A PROBABILITY MODEL FOR THEORY OF ORGANIZATION OF GROUPS WITH MULTI - VALUED RELATIONS BETWEEN PERSONS

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A PROBABILITY MODEL FOR THEORY OF ORGANIZATION OF GROUPS WITH MULTI-VALUED RELATIONS BETWEEN PERSONS

By

John Lucian Bagg

AN ABSTRACT

Submitted to the School of Advanced Graduate Studies
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JOHN LUCIAN BAGG ABSTRACT

This thesis is concerned with a probability model for group organization theory permitting multi-valued relations between persons. The most difficult problem in connection with this model was the development of a method of enumerating the elements in the universe of discourse.

The introduction briefly discusses the development of the matrix model presently being used in connection with sociometric tests. In addition the evolution of the extension to multi-valued relations is examined and a comparison is made between the difficulties surmounted in this study and those in the previous original study for the binary case.

The beginning portion of the next section sets up the machinery to be used in the three stage development leading to the main theorem. Due to the complexity of the problem it was considered important to indicate the evolution of the process which led to the general result.

The results of the first two stages are special cases of the main theorem. The main theorem provides a method for enumerating the total number of ways in which n persons can classify each of the other $(n-1)$ persons into one and only one of $(k+1)$ categories, with no restrictions on the number in any given category. This number is given in two ways, first as an expression involving untabulated hollow bipartitional functions, and second in terms of regular

bipartitional functions tabulated by David and Kendall[1].

This theorem makes possible the formulation of exact probability distributions for a large class of random variables. The last part of the thesis indicates a procedure for defining such random variables on a subspace of the sample space of all outcomes. Given a sociometric test, the nature of the test determines the universe of discourse for these random variables. Once this is decided, the definition of the random variable is a matter of expediency. For a large class of sociometric tests, e.g., those satisfying restrictions given previously, the exact distribution of these random variables can be found by methods developed in this paper.

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1 See reference [1].

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INTRODUCTION

0:1 The development of the matrix model in sociometric theory.

Many investigations, especially in the social sciences, involve the process of having individuals indicate one or more choices from a group of alternatives. The best known of this type of investigation is that referred to in present day literature as sociometry. Sociometry, introduced by Moreno [6], is a technique of measuring relationships in social groups by the simple expedient of asking individuals to indicate preferences for other individuals in the group.

As developed by Moreno [6], the sociometric test placed no restrictions on the persons within the group who may be chosen or rejected. Further, the subjects were permitted an unlimited number of choices or rejections. Modifications have since been introduced by some investigators which limit the number of choices or rejections made by each individual.

Until 1946 the only acceptable method of summarizing the results of a sociometric test was the sociogram, first described by Moreno [6]. The sociogram consisted of a configuration of directed lines emanating from and terminating on a finite set of points. On the basis of a single activity the members of a group (represented by the n points P_1, P_2, ..., P_n) would be asked to respond to the other members, the responses being represented by directed lines between ordered pairs of points. For example, with regard to the question "With whom would you like (or definitely not like) to work on a given detail?", if individual i liked to work with individual j then a directed solid line from P_i to P_j represented this response; if individual i definitely did not like to work with individual j then a directed dot-dash line from P_i to P_j represented this response; if no response was made then no line was drawn. This method of studying a group with regard to a single activity produced results; however, even though sociograms are widely used today there is yet no single convention for the construction of the diagram. As a result there are as many different diagrams for a given study as there are investigators, even though all are using the same data.

In 1946, Forsyth and Katz [3] presented sociologists with the matrix representation of responses between ordered pairs of individuals in a group on the basis of a single activity. Their representation of the responses was isomorphic to the sociogram in that there was a one-to-one correspondence between types of directed lines and types of elements in an $n \times n$ matrix $C = (c_{ij})$. That is, with regard to the question given previously, if individual i liked to work with individual j, then $c_{ij} = +$ (corresponding to a solid line); if individual i definitely did not like to work with individual j, then $c_{ij} = -$ (corresponding to a dot-dash line); otherwise c_{ij} was left blank (corresponding to no line). In this first matrix representation $c_{ii} \equiv X$, for all i, and no meaning is attached to the symbol.

The matrix representation has obvious advantages over the sociogram, both from the point of view of analysis and of the systematization

of the responses. This becomes very evident when the size of the group is increased and/or the number of types of responses is increased.

About a year later Katz [4] reformulated the matrix approach to the representation of responses between ordered pairs of individuals in a group. It was realized that reactions or responses may be expressed in various ways on different scales, but in any scale there is the possibility of no response at all. No response (usually termed "indifference") was now given recognition in the $n \times n$ matrix $C = (c_{ij})$ by $c_{ij} = 0$. Furthermore, the symbols $+$ and $-$ used formerly were now replaced by the scalars 1 and -1 respectively. In this paper by Katz the main diagonal elements were used as identifiers of the individuals in a group. At a later date the convention $c_{ii} \equiv 0$ was adopted and corresponding row and column indexes identified group members, assuming the members have been arbitrarily numbered. Thus, with regard to a single activity of a group, the ternary responses between ordered pairs of individuals were represented by an $n \times n$ matrix $C = (c_{ij})$, where $c_{ij} = 1, 0,$ or -1. The main purpose of this article was to demonstrate that certain matrix operations could be interpreted in a sociological sense. A special case of ternary responses is taken up in the appendix and a more general case in section 1.5 of this paper.

At this time no attempt was made to find probability distributions associated with certain classes of these matrices. In order to specify completely such distributions it was necessary to develop exact counting procedures, based on the assumption that all matrices in a given class were equally likely, and such procedures were yet to be found. The first success along this line is given in a paper by Katz and Powell [5]. Frequent reference will be made to this particular paper, since the main results which follow are extensions of their work.

This paper gives the number of locally restricted directed graphs. These graphs consist of t directed lines on n points or nodes; the lines are joins from one point to another. The graph of t lines on n points or nodes corresponds uniquely to an $n \times n$ matrix $C = (c_{ij})$. That is, $c_{ij} = 1$, if and only if a directed line connects node P_i to node P_j, otherwise $c_{ij} = 0$. Obviously $c_{ii} \equiv 0$ for all i. Thus the matrix exhibits the binary relations between ordered pairs of distinct individuals in a group of n individuals.

Associated with each node P_i is a non-negative number pair (r_i, s_i), where r_i is the number of lines emanating from P_i and s_i is the number of lines terminating on P_i. In the matrix C we then have the ith row total $r_i = \sum_{j=1}^{n} c_{ij}$, and the jth column total $s_j = \sum_{i=1}^{n} c_{ij}$. Furthermore, if the total number of directed lines is t, then $\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} s_j = t$. We let $\underline{r} = (r_1, r_2, \ldots, r_n)$ and $\underline{s} = (s_1, s_2, \ldots, s_n)$ be two n-part, non-negative, ordered partitions of t, thus $\underline{r}$ and $\underline{s}$ are respectively the marginal row and column totals of the matrix C.

With this notation Katz and Powell [5] noted that the number of locally restricted directed graphs, $\eta(\underline{r}, \underline{s})$, is identical with the number of distinct matrices C of zeroes and ones, subject to the local restrictions $\underline{r}$ and $\underline{s}$ plus the restriction that $c_{ii} \equiv 0$ for all i. Their main theorem expresses $\eta(\underline{r}, \underline{s})$ as a linear combination of Sukhatme's [8] functions $A(\underline{r}_\alpha, \underline{s}_\alpha)$, the latter being the number of ways in which the cells of a $\rho \times \sigma$ lattice (take $\rho = \sigma = n$) can be filled with zeroes and ones so that (1) the row totals from top to bottom form the partition $\underline{r}$ in some fixed order and (2) the column totals from left to right form the parts of the partition $\underline{s}$ in some fixed order. With this theorem it was possible to find exact probability distributions of random variables defined on the sample space given by the local restrictions.

0.2 Evolution of Problem

It is the purpose of this paper to extend the result of Katz and Powell [5] beyond the binary case. Their counting theorem can be applied only when the responses or reactions between the ordered pairs of distinct individuals in a group, with regard to a single criterion, fall into two categories or can be put on a two point scale. In a large number of sociometric investigations this is sufficient but it is obviously restrictive.

The first success in extending the results of Katz and Powell [5] will be found in the appendix. There a counting theorem is proved for a restricted ternary case of responses. That is, with reference to the question used at the beginning, each person in a group of n individuals must choose exactly one person with whom he would like to work and exactly one other with whom he would definitely not like to work, and the remaining (n-3) individuals are automatically put into the no response category. In the matrix representation of these responses, this implies that each row has exactly one $+1$ (representing the "like" category), exactly one -1 (representing the "not like" category), and (n-3) zeroes (representing the "no response" category). The corresponding sociogram, in this special case, has two lines emanating from each node P_i, one solid line (representing the "like" category), and one dot-dash line (representing the "not like" category). Since the method of proof for this special case is unique it was included in this paper. Also, without doubt, it provided an impetus and an insight toward the solution of the more general cases.

The next step was proving a counting theorem when the previous restrictions in the ternary case were relaxed. That is, each individual was permitted to classify any number of individuals in each of the three categories, subject only to the restriction that each person must classify each of the others into one and only one of the three categories. Theorem 1.5.4 of section 1.5 gives a method of counting the number of such matrices under certain other arbitrary but realistic restrictions.

Thus, subject to certain practical and realistic restrictions, this theorem will enable sociologists to find exact probability distributions of random variables defined over such sample spaces as

(t_1, t_2), where t_1 and t_2 are the total number of individuals classified into categories one and two respectively. Obviously, in this restricted ternary case, the total number classified in the third category is $n-1-t_1-t_2$. Similarly, let r_{iu} be the number of individuals classified by individual i in the uth category $(u = 1, 2; i = 1, 2, \ldots, n)$ and form the n-dimensional vectors $\underline{r}^1 = (r_{11}, r_{21}, \ldots, r_{n1})$ and $\underline{r}^2 = (r_{12}, r_{22}, \ldots, r_{n2})$. For any arbitrary but fixed vectors $\underline{r}^1$ and $\underline{r}^2$, subject only to the restrictions $r_{i1} + r_{i2} \leq (n-1)$ for each i, we can now find the exact probability distribution of $(\underline{s}^1, \underline{s}^2)$, which are the n-dimensional vectors representing the number of times each individual was classified into categories one and two respectively. Note that $r_{i1} + r_{i2} \leq (n-1)$, for all i, are necessary but not sufficient conditions to insure the realistic restriction that each person classify each of the others into one and only one category. It is admitted that the number of calculations, even for small groups, would be formidable but, at the same time, they are not impossible.

Achieving success in the ternary case the next case considered was not the case of four categories, as might be expected, but the general case of $(k+1)$, $k > 1$, categories. From a sociometric viewpoint this would cover <u>any</u> test satisfying the restriction that each person classify each of the others in one and only one category. For example, tests based on complete ranking or partial ranking of the members of a group could be employed, and using the counting theorem, exact probability distributions for certain random variables could be computed. There is a reason for referring to this case as $(k+1)$ categories. If the necessary restriction that each member of the group classify each of the others into one and only one category is a premise, then any individuals not classified by individual i in one of the first k categories would automatically be classified in the $(k+1)$st category. In some instances, depending on the criterion or test, the $(k+1)$st category will be the no response category, while in other cases this category will be of the same nature as the other k categories and there will not be a no response category, e.g. the case of complete ranking. The counting theorem for this general case of $(k+1)$ categories of response with regard to a single activity of a group will be found in section 1.6

0.3 A comparison with a previous study.

It is of interest to compare the problems and methods of attack in this paper with those of Katz and Powell [5]. Their main problem, with regard to the counting theorem, was to either find a direct combinatorial method of counting the matrices (subject to the local restrictions) or to find a method involving known functions. They accomplished the latter by proving that $\eta(\underline{r}, \underline{s})$ is a linear combination of Sukhatme's [8] Bipartitional Functions $A(\underline{r}_\alpha, \underline{s}_\alpha)$. As was pointed out earlier, the $A(\underline{r}_\alpha, \underline{s}_\alpha)$ function was concerned with matrices having zeroes or ones in any position, including the main diagonal. They accomplished this by first proving that $A(\underline{r}, \underline{s})$ is

a linear combination of $\eta(\underline{r}_\alpha, \underline{s}_\alpha)$, using Feller's [2] principle of inclusion and exclusion. Then, by taking the inverses of functions of certain operators, which they discovered, they were able to express $\eta(\underline{r}, \underline{s})$ as a linear combination of $A(\underline{r}_\alpha, \underline{s}_\alpha)$ functions.

In the first phases of the research in this paper to extend the results of Katz and Powell [5] an attempt was made to find a related tabulated function, similar to Sukhatme's Bipartitional Functions [8], to cope with the ternary responses. This search proved futile, however, and a different line of attack had to be taken. The method finally adopted in the ternary case was the forming of a product space which contained as a proper subspace the desired elements. The details of this will be found in section 1.2. Briefly described, the method is as follows. Using the methods of Katz and Powell [5] it is possible to form k independent sets of $n \times n$ matrices, one for each of the k categories (excluding the $(k+1)$st). The product space of these k independent sets is then a space of three dimensional, $n \times n \times k$, matrices. Since the k sets are independent, the resulting product space contains, in general, many $n \times n \times k$ matrices which violate the necessary restriction that each of the n members of a group classify each of the others in one and only one of the $(k+1)$ categories. The problem at this stage is then similar to that encountered by Katz and Powell [5]. Namely, a method of counting only those matrices in the product space which do not violate the necessary restriction had to be found. The general counting theorem will be found in section 1.5.

PART I

THEORY

1:1. Preliminaries.

This study is concerned with relations between ordered pairs of distinct individuals in a finite group of n individuals. It is assumed that there are $(k+1)$ distinct types of relations. We insist that one and only one of the $(k+1)$ relations exist between each ordered pair (i,j) of distinct individuals ($i \neq j$; $i,j = 1, 2, \ldots, n$). In most instances the $(k+1)$st relation will be known as the null relation. However, in some sociometric tests the $(k+1)$st relation will be of the same nature as the other k relations; in this event it would be treated in the same manner as the null relation. This would occur in sociometric tests in which it is mandatory that every individual be classified into non-null categories.

The totality of relations between all ordered pairs of distinct individuals can be represented in two ways which are isomorphic. One way is familiar to the social scientist and consists of graphs. The graphs have n points or nodes P_i $(i = 1, 2, \ldots, n)$. Between each pair of points P_i and P_j there is either one and only one of k different colored directed lines or there is no directed line, the absence of a line corresponding to the null relation. The second representation, the one with which we will be concerned, is isomorphic to the graphs briefly described above. It consists of $n \times n \times k$ matrices C with elements c_{iju} $(i,j = 1, 2, \ldots, n;\ u = 1, 2, \ldots, k)$, where $c_{iju} = 0$ or 1. We impose the restriction that $\sum_{u=1}^{k} c_{iju} = 0$ or 1 for each ordered pair (i,j) of distinct individuals and by convention we set $c_{iiu} \equiv 0$ for all i and u. If, for any ordered pair (i,j) of distinct individuals, one of the k distinct types of relations does not exist, then the null relation is mandatory and is represented by $c_{iju} = 0$ for every $u = 1, 2, \ldots, k$. If one of the k relations, say the uth, exists between the ordered pair (i,j), then $c_{iju} = 1$, $c_{ijv} = 0$ for $v \neq u$.

