TORSION OF AXIALLY NON-UNIFORM CIRCULAR SHAFTS-ITERATIVE FINITE DIFFERENCE SOLUTIONS

Thesis for the Degree of Ph. D.

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Dipak Kumar Bazaj

1964

This is to certify that the

thesis entitled

TORSION OF AXIALLY NON-UNIFORM
CIRCULAR SHAFTS-ITERATIVE FINITE DIFFERENCE
SOLUTIONS

presented by

DIPAK KUMAR BAZAJ

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mechanical Engineering

Laurence E. Malvern
Major professor
(Guidance Committee member)

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ABSTRACT

TORSION OF AXIALLY NON-UNIFORM
CIRCULAR SHAFTS - ITERATIVE
FINITE DIFFERENCE SOLUTIONS

by Dipak Kumar Bazaj

A method is developed to apply the iterative technique for solving finite difference equations to the torsion problem for axially non-uniform circular shafts. The method consists of making a general program which could be used for obtaining the shear stress distribution on an axial section for a variety of axial non-uniformities of the shaft.

A successive over-relaxation iterative method is used in the program and is found to converge rapidly enough so that one can solve this problem on a digital computer. A fairly fine mesh size can be used in the program, which can be varied according to the capacity of the computer and available time.

To demonstrate the use of the program we have considered some examples of collared, filleted and grooved shafts for the computation of shear stresses and stress concentration factors. The values of the stress functions for the collared shaft are compared with those obtained earlier by D. N. deG. Allen using a relaxation method. The stress concentration factors obtained for the filleted shafts are compared with the experimental values of L. S. Jacobson (electrical analogy

method) an

method) and of A. Weigand (precision strain gages).

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By

Dipak Kumar Bazaj

A THESIS

Submitted to
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My thanks are also due to the other members of the guidance committee: Dr. H. L. Womochel and Dr. D. W. Hall who have contributed to the academic background necessary for this work, as well as to the staff of the computer center for their programming help.

I am highly indebted to Dr. C. R. St.-Clair Jr., chairman of the Mechanical Engineering Department, for providing financial support in the form of graduate assistantship.

Finally, I dedicate this work to my wife, Manju, in appreciation of her patience, understanding and moral support.

TABLE OF CONTENTS

Table

LIST OF TABLES

Table

8A.

8B.

9A.

9B.

10A.

10B.

11A.

11B.

Table | Page

Figure

LIST OF FIGURES

Appendix

A. Table
She

B. Progr

LIST OF APPENDICES

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CHAPTER I

INTRODUCTION

Considerable progress has been made in solving plane elasticity problems, both by analytic methods and by finite difference methods. The elastic torsion problem for a non-uniform shaft has drawn much less attention, although the first formulation of such a problem was given as early as 1899. This is because of the difficulty of obtaining an analytic solution for such a problem and the absence until recent years of iterative schemes which could solve linear algebraic equations involved in the solution by a finite difference method. The present work applies recently developed iterative techniques to the finite difference solution of torsion problems for axially non-uniform shafts, and studies the practicability of the methods.

The study of the stress distribution and the maximum stress in shafts with non-uniform axial cross-section, subjected to torsional couple, is of considerable importance in the field of stress analysis, as it provides the stress concentration factors and the points of maximum and minimum stresses and thus is helpful in choosing appropriate factors of safety in the design criteria of machine parts. It can also be helpful in optimizing designs.

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Because of the difficulty of obtaining an analytic solution of this problem, all the work done until now has either used analog-experimental solutions or finite difference methods using graphical or relaxation techniques. However, these solutions always either lack generality or are deficient in accuracy or convenience. The relaxation method, in particular, requires practice, experience and patience to develop the skill required.

In recent years, the use of matrix iteration with high speed digital computers has made it possible to solve two-dimensional elasticity problems to the desired accuracy by the finite difference methods using a very fine grid. In this work a general program is prepared which can solve the torsion problem for a variety of axially non-uniform shafts.

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CHAPTER II

REVIEW OF LITERATURE

Michell (8, 1899) was the first to formulate equations defining the torsion problem for a solid of revolution. Föppl (3, 1905) undertook this problem independently in order to determine the maximum stress in filleted shafts. He suggested an indirect method of obtaining a displacement function v and then finding a contour s for which v is a solution by assuming a velocity potential function ϕ.

Föppl also used a hydrodynamic analogy assuming that most of the twisting moment is concentrated in a thin layer at the surface. This also was not very successful because of the nature of his assumption.

Willers (22, 1907) was the first to put the torsion problem in a form involving the stress function and the twist function. He used Runge's method of numerical integration to find the distribution of the shearing stress in fillets of a specifically dimensioned collared shaft.

Between 1912 and 1933 many papers* were published, most of which extended the mathematical theory and obtained the stress distribution in the mathematical solids of revolution such as ellipsoid, paraboloid, etc. In this connection

*For further reference on these see Higgins (5, 1945).

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the work of Timpe (17, 1912), Melan (9, 1920) and Neuber (10, 1933) are the basis of later development.

Of all the authors mentioned above, Willers was the only one who furnished a means of determining the stresses in a shaft of an arbitrary contour, using a time-consuming process of numerical integration of limited accuracy. To obtain a better method Thom and Orr (13, 1930) used a finite difference procedure involving the stress function. They did this first by estimating the stress function values at mesh points and then by calculating the value at the center of each square and repeating this back and forth.

More recently Southwell (13, 1946) and Allen (1, 1946) applied relaxation techniques to the finite difference method and obtained almost the same results as Thom and Orr did for a specifically dimensioned collared shaft. Their method is quicker, better formulated and much less vulnerable to arithmatical mistakes, than that of Thom and Orr, but still very time consuming, especially for one not an expert in relaxation techniques.

In addition to the analytical method of solution of the torsion problem, there have been attempts to solve it by analog-experimental methods. In this connection Wyszomirski's (23, 1914) work using fluid flow through a thin slit and Jacobson's work (6, 1925) and later Thum and Bautz's (15, 1934) work using an electrical analogy are important. Timoshenko and Goodier (16, 1951) give results of Jacobson's electrical analogy.

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CHAPTER III

FUNDAMENTALS

3.1 Elasticity Equations

The mathematical formulation of the torsion problem for circular shafts of variable diameter is given in most texts on elasticity. From the equations of equilibrium and compatibility for isotropic materials, a partial differential equation for a torsion stress function and another equation for a displacement function are derived.

Following the notations and derivation of Timoshenko and Goodier (16) the final form of the partial differential equation for the stress function ϕ in cylindrical coordinates with z axis along the axis of the shaft is:

$$\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r^3}\frac{\partial \phi}{\partial z}\right) = 0 \quad (1)$$

or $$\frac{\partial^2 \phi}{\partial r^2} - \frac{3}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (2)$$

The only non-zero stress components are $\tau_{r\theta}$ and $\tau_{\theta z}$, and these are related to the stress function ϕ by

$$\tau_{r\theta} = -\frac{1}{r^2}\frac{\partial \phi}{\partial z}$$

$$- (3)$$

$$\tau_{\theta z} = \frac{1}{r^2}\frac{\partial \phi}{\partial r}$$

A second equation, similar to equation (1) can be derived for the displacement function ψ, where ψ is the angle of rotat

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of rotation of an elemental ring of radius r in a cross-section of the shaft. The equation for ψ is

$$\frac{\partial}{\partial r}\left(r^3 \frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(r^3 \frac{\partial \psi}{\partial z}\right) = 0, \qquad (4)$$

and the relationship between the two functions is expressed by

$$\frac{1}{Gr^3}\frac{\partial \phi}{\partial r} = \frac{\partial \psi}{\partial z} \text{ and}$$

$$-\frac{1}{Gr^3}\frac{\partial \phi}{\partial z} = \frac{\partial \psi}{\partial r}. \qquad (5)$$

The boundary conditions for the function ϕ, obtained from the requirement that the boundary be stress free, require that function ϕ be constant on the boundary of an axial section of the shaft.

The magnitude of the torsional moment applied to the shaft is given by

$$T = 2\pi\,(\phi(a) - \phi(o)), \qquad (6)$$

where $\phi(a)$ and $\phi(o)$ are the values of the function ϕ at the boundary and at the axis, respectively.

Equations (2) and (6), with the help of the boundary condition, are sufficient to determine completely the stress function ϕ at every point in the shaft for a given twisting moment T, and the shear stresses can then be calculated with the help of equations (3) by differentiating the function ϕ with respect to r and z.

3.2 Finite Difference Equations

The partial differential equations (2) or (4) have been solved in closed form only for a few simple cases. For most practical shapes of the shafts we must resort to approximate methods. A

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methods. A convenient procedure for solving such problems on a digital computer is the finite difference method in which the equation is discretized. The section of the shaft is divided into a grid, and for each node in the grid a linear algebraic equation is derived from the original partial differential equation. Solution of these linear algebraic equations gives the value of the required function at each node point.

Following once again the notations and derivation of Timoshenko and Goodier (page 491) for a square grid, the difference equation corresponding to (2) is given by

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_o - \frac{3h}{2r_o}(\phi_1 - \phi_3) = o, \tag{7}$$

where o is the point for which the equation is formed. 1, 2, 3, 4 are the neighboring points to right, top, left and bottom of point o Figure (1), h is the mesh interval and r_o is the radius at point o.

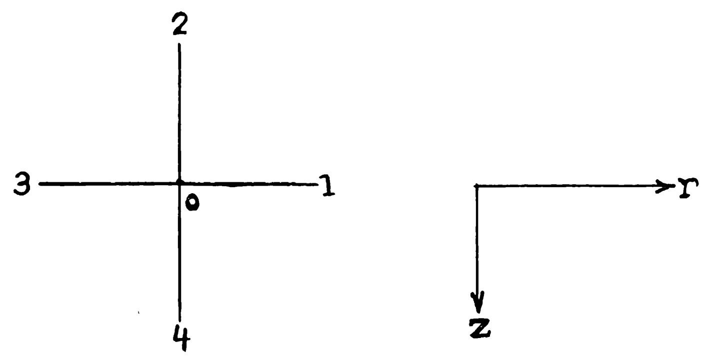

Fig. 1. Regular Star

3.3 Irregular Star

The finite difference equation (7) is suitable only for nodes with a constant h, i.e. for a regular star. Near a curved boundary there will be one or more neighboring nodes at a distance less than h. Such points have irregular stars, and for t

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and for them the resulting difference equation becomes more complicated. For example, consider the irregular star shown in Fig. (2). Points 1 and 4 lie on the boundary at distances

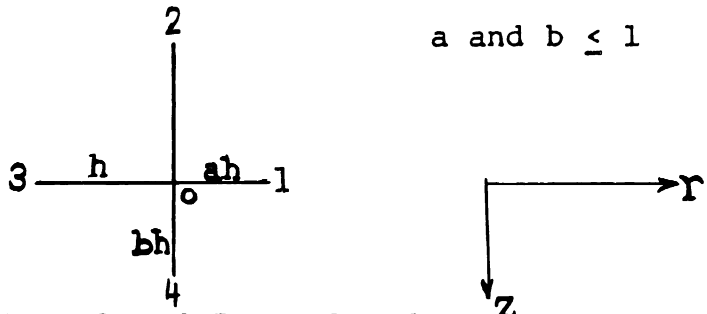

Fig. 2. Example of Irregular Star

ah and bh respectively from o. Using the same notations as before,

$$\left.\frac{\partial \phi}{\partial r}\right|_{o1} \simeq \frac{\phi_1-\phi_o}{ah}; \quad \left.\frac{\partial \phi}{\partial r}\right|_{3o} \simeq \frac{\phi_o-\phi_3}{h}$$

where $\left.\frac{\partial \phi}{\partial r}\right|_{o1}$ is the approximation of the derivative at the center of the interval o-1 and $\left.\frac{\partial \phi}{\partial r}\right|_{3o}$ is the approximation at the center of 3-o.

$$\left.\frac{\partial^2 \phi}{\partial r^2}\right|_{o} \simeq \frac{1}{\frac{h}{2}(1+a)}\left(\frac{\phi_1-\phi_o}{ah} - \frac{\phi_o-\phi_3}{h}\right) \quad \text{(a)}$$

Similarly

$$\left.\frac{\partial^2 \phi}{\partial z^2}\right|_{o} \simeq \frac{1}{\frac{h}{2}(1+b)}\left(\frac{\phi_4-\phi_o}{bh} - \frac{\phi_o-\phi_2}{h}\right) \quad \text{(b)}$$

Also

$$\left.\frac{\partial \phi}{\partial r}\right|_{o} \simeq \frac{1}{2}\left(\frac{\phi_1-\phi_o}{ah} + \frac{\phi_o-\phi_3}{h}\right)$$

or $\left.\frac{\partial \phi}{\partial r}\right|_{o} \simeq \frac{1}{2ah}\left(\phi_1 - a\phi_3 - (1-a)\,\phi_o\right).$

Substituting these in equation (2) we get the finite difference equation for an irregular star corresponding to equation (7).

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$$\left(\frac{2}{a(1+a)} - \frac{3}{2}\frac{h}{r_o a}\right)\phi_1 + \frac{2}{1+b}\phi_2 + \left(\frac{2}{1+a} + \frac{3}{2}\frac{h}{r_o}\right)\phi_3$$

$$+ \frac{2}{b(1+b)}\phi_4 - \left(\frac{2}{a} + \frac{2}{b} - \frac{3}{2}\frac{(1-a)h}{r_o a}\right)\phi_o = 0. \qquad (8)$$

If a=b=1, then equation (8) reduces to equation (7) for regular star.

An alternative method of obtaining the second partial derivative of ϕ with respect to r or z, i.e. equation (a) and (b), is by expanding the function ϕ (r, z) into a power series in the neighborhood of the point o. (See, for example, Wang, (20), page 138.) Thus, considering o as origin,

$$\phi\ (r, z) = \phi_o + a_1 r + a_2 z + a_3 r^2 + a_4 z^2$$
$$+ a_5 rz + - - -$$

At $r = o$, $z = -h$ we have $\phi = \phi_2$ and at $r = o$, $z = bh$, $\phi = \phi_4$.
Neglecting higher powers of r and z, it follows

$$\phi_2 \approx \phi_o - a_2 h + a_4 h^2$$

and $\phi_4 \approx \phi_o + a_2\,(bh) + a_4\,(bh)^2.$

Solving these equations simultaneously we get

$$a_2 \approx \frac{(\phi_4 - \phi_o) + b^2\,(\phi_o - \phi_2)}{bh(1+b)}$$

and

$$a_4 \approx \frac{(\phi_4 - \phi_o) + b\,(\phi_2 - \phi_o)}{bh^2(1+b)}$$

Similarly using point 1, o and 3,

$$a_1 \approx \frac{(\phi_1 - \phi_o) + a^2\,(\phi_o - \phi_3)}{ah\,(1+a)}$$

and

$$a_3 \approx \frac{(\phi_1 - \phi_o) - a\,(\phi_o - \phi_3)}{ah^2\,(1+a)}$$

At the point o (r = o, z = o)

$$\left.\frac{\partial^2 \phi}{\partial r^2}\right|_o = 2a_3 \approx \frac{2}{h(1+a)}\left[\frac{\phi_1 - \phi_o}{ah} - \frac{\phi_o - \phi_3}{h}\right]$$

and

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$$\left.\frac{\partial^2\phi}{\partial z^2}\right|_0 = 2a_4 \simeq \frac{2}{h(1+b)}\left[\frac{\phi_4-\phi_0}{bh} - \frac{\phi_0-\phi_2}{h}\right].$$

These are the same equations as (a) and (b).

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CHAPTER IV

METHOD OF ITERATION

4.1 Solution of Linear Algebraic Equations

Little is known concerning the extent of approximation obtained by solving the difference equations corresponding to a partial differential equation. It is therefore desirable to use an extremely fine mesh size in order to get a good approximation. However, since the number of nodes increases inversely as the square of the mesh interval, the number of linear algebraic equations to be solved also increases in that proportion. Thus the problem of solving difference equation presents a serious practical difficulty.

Methods for solving linear algebraic equations can be divided into two classes: direct and iterative. Direct methods such as Cramer's rule and Gaussian elimination methods are impracticable because of the size of the system. On the other hand the iterative method, which begins by assuming at each point an arbitrary value of the variable and then successively improves the values, yields the answer only as a limit of a sequence of calculations, each extending over the entire field, and therefore becomes time-consuming. In addition to the fact that the iterative methods can solve a large number of equations, they can usually take full advantage of numerous zeroes in the storage of matrix A, of the matrix equation

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$A\phi = B$, obtained from the set of linear algebraic equations. The iterative method also tends to minimize the round-off error, because of its self-correcting nature.

4.2 Gauss-Seidel Iterative Process

A well-known linear iterative process for approximating the solution of a set of simultaneous linear algebraic equations, is the method of Gauss-Seidel. This method is effected by selecting first arbitrary trial values for the set of variables and then improving these values gradually until the difference between the two successive values is less than a pre-determined number at every point. This method can be proved to be convergent for a strictly or irreducibly diagonally dominant matrix (for a statement and proof of the theorem see Varga (19), page 73).

A matrix A is irreducible if and only if its directed graph is strongly connected. (Varga, page 20.) As the nature of our equation (7) and (8) of Chapter III is such that a directed graph for any ordered pair of points is always strongly connected, our matrix A is always an irreducible matrix.

Also an irreducibly diagonally dominant matrix $A = a_{ij}$ is defined to be one in which

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ with strict inequality for at least}$$

one i (Varga, page 23). The strict inequality always holds in our case for all the points adjacent to the boundary.

