STATE-SPACE DESIGN AND OPTIMIZATION OF LINEAR TIME-INVARIANT SYSTEMS

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H. C. Koenig
Major professor

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ABSTRACT

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by Charles M. Bacon

State space concepts, traditionally applied to the analysis of nonlinear systems, are currently being extended to linear systems as well. For many years, the analysis and design of linear time-invariant systems was accomplished by Laplace transform and Fourier transform methods. However, the increased complexity of modern systems has emphasized the need for more effective design techniques, particularly ones which can be easily implemented on the digital computer.

This thesis presents a design technique which is applied directly to the state model

$$\dot{X}(t) = AX(t) + BE(t), \quad X(0) = X_o$$

$$Y(t) = CX(t) + DE(t)$$

Matrix equations, called fundamental design equations, are established which provide necessary and sufficient conditions for the state model to have a specified solution. These equations are written directly from the specified solution and furnish mathematical constraints on the matrices A, B, C, and

D. From these equations, the designer can generate a state model having a specified solution. The technique is applicable to vector input-vector output systems under either forced or unforced conditions. Any of those excitation functions traditionally used in s-domain design may be employed.

The relationship between the state model and the s-domain model is used to extend the fundamental design equations to the case where the design specifications are given in terms of a desired transfer function matrix rather than a desired time solution. This extension yields matrix equations in A, B, C and D which provide necessary and sufficient conditions for a state model to be equivalent to a specified transfer function matrix.

The fundamental design equations can be programmed directly on the digital computer. This leads immediately to a computer technique for parameter optimization. The least squared-error criteria is used together with the method of steepest descent to achieve optimization. A computer program implementing this technique is included. Several examples are included which illustrate the design and optimization methods.

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By

Charles M. Bacon

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I. INTRODUCTION

For over twenty years, the analysis and design of linear systems have been based on techniques derived from operational mathematics and associated transform methods. See, for example [1], [2], [3], [4], [5]. During this period, the frequency response characteristic, root locus plot and the analog computer have evolved as standard tools of the systems designer. Much to the disappointment of the engineer, the sophistication and capability of these tools has not kept pace with the increased complexity of the space-age systems. He has found that often the only acceptable system is the optimum system and that the "cut and try" procedures of ten years ago simply do not provide optimum designs. Faced with these problems, the systems designer has turned to the digital computer and the time-domain model in an attempt to develop more effective design techniques.

This thesis is concerned with the application of time-domain models to the design of linear time-invariant systems. The class of physical systems under consideration includes those having performance characteristics (voltages, currents, forces, pressures, displacements, etc.) which can be approximated by the equations

$$\dot{X}(t) = AX(t) + BE(t), \qquad X(0) = X_o \tag{1.1}$$

$$Y(t) = CX(t) + DE(t) \tag{1.2}$$

The vector time functions X(t), E(t) and Y(t) have dimension n, r and k, respectively, and the constant real matrices A, B, C and D are appropriately dimensioned.* The time derivative of X(t) is denoted by $\dot{X}(t)$ and the initial condition X_o is a constant n-vector.

Notationally, the set of equations (1.1) are called state equations and denote n linear constant-coefficient ordinary differential equations in first-derivative explicit form with initial condition vector X_o. The k linear algebraic equations (1.2) are called the output equations. The composite set (1.1-2) is referred to as the state model of the system and X(t), Y(t) and E(t) as the state vector, output vector and excitation vector, respectively.

By comparison, the most frequently used s-domain model takes the form

$$\bar{Y}(s) = \bar{M}(s)\bar{E}(s) \tag{1.3}$$

where $\bar{Y}(s)$ and $\bar{E}(s)$ are the Laplace transforms of Y(t) and E(t), respectively, and $\bar{M}(s)$ is called the transfer function matrix. The transforms $\bar{Y}(s)$, $\bar{E}(s)$ and $\bar{M}(s)$ are complex-

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*Throughout this thesis, upper case letters will denote matrices or vectors and lower case letters will denote scalars. All matrices and vectors are real (or real-valued functions) unless otherwise noted.

valued functions of the complex variable s. Provided the transforms exist, the s-domain model (1.3) is derivable directly from (1.1-2).

The total design problem embraces several separate tasks in which the mathematical model plays a central role. The model and its solution represent, respectively, the physical system and its response. In design, the specification of a desired system response equivalently specifies part or all of the solution of the model. To achieve the design, one must select the physical system possessing a model which has the specified solution. The usual order of events is to first derive the mathematical model from the specified solution and then attempt to realize this model in terms of a physical system.

The widespread use of the s-domain model in the design of scalar input-output systems stems from the fact that the form of the time response is directly related to the poles of the transfer function. Thus, given a desired system response, the s-domain model is quickly established. This property has been extended to multivariable (vector input-output) systems and methods are available for deriving the transfer function matrix from design specifications [6], [7], [8].

However, the realization of the s-domain model in terms of a physical system is, in general, quite difficult. Even for scalar input-output systems, only one or two

physical parameters can be analytically determined by the root-locus technique to yield a set of specified poles of the transfer function. Except in the network synthesis area, there have been no general analytical techniques advanced for the realization of vector input-output systems.

To offset the lack of general analytical methods, the designer has relied on the analog computer for the design of complex systems. However, the system parameters are found by a purely "cut and try" procedure and thus, it is not possible to achieve true optimal design with respect to many system parameters. In fact, since the word optimization implies an analytical method, it is doubtful if the analog computer will ever become effective in the optimal design of complex systems.

There is a need for an effective analytical method for linear system design which utilizes the capabilities of the digital computer. This thesis provides the first step in satisfying this need. The choice of the state model (1.1-2) over the s-domain model (1.3) as a basis for such a method is easily defended. The state model provides a more precise description of the system properties than does the s-domain model. Also, efficient techniques have been advanced for formulating the state model directly from the physical system without deriving the s-domain model [9], [10], [11], [12], [13]. Additionally, the state model is routinely used in the important areas of Liapunov stability

theory and optimal control [14], [15].

In Section II, a set of algebraic equations called fundamental design equations (FDE) are introduced. These equations provide necessary and sufficient constraints on the state model for the model to have a specified solution. The FDE allow the designer to derive the state model from the specified solution. These conditions are generally applicable to both forced and unforced systems with the excitation functions assuming any of the usual forms; step, ramp, sinusoid or any linear combination thereof. An additional set of necessary conditions are given which are easier to apply than the FDE. These can be used to quickly eliminate some state models which do not have the specified solution.

Kalman, Gilbert and others recently listed procedures for deriving the state model from a specified transfer function matrix $\overline{M}(s)$ [16], [17], [18]. The procedures are limited in two ways: (a) the poles of the entries of $\overline{M}(s)$ are assumed to be distinct and (b) the matrices A, B, C and D take only restricted forms, e.g., A is always diagonal and may have complex entries. Such forms do not normally occur when the state model is formulated directly from the physical system. Thus, some unknown transformation must be applied to the state variables before these procedures can be used in design.

In Section III, an extended set of fundamental design equations are shown to be necessary and sufficient for a state model to yield a specified transfer function matrix. These equations are not restricted by the multiplicity of the poles of the entries in $\overline{M}(s)$. Also, completely general forms of the matrices A, B, C and D in the state model can be derived.

An analytic technique for parameter optimization is proposed in Section IV. The method uses the digital computer and the numerical method of "steepest descent" to find the set of parameters yielding an "optimum" solution to the fundamental design equations. The "optimum" solution is defined by the least squared-error criterion. This technique represents the first step in applying the digital computer to optimal design via the state model.

Three appendices provide supplementary material including the computer program used in Section IV.

II. FUNDAMENTAL DESIGN EQUATIONS FROM TIME-DOMAIN SPECIFICATIONS

The theory regarding the existence and uniqueness of the solution to (1.1) is well established [19], [20]. If the r-vector E(t) is continuous for all $t \in T$, $T = \{t \mid 0 \leq t \leq t_1\}$, t_1 a non-zero constant, and the initial state X_o is finite, then (1.1) has the unique state solution

$$X(t) = e^{At}X_o + \int_o^t e^{A(t-u)}BE(u)du \qquad (2.1)$$

where $e^{At} = \Phi(t)$ is the n x n matrix function satisfying the homogeneous matrix system

$$\dot{\Phi}(t) = A\Phi(t) \quad , \quad \Phi(0) = U_n \qquad (2.2)$$

and U_n is the n-dimensional unit matrix. Using (2.1) in (1.2), the output solution of the state model for $t \in T$ is

$$Y(t) = Ce^{At}X_o + C\int_o^t e^{A(t-u)}BE(u)du + DE(t) \qquad (2.3)$$

Equation (2.3) provides a general expression for the output vector Y(t) in terms of the matrices A, B, C and D, the initial state X_o and the excitation vector E(t). Thus, if (1.1-2) is regarded as the mathematical model of a system,

Equation (2.3) gives the output performance characteristics as a function of the structure of the system, its initial state and the applied system excitation.

The particular design method developed in this thesis is based on an output specification of the form

$$Y(t) = N_s F_s(t) + N_e F_e(t) \qquad (2.4)$$

where $F_s(t)$ and $F_e(t)$ are real-valued vector functions having dimensions q_s and q_e, respectively, and the matrices N_s and N_e are $k \times q_s$ and $k \times q_e$, respectively.

If a desired system response is specified by (2.4) in terms of $F_s(t)$, $F_e(t)$, N_s and N_e, then (2.4) and (2.3) establish mathematical constraints on the matrices A, B, C, D and the vector X_o. Since the system parameters and system topology determine A, B, C, D, etc., these constraints represent restrictions on the system structure. The design procedure then utilizes these mathematical constraints to deduce the properties of the system.

The design procedure assumes that each component $e_i(t)$ of E(t) is representable as a finite sum

$$e_i(t) = \sum_{j=1}^{q} f_j(t)$$

where each $f_j(t)$ is the solution to some linear homogeneous constant coefficient differential equation of finite order.

This assumption does not place undue restrictions on the form of excitation. Typical system inputs such as the step, ramp and sinusoid are included without approximation and any input amenable to a Fourier series representation can be approximated by a finite set of such functions. Koenig and Tokad have utilized a similar restriction on E(t) to transform a set of nonhomogeneous differential equations to homogeneous form [21].

The main development requires the following definitions and lemma:

Definition 2.1

A set of real-valued scalar time functions $f_i(t)$, $i = 1,2,\ldots,q$, defined for all $t \in T$, $T = \{t \mid 0 \leq t \leq t_1\}$, t_1 a non-zero constant, are linearly independent on T if and only if the relation

$$\sum_{i=1}^{q} c_i f_i(t) = 0 \qquad t \in T \tag{2.5}$$

implies that every real scalar constant $c_i = 0$, $i = 1,2,\ldots,q$.

Definition 2.2

A real-valued q-vector function of time F(t) is a basis vector on T if and only if its components $f_i(t)$, $i = 1,2,\ldots,q$, are linearly independent and F(t) is the solution to

$$\dot{F}(t) = SF(t), \qquad F(0) = F_o \tag{2.6}$$

for some q x q matrix S. The homogeneous system (2.6) is called the basis system and S the basis matrix.

Lemma 2.1

Let F(t) be any basis vector of order q on T. Then for any n x q matrix M,

$$MF(t) = 0 \qquad t \in T \tag{2.7}$$

implies that M = 0.

Proof

Equation (2.7) implies

$$\sum_{j=1}^{q} m_{ij} f_j(t) = 0, \quad i=1,2,\ldots,n, \quad t \in T \tag{2.8}$$

But F(t) is a basis vector and from Definition 2.2, the components $f_j(t)$, j=1,2,...,q are linearly independent. Thus, by Definition 2.1, (2.8) implies that every $m_{ij} = 0$, i=1,2,...,n, j=1,2,...,q. The conclusion follows.

The following theorem provides the first step in synthesizing a system of differential equations having a specified solution.

Theorem 2.1

Hypothesis:

1) $T = \{t \mid 0 \leq t \leq t_1\}$, t_1 a non-zero constant.

2) $F_s(t)$ and $F_e(t)$ are vector functions of order q_s and q_e, respectively, satisfying, for all $t \in T$

$$\begin{bmatrix} \dot{F}_s(t) \\ \dot{F}_e(t) \end{bmatrix} = \begin{bmatrix} S_s & 0 \\ 0 & S_e \end{bmatrix} \begin{bmatrix} F_s(t) \\ F_e(t) \end{bmatrix} , \quad \begin{bmatrix} F_s(0) \\ F_e(0) \end{bmatrix} = \begin{bmatrix} F_{so} \\ F_{eo} \end{bmatrix} \tag{2.9}$$

where S_s and S_e are square submatrices of order q_s and q_e, respectively, and F_{so} and F_{eo} are constant q_s- and q_e-vectors, respectively.

3) The vector function

$$F(t) = \begin{bmatrix} F_s(t) \\ F_e(t) \end{bmatrix} \tag{2.10}$$

is a basis vector on T.

4) The r-vector function E(t) is defined on T by

$$E(t) = HF_e(t) \tag{2.11}$$

for some r x q_e matrix H.

5) The n-vector function $\overline{X}(t)$ is defined on T by

$$\overline{X}(t) = G_s F_s(t) + G_e F_e(t) \tag{2.12}$$

where the matrices G_s and G_e are n x q_s and n x q_e, respectively.

Conclusion:

Necessary and sufficient conditions for $\bar{X}(t)$ to be the solution on T of

$$\dot{X}(t) = AX(t) + BE(t),\ X(0) = X_o \tag{2.13}$$

provided such an equation system exists, are that

$$AG_s = G_sS_s, \tag{2.14}$$

$$AG_e + BH = G_eS_e \tag{2.15}$$

and

$$G_sF_{so} + G_eF_{eo} = X_o \tag{2.16}$$

The existence of the system (2.13) depends on the existence of matrices A and B satisfying (2.14) and (2.15).

Proof

Sufficiency: Assume that Matrices A and B exist satisfying (2.14-16). Taking the derivative of $\bar{X}(t)$ gives

$$\dot{\bar{X}}(t) = G_s\dot{F}_s(t) + G_e\dot{F}_e(t) \tag{2.17}$$

or, by virtue of the basis system (2.9),

$$\dot{\bar{X}}(t) = G_sS_sF_s(t) + G_eS_eF_e(t) \tag{2.18}$$

Using (2.14-15) and (2.11), (2.18) becomes

$$\begin{aligned}\dot{\bar{X}}(t) &= AG_sF_s(t) + (AG_e + BH)F_e(t)\\ &= A[G_sF_s(t) + G_eF_e(t)] + BHF_e(t)\\ &= A\bar{X}(t) + BE(t)\end{aligned} \tag{2.19}$$

To show that $\bar{X}(t)$ also satisfies the initial conditions, we set t=0 in (2.12) and apply (2.16) to obtain

$$\bar{X}(0) = G_s F_{so} + G_e F_{eo} = X_o \tag{2.20}$$

Thus, from (2.19-20), $\bar{X}(t)$ is the solution of (2.13) and sufficiency is established.