To establish notation, for each $u = 1, 2, \ldots, k$, we let

1.1.1
$$r_{iu} = \sum_{j=1}^{n} c_{iju} \quad \text{for each } i = 1, 2, \ldots, n, \text{ and}$$
$$s_{ju} = \sum_{i=1}^{n} c_{iju} \quad \text{for each } j = 1, 2, \ldots, n,$$

and we form the n-dimensional marginal row and column total vectors

1.1.2
$$\underline{r}^u = (r_{1u}, r_{2u}, \ldots, r_{nu}) \quad \text{and}$$
$$\underline{s}^u = (s_{1u}, s_{2u}, \ldots, s_{nu})$$

respectively, where the superscript u is an index[1]. Also, we let

$$t_u \equiv \sum_{i=1}^{n} r_{iu} = \sum_{j=1}^{n} s_{ju}$$

where the equality is an obvious consequence of the definitions.

---

[1]This should not lead to confusion for this notation will only be used in connection with the n-dimensional vectors $\underline{r}$ and $\underline{s}$.

In many sociometric tests each individual is asked to classify each of the others into one and only one of $(k+1)$ categories, $k \geq 1$, with no restrictions on the number of individuals to be classified in a given category. Using the notation above, we point out that r_{iu} is the total number of individuals classified by individual i into the uth category and that s_{ju} is the number of times that individual j was classified by other individuals into the uth category. In such sociometric tests it is of interest to know the exact probability distribution of the n-dimensional marginal total column vectors $\underline{s}^u$ $(u = 1, 2, \ldots, k)$ given the n-dimensional marginal row total vectors $\underline{r}^u$ $(u = 1, 2, \ldots, k)$. In order to determine the exact probability distribution (under the assumption that all outcomes in the sample space are equally likely) a method of counting the possible outcomes must be developed.

In the matrix representation this amounts to the problem of determining the total number of distinct $n \times n \times k$ matrices C with elements $c_{iju} = 0$ or 1 such that

1.1.3 $$\sum_{u=1}^{k} c_{iju} \leq 1 \quad \text{for each ordered pair } (i,j),\ i \neq j,\quad i,j = 1, 2, \ldots, n,$$

1.1.3a $$c_{iiu} \equiv 0 \quad \text{for } i = 1, 2, \ldots, n \text{ and } u = 1, 2, \ldots, k,$$

1.1.4 $$\sum_{j=1}^{n} c_{iju} = r_{iu},\quad \sum_{i=1}^{n} c_{iju} = s_{ju} \quad \text{for each } u = 1, 2, \ldots, k,$$

where r_{iu} and s_{ju} are respectively the ith and jth components of the given fixed n-dimensional marginal row and column total vectors $\underline{r}^u$ and $\underline{s}^u$.

In order to determine this number it was necessary to form a product space which contained as a subspace all the nxnxk matrices satisfying the restrictions given by 1.1.3 and 1.1.4 .

1:2 The product space.

Consider first a fixed but arbitrary relation, say the uth. For each fixed u (u = 1, 2, ..., k) there exists an nxn matrix which we denote by C_u. This matrix has elements $c_{iju} = 0$ or 1 ($i \neq j$; i,j = 1, 2, ..., n) and $c_{iiu} \equiv 0$ for i = 1, 2, ..., n. Associated with each C_u are two fixed n-dimensional marginal row and column total vectors $\underline{r}^u$ and $\underline{s}^u$ respectively, defined as in 1.1.4 . Thus, for fixed u, these matrices C_u exhibit binary relations between ordered pairs of individuals . Such a mathematical model was considered by Katz and Powell [5]. Using their methods we can find, for fixed but arbitrary $\underline{r}^u$ and $\underline{s}^u$, $\eta(\underline{r}^u, \underline{s}^u)$, the total number of distinct nxn matrices C_u with elements $c_{iju} = 0$ or 1, subject to the restrictions

1.2.1 $$c_{iiu} \equiv 0 \quad \text{for all } i = 1, 2, \ldots, n$$

and

1.2.2 $$\sum_{j=1}^{n} c_{iju} = r_{iu} \, , \quad \sum_{i=1}^{n} c_{iju} = s_{ju} \, ,$$

where r_{iu} and s_{ju} are respectively the ith and jth components of $\underline{r}^u$ and $\underline{s}^u$.

Thus, for each $u = 1, 2, \ldots, k$, there exists a space of $\eta(\underline{r}^u, \underline{s}^u)$ matrices C_u each subject to the restrictions 1.2.1 and 1.2.2. We now form the product space of the k spaces and obtain a space consisting of $\eta(\underline{r}^1, \underline{s}^1)\,\eta(\underline{r}^2, \underline{s}^2)\cdots\eta(\underline{r}^k, \underline{s}^k)$ distinct elements each of which is an nxnxk matrix C. We denote this product space by $\mathcal{C} = \{C\}$.

Many of the matrices in the product space will, in general, violate the restriction $\sum_{u=1}^{k} c_{iju} \leq 1$ for every ordered pair (i,j). We stated previously that the purpose of forming this product space was to have a space which contained as a subspace all those nxnxk matrices C satisfying the restrictions given by 1.1.3 and 1.1.4. We now show how to isolate this subspace.

The product space $\mathcal{C}$ can be decomposed into mutually exclusive and exhaustive subspaces $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots$, where $\mathcal{C}_m$ is that subspace of $\mathcal{C}$ containing all those nxnxk matrices for which $\sum_{u=1}^{k} c_{iju} > 1$ for exactly m distinct ordered pairs (i,j), $i \neq j$. It is easily seen that these subspaces $\mathcal{C}_m$ are mutually exclusive and they are obviously exhaustive for any m. Thus $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1 + \mathcal{C}_2 + \cdots$. When $m = 0$ this implies 1.1.3 and since 1.2.2 is identical with 1.1.4 it follows that $\mathcal{C}_0$ is the required subspace. We denote the number of nxnxk matrices in $\mathcal{C}_0$ by $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$.

Definition 1:2.1 Let $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$, $k \geq 1$, be the number of distinct nxnxk matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$.

In relation to the results of Katz and Powell [5], we point out that, for $k = 1$, $H(\underline{r}, \underline{s}) \equiv \eta(\underline{r}, \underline{s})$.

There is another number defined on the subspace $\mathcal{C}_0$ for which we shall have frequent use. This number, which is defined below, also involves a fixed but arbitrary set of M specified distinct off diagonal positions (i_m, j_m), $m = 1, 2, \ldots, M$. We denote this set by $\{i_m, j_m\}_M$.

Definition 1.2.2 $H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$, $k \geq 1$, is the number of distinct nxnxk matrices C which, in addition to belonging to the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$, have $\sum_{u=1}^{k} c_{iju} = 0$ for each pair $(i,j) \in \{i_m, j_m\}_M$.

Every theorem, lemma, and corollary in this paper is dependent upon operators ${}_u\alpha_i^j$. In the next section we will define these operators and will prove a most important lemma with regard to them.

1.3 The operators ${}_u\alpha_i^j$.

We require operators ${}_u\alpha_i^j$ $(u = 1, 2, \ldots, k;\ i,j = 1, 2, \ldots, n)$. These operators act independently on the n-dimensional vectors $\underline{r}^u$ and $\underline{s}^u$ and are defined by

$$
\begin{aligned}
& {}_u\alpha_i^j(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^{u-1}, \underline{s}^{u-1}, r_{1u}, \ldots, r_{iu}, \ldots, r_{nu}, s_{1u}, \\
& \qquad \ldots, s_{ju}, \ldots, s_{nu}, \underline{r}^{u+1}, \underline{s}^{u+1}, \ldots, \underline{r}^k, \underline{s}^k) \\
& \equiv (\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^{u-1}, \underline{s}^{u-1}, r_{1u}, \ldots, r_{iu}-1, \ldots, r_{nu}, \\
& \qquad s_{1u}, \ldots, s_{ju}-1, \ldots, s_{nu}, \underline{r}^{u+1}, \underline{s}^{u+1}, \ldots, \underline{r}^k, \underline{s}^k).
\end{aligned}
\tag{1.3.1}
$$

That is, letting $\sum_{i=1}^{n} r_{iu} = t_u$, as before, the effect of the operator ${}_u\alpha_i^j$ on $(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^u, \underline{s}^u, \ldots, \underline{r}^k, \underline{s}^k)$ is to replace the double partition $\underline{r}^u, \underline{s}^u$ of t_u by the double partition of $t_u - 1$ with the ith part of $\underline{r}^u$ and the jth part of $\underline{s}^u$ each reduced by unity.

For any fixed u $(u = 1, 2, \ldots, k)$ these operators are precisely the operators δ_i^j defined and introduced by Katz and Powell [5]. Hence, for fixed u, they have the same properties as the δ_i^j, namely, they are associative and commutative under addition and multiplication. Since they act independently on their respective n-dimensional marginal total vectors $\underline{r}^u$ and $\underline{s}^u$, they can be applied as operators on $(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$ in any order. Furthermore, the above properties serve to specify the effect of every sum of monomials of the form

$$
\begin{aligned}
& {}_1\alpha_{i_1}^{j_1} \cdots {}_1\alpha_{i_{m_1}}^{j_{m_1}} \; {}_2\alpha_{i_{m_1+1}}^{j_{m_1+1}} \cdots {}_2\alpha_{i_{m_1+m_2}}^{j_{m_1+m_2}} \cdots {}_k\alpha_{i_{m_1+\cdots+m_{k-1}+1}}^{j_{m_1+\cdots+m_{k-1}+1}} \cdots \\
& {}_k\alpha_{i_{m_1+\cdots+m_k}}^{j_{m_1+\cdots+m_k}}, \quad \text{where } m_v \geq 0,\ v = 1, 2, \ldots, k,
\end{aligned}
\tag{1.3.2}
$$

as an operator on $(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$. We further note that the number of non-trivial terms is finite for finite partitions of t_u $(u = 1, 2, \ldots, k)$ since, for a function G, we have for every u

$$\left({}_u\alpha_i^j\right)^{m+h} G(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$$

1.3.3

$$\equiv G\left\{\left({}_u\alpha_i^j\right)^{m+h}(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)\right\} \equiv 0 \, ,$$

if $h > 0$ and $m = \min(r_{iu}, s_{ju})$.

Equation 1.3.3 is written dually in order to exhibit the manner in which the ${}_u\alpha_i^j$ filter through a function G. Finally, it is to be noted that any identity among Sukhatme's [8] $A(\underline{r}^1, \underline{s}^1)$, Katz and Powell's [5] $\eta(\underline{r}^1, \underline{s}^1)$, $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$, and $H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$ is unaffected by the application of these operators, since the operation on any of these functions is an operation on the partitions involved.

We next establish inverses for certain operators.

Lemma 1.3.1 The operator $\left(1 + \sum_{u=1}^{h} {}_u\alpha_i^j\right)$, $1 \le h \le k$, has an inverse, left and right, given by

$$\left(1 + \sum_{u=1}^{h} {}_u\alpha_i^j\right)^{-1} \equiv 1 - \left(\sum_{u=1}^{h} {}_u\alpha_i^j\right) + \left(\sum_{u=1}^{h} {}_u\alpha_i^j\right)^2 - \cdots \, . \tag{1.3.4}$$

Proof: Using the associative and commutative properties of the operators, it is easily seen that

$$\left(\sum_{u=1}^{h} {}_u\alpha_i^j\right)^m \left(\sum_{u=1}^{h} {}_u\alpha_i^j\right)^n = \left(\sum_{u=1}^{h} {}_u\alpha_i^j\right)^{m+n} \, . \tag{1.3.5}$$

To show that it is a left inverse, we have

$$\left(1+\sum_{u=1}^{h} {}_{u}\alpha_i^j\right)^{-1}\left(1+\sum_{u=1}^{h} {}_{u}\alpha_i^j\right)(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$$

1.3.6

$$= \left(1+\sum_{u=1}^{h} {}_{u}\alpha_i^j\right)^{-1}\Big\{(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k) + \left(\sum_{u=1}^{h} {}_{u}\alpha_i^j\right)(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)\Big\}.$$

Applying the successive terms of the inverse, as given in the right member of 1.3.4, first to the first term within braces and then to the second, we obtain, using 1.3.5

$$\left(1+\sum_{u=1}^{h} {}_{u}\alpha_i^j\right)^{-1}\left(1+\sum_{u=1}^{h} {}_{u}\alpha_i^j\right)(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$$

$$= (\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k) - \left(\sum_{u=1}^{h} {}_{u}\alpha_i^j\right)(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$$

$$+\left(\sum_{u=1}^{h} {}_{u}\alpha_i^j\right)^2(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$$

$$-\left(\sum_{u=1}^{h} {}_{u}\alpha_i^j\right)^3(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k) + \cdots$$

$$+\left(\sum_{u=1}^{h} {}_{u}\alpha_i^j\right)(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$$

$$-\left(\sum_{u=1}^{h} {}_{u}\alpha_i^j\right)^2(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$$

$$+\left(\sum_{u=1}^{h} {}_{u}\alpha_i^j\right)^3(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k) - \cdots$$

$$= (\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k).$$

It follows that $\left(1+\sum_{u=1}^{h}\alpha_{i}^{j}\right)^{-1}$, $1 \leq h \leq k$, is a left inverse. A similar proof shows that it is also a right inverse and Lemma 1.3.1 is proved.

1.4 The three stages of development.

Up to the present time we have been developing a notation to be used for a general case of $(k+1)$ categories. The reader will no doubt be surprised to find that in the next section we consider the particular case $k = 2$. This is partly due to the method of development. The main purpose of this paper is to extend the counting theorem of Katz and Powell [5] which corresponds to the case $k = 1$.

A first natural step in extending such a result would be to consider the case $k = 2$ (three categories). As a matter of fact we commenced by considering a special case of $k = 2$, namely where each individual is asked to classify exactly one individual in category one, exactly one other individual in category two, and the remaining $(n-3)$ individuals in category three, the null category. This case is given in the appendix and the reader may benefit by first examining this case.

In most mathematical theories there is an underlying evolution process which is very evident here. In the appendix a counting theorem for the case described above is proved. This proof is constructed around an inherent feature of the special case. That is, since $r_{i1} = r_{i2} = 1$ for all $i = 1, 2, \ldots, n$, this implies that, for fixed i, there is <u>at most one</u> ordered pair (i,j) such that $c_{ij1} = c_{ij2} = 1$. In the appendix, $c_{ij1} = c_{ij2} = 1$ is referred to as a coincidence at (i,j).

In the next step, where the restrictions $r_{i1} = r_{i2} = 1$ are removed and $\underline{r}^1$, $\underline{s}^1$, $\underline{r}^2$, $\underline{s}^2$ are more general n-dimensional marginal total vectors, there is also an inherent feature. For any fixed ordered pair (i,j) of distinct individuals the <u>only</u> way in which the restriction, $c_{ij1} + c_{ij2} \leq 1$, can be violated is for $c_{ij1} = c_{ij2} = 1$. However, this can happen more than once in each row and therefore this case is different from the one considered in the appendix. This is the case taken up in the next section. The proofs are quite complicated, however, and for this reason we progress to the final result through a long series of Lemmas and Corollaries, hoping to preserve the continuity.

Finally we come to the general case, $k > 1$, in section 1.6. The proofs here are quite similar to those in section 1.5. However, again there is an inherent feature which distinguishes this case from the preceding cases. For a fixed ordered pair (i,j) and $k > 2$, the restriction $\sum_{u=1}^{k} c_{iju} \leq 1$ can be violated in $2^k - k - 1$ ways, i.e., the number of ways of choosing two or more things from among k different things. Thus, for fixed (i,j), instead of having only one way of violating the restriction, we have many ways and consequently the method of attack must be changed.

