A THERMOELASTIC PROBLEM IN THE HALF-SPACE

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By

James Leland Bailey

AN ABSTRACT

Submitted to the School of Graduate Studies of Michigan
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ABSTRACT

This thesis is concerned with the solution of the thermoelastic problems resulting from the sudden application and maintenance of the following temperature distribution on the surface of a half-space initially at zero temperature--the temperature equals a constant within a circle and zero elsewhere. The temperature distribution within the half-space exposed to the aforementioned surface temperature distribution is derived. Then under the assumption of a quasi-static condition, the stresses and displacements inside the half-space and on its boundary are determined. This quasi-static solution is obtained by transformation of the problem into the Laplace subsidiary space and solving it there by the introduction of a thermoelastic potential and the use of the Galerkin-Westergaard method specialized in the case of axial symmetry to determination of Love's function and then transforming the solution back into the original space. The quasi-static solution so derived is physically descriptive for all time with the exception of the first moment. To complete the physical description of stresses in the half-space, dynamic effects due to the stress wave emitted because of the suddeness of the application of this surface temperature distribution are taken into account in the stress solution for small values of time. This asymptotic dynamic solution is produced by means of the Laplace transformation and a thermoelastic potential. Finally, some numerical results pertinent to the steady-state solution

(a special case of the quasi-static solution) and asymptotic dynamic solution are computed and tabulated.

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I. Introduction.

In 1950 V. I. Danilovskaya [3][†] solved the one-dimensional thermoelastic problem with dynamic effects included which results from the sudden application of a constant temperature to the bounding plane of a half-space and in 1952 she [4] generalized this problem by maintenance of the medium instead of the surface at constant temperature. In 1955 M. A. Sadowsky [21] adopted a simplified model and neglected dynamic effects to obtain a solution for small values of time of the axially symmetric thermoelastic problem in the half-space which results from the application of a constant surface temperature within a circle and zero surface temperature outside this circle. Still more recently in January, 1957, E. Sternberg and E. L. McDowell [24] using the method of Green have solved the steady-state thermoelastic problem in the half-space and obtained explicit solutions for three temperature distributions: surface temperature equals a constant within a circle and zero outside, surface temperature equals a constant within a rectangle and zero outside, and surface temperature equals a hemispherical distribution within a circle and zero outside. Earlier in 1937 J. N. Goodier [10] had utilized the thermoelastic potential by means of which he reduced the general three-dimensional thermoelastic problem in the entire space to determination of the Newtonian potential for a mass distribution whose density coincides with the given temperature field. For domains other than the entire space, this potential yields only a particular solution of

[†] The numbers in square brackets refer to bibliography entries.

the thermoelastic equations which in general will not satisfy prescribed boundary conditions on the surface. Since for small deformations the thermoelastic equations are linear and the superposition principle is valid, it remains to find a solution of the classical homogeneous elasticity equations to superpose in order to adjust the boundary values to those prescribed. This latter solution can be obtained either by the Neuber-Papkovich [22, p. 328 ff.] or Galerkin-Westergaard [29, p. 119 ff.] methods. In problems with rotational symmetry the Galerkin-Westergaard method simplifies to solution for Love's displacement function [15, p. 274ff.] .

Within this framework, the results derived in this thesis are the following: (1) Given that the surface of a semi-infinite three-dimensional Hookean body bounded by a plane is suddenly subjected to a temperature distribution $T = T_0$ within a circular area and $T = 0$ outside this circular area and then maintained at that temperature distribution, the quasi-static solution is obtained. That is, the body is assumed to progress through successive equilibrium states slowly enough so that dynamic effects can be neglected. This assumption is plausible except at the first moment. From the quasi-static solution the corresponding steady-state solution is then derived. (2) Given the same physical situation as in (1), the solution taking into account the dynamic terms is obtained for small values of time. That is, in the second problem due regard is given the stress wave emitted at the first moment. It is known that the dynamic effects may be of great importance at the first moment but rapidly cease to be of consequence after that. (3) Numerical values of the steady-state stresses and displacements for thirty-one points on the axis of symmetry and the surface and of the asymptotic stresses for a twenty-seven point grid in a representative plane containing the axis of symmetry are computed and tabulated.

The first result is obtained by use of the Laplace transformation, a thermoelastic potential, and the Galerkin-Westergaard method specialized in this case of rotational symmetry to derivation of the proper Love's displacement function. For the second result, the thermoelastic potential in the Laplace subsidiary space giving the desired solution directly is exhibited. The numerical evaluation of the integral in the third result is accomplished by use of Simpson's rule and the computation is done on the Michigan State Illiac Computer, the Mistic.

Before initiating solution of the problems listed as (1), (2), and (3), the basic equations of thermoelasticity are reviewed, the boundary value problem is formulated, and certain general methods for solving it are discussed.

II. General Considerations.

Thermoelastic problems deal with the derivation of the stresses and displacements developed within bodies subjected to given heat distributions. If the temperature is not uniform throughout the body, the hotter parts of the body tend to expand more than the cooler parts, but since the body remains solid, this expansion is restricted. The restriction causes thermal stresses. These stresses and the resulting strains are assumed to be related through Hooke's law and displacement derivatives are assumed small. Problems in thermoelasticity involve sixteen unknowns--six independent strains, six independent stresses, three displacements, and temperature. For these sixteen unknowns, there are the following sixteen equations:

$$\sigma_{ij} = 2G\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\, e\, \delta_{ij} - \frac{1+\nu}{1-2\nu}\,\alpha T\,\delta_{ij}\right] \quad \text{with } \sigma_{ij} = \sigma_{ji},$$

$$2\,\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \quad \text{with } \varepsilon_{ij} = \varepsilon_{ji},$$

$$\frac{\partial \sigma_{1i}}{\partial x_1} + \frac{\partial \sigma_{2i}}{\partial x_2} + \frac{\partial \sigma_{3i}}{\partial x_3} = \rho\,\frac{\partial^2 u_i}{\partial t^2}, \tag{1}$$

$$\frac{\partial T}{\partial t} = a^2\,\nabla^2 T,$$

where $i, j = 1,2,3$, rectangular coordinates $(x,y,z) = (x_1, x_2, x_3)$, σ_{ij} = the stress on the surface x_i = constant in the direction of increasing x_j, G = the modulus of shear, ε_{ij} = the strain on the surface x_i = constant in the direction of increasing x_j, ν = the Poisson's ratio, e = the cubical dilatation, δ_{ij} = the Kronecker delta, α = the coefficient of linear expansion, T = the absolute temperature, u_i = the displacement in the direction of increasing x_i, ρ = the density, t = the time, a = the diffusion constant,

$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, and the body forces normally appearing in the third set of equations are neglected. In addition three other conditions of compatibility must be satisfied. These equations express mathematically the fact that the displacements are single-valued. The compatibility equations are usually written in the form of six second-order differential equations.

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}, \quad \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left[-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right],$$

(2)
$$\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} = 2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3}, \quad \frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_2}\left[-\frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1}\right],$$

$$\frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} = 2\frac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_1}, \quad \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left[-\frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2}\right].$$

Although it has been shown impossible to reduce these six equations to a lesser number of second-order differential equations, it is possible to write them as three differential equations of higher order. The set of sixteen equations and the compatibility equations can be reduced to four differential equations, the Navier equations and the heat equation, in the three displacements and temperature and the condition of single-valuedness of the displacements respectively.† This set of four equations is

(3)
$$\nabla^2 u_i + \frac{1}{1-2\nu}\frac{\partial e}{\partial x_i} - \frac{\rho}{G}\frac{\partial^2 u_i}{\partial t^2} = \frac{2(1+\nu)}{1-2\nu}\frac{\partial(\kappa T)}{\partial x_i},$$
$$\frac{\partial T}{\partial t} = a^2 \nabla^2 T.$$

† The coupling effect is neglected. The change in heat of the body due to its deformation is of second order with respect to the temperature induced by the temperature distribution on the surface.

where $i = 1,2,3$ and $e = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$. From these differential equations one can make three pertinent observations: all equations are linear so the superposition principle is valid, the fourth equation is independent of the preceding three, and each of the first three equations contains all three displacements since the cubical dilatation e contains them. The formulation of the thermoelastic boundary value problem is completed by specification of boundary conditions. These boundary conditions may be given directly in terms of the displacements on the surface, indirectly through specification of the stresses on the surface, or as a mixture by specification of the displacement over part of the surface and the stresses over the rest of the surface.

The first step in the solution of this boundary value problem is the determination of the temperature distribution within the body from the heat equation. With the temperature distribution known, the four differential equations are reduced to three nonhomogeneous differential equations for the three displacements. Because of the complicated nature of these three linear equations, a thermoelastic potential ϕ defined by the equations $2Gu_i = \frac{\partial \phi}{\partial x_i}$ is introduced in order to obtain a particular solution of the equations and simplify the remaining boundary value problem to one with homogeneous differential equations. By substitution of the appropriate ϕ expressions into the three equations for the displacements, it can be shown that ϕ will produce particular solutions if ϕ satisfies

(4) $$\nabla^2\phi - \frac{(1-2\nu)\rho}{2(1-\nu)G}\frac{\partial^2\phi}{\partial t^2} = \frac{2(1+\nu)G\alpha}{1-\nu}T .$$

By use of Hooke's law in terms of the displacements, the definition of ϕ in terms of the displacements, and equation (4), the following expressions are derived for the stresses correspondent to ϕ.

(5) $$\sigma_{ij} = \frac{\partial^2\phi}{\partial x_i \partial x_j} - \nabla^2\phi\,\delta_{ij} + \frac{\rho}{2G}\frac{\partial^2\phi}{\partial t^2}\delta_{ij},$$

with $i,j = 1,2,3$. In general these stresses will not be those specified in the boundary conditions. To adjust the stresses to those prescribed, a solution of the homogeneous equations is superposed. Since in the case of temperature independence the thermoelastic equations reduce to the classical equations of elasticity, the latter solution is obtained from the classical theory. That the final solution is unique up to a rigid motion, even with the neglected coupling effects included, has been shown by Weiner [28] .

After utilization of the thermoelastic potential to reduce the remaining boundary value problem to one independent of temperature, many physical problems of interest will admit a plane stress or strain condition making the Muskhelishvili technique and the powerful theory of complex variable applicable. If the temperature-independent problem will not admit either of these simplifying assumptions, the Neuber-Papkovich or Galerkin-Westergaard methods can be employed. Both of these techniques reduce the

Navier displacement equations to more familiar equations--the Neuber-Papkovich, to potential equations and the Galerkin-Westergaard, to biharmonic equations. In problems exhibiting axial symmetry, the latter method can be simplified to solution for Love's displacement function. The problem in this thesis, being of the axially symmetric type, is solved by use of Love's function.