Necessity: Assuming the hypothesis, let $\bar{X}(t)$ in (2.12) be solution of (2.13). It is necessary to show that (2.14-16) follow. Equation (2.16) follows immediately from (2.12) and the assumption that $\bar{X}(0) = X_o$. Substituting $\bar{X}(t)$ into both sides of (2.13) gives

$$\begin{aligned} \dot{\bar{X}} &= G_s \dot{F}_s(t) + G_e \dot{F}_e(t) \\ &= A\,[G_s F_s(t) + G_e F_e(t)] + BE(t) \end{aligned} \tag{2.21}$$

or by (2.9) and (2.11)

$$\begin{aligned} &G_s S_s F_s(t) + G_e S_e F_e(t) \\ &\quad = AG_s F_s(t) + AG_e F_e(t) + BHF_e(t) \end{aligned} \tag{2.22}$$

Thus

$$\begin{bmatrix} G_s S_s - AG_s & G_e S_e - AG_e - BH \end{bmatrix} \begin{bmatrix} F_s(t) \\ F_e(t) \end{bmatrix} = 0 \tag{2.23}$$

By assumption, (2.23) must hold for all $t \in T$. But F(t) defined by (2.10) is a <u>basis</u> vector on T. Thus, by Lemma 2.1,

the coefficient matrix in (2.23) must vanish. A column-wise expansion of this coefficient matrix establishes (2.14-15) and necessity is proved.

An alternate proof of Theorem 2.1 is based on the homogeneous form of the state model [21]. It is easy to verify that necessary and sufficient conditions for a vector function $\bar{X}(t) = GF(t)$ to be the solution to a homogeneous model $\dot{X}(t) = AX(t)$, $X(0) = X_o$, are

$$AG = GS \tag{2.24}$$

$$GF_o = X_o \tag{2.25}$$

where $F(t)$ is a basis vector with basis matrix S. With the assumption that the excitation vector $E(t) = HF_e(t)$, where $\dot{F}_e(t) = S_e F_e(t)$, $F_e(0) = F_{eo}$, the nonhomogeneous model

$$\dot{X}(t) = AX(t) + BE(t), \qquad X(0) = X_o \tag{2.26}$$

reduces to the homogeneous form

$$\begin{bmatrix} \dot{X}(t) \\ \dot{F}_e(t) \end{bmatrix} = \begin{bmatrix} A & BH \\ 0 & S_e \end{bmatrix} \begin{bmatrix} X(t) \\ F_e(t) \end{bmatrix}, \quad \begin{bmatrix} X(0) \\ F_e(0) \end{bmatrix} = \begin{bmatrix} X_o \\ F_{eo} \end{bmatrix} \tag{2.27}$$

Now, applying (2.24) and (2.25) to the homogeneous model (2.27), it follows that necessary and sufficient conditions for the vector function

$$\begin{bmatrix} X(t) \\ F_e(t) \end{bmatrix} = \begin{bmatrix} G_s & G_e \\ 0 & U \end{bmatrix} \begin{bmatrix} \overline{F}_s(t) \\ F_e(t) \end{bmatrix} \tag{2.28}$$

to be the solution of (2.27) are that

$$\begin{bmatrix} A & BH \\ 0 & S_e \end{bmatrix} \begin{bmatrix} G_s & G_e \\ 0 & U \end{bmatrix} = \begin{bmatrix} G_s & G_e \\ 0 & U \end{bmatrix} \begin{bmatrix} S_s & 0 \\ 0 & S_e \end{bmatrix} \tag{2.29}$$

and

$$\begin{bmatrix} G_s & G_e \\ 0 & U \end{bmatrix} \begin{bmatrix} F_{so} \\ F_{eo} \end{bmatrix} = \begin{bmatrix} X_o \\ F_{eo} \end{bmatrix} \tag{2.30}$$

where $\dot{F}_e(t) = S_e F_e(t)$, $F_e(0) = F_{eo}$ and the vector $F(t) = [F_s(t)\ F_e(t)]^T$ is a basis vector. The first sets of equations in (2.29) and (2.30) are identical to (2.14-16).

Theorem 2.1 is extended to the entire state model and the output solution Y(t) by the following theorem:

Theorem 2.2

Hypothesis:

1) Assume the hypothesis (1-5) of Theorem 2.1.

2) Define the k-vector $\overline{Y}(t)$ on T to be

$$\overline{Y}(t) = N_s F_s(t) + N_e F_e(t) \tag{2.31}$$

where the matrices N_s and N_e are $k \times q_s$ and $k \times q_e$, respectively.

Conclusion:

Necessary and sufficient conditions for $\bar{X}(t)$ and $\bar{Y}(t)$ to be, respectively, the state and output solutions, on T, of

$$\dot{X}(t) = AX(t) + BE(t), \quad X(0) = X_o \tag{2.32}$$

$$Y(t) = CX(t) + DE(t) \tag{2.33}$$

provided such an equation system exists, are that

$$AG_s = G_s S_s \tag{2.34}$$

$$AG_e + BH = G_e S_e \tag{2.35}$$

$$G_s F_{so} + G_e F_{eo} = X_o \tag{2.36}$$

$$CG_s = N_s \tag{2.37}$$

$$CG_e + DH = N_e \tag{2.38}$$

The existence of (2.32-33) depends on the existence of the matrices A, B, C and D satisfying (2.34-35) and (2.37-38).

Proof

Sufficiency: Assume that matrices A, B, C and D exist satisfying (2.34-38). Then by Theorem 2.1, the solution of (2.32) is $\bar{X}(t)$ as defined by (2.12) and from (2.33),

we have

$$Y(t) = C\bar{X}(t) + DE(t)$$

$$= CG_sF_s(t) + CG_eF_e(t) + DE(t)$$

Applying (2.37-38) and (2.11) gives

$$Y(t) = N_sF_s(t) + (CG_e + DH)F_e(t)$$

$$Y(t) = N_sF_s(t) + N_eF_e(t)$$

and sufficiency is shown.

Necessity: Assume that $\bar{X}(t)$ and $\bar{Y}(t)$ are the solutions of (2.32-33). By Theorem 2.1, (2.34-36) are necessary. Also, from (2.33),

$$\bar{Y}(t) = C\bar{X}(t) + DE(t) = N_sF_s(t) + N_eF_e(t) \qquad (2.39)$$

Using (2.11-12), (2.39) yields

$$C[G_sF_s(t) + G_eF_e(t)] + DHF_e(t) = N_sF_s(t) + N_eF_e(t)$$

or

$$\begin{bmatrix} CG_s - N_s & CG_e + DH - N_e \end{bmatrix} \begin{bmatrix} F_s(t) \\ \\ F_e(t) \end{bmatrix} = 0 \qquad (2.40)$$

By hypothesis, $F_s(t)$ and $F_e(t)$ form a basis vector. Hence, by Lemma 2.1, the coefficient matrix in (2.40) must vanish.

Thus,

$$CG_s = N_s$$

$$CG_e + DH = N_e$$

and necessity follows.

The theory of functions of matrices can also be used to establish the sufficiency proof of Theorem 2.2 by showing that the general output solution

$$Y(t) = Ce^{At}X_o + C\int_o^t e^{A(t-u)}BE(u)du + DE(t)$$

reduces to

$$Y(t) = N_sF_s(t) + N_eF_e(t)$$

when (2.34-38) are assumed. This particular development is primarily an exercise in functions of matrices and appears in Appendix A.

The necessary and sufficient conditions established in Theorems 2.1-2 can be used to find a state model having a specified solution. Because of their application to the design problem, the conditions (2.14-16) of Theorem 2.1 and (2.34-38) of Theorem 2.2 are hereafter referred to as fundamental design equations (FDE).

If only sufficient conditions are of interest, the hypothesis of Theorems 2.1-2 can be relaxed. For example,

the sufficiency proof of Theorem 2.1 does not require that component time functions of the vector

$$F(t) = \begin{bmatrix} F_s(t) \\ \\ F_e(t) \end{bmatrix}$$

be independent on the interval T. Thus the FDE (2.14-16) are <u>sufficient</u> for $\bar{X}(t)$ to be the solution of (2.13) even though $F_s(t)$ and $F_e(t)$ contain identical time functions. In a physical sense, this implies that the FDE still provide sufficient conditions even though the system excitation E(t) contains components identical to the natural modes of the system. A corresponding statement applies to Theorem 2.2.

Simpler necessary conditions can be derived, which hold regardless of the independence of the components of $F_s(t)$ and $F_e(t)$. For example, the FDE of Theorem 2.1 are satisfied by A, B and X_o only if

$$AX_o + BHF_{eo} = G_s S_s F_{so} + G_e S_e F_{eo} \qquad (2.41)$$

This vector equality merely represents the necessary condition $\dot{X}(t) = \dot{\bar{X}}(t)$ at t=0, which must hold if $\bar{X}(t)$ is the solution to (2.13).

Similarly, the FDE of Theorem 2.2 are satisfied by A, B, C, D and X_o only if

$$CAX_o + (DHS_e + CBH)F_{eo} = N_s S_s F_{so} + N_e S_e F_{eo} \qquad (2.42)$$

This vector equality can be obtained either by eliminating G_s and G_e from the FDE (2.34-38) or by using the appropriate matrices to express the necessary condition $\dot{Y}(t) = \bar{\dot{Y}}(t)$ at $t=0$.

The equations (2.41-2) are called <u>reduced necessary conditions</u> and they can be very useful in system design. For example, these conditions can be used to quickly check the matrices of a proposed state model to determine if the model can satisfy the FDE. Such a test immediately eliminates from further consideration those systems with state models which do not satisfy the reduced necessary conditions.

By inspection, the FDE of Theorems 2.1 and 2.2 can be written, respectively, in the augmented matrix forms

$$\begin{bmatrix} A & B & X_o \end{bmatrix} \begin{bmatrix} G_s & G_e & 0 \\ 0 & H & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} G_s & G_e \end{bmatrix} \begin{bmatrix} S_s & 0 & F_{so} \\ 0 & S_e & F_{eo} \end{bmatrix} \qquad (2.43)$$

and

$$\begin{bmatrix} A & B & X_o \\ C & D & 0 \end{bmatrix} \begin{bmatrix} G_s & G_e & 0 \\ 0 & H & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} G_s & G_e & 0 \\ 0 & 0 & U \end{bmatrix} \begin{bmatrix} S_s & 0 & F_{so} \\ 0 & S_e & F_{eo} \\ N_s & N_e & 0 \end{bmatrix} \qquad (2.44)$$

Both sets of FDE take the form

$$\mathcal{A}\mathcal{G}_1 = \mathcal{G}_2\,\mathcal{S} \tag{2.45}$$

where the matrix $\mathcal{A}$ contains those, and only those matrices, which define the time-domain model, $\mathcal{G}_1$ and $\mathcal{G}_2$ may contain both known and unknown submatrices and $\mathcal{S}$ contains only those matrices which are known and fixed by the design specifications. The submatrices G_s and G_e are unknown in (2.44) when the state solution $\bar{X}(t)$ is unspecified in Theorem 2.2.

In general, the solution of any set of FDE involves finding the matrices $\mathcal{A}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfying (2.45) for some given matrix $\mathcal{S}$. By virtue of the matrix product $\mathcal{A}\mathcal{G}_1$, the FDE generally exhibit second degree nonlinearity in the unknown variables. If in (2.44), C is known and nonsingular, then the unknown submatrices in $\mathcal{G}_1$ and $\mathcal{G}_2$ can be expressed uniquely in terms of the known matrices in $\mathcal{S}$ and the algebraic system (2.44) reduces to a linear system in the remaining unknown variables in $\mathcal{A}$.

The application of the fundamental design equations is illustrated in the following example:

Example 2.1

Find a third order state model

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \tag{2.46}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}$$

with zero initial state vector, for which the output solution Y(t) is

$$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-4t} + \sqrt{5}e^{-10t}\sin(5t + 63.5^{o}) + 1 + \sin 2t \\ \sqrt{5}e^{-10t}\sin(5t + 26.5^{o}) + 1 + \sin 2t \end{bmatrix} \tag{2.47}$$

when the excitation E(t) is

$$E(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} = \begin{bmatrix} .5 \\ -1 + 5\sin(2t + 143.2^{o}) \\ .25\cos 2t \end{bmatrix} \tag{2.48}$$

The excitation vector E(t) illustrates a variety of inputs including the step function and the sinusoid with and without phase shift. In the output vector, $y_1(t)$ and $y_2(t)$ are required to be identical except for the transient term e^{-4t} and the phase angle associated with sin5t. The steady-state portion of Y(t) is required to exhibit zero phase with respect to the sin2t excitation.

Comparing the various functions specified in Y(t) and those contributed by E(t), it follows that the order of the state model must be at least three, otherwise the transient portion of Y(t) can not be produced.

In order to present the given specifications on Y(t) and E(t) in the forms

$$Y(t) = N_s F_s(t) + N_e F_e(t) \tag{2.49}$$

and

$$E(t) = HF_e(t) \tag{2.50}$$

(2.47) and (2.48) are written as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} e^{-4t} \\ e^{-10t}\sin 5t \\ e^{-10t}\cos 5t \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \sin 2t \\ \cos 2t \end{bmatrix} \tag{2.51}$$

and

$$\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} = \begin{bmatrix} .5 & 0 & 0 \\ -1 & -4 & 3 \\ 0 & 0 & .25 \end{bmatrix} \begin{bmatrix} 1 \\ \sin 2t \\ \cos 2t \end{bmatrix} \tag{2.52}$$

The 3-vectors $F_s(t)$ and $F_e(t)$ are chosen so that $F_e(t)$ spans the function space of $E(t)$ and $F_s(t)$ and $F_e(t)$ together span the function space of $Y(t)$.

In this example, $F_s(t)$ and $F_e(t)$ comprise a basis vector for $t \geq 0$. The basis systems for $F_s(t)$ and $F_e(t)$ are easily obtained by differentiating the appropriate vector functions in (2.51). Thus

$$\dot{F}_s(t) = S_s F_s(t) \tag{2.53}$$

becomes

$$\frac{d}{dt} \begin{bmatrix} e^{-4t} \\ e^{-10t}\sin 5t \\ e^{-10t}\cos 5t \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -10 & 5 \\ 0 & -5 & -10 \end{bmatrix} \begin{bmatrix} e^{-4t} \\ e^{-10t}\sin 5t \\ e^{-10t}\cos 5t \end{bmatrix} \tag{2.54}$$

with

$$F_{so} = F_s(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and similarly

$$\dot{F}_e(t) = S_e F_e(t) \tag{2.55}$$

becomes

$$\frac{d}{dt}\begin{bmatrix} 1 \\ \sin 2t \\ \cos 2t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \sin 2t \\ \cos 2t \end{bmatrix} \tag{2.56}$$

with

$$F_{eo} = F_e(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Note that the basis matrices S_s and S_e are real and do not necessarily assume the Jordan canonical form.

By Theorem 2.2, the fundamental design equations

$$\begin{bmatrix} A & B & X_o \\ C & D & 0 \end{bmatrix}\begin{bmatrix} G_s & G_e & 0 \\ 0 & H & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} G_s & G_e & 0 \\ 0 & 0 & U \end{bmatrix}\begin{bmatrix} S_s & 0 & F_{so} \\ 0 & S_e & F_{eo} \\ N_s & N_e & 0 \end{bmatrix} \tag{2.57}$$

Provide necessary and sufficient conditions for the state model (2.46) to have the output solution Y(t) in (2.47) when the excitation E(t) is given by (2.48).