We now consider the case $k = 2$.

1:5 $H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$ as a bilinear form of $\eta(\underline{r}^1_\alpha, \underline{s}^1_\alpha)$ and $\eta(\underline{r}^2_\alpha, \underline{s}^2_\alpha)$

Consider now the case where there are three types of relations between ordered pairs of distinct individuals, $k = 2$. Let there be given two sets of arbitrary but fixed n-dimensional marginal row and column total vectors $\underline{r}^1$, $\underline{s}^1$ and $\underline{r}^2$, $\underline{s}^2$. These vectors are subject only to the restrictions imposed by the conditions that $c_{ij1} + c_{ij2} \leq 1$ for every ordered pair (i,j), $i \neq j$, $i,j = 1, 2, \ldots, n$. For this case, if a type 1 relation exists between the ordered pair (i,j) of distinct individuals, then $c_{ij1} = 1$; similarly for a type 2 relation, $c_{ij2} = 1$. If neither a type 1 nor a type 2 relation exists between the ordered pair (i,j) of distinct individuals then $c_{ij1} = c_{ij2} = 0$.

We wish to find $H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$, which by Definition 1:2.1 is the number of nxnx2 matrices C in $\mathcal{C}_0$. In general the product space $\mathcal{C}$ will contain nxnx2 matrices for which $c_{ij1} + c_{ij2} = 2$ for one or more ordered pairs (i,j). Such matrices violate the restriction that $c_{ij1} + c_{ij2} \leq 1$ for every ordered pair (i,j).

In order to find $H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$ we first consider a fixed but arbitrary pair (i,j), $i \neq j$, $i,j = 1, 2, \ldots, n$, in the nxnx2 matrices C of the subspace $\mathcal{C}_0$. Associated with this specified ordered pair (i,j) are two numbers c_{ij1} and c_{ij2}, i.e., $k = 2$, and since we are concerned with the subspace $\mathcal{C}_0$, the ordered number pair (c_{ij1}, c_{ij2}) can take on one and only one of the three values $(0, 0)$, $(0,1)$ and $(1, 0)$. Thus, with respect to the ordered pair (i,j) of distinct individuals, the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ can be divided into three mutually exclusive and exhaustive classes. This decomposition is given by the following lemma.

Lemma 1:5.1 With respect to the arbitrary but fixed ordered pair (i,j) of distinct individuals, the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ can be decomposed into three mutually exclusive and exhaustive classes represented by

$$1.5.1 \qquad H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2) = \left(1 + {}_1\alpha_i^j + {}_2\alpha_i^j\right) H_{\{i,j\}_1}(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2).$$

Proof: Expanding the right member of 1.5.1, the first term is, by Definition 1:2.2, the number of $n x n x 2$ matrices C in $\mathcal{C}_0$ for which $c_{ij1} = c_{ij2} = 0$. The second term is the number of $n x n x 2$ matrices C in $\mathcal{C}_0$ for which $c_{ij1} = 1$ and $c_{ij2} = 0$, since if a type one relation is fixed between the ordered pair (i,j) of distinct individuals then r_{i1} and s_{j1} must each be reduced by unity. This is precisely the effect of the operator ${}_1\alpha_i^j$ on $(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$. Similarly, the third term is the number of $n x n x 2$ matrices C in the subspace $\mathcal{C}_0$ for which $c_{ij1} = 0$ and $c_{ij2} = 1$. With respect to the ordered pair (i,j) this exhausts the possible pairs of values for the number pair (c_{ij1}, c_{ij2}) in the subspace $\mathcal{C}_0$ and hence the Lemma follows.

Applying the inverse, $\left(1 + {}_1\alpha_i^j + {}_2\alpha_i^j\right)^{-1}$, given by Lemma 1.3.1 with $h = 2$, to both members of 1.5.1, we have

Corollary 1:5.1.1 For an arbitrary but fixed ordered pair (i,j) of distinct individuals, the number of nxnx2 matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ for which $c_{ij1} = c_{ij2} = 0$ is given by

1.5.2 $$H_{\{i,j\}_1}(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2) \equiv \left(1 + {}_1\alpha_i^j + {}_2\alpha_i^j\right)^{-1} H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2).$$

The following corollary could be omitted completely. It is included to illustrate for single ordered pairs (i,j) the procedure to be followed later on for sets of ordered pairs (i,j). To be more explicit, Theorem 1.5.3 (near the end of this section) enumerates the number of nxnx2 matrices C in each of the subspaces $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\cdots$. The following corollary gives explicitly the number in $\mathcal{C}_1$; it is a special case of Corollary 1.5.2.2 which gives the number in $\mathcal{C}_m$ for $m = 1, 2, \cdots$.

The procedure is to fix a type 1 and a type 2 relation between the arbitrary but fixed ordered pair (i,j) of distinct individuals. If this is done for every possible ordered pair (i,j), $i \neq j$, and the sum is taken over all possible pairs, the result is

Corollary 1:5.1.2 The number of nxnx2 matrices C in the subspace $\mathcal{C}_1$ of the product space $\mathcal{C}$ is given by

1.5.3 $$\begin{aligned} &\sum_{i \neq j} {}_1\alpha_i^j \, {}_2\alpha_i^j \, H_{\{i,j\}_1}(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2) \\ &\equiv \sum_{i \neq j} {}_1\alpha_i^j \, {}_2\alpha_i^j \left(1 + {}_1\alpha_i^j + {}_2\alpha_i^j\right)^{-1} H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2) . \end{aligned}$$

Proof: By Definition 1:2.2 $H_{\{i,j\}_1}(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$ is the number of nxmx2 matrices C in the subspace $\mathcal{C}_0$ for which $c_{ij1} = c_{ij2} = 0$. We can assume without loss of generality that r_{i1}, s_{j1}, r_{i2}, and s_{j2} are greater than or equal to one. We now fix a type 1 relation between the ordered pair (i,j), thus making $c_{ij1} = 1$. Since r_{i1} and s_{j1} are respectively the number of type one relations in row i and column j, this one must come from these components of $\underline{r}^1$ and $\underline{s}^1$. Similary, if we fix $c_{ij2} = 1$, then r_{i2} and s_{j2} must each be reduced by unity. This removal is effected by applying the operator ${}_1\alpha_i^j\,{}_2\alpha_i^j$ to $(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$. It follows that ${}_1\alpha_i^j\,{}_2\alpha_i^j\, H_{\{i,j\}_1}(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$ is the number of nxmx2 matrices C in the product space $\mathcal{C}$ for which $c_{ij1} = c_{ij2} = 1$, and by definition each of these matrices belongs to $\mathcal{C}_1$. Applying Corollary 1.5.1.1 to the left member of 1.5.3 we obtain the right member for each ordered pair (i,j) and if the sum is taken over all possible locations we exhaust the cases for which $c_{ij1} = c_{ij2} = 1$. The corollary is an immediate consequence.

We now extend Lemma 1.5.1 to M specific distinct off-diagonal positions (i_m, j_m), $m = 1, 2, \ldots, M$, $i_m \neq j_m$, again using the notation $\{i_m, j_m\}_M$ for this set.

Lemma 1:5.2 With respect to the fixed but arbitrary set $\{i_m, j_m\}_M$ the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ can be decomposed into

3^M mutually exclusive and exhaustive classes enumerated by

1.5.4 $$H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2) = \left\{ \prod_{m=1}^{M} \left(1 + {}_1\alpha_{i_m}^{j_m} + {}_2\alpha_{i_m}^{j_m} \right) \right\} \times$$

$$H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2).$$

Proof: In the discussion which follows it should be understood that some of the classes may be vacuous.

It is obvious that there are 3^M distinct terms in the right member of 1.5.4. There is an isomorphism between these terms and the totality of number pairs (c_{ij1}, c_{ij2}) for $(i,j) \in \{i_m, j_m\}_M$. The subspace under consideration is $\mathcal{C}_0$ and therefore for each $(i,j) \in \{i_m, j_m\}_M$ we have $(c_{ij1}, c_{ij2}) = (0,0), (1,0)$ or $(0,1)$. Thus, for each of the M specified positions $(i,j) \in \{i_m, j_m\}_M$, the number pair (c_{ij1}, c_{ij2}) can assume any one of the three distinct values and it follows that there are 3^M number pairs on the specified set.

Consider the following general set of M number pairs on the specified set of positions $\{i_m, j_m\}_M$, namely, $c_{i_p j_p 1} = 1$, $c_{i_p j_p 2} = 0$, for $p = 1, 2, \ldots, m_1$; $c_{i_p j_p 1} = 0$, $c_{i_p i_p 2} = 1$, for $p = m_1+1, m_1+2, \ldots, m_1+m_2$; and $c_{i_p j_p 1} = c_{i_p j_p 2} = 0$, for $p = m_1+m_2+1, m_1+m_2+2, \ldots, m_1+m_2+m_3$, where $m_1+m_2+m_3 = M$. Since

$H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$ enumerates a set which, by Definition 1.2.2, has $c_{ij1} = c_{ij2} = 0$ for $(i, j) \in \{i_m, j_m\}_M$, we can proceed as follows. Fix a type one relation at the m_1 $(0 \leq m_1 \leq M)$ positions indicated above, then $r_{i_p 1}$ and $s_{j_p 1}$ must each be reduced by unity for $p = 1, 2, \ldots, m_1$; these reductions are effected by applying the operators ${}_1\alpha_{i_1}^{j_1} \cdots {}_1\alpha_{i_{m_1}}^{j_{m_1}}$ to $\underline{r}^1$, $\underline{s}^1$. Similarly, fix type two relations at the m_2 $(0 \leq m_2 \leq M - m_1)$ positions indicated above, then $r_{i_p 2}$ and $s_{j_p 2}$ must each be reduced by unity for $p = m_1+1, m_1+2, \ldots, m_1+m_2$; these reductions are effected by applying the operators ${}_2\alpha_{i_{m_1+1}}^{j_{m_1+1}} \cdots {}_2\alpha_{i_{m_1+m_2}}^{j_{m_1+m_2}}$ to $\underline{r}^2$, $\underline{s}^2$. Carrying out these reductions simultaneously, we would have

$$1.5.5 \qquad \left({}_1\alpha_{i_1}^{j_1} \cdots {}_1\alpha_{i_{m_1}}^{j_{m_1}} \, {}_2\alpha_{i_{m_1+1}}^{j_{m_1+1}} \cdots {}_2\alpha_{i_{m_1+m_2}}^{j_{m_1+m_2}}\right) H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2).$$

This is readily seen to be a unique term in the expansion of 1.5.4 for it is obtained from the M distinct factors by taking the specified ${}_1\alpha_{i_p}^{j_p}$ from m_1 of the M factors, taking the specified ${}_2\alpha_{i_p}^{j_p}$ from m_2 of the M factors (distinct from the preceding m_1), and taking 1 from the remaining $M - m_1 - m_2$ factors. Since this was done in complete generality and since the converse is easily demonstrated, it follows that there is a one-to-one correspondence and the lemma is proved.

Repeated application of Lemma 1.3.1 to both members of 1.5.4 gives us, without further proof

Corollary 1:5.2.1 The number of distinct $n \times n \times 2$ matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ for which $c_{ij1} = c_{ij2} = 0$, where $(i,j) \in \{i_m, j_m\}_M$ is given by

1.5.6 $$H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2) = \prod_{m=1}^{M}\left(1 + {}_1\alpha_{i_m}^{j_m} + {}_2\alpha_{i_m}^{j_m}\right)^{-1} \times H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2) \; .$$

We now proceed to Corollary 1.5.2.2 which is a generalization of Corollary 1.5.1.2. That is, between each ordered pair (i,j), $(i,j) \in \{i_m, j_m\}_M$, we fix both a type one and a type two relation.

Corollary 1:5.2.2 The number of distinct $n \times n \times 2$ matrices C in the product space $\mathcal{C}$ which have $c_{ij1} = c_{ij2} = 1$, for every ordered pair $(i,j) \in \{i_m, j_m\}_M$, and no coincidences elsewhere, is given by

1.5.7 $$\left\{\prod_{m=1}^{M} {}_1\alpha_{i_m}^{j_m}\, {}_2\alpha_{i_m}^{j_m}\right\} H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2) = \prod_{m=1}^{M}\left[\left({}_1\alpha_{i_m}^{j_m}\, {}_2\alpha_{i_m}^{j_m}\right)\left(1 + {}_1\alpha_{i_m}^{j_m} + {}_2\alpha_{i_m}^{j_m}\right)^{-1}\right] H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2).$$

Proof: In the discussion which follows it should be understood that some of the classes may be vacuous.

By Definition 1.2.2 $H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$ is the number of $n \times n \times 2$ matrices C in the subspace $\mathcal{C}_0$ for which

$c_{ij1} = c_{ij2} = 0$ for every $(i,j) \in \{i_m, j_m\}_M$, therefore we can proceed as follows. Fix a type one and a type two relation between each ordered pair $(i,j) \in \{i_m, j_m\}_M$ and it follows that $c_{ij1} = c_{ij2} = 1$ for each of the ordered pairs. If this is done, then the corresponding components of the n-dimensional marginal row and column total vectors $(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$ must each be reduced by unity, but this is precisely the effect of $\prod_{m=1}^{M} {}_1\alpha_{i_m}^{j_m} \, {}_2\alpha_{i_m}^{j_m}$ on $(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$. This proves the corollary with regard to the left member of 1.5.7 and the right member follows directly from Corollary 1.5.2.1.

The preceding corollary applies only to the specified $\{i_m, j_m\}_M$, $M \geq 1$. $\mathcal{C}_M$ is that subspace of $\mathcal{C}$ which contains all the matrices having $c_{ij1} + c_{ij2} > 1$ for exactly M positions (i,j), $i \neq j$, and no coincidences elsewhere. Thus matrices enumerated by 1.5.7 belong only to $\mathcal{C}_M$. It follows that if 1.5.7 is summed over all possible locations of the sets $\{i_m, j_m\}_M$, the result is the total number of matrices in the subspace $\mathcal{C}_M$.

For fixed M the number of distinct sets $\{i_m, j_m\}_M$ is the number of ways in which M distinct off-diagonal positions can be chosen from $n(n-1)$ positions, namely $\binom{n(n-1)}{M}$. We denote the sum over these $\binom{n(n-1)}{M}$ sets by $\sum_{\{i_m, j_m\}_M}$. Using this notation and the previous remarks we may write

Corollary 1.5.2.3 The number of distinct nxnx2 matrices C in the subspace $\mathcal{C}_M$ $(M \geq 1)$ of the product space $\mathcal{C}$ is given by

$$1.5.8 \qquad \sum_{\{i_m, j_m\}_M} \left\{ \left(\prod_{m=1}^{M} {}_1\alpha_{i_m}^{j_m}\, {}_2\alpha_{i_m}^{j_m} \right) H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2) \right\}$$

$$\equiv \sum_{\{i_m, j_m\}_M} \left\{ \prod_{m=1}^{M} \left({}_1\alpha_{i_m}^{j_m}\, {}_2\alpha_{i_m}^{j_m} \right) \left(1 + {}_1\alpha_{i_m}^{j_m} + {}_2\alpha_{i_m}^{j_m} \right)^{-1} \right\} H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$$

Proof: Each of the $\binom{n(n-1)}{M}$ terms in 1.5.8 is, for fixed M, of the form given in Corollary 1.5.2.2 and hence gives the number of nxnx2 matrices C in the product space $\mathcal{C}$ for which $c_{ij1} = c_{ij2} = 1$ for each ordered pair $(i,j) \in \{i_m, j_m\}_M$. Since each such matrix violates the restriction that $c_{ij1} + c_{ij2} \leq 1$ for exactly M ordered pairs (i,j), each belongs to $\mathcal{C}_M$ and since we are summing over all possible locations of the sets $\{i_m, j_m\}_M$ we exhaust the possibilities for the subspace $\mathcal{C}_M$. The corollary is an immediate consequence with regard to the left member and applying Corollary 1.5.2.1 to the left member, the right member follows and this completes the proof.