III. The Temperature Distribution.

The bounding plane surface of a half-space initially at zero temperature is subjected to the sudden application of the temperature distribution: $T = T_0$ within a circle of radius b and $T = 0$ outside this circle. After applied, the temperature distribution is maintained for all succeeding values of time. In this section the resulting temperature distribution in the half-space is determined. With the introduction of cylindrical coordinates (r, φ, z) and use of the rotational symmetry of the problem to eliminate the dependence on φ, the boundary value problem in $T = T(r,z,t)$ can be written

$$\frac{\partial T}{\partial t} = a^2 \nabla^2 T \quad \text{for } t,z > 0,$$

$$T(r,z,0) = 0 \quad \text{for all } r,z > 0,$$

$$(6) \qquad T(r,0,t) = T_0 \quad \text{for } r < b \text{ and } t > 0,$$

$$= 0 \quad \text{for } r > b \text{ and } t > 0,$$

$|T(r,z,t)| < M$ for all r,z, and t where M is a *sufficiently* large positive number. By means of a formal application of the Laplace transformation $L\{T(t)\} = T^*(p) = \int_0^\infty T(t)e^{-pt}\,dt$, this boundary value problem transforms into the following one in $T^* = T^*(r,z,p)$.

$$\nabla^2 T^* - \frac{p}{a^2} T^* = 0,$$

$$(7) \qquad T^*(r,0,p) = \frac{T_0}{p} \quad \text{for } r < b,$$

$$= 0 \quad \text{for } r > b,$$

$$|T^*(r,z,p)| < M \quad \text{for all } r,z, \text{ and } p.$$

The differential equation, a wave equation, can be separated in cylindrical coordinates to yield the solution

$$T^*(r,z,p) = \int_0^\infty A(p,\lambda)\, J_0(\lambda r)\, e^{-\gamma z} d\lambda,$$

where $\gamma(p,\lambda) = \frac{\sqrt{p + a^2\lambda^2}}{a}$ and where $A(p,\lambda)$ is to be determined from the boundary conditions. Use of an integral relation [26, p. 406]

$$(8) \qquad \int_0^\infty J_1(\lambda b)\, J_0(\lambda r)\, d\lambda \begin{aligned}[t] &= \frac{1}{b} \text{ for } r < b, \\ &= \frac{1}{2b} \text{ for } r = b, \\ &= 0 \text{ for } r > b, \end{aligned}$$

and the boundary conditions on $z = 0$ gives $A(p, \lambda) = \frac{bT_0 J_1(\lambda b)}{p}$.

Therefore

$$(9) \qquad T^*(r,z,p) = bT_0 \int_0^\infty J_1(\lambda b)\, J_0(\lambda r)\, \frac{e^{-\gamma z}}{p}\, d\lambda,$$

and [7, p. 246]

$$(10) \qquad T(r,z,t) = \frac{bT_0}{2} \int_0^\infty J_1(\lambda b)\, J_0(\lambda r) \cdot \left[e^{\lambda z} \operatorname{Erfc}\left(\frac{z + 2a^2\lambda t}{2a\sqrt{t}}\right) + e^{-\lambda z} \operatorname{Erfc}\left(\frac{z - 2a^2\lambda t}{2a\sqrt{t}}\right) \right] d\lambda .$$

It remains to show that this formal solution is the actual solution of the boundary value problem (6). By proving T to be represented by an integral uniformly convergent in t and z for $t, z \geq 0$ and then interchanging the limiting processes as t goes to zero and as z goes to zero with integration, the initial and boundary conditions can be verified. To show the differential equation satisfied, the uniform convergence of integral representations of T together with the first and second time derivatives and the first, second, and third space derivatives for $t, z > 0$ can be shown, differentiations and integrations can then be interchanged [1, p. 484], and the solution T can be verified by direct substitution. The integral for T can be shown uniformly convergent with respect to z for $z \geq 0$ by the Abel Theorem: If $\int_0^\infty f(\lambda)\, d\lambda$ converges and if for every value of z for $z \geq 0$, the function $v(\lambda, z)$ is non-negative, bounded for all z, λ and never increasing with λ, then $\int_0^\infty f(\lambda)\, v(\lambda, z)\, d\lambda$ is uniformly convergent with respect to z for $z \geq 0$. Taking $f(\lambda) = J_1(\lambda b)\, J_0(\lambda r)$ and $v(\lambda, z) = e^{\lambda z}\, \mathrm{Erfc}\left(\frac{z + 2a^2\lambda t}{2a\sqrt{t}}\right) + e^{-\lambda z}\mathrm{Erfc}\left(\frac{z - 2a^2\lambda t}{2a\sqrt{t}}\right)$, one notes from (8) that $\int_0^\infty f(\lambda)\, d\lambda$ converges, that for $t > 0$ $v(\lambda, z)$ is positive, bounded, and tending to zero as λ increases if $z \neq 0$ or equal 2 if $z = 0$, and that for $t = 0$ $v(\lambda, z) = 0$. It remains yet to show $v(\lambda, z)$ is a monotone function in λ for $z > 0$ and $t > 0$. Writing $v(\lambda, z) = \psi(\lambda, z) + \psi(-\lambda, z)$ where $\psi(\lambda, z) = e^{\lambda z}\, \mathrm{Erfc}\left(\frac{z + 2a^2\lambda t}{2a\sqrt{t}}\right)$ and using the relation $\mathrm{Erfc}(\xi) \simeq \frac{e^{-\xi^2}}{\sqrt{\pi}\,\xi} \cdot \left[1 - \frac{1}{2\xi^2} + \frac{1 \cdot 3}{(2\xi^2)^2} - \frac{1 \cdot 3 \cdot 5}{(2\xi^2)^3} \pm \cdots\right]$ [16, p. 126] valid for large $\xi > 0$ yields

$$\frac{\partial \psi(\lambda, z)}{\partial \lambda} = e^{\lambda z}\left[a\sqrt{t}\,\frac{d}{d\xi}\mathrm{Erfc}(\xi) + z\,\mathrm{Erfc}(\xi)\right],$$

$$= e^{\lambda z}\left[-\frac{2a\sqrt{t}}{\sqrt{\pi}}\, e^{-\xi^2} + \frac{z e^{-\xi^2}}{\sqrt{\pi}}\left(\frac{1}{\xi} - \frac{1}{2\xi^3} \pm \cdots\right)\right],$$

$$= e^{-\left(\frac{z^2}{4a^2 t} + a^2\lambda^2 t\right)}\left[-\frac{2a\sqrt{t}}{\sqrt{\pi}} + \frac{z}{\sqrt{\pi}}\left(\frac{1}{\xi} - \frac{1}{2\xi^3} \pm \cdots\right)\right].$$

where $\xi = \frac{z + 2a^2\lambda t}{2a\sqrt{t}}$ which shows $\frac{\partial\psi(\lambda,z)}{\partial\lambda} < 0$ for all sufficiently large λ. And

$$\frac{\partial\psi(-\lambda,z)}{\partial\lambda} = e^{-\lambda z}\left[\frac{2a\sqrt{t}}{\sqrt{\pi}}\, e^{-\eta^2} - z\,\mathrm{Erfc}(\eta)\right],$$
$$= e^{-\lambda z}\left[\frac{2a\sqrt{t}}{\sqrt{\pi}}\, e^{-\eta^2} - z\big(1 - \mathrm{Erf}(\eta)\big)\right] < 0 \text{ for all}$$

large λ where $\eta = \frac{z - 2a^2\lambda t}{2a\sqrt{t}}$ because Erf $(\eta) < 0$ for $\eta < 0$. Therefore $v(\lambda, z)$ is a monotone decreasing function after a sufficiently large λ for $z > 0$ and $t > 0$. Hence the integral for T is uniformly convergent with respect to z for $z \geq 0$. Repetition of a similar argument with z replaced by t shows uniform convergence with respect to t for $t \geq 0$ in the region $z \geq 0$. The integral for T can be shown uniformly convergent with respect to r for $r \geq 0$ in the region $z > 0$ by the Dirichlet-Hardy theorem: If $\int_a^\infty f(\lambda, r)\, d\lambda$ is bounded for all $\lambda > a$ and for $r \geq 0$, and if $v(\lambda)$ is bounded, positive, non-increasing, and tends to zero as λ approaches infinity, then $\int_a^\infty f(\lambda,r)\, v(\lambda)\, d\lambda$ is uniformly convergent for $r \geq 0$. It will suffice to show $\int_a^\infty J_1(\lambda b)\, J_0(\lambda r) \cdot \left[e^{\lambda z}\,\mathrm{Erfc}\left(\frac{z + 2a^2\lambda t}{2a\sqrt{t}}\right) + e^{-\lambda z}\,\mathrm{Erfc}\left(\frac{z - 2a^2\lambda t}{2a\sqrt{t}}\right)\right] d\lambda$ uniformly convergent where a is an arbitrarily large positive number. Here $f(\lambda, r) = J_1(\lambda b)\, J_0(\lambda r)\lambda$ and $v(\lambda) = \frac{e^{\lambda z}}{\lambda}\,\mathrm{Erfc}\left(\frac{z + 2a^2\lambda t}{2a\sqrt{t}}\right) + \frac{e^{-\lambda z}}{\lambda}\,\mathrm{Erfc}\left(\frac{z - 2a^2\lambda t}{2a\sqrt{t}}\right)$. Use of the asymptotic expansions of the Bessel functions gives $J_1(\lambda b)\, J_0(\lambda r)\lambda = \frac{1}{\pi\sqrt{br}}\left[\sin\lambda(b-r) - \cos\lambda(b+r)\right] + O\left(\frac{1}{\lambda^2}\right)$ [16, p. 32]. Then $\int_a^x f(\lambda,r)\, d\lambda = \frac{1}{\pi\sqrt{br}}\left[-\frac{\cos\lambda(b-r)}{b-r} - \frac{\sin\lambda(b+r)}{b+r}\right]_a^x + O\left(\frac{1}{\lambda}\right)$ is bounded for all $x > a$. For all $z > 0$, $v(\lambda)$ is bounded, positive, non-increasing, and tending to zero with increasing λ by an argument similar to the one used previously in connection with the application of the Abel theorem. Therefore $\int_0^\infty J_1(\lambda b)\, J_0(\lambda r)$ $\cdot \left[e^{\lambda z} \operatorname{Erfc}\left(\frac{z + 2a^2\lambda t}{2a\sqrt{t}}\right) + e^{-\lambda z} \operatorname{Erfc}\left(\frac{z - 2a^2\lambda t}{2a\sqrt{t}}\right)\right] d\lambda$ is uniformly convergent in r for $r \geq 0$ in the region $z > 0$. With the inclusion of the fact $\int_0^\infty J_1(\lambda b)\, J_1(\lambda r) \frac{d\lambda}{\lambda}$ converges [16, p. 50], the other integrals may be shown uniformly convergent in manners analogous to those already demonstrated.

IV. The Quasi-static Solution.

The temperature distribution derived in section III causes stresses and displacements in the half-space. In this section a quasi-static condition is assumed and the solution for those stresses and displacements is obtained. The assumption of the quasi-static condition allows the neglection of the dynamic terms appearing in the displacement equations (3). Under this assumption with the cylindrical coordinates introduced as in section III, the displacement equations take the form

$$\nabla^2 u - \frac{u}{r^2} + \frac{1}{1-2\nu}\frac{\partial e}{\partial r} = \frac{2(1+\nu)}{1-2\nu}\frac{\partial(\alpha T)}{\partial r},$$

$$(11) \qquad v = 0,$$

$$\nabla^2 w + \frac{1}{1-2\nu}\frac{\partial e}{\partial z} = \frac{2(1+\nu)}{1-2\nu}\frac{\partial(\alpha T)}{\partial z},$$

where u = displacement in the radial direction, v = displacement in the φ direction, w = displacement in the z direction, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$, $e = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}$, and T is given by equation (10). Since the surface plane is free from external loading, the normal stress σ_{zz} and the shear stress σ_{rz} must vanish on that boundary. Using the assumption from the linear theory that the boundary conditions are applied to the undeformed body and writing these boundary conditions on the surface $z=0$ gives

$$(12) \qquad \begin{aligned} \sigma_{zz}\Big|_{z=0} &= 0 = \frac{2G}{1-2\nu}\left[(1-\nu)\frac{\partial w}{\partial z} + \nu\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) - (1+\nu)\alpha T\right]_{z=0}, \\ \sigma_{rz}\Big|_{z=0} &= 0 = G\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right]_{z=0}. \end{aligned}$$

Although the latter equalities expressing the stresses in terms of the displacements do not enter into the solution of the boundary value problem in that form, they are included for completeness. The differential equations (11) and the boundary conditions (12) comprise the boundary value problem to be solved in this section.