Writing (2.57) as

$$\mathcal{A}\mathcal{G}_1 = \mathcal{G}_2\mathcal{S} \tag{2.58}$$

it is seen that all submatrices in $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{S}$ are specified with the exception of G_s and G_e. These may be selected arbitrarily so long as two conditions are met: (a) the initial condition vector must be zero and (b) there must exist at least one solution of (2.58) for the matrix $\mathcal{A}$. It is known that an infinite number of third order state models will produce the desired output solution Y(t). Selection of the particular matrices G_s and G_e determines a single state model having a particular state solution

$$X(t) = G_s F_s(t) + G_e F_e(t) \tag{2.59}$$

Thus, G_s and G_e are arbitrarily selected so that

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} e^{-4t} \\ e^{-10t}\sin 5t \\ e^{-10t}\cos 5t \end{bmatrix} + \begin{bmatrix} -1 & 2 & -1 \\ -3 & 2 & -1 \\ 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ \sin 2t \\ \cos 2t \end{bmatrix} \tag{2.60}$$

Equation (2.60) satisfies the initial condition $X_o = 0$, as required. All submatrices in $\mathcal{G}_1$ and $\mathcal{G}_2$ are now specified and it follows from the nonsingularity of G_s and H, that $\mathcal{G}_1^{-1}$ exists. Thus, from (2.58),

$$\mathcal{A} = \mathcal{G}_2 \, \mathcal{S} \, \mathcal{G}_1^{-1} \tag{2.61}$$

and for the selected G_s and G_e a unique solution for $\mathcal{A}$ exists. The matrix $\mathcal{G}_1$ is not difficult to invert because of its upper triangular submatrix form. Carrying out the indicated inverse (2.61) takes the detailed form

$$\begin{bmatrix} A & B & X_o \\ C & D & 0 \end{bmatrix} = \begin{bmatrix} G_s & G_e & 0 \\ 0 & 0 & U \end{bmatrix} \begin{bmatrix} S_s & 0 & F_{so} \\ 0 & S_e & F_{eo} \\ N_s & N_e & 0 \end{bmatrix} \begin{bmatrix} G_s^{-1} & -G_s^{-1}G_e H^{-1} & 0 \\ 0 & H^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or numerically,

$$\begin{bmatrix} A & B & X_o \\ C & D & 0 \end{bmatrix} = \left[\begin{array}{ccc|ccc|c} -16 & 6 & 2.5 & -15.75 & -4.875 & 74.5 & 0 \\ -12 & 2 & 2.5 & -31.75 & -4.875 & 74.5 & 0 \\ -20 & 10 & -10 & 41 & -9.5 & -78 & 0 \\ \hline 1.5 & -.5 & 1 & -1 & .5 & 14 & 0 \\ 4 & -2 & .5 & -2.25 & .875 & 5.5 & 0 \end{array}\right] \tag{2.62}$$

Thus, the third order state model

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -16 & 6 & 2.5 \\ -12 & 2 & 2.5 \\ -20 & 10 & -10 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} -15.75 & -4.875 & 74.5 \\ -31.75 & -4.875 & 74.5 \\ 41 & -9.5 & -78 \end{bmatrix} E(t)$$

(2.63)

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1.5 & -.5 & 1 \\ 4 & -2 & .5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} -1 & .5 & 14 \\ -2.25 & .875 & 5.5 \end{bmatrix} E(t)$$

with $X(0) = X_o = 0$ and $E(t)$ given by (2.48), has the specified output solution (2.47) and the state solution (2.60).

To generate a state model of order greater than three having the same output solution, the same general technique is followed. However, in this case, there is no matrix G_s which determines a unique state model. Indeed, even though G_s has maximum rank, there are always $n - q_s$ columns in each of the matrices A and C which are arbitrary.

Consider, for example, the problem of establishing a fourth-order state model having the same output solution and excitation as the third-order model above. For simplicity, pick G_s and G_e such that (2.59) takes the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} F_s(t) + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} F_e(t) \qquad (2.64)$$

Note that the initial condition $X_o = 0$ is satisfied.

With this choice of G_s, the matrices A and C which satisfy

$$AG_s = G_s S_s$$

$$CG_s = N_s$$

are not uniquely determined. Rather, one of the last two columns in each can be arbitrarily selected and are taken such that

$$A = \begin{bmatrix} -4 & 0 & -2 & 2 \\ 0 & -10 & 4 & 1 \\ 0 & -5 & -10 & 0 \\ 0 & -5 & -3 & 7 \end{bmatrix} \qquad (2.65)$$

$$C = \begin{bmatrix} 1 & 1 & -3 & 5 \\ 0 & 2 & 1.5 & -.5 \end{bmatrix} \tag{2.66}$$

Since H is nonsingular, the fundamental design equations

$$AG_e + BH = G_eS_e$$

$$CG_e + DH = N_e$$

can be solved for B and D to yield

$$B = \begin{bmatrix} -8 & 0 & 0 \\ -5 & -2.5 & 58 \\ -3.5 & -1.75 & -19 \\ -3.5 & -1.75 & -19 \end{bmatrix} \tag{2.67}$$

$$D = \begin{bmatrix} 4 & 0 & 8 \\ 2.5 & .25 & 1 \end{bmatrix} \tag{2.68}$$

It can be verified by substitution that the fourth-order state model defined by the matrices A, B, C and D given by

(2.65-68), with E(t) given by (2.48) has the state solution (2.64) and the output solution (2.47).

This example illustrates several properties of the fundamental design equations. First, the specifications are easily written in terms of the output solution Y(t), corresponding to a given excitation E(t). Second, the transient and steady-state portions of the output solution are independently written in the specification of Y(t) and remain separated throughout the computation. Third, any desired initial conditions can be easily satisfied. And, finally, when no constraints are placed on the state model by the physical system, as in this example, the solution of the fundamental design equations is accomplished by linear algebra alone.

If, for example, the physical realizability of a system requires that a subset of the entries in A, B, C or D assume specified values, then these a priori constraints are applied in (2.57) before the matrices G_s and G_e are selected. In fact, depending upon the extent of these realizability constraints, the matrices G_s and G_e may be partially or completely determined by the FDE. If the constraints are severe enough, it is possible that no choice of the matrices G_s and G_e will satisfy the FDE and only an approximate solution can be obtained.

The important point is that the FDE allow the designer to consider the physical properties of the system at the state model level rather than at the level of the corresponding s-domain model. This is a substantial improvement over the situation where the designer is given a transfer function matrix and he faces the task of realizing the matrix in terms of a particular physical system with certain unknown properties.

III. FUNDAMENTAL DESIGN EQUATIONS FROM S-DOMAIN AND FREQUENCY-DOMAIN MODELS

Many specifications, particularly in traditional control system design and network synthesis, are given in terms of transform models or frequency response characteristics. Also, the study of non-deterministic (random) processes depends heavily on the Fourier transform and the representation of signals in the frequency domain.

To establish fundamental design equations in terms of s-domain and frequency-domain specifications, consider the state model (1.1-2) with $X_o=0$ and $E(t) = X(t) = 0$, $t<0$. With these assumptions and the definition of the Laplace transform, it is easily shown that the state model has the unique equivalent s-domain model

$$\overline{Y}(s) = \overline{M}(s)\overline{E}(s) \qquad (3.1)$$

provided the Laplace transforms $\overline{E}(s)$ and $\overline{Y}(s)$ exist. The k x r matrix $\overline{M}(s)$ is called the <u>transfer function matrix</u> and, as shown as Appendix B, has the form

$$\overline{M}(s) = \overline{N}(s) + D \qquad (3.2)$$

where

$$\overline{N}(s) = C(sU-A)^{-1}B \qquad (3.3)$$

and $$N(t) = \mathcal{L}^{-1}\{\bar{N}(s)\} = Ce^{At}B \tag{3.4}$$

Given the matrices A, B, C and D which describe the unforced system, the matrix $\bar{M}(s)$ is uniquely determined by (3.2-3). However, the matrices A, B and C are not uniquely determined by a given $\bar{M}(s)$ by virtue of the nonunique matrix factors comprising the product $C(sU-A)^{-1}B$. Thus, there are many state models (or systems) corresponding to a given $\bar{M}(s)$, and the problem of system design is to select a realizable one.

Corresponding properties of the frequency response matrix can be easily shown. In fact, if the Fourier transforms of Y(t) and E(t) exist, then the state model (1.1-2) with zero initial conditions has the unique equivalent frequency-domain model

$$Y^*(w) = M^*(w)E^*(w) \tag{3.5}$$

where the frequency response matrix $M^*(w)$ is given by

$$M^*(w) = \bar{M}(s)\Big|_{s=jw} \tag{3.6}$$

Therefore, all of the preceding properties (3.2-4) listed for $\bar{M}(s)$ hold for $M^*(w)$ with s replaced by jw and the inverse Laplace transform replaced by the inverse Fourier transform. It should be noted that a necessary and sufficient condition for matrix function e^{At} to have a Fourier transform (in the strict sense, i.e., without impulse

functions) is that the eigenvalues of A have negative real parts. This property of A also defines a "strictly stable" system and thus indicates the restricted class of systems describable in terms of frequency-response characteristics. Because of the simple relationship (3.6), only the transfer function matrix $\overline{M}(s)$ is considered in the sequel.

A design or synthesis problem may take the following form: Given a desired transfer function matrix $\overline{M}(s)$, find a physical system having the specified $\overline{M}(s)$. For a vector-input system, the realization of a specified $\overline{M}(s)$ is more difficult than the realization of a specified time response to a given excitation vector E(t). For example, $\overline{M}(s)$ places constraints on the unforced system which are independent of the actual form of the excitation. Thus, $\overline{M}(s)$ specifies only that part of the output response which depends on the natural modes of the system. However, in a linear system, the natural modes are excited independently by each input and the total system response is the sum of the individual responses. A time-domain specification, as considered in Section II, only specifies the total response whereas $\overline{M}(s)$ independently fixes the contribution of each input to the total response. Thus, for an r-input system, the specification of $\overline{M}(s)$ is equivalent to a specification of r time responses corresponding to r independent excitations. This point is stressed later in the development.

Two lemmas and a definition precede the main theorem.

Lemma 3.1

Hypothesis:

1) $T = \{t \mid t \geq o\}$

2) F(t) is a basis vector of order q on T with an associated q^{th}-order basis system

$$\dot{F}(t) = SF(t), \quad F(0) = F_o \tag{3.7}$$

3) The n-vector function $\overline{X}(t)$ and the k-vector function $\overline{Y}(t)$ are defined on T by

$$\overline{X}(t) = GF(t) \tag{3.8}$$

$$\overline{Y}(t) = NF(t) \tag{3.9}$$

Conclusion:

Necessary and sufficient conditions for $\overline{X}(t)$ and $\overline{Y}(t)$ to be, respectively, the state and output solutions on T of

$$\begin{aligned} \dot{X}(t) &= AX(t), \quad X(0) = X_o \\ Y(t) &= CX(t) \end{aligned} \tag{3.10}$$

provided such an equation set exists, are that

$$AG = GS \tag{3.11}$$

$$GF_o = X_o \tag{3.12}$$

$$CG = N \tag{3.13}$$

The existence of (3.10) depends upon the existence of matrices A and C satisfying (3.11) and (3.13).

Proof

The conclusion follows directly from Theorem 2.2 with the following identities:

$$F_s(t) \equiv F(t)$$

$$S_s \equiv S$$

$$F_{so} \equiv F_o$$

$$G_s \equiv G$$

$$N_s \equiv N$$

and with $F_e(t)$, S_e, F_{eo}, G_e, N_e, B, D, H and E(t) set to zero.

Lemma 3.2

Hypothesis:

1) Assume hypothesis (1-2) of Lemma 3.1.
2) The n x r matrix function $\overline{\chi}(t)$ and the k x r matrix function $\overline{\gamma}(t)$ are defined on T by

$$\overline{\chi}(t) = [G_1F(t), G_2F(t), \ldots, G_rF(t)] \tag{3.14}$$

$$\overline{\mathcal{Y}}(t) = [N_1F(t),\ N_2F(t),\ldots,N_rF(t)] \qquad (3.15)$$

Conclusion:

Necessary and sufficient conditions for $\overline{\mathcal{X}}(t)$ and $\overline{\mathcal{Y}}(t)$ to be, respectively, the state and output matrix solutions on T of

$$\begin{aligned} \dot{\mathcal{X}}(t) &= A\mathcal{X}(t), \quad \mathcal{X}(0) = \mathcal{X}_o \\ \mathcal{Y}(t) &= C\mathcal{X}(t) \end{aligned} \qquad (3.16)$$

provided such an equation set exists, are that

$$AG_i = G_iS \qquad (3.17)$$

$$G_iF_o = X_{oi} \qquad (3.18)$$

$$CG_i = N_i \qquad (3.19)$$

$$i=1,2,\ldots,r$$

where X_{oi} is the i^{th} column of $\mathcal{X}_o$. The existence of (3.16) depends upon the existence of the matrices A and C satisfying (3.17) and (3.19).

Proof

The matrix differential equation system (3.16) is equivalent to the set of r vector differential equation systems

$$\begin{aligned} \dot{X}_i(t) &= AX_i(t), \quad X_i(0) = X_{oi} \\ Y_i(t) &= CX_i(t) \end{aligned} \qquad (3.20)$$

$i=1,2,\ldots,r$, where $X_i(t)$ and $Y_i(t)$ are, respectively, the i^{th} columns of $\mathcal{X}(t)$ and $\mathcal{Y}(t)$. From (3.14) and (3.15), we have

$$\bar{X}_i(t) = G_i F(t) \tag{3.21}$$

$$\bar{Y}_i(t) = N_i F(t) \tag{3.22}$$

Applying Lemma 3.1 successively to each of the r vector equation systems (3.20) and their corresponding proposed solutions (3.21) and (3.22), the conclusion follows.

Let the s-domain model (3.1) be given and assume the system to be found is characterized by a state model of the form (1.1-2) with $X_o=0$. The structure of the physical system is implicit in the matrices A, B, C and D and it is proposed that constraints on these matrices be found which are necessary and sufficient for the state model to be "equivalent" to the specified s-domain model. The term "equivalent" is defined as follows:

Definition 3.1

The state model

$$\dot{X}(t) = AX(t) + BE(t), \quad X(0) = 0$$

$$Y(t) = CX(t) + DE(t)$$

is said to be equivalent to the s-domain model

$$\bar{Y}(s) = \bar{M}(s)\bar{E}(s)$$

if and only if

$$\overline{M}(s) = C(sU-A)^{-1}B + D$$

It can be shown that there exists at least one equivalent state model of finite order if $\overline{M}(s)$ satisfies the following condition [16]: Every entry of $\overline{M}(s)$ is a rational function in s having the degree of the denominator finite and not less than the numerator.

Theorem 3.1

Hypothesis:

1) $T = \{t \mid t \geq 0\}$

2) The s-domain model

$$\overline{Y}(s) = \overline{M}(s)\overline{E}(s) \tag{3.23}$$

is given with

$$\overline{M}(s) = \overline{N}(s) + R \tag{3.24}$$

of order k x r. R is a constant matrix and $\overline{N}(s)$ has the inverse Laplace transform

$$N(t) = \mathcal{L}^{-1}\{\overline{N}(s)\}$$

$$= [N_1F(t),\ N_2F(t),\ldots,N_rF(t)] \tag{3.25}$$

for all $t \in T$ where N_i is a k x q matrix, i=1,2,...,r, and F(t) satisfies, for all $t \in T$

$$\dot{F}(t) = SF(t), \quad F(0) = F_o \tag{3.26}$$

for some q x q matrix S.