It was shown in section 1.2 that the total number of distinct nxnx2 matrices C in the product space $\mathcal{C}$ is $\eta(\underline{r}^1, \underline{s}^1)\,\eta(\underline{r}^2, \underline{s}^2)$, i.e., $k = 2$, where the η function is the function introduced by Katz and Powell [5]. At the same time we showed that the product space could be decomposed into mutually exclusive and exhaustive subspaces, that is $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1 + \mathcal{C}_2 + \cdots$. The following theorem gives a complete enumeration of the decomposition of the product space $\mathcal{C}$ into subspaces .

Theorem 1:5.3 If $\underline{r}^1$, $\underline{s}^1$ are any two n-part, non-negative, ordered partitions of t_1 and $\underline{r}^2$, $\underline{s}^2$ are any two n-part, non-negative, ordered partitions of t_2, then an exhaustive enumeration of the nxnx2 matrices C in each of the subspaces $\mathcal{C}_M$ ($M \geqslant 0$) of the product space $\mathcal{C}$ is given by

1.5.9
$$\eta(\underline{r}^1, \underline{s}^1)\,\eta(\underline{r}^2, \underline{s}^2) = \prod_{i \neq j}\left(1 + \frac{{}_1\alpha_i^j\,{}_2\alpha_i^j}{1 + {}_1\alpha_i^j + {}_2\alpha_i^j}\right) \times H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2).$$

Proof: Expanding the product in the right member of 1.5.9, we have

1.5.10
$$H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2) + \sum_{i \neq j}\left(\frac{{}_1\alpha_i^j\,{}_2\alpha_i^j}{1 + {}_1\alpha_i^j + {}_2\alpha_i^j}\right) H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$$
$$+ \sum_{\{i_m, j_m\}_2}\left\{\prod_{m=1}^{2}\left(\frac{{}_1\alpha_{i_m}^{j_m}\,{}_2\alpha_{i_m}^{j_m}}{1 + {}_1\alpha_{i_m}^{j_m} + {}_2\alpha_{i_m}^{j_m}}\right) H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)\right\}$$
$$+ \sum_{\{i_m, j_m\}_3}\left\{\prod_{m=1}^{3}\left(\frac{{}_1\alpha_{i_m}^{j_m}\,{}_2\alpha_{i_m}^{j_m}}{1 + {}_1\alpha_{i_m}^{j_m} + {}_2\alpha_{i_m}^{j_m}}\right) H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)\right\}$$
$$+ \quad \cdots .$$

The first term is by Definition 1.2.1 the number of distinct nxnx2 matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$. It can be seen from 1.3.4 of Lemma 1.3.1 that the inverse $(1 + {}_1\alpha_i^j + {}_2\alpha_i^j)^{-1}$ is the algebraic inverse and hence that the second term of 1.5.10 is the number of distinct nxnx2 matrices C in the subspace $\mathcal{C}_1$ by Corollary 1.5.1.2. The third and succeeding terms are precisely those of the right member of 1.5.8 of Corollary 1.5.2.3 for $M = 2, 3, \ldots$, respectively. It follows that the third and succeeding terms enumerate the number of distinct $n \times n \times 2$ matrices C in the respective subspaces $\mathcal{C}_2$, $\mathcal{C}_3$, etc. The theorem is an immediate consequnce.

The main problem under consideration is to find the number of distinct $n \times n \times 2$ matrices C in the subspace $\mathcal{C}_0$, which by Definition 1.2.1 is $H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$. For each factor in the right member of 1.5.9, it is easily seen that

$$1 + \frac{{}_1\alpha_i^j \, {}_2\alpha_i^j}{1 + {}_1\alpha_i^j + {}_2\alpha_i^j} \equiv \frac{(1 + {}_1\alpha_i^j)(1 + {}_2\alpha_i^j)}{(1 + {}_1\alpha_i^j + {}_2\alpha_i^j)} .$$

If this substitution is made in 1.5.9 and Lemma 1.3.1 is applied to both sides of 1.5.9 for each factor in the numerator and denominator after substitution, we obtain

Theorem 1:5.4 If $\underline{r}^1$, $\underline{s}^1$ are any two n-part, non-negative, ordered partitions of t_1 and $\underline{r}^2$, $\underline{s}^2$ are any two n-part, non-negative, ordered partitions of t_2, then the number of distinct $n \times n \times 2$ matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ is given by

$$1.5.11 \qquad H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2) = \prod_{i \neq j} \left\{ \frac{1 + {}_1\alpha_i^j + {}_2\alpha_i^j}{\left((1 + {}_1\alpha_i^j)(1 + {}_2\alpha_i^j)\right)} \right\} \eta(\underline{r}^1, \underline{s}^1)\,\eta(\underline{r}^2, \underline{s}^2),$$

where the operators ${}_1\alpha_i^j$ and ${}_2\alpha_i^j$ operate on $\eta(\underline{r}^1, \underline{s}^1)$ and $\eta(\underline{r}^2, \underline{s}^2)$, respectively.

Since there are no tables published which give the value of the $\eta(\underline{r}^i, \underline{s}^i)$ functions, the preceding theorem would hardly lend itself to computation. However, Katz and Powell [5] proved that $\eta(\underline{r}, \underline{s}) = \prod_{i=1}^{n}\left(1+\delta_i^i\right)^{-1} A(\underline{r}, \underline{s})$, where Sukhatme's [8] $A(\underline{r}, \underline{s})$ function is tabulated in reference [8] for partitions of t up to $t = 8$ and also in David and Kendall [1] for partitions of t up to $t = 12$. In section 1.3 we stated that the operators ${}_u\alpha_i^j$ were precisely the same as the δ_i^i introduced by Katz and Powell [5]. Therefore, for $u = 1, 2$, we have

$$1.5.12 \qquad \eta(\underline{r}^u, \underline{s}^u) = \prod_{i=1}^{n}\left(1+ {}_u\alpha_i^i\right)^{-1} A(\underline{r}^u, \underline{s}^u) .$$

Making this substitution in Theorem 1:5.4, we obtain the following corollary without further proof. It should be noted that in 1.5.13, below, the product in the denominator is now over all i and j.

Corollary 1.5.4.1 If $\underline{r}^1$, $\underline{s}^1$ are any two n-part, non-negative, ordered partitions of t_1 and $\underline{r}^2$, $\underline{s}^2$ are any two n-part, non-negative, ordered partitions of t_2, then the number of distinct nxnx2 matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ is given by

$$1.5.13 \qquad H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2) = \frac{\prod_{i \neq j}\left(1+{}_1\alpha_i^j+{}_2\alpha_i^j\right)}{\prod_{i,j}\left[\left(1+{}_1\alpha_i^j\right)\left(1+{}_2\alpha_i^j\right)\right]} A(\underline{r}^1, \underline{s}^1)A(\underline{r}^2, \underline{s}^2),$$

where the operators ${}_1\alpha_i^j$ and ${}_2\alpha_i^j$ operate on $A(\underline{r}^1, \underline{s}^1)$ and $A(\underline{r}^2, \underline{s}^2)$, respectively.

We now consider a special case of Theorem 1.5.4. This case is identical with the one considered in the appendix, as can be seen by comparing the following corollary with Theorem A.3.4 in the appendix. The following corollary is applicable to sociometric tests involving three categories ($k = 2$) where it is mandatory that each individual classify exactly one individual in category one, exactly one other individual in category two, and the remaining ($n-3$) individuals in the third category. For such tests, we have, in the matrix model, $\underline{r}^1 = \underline{r}^2 = (1, 1, \ldots, 1)$. We shall denote this n-dimensional vector of n ones by 1^n.

Every ${}_u\alpha_i^j$ involves both a row and column index, therefore it follows that any term which involves $\left({}_u\alpha_i^j\right)^m$, where $m \geq 2$, is trivial, since $r_{iu} = 1$ for all i. Therefore, it follows that

$$1.5.14 \qquad \frac{1 + {}_1\alpha_i^j + {}_2\alpha_i^j}{\left(1 + {}_1\alpha_i^j\right)\left(1 + {}_2\alpha_i^j\right)} = 1 - {}_1\alpha_i^j \, {}_2\alpha_i^j + R,$$

where all the terms in R are trivial since they involve powers of ${}_u\alpha_i^j$ greater than one.

Equation 1.5.15 of Corollary 1.4.5.2 below is written dually. First $H(1^n, \underline{s}^1, 1^n, \underline{s}^2)$ is expressed in terms of $\eta(\underline{r}^i, \underline{s}^i)$, $i = 1, 2$, by substituting from 1.5.14 in 1.5.11. In the second form, obtained by the additional substitution from 1.5.12, $H(1^n, \underline{s}^1, 1^n, \underline{s}^2)$ is expressed as a function of $A(\underline{r}^i, \underline{s}^i)$, $i = 1, 2$. We now give the corollary without further proof.

Corollary 1.5.4.2 If $r_{i1} = r_{i2} = 1$ ($i = 1, 2, \ldots, n$), and $\underline{s}^1$ and $\underline{s}^2$ are any two n-part, non-negative, ordered partitions of n, then the number of distinct $n \times m \times 2$ matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ is given by

1.5.15 $$H(1^n, \underline{s}^1, 1^n, \underline{s}^2) = \left\{ \prod_{i \neq j} \left(1 - {}_1\alpha_i^j \, {}_2\alpha_i^j\right) \right\} \eta(1^n, \underline{s}^1) \eta(1^n, \underline{s}^2),$$

$$= \left\{ \prod_{i \neq j} \left(1 - {}_1\alpha_i^j \, {}_2\alpha_i^j\right) \prod_{i=1}^{n} \left(1 - {}_1\alpha_i^i - {}_2\alpha_i^i + {}_1\alpha_i^i \, {}_2\alpha_i^i\right) \right\} \times$$

$$A(1^n, \underline{s}^1) A(1^n, \underline{s}^2) .$$

1.6 $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$ as a function of $\eta(\underline{r}^1_\alpha, \underline{s}^1_\alpha) \cdots \eta(\underline{r}^k_\alpha, \underline{s}^k_\alpha)$.

In this section Theorem 1.5.4 is generalized. The generalization is from three types of relations between ordered pairs of individuals, in a group of n individuals, to $(k+1)$, $k>1$, types of relations between ordered pairs of individuals.

It will be of interest to compare the method used in this section to find $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$, $k>1$, with the method used in the case $k = 2$ of section 1.5.

In both sections 1.5 and 1.6 there is the necessary restriction that $\sum_{u=1}^{k} c_{iju} = 0$ or 1 for every ordered pair (i,j), $i \neq j$. In section 1.5 there was associated with every ordered pair (i,j) of distinct individuals a number pair (c_{ij1}, c_{ij2}). It is easily seen that, for any (i,j), the only number pair which violates the restriction $\sum_{u=1}^{k} c_{iju} = 0$ or 1, $k = 2$, is the number pair $(1,1)$. However, when $k > 2$, there is associated with every ordered pair (i,j) of distinct individuals a k-dimensional vector $(c_{ij1}, c_{ij2}, \ldots, c_{ijk})$. It is obvious that the necessary restriction $\sum_{u=1}^{k} c_{iju} = 0$ or 1 is violated whenever two or more $c_{iju} = 1$, $u = 1, 2, \ldots, k$. It follows that, whereas when $k = 2$ there was only one possible violation, namely $(1,1)$, there is now a total of $2^k - k - 1$ different ways in which the necessary restriction could be violated for every pair (i,j), $i \neq j$. This implies that a method different from that used in section 1.5 will have to be used in this section to eliminate from the product space $\mathcal{C}$ all those matrices violating the necessary restriction.

Using the definitions and notations previously established we now proceed to Lemma 1.6.1 which is a generalization of Lemma 1.5.2.

Lemma 1.6.1 With respect to the arbitrary but fixed subset $\{i_m, j_m\}_M$ of M off diagonal positions, $M \geq 1$, the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ can be decomposed into $M+1$ mutually exclusive and exhaustive classes enumerated by

$$1.6.1 \qquad H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k) = \left\{ \prod_{m=1}^{M} \left(1 + \sum_{u=1}^{k} \alpha_{u\, i_m}^{j_m}\right) \right\} x$$

$$H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k).$$

Proof: Since there are $(k+1)$ distinct terms in each of the M distinct factors in the right member of 1.6.1, it is obvious that there are $(1+k)^M$ distinct terms in the expansion. The method of proof will be to divide the $(1+k)^M$ terms into $M+1$ subsets and prove that each of these $M+1$ subsets corresponds uniquely to a class of nxnxk matrices in the subspace $\mathcal{C}_0$. The $M+1$ classes will be shown to be mutually exclusive and exhaustive with regard to the arbitrary but fixed set $\{i_m, j_m\}_M$ of M distinct off diagonal positions.

By definitions 1.2.1 and 1.2.2, $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$ and $H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$ are defined on the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$. Therefore, for every ordered pair (i,j) of distinct individuals, $\sum_{u=1}^{k} c_{iju} = 0$ or 1. With this established it is obvious that there are k ways in which $\sum_{u=1}^{k} c_{iju} = 1$ and only one way in which $\sum_{u=1}^{k} c_{iju} = 0$ for each ordered pair (i,j), $i \neq j$. We shall carry out the decomposition of $\mathcal{C}_0$ into $M+1$ mutually exclusive and exhaustive classes and simultaneously show the unique correspondence of these classes to terms in 1.6.1.

The zero class, which we denote by $\mathcal{C}_{00}$, is that class containing all those nxnxk matrices for which $\sum_{u=1}^{k} c_{iju} = 0$ for each of the ordered pairs $(i,j) \in \{i_m, j_m\}_M$. It is not difficult to see that this class is represented in the expansion by taking 1 from each of the M factors in 1.6.1, thus obtaining $H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$. The first class, $\mathcal{C}_{01}$, contains all those nxnxk matrices for which $\sum_{u=1}^{k} c_{iju} = 1$ for exactly one of the ordered pairs $(i,j) \in \{i_m, j_m\}_M$. It is quite evident that this can be done in $\binom{M}{1}k$ ways since the single ordered pair can be chosen from M pairs in $\binom{M}{1}$ ways and the type of relation for this pair in any of k ways. This amounts to taking one from all but one of the M factors in 1.6.1 and from that factor choosing any one of the k different ${}_u\alpha_i^j$. Proceeding in this manner, we now consider the Rth class, $\mathcal{C}_{0R}$, $0 \leq R \leq M$. This class, $\mathcal{C}_{0R}$, contains all those nxnxk matrices for which $\sum_{u=1}^{k} c_{iju} = 1$ for exactly R of the ordered pairs $(i,j) \in \{i_m, j_m\}_M$. This can be done in $\binom{M}{R}k^R$ ways, since we can choose R ordered pairs from M ordered pairs in $\binom{M}{R}$ ways and once chosen we can choose the type of relation for each in k ways for a total of $\binom{M}{R}k^R$ ways. This amounts to taking a one from $M-R$ of the M factors in 1.6.1 and exactly one ${}_u\alpha_i^j$

(which can be chosen in any of k distinct ways) from each of the remaining R factors. These steps are reversible, hence the correspondence between classes $\mathcal{C}_{0R}$, $0 \leq R \leq M$, and terms in 1.6.1 is a one-to-one correspondence. There are obviously $M+1$ terms of the form $\binom{M}{R}k^R$ in $(1+k)^M$ and from the manner in which $\mathcal{C}_{0R}$ is defined, the classes are mutually exclusive and exhaustive. The lemma is an immediate consequence.