The set of linear equations is

$$\sum_{j=1}^{n} a_{ij} X_j = b_i \qquad i = 1, 2, \ldots n \qquad (1a)$$

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Starting with a trial vector $X_j^{(0)}$ or $X^{(0)}$ we improve these successively to $X^{(1)}$, $X^{(2)}$ etc., which converge to the solution vector X. The improvement is effected by cycling through the equations, replacing only the ith component of the trial vector by the value necessary to satisfy the ith equation.

The difference between the ordinary iterative scheme (Jacobi Method) and Gauss-Seidel method is that the latter uses in the process the improved values available of (i-1) previous components to improve the values of the ith component. This has an advantage, when working with the computer, that it does not require simultaneous storage of two sets of approximation $X^{(K+1)}$ and $X^{(K)}$ in the course of computation. It can also be shown that the rate of convergence of the Gauss-Seidel method for a symmetric matrix is greater than that of the Jacobi method (See, for example, Todd (18), page 404).

The basis of constructing an iterative scheme is provided by dividing the matrix A into a lower triangular matrix L, an upper triangular matrix R and a diagonal matrix D such that

$A = L + D + R.$ (2)

Assuming matrix A has no zero entries on its diagonal, we write

$DX = B - (L + R) X$ (3)

from which we derive

$$a_{ii}\, X_i^{(K)} = -\sum_{j=1}^{i-1} a_{ij} \cdot X_j^{(K)} - \sum_{j=i+1}^{n} a_{ij} \cdot X_j^{(K-1)} + b_i, \qquad (4)$$

where $i = 1, 2, \cdots n$, $K \geq 1$

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and finally obtain $X_j^{(K)}$ by dividing by a_{ii}.

$$X_i^{(K)} = -\left[\sum_{j=1}^{i-1} a_{ij}\, X_j^{(K)} + \sum_{j=i+1}^{n} a_{ij}\, X_j^{(K-1)} - b_i\right](1/a_{ii}).$$

This is the Gauss-Seidel iterative scheme. In this, now, we can introduce a relaxation factor ω to obtain a successive over or under relaxation iterative scheme.

$$X_i^{(K)} = (1-\omega)\, X_i^{(K-1)} + (\omega/a_{ii})\left[-\sum_{j=1}^{i-1} a_{ij}\, X_j^{(K)} - \sum_{j=i+1}^{n} a_{ij}\, X_j^{(K-1)} + b_i\right]. \quad (5)$$

This latter iterative scheme with $\omega>1$ is found to give a better rate of convergence in the present problem (See also Forsythe and Wasow (4), page 260).

4.3 Application of the Iterative Method

The importance of the iterative method of solving linear algebraic equations comes from the fact that it can take advantage of some of the special properties of the coefficient matrix A and the constant matrix B. These properties are common in matrices derived from most elliptic partial differential equations. The properties are

(a) Matrices A and B are usually of large order, but are sparse i.e. the non-zero elements are much less in number than the zero elements.

(b) The non-zero elements of A and B are easy to generate and therefore the coefficient matrices themselves do not require any storage place in the memory.

These properties can be well utilized in the iterative method, since for a Gauss-Seidel iterative scheme only one equation is required at a time, which can be generated just befor

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before its use. Thus the storage of the coefficient matrix A and the matrix B is completely eliminated and it is possible to solve a large number of linear algebraic equations.

4.4 Flow Chart

Ralston and Wilf (12) give a detailed analysis of the use of the Gauss-Seidel method in a computer. On the basis of their summary of the calculation procedure, the flow chart in Fig. (3) and a description of the flow chart follows.

Box 1: K is the counter which counts number of cycles of iterations. W is over-relaxation factor.

Box 2: i identifies the equation. $1 \leq i \leq n$ where n is the number of linear algebraic equations. ER is the error estimate summed over all point for the kth cycle of iteration.

Thus,

$$ER^{(K)} = \sum_{i=1}^{n} \left| X_i^{(K)} - X_i^{(K-1)} \right|$$

Box 3: Q_i is the value of the two summations under the square bracket in equation (5) of section 4.2.

Box 4: P_i is the final calculated value of $X_i^{(K)}$

Box 5: D is the difference value of X_i in Kth iteration and (K-1)th iteration.

Box 6: D is added to ER to get summation of box 2.

Box 7: X_i takes its value after Kth iteration i.e. $X_i^{(K)} = P_i$.

Circle A: If i=n which means all the n equations have gone through a cycle, then flow proceeds to box 9, otherwise the flow is directed to box 8.

Box 8: i is increased by 1 for iterating the next equation. Flow is directed back to box 3.

Circle B: If ER $\leq$ EC, then the X's have reached the desired accuracy and the output can be printed or called. However, if ER > EC then the flow is directed to Box 9 and another iteration cycle starts.

Box 9: K is advanced by one to start another cycle of iteration and the flow is back to box 2.

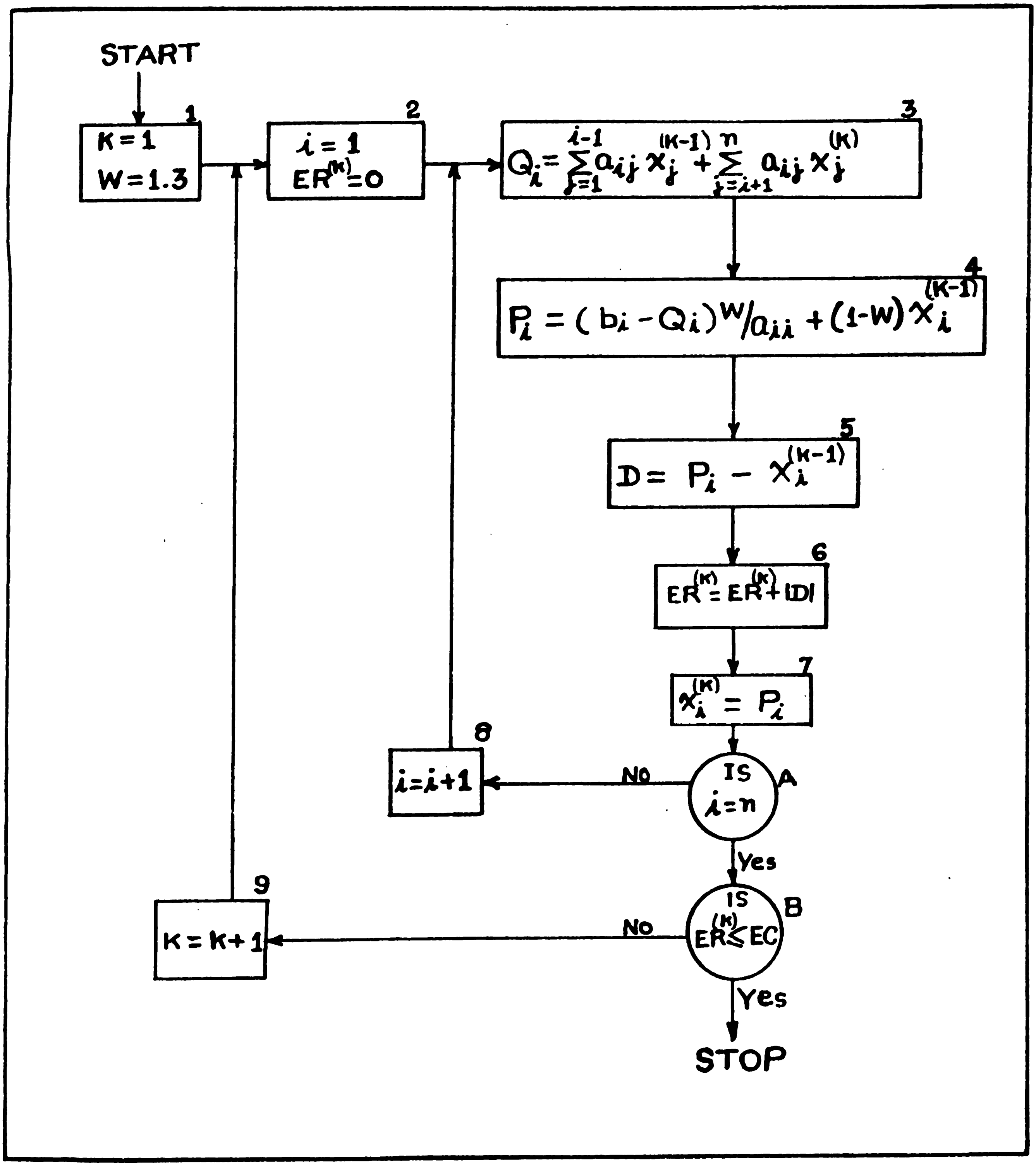

Fig. 3.--Flow chart for Gauss-Seidel iteration method

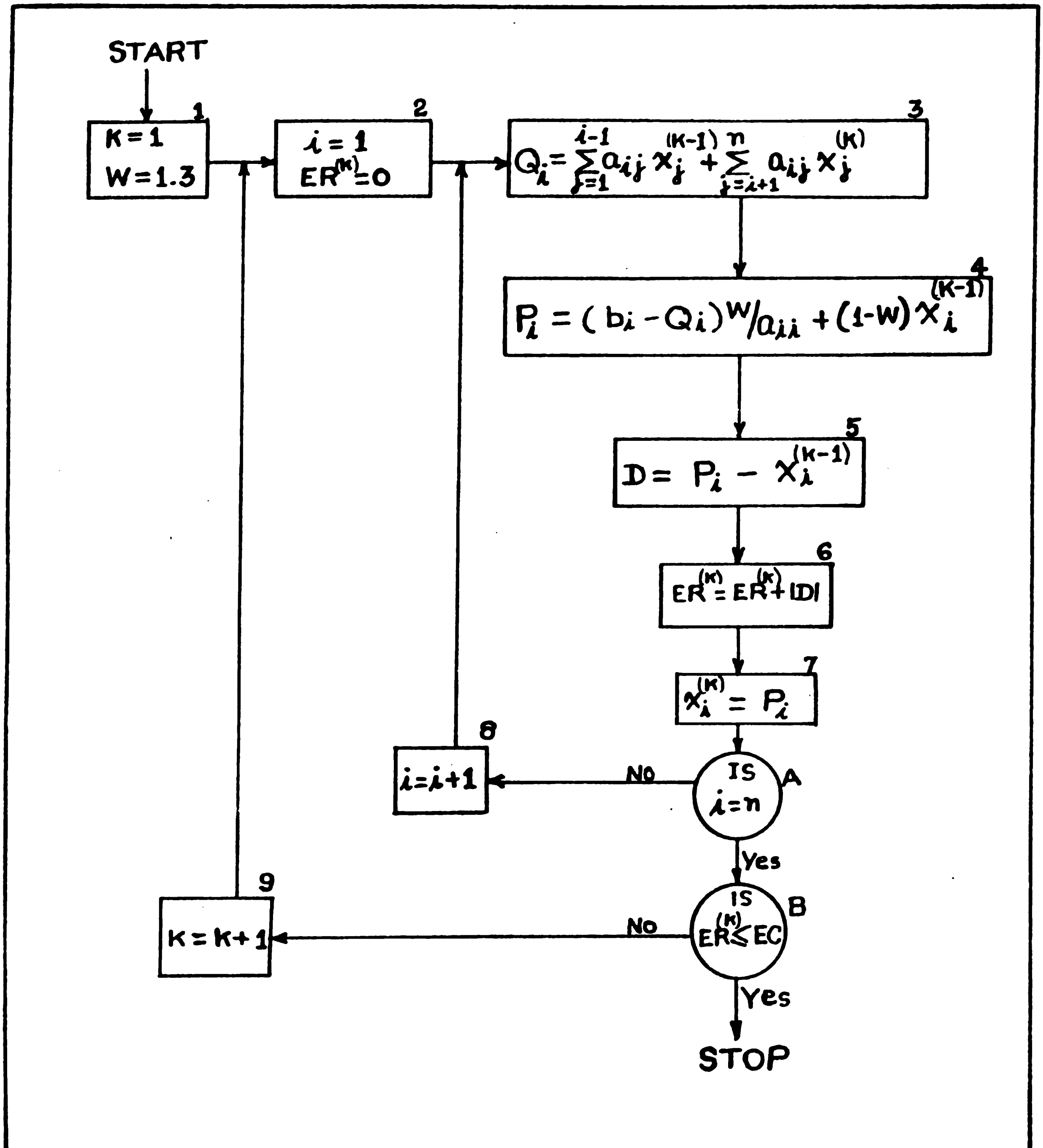

Fig. 3.--Flow chart for Gauss-Seidel iteration method

CHAPTER V

PROGRAMMING

5.1 Program Requirements

One of the objectives of this work is to make a general program which could solve for the stress distribution in any axially non-uniform shaft. To fulfill this object, this program must do the following things:

(1) Locate the mesh points and number them.

(2) Identify the mesh points having irregular stars and get the values of the factors a and b in equation 8 of Section 3.3.

(3) Generate and label the non-zero coefficients of matrix A as often as is needed.

(4) Modify the non-zero coefficients whenever an irregular star occurs.

(5) Solve the matrix equation AX=B by an iterative scheme of the type in Section 4.4.

(6) Generate the boundary values at ends of the shaft.

(7) Perform differentiation both in z and r direction and calculate the stress at each node.

(8) Print the results thus far calculated in proper order and place.

In addition, the program must have access to data providing the radius of the shaft at each section, values of the stress function at the center line and on the surface, and the mesh size.

5.2 Program Technique

To incorporate all the requirements of the program in Section 5.1, it is divided into three parts; two subroutines and a main program.

The first subroutine (named MATGN), which forms the main part of the program fulfills the requirements (1) to (6) of Section 5.1. This has three overlapping loops. The outermost loop is for the iteration cycle and this corresponds to the loop of the iteration scheme in Section 4.4.

The center loop generates the mesh by adding a mesh length to row (I-1) to get row I; then after obtaining the radius at the section where the I-th row occurs it generates the number of mesh points in this row.

The innermost loop is for points belonging to the same row. It performs the following functions:

(1) It selects the mesh points one at a time, starting from the center line,

(2) numbers them in succession,

(3) calculates the radial coordinate at the mesh point,

(4) determines the non-zero coefficients of the matrix A,

(5) determines the non-zero coefficient of the matrix B,

(6) if the mesh point is adjacent to the boundary curve and has an irregular star, then it modifies coefficients of the matrix A,

(7) it labels the neighboring four points, and

(8) iterates the Ith row according to the iteration scheme.

After all the mesh points in one row are considered, the row is advanced by one and the inner loop is repeated until the whole section is covered. This completes one iteration cycle. This is continued until iteration is completed as in Section 4.4.

The stress function is constant on the center line and constant also on the surface of the shaft. The difference between these constants is proportional to the torsional moment applied to the shaft (equation (6) of Section 3.1). In this work a constant value of moment, T, is chosen which gives the difference in the stress function at the center line and at the boundary as $T/2\pi$. Although it does not in principle make any difference whether we assume zero value of the stress function on the surface or on the center line, it does simplify the program if the zero value is chosen at the surface, because it facilitates the calculations with irregular stars.

It is also possible to divide subroutine MATGN into two separate subroutines. The first one generates the non-zero elements of matrices A and B and then stores them. Since there are at most five non-zero elements in each row of matrix A, it is possible to store them in five different unidimensional arrays. The row number of the element in these arrays remains the same as in the original matrix, and the names of the arrays indicate for which of the points 0, 1, 2, 3 and 4 (Fig. 1) the element is generated. The matrix B also needs, in this procedure, a storage place. The second subroutine picks up row by row one element from each array and iterates according to the scheme of section 4.4.

This procedure of dividing subroutine MATGN is found to take about 15% less computer time for approximately 1200 algebraic equations, although it sharply cuts down the capacity of the program because of the storage space required.

The second subroutine of the program (named BOUGEN) is for generating boundary values. On the basis of Saint-Venant's principle it is assumed that sufficiently far from the non-uniformity along z-direction, where the shaft is uniform, the stress function is independent of z. From our computations we noted that the value of the stress function is fairly independent of z at a distance greater than 0.75 the diameter on either side of the non-uniformity. The end boundaries for the solutions are therefore chosen beyond this distance and the boundary values at the mesh points on the ends are obtained from a mathematical solution for a uniform circular shaft.

An alternative method of obtaining the boundary values of the stress function is by a numerical solution. In this the boundary values at the mesh points are obtained by performing iterations in the r direction only. Since the differential equation here becomes z-independent, the terms ϕ_2 and ϕ_4 do not appear in equation (7) of section 3.2. This equation can now be written as $\phi_1 \left(1-\frac{3h}{2r_o}\right) -2\phi_o + \phi_3 \left(1+\frac{3h}{2r_o}\right) = 0.$

Southwell (13) and Allen (1) in use of relaxation method prefer to use the numerical solution for the boundary values. We have noted in this work that for the mesh size used, there is very little difference in the values of the stress functions by the two methods. For numerical solution, the subroutine BOUGEN has the same basic form as subroutine MATGN. It has, however, much fewer points and they all lie on a straight line.

The main program performs differentiation in the r and z directions. A three-point center derivative formula is used to obtain the derivative at each point lying inside the boundary curve. A one-sided three-point derivative formula is used to obtain derivatives on the surface. The latter is also used for points with irregular stars. From these derivatives, stresses are obtained by equation (3) of section 3.1, and the resultant shear stress is obtained at each point by a vector sum of the stresses in r and z directions. This part of the program also contains print statements to print the stress function and the shear stress in the same order and place as the mesh point on the section of the shaft.

5.3 Flow Chart for MATGN

As the subroutine MATGN forms the main part of the program, a step by step description of its flow chart (Fig. 4) is given below.