Conclusion:

Sufficient conditions for the n-th order state model

$$\dot{X}(t) = AX(t) + BE(t), \quad X(0) = 0 \tag{3.27}$$

$$Y(t) = CX(t) + DE(t) \tag{3.28}$$

to be equivalent to the s-domain model (3.23) are that there exist n x q matrices G_i, i=1,2,...,r, such that

$$AG_i = G_iS \tag{3.29}$$

$$G_iF_o = B_i \tag{3.30}$$

$$CG_i = N_i \tag{3.31}$$

$$D = R \tag{3.32}$$

where B_i is the i^{th} column of B. Conditions (3.29-32) are also necessary for the stated equivalence if the q-vector F(t) is a basis vector which spans the function space of the matrix function $e^{At}B$, i.e., if there exist n x q matrices G_i, i=1,2,...,r, such that

$$e^{At}B = [G_1F(t), \quad G_2F(t),...,G_rF(t)]$$

Proof

Sufficiency: Assume that there exist matrices G_i, i=1,2,...,r, such that (3.29-32) are satisfied. Solving the

state model (3.27-28) by Laplace transforms we obtain the transfer function matrix

$$\bar{M}_o(s) = C(sU-A)^{-1}B + D \tag{3.33}$$

By Definition 3.1, it must be shown that $\bar{M}_o(s)$ is identical to $\bar{M}(s)$ defined by (3.24-25). Since D=R by (3.32), it only remains to be shown that $\bar{N}(s)$ is identical to $C(sU-A)^{-1}B$. The inverse Laplace transform of $\bar{N}(s)$ is defined by (3.25) and the inverse Laplace transform of $C(sU-A)^{-1}B$ is

$$\mathcal{L}^{-1}\{C(sU-A)^{-1}B\} = Ce^{At}B \tag{3.34}$$

as shown in Appendix B.

Define

$$\mathcal{Y}(t) = Ce^{At}B \tag{3.35}$$

and note that this is the output solution of the matrix system

$$\dot{\mathcal{X}}(t) = A\mathcal{X}(t), \quad \mathcal{X}(0) = B \tag{3.36}$$

$$\mathcal{Y}(t) = C\mathcal{X}(t) \tag{3.37}$$

By Lemma 3.2, it follows that (3.29-31) are sufficient for $\mathcal{Y}(t)$ to have the form

$$\mathcal{Y}(t) = [N_1F(t), N_2F(t) \ldots N_rF(t)] \tag{3.38}$$

when B_i is identified with X_{oi}. Comparing (3.35) and (3.38) and utilizing the uniqueness of the Laplace transform, we have

$$\bar{N}(s) = C(sU-A)^{-1}B$$

and sufficiency is shown.

Necessity: Assume the state model (3.27-28) is equivalent to the s-domain model (3.23). Then by Definition 3.1,

$$\bar{M}(s) = \bar{N}(s) + R \qquad (3.39)$$

$$= C(sU-A)^{-1}B + D \qquad (3.40)$$

Equating corresponding powers of s on the right hand sides of (3.39) and (3.40) gives

$$C(sU-A)^{-1}B = \bar{N}(s) \qquad (3.41)$$

$$D = R \qquad (3.42)$$

and thus (3.32) is necessary. Taking the inverse Laplace transform of both sides of (3.41) gives, by virtue of (3.25),

$$Ce^{At}B = [N_1F(t), \quad N_2F(t), \ldots, N_rF(t)] \qquad (3.43)$$

The k x r matrix function $Ce^{At}B$ can be written as

$$Ce^{At}B = C\mathcal{X}(t) = \mathcal{Y}(t) \qquad (3.44)$$

where $\mathcal{X}(t)$ and $\mathcal{Y}(t)$ are, respectively, the state and output solutions to the matrix system

$$\begin{aligned} \dot{\mathcal{X}}(t) &= A\mathcal{X}(t), \quad \mathcal{X}(0) = B \\ \mathcal{Y}(t) &= C\mathcal{X}(t) \end{aligned} \tag{3.45}$$

Combining (3.44) and (3.43) it is necessary that

$$\mathcal{Y}(t) = C\mathcal{X}(t) = [N_1F(t), N_2F(t), \ldots, N_rF(t)] \tag{3.46}$$

By hypothesis, F(t) spans the function space of $e^{At}B = \mathcal{X}(t)$. Hence, there exist matrices G_i, i=1,2,...,r, such that

$$\mathcal{X}(t) = [G_1F(t), G_2F(t), \ldots, G_rF(t)] \tag{3.47}$$

But $\mathcal{X}(t)$ and $\mathcal{Y}(t)$ defined by (3.46-47) are the solutions of (3.45) and by Lemma 3.2, it is necessary that

$$\begin{aligned} AG_i &= G_iS \\ G_iF_o &= B_i \qquad i=1,2,\ldots,r \\ CG_i &= N_i \end{aligned}$$

and the conclusion follows.

By virtue of the direct relationship (3.6) between the transfer function matrix and the frequency response matrix, the proof of Theorem 3.1 can be notationally modified to establish a corresponding result for the frequency-domain

model (3.5).

Theorem 3.1 provides constraints on the system state model similar to those developed for time-domain specifications. But, in general, they are more restrictive. For example, the fundamental design equations (3.29-31) require that the matrices A, B, and C satisfy r sets of constraints whereas a time-domain specification, as considered in Section II, requires only one set to be satisfied. Hence, a specification of an equivalent transform model provides a more exact definition of the physical system than does a specification of a desired time response to a given excitation.

Kalman, Gilbert, and others have given algorithms for deriving certain restricted forms of the state model from a specified transfer function matrix [16], [17], [18]. These procedures are applicable when the entries of $\overline{M}(s)$ have simple poles and the resulting state model exhibits a matrix A which is always diagonal or takes the form of a hypercompanion matrix. Also, the entries of A, B and C may be complex. Such forms do not normally occur when the state model is formulated directly from the structure of a physical system. Thus, some unknown similarity transformation is required to relate these restricted forms to the model derived from the system. In their papers, Kalman and Gilbert also give a technique for finding the minimum order of the state model corresponding to a given $\overline{M}(s)$.

By comparison, the FDE of Theorem 3.1 allow multiple poles of the entries of $\bar{M}(s)$ and do not restrict the form of the matrices A, B and C in any way. Any of the particular forms obtained by Kalman and Gilbert may also be derived with the FDE by properly selecting F(t) and the matrices G_i, i=1,2...,r, so that A takes the proper form, i.e., diagonal or companion. This flexibility allows the designer to place constraints on the state model which are required by the physical realizability of the system.

Reduced necessary conditions for Theorem 3.1 are

$$CAB_i = N_i SF_o, \quad i=1,2,\ldots,r \tag{3.48}$$

$$D = R \tag{3.49}$$

The equations (3.48) are obtained by either eliminating G_i, i=1,2,...,r, from (3.29-31) or by differentiating (3.43), setting t = 0 and equating corresponding columns with $\dot{F}(0) = SF_o$. If a state model is equivalent to a specified transfer function matrix, the equations (3.48-49) must be satisfied by the matrices A, B, C and D.

An interesting application of the reduced necessary conditions occurs when the state model formulation of a physical system yields matrices A, B, C and D such that

$$CAB_i = 0, \quad i=1,2,\ldots,r \tag{3.50}$$

$$D = 0 \tag{3.51}$$

For example, this condition occurs for state models derived from a rather general class of L-C networks. Let the design specifications for the system be given in terms of the s-domain model

$$\bar{Y}(s) = \bar{M}(s)\bar{E}(s)$$

Upon comparing (3.48-49) with (3.50-51), we establish a necessary condition on $\bar{M}(s)$ if the state model is equivalent to $\bar{M}(s)$. That is, $\bar{M}(s)$ must decompose into

$$\bar{M}(s) = \bar{N}(s) + R \tag{3.52}$$

where

$$R = 0 \tag{3.53}$$

and

$$N(t) = \mathcal{L}^{-1}\{\bar{N}(s)\} = [N_1F(t), N_2F(t) \ldots N_rF(t)]$$

with $\dot{F}(t) = SF(t)$, $F(0) = F_o$ and the k x q matrices N_i satisfy

$$N_iSF_o = 0 \quad i=1,2,\ldots,r \tag{3.54}$$

This condition is illustrated in Example 3.1.

By direct comparison, it follows that the PDE of Theorem 3.1 can be written in the augmented matrix form

$$\begin{bmatrix} A & B_i \\ C & D_i \end{bmatrix}\begin{bmatrix} G_i & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} G_i & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} S & F_o \\ N_i & R_i \end{bmatrix}, \quad i=1,2,\ldots,r \tag{3.55}$$

where

$$B_i = i^{th} \text{ column of } B,$$
$$D_i = i^{th} \text{ column of } D,$$
$$R_i = i^{th} \text{ column of } R,$$

and $$N_i = k \times q \text{ matrix}$$

or equivalently,

$$\left[\begin{array}{cc} A & B \\ C & D \end{array}\right] \left[\begin{array}{c|c} G_i \ldots G_r & 0 \\ \hline 0 \ldots 0 & U \end{array}\right] =$$

$$= \left[\begin{array}{c|c} G_i \ldots G_r & 0 \\ \hline 0 \ldots 0 & U \end{array}\right] \left[\begin{array}{ccc|ccc} S & 0 \ldots & 0 & F_o & 0 \ldots & 0 \\ 0 & S \ldots & 0 & 0 & F_o \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \ldots & S & 0 & 0 \ldots & F_o \\ \hline N_1 & N_2 \ldots & N_r & R_1 & R_2 \ldots & R_r \end{array}\right] \quad (3.56)$$

Notice that (3.55-56) have the same general form (2.45) as the FDE of Theorems 2.1 and 2.2.

Example 3.1

Given the two-port L-C network

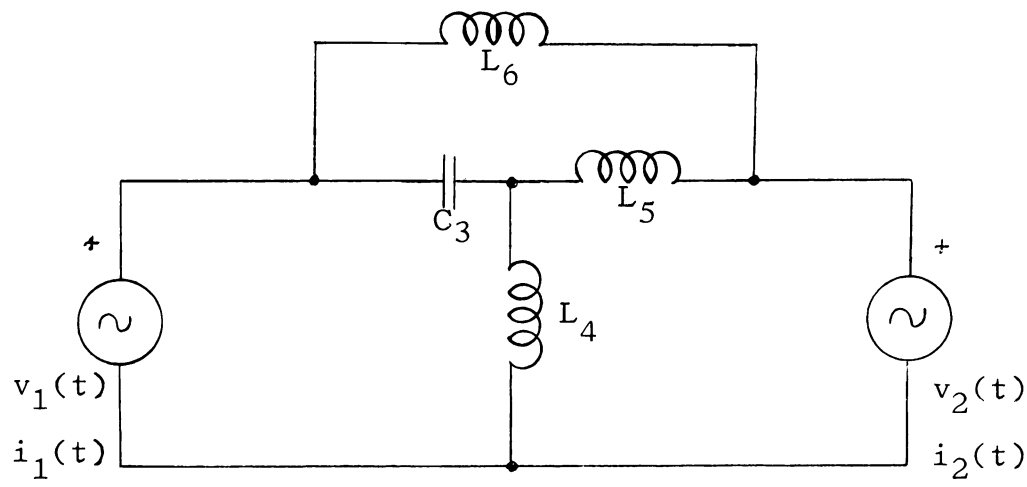

Figure 3.1 L-C Network

find the element values C_3, L_4, L_5 and L_6 such that the resulting admittance matrix $\overline{M}(s)$ of the network is

$$\overline{M}(s) = \begin{bmatrix} \dfrac{11s^2 + 280}{s(s^2 + 70)} & \dfrac{-6s^2 - 280}{s(s^2 + 70)} \\ \dfrac{-6s^2 - 280}{s(s^2 + 70)} & \dfrac{6s^2 + 380}{s(s^2 + 70)} \end{bmatrix} \tag{3.57}$$

The admittance matrix of the network is defined by

$$\overline{I}(s) = \overline{M}(s)\overline{V}(s) \tag{3.58}$$

where I(s) and V(s) are the Laplace transforms of

$I(t) = [i_1(t), i_2(t)]^T$ and $V(t) = [v_1(t), v_2(t)]^T$,

respectively.

The relation (3.58), together with $\bar{M}(s)$ given by (3.57) defines a desired s-domain model of the network. To determine the element values which yield this desired model, a state model of the network is to be found which is equivalent to the s-domain model in the sense of Definition 3.1. The state model of the network with excitation vector V(t) and output vector I(t) is easily derived. The result is

$$\frac{d}{dt}\begin{bmatrix} v_3(t) \\ i_4(t) \\ i_5(t) \\ i_6(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/C_3 & 1/C_3 & 0 \\ -1/L_4 & 0 & 0 & 0 \\ -1/L_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_3(t) \\ i_4(t) \\ i_5(t) \\ i_6(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/L_4 & 0 \\ 1/L_5 & -1/L_5 \\ 1/L_6 & -1/L_6 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \tag{3.59}$$

$$\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_3(t) \\ i_4(t) \\ i_5(t) \\ i_6(t) \end{bmatrix}$$

Utilizing the notation of Theorem 3.1 as it applies to the s-domain model (3.57-58) it follows that R=0 and therefore,

$$\bar{N}(s) = \bar{M}(s)$$

$$= \frac{s}{s^2 + 70}\begin{bmatrix} 7 & -2 \\ -2 & 4/7 \end{bmatrix} + \frac{1}{s}\begin{bmatrix} 4 & -4 \\ -4 & 38/7 \end{bmatrix}$$

Taking the inverse Laplace transform of $\bar{N}(s)$ yields

$$N(t) = [N_1 F(t), N_2 F(t)]$$

where

$$F(t) = \begin{bmatrix} \sin \sqrt{70}\, t \\ \cos \sqrt{70}\, t \\ 1 \end{bmatrix} \tag{3.60}$$

and

$$N_1 = \begin{bmatrix} 0 & 7 & 4 \\ 0 & -2 & -4 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} 0 & -2 & -4 \\ 0 & 4/7 & 38/7 \end{bmatrix} \tag{3.61}$$

Taking the derivative of F(t) defines the system

$$\frac{d}{dt}\begin{bmatrix} \sin \sqrt{70}\, t \\ \cos \sqrt{70}\, t \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{70} & 0 \\ -\sqrt{70} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \sin \sqrt{70}\, t \\ \cos \sqrt{70}\, t \\ 1 \end{bmatrix} \tag{3.62}$$

$$F(0) = F_o = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{3.63}$$

Thus, by Definition 2.2, F(t) is a basis vector with basis system (3.62-63).