Repeated application of Lemma 1.3.1 to both members of 1.6.1 gives us, without further proof

Corollary 1.6.1.1 The number of distinct $n \times n \times k$ matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ for which $\sum_{u=1}^{k} c_{iju} = 0$ for each ordered pair $(i,j) \in \{i_m, j_m\}_M$, $M \geq 1$, is given by

1.6.2
$$H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k) \equiv \prod_{m=1}^{M}\left(1 + \sum_{u=1}^{k} {}_u\alpha_{i_m}^{j_m}\right)^{-1} H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k) .$$

From this point on the proof given in this section differs from that given in section 1.5. We now wish to examine the ways in which $\sum_{u=1}^{k} c_{iju} > 1$. In the case $k = 2$, of section 1.5, this could happen in only one way, namely $c_{ij1} = c_{ij2} = 1$. With $k > 2$ the number of ways is $2^k - k - 1$, for each ordered pair (i,j). In this connection we now state the following definition.

Definition 1.6.1 We shall say we have a w-tuple coincidence at (i,j), $2 \leq w \leq k$, if $\sum_{u=1}^{k} c_{iju} = w$.

It was shown in section 1.2 that the product space $\mathcal{C}$ could be decomposed into mutually exclusive and exhaustive subspaces $\mathcal{C}_M$, $M \geq 0$, so that $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1 + \mathcal{C}_2 + \cdots$. At the same time $\mathcal{C}_M$ was defined to be that subspace of the product space $\mathcal{C}$ which contained all those nxnxk matrices C for which $\sum_{u=1}^{k} c_{iju} > 1$ for <u>exactly</u> M ordered pairs (i,j) of distinct individuals.

By Definition 1.2.2 $H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$ enumerates a subspace for which $\sum_{u=1}^{k} c_{iju} = 0$ for each ordered pair $(i,j) \in \{i_m, j_m\}_M$. We now proceed to fix a w-tuple coincidence $(2 \leq w \leq k)$ at one of the M positions given by the ordered pairs $(i,j) \in \{i_m, j_m\}_M$. Let the w-tuple coincidence at (i,j) consist of the relations u_v, $v = 1, 2, \ldots, w$, $2 \leq w \leq k$, then $c_{iju_v} = 1$ for $v = 1, 2, \ldots, w$. If this w-tuple is fixed at (i,j), then r_{iu_v} and s_{ju_v}, $v = 1, 2, \ldots, w$, must each be reduced by unity. This can only be done by the operator ${}_{u_1}\alpha_i^j \, {}_{u_2}\alpha_i^j \cdots {}_{u_w}\alpha_i^j$ applied to $(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$. It follows that

$$1.6.3 \qquad \left\{\prod_{v=1}^{w} {}_{u_v}\alpha_i^j\right\} H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$$

is the number of nxnxk matrices in the product space $\mathcal{C}$ which contain the particular w-tuple coincidence $c_{iju_1} = c_{iju_2} = \cdots = c_{iju_w} = 1$ at position (i,j). We point out that, for any fixed w, $(2 \leq w \leq k)$, there would actually be $\binom{k}{w}$ distinct ways of choosing a w-tuple coincidence. Therefore, to include all possibilities for the position (i,j), it is necessary to sum 1.6.3 over $u_1 < u_2 < \cdots < u_w$ for each w.

Proceeding in this manner we have the following corollary.

Corollary 1.6.1.2 For each w, $2 \le w \le k$

$$\sum_{u_1 < u_2 < \cdots < u_w} \left\{ \prod_{v=1}^{w} {}_{u_v}\alpha_i^j \right\} H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k) \tag{1.6.4}$$

is the number of nxnxk matrices in the product space $\mathcal{C}$ which contain a w-tuple coincidence at the position $(i,j) \in \{i_m, j_m\}_M$ and which have no coincidences elsewhere.

1.6.4 holds for each w $(2 \le w \le k)$ and therefore if the sum is taken over all w from 2 to k this will exhaust the possible types of w-tuple coincidences at (i,j). This summation can be simplified by substition from the following identity,

$$\prod_{u=1}^{k}\left(1 + {}_u\alpha_i^j\right) \equiv 1 + \sum_{u=1}^{k} {}_u\alpha_i^j + \sum_{u_1 < u_2} {}_{u_1}\alpha_i^j \, {}_{u_2}\alpha_i^j + \cdots + \sum_{u_1 < u_2 < \cdots < u_w} \left\{ \prod_{v=1}^{w} {}_{u_v}\alpha_i^j \right\} + \cdots + \prod_{u=1}^{k} {}_u\alpha_i^j . \tag{1.6.5}$$

From 1.6.5, we have immediately

$$\sum_{w=2}^{k} \sum_{u_1 < u_2 < \cdots < u_w} \left\{ \prod_{v=1}^{w} {}_{u_v}\alpha_i^j \right\} = \prod_{u=1}^{k}\left(1 + {}_u\alpha_i^j\right) - \left(1 + \sum_{u=1}^{k} {}_u\alpha_i^j\right). \tag{1.6.6}$$

Therefore, if 1.6.4 is summed on w from 2 to k, we have upon substitution from 1.6.6

Corollary 1.6.1.3 The total number of nxnxk matrices which have a w-tuple $(2 \le w \le k)$ coincidence at the position $(i,j) \in \{i_m, j_m\}_M$, $M \ge 1$, and no coincidences elsewhere, is given by

1.6.7 $$\left\{\prod_{u=1}^{k}\left(1+{}_{u}\alpha_{i}^{j}\right)-\left(1+\sum_{u=1}^{k}{}_{u}\alpha_{i}^{j}\right)\right\}H_{\{i_m,j_m\}_M}(\underline{r}^1,\underline{s}^1,\ldots,\underline{r}^k,\underline{s}^k).$$

It is quite evident from 1.6.7 that there are $2^k - k - 1$ possibilities for which $\sum_{u=1}^{k} e_{iju} > 1$. We point out that in $\prod_{u=1}^{k}\left(1+{}_{u}\alpha_{i}^{j}\right)$ there are $(1+1)^k = 2^k$ distinct terms and in $1+\sum_{u=1}^{k}{}_{u}\alpha_{i}^{j}$ there are $(k+1)$ distinct terms. We are merely choosing those resulting in w-tuple $(2\leq w\leq k)$ coincidences at (i,j). Corollary 1.6.1.3 applies to each position $(i,j)\in\{i_m,j_m\}_M$, $M\geq 1$. We now wish to enumerate the nxmxk matrices having a coincidence of multiplicity ≥ 2 at every ordered pair $(i,j)\in\{i_m,j_m\}_M$. This can be accomplished by taking the product of factors of the form given in 1.6.7 over all ordered pairs $(i,j)\in\{i_m,j_m\}_M$. Since it is desirable to express this as a function of $H(\underline{r}^1,\underline{s}^1,\ldots,\underline{r}^k,\underline{s}^k)$, we employ 1.6.2 of Corollary 1.6.1.1 for the right member of 1.6.8 below. We point out that the total number of possibilities of having some type of a w-tuple $(2\leq w\leq k)$ coincidence at every $(i,j)\in\{i_m,j_m\}_M$ is $(2^k - k - 1)^M$.

Corollary 1.6.1.4 The total number of nxmxk matrices which have a coincidence of multiplicty ≥ 2 at every $(i,j)\in\{i_m,j_m\}_M$, $M\geq 1$, and no coincidences elsewhere, is given by

1.6.8 $$\prod_{m=1}^{M}\left\{\prod_{u=1}^{k}\left(1+{}_{u}\alpha_{i_m}^{j_m}\right)-\left(1+\sum_{u=1}^{k}{}_{u}\alpha_{i_m}^{j_m}\right)\right\}H_{\{i_m,j_m\}_M}(\underline{r}^1,\underline{s}^1,\ldots,\underline{r}^k,\underline{s}^k)$$
$$=\prod_{m=1}^{M}\left[\left\{\prod_{u=1}^{k}\left(1+{}_{u}\alpha_{i_m}^{j_m}\right)-\left(1+\sum_{u=1}^{k}{}_{u}\alpha_{i_m}^{j_m}\right)\right\}\left\{1+\sum_{u=1}^{k}{}_{u}\alpha_{i_m}^{j_m}\right\}^{-1}\right]\times$$
$$H(\underline{r}^1,\underline{s}^1,\ldots,\underline{r}^k,\underline{s}^k).$$

The above corollary applies to the arbitrary but specified set of M off diagonal positions $\{i_m, j_m\}_M$, $M \geq 1$. For any fixed M this set can be chosen in $\binom{n(n-1)}{M}$ ways and we now wish to sum over all these possible sets $\{i_m, j_m\}_M$ for any fixed $M \geq 1$. We shall denote this sum, for fixed M, by $\sum_{\{i_m, j_m\}_M}$. Therefore, summing the right member of 1.6.8 in the manner just indicated, the following lemma is an immediate consequence of Corollary 1.6.1.4 and the definition of $\mathcal{C}_M$. Note that we are now dispensing with the auxiliary function $H_{\{i_m, j_m\}_M}(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$.

Lemma 1.6.2 For any $M \geq 1$, the total number of nxnxk matrices in the subspace $\mathcal{C}_M$ of the product space $\mathcal{C}$ is given by

1.6.9 $$\sum_{\{i_m, j_m\}_M} \prod_{m=1}^{M} \left[\left\{ \prod_{u=1}^{k} \left(1 + {}_u\alpha_{i_m}^{j_m}\right) - \left(1 + \sum_{u=1}^{k} {}_u\alpha_{i_m}^{j_m}\right) \right\} \left\{ 1 + \sum_{u=1}^{k} {}_u\alpha_{i_m}^{j_m} \right\}^{-1} \right] \times$$

$$H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k) .$$

We previously showed that the product space could be decomposed into mutually exclusive and exhaustive subspaces, i.e. $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1 + \mathcal{C}_2 + \cdots$. The preceding lemma gives the number in any of the subspaces $\mathcal{C}_M$, $M \geq 1$. Therefore, if 1.6.9 is summed on M, we would obtain a complete enumeration of the product space by subspaces $\mathcal{C}_M$, $M \geq 1$, with the exception of $\mathcal{C}_0$. The number in $\mathcal{C}_0$ is, by Definition 1.2.1, $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$. Hence, if the identity operator is added to the sum on M of 1.6.9, then the result would be a complete enumeration of the totality of the $\eta(\underline{r}^1, \underline{s}^1)\cdots\eta(\underline{r}^k, \underline{s}^k)$ $n \times m \times k$ matrices C in the product space $\mathcal{C}$ by subspaces $\mathcal{C}_M$, $M \geq 0$. Consider the following expansion

1.6.10 $$\prod_{i \neq j}\left\{1 + \frac{\prod_{u=1}^{k}\left(1+ {}_u\alpha_i^j\right) - \left(1+\sum_{u=1}^{k} {}_u\alpha_i^j\right)}{1+\sum_{u=1}^{k} {}_u\alpha_i^j}\right\} H(\underline{r}^1,\underline{s}^1,\ldots,\underline{r}^k,\underline{s}^k)$$

$$= H(\underline{r}^1,\underline{s}^1,\ldots,\underline{r}^k,\underline{s}^k)$$

$$+ \sum_{i \neq j}\left\{\frac{\prod_{u=1}^{k}\left(1+ {}_u\alpha_i^j\right) - \left(1+\sum_{u=1}^{k} {}_u\alpha_i^j\right)}{1+\sum_{u=1}^{k} {}_u\alpha_i^j}\right\} H(\underline{r}^1,\underline{s}^1,\ldots,\underline{r}^k,\underline{s}^k)$$

$$+ \sum_{\{i_m, j_m\}_2} \prod_{m=1}^{2}\left\{\frac{\prod_{u=1}^{k}\left(1+ {}_u\alpha_i^j\right) - \left(1+\sum_{u=1}^{k} {}_u\alpha_i^j\right)}{1+\sum_{u=1}^{k} {}_u\alpha_i^j}\right\} H(\underline{r}^1,\underline{s}^1,\ldots,\underline{r}^k,\underline{s}^k)$$

$$+ \cdots$$

$$+ \sum_{\{i_m, j_m\}_M} \prod_{m=1}^{M}\left\{\frac{\prod_{u=1}^{k}\left(1+ {}_u\alpha_i^j\right) - \left(1+\sum_{u=1}^{k} {}_u\alpha_i^j\right)}{1+\sum_{u=1}^{k} {}_u\alpha_i^j}\right\} H(\underline{r}^1,\underline{s}^1,\ldots,\underline{r}^k,\underline{s}^k)$$

$$+ \cdots .$$

The first term in right member of 1.6.10 is the number of matrices in $\mathcal{C}_0$. The second and succeeding terms are, by 1.6.9 of Lemma 1.6.2, the number of matrices in $\mathcal{C}_1$, $\mathcal{C}_2$, ..., $\mathcal{C}_M$, ... It follows that

the left member of 1.6.10 completely enumerates, by subspaces $\mathcal{C}_M$, $M \geq 0$, the totality of the $\eta(\underline{r}^1, \underline{s}^1)\cdots\eta(\underline{r}^k, \underline{s}^k)$ nxnxk matrices in the product space $\mathcal{C}$. The left member of 1.6.10 can be simplified since, for each factor $i \neq j$, we have

1.6.11 $$\left\{1 + \frac{\prod_{u=1}^{k}\left(1+\alpha_i^j\right) - \left(1+\sum_{u=1}^{k}\alpha_i^j\right)}{1+\sum_{u=1}^{k}\alpha_i^j}\right\} = \frac{\prod_{u=1}^{k}\left(1+\alpha_i^j\right)}{1+\sum_{u=1}^{k}\alpha_i^j} .$$

If 1.6.11 is substituted in 1.6.10 we have, without further proof

Theorem 1.6.3 Given k pairs $(\underline{r}^u, \underline{s}^u)$, $u = 1, 2, \ldots, k$, where $\underline{r}^u$ and $\underline{s}^u$ are any n-part, non-negative, ordered partitions of t_u, $t_u \geq 1$, then the totality of $\eta(\underline{r}^1, \underline{s}^1)\cdots\eta(\underline{r}^k, \underline{s}^k)$ nxnxk matrices C in the product space $\mathcal{C}$ defined by these partitions is completely enumerated by

1.6.12 $$\eta(\underline{r}^1, \underline{s}^1)\cdots\eta(\underline{r}^k, \underline{s}^k) = \prod_{i\neq j}\left\{\frac{\prod_{u=1}^{k}\left(1+\alpha_i^j\right)}{1+\sum_{u=1}^{k}\alpha_i^j}\right\} H(\underline{r}^1,\underline{s}^1,\ldots,\underline{r}^k,\underline{s}^k).$$

We are primarily concerned with the number of distinct nxnxk matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$, namely $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$. It was shown in Lemma 1.3.1 that, for $1 \leq h \leq k$, the inverse of $(1 + \sum_{u=1}^{h}\alpha_i^j)$ is the algebraic inverse. Each factor in right member of 1.6.12 is of the form given in Lemma 1.3.1, therefore repeated application of the lemma to both sides of 1.6.12 gives us, without further proof, the following theorem.