Box 1: In this an over-relaxation factor, W=1.3, and a counter K which counts the number of iterations performed, are introduced.

Box 2: ER is the error estimate and has the same meaning as ER in section 4.4. IM is the counter which identifies the boundary values supplied by subroutine BOUGEN.

Box 3: I and IR count and label the rows on the grid and the node points on the grid respectively.

Box 4: Z and ZB are the distances of rows I and I + 1 respectively from the top end of the section. Y and YB are the radii of the shaft at rows I and I + 1 respectively. These radii can either be read from data, or an equation of the boundary curve with respect to some origin on the center line of the shaft can be supplied to compute them.

Box 5: JN(I) is the number of mesh points on the Ith row.

Box 6: J is a counter which labels points lying in a row, starting from the center line.

Box 7: R is the radius at any point J. AA, AB, AC, AD and AE are the non-zero coefficients of matrix A for points 0, 1, 2, 3 and 4 (Fig. 1) respectively for mesh point IR. For points which lie adjacent to the boundaries these coefficients will be modified in boxes 8 to 14.

Circle A: This selects all the points lying in row 1 and directs their flow via box 8.

Box 8: IM is advanced by one each time this box is encountered. AB is set equal to zero and B takes the value BR(IM) supplied by subroutine BOUGEN. B, if not modified, later becomes a non-zero element of matrix B for point IR.

Circle B: This selects all the points lying in the last row and directs their flow via box 9.

Box 9: IM is once again advanced by one in this box every time it is encountered. AE is made zero and B=BR(IM) is supplied by BOUGEN as in box 8.

Circle C: This selects all the points lying adjacent to the center line and directs their flow to box 10.

Box 10: AD is made zero. A constant value of 0.398T is added to B. This is obtained from equation (7) of Section 3.3 and from the assumption that the stress function has a constant value of $T/2\pi$ on the center line.

Circle D: This selects all the points which lie adjacent to the boundary curve and also are the last points in the row and directs their flow to box 11.

Box 11: AB is set equal to zero.

Circle E: If the last point in a row has an irregular star, i.e., if $Y-R \neq h$ then the flow is directed to box 12.

Box 12: In this the values of factors a and b are obtained as follows:

$$a = Y-R \text{ and}$$

from the equation $z=f(y)$ of the boundary the value ZL is determined where $Y=R$. Then

$$b = ZL-Z.$$

If b is found greater or equal to one, then its value is taken as one. Values of AA, AB, AC, AD and AE are modified according to equation (8) of section 3.3.

Circle F: If the number of points in row I is greater than that in row I+1 and if b is less than or equal to one then the flow is directed to box 13.

Box 13: AE is set equal to zero.

Circle G: If any of the internal points, where $J = JN(I)$ are adjacent to the boundary, then $J > JN(I + 1)$ and the flow is directed to box 14.

Box 14: AE is set equal to zero.

Box 15: This identifies the number of the neighboring points 1, 2, 3, 4 of IR by giving them number KR, LR, MR and NR respectively.

Box 16: From this box onwards the iteration process starts and the steps are very similar to that of Section 4.4. In this box :

$$Q = Q_i = \sum_{\substack{j = 1 \\ j \neq i}}^{n} A_{ij} X_i$$

Since all the A_{ij}'s except those corresponding to AB, AC, AD, and AE are zero we get

$$Q = AB \cdot X(KR) + AC \cdot X(LR) + AD \cdot X(MR) + AE \cdot X(NR).$$

Box 17: In this

$$P = P_i = (B - Q)\ W/AA + (1-W)\ X(IR)$$

which follows directly from section 4.4.

Box 18: D is the error estimate for point IR and since P is the latest value of X(IR) from equation (5) of section 4.4

$$D = X(IR) - P.$$

Box 19: ER is the sum of the error estimate for all the points $ER = ER + |D|$.

Box 20: Here iteration of point IR is over. J is advanced by one to the next point in the row. X(IR) takes its latest value P. B is set equal to zero and IR is advanced by one.

Circle H: If J > JN(I) i.e. if the last point in the row is already considered then the flow is directed to box 21, otherwise the flow is directed to box 7.

Box 21: I is advanced by one to change over to next row.

Circle K: If I is greater than the total number of rows by one, which means that all the section has been considered, then flow is directed to box 22. Otherwise the flow is to box 4.

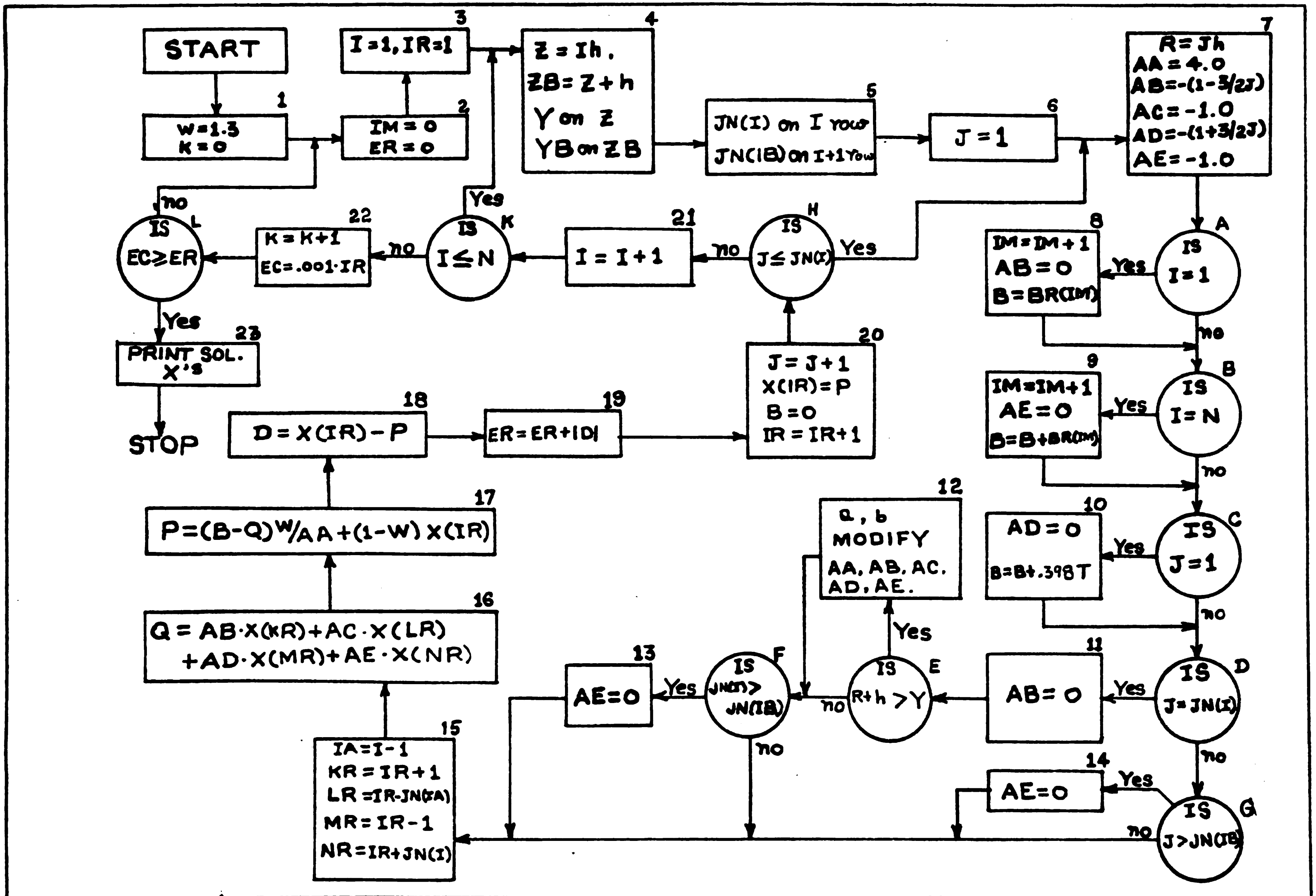

Fig. 4.--Flow chart for subroutine MATGN

Box 22: K is advanced by one, indicating completion of one more iterative cycle. EC is the error criterion, which in this case is assumed to be 0.001(IR).

Circle L: If ER $\leq$ EC i.e. if the desired accuracy of X's has been attained, the flow proceeds to box 23. If ER > EC, the flow proceeds to box 2 where IM and ER are set to zero and another complete cycle of iteration begins.

Box 23: The values of X can be printed or called by some other subroutine. K also may be printed to get the total number of iterative cycles.

Later in the work an alternative method of obtaining the error estimate ER is also tried. In this ER is defined as

$$ER^{(K)} = \max_i \left| \frac{X^{(K-1)} - X^{(K)}}{X^{(K)}} \right|,$$

for Kth iteration. (For previous definition see section 4.4, box 2). Boxes 18 and 19 are modified to make these changes as shown in Fig. (5). The explanation of the figure follows.

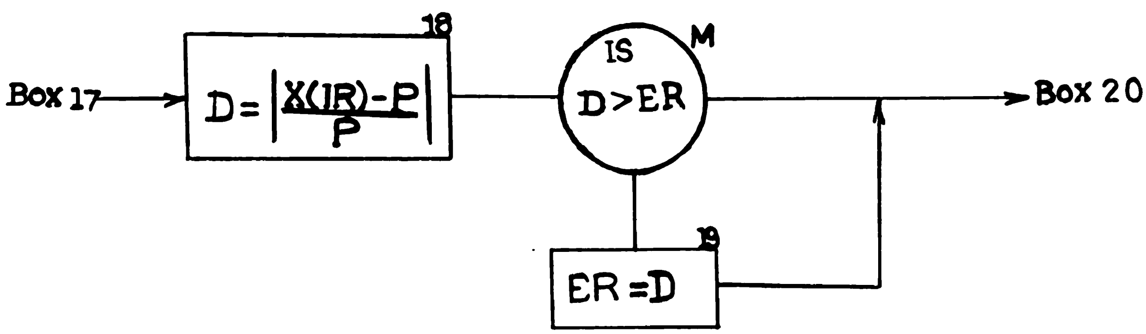

Fig. 5. Alternative method of obtaining ER.

Box 18: Here D is the ratio of the difference between the two successive values of X and the latest value of X at a point.

Circle M: In this if D > ER then in box 19 ER is set equal to D. In this way after all the points are considered, ER has the maximum value of D.

The maximum value of ER over all the points is compared with EC the error criterion as before in circle L, Fig. (4). However, here EC is given a constant value 0.0001.

5.4 Flow Chart of Subroutine BOUGEN

As stated earlier, subroutine BOUGEN, Fig. (6), is basically the same as MATGN and, therefore, we do not give here a step by step description of its flow chart, which is self explanatory once the flow chart of MATGN is understood. However, some of the variables which do not appear in the subroutine MATGN are described here. Also, since two sets of values of the stress functions are to be generated in this subroutine, one for each end of the shaft, there are statements which make a shift to the next set after the first is calculated. The calculated values of the two sets are stored under the same array name in the order of their calculation and in the order they are required by subroutine MATGN. The variable ID in box 1 takes the value of IR in box 17 when the first set of calculations is over and then in the second set the labeling of points starts from ID + 1 in box 4 and 6. Circles A and F determine the shift from one set to another. L is the value of large radius and S is that of small radius at the ends. T corresponds to variable B in subroutine MATGN. IG determines number of points considered in one set for the purpose of calculating the error criterion. If the alternative definition of ER is used, a modification similar to subroutine MATGN (Fig. 5) is required in boxes 13 and 14.

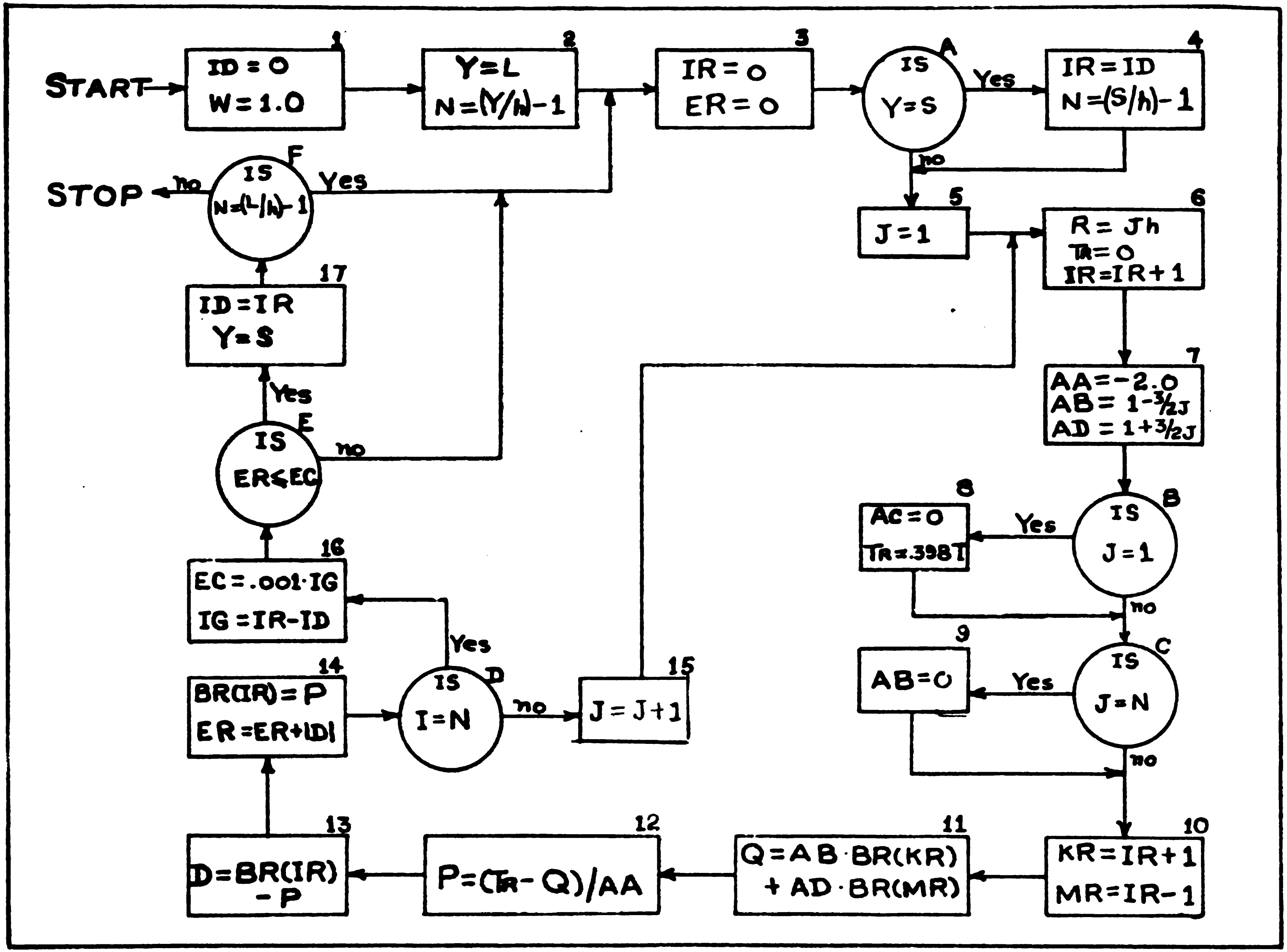

Fig. 6.--Flow chart for subroutine BOUGEN

CHAPTER VI

EXAMPLES OF SOME NON-UNIFORM SHAFTS

6.1 Examples

In this chapter, we shall apply the programming technique of Chapter V to prepare a program in the Fortran language (Appendix B) to be run on a CDC 3600 digital computer to solve some specific torsional problems. Three varieties of shafts having axial non-uniformities are considered:

(1) Collared shafts,

(2) Filleted shafts, and

(3) Grooved shafts.

Although only the three most usual types of non-uniformities encountered in practice are considered, the program of Appendix B is very general and could be used for any other types of non-uniformities, such as non-circular fillets, grooves, etc. With a memory capacity of 32000 (capacity of the M.S.U. CDC 3600 computer), as many as 18000 points can be considered. This, however, will not leave any place in the memory for storing the stress values. But, since for a central three point differentiation only three rows of values at mesh points are required at a time, it is possible to store the stresses at the same place in the memory as the stress function. This can be done either by use of a slow speed memory (magnetic tape) or by substituting the stress values in places where the values of the stress function have no further use. This is not necessary in any of the examples solved in the present work, since the maximum number of points considered is not more than 1500.

A mesh size of L/24, where L is the maximum radius, is used in all the examples except the collared shaft where the size used is S/16, where S is the radius of the shaft. The different mesh size for the collared shaft is used for comparing the result of this work with that of Allen (1), who used a relaxation technique. The relaxation solution is given by Southwell (13, page 153).

6.2 Collared shaft

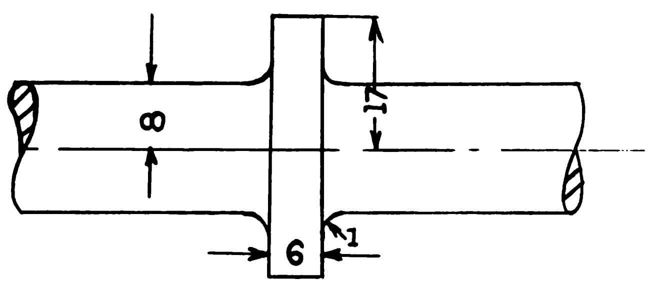

Fig. 7. The collared shaft

The specifically dimensioned example used by Allen is shown in Fig. 7. He used a value of the stress function on the boundary as 4096* and zero value at the center line. The program in Appendix B was therefore modified to suit this requirement. The values of the stress function obtained by the two methods are tabulated in Tables 3A and 3B. Only the values

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*Allen used this value to compare his results with that of Thom and Orr (14).

for points corresponding to mesh size S/8 are tabulated due to space limitations. The maximum disagreement between two sets at any point is not more than 4%. The third Table 3C gives the calculated values of the stress. Neither Allen nor Southwell give these values in their publications.