The state model (3.59) is equivalent to the specified s-domain model (3.58) if the fundamental design equations of Theorem 3.1 are satisfied for some realizable values of C_3, L_4, L_5 and L_6. Note that since F(t) is a 3-vector, it may not span the function space of e^{At} and thus, we cannot conclude that the FDE provide necessary conditions for the equivalence of the state model and the s-domain model. However, the reduced necessary conditions can be applied. Thus, from (3.48-49), the state model is equivalent to the s-domain model only if the matrices A, B, C and D satisfy

$$CAB_i = N_i S F_o, \quad i=1,2 \tag{3.64}$$

$$D = R \tag{3.65}$$

Condition (3.65) is satisfied. Also from (3.61-63)

$$N_1 S F_o = \begin{bmatrix} 0 & 7 & 4 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{70} & 0 \\ -\sqrt{70} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Similarly,

$$N_2SF_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, if (3.64) is to be satisfied, the matrix product CAB must vanish. From (3.59) we have

$$CAB = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1/C_3 & 1/C_3 & 0 \\ -1/L_4 & 0 & 0 & 0 \\ -1/L_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1/L_4 & 0 \\ 1/L_5 & -1/L_5 \\ 1/L_6 & -1/L_6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, the state model satisfies the reduced necessary conditions (3.64-65). Since these conditions are satisfied regardless of the choice of C_3, L_4, L_5 and L_6, a general necessary condition on $\overline{M}(s)$ is established. That is, every L-C network having the topology of Figure 3.1 will have an admittance matrix $\overline{M}(s)$ only if $\overline{M}(s)$ yields matrices R, N_1, N_2, S and F_o such that R=0 and $N_iSF_o = 0$, i=1,2.

Proceeding now to the formulation of the FDE, the augmented form (3.56), written for i=2, gives

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} G_1 & G_2 & 0 \\ 0 & 0 & U \end{bmatrix} = \begin{bmatrix} G_1 & G_2 & 0 \\ 0 & 0 & U \end{bmatrix} \begin{bmatrix} S & 0 & F_o & 0 \\ 0 & S & 0 & F_o \\ N_1 & N_2 & R_1 & R_2 \end{bmatrix} \qquad (3.66)$$

or, in detail

$$\left[\begin{array}{cccc:cc} 0 & 1/C_3 & 1/C_3 & 0 & 0 & 0 \\ -1/L_4 & 0 & 0 & 0 & 1/L_4 & 0 \\ -1/L_5 & 0 & 0 & 0 & 1/L_5 & -1/L_5 \\ 0 & 0 & 0 & 0 & 1/L_6 & -1/L_6 \\ \hdashline 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{array}\right] \left[\begin{array}{cccccc:c} g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} & \\ \hdashline & & & & & & 1\ 0 \\ & 0 & & & 0 & & \\ & & & & & & 0\ 1 \end{array}\right] =$$

$$= \left[\begin{array}{cccccc:c} g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} & \\ \hdashline & & & & & & 1\ 0 \\ & 0 & & & 0 & & \\ & & & & & & 0\ 1 \end{array}\right] \left[\begin{array}{ccc:ccc:cc} 0 & \sqrt{70} & 0 & & & & 0 & \\ -\sqrt{70} & 0 & 0 & & 0 & & 1 & 0 \\ 0 & 0 & 0 & & & & 1 & \\ \hdashline & & & 0 & \sqrt{70} & 0 & & 0 \\ & 0 & & -\sqrt{70} & 0 & 0 & 0 & 1 \\ & & & 0 & 0 & 0 & & 1 \\ \hdashline 0 & 7 & 4 & 0 & -2 & -4 & & \\ & & & & & & 0 & 0 \\ 0 & -2 & -4 & 0 & 4/7 & 38/7 & & \end{array}\right]$$

(3.67)

The nonlinear algebraic system (3.67) is solved, subject to the constraint that the network parameters C_3, L_4, L_5 and L_6 be positive. Although the system represents 44 nontrivial equations in 28 unknowns, the solution is quite easy to achieve by elimination. This is due to two factors: First, the equations representing $CG_i = N_i$, i=1,2, are independent of the network parameters and thus allow an immediate elimination of 12 unknown entries in G_1 and G_2. Second, the degree of nonlinearity exhibited by the system is low since only second order cross-products of unknowns appear.

The details of the solution are not included but it can be verified that (3.67) is satisfied with

$$\begin{aligned} C_3 &= 0.1 \\ L_4 &= 0.2 \\ L_5 &= 0.5 \\ L_6 &= 0.25 \end{aligned} \tag{3.68}$$

and

$$[G_1 \mid G_2] = \left[\begin{array}{ccc|ccc} \sqrt{70} & 0 & 0 & \frac{-2\sqrt{70}}{7} & 0 & 0 \\ 0 & 5 & 0 & 0 & -10/7 & 10/7 \\ 0 & 2 & 0 & 0 & -4/7 & -10/7 \\ 0 & 0 & 4 & 0 & 0 & -4 \end{array}\right] \tag{3.69}$$

Thus, the network having the specified admittance matrix (3.57) is shown in Figure 3.1 with element values (3.68). The significance of the matrices G_1 and G_2 given in (3.69) is apparent from the conditions

$$\left.\begin{aligned} AG_i &= G_iS \\ G_iF_o &= B_i \end{aligned}\right\} i=1,2 \tag{3.70}$$

which are satisfied by matrices A and B of the state equations. However, it follows from Lemma 3.2 that the conditions (3.70) are necessary and sufficient for the matrix system

$$\dot{\mathcal{X}}(t) = A\mathcal{X}(t), \quad \mathcal{X}(0) = B$$

to have the solution

$$\mathcal{X}(t) = [G_1F(t), G_2F(t)]$$

Therefore, G_1 and G_2 define the state responses of the network when the initial states are B_1 and B_2, respectively, with $V(t) = 0$.

Although this example is based on a simple L-C network, the technique used can be extrapolated to the more difficult problem of realizing a network of unknown topology. The only existing technique for establishing equations which relate a general R-L-C network to a specified transfer function matrix $\overline{M}(s)$ is to actually derive $\overline{M}(s)$ from the network while holding the parameters in literal form. Equating the derived form and the specified $\overline{M}(s)$ establishes nonlinear algebraic equations in the network parameters. This system has fewer equations than a corresponding set of FDE but the degree of nonlinearity is greater.

One cannot conclude that the FDE are easier to solve but there is no doubt that the FDE are established with considerably less effort for a variety of candidate networks. The specified transfer function matrix $\overline{M}(s)$ is inverted only once to yield the time-domain quantities F(t) and N_i, i=1,2,...,r. Then, for each trial network, the state model is formulated with all network parameters in literal form. For an arbitrary network, the state model can always be established with less manipulation than can the corresponding s-domain model. The reduced necessary conditions can be applied to eliminate those networks which cannot yield the given transfer function matrix. This test may also provide necessary values of some network parameters or necessary relationships between the parameters.

The parameter values can only be determined by solving the FDE, provided a solution exists. Even when no exact solution exists, there is practical significance in finding the set of parameter values which yield the "best" solution in some sense. This phase of the design problem is considered in Section IV.

IV. PARAMETER OPTIMIZATION

The fundamental design equations in Section II and III define constraints on the state model (1.1-2) which, if satisfied, assure the designer that the state model will have a specified solution. The desired solution can be expressed in the time domain or in terms of an equivalent transform model.

To be of practical use, the fundamental design equations must help the designer select a physical system having a desired response. This means finding a system which satisfies the FDE or, more exactly, finding a system having a state model which satisfies the FDE.

Often, the physical realizability of a system will not allow the FDE to be exactly satisfied. Thus, a frequently posed problem in system design is the following: Let a physical system have fixed topology but several unspecified scalar parameters which are available for variation. It is desired to find the values of these parameters which provide the "best" design in terms of some desired system response.

To be more precise, let a physical system be completely determined except for m real scalar parameters $p_1, p_2, \ldots, p_m$ which are available for variation in the design

problem. Let every parameter p_i satisfy

$$p_i' \leq p_i \leq p_i'' \tag{4.1}$$

for some choice of finite constants p_i' and p_i''. This choice is dictated by the physical realizability of the parameter.

Define the m-vector $\mathcal{P} = [p_1, p_2, \ldots, p_m]^T$ as the parameter vector which ranges over a subset Π of the real m-dimensional Euclidean vector space. The set Π is defined as the set of all m-tuples $(p_1, p_2, \ldots, p_m)$ such that p_i satisfies (4.1), i=1,2,...,m. The set Π is called the realizable parameter set and is a compact set by virtue of being closed and bounded [22].

Any mathematical model of the system will depend on the parameter vector $\mathcal{P}$. Assume the system has a state model for all $\mathcal{P} \in \Pi$ of the form

$$\dot{X}(t) = A(\mathcal{P})X(t) + B(\mathcal{P})E(t), \quad X(0) = X_o(\mathcal{P}) \tag{4.2}$$

$$Y(t) = C(\mathcal{P})X(t) + D(\mathcal{P})E(t) \tag{4.3}$$

where the matrices A, B, C, D and X_o are functions of the vector $\mathcal{P}$. For example, $A(\mathcal{P})$ denotes a matrix function with typical entry

$$a_{ij}(\mathcal{P}) = a_{ij}(p_1, p_2, \ldots, p_m) \quad i,j=1,2,\ldots,n$$

where $a_{ij}(\mathcal{P})$ is assumed to be a nonlinear continuous function

of its arguments. In general, we assume that all entries of $A(\mathcal{P})$, $B(\mathcal{P})$, $C(\mathcal{P})$, $D(\mathcal{P})$ and $X_o(\mathcal{P})$ are continuous functions of $\mathcal{P}$ for all $\mathcal{P} \in \Pi$. In fact, the realizable parameter set can always be suitably restricted to assure this condition.

Let the design specifications on the system response be given either in the time domain or in terms of an equivalent s-domain or frequency domain model. From the results of the previous sections, these specifications lead to a set of fundamental design equations which can be written in the augmented matrix form

$$\mathcal{A}(\mathcal{P})\mathcal{G}_1 = \mathcal{G}_2 \mathcal{S} \qquad (4.4)$$

Note that all matrix functions of $\mathcal{P}$ appearing in the state model (4.2-3) are contained in the augmented matrix $\mathcal{A}(\mathcal{P})$. Define the h-vector $\mathcal{G}$ which contains all unknown entries of $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $\mathcal{G}$ range over a compact subset Γ of the real h-dimensional Euclidean space. There is no loss of generality in requiring the compactness of Γ since the physical properties of the system also require matrices $\mathcal{G}_1$ and $\mathcal{G}_2$ with bounded entries.

The parameter design problem now takes the following form: Find the vectors $\mathcal{P}_o \in \Pi$ and $\mathcal{G}_o \in \Gamma$ such that (4.4) is satisfied; if not exactly, then in some "optimum" sense. Let the matrix products $\mathcal{A}(\mathcal{P})\mathcal{G}_1$ and $\mathcal{G}_2 \mathcal{S}$ have dimension r x q and define the r x q matrix

$$\mathcal{Z} = \mathcal{A}(\mathcal{P})\,\mathcal{G}_1 - \mathcal{G}_2\,\mathcal{S} \tag{4.5}$$

Consider the scalar function

$$u(\mathcal{P},\mathcal{G}) = \sum_{i=1}^{r}\sum_{j=1}^{q} z_{ij}^{2} \tag{4.6}$$

where z_{ij} is the typical entry of $\mathcal{Z}$. The domain of u is $\Theta = \Pi \times \Gamma$, the cartesian product of the two compact sets Π and Γ and is therefore compact [23]. The range of u is the non-negative real axis.

Definition 4.1

The optimum parameter vector $\mathcal{P}_o$ is defined as that vector $\mathcal{P} \in \Pi$ which, together with some vector $\mathcal{G}_o \in \Gamma$, minimizes the function $u(\mathcal{P},\mathcal{G})$ over all vectors $\mathcal{P} \in \Pi$ and $\mathcal{G} \in \Gamma$. That is,

$$u(\mathcal{P}_o, \mathcal{G}_o) = \min_{\substack{\mathcal{P}\in\Pi \\ \mathcal{G}\in\Gamma}} u(\mathcal{P},\mathcal{G}) \tag{4.7}$$

The state model (4.2-3) with $\mathcal{P} = \mathcal{P}_o$ is called the optimum state model and the corresponding physical system the optimum system.

The compactness of the set Θ assures the designer that there always exists at least one pair of vectors $\mathcal{P} \in \Pi$

and $\mathcal{G} \in \Gamma$ for which $u(\mathcal{P}, \mathcal{G})$ takes a minimum value. This assertion, not proved here, follows from the continuity of $u(\mathcal{P}, \mathcal{G})$ on the compact set Θ [23].

It follows from Definition 4.1, together with the properties of the fundamental design equations, that if $u(\mathcal{P}_o, \mathcal{G}_o) = 0$, then the optimum parameter vector yields an exact design. If, on the other hand, the optimum parameter vector $\mathcal{P}_o$ yields $u(\mathcal{P}_o, \mathcal{G}_o) = \gamma > 0$, then, exact design is not achieved and an important question arises. How "close" is the response of the optimum system to the specified response? Under these conditions, the r x q matrix defined by (4.5) can be considered as an error matrix but the relationship between this matrix and the corresponding time-domain error in the resulting solution has not been established. However, this need not be considered as a great disadvantage since the widely-used mean squared-error criteria is not directly related to a time-domain error expression. That is, a given value of the mean squared-error for a particular system response does not disclose the distribution of the error over the time interval.

As a practical alternative, the solution to the optimum state model can always be obtained and compared with the desired solution. If the difference in the two solutions is unacceptable, the designer is at least assured that the proposed physical system must be altered topologically in order to achieve the desired response by this method. This

information, although negative in character, is still useful in practical design problems. Also, such conclusions cannot be drawn immediately from parameter design techniques based on the analog computer.

Except in simple cases, the most efficient procedure for accomplishing the minimization of the function $u(\mathcal{P}, \mathcal{G})$ is by numerical techniques. The FDE (4.4) can be programmed directly on the digital computer which then forms the function $u(\mathcal{P}, \mathcal{G})$ and performs the minimization. Appendix C lists the FORTRAN program GRADNN which accepts equations of the form (4.4) and uses the method of steepest descent to find vectors $\mathcal{P}_o$ and $\mathcal{G}_o$ minimizing $u(\mathcal{P}, \mathcal{G})$. Example 4.1 illustrates the parameter optimization technique and the use of the program GRADNN.

Example 4.1

A physical system, with fixed structure except for three real parameters p_1, p_2 and p_3, has the state model

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 30p_3 + p_2^2 - 2 & 13.75 & -40p_1 \\ 56 & -5p_1^2 & 5-30p_2 \\ 75p_2 - p_1(10p_3+.5) & 36.25 & -80 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -p_3-6 & -50 \\ -18.3 & 70 \\ -71.4p_1 & -30 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \qquad (4.8)$$

$$X_o = \begin{bmatrix} x_1(o) \\ x_2(o) \\ x_3(o) \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix} \tag{4.9}$$

The system has two inputs with excitation

$$E(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{bmatrix} 10 \\ .5t \end{bmatrix}, \quad t \geq 0 \tag{4.10}$$

$$= 0, \text{ otherwise}$$

The parameters p_1, p_2 and p_3 are to be determined so that the state variables $x_1(t)$ and $x_2(t)$ have solutions specified by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 3 & -10 & -1 \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} e^{-20t} \\ e^{-20t}\sin 5t \\ e^{-20t}\cos 5t \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 0 & 2 \\ g_{34} & g_{35} \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} \tag{4.11}$$

Except as to form, the time variation of $x_3(t)$ is unspecified. The real constants g_{3j}, $j=1,2,\ldots,5$, can assume any finite values consistent with the given specifications on $x_1(t)$ and $x_2(t)$.