Theorem 1.6.4 Given k pairs $(\underline{r}^u, \underline{s}^u)$, $u = 1, 2, \ldots, k$, where

$\underline{r}^u$ and $\underline{s}^u$ are any n-part, non-negative, ordered partitions of t_u, $t_u \geq 1$, then the number of distinct nxnxk matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ is given by

1.6.13 $$H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k) = \prod_{i \neq j} \left\{ \frac{1 + \sum_{u=1}^{k} {}_u\alpha_i^j}{\prod_{u=1}^{k}\left(1 + {}_u\alpha_i^j\right)} \right\} \eta(\underline{r}^1, \underline{s}^1) \cdots \eta(\underline{r}^k, \underline{s}^k),$$

where ${}_u\alpha_i^j$ operates on $(\underline{r}^u, \underline{s}^u)$, u = 1, 2, ..., k.

In the form given by 1.6.13 we note that $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$ is expressed as a function of the $\eta(\underline{r}_\alpha, \underline{s}_\alpha)$ introduced by Katz and Powell [5]. At the present time no published tables are available for their function $\eta(\underline{r}_\alpha, \underline{s}_\alpha)$. Therefore, as in Section 1.5, we employ their counting theorem, namely

$$\eta(\underline{r}^u, \underline{s}^u) = \prod_{i=1}^{n}\left(1 + {}_u\alpha_i^i\right)^{-1} A(\underline{r}^u, \underline{s}^u),$$

for each u = 1, 2, ..., k. If this substitution is made for each $\eta(\underline{r}^u, \underline{s}^u)$ given in 1.6.13, then Theorem 1.6.4 can be written in the form given in the following corollary.

Corollary 1.6.4.1 Given k pairs $(\underline{r}^u, \underline{s}^u)$, u = 1, 2, ..., k, where $\underline{r}^u$ and $\underline{s}^u$ are any n-part, non-negative, ordered partitions of t_u, $t_u \geq 1$, then the number of distinct nxnxk matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ is given by

1.6.14 $$H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k) = \frac{\prod_{i \neq j}\left(1 + \sum_{u=1}^{k} {}_u\alpha_i^j\right)}{\prod_{i,j}\left(1 + {}_u\alpha_i^j\right)} A(\underline{r}^1, \underline{s}^1) \cdots (\underline{r}^k, \underline{s}^k),$$

where ${}_u\alpha_i^j$ operates on $(\underline{r}^u, \underline{s}^u)$ $u = 1, 2, \ldots, k$.

We call the readers attention to the product in the denominator of 1.6.14 which is over all pairs (i,j), including (i,i). Tables of $A(\underline{r}, \underline{s})$ are available for partitions of t up to $t = 8$ in Sukhatme [8]. Furthermore, Sukhatme [8] gives an algorism for finding any $A(\underline{r}, \underline{s})$. David and Kendall [1] give tables of $A(\underline{r}, \underline{s})$ for partitions of t up to $t = 12$.

It can now be seen that Corollary 1.5.4.1 in section 1.5 is a special case of Corollary 1.6.4.1, namely the case $k = 2$. Also, Theorem A.3.4 in the appendix is a special case of Corollary 1.6.4.1, namely $k = 2$ and $\underline{r}^1 = \underline{r}^2 = (1, 1, \ldots, 1)$. In this connection we point out that unpublished tables have been completed for $\eta(1^n, \underline{s})$ for values of n through 16 under the direction of Katz. These tables give the value of $\eta(1^n, \underline{s})$ for any $\underline{s}$, $\underline{s}$ being n-part, non-negative, ordered partitions of n. It is for this reason that we next consider a special form of Theorem 1.6.4. The following corollary can be used in sociometric tests in which it is mandatory that each person classify exactly one other person in each of the k categories, the remaining $n-k-1$ being automatically classified in the $(k+1)$st category. For such tests $\underline{r}^u = (1, 1, \ldots, 1)$, for $u = 1, 2, \ldots, k$. It follows that, in the expansion of 1.6.13, any terms which contain a factor of the form $\left({}_u\alpha_i^j\right)^m$, where $m > 1$, are trivial. We now consider the expansion of 1.6.13 for this special case.

For fixed i and j $\dfrac{1 + \sum_{u=1}^{k} {}_u\alpha_i^j}{\prod_{u=1}^{k}\left(1 + {}_u\alpha_i^j\right)}$ can be expressed as elementary symmetric functions of the ${}_u\alpha_i^j$ by letting

$$a_r = \sum {}_{u_1}\alpha_i^j \; {}_{u_2}\alpha_i^j \cdots {}_{u_r}\alpha_i^j \qquad u_1 \neq u_2 \neq \cdots \neq u_r \,.$$

Using this notation we have

1.6.15 $$\frac{1 + \sum_{u=1}^{k} {}_u\alpha_i^j}{\prod_{u=1}^{k}\left(1 + {}_u\alpha_i^j\right)} = \frac{1 + a_1}{1 + a_1 + a_2 + \cdots + a_k}\,.$$

Carrying out the division in a formal manner, we have

1.6.16 $$1 + a_1 + \cdots + a_k \;\overline{\big)\, 1 + a_1} \quad \begin{array}{l} 1 - a_2 \\ \hline 1 + a_1 \\ 1 + a_1 + a_2 + a_3 + \cdots + a_r + \cdots + a_k \\ \hline \quad -a_2 - a_3 - \cdots - a_r - \cdots - a_k \end{array}\,.$$

Next we must subtract from the first remainder $-a_2(1 + a_1 + \cdots + a_{k-2})$. In this expansion we wish to retain only elementary symmetric functions. From Sukhatme [8] we find that in the expansion of $a_u a_v$, in terms of monomial symmetric functions, that the only term involving an elementary symmetric function is $G(1^{u+v}) = a_{u+v}$ and its coefficient is $\binom{u+v}{v}$. It follows that the coefficient of a_r in the second remainder is $-1+\binom{r}{2}$, furthermore the next term in the quotient is $2a_3$, i.e. $-1+\binom{3}{2} = 2$. If we accumlate the coefficient of a_r, for the third remainder, this coefficient will be $-1+\binom{r}{2} - 2\binom{r}{3}$, for the fourth remainder $-1+\binom{r}{2} - 2\binom{r}{3} + 3\binom{r}{4}$, etc. Proceeding in this manner we find that the coefficient of a_r in the quotient is $(-)^{r-1}(r-1)$, and therefore we have from 1.6.15

1.6.17 $$\frac{1 + \sum_{u=1}^{k} {}_u\alpha_i^j}{\prod\left(1 + {}_u\alpha_i^j\right)} = 1 - a_2 + 2a_3 - 3a_4 + \cdots (-)^{k-1}(k-1)a_k + R,$$

where the remainder R involves only trivial terms. We shall write the right member of 1.6.17 in the form $1 + \mathcal{Q}_i^j + R$ in the interest of brevity of notation. Using this notation for each of the factors in 1.6.13, the special case of Theorem 1.6.4 can be written as

Corollary 1.6.4.2 If $r_{iu} = 1$, for $i = 1, 2, \ldots, n$ and $u = 1, 2, \ldots, k$, and if $\underline{s}^u$, $u = 1, 2, \ldots, k$, are any n-part, non-negative, ordered partitions of n, then the number of nxnxk matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$ is given by

1.6.18 $$H(1^n, \underline{s}^1, \ldots, 1^n, \underline{s}^k) = \prod\left(1 + a_i^j\right)\eta(1^n, \underline{s}^1)\cdots\eta(1^n, \underline{s}^k),$$

where $1 + a_i^j$ is the non-trivial part of 1.6.17.

We point out that this corollary could be used without reference to tables if Theorem 1.2, page 29, in the dissertation of Powell [7], is employed. His theorem expresses $\eta(1^n, \underline{s})$ as a function of symmetric functions of the non-zero s_i in $\underline{s}$.

In the next section we shall consider applications of the results of this section.

PART II

APPLICATIONS

2.1 Preliminaries

The main result of section 1.6 gives a method of finding the number, $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$, of nxnxk matrices C defined by 1.1.3 and 1.1.4. Thus, for any k pairs of fixed n-dimensional marginal row and column total vectors $(\underline{r}^u, \underline{s}^u)$ $u = 1, 2, \ldots, k$, we have a space $\mathcal{C}_0$ consisting of $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$ matrices C.

The definition of these matrices was formulated by considering the possible outcomes of a class of sociometric tests. This class consists of tests in which each member of the group must classify each of the others into one and only one of $(k+1)$ distinct categories, $k \geqslant 1$, the categories being dependent on the criterion of the test and the number classified in each category possibly restricted by the experimenter. A large number of sociometric tests are so characterized.

We now wish to consider some of the statistical aspects of the preceding results. We will define random variables within the realm of these tests and examine a particular hypothetical random variable.

2.2 Random variables defined on the matrices C.

Usually, the number of individuals (n) and the number of categories (k+1) are taken to be finite. It then follows that the space of all possible outcomes of any given sociometric test can be represented by a finite number of $n \times k$ matrices C. For any given test, let us call this space of all possible outcomes the sample space. We denote the total number of outcomes by N.

For this sample space we shall adopt the usual convention and consider all possible outcomes as equally likely. Using this convention we assign probability of 1/N to each of the N outcomes, resulting in a uniform probability measure on the sample space. In some relatively rare circumstances we may wish to assign uniform probability measure on subsets of such a sample space; we here avoid this complication.

Given this sample space of N $n \times k$ matrices C we can now define various random variables over these N matrices. A random variable is a single valued, measure preserving mapping from its domain to its range space. The domain in this case is the sample space. The range space for the random variables we are considering will be appropriate subsets of the real line.

Thus, for a given random variable X, each of the N matrices in the sample space has an image on the real line, this image being the value of the random variable X for the particular matrix. In general this mapping is many to one, that is, there are many matrices C resulting in the same value of the random variable X. This mapping induces a probability measure on the random variable since, for a given X = x, the probability measure of x is the probability measure of the inverse image of x in the domain space. It follows that if the domain has uniform probability measure on it, then, using the counting procedures developed in section 1.6, we can find the probability distribution of the random variable X.

For any given sociometric test, the instructions with regard to making responses to the other members of the group will determine the sample space on which random variables will be defined; in other words, will delimit the universe of discourse. We will consider a few of these sample spaces.

2.3 Some sample spaces defined by test instructions.

In the discussion which follows, we assume that the restrictions on the $n \times n \times k$ matrices C given by 1.1.3 and 1.1.4 are satisfied.

Suppose the instructions to a sociometric test state that each person must classify d of the others in a first group, d more in a second group, etc., for a total of k categories, $kd \leq n-1$, and the remaining $n-kd-1$ members are classified in the $(k+1)$st category. For fixed d and k the corresponding natural sample space consists of all $n \times n \times k$ matrices C having $\underline{r}^u = (d, d, \ldots, d)$ for $u = 1, 2, \ldots, k$. For each of the sets of k n-dimensional vectors $\underline{s}^u$, $u = 1, 2, \ldots, k$, we can find $H(d^n, \underline{s}^1, d^n, \underline{s}^2, \ldots, d^n, \underline{s}^k)$ and thus the number in a subspace $\mathcal{C}_0$ of the sample space. There are as many such subspaces $\mathcal{C}_0$ in the sample space as there are distinct sets of k n-dimensional vectors $\underline{s}^u$, $u = 1, 2, \ldots, k$. This sample space then consists of a collection of subspaces $\mathcal{C}_0$, each of the type enumerated by some $H(d^n, \underline{s}^1, \ldots, d^n, \underline{s}^k)$.

If the previous instructions are altered slightly we obtain a different sample space. Assume we would like to divide the group into categories containing unequal numbers of members. For instance, the group may have k projects to complete simultaneously and the uth project requires v_u members. The instructions request each member to choose v_u others for the uth project. The marginal row total vectors are then of the form $\underline{r}^u = (v_u, v_u, \ldots, v_u)$, $u = 1, 2, \ldots, k$, where $\sum_{u=1}^{k} v_u \leq n-1$ is the only restriction on the v_u's.

The sample space determined by these instructions is the one consisting of all those nxnxk matrices C having $\underline{r}^u = (v_u, v_u, \ldots, v_u)$, $u = 1, 2, \ldots, k$, where the v_u are a fixed set of positive integers such that $\sum_{u=1}^{k} v_u \leq n-1$. We point out that if $v_u = d$ for all u, then this sample space is equivalent to the preceding sample space. The same remarks now apply to sets of k n-dimensional vectors $\underline{s}^u$, $u = 1, 2, \ldots, k$, as in the preceding sample space, except, of course, that we no longer have $r_{iu} = d$, for all i and u. We next consider a case in which the choices made by the members are unrestricted.

In many situations members of a group may be asked to classify other members on a qualitative basis. That is, the criterion of the test may deal with some individual trait such as loyalty or integrity; in such cases the categories would be characterized by such words as superior, excellent, good, etc. Classifications are then a matter of personal judgment and the instructions would have to permit unrestricted choices for the various categories. The marginal row total vectors are then random variables. The sample space is now the space of all $n \times n \times k$ matrices C. For each set of marginal row and column total vectors $(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$ we have a subspace $\mathcal{C}_0$ of the sample space which is enumerated by $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$. The total number of $n \times n \times k$ matrices C in this sample space is, by elementary combinatorial considerations, $(k+1)^{n(n-1)}$. Before proceeding to the hypothetical example we consider one more case.

Assume a sociometric test has been administered and the results recorded. This would result in a fixed matrix C with fixed c_{iju}, r_{iu}, and s_{ju}. We may wish to confine attention to only those matrices having the same fixed $\underline{r}^u$ and $\underline{s}^u$. In this case, the appropriate sample space would contain $H(\underline{r}^1, \underline{s}^1, \ldots, \underline{r}^k, \underline{s}^k)$ $n \times n \times k$ matrices C, where the $\underline{r}^u$ and $\underline{s}^u$, $u = 1, 2, \ldots, k$, are those recorded in the test.

It can be seen that the sample spaces to be used depend heavily upon the instructions given for the sociometric test. Furthermore, the investigator may wish to alter his frame of reference by choosing one or other subset of all matrices C as his sample space. We next consider a particular example in which we show the complete process, including definition of the random variable to conform with the nature of the sociometric investigation.

2.4 A hypothetical example.

Suppose we are interested in whether all, or most, or few of the members are "accepted" by the rest of the group.