As a first step in the solution, a thermoelastic potential $\phi = \phi(r,z,t)$ is introduced in order to derive a particular solution and thus reduce the differential equations to homogeneous form. The thermoelastic potential ϕ defined through the relations $2Gu_i = \frac{\partial \phi}{\partial x_i}$ yields particular solutions to the differential equations (11) if ϕ satisfies the equation

$$(13) \qquad \nabla^2 \phi = \frac{2(1+\nu)G\alpha}{1-\nu} T .$$

Because of the simpler form of T in the Laplace subsidiary space, the equation (13) is transformed there with $L\{\phi(t)\} = \phi^*(p) = \int_0^\infty \phi(t) \; e^{-pt} \, dt.$

$$(14) \qquad \nabla^2 \phi^* = \frac{2(1+\nu)G\alpha}{1-\nu} T^*.$$

According to the heat equation, $\frac{\partial T}{\partial t} = a^2 \nabla^2 T$ and $pT^* = a^2 \nabla^2 T^*$. Substituting from this relation into (14) yields

$$\nabla^2 \phi^* = \nabla^2 \left[\frac{2(1+\nu)G\alpha a^2}{1-\nu} \; \frac{T^*}{p} \right] .$$

Then a particular solution of (14) is $\phi_1^* = \frac{2(1+\nu)G\alpha a^2}{1-\nu} \frac{T^*}{p}$. From use of Hooke's law and the defining relation for ϕ, the following equations relating the stresses and the thermoelastic potential are derived.

$$\bar{\sigma}^*_{rr} = \frac{\partial^2\phi^*}{\partial r^2} - \nabla^2\phi^*,$$

(15)
$$\bar{\sigma}^*_{\varphi\varphi} = \frac{1}{r}\frac{\partial\phi^*}{\partial r} - \nabla^2\phi^*,$$

$$\bar{\sigma}^*_{rz} = \frac{\partial^2\phi^*}{\partial r\,\partial z},$$

$$\bar{\sigma}^*_{zz} = \frac{\partial^2\phi^*}{\partial z^2} - \nabla^2\phi^*.$$

Substituting ϕ^*_1 into equations (15) gives the stresses in the form of integrals divergent on the surface $z = 0$. For this reason a harmonic function correspondent to the steady-state temperature distribution must be superposed to secure convergence. $\phi^*_2 = -\frac{2(1+\nu)G\alpha a^2\, bT_0}{1-\nu}\int_0^\infty J_1(\lambda b)\, J_0(\lambda r)\,\frac{e^{-\lambda z}}{p^2}\,d\lambda$ is such a function.† The solution for the thermoelastic potential in the subsidiary space is

(16) $$\phi^*(r,z,p) = C_1\int_0^\infty J_1(\lambda b)\, J_0(\lambda r)\,\frac{e^{-\gamma z} - e^{-\lambda z}}{p^2}\,d\lambda,$$

where $C_1 = \frac{2(1+\nu)G\alpha a^2\, bT_0}{1-\nu}$. The stresses in the subsidiary space due to this potential are given by equations (15).

† This selection of ϕ^*_2 is motivated by solution of the steady-state problem not included separately here since it is obtained subsequently as a special case of the quasi-static solution by letting t approach infinity.

$$\overline{\sigma}_{rr}^* = C_1 \int_0^\infty \left[J_1(\lambda b)\ \frac{J_1(\lambda r)}{\lambda r}\ \frac{e^{-\gamma z} - e^{-\lambda z}}{p^2} \lambda^2 - J_1(\lambda b)\ J_0(\lambda r)\ \frac{\gamma^2 e^{-\gamma z} - \lambda^2 e^{-\lambda z}}{p^2} \right] d\lambda,$$

$$\overline{\sigma}_{\varphi\varphi}^* = -C_1 \int_0^\infty \left[J_1(\lambda b)\ \frac{J_1(\lambda r)}{\lambda r}\ \frac{e^{-\gamma z} - e^{-\lambda z}}{p^2} \lambda^2 + J_1(\lambda b)\ J_0(\lambda r)\ \frac{e^{-\gamma z}}{a^2 p} \right] d\lambda,$$

(17)

$$\overline{\sigma}_{rz}^* = C_1 \int_0^\infty J_1(\lambda b)\ J_1(\lambda r)\ \frac{\gamma e^{-\gamma z} - \lambda e^{-\lambda z}}{p^2} \lambda \ d\lambda,$$

$$\overline{\sigma}_{zz}^* = C_1 \int_0^\infty J_1(\lambda b)\ J_0(\lambda r)\ \frac{e^{-\gamma z} - e^{-\lambda z}}{p^2} \lambda^2 \ d\lambda.$$

The physical problem demands that the stresses be continuous in the half-space and that the normal and shear stresses on the plane $z = 0$ vanish. That these $\overline{\sigma}^*$ and consequently the $\overline{\sigma}$ are continuous in the half-space can be shown by proving the integrals uniformly convergent in the same manner as that used in the latter part of section III. Having shown the uniform convergence with respect to z, the limiting process as z goes to zero can be interchanged with integration to note $\overline{\sigma}_{zz}^*\Big]_{z=0} = 0$ and $\overline{\sigma}_{rz}^*\Big]_{z=0} = C_1 \int_0^\infty J_1(\lambda b)\ J_1(\lambda r)\ \frac{\gamma - \lambda}{p^2} \lambda \ d\lambda$. As an example of the method for showing these integrals uniformly convergent and also to show $\overline{\sigma}_{rz}^*$ exists on the boundary $z = 0$, $\overline{\sigma}_{rz}^*$ is proved a continuous function of r and z in the half-space and on the boundary. This proposition includes $\overline{\sigma}_{rz}^*$ having a finite value on the bounding surface $z = 0$. The Dirichlet-Hardy theorem as stated in section III with $f(\lambda, r) = J_1(\lambda b) \quad J_1(\lambda r)\lambda$ and $v(\lambda) = \gamma e^{-\gamma z} - \lambda e^{-\lambda z}$ is applicable; and as in section III, since the part of the integral from 0 to a need not be considered, the integral from a to ∞ where a is an arbitrarily large positive number is used. The asymptotic expansion of $f(\lambda,r)$ is $\frac{1}{\pi\sqrt{br}}\left[\cos\lambda(b-r) - \sin\lambda(b+r)\right] + O(\frac{1}{\lambda^2})$ [16, p. 32] and the integral $\int_a^x f(\lambda,r)\,d\lambda = \frac{1}{\pi\sqrt{br}}\left[\frac{\sin\lambda(b-r)}{b-r} + \frac{\cos\lambda(b+r)}{b+r}\right]_a^x + O(\frac{1}{\lambda})$ is bounded for all $x > a$. $v(\lambda)$ is bounded, monotone, and tends to zero with increasing λ for $z \geq 0$. Therefore the integral representing $\overline{\sigma}_{rz}^*$ is uniformly convergent in r for $r \geq 0$ in the region $z \geq 0$ and $\overline{\sigma}_{rz}^*$ is continuous in r for $r \geq 0$ and $z \geq 0$. The Abel test as stated in section III with $f(\lambda) = \frac{J_1(\lambda b)\, J_1(\lambda r)}{\lambda}$ and $v(\lambda,z) = \lambda^2(e^{-\gamma z} - e^{-\lambda z})$ shows uniform convergence with respect to z for $z \geq 0$. Therefore $\overline{\sigma}_{rz}^*$ is a continuous function of z for $z \geq 0$ having the value $C_1 \int_0^\infty J_1(\lambda b)\, J_1(\lambda r)\, \frac{\gamma-\lambda}{p^2}\,\lambda\, d\lambda$ on the surface $z = 0$. Since the normal and shear stresses in the original space and thus the same stresses in the subsidiary space must vanish, the subsidiary shear stress derived from the thermoelastic potential must be adjusted to zero by the superposition of stresses obtained from a Love's function L. Because of the axial symmetry of the problem, Love's function gives a general solution of the homogeneous elasticity equations and therefore such a function is known to exist. The boundary value problem in Love's function $L = L(r,z,t)$ can be written in the subsidiary space in $L^* = L^*(r,z,p)$ as

$$\nabla^4 L^* = 0,$$

(18) $$\overline{\overline{\sigma}}{}^{*\prime}_{zz}\Big]_{z=0} = 0 = \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left[(2-\nu)\ \nabla^2 L^* - \frac{\partial^2 L^*}{\partial z^2}\right]_{z=0},$$

$$\overline{\overline{\sigma}}{}^{*\prime}_{rz}\Big]_{z=0} = -\frac{C_1(\gamma-\lambda)\lambda\ J_1(\lambda b)}{p^2} J_1(\lambda r) = \frac{2G}{1-2\nu}\frac{\partial}{\partial r}\left[(1-\nu)\ \nabla^2 L^* - \frac{\partial^2 L^*}{\partial z^2}\right]_{z=0},$$

where $\overline{\overline{\sigma}}{}^{*} = \int_0^\infty \overline{\overline{\sigma}}{}^{*\prime}\, d\lambda$. This function in the subsidiary space can be obtained by combination of two functions, $L_1^* = z\ J_0(\lambda r)\ e^{-\lambda z}$ and $L_2^* = J_0(\lambda r)\ e^{-\lambda z}$.

(19) $L^*(r,z,p) = A(p,\lambda)\ L_1^*(r,z,p) + B(p,\lambda)\ L_2^*(r,z,p)$,

where $A(p,\lambda)$ and $B(p,\lambda)$ are to be determined such that the boundary conditions are satisfied. Substitution of L^* into the boundary conditions gives the following two equations for the two unknowns A and B.