The vector functions $F_s(t)$ and $F_e(t)$ are identified by

$$\begin{bmatrix} F_s(t) \\ F_e(t) \end{bmatrix} = \begin{bmatrix} e^{-20t} \\ e^{-20t}\sin 5t \\ e^{-20t}\cos 5t \\ \hline 1 \\ t \end{bmatrix} \qquad (4.12)$$

where

$$\frac{d}{dt}\begin{bmatrix} F_s(t) \\ F_e(t) \end{bmatrix} = \begin{bmatrix} S_s & 0 \\ 0 & S_e \end{bmatrix}\begin{bmatrix} F_s(t) \\ F_e(t) \end{bmatrix}, \quad \begin{bmatrix} F_{so} \\ F_{eo} \end{bmatrix} = \begin{bmatrix} F_s(o) \\ F_e(o) \end{bmatrix}$$

$$= \left[\begin{array}{ccc:cc} -20 & 0 & 0 & & \\ 0 & -20 & 5 & \multicolumn{2}{c}{O} \\ 0 & -5 & -20 & & \\ \hdashline & & & 0 & 0 \\ \multicolumn{3}{c:}{O} & & \\ & & & 1 & 0 \end{array}\right]\begin{bmatrix} F_s(t) \\ F_e(t) \end{bmatrix} \qquad (4.13)$$

$$\begin{bmatrix} F_{so} \\ F_{eo} \end{bmatrix} = \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ \hline 1 \\ 0 \end{array}\right] \tag{4.14}$$

By Definition 2.2, (4.12) is a basis vector for $t \geq 0$ with basis system (4.13-14). In addition, the excitation vector (4.10) has the representation

$$\begin{aligned} E(t) &= HF_e(t) \\ &= \begin{bmatrix} 10 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} \end{aligned} \tag{4.15}$$

By Theorem 2.1, the state model (4.8-9) has the solution (4.11) for all $t \geq 0$ if and only if the fundamental design equations are satisfied. From (2.43), these are

$$\begin{bmatrix} A & B & X_o \end{bmatrix} \begin{bmatrix} G_s & G_e & 0 \\ 0 & H & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} G_s & G_e \end{bmatrix} \begin{bmatrix} S_s & 0 & F_{so} \\ 0 & S_e & F_{eo} \end{bmatrix}$$

or in detail,

$$\left[\begin{array}{ccc|cc|c} 30p_3 + p_2^2 - 2 & 13.75 & -40p_1 & -p_3 - 6 & -50 & 6 \\ 56 & -5p_1^2 & 5 - 30p_2 & -18.3 & 70 & 2 \\ 75p_2 - p_1(10p_3 + .5) & 36.25 & -80 & -71.4p_1 & -30 & 6 \end{array}\right] .$$

$$\cdot \left[\begin{array}{ccc|cc|c} -1 & 2 & 3 & 4 & -2 & \\ 3 & -10 & -1 & 0 & 2 & 0 \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & \\ \hline & & & 10 & 0 & \\ & 0 & & & & 0 \\ & & & 0 & .5 & \\ \hline & 0 & & & 0 & 1 \end{array}\right] =$$

$$\left[\begin{array}{ccc|cc} -1 & 2 & 3 & 4 & -2 \\ 3 & -10 & -1 & 0 & 2 \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} \end{array}\right] \left[\begin{array}{ccc|cc|c} -20 & 0 & 0 & & & 1 \\ 0 & -20 & 5 & 0 & & 0 \\ 0 & -5 & -20 & & & 1 \\ \hline & & & 0 & 0 & 1 \\ & 0 & & & & \\ & & & 1 & 0 & 0 \end{array}\right] \qquad (4.16)$$

The FDE of (4.16) exhibit the general form

$$\mathcal{A}(\mathcal{P})\, \mathcal{G}_1 = \mathcal{G}_2\, \mathcal{S} \qquad (4.17)$$

where the parameter vector $\mathcal{P}$ is a 3-vector defined as

$$\mathcal{P} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \qquad (4.18)$$

Let the realizable parameter bounds be

$$.1 \leq p_i \leq 10 \ , \ i=1,2,3 \tag{4.19}$$

and let the unknown entries in $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy

$$\left|g_{3j}\right| \leq 100 \ , \ j=1,2,\ldots,5 \tag{4.20}$$

In the language of Definition 4.1, the realizable parameter set $\mathcal{T}$ is compact and the 5-vector $\mathcal{G}$, containing g_{3j}, $j=1,2,\ldots5$, ranges over a compact set Γ. The design problem reduces to finding the optimum parameter vector $\mathcal{P}_o \in \mathcal{T}$ which, together with some vector $\mathcal{G}_o \in \Gamma$, minimizes the function

$$u(\mathcal{P},\mathcal{G}) = \sum_{i=1}^{3} \sum_{j=1}^{6} z_{ij}^{2} \tag{4.21}$$

where z_{ij} is the typical entry of the 3 x 6 difference matrix

$$\mathcal{Z} = \mathcal{A}(\mathcal{P})\mathcal{G}_1 - \mathcal{G}_2 \mathcal{S} \tag{4.22}$$

The minimization of $u(\mathcal{P},\mathcal{G})$ is accomplished by using program GRADNN listed in Appendix C. This program requires that initial values $\mathcal{P}'$ and $\mathcal{G}'$ of the vectors $\mathcal{P}$ and $\mathcal{G}$ be assumed. Let

$$\mathcal{P}' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{4.23}$$

and take $\mathcal{G}'$ such that $g_{3j}=1$, $j=1,2,\ldots,5$. With these values, the initial value of $u(\mathcal{P},\mathcal{G})$, as calculated by program GRADNN, is approximately 476,000. Using the initial parameter values (4.23), the solution to the state model (4.8-9) is approximately

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 8.00 & 13.50 & -22.71 \\ -1.51 & -9.79 & -46.60 \\ 15.21 & -2.80 & -40.40 \end{bmatrix} \begin{bmatrix} e^{-49.6t} \\ e^{-3.2t}\sin 18.1t \\ e^{-3.2t}\cos 18.1t \end{bmatrix}$$

$$+ \begin{bmatrix} 21.51 & -2.19 \\ 49.78 & -2.37 \\ 31.02 & -3.03 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} \qquad (4.24)$$

The solution $x_1(t)$ and $x_2(t)$ differ significantly from the specified solutions given in (4.11).

An optimization of the parameters by program GRADNN yields the optimum parameter vector

$$\mathcal{P}_o = \begin{bmatrix} 0.5008 \\ 1.5003 \\ 0.7002 \end{bmatrix} \qquad (4.25)$$

with

$$\mathcal{G}_o = \begin{bmatrix} -0.0010 \\ -1.9979 \\ 5.0007 \\ 0.9930 \\ -2.0003 \end{bmatrix} \qquad (4.26)$$

and with $u(\mathcal{P}_o, \mathcal{G}_o) = 0.1370$; all values being corrected to four decimal places. The optimum state model is obtained by using the optimum parameter vector (4.25) in the state model (4.8-9). The solution of this model is approximately

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -1.08 & 1.53 & 3.03 \\ 2.93 & -9.72 & -0.68 \\ -0.18 & -2.58 & 5.35 \end{bmatrix} \begin{bmatrix} e^{-19.4t} \\ e^{-20.3t}\sin 5.94t \\ e^{-20.3t}\cos 5.94t \end{bmatrix}$$

$$+ \begin{bmatrix} 4.00 & -1.98 \\ -0.04 & 2.02 \\ 0.96 & -1.97 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} \qquad (4.27)$$

The solutions for $x_1(t)$ and $x_2(t)$ correspond closely to those specified in (4.11).

Using the CDC 3600 computer, the total computation time for the optimization was three minutes and twenty-one seconds with 900 iterations being executed. The solutions (4.24) and (4.27) were also generated on the computer by using a modified version of program GRADNN. This solution technique is discussed in Appendix C.

It is difficult to compare the present parameter optimization technique with existing methods because there is essentially only one alternative approach which exhibits comparable generality and this method is not widely used.

This approach first requires the derivation of the s-domain model from the system while holding the parameters in literal form. Next, the time domain specifications on the output response are transformed into the s-domain to yield a specified s-domain model. Equating the two s-domain models defines a system of nonlinear algebraic equations which must be solved to yield the optimum parameter set. In general, this system contains fewer equations and unknowns than the corresponding FDE but the degree of nonlinearity is considerably higher. Thus, in general, it cannot be concluded that the s-domain method generates equations which enjoy a more efficient computer solution. Also, as pointed out in Section III, the formulation of the s-domain model from a physical system, while holding parameters in literal form, requires considerably more algebraic manipulation than the formulation of the corresponding state model. Moreover, the computer cannot be used to generate the s-domain model in literal form so that this additional manipulation represents an inefficient use of the designer's time.

V. CONCLUSION

The expanding complexity of modern-day systems has exposed a basic weakness in the traditional design techniques. The transform model and the analog computer, once thought to be adequate design tools, now leave much to be desired. In the search for more effective techniques, many engineers have turned to the digital computer.

In retrospect, it is clear that too much effort has been devoted to implementing existing design procedures on the computer. Consequently, no new design techniques have evolved which are truly general in application, fundamentally different from existing methods and which offer promise in extending the role of the digital computer in system design.

This thesis establishes an approach to the design of linear time-invariant systems which is fundamentally different from any technique proposed thus far. Design specifications, expressed either in the time domain as a desired time solution or in the s-domain as a desired transfer function matrix, are used to define fundamental design equations. These equations are algebraic and provide necessary and sufficient conditions for the state model (1.1-2) to satisfy the given design criteria. Both forced and unforced

systems are treated with excitation functions assuming any form obtained as the solution to a homogeneous constant-coefficient differential equation of finite order. This includes all non-impulsive inputs normally used in s-domain design.

In addition to providing necessary and sufficient conditions, the fundamental design equations can be solved to yield a state model satisfying the design criteria. When no requirements are placed on the form of the state model, the fundamental design equations yield arbitrary forms of the model including those recently shown by Kalman and Gilbert. If restrictions are placed on certain entries of the state model, the fundamental design equations incorporate these restrictions directly as algebraic constraints on the unknown entries. The restrictions on the form of the state model can be changed at will and without additional manipulation.

Reduced necessary conditions are given which are useful in eliminating those state models which will not satisfy the design specifications. These conditions, together with the fundamental design equations provide a new and interesting approach to the classical problem of network synthesis.

For some time, optimal design has been merely a concept rather than a practical reality. The analog computer yields only "cut and try" designs and no acceptable analytical technique for parameter optimization, using the

s-domain model, has been proposed. Parameter optimization techniques based on the s-domain model are restricted because of the difficulty in relating the system parameters to the model. On the other hand, the state model is inherently more precise and many system parameters can be carried directly into the model.

In Section IV, a parameter optimization technique is proposed which exploits both the descriptive properties of the state model and the iterative capabilities of the digital computer. The technique allows the fundamental design equations to be programmed directly on the computer without further manipulation and the standard numerical method of "steepest descent" is used to achieve the optimization. A computer program is given which implements this optimization technique.

APPENDIX A

ALTERNATE PROOF OF SUFFICIENT CONDITIONS

This section contains an alternate sufficiency proof of Theorem 2.2. The proof rests on the theory of functions of matrices and its application to the solution of linear constant coefficient differential equations [20], [24], [25]. The following lemma is required in the main theorem.

Lemma A.1

Let the matrix relationship

$$AG = GS \tag{A.1}$$

hold between an m x q matrix G and the square matrices A and S. Then

$$f(A)G = Gf(S) \tag{A.2}$$

also holds for any function of the matrices A and S that is representable as the sum of a convergent matrix power series.

Proof

Let f(x) be any scalar function representable as the sum of a convergent power series

$$f(x) = \sum_{n=o}^{\infty} k_n x^n \tag{A.3}$$

where k_n, n=0,1,..., are scalar constants.

Then

$$f(A) = \sum_{n=o}^{oo} k_n A^n \qquad (A.4)$$

and

$$f(S) = \sum_{n=o}^{oo} k_n S^n \qquad (A.5)$$

where A^o and S^o are defined as unit matrices of order m and q, respectively.

Since AG = GS, it follows that

$$A(AG) = A(GS) = (AG)S = (GS)S$$

or $$\therefore A^2G = GS^2$$

Similarly,

$$A^3G = GS^3$$

$$\cdots\cdots\cdots$$

$$A^nG = GS^n$$

for all n. Thus,

$$k_nA^nG = k_nGS^n \qquad (A.6)$$

for all n and therefore

$$\sum_{n=o}^{\infty} k_n A^n G = \sum_{n=o}^{\infty} k_n G S^n$$

or

$$\left[\sum_{n=o}^{\infty} k_n A^n\right] G = G\left[\sum_{n=o}^{\infty} k_n S^n\right]$$

The conclusion follows from (A.4-5).

Theorem A.1

Hypothesis:

1) $T = \{t \mid o \leq t \leq t_1\}$, t_1 a non-zero constant.

2) $F_s(t)$ and $F_e(t)$ are vector functions of order q_s and q_e, respectively, satisfying,

$$\dot{F}_s(t) = S_s F_s(t), \quad F_s(0) = F_{so} \tag{A.7}$$

$$\dot{F}_e(t) = S_e F_e(t), \quad F_e(0) = F_{eo} \tag{A.8}$$

on T, where S_s and S_e are square matrices of order q_s and q_e, respectively and F_{so} and F_{eo} are constant q_s- and q_e-vectors, respectively.

3) The r-vector function E(t) is defined on T by

$$E(t) = HF_e(t) \tag{A.9}$$

where H is an r x q_e matrix.

4) The k-vector function $\overline{Y}(t)$ is defined on T by

$$\overline{Y}(t) = N_s F_s(t) + N_e F_e(t) \tag{A.10}$$

where N_s and N_e are matrices of order k x q_s and k x q_e, respectively.