Let the sociometric test be one in which each individual is asked to make a first, a second and a third choice from among the other (n-1) individuals on the basis of some criterion. The (n-4) not selected by an individual are thus automatically classified in the fourth category by that individual. Let $\underline{s}^1$, $\underline{s}^2$, $\underline{s}^3$ be the n-dimensional marginal column total vectors representing the numbers of first, second and third choices received, respectively. From the nature of the sociometric test employed, we have $r_{iu} = 1$ for $i = 1, 2, \ldots, n$ and $u = 1, 2, 3$. It follows that the sample space consists of all those nxnx3 hollow matrices C with marginal row total vectors $\underline{r}^1 = \underline{r}^2 = \underline{r}^3 = (1, 1, \ldots, 1)$. The total number of matrices in the sample space can be found in two ways. First, from elementary combinatorial considerations, we see that the ith individual can make his first choice in (n-1) ways, next, his second in (n-2) ways, and, finally, his third in (n-3) ways; the remaining (n-4) choices are then determined. It follows that the total number of ways is $[(n-1)(n-2)(n-3)]^n$, since the n individuals make their choices independently. Second, the total number could be found by accumulating $H(1^n, \underline{s}^1, 1^n, \underline{s}^2, 1^n, \underline{s}^3)$ for all possible distinct sets of column total vectors $\underline{s}^1$, $\underline{s}^2$ and $\underline{s}^3$. (Since $\underline{r}^u = (1, 1, \ldots, 1)$ for $u = 1, 2, 3$, this task could be simplified considerably by considering permutations of the s_{ju}.) Thus, the sample space contains $[(n-1)(n-2)(n-3)]^n$ nxnx3 matrices C and for enumeration purposes these can be classified into disjoint subspaces $\mathcal{C}_0$ enumerated by $H(1^n, \underline{s}^1, 1^n, \underline{s}^2, 1^n, \underline{s}^3)$ for distinct sets of $\underline{s}^1$, $\underline{s}^2$, $\underline{s}^3$. To each of these matrices we assign a probability measure of one divided by $\left[(n-1)(n-2)(n-3)\right]^n$. This sample space is the domain for the random variable X which must now be specified.

We may choose to define acceptance of an individual by consideration of a weighted sum of choices. A weight of three is given to a first choice, a weight of two to a second choice, and a weight of one to a third choice. Thus, for the jth individual we have a weighted sum of choices received, namely $3s_{j1} + 2s_{j2} + s_{j3}$. From the nature of the responses we have $\sum_{j=1}^{n} s_{ju} = n$, for each u = 1, 2, 3; therefore the expected value of s_{ju}, for every j and u, is one. It follows that the expected value of the weighted sum for individual j is 6. We shall say that individual j is not fully accepted if his weighted sum is less than the expectation.

Let X be the number of such individuals in a group of n individuals, where individual j is not fully accepted if the weighted sum of his choices received is less than six. The domain of this random variable is the set of $\left[(n-1)(n-2)(n-3)\right]^n$ nxnx3 matrices C in the sample space. The range is the set of non-negative integers between 0 and (n-1), inclusive. One obvious way in which all members can be accepted (X = 0) is $s_{ju} = 1$, for all u and j. The value (n-1) is assumed by X in the following manner. Let any individual receive (n-1) first choices, he is obviously accepted, the remaining first choice and the n second and n third choices are easily distributed among the other individuals so that weighted sum is five or less; this is true only if $n \geqslant 4$.

The single valued, measure preserving mapping is evident by inspection. Given any nxnxk matrix C in the domain (sample space) we can determine the value of X for this matrix by examining, for each j, $3s_{j1} + 2s_{j2} + s_{j3}$ and take X to be the number of j for which this weighted sum is less than six. This is a many to one mapping. For each X in the range space, $0 \leq X \leq n-1$, $(n \geqslant 4)$, there is a class of nxnx3 matrices in the sample space. The probability that X = x, is the probability of the inverse image of x in the domain. Thus, the probability distribution in the range space is induced by the single valued measure preserving mapping. In actual practice it would be necessary to find $H(1^n, \underline{s}^1, 1^n, \underline{s}^2, 1^n, \underline{s}^3)$ for all distinct sets of three n-dimensional marginal column total vectors $\underline{s}^1$, $\underline{s}^2$ and $\underline{s}^3$. The random variable X assumes a value for each set. This procedure leads to the probability distribution of the random variable X.

Thus, once the universe of discourse has been decided upon, many random variables can be defined by procedures similar to that given above. In closing we remark that meaningful random variables can best be selected through the cooperation of sociologists and mathematical statisticians. The immediate future applications of the results of this thesis will probably be in this field.

SUMMARY

The problem considered in this paper was the extension of a result originally obtained by Katz and Powell [5]. They were concerned with the one dimensional theory of group organization as a complex of irreflexive binary relationships, taking values of 0 and 1, between the pairs of individuals. In this paper the binary restriction is removed and replaced by $(k+1)$-nary relationships, $k > 1$.

The problem was solved in three successive stages. The successive stages will be found in the appendix, section 1.5, and section 1.6, respectively. A hypothetical example which demonstrates the application of the main theorem is given in Chapter 2.

The first two stages are special cases of the third stage, although the methods of proof in each of the three stages are unique. It was considered important to indicate the evolution of the process which led to the general result. It is doubtful that the general result would have been attained so soon by a direct attack. The success at each stage not only served as a natural impeller to the next stage, but also provided clues for the method of attack in the following stage. The three stages are described below.

1) Ternary relationships between ordered pairs of distinct individuals; each individual chooses exactly one for category one, exactly one other for category two, and the remaining n-3 are automatically classified in the third category.

2) Ternary relationships between ordered pairs of distinct individuals; each individual classifies each of the others into one and only one of the three categories.

3) $(k+1)$nary relationships $(k > 1)$ between ordered pairs of distinct individuals; each individual classifies each of the others into one and only one of the $(k+1)$ categories.

These relationships between ordered pairs of individuals have matrix representations, this method of representation being introduced by Forsyth and Katz [3] in 1946. In all three stages the matrices are three dimensional, namely nxnxk for groups of n-individuals and $(k+1)$ relationships. The third dimension of the matrix, k, is always one less than the number of relations because of the $(k+1)$st relationship (null relationship) which is mandatory for every ordered pair not having one of the other k relationships. The number of distinct matrices satisfying the restrictions is given by $H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2, \ldots, \underline{r}^k, \underline{s}^k)$, where $\underline{r}^u$ and $\underline{s}^u$ are n-dimensional vectors. In relation to the result of Katz and Powell [5], $H(\underline{r},\underline{s}) = \eta(\underline{r},\underline{s})$.

The first stage results in a theorem which gives $H(1^n,\underline{s}^1,1^n,\underline{s}^2)$ as a bilinear function of $\eta(1^n, \underline{s}^1)\,\eta(1^n, \underline{s}^2)$. This proof, found in the appendix, centers on the fact that $\underline{r}^1 = \underline{r}^2 = (1, 1, \ldots, 1)$, hence there can be at most one violation of the restriction that one and only one relationship exists between each ordered pair. A permutation function of row and column indices is the distinctive feature of the proof.

The second stage, section 1.5, results in a theorem which expresses

$H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$ as a bilinear function of $\eta(\underline{r}^1, \underline{s}^1)\eta(\underline{r}^2, \underline{s}^2)$. This proof has the same feature as the proof in the appendix, namely there is only one way to violate the restriction that one and only one relation exists between each ordered pair. However, in this case there can be one or more violations in a row. Introduction of a new function, $H\{i_m, j_m\}_M(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$ circumvents this difficulty.

The third and last stage, section 1.6, expresses $H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2, \ldots, \underline{r}^k, \underline{s}^k)$ as a function of $\eta(\underline{r}^1, \underline{s}^1)\eta(\underline{r}^2, \underline{s}^2)\cdots\eta(\underline{r}^k, \underline{s}^k)$. The essential difference between this and the preceding cases is the restriction that one and only one relationship exists between each ordered pair can be violated in $2^k - k - 1$ ways for each ordered pair of distinct individuals. The machinery used in section 1.5 is adapted to handle this case.

In each of the above stages, $H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2, \ldots, \underline{r}^k, \underline{s}^k)$ is also expressed as a function of Suknatme's [8] bipartitional functions $A(\underline{r}^1, \underline{s}^1)A(\underline{r}^2, \underline{s}^2)\cdots A(\underline{r}^k, \underline{s}^k)$. This is made possible by Katz and Powell's [5] theorem which expresses $\eta(\underline{r}, \underline{s})$ as a function of $A(\underline{r}, \underline{s})$.

This is very fortunate indeed since $A(\underline{r}, \underline{s})$ is tabulated and published for partitions up through eight in Suknatme [8] and for partitions up through twelve in David and Kendall [1]. It is extremely doubtful that tables of $\eta(\underline{r}, \underline{s})$ will ever be published except for very special cases, in particular, cases where $\underline{r}$ is of the form $(d, d, \ldots, d)$, where d is a small positive integer. One reason for this is that the ordering in the n-dimensional vectors $\underline{r}$ and $\underline{s}$ is important, and this fact alone would tend to make excessively long tables. Tables are being constructed at the present time for $\eta(1^n, \underline{s})$ for values of n up through 16. These tables are being constructed under the direction of Katz and, if published, will be useful for any cases involving $H(1^n, \underline{s}^1, 1^n, \underline{s}^2, \ldots, 1^n, \underline{s}^k)$, i.e. $\underline{r}^1 = \underline{r}^2 = \underline{r}^3 = \cdots = \underline{r}^k = (1, 1, \ldots, 1)$.

In relation to sociometric testing the extension can be beneficial. Prior to this extension the application to sociometric testing was confined to criteria which had only two categories of response. The extension is applicable to any number of responses between individuals on the basis of a single criterion. There could be, for example, a criterion in which each member was to rank, partly or completely, the remaining members. The method is also adaptable to criteria in which the responses have one or more scalar values.

For fixed n and k the total sample space of all $n \times n \times k$ matrices, satisfying the given restrictions, contains $(k+1)^{n(n-1)}$ distinct matrices. Although random variables can be defined on this space the more interesting sample spaces are subspaces of this space. These subspaces become the universes of discourse for defining random variables which are single valued, measure preserving mappings from the domain to range space of the variable, the range space being a subset of the real line.

Given any sample space there is a large class of random variables which could be defined. It is not the purpose of this paper to define these random variables. However, exact distributions can be obtained in many interesting cases by employing the results of this paper. We say this in a rather light vein; the actual task of computation, even for small n and k, would be a long exacting process, though not an impossible task.

The mechanics of constructing a random variable, given a universe of discourse, are discussed in Chapter 2. A hypothetical example is given as an illustration.

Prior to concluding remarks, we point out that the results of this study may be adapted to problems in communications networks. A simple example occurs in the enumeration of (or testing whether there exist any) possible networks which may be formed among stations containing specified numbers of several kinds of outgoing and incoming trunk lines.

In conclusion, this study has developed a probability model for group organization theory with multi-valued relations between persons. It is now possible to obtain exact probability distributions for meaningful random variables. The immediate future application of the results would appear to be in the field of sociometry. This will require close cooperation between sociologists and mathematical statisticians.

APPENDIX

A.1 Preliminaries

The first successful attempt to extend the counting theorem of Katz and Powell [5] from binary to ternary respones is given below. In the terminology of section 1.1 the case under consideration is $H(1^n, \underline{s}^1, 1^n, \underline{s}^2)$ where the row totals for categories one and two are all one.

It is only natural that the first case examined would be an especially simple case of three categories of response. This special case is applicable to sociometric tests of the following type. Each individual in a group of n individuals must classify exactly one individual in category one, exactly one individual in category two, and the remaining (n-3) individuals into category three.

In the corresponding matrix representation (using notation of Section 1.1) it follows that for fixed i, $c_{ij1} = 1$ for exactly one j (corresponding to category one), $c_{ij2} = 1$ for exactly one other j (corresponding to category two), and the remaining (n-3) positions in the ith row have $c_{ij1} = c_{ij2} = 0$. By convention $c_{ii1} = c_{ii2} \equiv 0$ for all i. This implies $r_{i1} = r_{i2} = 1$ for all i, since $r_{iu} = \sum_{i=1}^{n} c_{iju}$ for each $u = 1, 2$. From the mandatory restriction given by $\underline{r}^1 = (r_{11}, r_{21}, \ldots, r_{n1}) = \underline{r}^2 = (r_{12}, r_{22}, \ldots, r_{n2}) = (1, 1, \ldots, 1)$ it follows that $\underline{s}^1 = (s_{11}, s_{21}, \ldots, s_{n1})$ and $\underline{s}^2 = (s_{12}, s_{22}, \ldots, s_{n2})$ are n-part, non-negative, ordered partitions of n ($t_1 = t_2 = n$ in the notation of section 1.1). Thus, in the $n \times n \times 2$ matrices $C = (c_{iju})$, we have $c_{iju} = 0$ or 1 $(i \neq j;\ i,j = 1, 2, \ldots, n;\ u = 1, 2)$, and $c_{iiu} \equiv 0$ for all i and u. Associated with each matrix C are n-dimensional marginal total vectors $(1^n, \underline{s}^1, 1^n, \underline{s}^2)$; we have denoted $\underline{r}^1 = \underline{r}^2 = (1, 1, \ldots, 1)$ by 1^n. Since each individual classifies each of the others into one and only one of the three categories, it follows that, in the matrix representation, we have the restriction $c_{ij1} + c_{ij2} = 0$ or 1 for every ordered pair (i,j) of distinct individuals.

An attempt was made to find a related function, similar to one of Sukhatme's [8] bipartitional functions, to cope with the matrices defined above. This effort was fruitless. However, it was realised that if the n-dimensional marginal total vectors $(1^n, \underline{s}^1)$ and $(1^n, \underline{s}^2)$ were considered independently, then, using the methods of Katz and Powell [5], there existed two spaces of $n \times n$ matrices C consisting of $\eta(1^n, \underline{s}^1)$ and $\eta(1^n, \underline{s}^2)$ matrices, respectively. Furthermore, the product space $\mathcal{C}$ of these two spaces contained a proper subspace consisting of all the matrices satisfying the given sociometric test. However, in general there were many instances in which $c_{ij1} + c_{ij2} = 2$ for one or more ordered pairs (i,j) and such matrices violate the restriction $c_{ij1} + c_{ij2} = 0$ or 1. These matrices were referred to as coincidence matrices since, at one or more positions (i,j), there were two ones coinciding. The problem then was to algebraically decompose the product space $\mathcal{C}$ in such a manner that the required matrices could be counted. In order to do this we had to employ operators δ_i^j which we now define.

A.1.1 $$\delta_i^j(r_{11}, \ldots, r_{i1}, \ldots, r_{n1}, s_{11}, \ldots, s_{j1}, \ldots, s_{n1}, r_{12}, \ldots, r_{i2}, \ldots, r_{n2}, s_{12}, \ldots, s_{j2}, \ldots, s_{n2})$$
$$\equiv (r_{11}, \ldots, r_{i1}-1, \ldots, r_{n1}, s_{11}, \ldots, s_{j1}-1, \ldots, s_{n1}, r_{12}, \ldots, r_{i2}-1, \ldots, r_{n2}, s_{12}, \ldots, s_{j2}-1, \ldots, s_{n2}).$$

The effect of the operator δ_i^j is to replace the four n-part, non-negative, ordered partitions of n by four n-part, non-negative, ordered partitions of (n-1). That is, the operator δ_i^j simultaneously reduces the ith component of $\underline{r}^1$ and $\underline{r}^2$ and simultaneously reduces the jth component of $\underline{s}^1$ and $\underline{s}^2$ each by unity. Thus, δ_i^j operates simultaneously on two pairs of n-dimensional marginal row and column total vectors $(\underline{r}^1, \underline{s}^1)$ and $(\underline{r}^2, \underline{s}^2)$ in precisely the same manner as the δ_i^j introduced by Katz and Powell [5] operates on one pair of n-dimensional marginal row and column total vectors $(\underline{r}, \underline{s})$. It is easily shown that if $(\underline{r}, \underline{s})$ (in section 4 of reference [5]) is replaced by $(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$ and if δ_i^j is defined as in A.1.1 above, then the latter δ_i^j has all the properties of the δ_i^j introduced by Katz and Powell [5]. In particular, we desire to use the inverse of $(1+\delta_i^j)$ which they proved was the algebraic inverse. We state this in the form of a lemma.