Since the collar is symmetrical about the plane perpendicular to the center line and passing through the middle of the collar, $\phi_2 = \phi_4$ in equation (7) of section 3.2 for all the points lying on the plane of symmetry. Thus in all the cases where such a symmetry occurs, one needs to compute the values of the stress function only for the points which lie on one side of the plane of symmetry.

6.3 Filleted Shafts.

Filleted shafts are those which have a step and a fillet of a specified radius to avoid sharp corners and consequently high stress concentration.

The extent of the stress concentration is directly related to the diameter at the two ends and the radius of the fillet. It is possible to see the effect of the fillet radius on the general distribution of the shear stress, since we have now a program which gives the shear stresses at each point.

As an example, a set of four shafts are considered with the ratio of their large radius L and small radius S equal to 1.5 in all four shafts. The only dimension varied is r, the radius of the fillet. The value 2r/S is taken as, .25, .5, .75 and 1.0. In doing the calculations L was chosen 3 units, S was chosen 2 units and the value of torsional moment T was taken as 162π. The values of the stress and the stress function were obtained by substituting these values in the program and are tabulated in Tables 4 through 7 for each of the shafts.

Although the values of the stress and the stress function tabulated are based on the arbitrary values of S, L, and T, these numbers can also be used for any other set of values of S_2, L_2 and T_2 to give the shear stress and the stress function, so long as L/S and 2r/S remain constant. This can be done by use of what may be called the stress factor and the stress function factor.

The maximum shear stress in a uniform shaft of the same radius as the minimum radius of the shaft under consideration can be calculated from the mathematical solution and is given by

$$\tau_m = \frac{2T}{\pi S^3} \qquad (1)$$

From this the value of the maximum stress for $T = 162\pi$ and $S_1 = 2.0$ is 40.5. Now we define the stress factor k to be the ratio of the tabulated stress at any one point to that of the maximum stress τmax of equation (1). This factor remains constant for a particular set of L/S and 2r/S. From this it is possible to determine the shear stress at any point for any other set of values of T, L, and S. Thus,

$$k = \frac{\tau_2}{\tau_{m2}} = \frac{\tau_1}{40.5}$$

$$\text{or } \tau_2 = \frac{\tau_{m2}}{40.5} \cdot \tau_1 \qquad (2)$$

where τ_2 is the required value of the shear stress at any point for S_2, L_2 and T_2, τ_{m2} is the maximum value of the shear stress from equation (1) for S_2, L_2 and T_2 and τ_1 is the tabulated value of the shear stress at that point.

A similar conversion formula can be obtained for the stress function by defining a stress function factor m. This factor is defined to be the ratio of the stress function at a point to the maximum value of the stress function on the section of the shaft. Thus,

$$m = \frac{\phi}{\phi max} = \text{constant}, \qquad (3)$$

where $\phi max = \frac{T}{2\pi} + \phi minimum$ (Equation (6) of section 3.1). If $T = 162\pi$ then $\phi max = 81$ for ϕ minimum = 0. From this we get

$$\phi_2 = \frac{\phi max}{81} \quad \phi_1 \qquad (4)$$

where ϕ_2 is the required value of the stress function at any point for S_2, L_2 and T_2, ϕ_1 is the tabulated value of the stress function at the same point and ϕmax is the maximum value of the stress function for S_2, L_2 and T_2.

Figures 8, 9, 10 and 11 give a general distribution of the shear stress and the stress function over a cross section of the shafts on the basis of the tabulated results. In these figures level lines of constant stress and stress functions are plotted.

6.4 Grooved Shafts

In this case four shafts with grooves having semi-circular bottoms are considered. The same values are chosen for L/S (1.5) and 2r/S (.25, .5, .75 and 1.0), where r, this time, is the radius of semi-circular groove bottom. For convenience of calculation the values of S, L and T, again, are taken to be 2 units, 3 units and 162π respectively. The stress and the stress functions corresponding to these values are tabulated

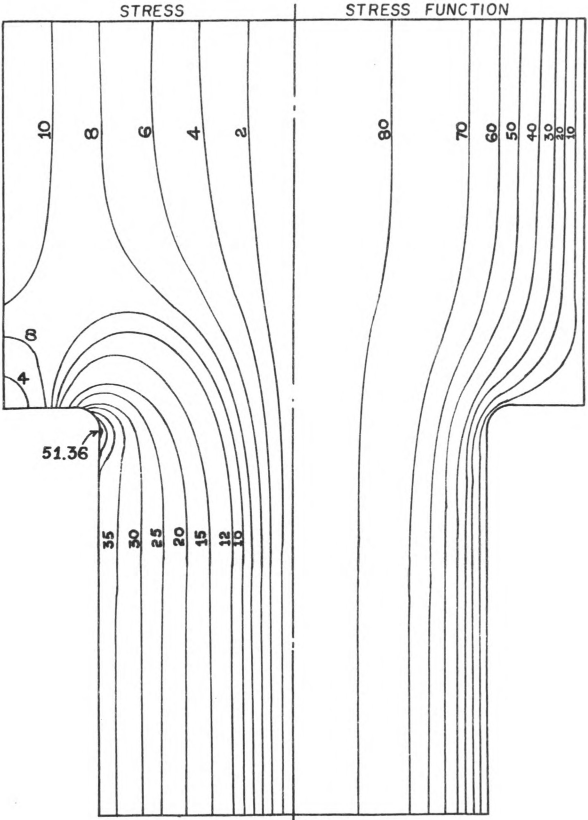

Fig. 8.--Filleted shaft, 2r/S = 0.25

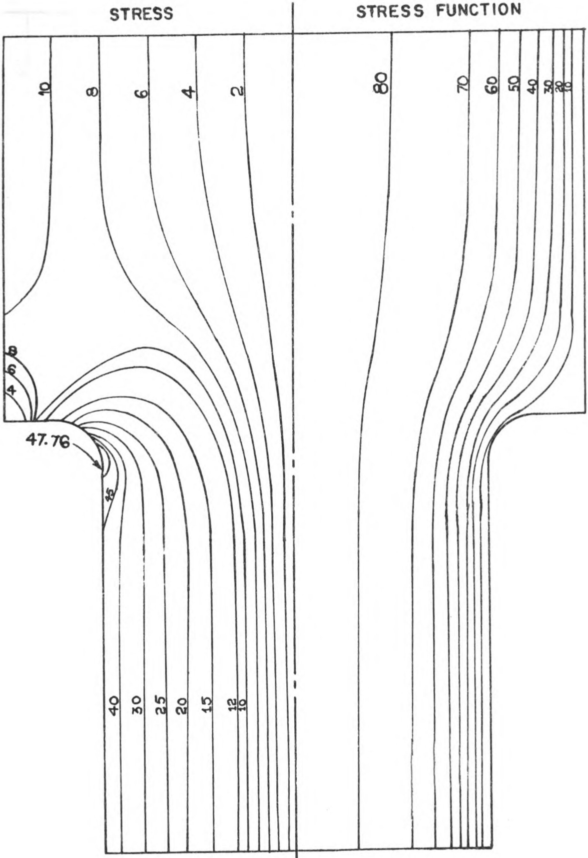

Fig. 9.--Filleted shaft, 2r/S = 0.5

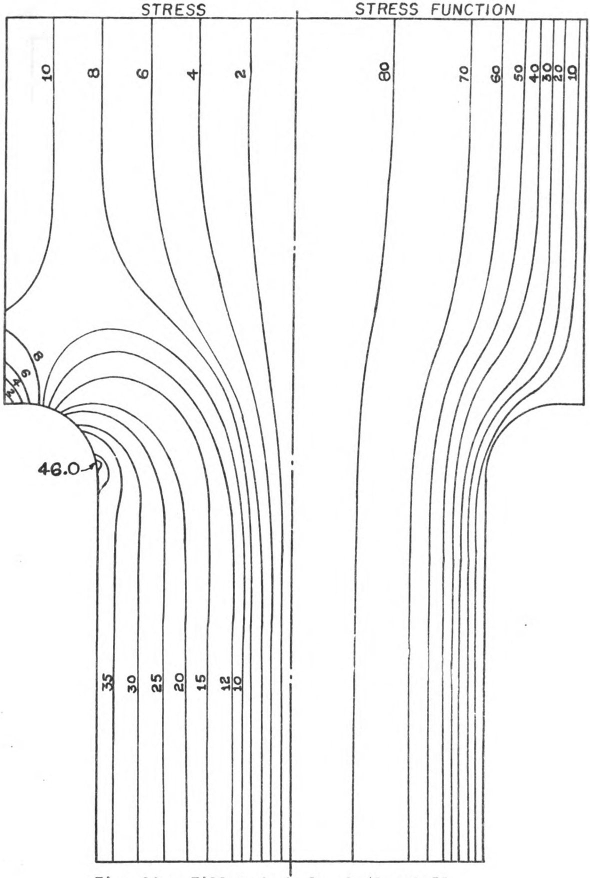

Fig. 10.--Filleted shaft, 2r/S = 0.75

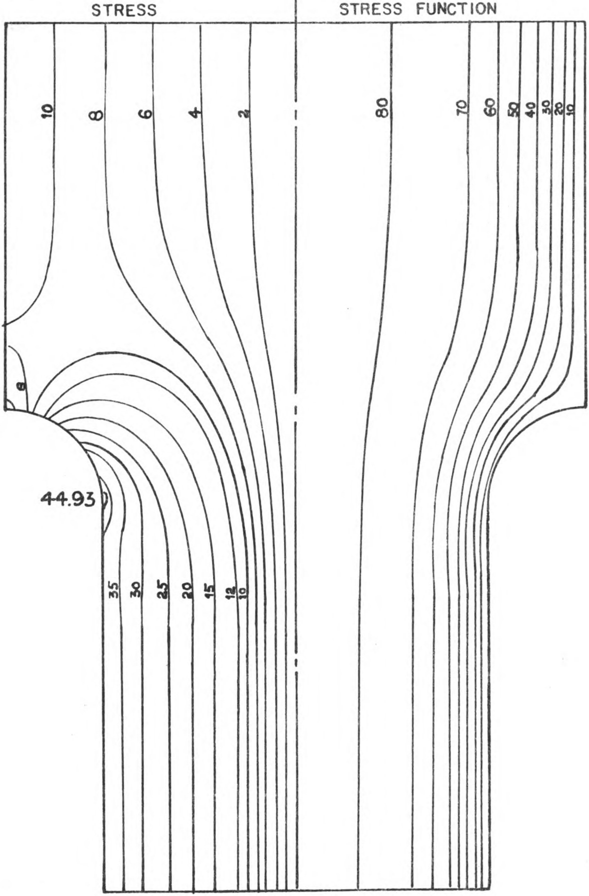

Fig. 11a--Filleted shaft, 2r/S = 1.0

in tables 8 through 11. Since all the dimensions of the grooved shaft are the same as those of the filleted shaft, the same equations (1-4) apply here for getting the shear stress and the stress function for different sets of values of S, L and T. Figures 12 through 15 give the constant stress and the constant stress function level lines corresponding to the tabulated values.

6.5 Running Time and Number of Iterations

The running time on the CDC 3600 digital computer for the programs and the number of iterations, for the examples considered here varied with the type of shaft. There is only a slight variation for the shafts of the same type. Table 1 gives the actual running time and number of iterations for each of the filleted and grooved shafts considered. For the example of the collared shaft, the time taken is 5 minutes, 24 seconds and the number of iterations 142.

Table 1. Running Time and Number of Iterations

	Grooved			Filleted		
	Time		No. of	Time		No. of
2r/d	Min.	Sec.	Iterations	Min.	Sec.	Iterations
.25	6	21	164	6	43	150
.5	6	24	163	7	14	142
.7	6	40	163	7	30	151
1.0	6	40	159	7	14	151

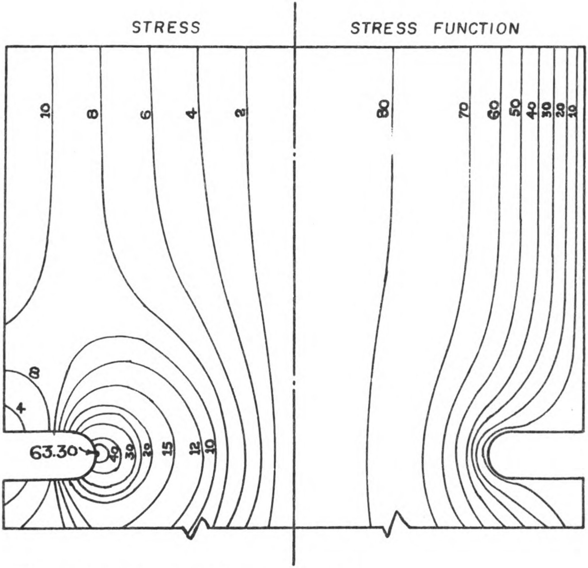

Fig. 12.--Grooved shaft, 2r/S = 0.25

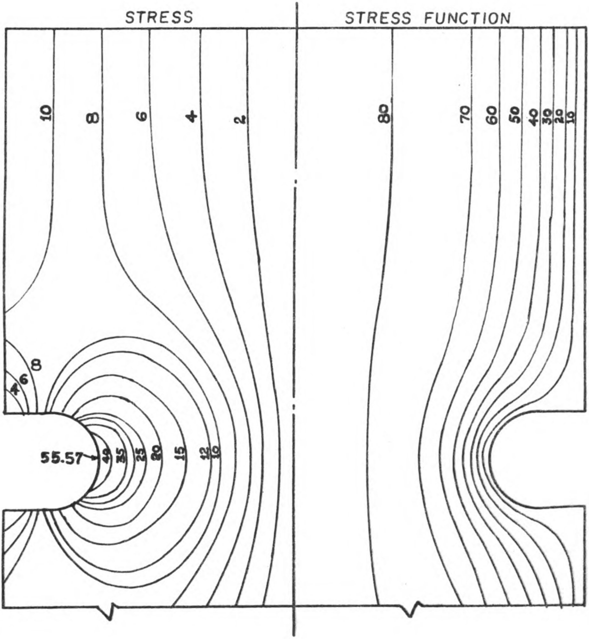

Fig. 13.--Grooved shaft, $2r/S = 0.5$

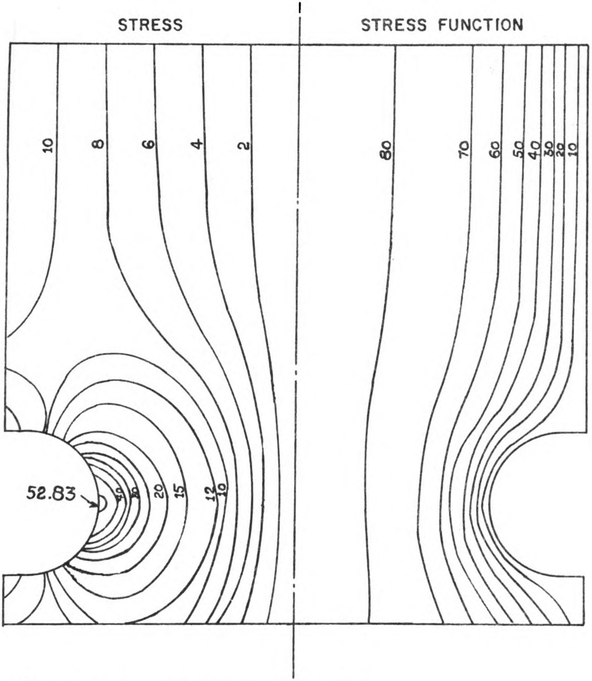

Fig. 14.--Grooved shaft, 2r/S = 0.75

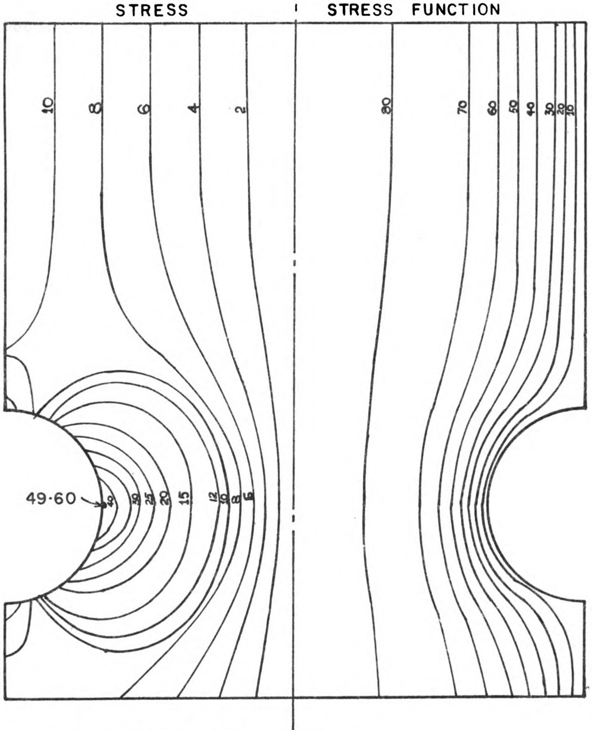

Fig. 15.--Grooved shaft, 2r/S = 1.0

The running time and the numbers of iterations, of course, depends on EC the error criterion (see 5.3). The smaller is the value the EC the larger will be the number of iterations and consequently the larger will be the running time.

