle$	Subgroup generated by
$\{ \ \}$	Set whose members are
$\{x \mid P\}$	Set of all x such that P is true
$\lvert G \rvert$	Number of elements in G
$\lvert g \rvert$	Order of the element g

III. Groups and Sets, and Miscellaneous:

$A(G)$	Automorphism group of G
$I(G)$	Inner automorphism group of G
$\Delta(G)$	Dilation group of G
s	Chain of subgroups of the group G; $s: G = G_0 > G_1 > G_2 > \cdots > G_n = 1$
$S_0(s)$	$\{\theta \in A(G) \mid G_i^{\theta} = G_i \text{ for } i = 1,2,\cdots, n\}$
$S_i(s)$	$\{\theta \in S_{i-1} \mid (G_i x)^{\theta} = G_i x\}$
$C(G)$	Class of all composition series of G
$B_0(G)$	Set of all automorphisms of G fixing all subnormal subgroups
$B_i(G)$	$\bigcap_{s \in C(G)} S_i(s)$
$D(G)$	Class of all chief series
$A_0(G)$	Set of all automorphisms fixing all normal subgroups
$A_i(G)$	$\bigcap_{s \in D(G)} S_i(s)$
$M(G;E)$	$\langle g^{\theta} g^{-1} \mid g \in G;\ \theta \in E;\ \text{where } E \preceq A(G)\rangle$
$F(G;E)$	$\{g \in G \mid g^{\theta} = g \text{ for all } \theta \in E; \text{ where } E \preceq A(G)\}$
N	Norm of G
$\overline{N}$	$\{g \in G \mid \alpha_g \in B_0;\ \alpha_g$ is the inner automorphism induced by $g\}$
$Z(G)$	Center of G
$C_G(H)$	Centralizer of H in G
$N_G(H)$	Normalizer of H in G
$\Phi(G)$	Frattini subgroup of G
$F^*(G)$	Fitting subgroup of G
S_n	Symmetric group of degree n.

BIBLIOGRAPHY

1. Hall, P., Some Sufficient Conditions for a Group to be Nilpotent, Illinois Journal of Mathematics, vol. 2 (1958), pp. 787-801.

2. Huppert, B., Zur Sylowstructur Auflösbarer Gruppen, Archives der Mathematic, vol. 12 (1961), pp. 161-169.

3. Jäschke, K. H., Dilatationen Endlicher Gruppen. (Doctoral Thesis: Kiel 1963).

4. Polemini, A., A Study of Automorphisms and Chains of a Finite Group. (Doctoral Thesis: Michigan State University, 1965).

5. Scott, W. R., Group Theory (New Jersey: Prentice-Hall Inc., 1965).

6. Taunt, D. R., On A-Groups, Proceedings of the Cambridge Philosophical Society, vol. 45 (1949, pp. 24-42).

APPENDIX

THEOREMS

Theorem T-1:

If G is a finite group, the following are equivalent:

(i) G is nilpotent.

(ii) If M is a maximal subgroup of G, then $M \lhd G$.

(iii) $H < G$, then $H \subsetneq N_G(H)$.

(iv) G is a direct product of its Sylow subgroups.

Theorem T-2:

If G is a finite p-group and α is an automorphism of G, inducing the identity automorphism on $G/\phi(G)$, then $|\alpha| = p^i$ for some positive integer i.

Theorem T-3:

If A and B are normal nilpotent subgroups of a group G, then AB is also a normal nilpotent subgroup of G.

Theorem T-4:

If G is a solvable group having a maximum nilpotent characteristic subgroup H, then $H \geqslant C_G(H)$.

Theorem T-5: (P. Hall, 1)

If G is a group and $s: G = G_o > G_1 > \cdots > G_n = 1$ is a chain terminating in the identity and $A(s)$ is the stability group of s, then:

(i) $A(s)$ is nilpotent of class $\leq 1/2\, n(n-1)$.

(ii) If s is a normal chain, then $A(s)$ is nilpotent of class $\leq n-1$.

Theorem T-6:

If H is a normal Hall subgroup of a group G, then H has a complement.

Theorem T-7:

If θ is an automorphism of G, then the following are equivalent:

(i) θ is central

(ii) θ permutes with all inner automorphisms.

Theorem T-8:

If G is a p-group and θ is an automorphism of G such that $(|\theta|,p) = 1$, and θ fixes all normal subgroups of G, then the upper and lower central series coincide. In other words, if $1 = Z_0 \leq Z_1 = Z(G) \leq \cdots \leq Z_n = G$ is the upper central series and $Z^0 = G \geq Z' \geq \cdots \geq Z^n = 1$, is the lower central series of G, then $Z_i = Z^{n-i}$ for all i.

Theorem T-9: (5, pp. 253-254)

A group G is Hamiltonian iff $G = A \times B \times C$, where A is a quaternion group, B is an elementary abelian 2-group, and D is a periodic abelian group with all elements of odd order.