(20)
$$0 = \frac{2G}{1-2\nu}\lambda^2\left[(1-2\nu)\ A + \lambda B\right]\ J_0(\lambda r),$$

$$-\frac{C_1(\gamma-\lambda)\lambda\ J_1(\lambda b)}{p^2} J_1(\lambda r) = \frac{2G}{1-2\nu}\lambda^2\left[-2\nu A + \lambda B\right]\ J_1(\lambda r).$$

From equations (20)

(21)
$$A(p,\lambda) = \frac{(1+\nu)(1-2\nu)\alpha\, a^2 b T_0}{1-\nu}\,\frac{(\gamma-\lambda)\ J_1(\lambda b)}{\lambda p^2},$$

$$B(p,\lambda) = -\frac{(1-2\nu)}{\lambda} A(p,\lambda).$$

The stresses in the subsidiary space due to Love's function are given by [19, p. 73].

$$
\overline{\overline{\sigma}}{}^*_{rr} = \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left[\nu\nabla^2 L^* - \frac{\partial^2 L^*}{\partial r^2}\right],
$$

(22)
$$
\overline{\overline{\sigma}}{}^*_{\varphi\varphi} = \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left[\nu\nabla^2 L^* - \frac{1}{r}\,\frac{\partial L^*}{\partial r}\right],
$$

$$
\overline{\overline{\sigma}}{}^*_{rz} = \frac{2G}{1-2\nu}\frac{\partial}{\partial r}\left[\,(1-\nu)\,\nabla^2 L^* - \frac{\partial^2 L^*}{\partial z^2}\right],
$$

$$
\overline{\overline{\sigma}}{}^*_{zz} = \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left[\,(2-\nu)\,\nabla^2 L^* - \frac{\partial^2 L^*}{\partial z^2}\right].
$$

Use of equations (19), (21), and (22) and integration over λ yields the stresses necessary to adjust the normal and shear stresses from the thermoelastic potential to zero on the unloaded plane $z = 0$.

$$
\overline{\overline{\sigma}}{}^*_{rr} = \frac{C_1}{p^2}\int_0^\infty \Big[-(\gamma-\lambda)(2-\lambda z-2\nu)\; J_1(\lambda b)\,\frac{J_1(\lambda r)}{\lambda r}
+ (\gamma-\lambda)(2-\lambda z)\; J_1(\lambda b)\; J_0(\lambda r)\Big]\, e^{-\lambda z}\lambda\, d\lambda,
$$

$$
\overline{\overline{\sigma}}{}^*_{\varphi\varphi} = \frac{C_1}{p^2}\int_0^\infty \Big[(\gamma-\lambda)(2-\lambda z-2\nu)\; J_1(\lambda b)\,\frac{J_1(\lambda r)}{\lambda r}
+ 2\nu(\gamma-\lambda)\; J_1(\lambda b)\; J_0(\lambda r)\Big]\, e^{-\lambda z}\lambda\, d\lambda,
$$

(23)

$$
\overline{\overline{\sigma}}{}^*_{rz} = \frac{C_1}{p^2}\int_0^\infty (\gamma-\lambda)(\lambda z-1)\; J_1(\lambda b)\; J_1(\lambda r)\; e^{-\lambda z}\lambda\, d\lambda,
$$

$$
\overline{\overline{\sigma}}{}^*_{zz} = \frac{C_1 z}{p^2}\int_0^\infty (\gamma-\lambda)\; J_1(\lambda b)\; J_0(\lambda r)\; e^{-\lambda z}\lambda^2\, d\lambda.
$$

The final stresses in the subsidiary space are obtained by addition of the stresses derived from the potential and those from Love's function. These final stresses in the subsidiary space are

$$
\begin{aligned}
\sigma_{rr}^{*} &= C_1 \int_0^{\infty} \Big\{ J_1(\lambda b)\, \frac{J_1(\lambda r)}{\lambda r} \Big[\frac{\lambda^2 e^{-\gamma z}}{p^2} + \big\langle \lambda(\lambda z + 2\nu - 2)(\gamma - \lambda) - \lambda^2 \big\rangle \frac{e^{-\lambda z}}{p^2} \Big] \\
&\qquad + J_1(\lambda b)\, J_0(\lambda r) \Big[-\frac{\gamma^2 e^{-\gamma z}}{p^2} + \big\langle \lambda(2 - \lambda z)(\gamma - \lambda) + \lambda^2 \big\rangle \frac{e^{-\lambda z}}{p^2} \Big] \Big\} \, d\lambda, \\
\sigma_{\varphi\varphi}^{*} &= C_1 \int_0^{\infty} \Big\{ J_1(\lambda b)\, \frac{J_1(\lambda r)}{\lambda r} \Big[\frac{-\lambda^2 e^{-\gamma z}}{p^2} + \big\langle \lambda(2 - \lambda z - 2\nu)(\gamma - \lambda) + \lambda^2 \big\rangle \frac{e^{-\lambda z}}{p^2} \Big] \\
&\qquad + J_1(\lambda b)\, J_0(\lambda r) \Big[-\frac{e^{-\gamma z}}{a^2 p} + 2\nu\lambda(\gamma - \lambda) \frac{e^{-\lambda z}}{p^2} \Big] \Big\} \, d\lambda, \qquad (24) \\
\sigma_{rz}^{*} &= C_1 \int_0^{\infty} J_1(\lambda b)\, J_1(\lambda r) \Big[\frac{\lambda\gamma e^{-\gamma z}}{p^2} + \big\langle \lambda(\lambda z - 1)(\gamma - \lambda) - \lambda^2 \big\rangle \frac{e^{-\lambda z}}{p^2} \Big] \, d\lambda, \\
\sigma_{zz}^{*} &= C_1 \int_0^{\infty} J_1(\lambda b)\, J_0(\lambda r) \Big[\frac{\lambda^2 e^{-\gamma z}}{p^2} + \big\langle \lambda^2 z(\gamma - \lambda) - \lambda^2 \big\rangle \frac{e^{-\lambda z}}{p^2} \Big] \, d\lambda.
\end{aligned}
$$

In order to complete the stress determination, the final stresses in the subsidiary space must be transformed back into the original space. This is done by interchanging the inverse Laplace operator with integration. In order to justify this interchange of limiting processes, the integrals can be shown uniformly convergent in p by the Abel theorem in a manner analogous to that used in the latter part of section III. This inverse transformation involves terms of five types. These transformations are listed below and derived subsequently.

$$
\begin{aligned}
&(a)\quad L^{-1}\Big\{\frac{1}{p^2}\Big\} = t, \\
&(b)\quad L^{-1}\Big\{\frac{e^{-\gamma z}}{p}\Big\} = L^{-1}\Big\{\frac{e^{-\frac{\sqrt{p+a^2\lambda^2}}{a} z}}{p}\Big\} = \tfrac{1}{2}\Big[e^{\lambda z}\,\mathrm{Erfc}\Big(\frac{z+2a^2\lambda t}{2a\sqrt{t}}\Big) \\
&\qquad\qquad + e^{-\lambda z}\,\mathrm{Erfc}\Big(\frac{z-2a^2\lambda t}{2a\sqrt{t}}\Big)\Big], \\
&(c)\quad L^{-1}\Big\{\frac{\gamma}{p^2}\Big\} = \frac{1}{a} L^{-1}\Big\{\frac{\sqrt{p+a^2\lambda^2}}{p^2}\Big\} = \frac{1+2a^2\lambda^2 t}{2a^2\lambda}\,\mathrm{Erf}(a\lambda\sqrt{t}) + \frac{1}{a}\sqrt{\frac{t}{\pi}}\; e^{-a^2\lambda^2 t}, \\
(25)\quad &(d)\quad L^{-1}\Big\{\frac{e^{-\gamma z}}{p^2}\Big\} = L^{-1}\Big\{\frac{e^{-\frac{\sqrt{p+a^2\lambda^2}}{a} z}}{p^2}\Big\}, \\
&\qquad = \frac{z+2a^2\lambda t}{4a^2\lambda}\, e^{\lambda z}\,\mathrm{Erfc}\Big(\frac{z+2a^2\lambda t}{2a\sqrt{t}}\Big) - \frac{z-2a^2\lambda t}{4a^2\lambda}\, e^{-\lambda z} \\
&\qquad\quad \cdot \mathrm{Erfc}\Big(\frac{z-2a^2\lambda t}{2a\sqrt{t}}\Big),
\end{aligned}
$$

$$\text{(e)}\quad L^{-1}\left\{\frac{\gamma e^{-\gamma z}}{p^2}\right\} = \frac{1}{a} L^{-1}\left\{\frac{\sqrt{p+a^2\lambda^2}\; e^{-\frac{\sqrt{p+a^2\lambda^2}\,z}{a}}}{p^2}\right\} = -\frac{1+\lambda z+2a^2\lambda^2 t}{4a^2\lambda}\, e^{\lambda z} \cdot \operatorname{Erfc}\left(\frac{z+2a^2\lambda t}{2a\sqrt{t}}\right) + \frac{1-\lambda z+2a^2\lambda^2 t}{4a^2\lambda}\, e^{-\lambda z} \cdot \operatorname{Erfc}\left(\frac{z-2a^2\lambda t}{2a\sqrt{t}}\right) + \frac{1}{a}\sqrt{\frac{t}{\pi}}\; e^{-\left(a^2\lambda^2 t + \frac{z^2}{4a^2 t}\right)}.$$

Transformation (a) is an elementary transformation tabulated directly. Transformation (b) can be obtained from the tabulated transformation for $2(p+\beta)^{-1}\ e^{-\alpha^{1/2}p^{1/2}}$ [7, p. 246] by means of the shifting theorem: $f^*(p-a)$ in the subsidiary space corresponds to $e^{at}f(t)$ in the original space. Transformation (c) can be derived by use of the tabulated transformations for $\frac{1}{p}$ and $\frac{\sqrt{p+a^2\lambda^2}}{p}$ [7, p. 235] and the convolution theorem: $f_1^*(p)\cdot f_2^*(p)$ in the subsidiary space corresponds to $\int_0^t f_1(t-\tau)\, f_2(\tau)\, d\tau$ in the original space. With the substitution of $f_1(t) = L^{-1}\left\{\frac{1}{p}\right\} = 1$ and $f_2(t) = L^{-1}\left\{\frac{\sqrt{p+a^2\lambda^2}}{p}\right\} = \frac{e^{-a^2\lambda^2 t}}{\sqrt{\pi t}} + a\lambda \operatorname{Erf}(a\lambda\sqrt{t})$, $L^{-1}\left\{\frac{\sqrt{p+a^2\lambda^2}}{p^2}\right\} = \int_0^t f_2(\tau)\, d\tau = \frac{1}{\sqrt{\pi}}\int_0^t \frac{e^{-a^2\lambda^2\tau}}{\sqrt{\tau}}\, d\tau + \frac{2a\lambda}{\sqrt{\pi}}\int_0^t d\tau \int_0^{a\lambda\sqrt{\tau}} e^{-\xi^2}\, d\xi$. The first of these integrals can be evaluated by a change of variable $\xi = \sqrt{\tau}$; the second, by the Dirichlet formula: $\int_a^b dx \int_a^x f(x,y)\, dy = \int_a^b dy \int_y^b f(x,y)\, dx$. In the evaluation of the second integral,

$$\int_0^t d\tau \int_0^{a\lambda\sqrt{\tau}} e^{-\xi^2}\, d\xi = \int_0^{a\lambda\sqrt{t}} \frac{2\eta\, d\eta}{a^2\lambda^2} \int_0^{\eta} e^{-\xi^2}\, d\xi = \frac{2}{a^2\lambda^2}\int_0^{a\lambda\sqrt{t}} d\eta \int_0^{\eta} \eta e^{-\xi^2}\, d\xi$$

$$= \frac{2}{a^2\lambda^2}\int_0^{a\lambda\sqrt{t}} d\xi \int_{\xi}^{a\lambda\sqrt{t}} \eta e^{-\xi^2}\, d\eta = \frac{1}{a^2\lambda^2}\int_0^{a\lambda\sqrt{t}} \eta^2\Big]_{\eta=\xi}^{a\lambda\sqrt{t}} e^{-\xi^2}\, d\xi$$

$$= \frac{\sqrt{\pi}\,t}{2}\operatorname{Erf}(a\lambda\sqrt{t}) - \frac{1}{a^2\lambda^2}\int_0^{a\lambda\sqrt{t}} \xi^2 e^{-\xi^2}\, d\xi.$$