In all the calculations, here, the error estimate ER as defined in Section 4.4 is taken to be

$$ER^{(K)} = \sum_{i=1}^{n} \left| X_i^{(K)} - X_i^{(K-1)} \right| \text{ for kth iteration.}$$

Another definition of $ER^K = \max_i \left| \dfrac{X_i^{(K)} - X_i^{(K-1)}}{X_i^{(K)}} \right|$ was also tried with the value of EC = .0001. It was found that the two methods take nearly the same time and the same number of iterations.

6.6 Stress Concentration Factors

From the values of the shear stresses obtained for the filleted and the grooved shafts, the stress concentration factors are calculated. The stress concentration factor K is defined to be the ratio of maximum stress in the shaft to the maximum stress in a shaft of uniform radius, the radius of the uniform shaft being the minimum radius of the non-uniform shaft under consideration. Thus,

$$K = \frac{\tau_{max}}{\tau_m}$$

where τ_m is the same as in equation (1) of Section 6.3.

Table 2 gives the calculated values of the stress concentration factors in each case. An additional value of K corresponding to 2r/S = .125 for both types of the shaft is also tabulated.

Table 2. Stress Concentration Factors for Grooved and Filleted Shafts

	Grooved		Filleted	
2r/S	Max^{m} stress	K	Max^{m} stress	K
.125	74.93	1.85	66.4	1.64
.25	63.20	1.56	51.36	1.28
.5	55.57	1.37	47.76	1.177
.75	52.83	1.30	46.0	1.136
1.0	49.60	1.22	44.93	1.11

In Figure 15 is plotted the stress concentration factor K versus 2r/S for the grooved and the filleted shafts. The dotted line in the figure shows the experimental (methods of electrical analogy and precision strain gages) values of the factor K for a filleted shaft based on the work of Jacobson and of Weigand and given by Peterson (11). Peterson has not given the stress concentration factor values for the grooved shafts of the type solved here.

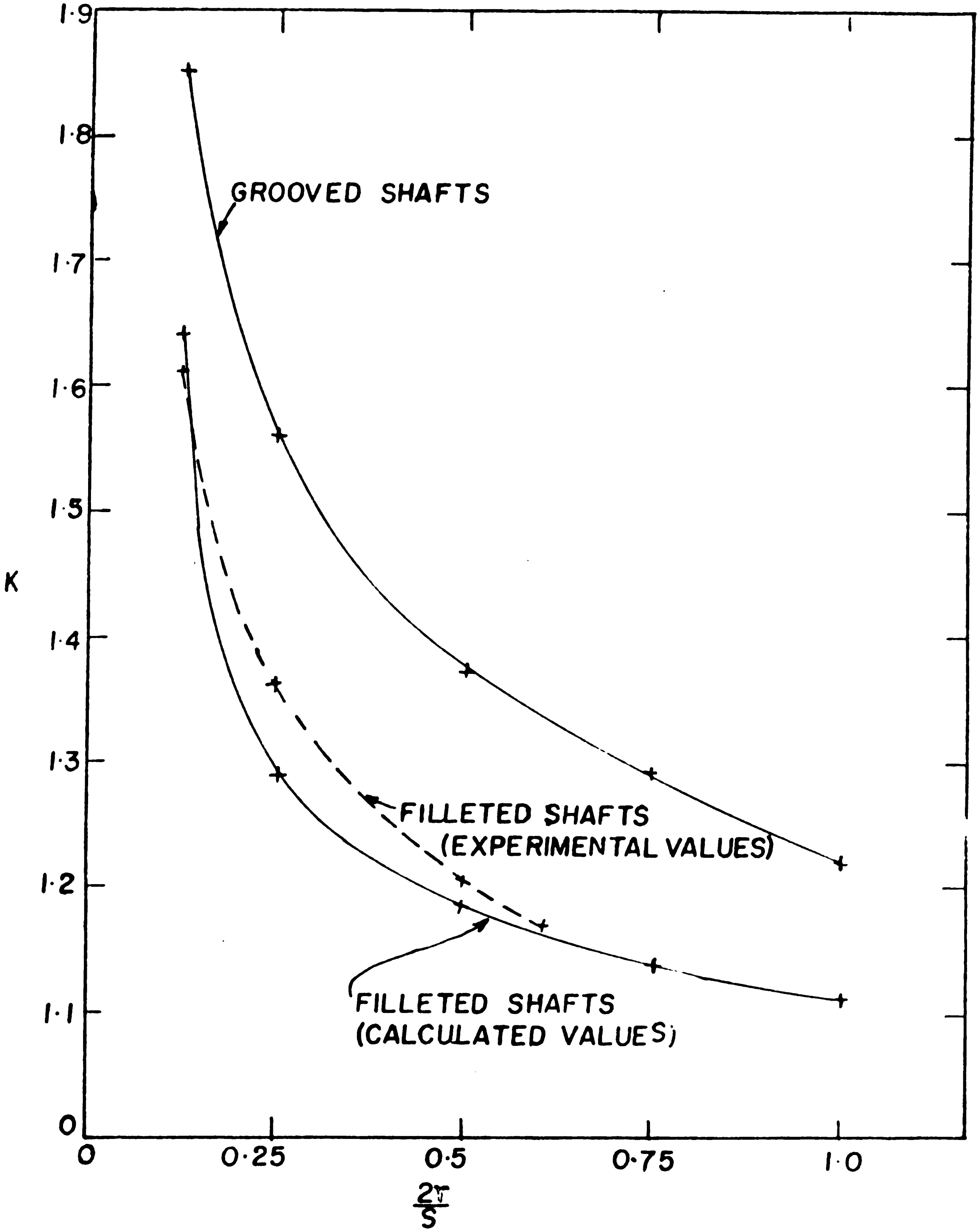

Fig. 16.--Stress concentration factors for grooved and filleted shafts.

CHAPTER VII

CONCLUSION

The iterative technique is found useful for solving the finite difference equations of axially non-uniform circular shafts subjected to a torsional moment. It is found that the method of successive over-relaxation converges rapidly enough so that one can solve this problem on a digital computer. The results obtained by this method for a collared shaft agree closely with those obtained by the relaxation method. The method of iteration, however, is much more convenient and takes far less time. This method allows for all types of non-uniformities in circular shafts without any basic changes in the programming. A fine grid can be used for the solution, restricted only by the capacity of the computer and the available time.

From the figures of stress distribution for filleted and grooved shafts, it can be seen that the maximum stress concentration occurs at the place where the cross-section is minimum. As one goes away from the non-uniformity the stress levels off to the value for the uniform cross-section shaft, which is similar to the case of pure tension or compression. In the case of torsion, however, the actual value of the stress concentration factor is much less than that in the case of tension or compression. The calculated values of the stress concentration factors obtained by the present method are slightly smaller (maximum difference is approximately 6%) than those obtained by the methods of Jacobson (6) based on electrical analogy and of Weigand (21) based on precision strain gages for filleted shafts.

We have by example shown that the finite difference method with the help of the iterative technique furnishes a feasible and convenient way to determine the stress distribution for axially non-uniform circular shafts under torsional couples and that it agrees well in one case with the experimental values.

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APPENDIX A

Table 3A. Calculated Stress Function Values for the Collared Shaft

.3	9.5	52	165	393	772	1306	1934	2533	3015	3368	3614	3782	3896	3977	4039
.3	9.7	53	170	407	807	1385	2072	2687	3144	3461	3677	3824	3923	3993	4046
.3	10.	56	182	443	903	1619	2567	3190	3523	3724	3852	3938	3996	4036	4067
.3	11	61	198	491	1030	1941	3331								
.3	12	65	214	536	1143	2209									
.3	12	69	227	569	1213	2320									
.4	13	72	236	590	1250	2361									
.4	13	74	242	602	1268	2379									
.4	13	75	245	608	1277	2386									
.4	14	75	246	612	1281	2390									
.4	14	76	247	614	1284	2392									
.4	14	76	248	615	1285	2393									

Table 3B. Stress Function Values for the Collared Shaft by Allen

0	10	52	160	373	725	1223	1820	2418	2921	3300	3566	3748	3873	3962	4031
0	10	53	165	386	757	1292	1938	2563	3053	3399	3634	3794	3903	3979	4039
0	11	57	176	423	851	1504	2330	3048	3454	3683	3827	3921	3984	4028	4063
0	11	61	193	473	986	1844	3185								
0	12	65	210	523	1116	2165									
0	13	69	224	561	1199	2305									
0	14	72	234	586	1243	2357									
0	14	74	240	600	1264	2375									
0	14	75	243	607	1275	2385									
0	14	76	246	611	1281	2390									
0	14	76	247	614	1284	2392									
0	14	76	248	615	1286	2394									

Table 3C. Calculated Shear Stress Values for the Collared Shaft

2.6	5.3	8.0	10.6	12.2	12.7	11.9	9.6	7.3	4.6	2.5	1.4	.84	.5	.3	.2	.18
2.7	5.6	8.3	10.7	12.6	13.8	13.6	11.5	7.7	4.6	2.6	1.5	.90	.5	.3	.2	.13
2.8	5.9	8.8	11.6	14.2	16.5	18.4	18.5	10.3	5.5	3.0	1.7	.90	.5	.3	.2	.1
3.0	6.3	9.6	12.9	16.2	19.9	24.3	24.3	14.7	6.8	3.5	1.1	.69	.3	.2	.07	0
3.4	7.2	11.0	14.9	19.1	23.7	28.9	33.1									
3.5	7.5	11.4	15.4	19.6	24.1	28.6	31.9									
3.6	7.7	11.7	15.7	19.9	24.2	28.4	31.7									
3.7	7.8	11.8	15.9	20.0	24.2	28.3	31.6									
3.8	7.9	12.0	16.0	20.1	24.2	28.3	31.6									
3.8	7.9	12.0	16.0	20.1	24.2	28.3	31.6									

Table 4A. Stress Function Values for Filleted Shaft, 2r/S = 0.25

ROW	2	81.00	80.95	80.70	80.03	78.59	75.97	71.65	65.03	55.40	41.96	23.83
ROW	4	81.00	80.95	80.70	80.02	78.58	75.96	71.64	65.02	55.38	41.95	23.82
ROW	6	81.00	80.95	80.70	80.02	78.58	75.95	71.63	65.00	55.36	41.93	23.81
ROW	8	81.00	80.95	80.70	80.02	78.57	75.94	71.61	64.97	55.34	41.91	23.79
ROW	10	81.00	80.95	80.70	80.01	78.56	75.92	71.58	64.94	55.30	41.88	23.77
ROW	12	81.00	80.94	80.69	80.00	78.55	75.90	71.55	64.90	55.26	41.84	23.75
ROW	14	81.00	80.94	80.69	80.00	78.53	75.87	71.51	64.85	55.20	41.79	23.72
ROW	16	81.00	80.94	80.69	79.99	78.50	75.83	71.44	64.77	55.11	41.71	23.67
ROW	18	81.00	80.94	80.68	79.97	78.47	75.76	71.35	64.65	54.98	41.59	23.60
ROW	20	81.00	80.94	80.67	79.94	78.41	75.67	71.21	64.47	54.79	41.42	23.48
ROW	22	81.00	80.94	80.66	79.91	78.33	75.52	70.99	64.19	54.48	41.14	23.31
ROW	24	81.00	80.94	80.64	79.85	78.21	75.31	70.66	63.76	54.01	40.71	23.04
ROW	26	81.00	80.93	80.61	79.77	78.03	74.98	70.16	63.10	53.28	40.04	22.62
ROW	28	81.00	80.92	80.58	79.65	77.76	74.48	69.39	62.08	52.13	39.00	21.96
ROW	30	81.00	80.91	80.52	79.48	77.36	73.73	68.21	60.49	50.33	37.35	20.91
ROW	32	81.00	80.90	80.44	79.23	76.78	72.62	66.41	58.01	47.48	34.74	19.27
ROW	34	81.00	80.88	80.34	78.91	75.98	71.01	63.66	54.10	42.93	30.60	16.70
ROW	36	81.00	80.86	80.22	78.49	74.92	68.74	59.53	47.81	35.50	24.06	12.77
ROW	38	81.00	80.83	80.08	78.00	73.62	65.76	53.47	37.36	22.92	13.98	7.12
ROW	40	80.99	80.81	79.93	77.49	72.21	62.35	45.51	19.23			
ROW	42	80.99	80.79	79.80	77.03	70.96	59.37	38.54				
ROW	44	80.99	80.77	79.69	76.67	70.07	57.51	35.61				
ROW	46	80.99	80.75	79.62	76.43	69.51	56.54	34.52				
ROW	48	80.99	80.74	79.57	76.28	69.20	56.06	34.07				
ROW	0	80.99	80.74	79.53	76.19	69.02	55.82	33.86				
ROW	52	80.99	80.73	79.52	76.15	68.93	55.70	33.76				
ROW	54	80.99	80.73	79.51	76.12	68.88	55.63	33.71				
ROW	56	80.99	80.73	79.50	76.10	68.86	55.60	33.69				
ROW	58	80.99	80.73	79.50	76.10	68.85	55.59	33.68				
ROW	60	80.99	80.73	79.50	76.09	68.84	55.58	33.67				
ROW	62	80.99	80.73	79.50	76.09	68.83	55.57	33.67				
ROW	64	80.99	80.73	79.50	76.09	68.83	55.57	33.66				
ROW	66	80.99	80.73	79.49	76.09	68.83	55.57	33.66				

Table 4B. Shear Stress Values for Filleted Shaft, 2r/S = 0.25

.95	2.00	3.02	4.03	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.96	2.01	3.03	4.04	5.05	6.05	7.04	8.04	9.03	10.03	11.02	11.97
.96	2.02	3.04	4.05	5.06	6.06	7.05	8.04	9.03	10.02	11.02	11.96
.97	2.03	3.05	4.07	5.07	6.07	7.06	8.04	9.03	10.02	11.01	11.95
.97	2.04	3.07	4.08	5.09	6.08	7.06	8.05	9.03	10.01	11.01	11.94
.98	2.05	3.09	4.11	5.11	6.10	7.08	8.05	9.03	10.01	11.00	11.93
.99	2.07	3.11	4.13	5.14	6.12	7.09	8.06	9.03	10.00	10.98	11.91
1.00	2.10	3.15	4.18	5.18	6.16	7.12	8.07	9.03	9.99	10.96	11.89
1.02	2.14	3.20	4.24	5.24	6.21	7.16	8.09	9.03	9.97	10.93	11.85
1.05	2.19	3.28	4.33	5.33	6.29	7.22	8.13	9.03	9.94	10.88	11.79
1.10	2.28	3.40	4.47	5.47	6.42	7.31	8.18	9.03	9.90	10.81	11.70
1.16	2.41	3.58	4.68	5.69	6.62	7.47	8.27	9.04	9.84	10.70	11.56
1.25	2.59	3.84	4.99	6.02	6.93	7.71	8.41	9.07	9.75	10.52	11.34
1.38	2.86	4.23	5.46	6.54	7.43	8.12	8.67	9.13	9.62	10.25	10.99
1.57	3.24	4.78	6.16	7.32	8.22	8.81	9.12	9.27	9.44	9.82	10.44
1.81	3.76	5.55	7.16	8.50	9.47	9.97	9.97	9.60	9.21	9.16	9.57
2.13	4.43	6.58	8.55	10.23	11.44	11.96	11.57	10.36	8.97	8.16	8.23
2.51	5.25	7.87	10.37	12.63	14.44	15.36	14.71	12.09	8.87	6.73	6.22
2.94	6.18	9.37	12.54	15.69	18.69	21.09	21.31	16.04	9.14	4.96	3.41
3.38	7.14	10.91	14.83	19.05	23.87	29.89	37.03	23.90	9.98	4.01	0.00
3.78	8.01	12.30	16.84	21.91	28.07	37.07	51.36				
4.12	8.71	13.38	18.31	23.70	29.84	36.90	43.54				
4.36	9.23	14.13	19.23	24.63	30.39	36.37	41.63				
4.53	9.57	14.60	19.76	25.08	30.55	36.06	40.96				
4.64	9.78	14.88	20.05	25.29	30.60	35.91	40.67				
4.71	9.91	15.04	20.20	25.40	30.62	35.83	40.53				
4.75	9.98	15.13	20.28	25.45	30.62	35.79	40.46				
4.77	10.02	15.18	20.32	25.47	30.62	35.76	40.43				
4.78	10.04	15.20	20.35	25.48	30.62	35.75	40.41				
4.79	10.05	15.22	20.36	25.49	30.62	35.75	40.40				
4.79	10.06	15.22	20.36	25.49	30.62	35.74	40.40				
4.79	10.06	15.23	20.37	25.50	30.62	35.74	40.40				
4.79	10.06	15.23	20.37	25.50	30.62	35.74	40.40				