Conclusion:

Sufficient conditions for $\overline{Y}(t)$ to be the output solution, on T, of

$$\dot{X}(t) = AX(t)+BE(t), \quad X(0) = X_o \tag{A.11}$$

$$Y(t) = CX(t)+DE(t) \tag{A.12}$$

are that there exist matrices G_s and G_e such that

$$AG_s = G_sS_s \tag{A.13}$$

$$AG_e + BH = G_eS_e \tag{A.14}$$

$$G_sF_{so} + G_eF_{eo} = X_o \tag{A.15}$$

$$CG_s = N_s \tag{A.16}$$

$$CG_e + DH = N_e \tag{A.17}$$

Proof

Assume that there exist matrices G_s and G_e such that A, B, C and D satisfy (A.13-17). The general state and output solutions of (A.11-12) are, respectively

$$X(t) = e^{At}X_o + \int_o^t e^{A(t-u)}BE(u)du \tag{A.18}$$

and

$$Y(t) = Ce^{At}X_o + C\int_o^t e^{A(t-u)}BE(u)du + DE(t) \qquad (A.19)$$

It must be shown that (A.19) reduces to (A.10) by virtue of conditions (A.13-17) and the hypothesis. It is first shown that X(t) given by (A.18) reduces to

$$X(t) = G_sF_s(t) + G_eF_e(t) \qquad (A.20)$$

Substituting (A.9) into (A.18) and using (A.14), X(t) takes the form

$$X(t) = e^{At}X_o + \int_o^t e^{A(t-u)}BHF_e(u)du$$

or

$$X(t) = e^{At}X_o + \int_o^t e^{A(t-u)}[G_eS_e - AG_e]F_e(u)du$$

Since $e^{A(t-u)} = e^{At}e^{-Au}$, it follows that

$$X(t) = e^{At}\left[X_o + \int_o^t e^{-Au}[G_eS_e - AG_e]F_e(u)du\right] \qquad (A.21)$$

Applying (A.8) and the property that the matrices A and e^{-Au} commute, the integrand in (A.21), reduces to

$$e^{-Au}[G_eS_e - AG_e]F_e(u) =$$

$$= e^{-Au}G_eS_eF_e(u) - e^{-Au}AG_eF_e(u) \qquad (A.22)$$

$$= e^{-Au}G_e\dot{F}_e(u) - Ae^{-Au}G_eF_e(u)$$

The commutative property follows directly from the power series representation of e^{-Au} as

$$e^{-Au} = U - Au + \frac{A^2u^2}{2!} - \frac{A^3u^3}{3!} + \ldots \qquad (A.23)$$

where U is the n-dimensional unit matrix. This series converges uniformly in any finite interval and is continuous in that interval. Also, the differentiated series converges uniformly and it is seen from (A.23) that

$$\frac{d}{du}\, e^{-Au} = -Ae^{-Au} \qquad (A.24)$$

Using (A.24) in (A.22), the integrand of (A.21) becomes

$$e^{-Au}[G_eS_e - AG_e]F_e(u) = \frac{d}{du}\,[e^{-Au}G_eF_e(u)] \qquad (A.25)$$

Using this result in (A.21) gives

$$X(t) = e^{At}\left[X_o + \int_o^t \frac{d}{du}\,[e^{-Au}G_eF_e(u)]du\right] \qquad (A.26)$$

$$= e^{At}\left[X_o + [e^{-Au}G_eF_e(u)]\Big|_o^t\right]$$

$$= e^{At}\left[X_o + e^{-At}G_eF_e(t) - G_eF_e(0)\right] \qquad (A.27)$$

where from (A.23), $e^{-Au}\Big|_{u=o} = U$. Substituting (A.15) into (A.27) yields

$$X(t) = e^{At}\left[G_sF_{so} + G_eF_{eo} + e^{-At}G_eF_e(t) - G_eF_{eo}\right]$$
$$= e^{At}\, G_sF_{so} + G_eF_e(t) \qquad \text{(A.28)}$$

Now (A.13) together with Lemma A.1 imply that $f(A)G_s = G_sf(S_s)$, where $f(A)$ and $f(S_s)$ denote any function of the matrices A and S_s which is representable as a convergent matrix power series. The matrix function e^{At} satisfies these conditions and therefore,

$$e^{At}G_s = G_se^{S_st} \qquad \text{(A.29)}$$

Substituting (A.29) into (A.28) and noting that (A.7) has the solution $F_s(t) = e^{S_st}F_{so}$, it follows that

$$X(t) = G_se^{S_st}F_{so} + G_eF_e(t)$$

or finally,

$$X(t) = G_sF_s(t) + G_eF_e(t) \qquad \text{(A.30)}$$

It has been shown that (A.18) reduces to (A.20). To complete the proof, use this identity to write (A.19) as

$$Y(t) = C[G_sF_s(t) + G_eF_e(t)] + DE(t) \qquad \text{(A.31)}$$

From (A.9) and (A.16-17), (A.31) becomes

$$Y(t) = CG_sF_s(t) + [CG_e + DH]F_e(t)$$

or

$$Y(t) = N_s F_s(t) + N_e F_e(t)$$

and the conclusion follows.

APPENDIX B

TRANSFORM MODELS OF LINEAR TIME-INVARIANT SYSTEMS

This section serves as a reference for the development given in Section III. There, several properties are assumed which involve the time-domain, s-domain, and real frequency-domain models. These properties are easily derived from the fundamental definitions of linear time-invariant systems and the theory of Laplace and Fourier transforms. By virtue of the large number of references on Laplace and Fourier transform theory, a detailed discussion of these topics in this section is neither necessary nor appropriate [26], [27], [28].

Let a linear time-invariant system have the state model

$$\dot{X}(t) = AX(t) + BE(t), \quad X(0) = X_o \tag{B.1}$$

$$Y(t) = CX(t) + DE(t) \tag{B.2}$$

where X(t), E(t) and Y(t) are vector functions of time having order n, r and k, respectively, and A, B, C and D are constant real matrices of appropriate dimension.

Assume that X(t), Y(t) and E(t) have corresponding vector Laplace transforms $\overline{X}(s)$, $\overline{Y}(s)$ and $\overline{E}(s)$. The one-sided Laplace transform is implied here and the condition

$$E(t) = X(t) = 0, \quad t < 0 \tag{B.3}$$

assures a unique correspondence between the time functions and their respective transforms.

The k x r transfer function matrix $\bar{M}(s)$ of a given system is defined to be the matrix relating the transform of the excitation vector to the transform of the output vector as

$$\bar{Y}(s) = \bar{M}(s)\bar{E}(s) \tag{B.4}$$

Under the assumption that the s-domain model (B.4) and the state model (B.1-2) characterize the same system, we desire to derive the form of $\bar{M}(s)$ and thereby develop certain useful relationships between the two models. It is immediately noted that the two models cannot yield identical system responses unless the initial state X_o in (B.1) is zero. For if not, the unforced ($E(t) = 0$, $t \geq o$) solutions of the two models do not coincide.

The most direct approach for developing the form of $\bar{M}(s)$ is to solve the state model by Laplace transforms. Transformation of (B.1-2) yields

$$s\bar{X}(s) = A\bar{X}(s) + B\bar{E}(s) \tag{B.5}$$

$$\bar{Y}(s) = C\bar{X}(s) + D\bar{E}(s) \tag{B.6}$$

under the assumption $X_o = 0$. Solving (B.5) for $\bar{X}(s)$ and substituting this result into (B.6), we obtain

$$\bar{Y}(s) = [C(sU - A)^{-1}B + D]\bar{E}(s) \qquad (B.7)$$

where U is the n-dimensional unit matrix.

Comparing (B.7) with (B.4), it follows that a system having the state model (B.1-2) has a unique transfer function matrix $\bar{M}(s)$ defined by

$$\bar{M}(s) = \bar{N}(s) + D \qquad (B.8)$$

where

$$\bar{N}(s) = C(sU - A)^{-1}B \qquad (B.9)$$

Bringing to bear the well-known properties of the matrix $(sU - A)^{-1}$, it follows that the k x r matrix $\bar{N}(s)$ takes the form

$$\bar{N}(s) = \frac{1}{d(s)} [C \text{ adj}(sU - A)B]$$

where d(s) is the characteristic polynomial of A and adj(sU - A) denotes the adjoint of the matrix (sU - A); sometimes called the "conjoint" of A [25]. Also, $\bar{N}(s)$ has entries which are rational functions in s. Let

d_n = degree of numerator polynomial

d_d = degree of denominator polynomial

Then for every entry of $\bar{N}(s)$, the inequality

$$d_n < d_d \leq n \qquad (B.10)$$

holds, where n is the order of the state vector. Since D is a constant k x r matrix, (B.8) implies that every entry in $\overline{M}(s)$ is also a rational function in s for which (B.10) holds with the additional possibility that $d_n = d_d$.

Although, the transfer function matrix $\overline{M}(s)$ is uniquely determined by a given state model, the converse is obviously not true by virtue of the non-unique factorization of the matrix product (B.9).

The inverse Laplace transform of the matrix $\overline{N}(s)$ has a characteristic form which is very useful. Consider the general output solution of the state model (B.1-2) with $X_o = 0$:

$$Y(t) = C \int_o^t e^{A(t-u)} BE(u)du + DE(t) \qquad (B.11)$$

Noting that the integral of (B.11) takes the convolution form, the Laplace transform of Y(t) gives

$$\begin{aligned} \overline{Y}(s) &= \overline{W}(s)\overline{E}(s) + D\overline{E}(s) \\ &= [\overline{W}(s) + D]\overline{E}(s) \end{aligned} \qquad (B.12)$$

where

$$\overline{W}(s) = \mathcal{L}\{Ce^{At}B\} \qquad (B.13)$$

Comparing (B.12-13) with (B.7-9), it follows that

$$\overline{N}(s) + D = \overline{W}(s) + D$$

or

$$\bar{N}(s) = C(sU - A)^{-1}B = \bar{W}(s) = \mathcal{L}\{Ce^{At}B\}$$

Inverting $\bar{N}(s)$ gives

$$\begin{aligned} N(t) &= \mathcal{L}^{-1}\{\bar{N}(s)\} \\ &= \mathcal{L}^{-1}\{C(sU - A)^{-1}B\} \\ &= Ce^{At}B \end{aligned} \qquad (B.14)$$

The equivalence (B.14) provides a useful relationship between the s-domain and time-domain models. This result is exploited in Section III.

The k x r <u>frequency response matrix</u> $M^*(w)$ of a given system is defined to be the matrix relating the Fourier transform $E^*(w)$ of the excitation vector $E(t)$ to the Fourier transform $Y^*(w)$ of the output vector $Y(t)$ as

$$Y^*(w) = M^*(w)E^*(w) \qquad (B.15)$$

provided the transforms exist.

It can be shown that if a scalar time function $f(t) = 0$ for $t < 0$ and has the Laplace transform $\bar{f}(s)$ and the Fourier transform $f^*(w)$, then

$$f^*(w) = \bar{f}(s)\Big|_{s=jw} \qquad (B.16)$$

This property follows directly from the definitions of the

transforms [27]. Noting that condition (B.3) implies that $Y(t) = 0$ for $t < 0$, the equality (B.16) can be extended to the frequency-domain model (B.15) to give

$$Y^*(w) = \bar{Y}(s)\Big|_{s = jw} \tag{B.17}$$

and

$$E^*(w) = \bar{E}(s)\Big|_{s = jw} \tag{B.18}$$

Hence, we have

$$M^*(w) = \bar{M}(s)\Big|_{s = jw} \tag{B.19}$$

Thus, all the results derived for the transfer function matrix $\bar{M}(s)$ also hold for the frequency response matrix when s is replaced by jw and the Laplace transform is replaced by the Fourier transform. That is, if a system has the state model (B.1-2) with $X_o=0$, then the corresponding frequency response matrix $M^*(w)$ takes the form

$$M^*(w) = N^*(w) + D \tag{B.20}$$

where

$$N^*(w) = C(jwU - A)^{-1} B \tag{B.21}$$

and

$$N(t) = \mathcal{F}^{-1}\{N^*(w)\} = Ce^{At}B \tag{B.22}$$

The results above may be obtained in an alternative fashion by invoking an equivalent definition of the frequency response matrix. Let the excitation vector in (B.1-2) be defined as

$$E(t) = E_w e^{jwt}, \quad t \geq 0 \qquad (B.23)$$

$$= 0, \text{ otherwise}$$

where E_w is an r-vector with complex entries and e^{jwt} is a complex-valued scalar function with $j \equiv \sqrt{-1}$. Under these conditions the steady-state output vector $Y_{ss}(t)$ takes the form

$$Y_{ss}(t) = Y_w e^{jwt} \qquad (B.24)$$

where Y_w is a k-vector with complex entries.

An equivalent definition of the frequency-response matrix is that k x r matrix relating the complex magnitude vector E_w to the complex magnitude vector Y_w as

$$Y_w = M^*(w)E_w \qquad (B.25)$$

Using this definition, the properties (B.20) and (B.22) of $M^*(w)$ can be demonstrated easily. Consider the general output solution of the state model (B.1-2) with $X_o = 0$:

$$Y(t) = C \int_o^t e^{A(t-u)} BE(u)du + DE(t) \qquad (B.26)$$

With a change of variable $x = t - u$, (B.26) becomes

$$Y(t) = \int_o^t Ce^{Ax}BE(t - x)dx + DE(t) \qquad (B.27)$$

Substituting (B.23) into (B.27) gives

$$\begin{aligned} Y(t) &= \int_o^t Ce^{Ax}BE_w e^{jw(t - x)}dx + DE_w e^{jwt} \\ &= e^{jwt}\left[\int_o^{\infty} Ce^{Ax}BE_w e^{-jwx}dx \right. \\ &\quad \left. - \int_t^{\infty} Ce^{Ax}BE_w e^{-jwx}dx\right] + DE_w e^{jwt} \end{aligned} \qquad (B.28)$$

Provided the system is strictly stable, the matrix function $e^{At} \rightarrow 0$ as $t \rightarrow \infty$. This occurs if and only if the eigenvalues of A have negative real parts. In this case, the first integral in (B.28) exists and is independent of time. The second integral represents the transient part of the solution and clearly approaches zero as $t \longrightarrow \infty$. Thus, the steady-state solution $Y_{ss}(t)$ is

$$Y_{ss}(t) = \left[\int_o^{\infty} Ce^{Ax}Be^{-jwx}dx + D\right] E_w e^{jwt} \qquad (B.29)$$

Comparing (B.29) and (B.24), it follows that

$$Y_w = \left[\int_o^\infty Ce^{Ax} Be^{-jwx} dx + D \right] E_w$$

and therefore from (B.25)

$$M^*(w) = N^*(w) + D$$

where

$$N^*(w) = \int_o^\infty Ce^{Ax} Be^{-jwx} dx$$

$$= \mathcal{F}\{Ce^{At} B\}$$

We note that because of their complex representation, the excitation vector and steady-state output vector defined by (B.23-24) do not have physical significance. In practice, we consider only their real parts, which specify the steady-state oscillations (of radian frequency w) at the input and output of the system. It is precisely this fact, coupled with the definition (B.25), which allows laboratory measurement of the frequency-response matrix M*(w) of a strictly stable system.

APPENDIX C

PROGRAM GRADNN

The computer program GRADNN, described in this appendix, utilizes the method of steepest descent to find the vectors $\mathcal{P}_o$ and $\mathcal{G}_o$ minimizing the scalar function

$$u(\mathcal{P},\mathcal{G}) = \sum_{i=1}^{r}\sum_{j=1}^{q} z_{ij}^{2} \qquad (C.1)$$

where z_{ij} is the typical entry of the r x q matrix

$$\mathcal{Z} = \mathcal{A}(\mathcal{P})\,\mathcal{G}_1 - \mathcal{G}_2\,\mathcal{S} \qquad (C.2)$$

and the vector $\mathcal{G}$ contains the unknown entries of $\mathcal{G}_1$ and $\mathcal{G}_2$.