The latter integral, $\int_0^{a\lambda\sqrt{t}} \xi^2 e^{-\xi^2}\, d\xi$, can be evaluated by inserting a parameter z to define $F(z) = \int_0^{a\lambda\sqrt{t}} \xi^2 e^{-z\xi^2}\, d\xi$, observing $F(z) = f'(z)$ where

$f(z) = -\int_0^{a\lambda\sqrt{t}} e^{-z\xi^2}\, d\xi = -\frac{\sqrt{\pi}}{2\sqrt{z}} \operatorname{Erf}(a\lambda\sqrt{zt})$, and noting F(1) to be the integral in question. Combination and simplification yields transformation (c). Use of the shifting theorem and changes of variable $\alpha = a^2\lambda^2$ and $x = \frac{z}{a}$ reduces the evaluation of transformation (d) to evaluation of $L^{-1}\left\{\frac{e^{-\sqrt{p}\,x}}{(p-\alpha)^2}\right\}$. By writing $f^*(p)$

$= \frac{e^{-\sqrt{p}\,x}}{(p-\alpha)^2} = \frac{e^{-\sqrt{p}\,x}}{p^2} + 2\alpha\frac{1}{p}\frac{e^{-\sqrt{p}\,x}}{(p-\alpha)^2} - \alpha^2\frac{1}{p^2}\frac{e^{-\sqrt{p}\,x}}{(p-\alpha)^2}$, employing the tabulated transformation for $\frac{e^{-\sqrt{p}\,x}}{p^2}$ [7, p. 245], and using the operational formula: $\frac{1}{p^n} f^*(p)$ in the subsidiary space corresponds to $\left(\int_0^t d\tau\right)^n f(\tau)$ in the original space, the evaluation of the simplified transformation (d) is reduced to solution of the following differential equation: $f''(t) - 2\alpha f'(t) + \alpha^2 f(t)$

$= \frac{x}{2\sqrt{\pi}}\,\frac{e^{-x^2/4t}}{t^{3/2}}$. The general solution of this differential equation is the sum of the homogeneous solution, $f(t) = (A+Bt)\,e^{\alpha t}$ where A and B are thus far arbitrary, and the particular solution, $f(t) = (C+Dt)\,e^{\alpha t}$ where $C = \frac{x}{4\sqrt{\alpha}}\Big[2\sinh\sqrt{\alpha}\,x - e^{-\sqrt{\alpha}\,x}\operatorname{Erf}\left(\sqrt{\alpha t} - \frac{x}{2\sqrt{t}}\right)$

$- e^{\sqrt{\alpha}\,x}\operatorname{Erf}\left(\sqrt{\alpha t} + \frac{x}{2\sqrt{t}}\right)\Big]$ and $D = \cosh\sqrt{\alpha}\,x + \frac{1}{2}\Big[e^{-\sqrt{\alpha}\,x}$

$\cdot\operatorname{Erf}\left(\sqrt{\alpha t} - \frac{x}{2\sqrt{t}}\right) - e^{\sqrt{\alpha}\,x}\operatorname{Erf}\left(\sqrt{\alpha t} + \frac{x}{2\sqrt{t}}\right)\Big]$. Instrumental in the derivation of the particular solution by the method of variation of parameters is Horenstein's evaluation [12] of two integrals:

$$I_1 = \int_0^t x^{-3/2}\exp\left(-\frac{a^2}{x} - b^2 x\right) dx = \frac{\sqrt{\pi}}{a}\cosh 2ab + \frac{\sqrt{\pi}}{2a}$$
$$\cdot\left[e^{-2ab}\operatorname{Erf}\left(b\sqrt{t} - \frac{a}{\sqrt{t}}\right) - e^{2ab}\operatorname{Erf}\left(b\sqrt{t} + \frac{a}{\sqrt{t}}\right)\right],$$

$I_2 = \int_0^t x^{-1/2}\exp\left(-\frac{a^2}{x} - b^2 x\right) dx = -\frac{1}{2b}\frac{dI_1}{db}$, where I_1 is given in the preceding equation. To find the initial conditions f(0) and

$f'(0)$ in order to evaluate A and B, the theorem [5, vol. 2, p. 45] --if $\lim_{t \to 0} f(t)$ exists then $\lim_{p \to \infty} p\, f^*(p) = \lim_{t \to 0} f(t)$--is applicable. By employing the fact that the solution $f(t) = (A+Bt)\, e^{\alpha t} + (C+Dt)\, e^{\alpha t}$ where A and B are arbitrary and where C and D are known functions bounded as t approaches zero, one can note that $\lim_{t \to 0} f(t)$ exists. This theorem yields the result $f(0) = f'(0) = 0$. Use of these initial conditions shows $A = B = 0$. The evaluation of $L^{-1}\left\{ \frac{e^{-\sqrt{p}\, x}}{(p-\alpha)^2} \right\}$ is thus completed. A second application of the shifting theorem yields the transformation (d). The last of the transformations, transformation (e), is obtained through differentiation of transformation (d) with respect to $\frac{z}{a}$.

The inverse Laplace transformation of the stresses in the subsidiary space from equations (24) by means of equations (25) completes the stress determination in the quasi-static case.

$$
\begin{aligned}
\sigma_{rr} = C_1 \int_0^\infty \Big\{ & J_1(\lambda b) \frac{J_1(\lambda r)}{\lambda r} \Big[\lambda^2 L^{-1}\Big\{ \frac{e^{-\gamma z}}{p^2} \Big\} + \lambda(\lambda z + 2\nu - 2) e^{-\lambda z} L^{-1}\Big\{ \frac{\gamma}{p^2} \Big\} \\
& - \lambda^2 (\lambda z + 2\nu - 1) e^{-\lambda z} L^{-1}\Big\{ \frac{1}{p^2} \Big\} \Big] + J_1(\lambda b)\, J_0(\lambda r) \Big[-\frac{1}{a^2} L^{-1}\Big\{ \frac{e^{-\gamma z}}{p} \Big\} \\
& - \lambda^2 L^{-1}\Big\{ \frac{e^{-\gamma z}}{p^2} \Big\} - \lambda(\lambda z - 2) e^{-\lambda z} L^{-1}\Big\{ \frac{\gamma}{p^2} \Big\} \\
& + \lambda^2 (\lambda z - 1) e^{-\lambda z} L^{-1}\Big\{ \frac{1}{p^2} \Big\} \Big] \Big\}\, d\lambda,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\sigma_{\varphi\varphi} = C_1 \int_0^\infty \Big\{ & J_1(\lambda b) \frac{J_1(\lambda r)}{\lambda r} \Big[-\lambda^2 L^{-1}\Big\{ \frac{e^{-\gamma z}}{p^2} \Big\} - \lambda(\lambda z + 2\nu - 2) e^{-\lambda z} L^{-1}\Big\{ \frac{\gamma}{p^2} \Big\} \\
& + \lambda^2 (\lambda z + 2\nu - 1) e^{-\lambda z} L^{-1}\Big\{ \frac{1}{p^2} \Big\} \Big] + J_1(\lambda b)\, J_0(\lambda r) \Big[-\frac{1}{a^2} L^{-1}\Big\{ \frac{e^{-\gamma z}}{p} \Big\} \\
& + 2\nu\lambda e^{-\lambda z} L^{-1}\Big\{ \frac{\gamma}{p^2} \Big\} - 2\nu\lambda^2 e^{-\lambda z} L^{-1}\Big\{ \frac{1}{p^2} \Big\} \Big] \Big\}\, d\lambda,
\end{aligned}
$$

$$\sigma_{rz}=C_1\int_0^\infty J_1(\lambda b)\ J_1(\lambda r)\Big[\lambda L^{-1}\Big\{\frac{\gamma e^{-\gamma z}}{p^2}\Big\}+\lambda(\lambda z-1)e^{-\lambda z}L^{-1}\Big\{\frac{\gamma}{p^2}\Big\}$$

$$-\lambda^3 z\, e^{-\lambda z}L^{-1}\Big\{\frac{1}{p^2}\Big\}\Big]\ d\lambda,$$

$$\sigma_{zz}=C_1\int_0^\infty J_1(\lambda b)\ J_0(\lambda r)\Big[\lambda^2 L^{-1}\Big\{\frac{e^{-\gamma z}}{p^2}\Big\}+\lambda^2 z\, e^{-\lambda z}L^{-1}\Big\{\frac{\gamma}{p^2}\Big\}$$

$$-\lambda^2(\lambda z+1)e^{-\lambda z}L^{-1}\Big\{\frac{1}{p^2}\Big\}\Big]\ d\lambda.$$

That the shear and normal stress on the plane $z=0$ vanish can be verified by substitution $z=0$ in equations (26). By use of equations (8), (25), and (26), one can observe that for t approaching zero on the surface $z=0$

$$\sigma_{rr}=\sigma_{\varphi\varphi}=-\frac{2(1+\nu)G\alpha T_0}{1-\nu}\quad \text{for } r<b,$$

$$=-\frac{(1+\nu)G\alpha T_0}{1-\nu}\quad \text{for } r=b,$$

(27)

$$=0\quad \text{for } r>b,$$

$$\sigma_{rz}=\sigma_{zz}=0\ .$$

This result is of interest because it was conjectured earlier on the basis of classical work on the quenching of spheres and cylinders. The stress solution of the corresponding steady-state problem is obtained by letting t approach infinity in the quasi-static stress solution given by equations (26). After a justifiable interchange of limiting processes, the steady-state solution becomes

(28)

$$\sigma_{\varphi\varphi}=-2(1+\nu)G\alpha bT_0\int_0^\infty\Big[J_1(\lambda b)\ J_0(\lambda r)-J_1(\lambda b)\ \frac{J_1(\lambda r)}{\lambda r}\Big]e^{-\lambda z}\ d\lambda,$$

$$=-2(1+\nu)G\alpha bT_0\int_0^\infty J_1(\lambda b)\ \frac{d\ J_0(\lambda r)}{dr}\ e^{-\lambda z}\ \frac{d\lambda}{\lambda},$$

$$\sigma_{rr} = -2(1+\nu)G\alpha bT_0 \int_0^\infty J_1(\lambda b)\,\frac{J_1(\lambda r)}{\lambda r}\,e^{-\lambda z}\,d\lambda,$$

$$\sigma_{rz} = \sigma_{zz} = 0\ .$$

The displacements are computed as the sum of displacements due to the thermoelastic potential plus those due to Love's function. As differentiation under the integral sign can be permitted in this case, the displacements in the subsidiary space due to the potential are

$$\bar{u}^* = \frac{1}{2G}\frac{\partial \phi^*}{\partial r} = -\frac{C_1}{2G}\int_0^\infty J_1(\lambda b)\,J_1(\lambda r)\,\frac{e^{-\gamma z} - e^{-\lambda z}}{p^2}\,\lambda\,d\lambda,$$