Table 5A. Stress Function Values for Filleted Shaft, $2r/S = 0.5$

ROW 2	81.00	80.95	80.70	80.02	78.58	75.96	71.63	65.00	55.37	41.94	23.81
ROW 4	81.00	80.95	80.70	80.01	78.57	75.93	71.59	64.96	55.32	41.89	23.78
ROW 6	81.00	80.94	80.69	80.00	78.55	75.90	71.55	64.90	55.25	41.83	23.75
ROW 8	81.00	80.94	80.69	79.99	78.52	75.85	71.48	64.82	55.17	41.76	23.70
ROW 10	81.00	80.94	80.68	79.98	78.49	75.79	71.39	64.70	55.05	41.65	23.63
ROW 12	81.00	80.94	80.68	79.95	78.44	75.71	71.27	64.54	54.87	41.49	23.53
ROW 14	81.00	80.94	80.67	79.92	78.37	75.58	71.08	64.30	54.60	41.25	23.38
ROW 16	81.00	80.94	80.65	79.87	78.26	75.39	70.80	63.93	54.19	40.88	23.14
ROW 18	81.00	80.93	80.63	79.80	78.11	75.11	70.37	63.37	53.56	40.30	22.78
ROW 20	81.00	80.93	80.59	79.70	77.87	74.69	69.71	62.50	52.58	39.39	22.20
ROW 22	81.00	80.92	80.55	79.55	77.54	74.06	68.71	61.15	51.03	37.96	21.29
ROW 24	81.00	80.91	80.48	79.35	77.05	73.13	67.20	59.05	48.58	35.67	19.82
ROW 26	81.00	80.89	80.39	79.07	76.37	71.78	64.92	55.76	44.66	31.97	17.47
ROW 28	81.00	80.87	80.28	78.70	75.46	69.87	61.52	50.60	38.26	25.95	13.70
ROW 30	81.00	80.85	80.15	78.26	74.30	67.33	56.60	42.40	27.46	16.05	7.91
ROW 32	81.00	80.82	80.01	77.76	72.97	64.22	49.96	29.49	7.72		
ROW 34	80.99	80.80	79.87	77.28	71.65	61.03	42.50	11.03			
ROW 36	80.99	80.78	79.75	76.87	70.56	58.54	37.23				
ROW 38	80.99	80.76	79.66	76.56	69.82	57.08	35.12				
ROW 40	80.99	80.75	79.59	76.36	69.37	56.32	34.32				
ROW 42	80.99	80.74	79.55	76.24	69.12	55.95	33.98				
ROW 44	80.99	80.74	79.53	76.17	68.98	55.76	33.82				
ROW 46	80.99	80.73	79.51	76.13	68.91	55.67	33.74				
ROW 48	80.99	80.73	79.50	76.11	68.87	55.62	33.70				
ROW 50	80.99	80.73	79.50	76.10	68.85	55.60	33.68				
ROW 52	80.99	80.73	79.50	76.09	68.84	55.58	33.67				
ROW 54	80.99	80.73	79.50	76.09	68.84	55.58	33.67				
ROW 56	80.99	80.73	79.50	76.09	68.83	55.57	33.67				
ROW 58	80.99	80.73	79.50	76.09	68.83	55.57	33.66				
ROW 60	80.99	80.73	79.49	76.09	68.83	55.57	33.66				
ROW 62	80.99	80.73	79.49	76.09	68.83	55.57	33.66				
ROW 64	80.99	80.73	79.49	76.09	68.83	55.56	33.66				

Table 5B. Shear Stress Values for Filleted Shaft, 2r/S = 0.5

.97	2.03	3.06	4.08	5.08	6.07	7.06	8.04	9.03	10.02	11.01	11.95
.98	2.05	3.09	4.11	5.11	6.10	7.08	8.05	9.03	10.01	10.99	11.93
.99	2.08	3.13	4.15	5.15	6.13	7.10	8.07	9.03	10.00	10.97	11.90
1.01	2.12	3.18	4.21	5.21	6.18	7.14	8.08	9.03	9.98	10.94	11.87
1.04	2.17	3.25	4.29	5.29	6.26	7.19	8.11	9.03	9.95	10.90	11.82
1.08	2.25	3.35	4.41	5.41	6.37	7.28	8.16	9.03	9.92	10.84	11.74
1.13	2.36	3.51	4.59	5.60	6.54	7.41	8.23	9.04	9.87	10.74	11.61
1.21	2.51	3.73	4.86	5.88	6.80	7.62	8.36	9.06	9.79	10.59	11.42
1.32	2.74	4.06	5.26	6.32	7.22	7.96	8.57	9.12	9.68	10.35	11.11
1.48	3.07	4.53	5.84	6.97	7.88	8.53	8.96	9.24	9.52	9.98	10.63
1.70	3.51	5.19	6.69	7.96	8.92	9.49	9.67	9.53	9.33	9.39	9.85
1.97	4.10	6.07	7.87	9.40	10.53	11.10	10.99	10.21	9.16	8.50	8.61
2.32	4.83	7.22	9.44	11.42	12.96	13.79	13.49	11.81	9.22	7.17	6.65
2.72	5.70	8.58	11.39	14.06	16.44	18.13	18.29	15.63	10.09	5.48	3.74
3.15	6.63	10.08	13.58	17.19	20.94	24.69	27.50	21.30	12.61	4.62	0.00
3.57	7.54	11.54	15.73	20.30	25.64	32.78	43.71				
3.93	8.33	12.78	17.50	22.70	28.83	36.98	47.76				
4.23	8.95	13.72	18.72	24.11	30.08	36.66	42.68				
4.44	9.38	14.34	19.47	24.83	30.46	36.23	41.33				
4. 8	9.66	14.73	19.89	25.18	30.57	35.99	40.83				
4.67	9.84	14.95	20.12	25.34	30.61	35.87	40.61				
4.72	9.94	15.08	20.24	25.42	30.62	35.81	40.50				
4.76	10.00	15.15	20.30	25.46	30.62	35.78	40.45				
4.77	10.03	15.19	20.33	25.48	30.62	35.76	40.42				
4.78	10.05	15.21	20.35	25.49	30.62	35.75	40.41				
4.79	10.05	15.22	20.36	25.49	30.62	35.75	40.40				
4.79	10.06	15.22	20.36	25.50	30.62	35.74	40.40				
4.79	10.06	15.23	20.37	25.50	30.62	35.74	40.40				
4.79	10.07	15.23	20.37	25.50	30.62	35.74	40.40				
4.79	10.07	15.23	20.37	25.50	30.62	35.74	40.39				
4.80	10.07	15.24	20.37	25.50	30.62	35.74	40.39				
4.80	10.08	15.24	20.38	25.50	30.62	35.74	40.39				

Table 6A. Stress Function Values for Filleted Shaft, 2r/S ≈ 0.75

ROW	2	81.00	80.95	80.70	80.03	78.59	75.97	71.66	65.03	55.40	41.97	23.83
ROW	4	81.00	80.95	80.70	80.02	78.59	75.96	71.65	65.02	55.39	41.95	23.82
ROW	6	81.00	80.95	80.70	80.02	78.58	75.95	71.63	65.00	55.37	41.94	23.81
ROW	8	81.00	80.95	80.70	80.02	78.57	75.94	71.62	64.98	55.35	41.92	23.80
ROW	10	81.00	80.95	80.70	80.01	78.57	75.93	71.60	64.96	55.32	41.90	23.79
ROW	12	81.00	80.95	80.70	80.01	78.56	75.91	71.57	64.93	55.29	41.87	23.77
ROW	14	81.00	80.94	80.69	80.00	78.54	75.89	71.54	64.89	55.25	41.83	23.74
ROW	16	81.00	80.94	80.69	79.99	78.52	75.86	71.49	64.83	55.18	41.77	23.71
ROW	18	81.00	80.94	80.69	79.98	78.50	75.81	71.43	64.74	55.09	41.69	23.66
ROW	20	81.00	80.94	80.68	79.96	78.46	75.75	71.33	64.62	54.95	41.56	23.57
ROW	22	81.00	80.94	80.67	79.94	78.40	75.65	71.17	64.42	54.73	41.36	23.45
ROW	24	81.00	80.94	80.66	79.90	78.32	75.49	70.94	64.12	54.39	41.05	23.26
ROW	26	81.00	80.93	80.64	79.84	78.19	75.26	70.58	63.65	53.87	40.57	22.95
ROW	28	81.00	80.93	80.61	79.76	78.00	74.91	70.04	62.92	53.04	39.81	22.46
ROW	30	81.00	80.92	80.57	79.63	77.72	74.39	69.22	61.80	51.74	38.59	21.68
ROW	32	81.00	80.91	80.52	79.46	77.31	73.62	67.97	60.06	49.69	36.63	20.40
ROW	34	81.00	80.90	80.44	79.22	76.75	72.50	66.10	57.36	46.39	33.42	18.30
ROW	36	81.00	80.88	80.35	78.91	75.98	70.93	63.34	53.18	41.07	28.07	14.78
ROW	38	81.00	80.86	80.23	78.52	74.98	68.80	59.38	46.76	32.32	18.76	8.92
ROW	40	81.00	80.84	80.10	78.06	73.78	66.11	53.97	37.07	17.80	3.26	
ROW	42	80.99	80.81	79.96	77.58	72.48	63.04	47.29	23.25			
ROW	44	80.99	80.79	79.83	77.13	71.25	60.14	40.67	7.25			
ROW	46	80.99	80.77	79.72	76.76	70.29	58.02	36.49				
ROW	48	80.99	80.76	79.63	76.49	69.66	56.81	34.84				
ROW	50	80.99	80.75	79.58	76.32	69.28	56.19	34.20				
ROW	52	80.99	80.74	79.54	76.22	69.07	55.88	33.92				
ROW	54	80.99	80.74	79.52	76.16	68.96	55.73	33.79				
ROW	56	80.99	80.73	79.51	76.13	68.90	55.65	33.73				
ROW	58	80.99	80.73	79.50	76.11	68.87	55.61	33.70				
ROW	60	80.99	80.73	79.50	76.10	68.85	55.59	33.68				
ROW	62	80.99	80.73	79.50	76.09	68.84	55.58	33.67				
ROW	64	80.99	80.73	79.50	76.09	68.84	55.57	33.67				
ROW	66	80.99	80.73	79.50	76.09	68.83	55.57	33.66				
ROW	68	80.99	80.73	79.49	76.09	68.83	55.57	33.66				

Table 6B. Shear Stress Values for Filleted Shaft, 2r/S = 0.75

.95	2.00	3.02	4.03	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.96	2.01	3.03	4.04	5.05	6.04	7.04	8.04	9.03	10.03	11.03	11.97
.96	2.01	3.04	4.05	5.05	6.05	7.05	8.04	9.03	10.02	11.02	11.96
.96	2.02	3.05	4.06	5.06	6.06	7.05	8.04	9.03	10.02	11.02	11.96
.97	2.03	3.06	4.07	5.08	6.07	7.06	8.04	9.03	10.02	11.01	11.95
.98	2.04	3.07	4.09	5.09	6.08	7.07	8.05	9.03	10.01	11.00	11.94
.98	2.06	3.09	4.11	5.11	6.10	7.08	8.05	9.03	10.01	10.99	11.93
.99	2.08	3.12	4.14	5.14	6.13	7.10	8.06	9.03	10.00	10.98	11.91
1.01	2.10	3.16	4.19	5.19	6.17	7.13	8.08	9.03	9.98	10.96	11.88
1.03	2.14	3.22	4.25	5.25	6.22	7.17	8.10	9.03	9.97	10.92	11.84
1.06	2.21	3.30	4.35	5.35	6.31	7.24	8.14	9.03	9.94	10.87	11.77
1.10	2.29	3.42	4.49	5.50	6.45	7.34	8.20	9.04	9.89	10.79	11.67
1.17	2.43	3.61	4.71	5.74	6.67	7.52	8.30	9.06	9.83	10.66	11.51
1.26	2.62	3.88	5.04	6.09	7.01	7.80	8.48	9.10	9.74	10.46	11.25
1.39	2.89	4.27	5.53	6.63	7.55	8.27	8.80	9.21	9.61	10.14	10.83
1.57	3.26	4.82	6.23	7.44	8.39	9.04	9.37	9.45	9.45	9.64	10.15
1.81	3.76	5.57	7.21	8.62	9.69	10.33	10.44	10.03	9.33	8.85	9.02
2.12	4.40	6.55	8.54	10.28	11.64	12.42	12.39	11.38	9.52	7.67	7.15
2.48	5.18	7.77	10.24	12.51	14.44	15.73	15.91	14.42	11.10	6.14	4.14
2.89	6.06	9.17	12.25	15.28	18.17	20.66	22.02	20.71	11.75	5.44	0.00
3.31	6.97	10.63	14.39	18.33	22.58	27.28	35.61	22.31			
3.70	7.83	12.00	16.37	21.15	26.71	34.06	44.62				
4.04	8.55	13.12	17.94	23.19	29.25	36.84	45.99				
4.30	9.10	13.94	18.99	24.36	30.21	36.50	42.21				
4.49	9.48	14.48	19.62	24.95	30.50	36.14	41.15				
4.61	9.73	14.81	19.97	25.23	30.59	35.95	40.75				
4.69	9.87	15.00	20.16	25.37	30.61	35.85	40.57				
4.74	9.96	15.10	20.26	25.43	30.62	35.80	40.48				
4.76	10.01	15.16	20.31	25.46	30.62	35.77	40.44				
4.78	10.03	15.19	20.34	25.48	30.62	35.76	40.42				
4.78	10.05	15.21	20.35	25.49	30.62	35.75	40.41				
4.79	10.06	15.22	20.36	25.49	30.62	35.75	40.40				
4.79	10.06	15.23	20.37	25.50	30.62	35.74	40.40				

Table 7A. Stress Function Values for the Filleted Shaft, 2r/S = 1.0

ROW 2	81.00	80.95	80.70	80.03	78.59	75.97	71.66	65.03	55.40	41.97	23.83
ROW 4	81.00	80.95	80.70	80.02	78.59	75.97	71.65	65.02	55.39	41.95	23.82
ROW 6	81.00	80.95	80.70	80.02	78.58	75.96	71.63	65.01	55.37	41.94	23.81
ROW 8	81.00	80.95	80.70	80.02	78.57	75.95	71.62	64.99	55.35	41.92	23.80
ROW 10	81.00	80.95	80.70	80.01	78.57	75.93	71.60	64.97	55.33	41.90	23.79
ROW 12	81.00	80.95	80.70	80.01	78.56	75.92	71.58	64.94	55.30	41.88	23.78
ROW 14	81.00	80.94	80.69	80.01	78.55	75.90	71.55	64.91	55.27	41.85	23.75
ROW 16	61.00	80.94	80.69	80.00	78.53	75.88	71.52	64.86	55.22	41.80	23.73
ROW 18	81.00	80.94	80.69	79.99	78.51	75.84	71.46	64.79	55.14	41.73	23.68
ROW 20	81.00	80.94	80.68	79.97	78.48	75.78	71.38	64.69	55.03	41.63	23.62
ROW 22	81.00	80.94	80.68	79.95	78.44	75.70	71.26	64.53	54.85	41.47	23.52
ROW 24	81.00	80.94	80.67	79.92	78.37	75.58	71.08	64.29	54.59	41.23	23.37
ROW 26	81.00	80.94	80.65	79.88	78.26	75.40	70.79	63.92	54.17	40.85	23.12
ROW 28	81.00	80.93	80.63	79.81	78.11	75.12	70.36	63.34	53.51	40.24	22.73
ROW 30	81.00	80.93	80.60	79.71	77.89	74.70	69.70	62.45	52.49	39.28	22.11
ROW 32	81.00	80.92	80.55	79.57	77.56	74.08	68.71	61.07	50.86	37.72	21.10
ROW 34	81.00	80.91	80.49	79.37	77.10	73.19	67.22	58.94	48.26	35.17	19.40
ROW 36	81.00	80.89	80.41	79.11	76.46	71.91	65.03	55.66	44.09	30.92	16.50
ROW 38	81.00	80.87	80.31	78.77	75.62	70.16	61.86	50.65	37.32	23.63	11.33
ROW 40	81.00	80.85	80.18	78.37	74.57	67.87	57.47	43.16	26.29	10.75	1.60
ROW 42	81.00	80.83	80.05	77.91	73.35	65.11	51.79	32.48	8.42		
ROW 44	80.99	80.81	79.92	77.44	72.09	62.15	45.32	18.81			
ROW 46	80.99	80.79	79.79	77.01	70.96	59.51	39.48	5.31			
ROW 48	80.99	80.77	79.69	76.68	70.10	57.66	36.02				
ROW 50	80.99	80.75	79.62	76.44	69.55	56.63	34.65				
ROW 52	80.99	80.74	79.57	76.29	69.22	56.10	34.12				
ROW 54	80.99	80.74	79.54	76.20	69.04	55.84	33.88				
ROW 56	80.99	80.74	79.52	76.15	68.94	55.71	33.77				
ROW 58	80.99	80.73	79.51	76.12	68.89	55.64	33.72				
ROW 60	80.99	80.73	79.50	76.11	68.86	55.61	33.69				
ROW 62	80.99	80.73	79.50	76.10	68.85	55.59	33.68				
ROW 64	80.99	80.73	79.50	76.09	68.84	55.58	33.67				
ROW 66	80.99	80.73	79.50	76.09	68.83	55.57	33.67				
ROW 68	80.99	80.73	79.50	76.09	68.83	55.57	33.66				
ROW 70	80.99	80.73	79.49	76.09	68.83	55.57	33.66				