The method of steepest descent is widely used in the solution of simultaneous nonlinear algebraic equations [29], [30]. The method may require a large number of iterations but it has a distinct advantage over other methods (Newton's Method, for example) in that it always converges. Essentially, the method finds the vector $X = [x_1, x_2, \ldots, x_n]^T$ which yields a minimum value of the function $\Phi(X) = \Phi(x_1, x_2, \ldots, x_n)$, provided a minimum exists. To accomplish this, a convergent sequence of vectors $X^{(k)}$, k=1,2,3,..., are generated with

components defined by

$$x_1^{(k+1)} = x_1^{(k)} - \lambda_k \frac{\partial \Phi(X^{(k)})}{\partial x_1}$$

$$x_2^{(k+1)} = x_2^{(k)} - \lambda_k \frac{\partial \Phi(X^{(k)})}{\partial x_2} \qquad (C.3)$$

$$x_n^{(k+1)} = x_n^{(k)} - \lambda_k \frac{\partial \Phi(X^{(k)})}{\partial x_n}$$

where the notation $\frac{\partial \Phi(X^{(k)})}{\partial x_i}$ denotes the partial derivative $\frac{\partial \Phi(X)}{\partial x_i}$ evaluated at $X = X^{(k)}$.

The scalar constant λ_k is obtained as a root of the equation in λ:

$$\frac{\partial}{\partial \lambda} R(\lambda) = \frac{\partial}{\partial \lambda} \Phi \left[x_1^{(k)} - \lambda \frac{\partial \Phi(X^{(k)})}{\partial x_1} , \ldots \right.$$

$$\left. \ldots, x_n^{(k)} - \lambda \frac{\partial \Phi(X^{(k)})}{\partial x_n} \right] = 0 \qquad (C.4)$$

Geometrically, the recursion formulae (C.3) transfer a given point $X^{(k)} = [x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}]^T$ on the hypersurface $\Phi(X^{(k)}) = \text{constant}$ to a second point $X^{(k+1)}$ located on the vector defined by the gradient of $\Phi(X)$ at the point $X^{(k)}$. The root λ_k of (C.4) minimizes $\Phi(X)$ along this gradient and thus the condition $\Phi(X^{(k+1)}) < \Phi(X^{(k)})$ holds at each iteration.

The formulation and solution of (C.4) is usually difficult. In lieu of this operation, program GRADNN utilizes an approximation technique given by Booth [30]. With this method, a quadratic approximation to $R(\lambda)$ in (C.4) is generated by evaluating $R(\lambda)$ at the points λ_0, λ_1, and λ_2 defined by

$$\begin{aligned} \lambda_0 &= 0 \\ \lambda_1 &= \frac{\lambda_2}{2} \\ \lambda_2 &= \frac{\underline{\Phi}(X^{(k)})}{\sum_{i=1}^{n} \left[\frac{\partial \underline{\Phi}(X^{(k)})}{\partial x_i} \right]^2} \end{aligned} \qquad (C.5)$$

The value λ_2 is obtained by setting to zero the first two terms of the Taylor expansion of $\underline{\Phi}(X^{(k)} + \lambda X)$. Minimizing the quadratic approximation with respect to λ yields

$$\lambda_k = \frac{\lambda_2[R(\lambda_2) - 4R(\lambda_1) + 3R(\lambda_0)]}{4[R(\lambda_2) - 2R(\lambda_1) + R(\lambda_0)]} \qquad (C.6)$$

This value is taken as the approximation to the root of (C.4) and used in the recursion formulae (C.3).

The major disadvantage in the method of steepest descent is that the method may converge to a relative minimum of $\underline{\Phi}(X)$ rather than its absolute minimum. If this occurs, and it is detected, then the procedure must be repeated with initial values which are closer to those yielding

the absolute minimum.

From practical experience with program GRADNN, it appears that some difficulties with relative minima can be avoided by selecting initial values of $\mathcal{P}$, $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{S}$ for which $u(\mathcal{P},\mathcal{G}) = 0$. This is always possible since a given value of $\mathcal{P}$ establishes a particular state model with a particular solution. This solution determines corresponding values of $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{S}$. Starting with these values, $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{S}$ are slowly incremented toward the desired values while simultaneously minimizing $u(\mathcal{P},\mathcal{G})$. The program listed below can be easily modified to implement this scheme.

Program GRADNN can also be used to generate an approximate analytic solution to the state model (1.1-2). The vector function $F_s(t)$, together with S_s and F_{so}, can be written directly from a knowledge of the eigenvalues of the matrix A in the state model. The values S_e, F_{eo} and H are determined by the excitation vector. Referring to the FDE of (2.43), all entries are known except for those in G_s and G_e. Letting the vector $\mathcal{G}$ contain these unknown entries, the function $u(\mathcal{P},\mathcal{G})$ in (C.1) becomes a function of $\mathcal{G}$ alone. With only minor alteration, program GRADNN can be used to determine the vector $\mathcal{G}_o$ such that $u(\mathcal{G}_o) = 0$. This defines the state solution X(t) of the state model from which the output solution Y(t) can be obtained immediately.

To use the program GRADNN listed below, one must provide a set of statements defining the matrix function $\mathcal{A}(\mathcal{P})$ in (C.2). The 2-dimensional array A denotes $a_{ij}(\mathcal{P})$ while the 3-dimensional array AP corresponds to $\frac{\partial}{\partial p_k} a_{ij}(\mathcal{P})$. These statements must immediately follow statement number 42.

In addition, a data deck must be provided as follows: The first data card contains five integers which define respectively, the row and column dimensions of $\mathcal{A}(\mathcal{P})$, the column dimensions of $\mathcal{G}_1$ and $\mathcal{G}_2$ and the dimension of the parameter vector $\mathcal{P}$. The format of this card is given by statement 10 while statement 20 gives the format of all remaining data cards.

The next group of cards, KN in number, defines the initial (or known) values of the matrices $\mathcal{G}_1$ and $\mathcal{G}_2$. The 2-dimensional array G stores these values. The particular program listed below requires that $\mathcal{G}_2$ be the leading submatrix of $\mathcal{G}_1$ or vice versa. All sets of PDE, with the exception of (2.44), satisfy this requirement and the program is easily modified to accommodate this set also.

The next group of N data cards determines the 2-dimensional array GT. This array defines each entry of $\mathcal{G}_1$ and $\mathcal{G}_2$ as either a variable in the iteration or a fixed constant as follows: GT(i,j) = 1 implies that the g_{ij} is to be iterated; GT(i,j) = 0 implies that g_{ij} is known and fixed.

The known matrix $\mathcal{S}$, represented by the 2-dimensional

array S is defined by the next set of NQ cards. The last data card contains the initial values of the parameters which are stored in the 1-dimensional array P.

The output from the computer includes the initial value of Q which represents the function $u(\mathcal{P}, \mathcal{G})$ in the program. Also, the current value of Q, together with the arrays A, P and G are printed after each 100 iterations. The total number of iterations is determined, in hundreds, by the index of the variable MMM. The program automatically replaces any negative parameter value with the value 10^{-8}. Other bounds, both upper and lower, are easily included just above statement 144.

The program accepts matrices with dimensions up to 20 x 20 and 20 parameters can be included. For problems similar to Example 4.1, the program executes approximately 1000 iterations/minute on the CDC 3600 computer.

Program GRADNN

```
   DIMENSION A(20,20), AP(20,20,20), QP(20), G(20,20), S(20,20),
  1 Y(20,20), P(20), QG(20,20), GT(20,20), PS(20), GS(20,20)
   READ 10, N,KN,KQ,NQ,NP
10 FORMAT (512)
   DO 15 I=1,KN
15 READ 20, (G(I,J), J=1,KQ)
20 FORMAT (8F10.3)
   DO 25 I=1,N
25 READ 20, (GT(I,J), J=1,NQ)
   DO 30 I=1,NQ
30 READ 20, (S(I,J), J=1, KQ)
   READ 20, (P(J), J=1,NP)
33 FORMAT (1H ,7(E14.8,2X)
34 FORMAT (1H ,E14.8)
```

```
      DS = 1
      DO 240 MMM = 1, 10
      QS = 10.**20.
      JJ = 1
      MM = 0
   42 CONTINUE
C     LIST ARRAYS A(I,J) AND AP(K,I,J)
      Q = 0.
      DO 50 I=1,N
      DO 50 J=1,KQ
      T = 0
      DO 45 K=1,KN
   45 T = T + A(I,K)*G(K,J)
      S = 0.
      DO 48 K=1,NQ
   48 S = S + G(I,K)*S(K,J)
      Y(I,J) = T - S
   50 Q = Q + (T-S)*(T-S)
      IF(JJ) 52,53,51
   51 IF(Q-QS) 55,59,59
   52 Q2 = Q
      GR = .5*GR
      JJ = 0
      GO TO 135
   53 GR = DS*(Q2-4.*Q+3.*QS)
      GR = GR/4.*(Q2-2.*Q+QS)
      IF(GR) 54,54,58
   54 GR = .0001
   58 JJ = 1
      GO TO 135
   59 GR = .5*GR
      GO TO 135
   55 QS = Q
      JJ = -1
      DO 56 J=1,NP
   56 PS(J) = P(J)
      DO 57 I=1,KN
      DO 57 J=1,KQ
   57 GS(I,J) = G(I,J)
      IF(MM) 60,60,75
   60 PRINT 65, Q
   65 FORMAT (5HOQ = ,E14.8)
   75 MM = MM + 1
      IF(MM-100) 80,80,145
   80 CONTINUE
      QD = 0.
      DO 105 K=1, NP
      S = 0.
      DO 92 I=1,N
      DO 92 J=1,KQ
      T = 0.
```

```
      DO 91 KK=1,KN
   91 T = T + AP(K,I,KK)*G(KK,J)
   92 S = S + T*Y(I,J)
      QP(K) = 2.*S
      QD = QD + 4.*S*S
  105 CONTINUE
      DO 130 KR = 1,N
      DO 130 KC = 1,NQ
      IF (GT(KR,KC)) 110,130,110
  110 S = 0.
      DO 115 I=1,N
  115 S = S + Y(I,KC)*A(I,KR)
      DO 120 J=1,KQ
  120 S = S - Y(KR,J)*S(KC,J)
      QG(KR,KC) = 2.*S
      QD = QD + 4.*S*S
  130 CONTINUE
      GR = DS*Q/QD
  135 CONTINUE
      DO 140 I=1,N
      DO 140 J=1,NQ
  140 G(I,J) = GS(I,J) - GR*QG(I,J)
      DO 144 K=1,NP
      P(K) = PS(K) - GR*QP(K)
      IF(P(K)) 143,143,144
  143 P(K) = (10.)**(-8.)
  144 CONTINUE
      GO TO 42
  145 CONTINUE
      PRINT 200, Q
  200 FORMAT (5HOQ = ,E14.8)
      PRINT 205
  205 FORMAT  (16HOP IS THE VECTOR)
      DO 210 J=1,NP
  210 PRINT 34, P(J)
      PRINT 215
  215 FORMAT (16HOG IS THE MATRIX)
      DO 220 I=1,KN
  220 PRINT 33, (G(I,J), J=1,KQ)
      PRINT 230
  230 FORMAT (16 HOA IS THE MATRIX)
      DO 235 I=1,N
  235 PRINT 33, (A(I,J), J=1,KN)
  240 CONTINUE
      END
      END
C     ADD 1 BLANK CARD PLUS DATA CARDS
```

REFERENCES

1. Gardner, M. F., and Barnes, J. L. Transients in Linear Systems. John Wiley and Sons, New York, 1942.

2. Guillemin, E. A. Mathematics of Circuit Analysis. John Wiley and Sons, Inc., New York, 1949.

3. Truxal, J. G. Automatic Feedback Control System Synthesis. McGraw-Hill Book Co., Inc., New York, 1955.

4. Koenig, H. E., and Blackwell, W. A. Electromechanical System Theory. McGraw-Hill Book Co., Inc., 1961.

5. Kuo, B. C. Automatic Control System. Prentice-Hall, Inc., 1962.

6. Freeman, H. "A Synthesis Method for Multipole Control Systems," AIEE Transactions, Vol. 76, Part II, 1957, pp. 28-31.

7. Kavanaugh, J. J. "Multivariable Control Systems Synthesis," AIEE Transactions, Vol. 77, Part II, 1958, pp. 425-429.

8. Shipley, P. P. "A Unified Approach to Synthesis of Linear Systems," IEEE Transactions on Automatic Control, Vol. AC-8, April, 1963, pp. 114-120.

9. Baskow, T. "The A Matrix, New Network Description," IRE Transactions on Circuit Theory, Vol. CT-4, No. 3, September 1957, pp. 117-119.

10. Desoer, C. A. "Modes in Linear Circuits," IRE Transactions on Circuit Theory, Vol. CT-7, September 1960, pp. 211-223.

11. Koenig, H. E., Tokad, Y., and Kesavan, H. K. Analysis of Discrete Physical Systems. McGraw-Hill Book Co., Inc., New York, in publication.

12. Koenig, H. E., and Tokad, Y. "State Models of Systems of Multi-Terminal Linear Components," IRE International Convention Record, 1964.

13. Wirth, J. L. "Time-domain Models of Physical Systems and Existence of Solutions," Ph.D. Thesis, Michigan State University, 1962.

14. Kalman, R. E., and Bertram, J. E. "General Synthesis Procedure for Computer Control and Single- and Multi-Loop Linear Systems," AIEE Transactions, Vol. 77, Part II, 1958, pp. 602-9.

15. _____. "Control System Analysis and Design via the 'Second Method' of Liapunov," Journal of Basic Engineering, ASME Transactions, June, 1960, pp. 371-400.

16. Kalman, R. E. "Mathematical Description of Linear Dynamical Systems," Journal of the Society of Industrial Applied Mathematics, Series A: Control, Vol. 1, No. 2, 1963, pp. 152-192.

17. Gilbert, E. G. "Controllability and Observability in Multivariable Control Systems," Journal of the Society of Industrial Applied Mathematics, Series A: Control, Vol. 1, No. 2, 1963, pp. 128-151.

18. Zadeh, L. A., and Desoer, C. A. Linear System Theory. McGraw-Hill Book Company, Inc., New York, 1963.

19. Coddington, E. A., and Levinson, N. Theory of Ordinary Differential Equations. McGraw-Hill Book Company, Inc., New York, 1953.

20. Bellman, R. Stability Theory of Differential Equations. McGraw-Hill Book Company, Inc., New York, 1953.

21. Koenig, H. E., and Tokad, Y. "A Homogeneous Equivalent of Nonhomogeneous Linear State Models and Its Application to the Analysis of Continuous and Discrete-state Systems," (unpublished paper), Michigan State University, 1963.

22. Olmsted, J. M. H. Advanced Calculus. Appleton-Century-Crofts, Inc., New York, 1961.

23. Kuratowski, K. Introduction to Set Theory and Topology. Pergaman Press, Ltd., London, 1962.

24. Gantmacher, F. R. The Theory of Matrices. Vol. 1. Chelsea Publishing Company, New York, 1960.

25. Frame, J. S. "Matrix Functions and Applications," IEEE Spectrum, March-April, 1964.

26. Goldman, S. Transformation Calculus and Electrical Transients. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1949.

27. Aseltine, J. A. Transform Method in Linear System Analysis. McGraw-Hill Book Company, Inc., New York, 1958.

28. Lepage, W. R. Complex Variables and the Laplace Transform for Engineers. McGraw-Hill Book Company, Inc., New York, 1961.

29. Zaguskin, V. L. Handbook of Numerical Methods for the Solution of Algebraic and Transcendental Equations. Pergamon Press, New York, 1961.

30. Booth, A. D. Numerical Methods. Butterworths Scientific Publications, London, 1957.

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