(29)

$$\bar{w}^* = \frac{1}{2G}\frac{\partial \phi^*}{\partial z} = -\frac{C_1}{2G}\int_0^\infty J_1(\lambda b)\,J_0(\lambda r)\,\frac{\gamma e^{-\gamma z} - \lambda e^{-\lambda z}}{p^2}\,d\lambda,$$

which, after justification of the interchange of the inverse Laplace operator and integration, yield in the original space

$$\bar{u} = -\frac{C_1}{2G}\int_0^\infty J_1(\lambda b)\,J_1(\lambda r)\left[\,L^{-1}\left\{\frac{e^{-\gamma z}}{p^2}\right\} - e^{-\lambda z}L^{-1}\left\{\frac{1}{p^2}\right\}\right]\lambda\,d\lambda,$$

(30)

$$\bar{w} = -\frac{C_1}{2G}\int_0^\infty J_1(\lambda b)\,J_0(\lambda r)\left[\,L^{-1}\left\{\frac{\gamma e^{-\gamma z}}{p^2}\right\} - \lambda\,e^{-\lambda z}L^{-1}\left\{\frac{1}{p^2}\right\}\right]d\lambda.$$

In the subsidiary space the displacements due to Love's function [19, p. 73] are

$$\bar{\bar{u}}^* = -\frac{1}{1-2\nu}\frac{\partial^2 L^*}{\partial r\,\partial z} = \frac{C_1}{2G}\int_0^\infty J_1(\lambda b)\,J_1(\lambda r)\,(2-2\nu-\lambda z)\,\frac{(\gamma-\lambda)e^{-\lambda z}}{p^2}\,d\lambda,$$

(31)

$$\bar{\bar{w}}^* = \frac{1}{1-2\nu}\left[\,2(1-\nu)\,\nabla^2 L^* - \frac{\partial^2 L^*}{\partial z^2}\right],$$

$$= \frac{C_1}{2G}\int_0^\infty J_1(\lambda b)\,J_0(\lambda r)\,(2\nu-1-\lambda z)\,\frac{(\gamma-\lambda)e^{-\lambda z}}{p^2}\,d\lambda,$$

which, after valid interchange of limiting processes, yield in the original space

$$\overline{\overline{u}} = \frac{C_1}{2G}\int_0^\infty J_1(\lambda b)\ J_1(\lambda r)\ (2-2\nu-\lambda z)e^{-\lambda z}\left(L^{-1}\left\{\frac{\gamma}{p^2}\right\} - \lambda L^{-1}\left\{\frac{1}{p^2}\right\}\right)\ d\lambda,$$

(32)

$$\overline{\overline{w}} = -\frac{C_1}{2G}\int_0^\infty J_1(\lambda b)\ J_0(\lambda r)\ (1-2\nu+\lambda z)e^{-\lambda z}\left(L^{-1}\left\{\frac{\gamma}{p^2}\right\} - \lambda L^{-1}\left\{\frac{1}{p^2}\right\}\right)\ d\lambda.$$

Combination of results (30) and (32) determines the final displacements in the quasi-static case.

$$u = \overline{u} + \overline{\overline{u}} = \frac{(1+\nu)\alpha a^2 bT_0}{1-\nu}\int_0^\infty J_1(\lambda b)\ J_1(\lambda r)\left[(2-2\nu-\lambda z)e^{-\lambda z}L^{-1}\left\{\frac{\gamma}{p^2}\right\}\right.$$

$$\left. -\lambda L^{-1}\left\{\frac{e^{-\gamma z}}{p^2}\right\} - \lambda(1-2\nu-\lambda z)e^{-\lambda z}L^{-1}\left\{\frac{1}{p^2}\right\}\right] d\lambda,$$

(33)

$$w = \overline{w} + \overline{\overline{w}} = \frac{(1+\nu)\alpha a^2 bT_0}{1-\nu}\int_0^\infty J_1(\lambda b)\ J_0(\lambda r)\left[(-1+2\nu-\lambda z)e^{-\lambda z}L^{-1}\left\{\frac{\gamma}{p^2}\right\}\right.$$

$$\left. - L^{-1}\left\{\frac{\gamma e^{-\gamma z}}{p^2}\right\} + \lambda(2-2\nu+\lambda z)e^{-\lambda z}L^{-1}\left\{\frac{1}{p^2}\right\}\right] d\lambda.$$

The displacements in the corresponding steady-state problem obtained from equations (33) by letting t approach infinity are

$$u = (1+\nu)\alpha bT_0\int_0^\infty J_1(\lambda b)\ J_1(\lambda r)\ e^{-\lambda z}\frac{d\lambda}{\lambda},$$

(34)

$$w = -(1+\nu)\alpha bT_0\int_0^\infty J_1(\lambda b)\ J_0(\lambda r)\ e^{-\lambda z}\frac{d\lambda}{\lambda}.$$

On the axis of symmetry and on the surface with $\xi = \frac{z}{b}$, $\eta = \frac{r}{b}$, and $x = \lambda z$, for $r=0$ the steady-state stresses and displacements of equations (28) and (34) can be written [16, p. 47]

$$\sigma_{rr} = C_2 \frac{\sqrt{\xi^2+1} - \xi}{2\sqrt{\xi^2+1}},$$

(35)
$$\sigma_{\varphi\varphi} = C_2 \frac{\sqrt{\xi^2+1} - \xi}{2\sqrt{\xi^2+1}},$$

$$\sigma_{rz} = \sigma_{zz} = 0,$$

$$u = 0,$$

$$w = C_3(\xi - \sqrt{\xi^2+1}),$$

and for $z=0$ [16, pp. 49-50],

$$\sigma_{rr} = C_2 \begin{cases} \frac{1}{2} & \text{for } \eta \leq 1, \\ \frac{1}{2\eta^2} & \text{for } \eta > 1, \end{cases}$$

$$\sigma_{\varphi\varphi} = C_2 \begin{cases} \frac{1}{2} & \text{for } \eta < 1, \\ 0 & \text{for } \eta = 1, \\ -\frac{1}{2\eta^2} & \text{for } \eta > 1, \end{cases}$$

(36)
$$\sigma_{rz} = \sigma_{zz} = 0,$$

$$u = C_3 \begin{cases} \frac{\eta}{2} & \text{for } \eta \leq 1, \\ \frac{1}{2\eta} & \text{for } \eta > 1, \end{cases}$$

$$w = -C_3 \begin{cases} {}_2F_1(\frac{1}{2}, -\frac{1}{2}; 1; \eta^2) & \text{for } \eta < 1, \\ \frac{2}{\pi} & \text{for } \eta = 1, \\ \frac{1}{2\eta}\, {}_2F_1(\frac{1}{2}, \frac{1}{2}; 2; \frac{1}{\eta^2}) & \text{for } \eta > 1, \end{cases}$$

where $C_2 = -2(1+\nu)G\alpha T_o$, $C_3 = (1+\nu)\alpha bT_o$, ${}_2F_1(a,b;c:z)$

$$= 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots\cdot$$

V. The Dynamic Effects at the First Moment.

The temperature distribution of section III causes a stress wave to be emitted through the half-space. At the first moment such stress waves can be of considerable consequence. As this stress wave is obviously a dynamic effect, the dynamic terms of the Navier displacement equations must be retained. In this section an asymptotic stress solution of the boundary value problem consisting of the Navier equations with dynamic terms included and the boundary conditions implied by the vanishing of the normal and shear stresses over the surface $z = 0$ is obtained for small values of time. The boundary value problem can be stated

$$\nabla^2 u - \frac{u}{r^2} + \frac{1}{1-2\nu}\frac{\partial e}{\partial r} - \frac{\rho}{G}\frac{\partial^2 u}{\partial t^2} = \frac{2(1+\nu)}{1-2\nu}\frac{\partial(\alpha T)}{\partial r},$$

$$v = 0,$$

$$\nabla^2 w + \frac{1}{1-2\nu}\frac{\partial e}{\partial z} - \frac{\rho}{G}\frac{\partial^2 w}{\partial t^2} = \frac{2(1+\nu)}{1-2\nu}\frac{\partial(\alpha T)}{\partial z}, \tag{37}$$

$$\sigma_{zz}\Big|_{z=0} = 0 = \frac{2G}{1-2\nu}\left[(1-\nu)\frac{\partial w}{\partial z} + \nu\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) - (1+\nu)\alpha T\right]_{z=0},$$

$$\sigma_{rz}\Big|_{z=0} = 0 = G\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right]_{z=0},$$

where all symbols are defined as in equations (1) and (11) and where as in equations (12) the boundary conditions are written in terms of displacements for completeness but are not utilized in that form. A particular solution of the differential equations of (37) can be obtained with the aid of the thermoelastic potential defined as in section IV from the relations $2Gu_i = \frac{\partial \phi}{\partial x_i}$. The potential so defined provides particular solutions if it satisfies the following equation:

$$(38) \qquad \nabla^2\phi - C_4 \frac{\partial^2 \phi}{\partial t^2} = \frac{2(1+\nu)G\alpha}{1-\nu} \; T,$$

where $C_4 = \frac{(1-2\nu)\rho}{2(1-\nu)G}$. To eliminate the time derivative and simplify the temperature expression, equation (38) is transformed into the Laplace subsidiary space.

$$(39) \qquad \nabla^2\phi^* - C_4 \, p^2 \phi^* = \frac{2(1+\nu)G\alpha}{1-\nu} \; T^*,$$

where use has been made of the fact that at $t = 0$, $\phi = \frac{\partial \phi}{\partial t} = 0$. Since T satisfies the heat equation $\frac{\partial T}{\partial t} = a^2 \nabla^2 T$, $pT^* = a^2 \nabla^2 T^*$. The latter equality suggests $\phi_1^* = KT^*$ as a particular solution of (39). K is determined by substitution of ϕ_1^* into equation (39) to be

$$(40) \quad K = \frac{2(1+\nu)G\alpha a^2}{(1-\nu)p - (1-\nu)C_4 a^2 p^2} = \frac{4(1+\nu)G^2\alpha\, a^2}{2(1-\nu)Gp - (1-2\nu)\rho\, a^2 p^2}.$$

The relationship between the thermoelastic potential and the stresses is given by equations (5). In the subsidiary space these relationships written in cylindrical coordinates (r, φ, z) for this axially symmetric case are

$$(41) \qquad \begin{aligned} \bar{\sigma}^*_{rr} &= \frac{\partial^2 \phi^*}{\partial r^2} - \nabla^2\phi^* + \frac{\rho p^2}{2G}\, \phi^*, \\ \bar{\sigma}^*_{\varphi\varphi} &= \frac{1}{r}\frac{\partial \phi^*}{\partial r} - \nabla^2\phi^* + \frac{\rho p^2}{2G}\, \phi^*, \\ \bar{\sigma}^*_{rz} &= \frac{\partial^2 \phi^*}{\partial r \partial z}, \end{aligned}$$

$$\bar{\sigma}_{zz}^* = \frac{\partial^2 \phi^*}{\partial z^2} - \nabla^2 \phi^* + \frac{\rho p^2}{2G}\phi^*,$$

where again use has been made of the fact that $\phi = \frac{\partial \phi}{\partial t} = 0$ at $t = 0$. As in the quasi-static case, ϕ_1^* yields stresses in the form of integrals divergent on the surface $z = 0$. For this reason it is necessary to superpose on ϕ_1^* a solution of the homogeneous wave equation $\nabla^2\phi^* - C_4 p^2 \phi^* = 0$ in order to make the stress integrals convergent for $z = 0$. Separation of the wave equation in cylindrical coordinates in the case of axial symmetry yields a solution of the form $\phi_2^* = \int_0^\infty A(p,\lambda)\, J_0(\lambda r)\, e^{-\beta z}\, d\lambda$ where $A(p,\lambda)$ is arbitrary and $\beta = \sqrt{C_4 p^2 + \lambda^2}$. With a convenient choice of $A(p,\lambda)$,

$$(42) \qquad \phi^* = \phi_1^* + \phi_2^* = \int_0^\infty C_5 J_0(\lambda r)\, (e^{-\gamma z} - e^{-\beta z})\, d\lambda,$$

where $C_5 = \frac{K b T_0\, J_1(\lambda b)}{p} = \frac{4(1+\nu)G^2 \alpha\, a^2 b T_0\, J_1(\lambda b)}{2(1-\nu)Gp^2 - (1-2\nu)\rho\, a^2 p^3}$. Use of equations (41) gives the stresses in the subsidiary space resulting from the potential ϕ^*.