Table 7B. Shear Stress Values for the Filleted Shaft, 2r/S = 1.0

.95	2.00	3.02	4.03	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.96	2.01	3.03	4.04	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.96	2.01	3.04	4.05	5.05	6.05	7.04	8.04	9.03	10.02	11.02	11.96
.96	2.02	3.05	4.06	5.06	6.06	7.05	8.04	9.03	10.02	11.02	11.96
.97	2.03	3.06	4.07	5.07	6.07	7.06	8.04	9.03	10.02	11.01	11.95
.97	2.04	3.07	4.09	5.09	6.08	7.06	8.05	9.03	10.01	11.01	11.94
.98	2.05	3.09	4.10	5.11	6.09	7.07	8.05	9.03	10.01	11.00	11.93
.99	2.07	3.11	4.13	5.13	6.12	7.09	8.06	9.03	10.00	10.99	11.92
1.00	2.09	3.14	4.16	5.16	6.15	7.11	8.07	9.03	9.99	10.97	11.90
1.02	2.12	3.18	4.21	5.22	6.19	7.15	8.09	9.03	9.98	10.94	11.86
1.04	2.17	3.25	4.29	5.30	6.26	7.20	8.12	9.03	9.95	10.90	11.81
1.08	2.24	3.35	4.41	5.42	6.37	7.28	8.16	9.04	9.92	10.83	11.73
1.13	2.35	3.50	4.58	5.60	6.54	7.42	8.24	9.05	9.87	10.73	11.60
1.20	2.50	3.72	4.85	5.88	6.81	7.64	8.38	9.08	9.79	10.57	11.39
1.31	2.72	4.03	5.23	6.31	7.23	8.00	8.62	9.15	9.68	10.32	11.06
1.46	3.03	4.48	5.80	6.95	7.89	8.59	9.05	9.33	9.55	9.92	10.52
1.66	3.44	5.10	6.60	7.89	8.91	9.57	9.84	9.74	9.43	9.28	9.60
1.92	3.99	5.93	7.70	9.24	10.44	11.17	11.29	10.69	9.51	8.32	8.01
2.24	4.67	6.98	9.14	11.08	12.67	13.70	13.87	12.84	10.43	7.14	5.13
2.62	5.48	8.24	10.92	13.46	15.73	17.49	18.28	17.35	13.84	7.25	0.00
3.02	6.36	9.65	12.94	16.26	19.57	22.74	25.30	26.60			
3.43	7.25	11.06	15.01	19.18	23.78	29.07	35.43				
3.81	8.06	12.35	16.85	21.75	27.41	34.71	44.41				
4.12	8.72	13.37	18.25	23.53	29.51	36.71	44.93				
4.36	9.22	14.10	19.17	24.53	30.29	36.40	41.90				
4. 3	9.56	14.58	19.72	25.03	30.52	36.09	41.04				
4.64	9.77	14.87	20.02	25.27	30.59	35.92	40.70				
4.70	9.90	15.03	20.19	25.39	30.61	35.83	40.54				
4.74	9.97	15.12	20.27	25.44	30.62	35.79	40.47				
4.77	10.02	15.17	20.32	25.47	30.62	35.77	40.43				
4.78	10.04	15.20	20.34	25.48	30.62	35.75	40.42				
4.79	10.05	15.21	20.36	25.49	30.62	35.75	40.41				
4.79	10.06	15.22	20.36	25.49	30.62	35.74	40.40				
4.79	10.06	15.23	20.37	25.50	30.62	35.74	40.40				

Table 8A. Stress Function Values for the Grooved Shaft, $2r/S = 0.25$

ROW 2	81.00	80.95	80.70	80.03	78.59	75.98	71.67	65.05	55.42	41.98	23.83
ROW 4	81.00	80.95	80.70	80.03	78.59	75.98	71.66	65.04	55.41	41.97	23.83
ROW 6	81.00	80.95	80.70	80.03	78.59	75.98	71.66	65.04	55.41	41.97	23.83
ROW 8	81.00	80.95	80.70	80.03	78.59	75.97	71.65	65.03	55.40	41.96	23.83
ROW 10	81.00	80.95	80.70	80.02	78.59	75.97	71.65	65.02	55.39	41.96	23.82
ROW 12	81.00	80.95	80.70	80.02	78.58	75.96	71.64	65.01	55.38	41.95	23.82
ROW 14	81.00	80.95	80.70	80.02	78.58	75.95	71.63	65.00	55.37	41.94	23.81
ROW 16	81.00	80.95	80.70	80.02	78.57	75.95	71.62	64.99	55.35	41.92	23.80
ROW 18	81.00	80.95	80.70	80.01	78.57	75.93	71.60	64.96	55.33	41.90	23.79
ROW 20	81.00	80.95	80.70	80.01	78.56	75.92	71.58	64.93	55.29	41.87	23.77
ROW 22	81.00	80.94	80.69	80.00	78.54	75.89	71.54	64.88	55.24	41.82	23.74
ROW 24	81.00	80.94	80.69	79.99	78.52	75.85	71.48	64.81	55.16	41.75	23.69
ROW 26	81.00	80.94	80.68	79.98	78.49	75.79	71.39	64.70	55.04	41.64	23.63
ROW 28	81.00	80.94	80.68	79.96	78.44	75.71	71.26	64.53	54.85	41.47	23.52
ROW 30	81.00	80.94	80.67	79.92	78.36	75.57	71.06	64.27	54.56	41.21	23.35
ROW 32	81.00	80.94	80.65	79.87	78.25	75.37	70.75	63.86	54.11	40.80	23.09
ROW 34	81.00	80.93	80.62	79.79	78.08	75.06	70.27	63.23	53.40	40.15	22.68
ROW 36	81.00	80.93	80.59	79.68	77.83	74.59	69.54	62.26	52.30	39.14	22.04
ROW 38	81.00	80.92	80.54	79.53	77.46	73.90	68.43	60.73	50.56	37.53	21.02
ROW 40	81.00	80.90	80.47	79.32	76.95	72.88	66.73	58.35	47.78	34.97	19.40
ROW 42	81.00	80.89	80.39	79.04	76.25	71.42	64.16	54.58	43.32	30.87	16.84
ROW 44	81.00	80.88	80.30	78.72	75.38	69.43	60.31	48.49	35.97	24.34	12.91
ROW 46	81.00	80.86	80.21	78.38	74.40	66.95	54.78	38.27	23.35	14.19	7.21
ROW 48	81.00	80.85	80.14	78.12	73.55	64.51	47.89	20.16			
ROW 50	81.00	80.85	80.12	78.01	73.20	63.37	43.74				

Table 8B. Shear Stress Values for the Grooved Shaft, 2r/S = 0.25

.95	1.99	3.01	4.03	5.03	6.03	7.03	8.03	9.03	10.03	11.03	11.97
.95	2.00	3.02	4.03	5.03	6.03	7.03	8.03	9.03	10.03	11.03	11.97
.95	2.00	3.02	4.03	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.95	2.00	3.02	4.03	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.96	2.00	3.03	4.04	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.96	2.01	3.03	4.04	5.05	6.05	7.04	8.04	9.03	10.03	11.02	11.97
.96	2.01	3.04	4.05	5.05	6.05	7.05	8.04	9.03	10.02	11.02	11.96
.96	2.02	3.05	4.06	5.06	6.06	7.05	8.04	9.03	10.02	11.02	11.96
.97	2.03	3.06	4.07	5.07	6.07	7.06	8.04	9.03	10.02	11.01	11.95
.97	2.04	3.07	4.09	5.09	6.08	7.07	8.05	9.03	10.01	11.00	11.94
.98	2.05	3.09	4.11	5.12	6.10	7.08	8.06	9.03	10.01	10.99	11.92
.99	2.08	3.12	4.15	5.15	6.14	7.11	8.07	9.03	10.00	10.97	11.90
1.01	2.11	3.17	4.20	5.21	6.19	7.14	8.09	9.03	9.98	10.94	11.87
1.04	2.17	3.25	4.29	5.29	6.26	7.20	8.12	9.03	9.95	10.90	11.81
1.08	2.25	3.36	4.42	5.43	6.38	7.29	8.17	9.04	9.92	10.83	11.72
1.14	2.36	3.52	4.61	5.63	6.57	7.44	8.25	9.05	9.86	10.72	11.58
1.22	2.53	3.76	4.91	5.95	6.87	7.68	8.40	9.08	9.77	10.55	11.37
1.34	2.77	4.11	5.34	6.43	7.34	8.07	8.65	9.14	9.65	10.28	11.03
1.49	3.10	4.60	5.97	7.16	8.09	8.74	9.10	9.29	9.48	9.87	10.49
1.70	3.54	5.27	6.86	8.24	9.28	9.87	9.94	9.62	9.26	9.22	9.64
1.95	4.08	6.12	8.06	9.79	11.14	11.81	11.55	10.40	9.03	8.23	8.30
2.22	4.68	7.11	9.53	11.88	13.93	15.16	14.74	12.19	8.96	6.80	6.28
2.48	5.26	8.10	11.13	14.39	17.82	20.86	21.60	16.31	9.27	5.02	3.45
2.66	5.69	8.88	12.45	16.73	22.25	29.85	38.64	24.41	10.11	4.07	0.00
2.75	5.85	9.01	12.95	17.75	24.70	37.21	63.30				

Table 9A. Stress Function Values for the Grooved Shaft, 2r/S = 0.5

ROW 2	81.00	80.95	80.70	80.03	78.59	75.98	71.67	65.05	55.41	41.98	23.83
ROW 4	81.00	80.95	80.70	80.03	78.59	75.98	71.66	65.04	55.41	41.97	23.83
ROW 6	81.00	80.95	80.70	80.03	78.59	75.97	71.66	65.04	55.41	41.97	23.83
ROW 8	81.00	80.95	80.70	80.03	78.59	75.97	71.65	65.03	55.40	41.96	23.83
ROW 10	81.00	80.95	80.70	80.02	78.59	75.97	71.65	65.02	55.39	41.95	23.82
ROW 12	81.00	80.95	80.70	80.02	78.58	75.96	71.64	65.01	55.38	41.95	23.82
ROW 14	81.00	80.95	80.70	80.02	78.58	75.95	71.63	65.00	55.37	41.94	23.81
ROW 16	81.00	80.95	80.70	80.02	78.57	75.95	71.62	64.99	55.35	41.92	23.80
ROW 18	81.00	80.95	80.70	80.02	78.57	75.94	71.60	64.97	55.33	41.90	23.79
ROW 20	81.00	80.95	80.70	80.01	78.56	75.92	71.58	64.94	55.30	41.88	23.77
ROW 22	81.00	80.94	80.69	80.01	78.55	75.90	71.55	64.90	55.26	41.83	23.75
ROW 24	81.00	80.94	80.69	80.00	78.53	75.86	71.50	64.83	55.19	41.77	23.71
ROW 26	81.00	80.94	80.69	79.98	78.50	75.81	71.42	64.74	55.08	41.68	23.65
ROW 28	81.00	80.94	80.68	79.96	78.45	75.74	71.31	64.59	54.92	41.53	23.55
ROW 30	81.00	80.94	80.67	79.93	78.39	75.62	71.13	64.36	54.66	41.30	23.41
ROW 32	81.00	80.94	80.65	79.89	78.29	75.44	70.86	64.00	54.27	40.94	23.18
ROW 34	81.00	80.93	80.63	79.82	78.14	75.17	70.44	63.45	53.65	40.37	22.82
ROW 36	81.00	80.93	80.60	79.72	77.92	74.76	69.80	62.60	52.68	39.48	22.25
ROW 38	81.00	80.92	80.56	79.58	77.59	74.15	68.83	61.28	51.16	38.06	21.35
ROW 40	81.00	80.91	80.50	79.39	77.13	73.25	67.36	59.22	48.74	35.79	19.89
ROW 42	81.00	80.89	80.42	79.13	76.50	71.97	65.15	55.99	44.86	32.11	17.54
ROW 44	81.00	80.88	80.32	78.81	75.66	70.17	61.86	50.91	38.50	26.10	13.78
ROW 46	81.00	80.86	80.22	78.43	74.65	67.83	57.14	42.82	27.70	16.17	7.96
ROW 48	81.00	80.84	80.12	78.06	73.57	65.10	50.86	30.00	7.81		
ROW 50	81.00	80.83	80.04	77.78	72.70	62.65	44.17	11.46			
ROW 52	81.00	80.83	80.01	77.67	72.35	61.62	40.95				

Table 9B. Shear Stress Values for the Grooved Shaft, 2r/S = 0.5

.95	2.00	3.02	4.03	5.03	6.03	7.03	8.03	9.03	10.03	11.03	11.97
.95	2.00	3.02	4.03	5.03	6.03	7.03	8.03	9.03	10.03	11.03	11.97
.95	2.00	3.02	4.03	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.95	2.00	3.02	4.03	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.96	2.01	3.03	4.04	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.96	2.01	3.03	4.04	5.05	6.05	7.04	8.04	9.03	10.03	11.02	11.97
.96	2.01	3.04	4.05	5.06	6.05	7.05	8.04	9.03	10.02	11.02	11.96
.96	2.02	3.05	4.06	5.06	6.06	7.05	8.04	9.03	10.02	11.02	11.96
.97	2.03	3.05	4.07	5.07	6.07	7.06	8.04	9.03	10.02	11.01	11.95
.97	2.04	3.07	4.08	5.09	6.08	7.07	8.05	9.03	10.01	11.01	11.94
.98	2.05	3.09	4.11	5.11	6.10	7.08	8.05	9.03	10.01	10.99	11.93
.99	2.07	3.12	4.14	5.14	6.13	7.10	8.06	9.03	10.00	10.98	11.91
1.01	2.10	3.16	4.19	5.19	6.17	7.13	8.08	9.03	9.98	10.95	11.88
1.03	2.15	3.22	4.26	5.27	6.24	7.18	8.11	9.03	9.96	10.91	11.83
1.06	2.22	3.32	4.37	5.38	6.34	7.26	8.15	9.03	9.93	10.85	11.75
1.12	2.32	3.46	4.55	5.56	6.50	7.39	8.23	9.04	9.88	10.76	11.63
1.19	2.47	3.68	4.80	5.83	6.76	7.59	8.35	9.07	9.80	10.61	11.44
1.30	2.69	3.99	5.19	6.25	7.17	7.93	8.56	9.12	9.69	10.37	11.14
1.44	2.99	4.43	5.75	6.89	7.81	8.50	8.95	9.25	9.54	10.00	10.66
1.64	3.40	5.05	6.55	7.83	8.83	9.44	9.65	9.54	9.36	9.43	9.89
1.88	3.93	5.86	7.64	9.20	10.40	11.04	10.98	10.23	9.19	8.53	8.65
2.17	4.56	6.86	9.07	11.10	12.75	13.70	13.50	11.85	9.26	7.21	6.68
2.48	5.24	7.99	10.76	13.53	16.11	18.04	18.37	15.74	10.16	5.52	3.76
2.77	5.89	9.09	12.50	16.25	20.37	24.67	27.87	20.41	12.72	4.80	0.00
2.98	6.36	9.91	13.88	18.58	24.60	33.07	45.51				
3.17	8.00	11.25	14.44	19.91	26.50	37.80	55.57				

Table 10A. Stress Function Values for the Grooved Shaft, 2r/S = 0.75

ROW 2	81.00	80.95	80.70	80.03	78.59	75.98	71.67	65.05	55.41	41.98	23.83
ROW 4	81.00	80.95	80.70	80.03	78.59	75.98	71.66	65.04	55.41	41.97	23.83
ROW 6	81.00	80.95	80.70	80.03	78.59	75.97	71.66	65.04	55.41	41.97	23.83
ROW 8	81.00	80.95	80.70	80.03	78.59	75.97	71.65	65.03	55.40	41.96	23.83
ROW 10	81.00	80.95	80.70	80.02	78.59	75.97	71.65	65.02	55.39	41.96	23.82
ROW 12	81.00	80.95	80.70	80.02	78.58	75.96	71.64	65.01	55.38	41.95	23.82
ROW 14	81.00	80.95	80.70	80.02	78.58	75.96	71.63	65.00	55.37	41.94	23.81
ROW 16	81.00	80.95	80.70	80.02	78.58	75.95	71.62	64.99	55.36	41.93	23.80
ROW 18	81.00	80.95	80.70	80.02	78.57	75.94	71.61	64.97	55.34	41.91	23.79
ROW 20	81.00	80.95	80.70	80.01	78.56	75.93	71.59	64.95	55.31	41.89	23.78
ROW 22	81.00	80.95	80.69	80.01	78.55	75.91	71.56	64.91	55.27	41.85	23.76
ROW 24	81.00	80.94	80.69	80.00	78.54	75.88	71.52	64.86	55.22	41.80	23.72
ROW 26	81.00	80.94	80.69	79.99	78.51	75.84	71.46	64.78	55.13	41.72	23.67
ROW 28	81.00	80.94	80.68	79.97	78.47	75.77	71.36	64.65	54.99	41.59	23.59
ROW 30	81.00	80.94	80.67	79.95	78.42	75.67	71.21	64.46	54.77	41.40	23.47
ROW 32	81.00	80.94	80.66	79.91	78.33	75.52	70.98	64.16	54.44	41.09	23.28
ROW 34	81.00	80.94	80.64	79.85	78.21	75.29	70.63	63.70	53.92	40.62	22.97
ROW 36	81.00	80.93	80.62	79.77	78.02	74.95	70.09	62.98	53.10	39.86	22.49
ROW 38	81.00	80.92	80.58	79.65	77.75	74.43	69.28	61.87	51.81	38.65	21.71
ROW 40	81.00	80.91	80.52	79.48	77.36	73.68	68.05	60.15	49.77	36.70	20.44
ROW 42	81.00	80.90	80.45	79.26	76.81	72.60	66.21	57.47	46.50	33.50	18.34
ROW 44	81.00	80.89	80.37	78.96	76.07	71.07	63.50	53.34	41.19	28.15	14.82
ROW 46	81.00	80.87	80.26	78.60	75.14	69.03	59.63	46.97	32.46	18.83	8.95
ROW 48	81.00	80.85	80.15	78.21	74.06	66.50	54.37	37.35	17.91	3.27	
ROW 50	81.00	80.83	80.04	77.83	72.96	63.74	47.98	23.58			
ROW 52	81.00	80.82	79.97	77.55	72.12	61.45	41.97	7.49			
ROW 54	80.99	80.81	79.94	77.44	71.79	60.53	39.41				