Lemma A.1 The operator $(1+\delta_i^j)$ has an inverse, left and right, given by

A.1.2 $$(1+\delta_i^j)^{-1} = 1 - \delta_i^j + (\delta_i^j)^2 - (\delta_i^j)^3 + \cdots ,$$

where δ_i^j is the operator defined by A.1.1.

There is a very special feature of the product space $\mathcal{C}$ around which the proofs are formulated. This feature is that $r_{i1} = r_{i2} = 1$ for all $i = 1, 2, \ldots, n$. Hence, in the product space $\mathcal{C}$, there is at most one coincidence ($c_{ij1} = c_{ij2} = 1$) in each row. Furthermore, $\delta_i^j(1^n, \underline{s}^1, 1^n, \underline{s}^2)$ reduces both r_{i1} and r_{i2} to zero and the resulting matrix can have no ones in the ith row. With regard to coincidences, it is also quite obvious that the maximum number of coincidences in jth column is equal to the minimum of s_{j1} and s_{j2}. We shall have occasion to refer to this and therefore we make the following definition. Let $s_j^* = \min(s_{j1}, s_{j2})$ and let $\underline{s}^* = (s_1^*, s_2^*, \ldots, s_n^*)$. It follows that $\underline{s}^*$ gives the maximum number of coincidences in each of the columns.

Since there is at most one coincidence in each row of the matrices in the product space $\mathcal{C}$ we can decompose this space on this basis.

Definition A.1 Let $\mathcal{C}_m$ be the subspace of the product space $\mathcal{C}$ which has coincidences ($c_{ij1} = c_{ij2} = 1$) in exactly m rows.

Using this definition it follows that $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1 + \mathcal{C}_2 + \cdots$, and that this decomposition is into mutually exclusive and exhaustive subspaces.

A.2 Method of attack.

Each matrix in the product space C can be expressed as the sum of two matrices A and B. The matrices A will contain only coincidences and the matrices B will contain no coincidences. For example, if C is a matrix in the product space $\mathcal{C}$ having coincidences only in the rows $i_1 < i_2 < \cdots < i_M$, then A is a matrix with these respective coincidences and B is the matrix defined by C - A.

An expression will be given which enumerates the number of matrices in each of the subspaces $\mathcal{C}_m$, $m \geq 0$. This will involve a function $H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$ which we now define.

Definition A.2.1 Let $H(\underline{r}^1, \underline{s}^1, \underline{r}^2, \underline{s}^2)$ be the total number of distinct matrices in the product space $\mathcal{C}$ which have $c_{ij1} + c_{ij2} =$ 0 or 1 for every ordered pair (i,j), $i \neq j$.

Since any matrix which has m coincidences must have these m coincidences in m distinct rows we shall have frequent reference to subsets involving an arbitrary but specified number of rows. We now define such subsets and it should be noted that the definition has to do with row and column indices and not ordered pairs (i,j).

Definition A.2.2 Let $(i_m, j_m)_M$ denote an arbitrary but specified set of M distinct row indices $i_1 < i_2 < \cdots < i_M$, and M column indices $j_1, j_2, \ldots$, not necessarily distinct.

Since we imply that the M column indices j_m are not necessarily distinct, we ought to specifiy the number of times each is used. Therefore we let w_j be the multiplicity of the column index j in the arbitrary but specified set of M column indices $j_1, j_2, \ldots$. For any given set $(i_m, j_m)_M$ then, it follows that $\sum_j w_j = M$. Since w_j is used in connection with the coincidence matrices, and since there are at most s_j^* coincidences in the jth column, we have $w_j \leq s_j^*$ for $j = 1, 2, \ldots, n$.

For any given $(i_m, j_m)_M$ the number of matrices A is a function of the permutations of the j_m with respect to an arbitrary ordering of the i_m. We assume, without loss of generality, that the ordering of the i_m is $i_1 < i_2 < \cdots < i_M$. We are concerned only with permutations for which $i_m \neq j_m$ and we denote the total number of such permutations by $P\begin{bmatrix} j_1 \cdots & j_M \\ i_1 & i_M \end{bmatrix}$.

Definition A.2.3 Let $P\begin{bmatrix} j_1 \cdots & j_M \\ i_1 & i_M \end{bmatrix}$ be the total number of distinct permutations of the j_m with respect to a fixed ordered of the i_m with no $j_m = i_m$.

Thus, using Definition A.2.3, every set $(i_m, j_m)_M$ can be associated with a unique $P\begin{bmatrix} j_1 \cdots & j_M \\ i_1 & i_M \end{bmatrix}$. Since these permutations are distinct and $i_m \neq j_m$, each permutation gives M pairs (i_m, j_m) of off diagonal positions. It follows that there are exactly $P\begin{bmatrix} j_1 \cdots & j_M \\ i_1 & i_M \end{bmatrix}$ distinct sets of M pairs of off diagonal positions associated with every $(i_m, j_m)_M$.

Using the notation and definitions given above, we now proceed to find $H(1^n, \underline{s}^1, 1^n, \underline{s}^2)$.

A.3 $H(1^n, \underline{s}^1, 1^n, \underline{s}^2)$ as a bilinear combination of $\eta(1^n, \underline{s}^1)$ and $\eta(1^n, \underline{s}^2)$.

We have shown that every matrix C in the product space $\mathcal{C}$ can be written as the sum of two matrices A and B. These matrices A and B will now be defined algebraically.

For every $(i_m, j_m)_M$ there exists a set of exactly $P\begin{bmatrix} j_1 \cdots & j_M \\ i_1 & i_M \end{bmatrix}$ matrices A having coincidences ($c_{ij1} = c_{ij2} = 1$) in the M specified rows $i_1 < i_2 < \cdots < i_M$. Each matrix A is formed by putting $c_{ij1} = c_{ij2} = 1$ for each of the M pairs (i_m, j_m) given by one of the $P\begin{bmatrix} j_1 \cdots & j_M \\ i_1 & i_M \end{bmatrix}$ permutations, otherwise $c_{ij1} = c_{ij2} = 0$. Since the permutations are distinct it follows that the matrices are distinct and there are obviously $P\begin{bmatrix} j_1 \cdots & j_M \\ i_1 & i_M \end{bmatrix}$ such matrices. The two sets of marginal row and column total vectors associated with each A (for the given set $(i_m, j_m)_M$) are $\left[(1^n, \underline{s}^1, 1^n, \underline{s}^2) - \delta^{j_1}_{i_1} \cdots \delta^{j_M}_{i_M}(1^n, \underline{s}^1, 1^n, \underline{s}^2)\right]$ where the row and column indices i_m and j_m respectively are those of the set $(i_m, j_m)_M$. Since the δ^j_i are associative, the ordering of the row and column indices is immaterial.

The matrices B will now be defined for every set $(i_m, j_m)_M$. Given a set $(i_m, j_m)_M$, the matrices B associated with the set have marginal row and column total vectors given by $\delta^{j_1}_{i_1} \cdots \delta^{j_M}_{i_M} (1^n, \underline{s}^1, 1^n, \underline{s}^2)$. Since the matrices B are to have no coincidences we insist that $c_{ij1} + c_{ij2} = 0$ or 1. By Definition A.2.1, the total number of such matrices is $H\left\{\delta^{j_1}_{i_1}\cdots\delta^{j_M}_{i_M}(1^n, \underline{s}^1, 1^n, \underline{s}^2)\right\} = \delta^{j_1}_{i_1}\cdots\delta^{j_M}_{i_M} H(1^n, \underline{s}^1, 1^n, \underline{s}^2)$. It is obvious, from the manner in which A and B were defined, that every $A + B$ has marginal row and column total vectors $(1^n, \underline{s}^1, 1^n, \underline{s}^2)$ and thus belongs to the product space $\mathcal{C}$. We summarize the above results in the following lemma.

Lemma A.3.1 For every set $(i_m, j_m)_M$, as given by Definition A.2.2, there exists a total of $P\begin{bmatrix} j_1 \cdots j_M \\ i_1 \cdots i_M \end{bmatrix}\left(\delta^{j_1}_{i_1}\cdots\delta^{j_M}_{i_M}\right) H(1^n, \underline{s}^1, 1^n, \underline{s}^2)$ distinct nxnx2 matrices C. Each of these matrices has marginal row and column total vectors $(1^n, \underline{s}^1, 1^n, \underline{s}^2)$ and thus belongs to the product space $\mathcal{C}$. Each has M coincidences in the M specified rows of $(i_m, j_m)_M$, and these M coincidences are distributed columnwise in accordance with the indices j_m of $(i_m, j_m)_M$. There are no other coincidences.

The above lemma applies to the specified column indices in $(i_m, j_m)_M$. It was noted previously that the j_m are not distinct and that w_{j_m} represents the multiplicity of the column index j_m. We now propose to sum over all possible sets of M column indices while keeping the row indices $i_1 < i_2 < \cdots < i_M$ fixed. In every case $\sum w_{j_m} = M$, since w_{j_m} is the multiplicity of j_m in $(i_m, j_m)_M$. This is a practical restriction. Theoretically this restriction on the multiplicity is unnecessary since, if the multiplicity of j_m is greater than $s^*_{j_m}$, it follows that $\delta^{j_1}_{i_1}\cdots\delta^{j_M}_{i_M} H(1^n, \underline{s}^1, 1^n, \underline{s}^2) \equiv 0$. We represent this sum over all possible sets of M column indices by

$$\sum_{\substack{(j_m)_M \\ i_m \neq j_m}}$$

where the row indices $i_1 < i_2 < \cdots < i_M$ remain fixed and ordered.

Lemma A.3.2 The total number of nxnx2 matrices C in the product space $\mathcal{C}$ with M coincidences in the M specified rows $i_1 < i_2 < \cdots < i_M$ is given by

A.3.1 $$\sum_{\substack{(j_m)_M \\ i_m \neq j_m}} \delta_{i_1}^{j_1} \cdots \delta_{i_M}^{j_M} H(1^n, \underline{s}^1, 1^n, \underline{s}^2).$$

Proof: Given any one of the sets of M column indices $(j_m)_M$, the j_m not necessarily distinct, the sum over $i_m \neq j_m$ has exactly $P\begin{bmatrix} j_1 \cdots j_M \\ i_1 \cdots i_M \end{bmatrix}$ terms in it. Hence, for fixed sets $(j_m)_M$, this could be written $\sum_{(j_m)_M} P\begin{bmatrix} j_1 \cdots j_M \\ i_1 \cdots i_M \end{bmatrix} \delta_{i_1}^{j_1} \cdots \delta_{i_M}^{j_M} H(1^n, \underline{s}^1, 1^n, \underline{s}^2)$, since the δ_i^j are associative. Each term is then of the form given in Lemma A.3.1 and since we exhaust the possible locations of M coincidences columnwise, the lemma follows.

This gives us the number of nxnx2 matrices C in the product space $\mathcal{C}$ having exactly M coincidences in the M specified rows. If we now sum over all locations of M specified rows, since there is at most one coincidence in each row, we will obtain the total number of matrices in the subspace $\mathcal{C}_M$ of the product space $\mathcal{C}$. Without further proof, we have

Corollary A.3.2.1 The total number of nxnx2 matrices C in the subspace $\mathcal{C}_M$ of the product space $\mathcal{C}$ is given by

A.3.2 $$\sum_{\substack{i_1 < i_2 < \cdots < i_M \\ i_m \neq j_m}} \delta_{i_1}^{j_1} \cdots \delta_{i_M}^{j_M} H(1^n, \underline{s}^1, 1^n, \underline{s}^2) .$$

It was shown previously that the product space $\mathcal{C}$ could be decomposed into mutually exclusive and exhaustive subspaces $\mathcal{C}_M$, and also that the total number of nxnx2 matrices C in the product space $\mathcal{C}$ is given by $\eta(1^n, \underline{s}^1)\,\eta(1^n, \underline{s}^2)$. This gives us

Theorem A.3.3 If $r_{i1} = r_{i2} = 1$ for all $i = 1, 2, \ldots, n$ and $\underline{s}^1$ and $\underline{s}^2$ are any two n-part, non-negative, ordered partitions of n, then an exhaustive enumeration of the total number of nxnx2 matrices C in the product space $\mathcal{C}$, by subspaces $\mathcal{C}_M$, is given by

A.3.3 $$\eta(1^n, \underline{s}^1)\,\eta(1^n, \underline{s}^2) = \prod_{i \neq j} \left\{1 + \delta_i^j\right\} H(1^n, \underline{s}^1, 1^n, \underline{s}^2).$$

Proof: Expanding the product on the right, we obtain

A.3.4 $$H(1^n, \underline{s}^1, 1^n, \underline{s}^2) + \sum_{i \neq j} \delta_i^j H(1^n, \underline{s}^1, 1^n, \underline{s}^2)$$
$$+ \sum_{\substack{i_1 < i_2 \\ i_m \neq j_m}} \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} H(1^n, \underline{s}^1, 1^n, \underline{s}^2) + \cdots$$
$$+ \sum_{\substack{i_1 < i_2 < \cdots < i_M \\ i_m \neq j_m}} \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \cdots \delta_{i_M}^{j_M} H(1^n, \underline{s}^1, 1^n, \underline{s}^2) + \cdots .$$

The first term in A.3.4 is by definition the number of nxnx2 matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$. By the preceding Corollary A.3.2.1, the successive summations thereafter give the total number in the subspaces $\mathcal{C}_1$, $\mathcal{C}_2$, ..., respectively. The theorem is an immediate consequence.

The primary objective was to find the number of matrices C in the subspace $\mathcal{C}_0$, namely $H(1^n, \underline{s}^1, 1^n, \underline{s}^2)$. We now employ Lemma A.1 regarding inverse of $(1+\delta_i^j)$. In this special case $r_{i1} = r_{i2} = 1$, for all i, therefore $(1+\delta_i^j)^{-1}$ reduces to $1 - \delta_i^j + R$, where all the terms in R are trivial since they involve $\left(\delta_i^j\right)^m$, $m>1$. Applying $\prod_{i\neq j} (1+\delta_i^j)^{-1} = \prod_{i\neq j} (1-\delta_i^j)$ to both members of A.3.3, we have immediately

Theorem A.3.4 If $r_{i1} = r_{i2} = 1$ for all $i = 1, 2, \ldots, n$ and $\underline{s}^1$ and $\underline{s}^2$ are any two n-part, non-negative, ordered partitions of n, then the total number of nxnx2 matrices C in the subspace $\mathcal{C}_0$ of the product space $\mathcal{C}$, is given by

A.3.5. $$H(1^n, \underline{s}^1, 1^n, \underline{s}^2) = \prod_{i\neq j} \left\{1 - \delta_i^j\right\} \eta(1^n, \underline{s}^1)\, \eta(1^n, \underline{s}^2),$$

where δ_i^j operates simultaneously on $(1^n, \underline{s}^1)$ and $(1^n, \underline{s}^2)$.

We note, in closing, that this is a special case of Theorem 1.5.4, namely $r_{i1} = r_{i2} = 1$ for all i, but remark that it was included because of the distinctness of the proof.

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