$$\bar{\sigma}_{rr}^* = \int_0^\infty C_5 \Big[\Big\langle\Big(-\gamma^2 + \frac{\rho p^2}{2G}\Big)e^{-\gamma z} - \Big(-\beta^2 + \frac{\rho p^2}{2G}\Big)e^{-\beta z}\Big\rangle J_0(\lambda r) + \frac{\lambda}{r}(e^{-\gamma z} - e^{-\beta z})\, J_1(\lambda r)\Big]\, d\lambda,$$

(43)
$$\bar{\sigma}_{\varphi\varphi}^* = \int_0^\infty C_5 \Big[\Big\langle\Big(\lambda^2 - \gamma^2 + \frac{\rho p^2}{2G}\Big)e^{-\gamma z} - \Big(\lambda^2 - \beta^2 + \frac{\rho p^2}{2G}\Big)e^{-\beta z}\Big\rangle J_0(\lambda r) - \frac{\lambda}{r}(e^{-\gamma z} - e^{-\beta z})\, J_1(\lambda r)\Big]\, d\lambda,$$

$$\bar{\sigma}_{rz}^* = \int_0^\infty C_5 (\gamma e^{-\gamma z} - \beta\, e^{-\beta z})\, J_1(\lambda r)\, \lambda\, d\lambda,$$

$$\bar{\sigma}_{zz}^* = \int_0^\infty C_5 \Big\langle\Big(\lambda^2 + \frac{\rho p^2}{2G}\Big)e^{-\gamma z} - \Big(\lambda^2 + \frac{\rho p^2}{2G}\Big)e^{-\beta z}\Big\rangle J_0(\lambda r)\, d\lambda,$$

Since in this section only small values of time in the original space are involved, only large values of p in the subsidiary space need be considered [9] . For large values of p, it will now be shown that the normal and shear stresses vanish on the surface $z = 0$. For large p, $C_5 \simeq D \dfrac{J_1(\lambda b)}{p^3}$ where D

$= -\dfrac{4(1+\nu)G^2\alpha b T_0}{(1-2\nu)\rho}$. $a\gamma$ and β can be expanded binomially to give

$$a\gamma = (p+a^2\lambda^2)^{1/2} = \left[(\sqrt{p}+a\lambda)^2 - 2a\lambda\sqrt{p}\right]^{1/2} = (\sqrt{p}+a\lambda) \cdot\left[1 - \frac{a\lambda\sqrt{p}}{(\sqrt{p}+a\lambda)^2} - \cdot\cdot\cdot\right],$$

$$\beta = (C_4p^2+\lambda^2)^{1/2} = \left[(\sqrt{C_4}\,p+\lambda)^2 - 2\lambda\sqrt{C_4}\,p\right]^{1/2} = (\sqrt{C_4}\,p+\lambda) \cdot\left[1 - \frac{\lambda\sqrt{C_4}\,p}{(\sqrt{C_4}\,p+\lambda)^2} - \cdot\cdot\cdot\cdot\right].$$

Both of these expansions converge for all positive p and λ since the form $\dfrac{2\xi\eta}{(\xi+\eta)^2} \leq \tfrac{1}{2} < 1$. Now it will be shown that for p sufficiently large, γ and β may be replaced in the integrals by $\dfrac{\sqrt{p}}{a}+\lambda$ and $\sqrt{C_4}\,p+\lambda$ respectively. From the binomial expansions of $a\gamma$ and β and the fact that a constant times the geometric series $\sum_{n=1}^{\infty} \dfrac{1}{(\sqrt{p})^n} = \dfrac{1}{\sqrt{p}-1}$ is greater than those parts of the binomial expansions to be neglected, it is observed possible to select p so large as to make $\left|\gamma - (\dfrac{\sqrt{p}}{a}+\lambda)\right|$ and $\left|\beta - (\sqrt{C_4}\,p+\lambda)\right|$ and therefore $\left|e^{-\gamma z} - e^{-(\frac{\sqrt{p}}{a}+\lambda)z}\right|$ and $\left|e^{-\beta z} - e^{-(\sqrt{C_4}p+\lambda)z}\right|$ as small as desired. The $\bar{\sigma}^*$ integrals can be shown uniformly convergent in p by use of the Abel theorem in a manner analogous to that used in section III. Then for any ε there exists an x such that $\left|\bar{\sigma}^* - \int_0^x \bar{\sigma}^{*\prime}\, d\lambda\right| < \varepsilon$

where $\bar{\sigma}^{*\prime}$ is the integrand of the $\bar{\sigma}^{*}$ integral and by the justifiable interchange of limiting processes $\left| \bar{\sigma}^{*} - \int_0^{x} [\bar{\sigma}^{*\prime}\right.$ with $\gamma \rightarrow \frac{\sqrt{p}}{a} + \lambda$ and $\left.\beta \rightarrow \sqrt{C_4}p + \lambda]\, d\lambda \right| < \varepsilon$ for sufficiently large p. Substitution of the asymptotic expansions for C_5, γ, and β into equations (43) and retention of only the higher order p terms gives

$$\bar{\sigma}_{rr}^{*} \simeq \bar{\sigma}_{\varphi\varphi}^{*} \simeq \frac{D\rho}{2Gp}\left(e^{-\frac{z}{a}\sqrt{p}} - \frac{\nu}{1-\nu}e^{-\sqrt{C_4}zp}\right) I(r,z),$$

$$(44) \qquad \bar{\sigma}_{rz}^{*} \simeq 0,$$

$$\bar{\sigma}_{zz}^{*} \simeq \frac{D\rho}{2Gp}\left(e^{-\frac{z}{a}\sqrt{p}} - e^{-\sqrt{C_4}zp}\right) I(r,z),$$

where $I(r,z) = \int_0^{\infty} J_1(\lambda b)\, J_0(\lambda r) e^{-\lambda z}\, d\lambda$. The transformations necessary to transform the subsidiary stresses to the original space are [7, pp. 245 and 241]

$$L^{-1}\left\{\frac{e^{-\frac{z}{a}\sqrt{p}}}{p}\right\} = \operatorname{Erfc}\left(\frac{z}{2a\sqrt{t}}\right),$$

$$(45) \quad L^{-1}\left\{\frac{e^{-\sqrt{C_4}zp}}{p}\right\} = \begin{cases} 0, & 0 < t < \sqrt{C_4}z = \sqrt{\dfrac{(1-2\nu)\rho}{2(1-\nu)G}}\, z, \\ 1, & t > \sqrt{C_4}z. \end{cases}$$

The final stresses in the original space for small t corresponding to large values of p in the subsidiary space are

$$(46) \quad \sigma_{rr} \simeq \sigma_{\varphi\varphi} \simeq -\frac{2(1+\nu)G\alpha bT_0}{(1-2\nu)}\left[\operatorname{Erfc}\left(\frac{z}{2a\sqrt{t}}\right) - \begin{cases} 0, & t < \sqrt{C_4}z \\ \frac{\nu}{1-\nu}, & t > \sqrt{C_4}z \end{cases}\right] I(r,z),$$

$$\sigma_{rz} \simeq 0,$$

(46)
$$\sigma_{zz} \simeq -\frac{2(1+\nu)G\alpha bT_0}{(1-2\nu)} \left[\mathrm{Erfc}\left(\frac{z}{2a\sqrt{t}}\right) - \begin{cases} 0, & t < \sqrt{C_4}\,z \\ 1, & t > \sqrt{C_4}\,z \end{cases} \right] I(r,z) .$$

At this point it can be observed that the asymptotic normal and shear stresses are zero on the surface $z = 0$. Since, within the approximations made forlarge p, the stresses due to the thermoelastic potential satisfy the boundary conditions, no stresses need to be superposed as in the quasi-static solution. In this case the thermoelastic potential has produced the asymptotic stress solution for small t to boundary value problem (35).

With $\xi = \frac{z}{b}$, $\eta = \frac{r}{b}$, and $x = \lambda z$, for $r = 0$ the dynamic stresses for small values of time given by equations (46) can be written [16, p. 47]

(47)
$$\sigma_{rr} \simeq \sigma_{\varphi\varphi} \simeq \frac{\sqrt{\xi^2+1} - \xi}{\sqrt{\xi^2+1}} E(\xi, t),$$
$$\sigma_{rz} \simeq 0,$$
$$\sigma_{zz} \simeq \frac{\sqrt{\xi^2+1} - \xi}{\sqrt{\xi^2+1}} F(\xi, t) \ ;$$

for $z = 0$ [16, p. 50],

(48)
$$\sigma_{rr} \simeq \sigma_{\varphi\varphi} \simeq C_6 \frac{1-2\nu}{1-\nu} \begin{cases} 1, & \eta < 1, \\ \frac{1}{2}, & \eta = 1, \text{ for } t>0, \\ 0, & \eta > 1, \end{cases}$$
$$\sigma_{rz} \simeq 0,$$
$$\sigma_{zz} \simeq 0 \ , \text{ for } t>0 \ ;$$

and for neither r nor z equal zero,

$$\sigma_{rr} \simeq \sigma_{\varphi\varphi} \simeq \frac{E(\xi,t)}{\xi} I(\xi,\eta),$$

(49) $$\sigma_{rz} \simeq 0,$$

$$\sigma_{zz} \simeq \frac{F(\xi,t)}{\xi} I(\xi,\eta),$$

where $C_6 = -\frac{2(1+\nu)G\alpha T_0}{1-2\nu}$, $E(\xi,t) = C_6\left[\mathrm{Erfc}\left(\frac{b\xi}{2a\sqrt{t}}\right) - \begin{cases} 0 & , t < \sqrt{C_4}\, b\xi \\ \frac{\nu}{1-\nu} & , t > \sqrt{C_4}\, b\xi \end{cases}\right]$, $F(\xi,t) = C_6\left[\mathrm{Erfc}\left(\frac{b\xi}{2a\sqrt{t}}\right) - \begin{cases} 0 & , t < \sqrt{C_4}\, b\xi \\ 1 & , t > \sqrt{C_4}\, b\xi \end{cases}\right]$,

and $I(\xi,\eta) = \int_0^\infty J_1\left(\frac{1}{\xi}x\right) J_0\left(\frac{\eta}{\xi}x\right) e^{-x}\, dx$. It can be noted that equations (48) describing the surface stresses in the dynamic case for small values of time are the same as equations (27) describing the surface stresses in the quasi-static case for time approaching zero. That is, the surface stresses for time approaching zero are given correctly by the quasi-static solution. This result is in agreement with the work of Danilovskaya [3 and 4] .