Table 10B. Shear Stress Values for the Grooved Shaft, 2r/S = 0.75

.95	2.00	3.01	4.03	5.03	6.03	7.03	8.03	9.03	10.03	11.03	11.97
.95	2.00	3.02	4.03	5.03	6.03	7.03	8.03	9.03	10.03	11.03	11.97
.95	2.00	3.02	4.03	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.95	2.00	3.02	4.03	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.96	2.00	3.03	4.04	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.96	2.01	3.03	4.04	5.05	6.05	7.04	8.04	9.03	10.03	11.02	11.97
.96	2.01	3.04	4.05	5.05	6.05	7.05	8.04	9.03	10.02	11.02	11.96
.96	2.02	3.04	4.06	5.06	6.06	7.05	8.04	9.03	10.02	11.02	11.96
.97	2.02	3.05	4.07	5.07	6.06	7.05	8.04	9.03	10.02	11.01	11.95
.97	2.03	3.06	4.08	5.08	6.08	7.06	8.05	9.03	10.02	11.01	11.95
.98	2.04	3.08	4.10	5.10	6.09	7.07	8.05	9.03	10.01	11.00	11.93
.99	2.06	3.10	4.12	5.13	6.11	7.09	8.06	9.03	10.00	10.99	11.92
1.00	2.09	3.14	4.17	5.17	6.15	7.12	8.07	9.03	9.99	10.96	11.89
1.02	2.13	3.19	4.23	5.23	6.21	7.16	8.10	9.03	9.97	10.93	11.85
1.05	2.19	3.27	4.32	5.33	6.29	7.23	8.13	9.03	9.94	10.88	11.78
1.09	2.27	3.40	4.47	5.48	6.43	7.33	8.19	9.04	9.90	10.80	11.68
1.15	2.40	3.58	4.68	5.71	6.65	7.50	8.30	9.06	9.84	10.67	11.52
1.25	2.59	3.84	5.01	6.06	6.98	7.78	8.48	9.11	9.75	10.47	11.26
1.37	2.85	4.22	5.48	6.59	7.52	8.25	8.79	9.21	9.62	10.16	10.85
1.55	3.21	4.75	6.16	7.38	8.35	9.02	9.37	9.46	9.46	9.66	10.17
1.77	3.68	5.47	7.11	8.53	9.63	10.29	10.43	10.04	9.35	8.87	9.05
2.05	4.27	6.38	8.37	10.14	11.55	12.38	12.39	11.40	9.54	7.69	7.17
2.36	4.96	7.49	9.95	12.27	14.29	15.68	15.93	14.47	11.14	6.16	4.16
2.69	5.69	8.68	11.74	14.86	17.93	20.63	22.13	20.84	11.47	5.46	0.00
2.99	6.35	9.81	13.51	17.59	22.18	27.35	32.60	7.52			
3.20	6.83	10.63	14.85	19.78	25.96	34.44	44.24				
3.80	7.40	12.11	15.32	20.61	27.59	37.6	52.83				

Table 11A. Stress Function Values for the Grooved Shaft, 2r/S = 1.0

ROW	2	81.00	80.95	80.70	80.03	78.59	75.98	71.66	65.04	55.41	41.97	23.83
ROW	4	81.00	80.95	80.70	80.03	78.59	75.97	71.66	65.03	55.40	41.96	23.83
ROW	6	81.00	80.95	80.70	80.02	78.59	75.97	71.65	65.02	55.39	41.95	23.82
ROW	8	81.00	80.95	80.70	80.02	78.58	75.96	71.64	65.01	55.38	41.94	23.81
ROW	10	81.00	80.95	80.70	80.02	78.58	75.95	71.62	64.99	55.36	41.93	23.80
ROW	12	81.00	80.95	80.70	80.02	78.57	75.94	71.60	64.97	55.33	41.90	23.79
ROW	14	81.00	80.95	80.70	80.01	78.56	75.92	71.58	64.94	55.30	41.87	23.77
ROW	16	81.00	80.94	80.69	80.00	78.55	75.90	71.54	64.89	55.25	41.83	23.74
ROW	18	81.00	80.94	80.69	80.00	78.52	75.86	71.49	64.82	55.18	41.76	23.70
ROW	20	81.00	80.94	80.69	79.98	78.49	75.81	71.41	64.72	55.06	41.66	23.64
ROW	22	81.00	80.94	80.68	79.96	78.45	75.73	71.29	64.57	54.89	41.51	23.54
ROW	24	81.00	80.94	80.67	79.93	78.38	75.60	71.11	64.33	54.62	41.26	23.39
ROW	26	81.00	80.94	80.65	79.88	78.28	75.42	70.82	63.96	54.21	40.88	23.14
ROW	28	81.00	80.93	80.63	79.82	78.13	75.14	70.40	63.38	53.56	40.28	22.76
ROW	30	81.00	80.93	80.60	79.72	77.91	74.73	69.74	62.50	52.53	39.31	22.13
ROW	32	81.00	80.92	80.56	79.58	77.58	74.12	68.76	61.13	50.91	37.76	21.12
ROW	34	81.00	80.91	80.50	79.39	77.13	73.24	67.29	59.01	48.33	35.22	19.43
ROW	36	81.00	80.89	80.42	79.14	76.51	71.99	65.11	55.75	44.16	30.97	16.53
ROW	38	81.00	80.88	80.32	78.82	75.70	70.28	61.99	50.77	37.41	23.68	11.35
ROW	40	81.00	80.86	80.21	78.44	74.70	68.06	57.67	43.32	26.38	10.77	1.60
ROW	42	81.00	80.84	80.10	78.03	73.59	65.44	52.12	32.68	8.47		
ROW	44	81.00	80.82	79.99	77.65	72.51	62.74	45.88	19.03			
ROW	46	80.99	80.81	79.92	77.38	71.70	60.62	40.56	5.46			
ROW	48	80.99	80.81	79.89	77.28	71.40	59.80	38.44				

Table 11B. Shear Stress Values for the Grooved Shaft, 2r/S = 1.0

.95	2.00	3.02	4.03	5.03	6.04	7.03	8.03	9.03	10.03	11.03	11.97
.95	2.00	3.02	4.03	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.96	2.00	3.03	4.04	5.04	6.04	7.04	8.03	9.03	10.03	11.03	11.97
.96	2.01	3.03	4.05	5.05	6.05	7.04	8.04	9.03	10.03	11.02	11.96
.96	2.02	3.04	4.06	5.06	6.06	7.05	8.04	9.03	10.02	11.02	11.96
.97	2.02	3.05	4.07	5.07	6.07	7.06	8.04	9.03	10.02	11.01	11.95
.97	2.04	3.07	4.08	5.09	6.08	7.07	8.05	9.03	10.01	11.01	11.94
.98	2.05	3.09	4.11	5.11	6.10	7.08	8.05	9.03	10.01	10.99	11.93
.99	2.07	3.12	4.14	5.15	6.13	7.10	8.07	9.03	10.00	10.98	11.91
1.01	2.11	3.16	4.19	5.20	6.18	7.14	8.08	9.03	9.98	10.95	11.87
1.03	2.15	3.23	4.27	5.28	6.25	7.19	8.11	9.03	9.96	10.91	11.82
1.07	2.23	3.33	4.39	5.39	6.36	7.27	8.16	9.04	9.92	10.84	11.74
1.12	2.33	3.48	4.56	5.58	6.53	7.41	8.24	9.05	9.87	10.74	11.61
1.19	2.48	3.69	4.82	5.86	6.79	7.63	8.38	9.08	9.80	10.58	11.40
1.30	2.70	4.00	5.20	6.28	7.21	7.98	8.62	9.16	9.69	10.33	11.07
1.44	3.00	4.44	5.76	6.91	7.86	8.58	9.05	9.33	9.56	9.93	10.53
1.64	3.40	5.04	6.54	7.84	8.87	9.55	9.84	9.74	9.44	9.29	9.61
1.89	3.92	5.84	7.61	9.17	10.39	11.15	11.28	10.70	9.53	8.33	8.02
2.18	4.56	6.84	9.00	10.97	12.60	13.67	13.87	12.86	10.45	7.16	5.14
2.52	5.29	8.00	10.68	13.26	15.61	17.46	18.31	17.40	13.88	7.25	0.00
2.86	6.04	9.23	12.52	15.92	19.39	22.74	25.42	26.71			
3.16	6.71	10.36	14.26	18.57	23.47	29.17	35.61				
3.37	7.18	11.16	15.54	20.61	26.83	35.10	46.10				
4.00	9.11	12.52	16.00	21.31	28.19	37.28	49.60				

APPENDIX B

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      PROGRAM ITSLN4
      DIMENSION X(8000),JN(500),AH(100),BH(100),BR(100),
     1ZB(500),YB(8000),ER(40)
      COMMON X,JN,IR,I,AH,BH,IGR,BR
      NH=8
  900 FORMAT (3HROW,I3,11(F6.2))
  920 FORMAT(12(F6.2))
      NG=5
      DN=1.0
      W=1.3
      IGS=0
      NF=0
      CALL BOUGEN (NH,NC)
      CALL MATGN1 (NC,DN,NF,NG,NH,W)
      K=I
  100 I=2
      LS=25
      L=LS+JN(I)-2
  103 PRINT 900,(I,(X(J),J=LS,L,2))
      I=I+2
      IB=I-2
      IA=I-1
      LS=LS+JN(IB)+JN(IA)
      L=LS+JN(I)-2
  106 IF(I-K)103,103,107
  107 I=1
 1000 HN=FLOATF(NH)/2.0
      IR=JN(I)
      KL=K-1
      DO 1006 I=2,KL
  398 R=0.0
      NP=JN(I)
      IA=I-1
      DO 1006 J=1,NP
      IR=IR+1
      R=FLOATF(J)/FLOATF(NH)
      KR=IR+1
      LR=IR-JN(IA)
      MR=IR-1
      NR=IR+JN(I)
      TX=(X(MR)-X(KR))*HN
      TY=(X(LR)-X(NR))*HN
      IF(J-1)1003,1003,1004
 1003 TX=(81.0-X(KR))*HN
 1004 IF(J-JN(I))1006,1005,1005
 1005 TX=X(MR)*HN
      ZB(I)=(4.0*X(IR)-X(MR))*HN/(R+1.0/FLOATF(NH))**2
 1006 YB(IR)=(SQRTF(TX**2+TY**2))/(R**2)
      K=K-2
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(CONTINUED ON NEXT PAGE

```
  200 I=2
      LS=25
      L=LS+JN(I)-2
  203 PRINT920,(YB(J),J=LS,L,2),ZB(I)
      I=I+2
      IB=I-2
      IA=I-1
      LS=LS+JN(IA)+JN(IB)
      L=LS+JN(I)-2
  206 IF(I-K)203,203,207
  207 CONTINUE
 1017 CONTINUE
      END
      SUBROUTINE MATGN1 (NC,DN,NF,NG,NH,W)
      DIMENSION X(8000),JN(500),AH(100),BH(100),BR(100)
      COMMON X,JN,IR,I,AH,BH,IGR,BR
  900 FORMAT (18H NO OF ITERATIONS=,I3)
  350 FORMAT(1H0,3HAA=,F15.6,3HAC=,F15.6,3HAD=,F15.6)
  250 FORMAT(1H0,3HAH(,I2,2H)=,F15.6,3HBH(,I2,2H)=,F15.6)
      DO 460 L1=1,8000
  460 X(L1)=0.0
      KNT=0
      CN =FLOATF(NC)
      HN=FLOATF(NH)
      FN=FLOATF(NF)
      GN=FLOATF(NG)
  470 ER=0.0
      IGR=0
      IM=0
      Z=1.0/HN
      I=1
      IR=0
  300 IF(Z-GN+1.0/HN)301,312,302
  301 Y=3.0
      YB=Y
      CO=0.0
      CURVE=0.
      GO TO 306
  312 Y=3.0
      YB=3.0
      GO TO 306
  302 IF(Z-GN-DN)303,310,304
  303 ZA=Z-GN
      ZB=ZA+1.0/HN
      CO=0.0
C     CO=0.IF Y=3. ATZ=GN.CO=NON ZERO IF Y ISNOT 3. AT Z=GN
      CURVE=1.
      GO TO 311
  310 ZA=Z-GN
```

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```
      ZB=ZA
      CURVE=0.
  311 Y=-(SQRTF(1.0-(ZA-1.)**2))+3.0
      YB=-(SQRTF(1.0-(ZB-1.)**2))+3.0
      GO TO 306
  304 IF(Z-GN-DN-FN)305,305,6
  305 Y=Y
      YB=Y
      CURVE=0.
  306 JN(I)=0
      R=1.0/HN
  307 IF(Y-R)409,409,308
  308 R=R+1.0/HN
      JN(I)=JN(I)+1
      GO TO 307
  409 NP=JN(I)
      PN=FLOATF(NP)
      DO 200 J=1,NP
      IR=IR+1
      R=FLOATF(J)/HN
      AA=4.0
      AB=-(1.0-(3.0/(2.0*HN*R)))
      AC=-1.0
      AD=-(1.0+(3.0/(2.0*HN*R)))
      AE=-1.0
      IF(I-1)1,1,2
    1 AC=0.0
      IM=IM+1
      B=BR(IM)+B
    2 IF(J-1)3,3,4
    3 AD=0.0
      B=B+81.0*2.5
    4 IF(Z-GN-DN-FN)7,5,5
    5 AE=0.0
      IM=IM+1
      AC=-2.0
    7 IA=I-1
      IB=I+1
      JN(IB)=0
      RB=1.0/HN
   30 IF(YB-RB)32,32,31
   31 RB=RB+1.0/HN
      JN(IB)=JN(IB)+1
      GO TO 30
   32 IF(J-JN(IB))10,704,702
  704 IF(J-JN(I))10,703,10
  702 IF(J-JN(I)) 701,703,10
  703 IF(CURVE)9,9,22
  701 IF(CURVE)708,709,708
```

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```
709 AE=0.0
    GO TO 10
  9 AB=0.
    IF(CO-2.0)10,710,10
710 AE=0.0
    GO TO 10
 22 IF(JN(I)-JN(IB))33,33,34
 33 AB=0.0
    IGR=IGR+1
    BH(IGR)=1.0
    AE=-1.0
    GO TO 35
 34 AB=0.0
708 AE=0.0
    IGR=IGR+1
399 BH(IGR)=-(SQRTF(1.0-(R-3.0)**2))+1.0
    BH(IGR)=(BH(IGR)-ZA)*FLOATF(NH)
 35 IF(J-JN(I))705,706,706
705 AH(IGR)=1.0
    GO TO 707
706 AH(IGR)=(Y-R)*HN
707 CONTINUE
    AA=(2./AH(IGR)+2./BH(IGR)-(3.*(1.-AH(IGR)))/(J*2.
   1*AH(IGR))
    AC=-(2./(1+BH(IGR)))
    AD=-(2./(1+AH(IGR)+3./(2.*J))
    IF(KNT-1)23,24,24
 23 PRINT 250,IGR,AH(IGR),IGR,BH(IGR)
    PRINT 350,AA,AC,AD
 24 CONTINUE
 10 IF(I-1)11,11,12
 11 LR=0
    GO TO 13
 12 LR=IR-JN(IA)
 13 KR=IR+1
    IF(J-1)14,14,15
 14 MR=1
    GO TO 16
 15 MR=IR-1
 16 NR=IR+JN(I)
 20 Q=0.0
    Q=AB*X(KR)+AC*X(LR)+AD*X(MR)+AE*X(NR)
    P=(B-Q)*(W/AA)+(1.0-W)*X(IR)
    D=(X(IR)-P)/P
    IF(ER-ABSF(D))71,72,72
 71 ER=ABSF(D)
 72 X(IR)=P
200 B=0.0
    Z=Z+1.0/HN
```

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```
      I=I+1
      GO TO 300
    6 KNT=KNT+1
      IF(.0001-ER)470,21,21
   21 PRINT 900,KNT
      END
      SUBROUTINE BOUGEN (NH,NC)
      DIMENSION X(8000),JN(500),AH(100),BH(100),BR(100)
      COMMON X,JN,IR,I,AH,BH,IGR,BR
      ID=0
      DO 150 JA=1,100
  150 BR(JA)=0.0
      W=1.0
      HN=FLOATF(NH)
      Y=3.0
      N=23
    1 IR=0
      ER=0.0
      IF(Y-2.0)2,9,2
    9 IR=ID
      N=15
    2 DO 7 J=1,N
      R=FLOATF(J)/HN
      T=0.
      IR=IR+1
      AA=2.0
      AB=-(1.0-(3.0/(2.0*HN*R)))
      AC=-(1.0+(3.0/(2.0*HN*R)))
      IF(J-1)4,3,4
    3 AC=0.0
      T=81.*2.5
    4 GO TO 6
    6 KR=IR+1
      MR=IR-1
      Q=0.0
      Q=AB*BR(KR)+AC*BR(MR)
      P=(T-Q)*(W/AA)+(1.0-W)*BR(IR)
      D=(BR(IR)-P)/P
      IF(ER-ABSF(D))71,72,72
   71 ER=ABSF(D)
   72 CONTINUE
    7 BR(IR)=P
      IF(.0001-ER)1,8,8
    8 Y=2.0
      ID=IR
      IF(N-15)1,10,1
   10 PRINT 100,(BR(J),J=1,IR)
  100 FORMAT(1H0,17(F7.1))
      END
      END
```