VI. Numerical Results.

In this section the steady-state stresses and displacements of equations (35) and (36) are tabulated for thirty-one equally spaced points on the axis of symmetry and on the surface in a representative plane containing the axis of symmetry, the integral $I = \int_0^\infty J_1(\frac{1}{\zeta}x) \cdot J_0(\frac{\eta}{\zeta}x)e^{-x}\,dx$ is evaluated for fourteen of twenty-seven grid points in this plane, and the dynamic stresses for small values of time given by equations (47), (48), and (49) are tabulated for these twenty-seven grid points in terms of $E(\zeta, t)$, and $F(\zeta, t)$. Because of axial symmetry the evaluation of I and the asymptotic dynamic stresses actually involves computation at only seven and sixteen points respectively. The sixteen grid points for which the asymptotic dynamic stresses are computed are shown below in figure 1. The seven points in this grid neither on the axis of symmetry nor on the surface are those at which I is computed.

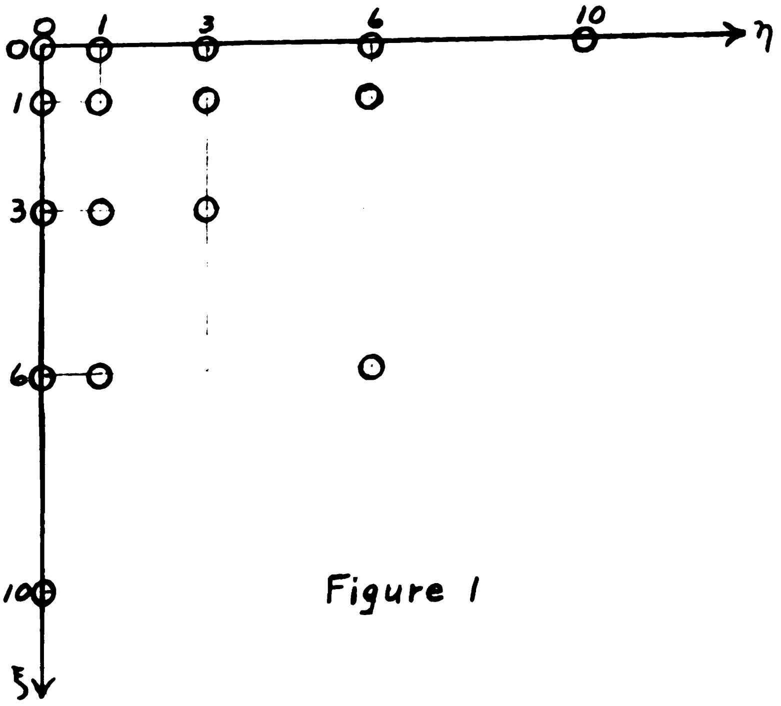

Figure 1

The values of the steady-state stresses and displacements for the indicated points on the axis and on the surface are given to four-digit accuracy in table 1. Since the steady-state normal and shear stresses are zero throughout, they are not included in this tabulation.

Table 1

Steady-state Solution

On the surface:

	0	±1	±2	±3	±4	±5	±6	±7	±8	±9	±10 η
$\frac{\sigma_{rr}}{c_2}$	.5000	.5000	.1250	.0556	.0313	.0200	.0139	.0102	.0078	.0062	.0050
$\frac{\sigma_{\theta\theta}}{c_2}$	.5000	0	-.1250	-.0556	-.0313	-.0200	-.0139	-.0102	-.0078	-.0062	-.0050
$\frac{u}{c_3}$	0	.5000	.2500	.1667	.1250	.1000	.0833	.0714	.0625	.0556	.0500
$\frac{w}{c_3}$	-1.0000	-.6366	-.2587	-.1691	-.1260	-.1005	-.0836	-.0716	-.0626	-.0556	-.0501

On the axis:

	0	1	2	3	4	5	6	7	8	9	10 ξ
$\frac{\sigma_{rr}}{c_2}$	.5000	.1464	.0528	.0257	.0149	.0097	.0068	.0050	.0039	.0031	.0025
$\frac{\sigma_{\theta\theta}}{c_2}$	.5000	.1464	.0528	.0257	.0149	.0097	.0068	.0050	.0039	.0031	.0025
$\frac{u}{c_3}$	0	0	0	0	0	0	0	0	0	0	0
$\frac{w}{c_3}$	-1.0000	-.4142	-.2361	-.1623	-.1231	-.0990	-.0828	-.0711	-.0623	-.0554	-.0499

The value of I is approximated for the seven inner points of figure 1 by terminating the integration at $x = 12$ with a maximum error less than $(3.7)10^{-6}$ since $\left|J_1(\frac{1}{\xi}x)\ J_0(\frac{\eta}{\xi}x)\right| < .6$ and $\int_{12}^{\infty} e^{-x}\,dx < (6.1)10^{-6}$. The integral terminated at $x = 12$ is then approximated by Simpson's rule:

$$\int_0^{12} f(x)\,dx = \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 4f_{n-3} + 2f_{n-2} + 4f_{n-1} + f_n) - \frac{nh^5}{180} f^{IV}(\xi)$$

where h = the length of one subinterval = .03, n = the even number designating the number of subintervals = 400, $f_k = f(x+kh)$, and $0 < \xi < 12$. The maximum error in approximating $I(\xi,\eta)$ can be bounded in the following manner. Let the integrand $f(x)$ $= g(x)\ J_0(\frac{\eta}{\xi}x)$ where $g(x) = J_1(\frac{1}{\xi}x)\ e^{-x}$. Since, for all the inner points marked in figure 1, the absolute values of both $J_1(\frac{1}{\xi}x)$ and e^{-x} as well as all their derivatives are bounded by one, $\frac{d^n g}{dx^n} < 2^n$. Because $\left|\frac{d^n}{dx^n} J_0(x)\right| < 1$, $\left|\frac{d^n}{dx^n} J_0(\frac{\eta}{\xi}x)\right| < (\frac{\eta}{\xi})^n$. Therefore $\frac{nh^5}{180} f^{IV}(x) < (5.4)10^{-8}\left[16 + 32\frac{\eta}{\xi} + 24(\frac{\eta}{\xi})^2 + 8(\frac{\eta}{\xi})^3 + (\frac{\eta}{\xi})^4\right]$ for all $x > 0$. Hence one can easily show that at all the marked inner points of figure 1 except $\xi = 1$ and $\eta = 6$ the maximum total error inherent in this approximating scheme for $I(\xi,\eta)$ is less than $(4)10^{-5}$ and that at $\xi = 1$ and $\eta = 6$ this error is less than $(2.3)10^{-4}$. Round-off error can enter the computation from the approximation in the tabulation of the Bessel and exponential functions and from the operations performed within the computer. The round-off error due to approximating the Bessel and exponential functions is held to a minimum by use of a minimum of nine digits in their tabulation [23]. It is easily shown that for these seven grid points this round-off error is less than 10^{-5}. The round-off error within the computer has been checked by instructing the computer to make the calculation once rounding down and again rounding up. Comparison of these results ascertains the maximum round-off error within the computer also to be less than 10^{-5} at all seven points. In other words the round-off error cannot be of any consequence in the four-digit tabulations which follow.

The value of I at the seven aforementioned grid points has been computed with the Michigan State Illiac Computer, the Mistic, by the following program:

Program†:

1. Four hundred values of e^{-x} for $x = .03$ to $x = 12$ in intervals of .03 are computed and stored in the Mistic.
2. Four hundred values of $J_1(\frac{1}{\xi} x)$ for $x = .03$ to $x = 12$ in intervals of .03 and for ξ fixed are input.
3. One value of $J_0(\frac{\eta}{\xi} x)$ for $x = .03$ then $x = .06$ . . . up to $x = 12$ and for $\frac{\eta}{\xi}$ fixed is input and one f_0, f_1, . . . f_k. . . f_{400} where $f_k = f(x + .03k)$ is computed.
4. The Simpson's rule routine which evaluates the integral up to the i^{th} summand is "jumped into".
5. The program returns to step three fourhundred times. Then step six is executed.
6. The result is output.
7. Thevalue of η is raised to the next higher value across the grid.

---

† This program was written by Dr. Gerard P. Weeg.

8. If all values of η for a given ξ are not yet used, the program returns to step three; if they are all used, then the program continues to step nine.
9. The ξ value is raised to the next higher value down the grid. If all ξ values in the grid are used, the program stops the computer; if not, then the program takes η at its lowest value and returns to step two.

The values of I determined from this program are written in table 2, All digits tabulated are significant.

		$\eta = 1$	$\eta = 3$	$\eta = 6$
$\xi = 1$	I =	.1787	.0174	.002
$\xi = 3$	I =	.1346	.0595	
$\xi = 6$	I =	.0785		.0295

Table 2

By use of table 2 and equations (47), (48), and (49), the asymptotic dynamic stresses are written in table 3. In table 3 E(n,t) and F(n,t) defined on page 36 are abbreviated as E_n and F_n respectively. The t in the arguments of E and F is taken as positive in table 3. Because the shear stress vanishes throughout, it is not included in table 3. As in tables 1 and 2, all digits tabulated are significant.

ξ \ η		0	±1	±3	±6	±10
0	$\frac{\sigma_{rr}}{E_0} \simeq \frac{\sigma_{\varphi\varphi}}{E_0} \simeq$	1.0000	.5000	0	0	0
	$\frac{\sigma_{zz}}{F_0} \simeq$	0	0	0	0	0
1	$\frac{\sigma_{rr}}{E_1} \simeq \frac{\sigma_{\varphi\varphi}}{E_1} \simeq$	.2929	.1787	.0174	.002	
	$\frac{\sigma_{zz}}{F_1} \simeq$	.2929	.1787	.0174	.002	
3	$\frac{\sigma_{rr}}{E_3} \simeq \frac{\sigma_{\varphi\varphi}}{E_3} \simeq$	.0513	.0449	.0198		
	$\frac{\sigma_{zz}}{F_3} \simeq$	.0513	.0449	.0198		
6	$\frac{\sigma_{rr}}{E_6} \simeq \frac{\sigma_{\varphi\varphi}}{E_6} \simeq$	.0136	.0131		.0049	
	$\frac{\sigma_{zz}}{F_6} \simeq$	.0136	.0131		.0049	
10	$\frac{\sigma_{rr}}{E_{10}} \simeq \frac{\sigma_{\varphi\varphi}}{E_{10}} \simeq$	.0050				
	$\frac{\sigma_{zz}}{F_{10}} \simeq$	.0050				

Table 3

VII. Conclusion.

The basic theory of the thermoelastic problem has been reviewed and the temperature distribution within a semi-infinite, three-dimensional half-space bounded by a plane due to a temperature distribution of $T =$ constant within a circular area and $T = 0$ outside maintained on the surface has been obtained. The original results of this thesis include the derivation of the quasi-static stress and displacement distributions within the half-space and on the boundary and the derivation of the dynamic stress distribution within the half-space and on the boundary for small values of time due to the previously indicated temperature distribution. In addition numerical results relevant to the steady-state and asymptotic dynamic solutions have been tabulated.

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