

A STUDY OF THE COGNITIVE
DEVELOPMENT OF LENGTH
AND AREA MEASUREMENT

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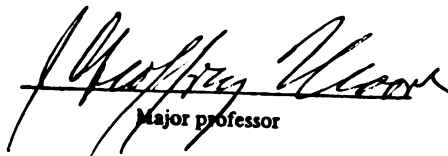
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ABSTRACT

A STUDY OF THE COGNITIVE DEVELOPMENT OF LENGTH AND AREA MEASUREMENT

By

Robert J. Kosanovich

It was the stated purpose of this study to investigate the cognitive development of length and area measurement relative to four common component properties (congruence, conservation, additivity, and unit measure) and chronological age. The intent was to conduct a comparative investigation to lend support to one of the two contrasting points of view identified to be: (1) There is no difference between the ages at which a child attains corresponding levels of understanding relative to length and area measurement and that both of these concepts are finally attained at approximately the same age; (2) There is a difference between the ages at which a child attains corresponding levels of understanding relative to length and area measurement and that a child finally attains length measurement prior to area measurement.

The need for this study was based on the conflicting conclusions made from two separate investigations: (1) Piaget, Inhelder, and Szeminska concluded that there is a simultaneous development of and final attainment of the cognition of length and area measurement; (2) Beilin and Franklin concluded that the component properties of length measurement are understood prior to corresponding properties of area measurement and that the cognition of length and area measurement are finally attained in that order.

Pilot Study and Sample

The population for the study was the student body of a public elementary school in a northern Michigan city serving a middle class neighborhood. Prior to the actual study, a pilot study was conducted to determine the age groups to be used and to refine the tasks. As a result of the pilot study, twenty children in each of the five age groups (age seven through eleven) were randomly selected to be included in the sample.

Collection of the Data

Each subject was given nine tasks. The first was a vocabulary task of measurement terms used to determine inclusion in the final sample. Four length measurement tasks (concerning the properties of congruence, conservation, additivity, and unit measure) corresponding to four area measurement tasks were given to each child to determine their level of cognitive development.

Analysis of Data

Two research hypotheses were developed for study. Operational hypotheses derived from the research hypotheses were submitted to test. A seven step inference process was employed to determine whether the operational hypotheses should be accepted or rejected. The Chi-square test for independence and the Phi-coefficient were used as test statistics. The Phi-coefficient was then used as an indicator of the association between scores on the length measurement tasks and scores on corresponding area measurement tasks.

Research Hypothesis I

I. The cognitive development of length measurement is simultaneous to the cognitive development of area measurement relative to the properties of congruence, conservation, additivity, and unit measure.

This research hypothesis was transformed into twenty operational hypotheses relating a measurement property to chronological age. Seventeen of the twenty hypotheses were accepted. Four summarizing operational hypotheses across all ages were formed for each measurement property. Each hypothesis was submitted to test and accepted. Thus, there is evidence to indicate that there is a simultaneous cognitive development of length and area measurement.

Research Hypothesis II

II. The understanding of length and area measurement are attained simultaneously.

This hypothesis was tested for each of the five age groups and accepted for all but the nine-year old group, although 75 per cent of the nine-year olds were scored the same regarding the final attainment of length and area measurement. A summarizing hypothesis across all ages relating the final attainment of these two measurement concepts was statistically tested and accepted. Thus, there is evidence to indicate that there is a simultaneous final attainment of the understanding of length and area measurement.

A STUDY OF THE COGNITIVE
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AND AREA MEASUREMENT

By

Robert J. Kosanovich

A THESIS

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DEDICATION

To my wife
Mary Louise
and children
Tad, Keith, Marcia

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CHAPTER I

THE PROBLEM AND ITS SETTING

Introduction to the Problem

One of the most significant developments that has occurred recently in the school mathematics curriculum has been the inclusion of a considerable amount of geometric material throughout the program. Only a few years ago practically all of the geometry being taught was concentrated at the tenth grade. Now, in the more updated curricula, it is being taught at all levels.

The Cambridge Conference on School Mathematics has suggested that geometry be studied with arithmetic and algebra from kindergarten on with the aims of developing the planar and spatial intuition of the pupil, affording a source of visualization for arithmetic and algebra, and to serve as a model for that branch of natural science which investigates physical space by mathematical models.¹ One of the beneficial results of introducing geometry at an early stage is that it provides for a more fundamental development of the nature of

¹Cambridge Conference on School Mathematics, Goals for School Mathematics (Boston: Houghton Mifflin Company, 1963), p. 33.

measurement and the measuring process.¹ It is the measuring process, specifically length and area measurement, that will be considered in this paper.

Almy, Chittenden, and Miller contend "that the success of the various new programs in mathematics . . . is largely dependent on their appropriateness for the conceptual abilities of the children receiving instruction."² This statement indicates the importance to the design of mathematics curricula of data regarding the cognitive development of mathematical concepts in children. Gibney and Houle indicate that emphasis on the cognitive development of mathematical concepts in children is lacking: "Geometry is an area of mathematics that has received much attention and space in contemporary mathematics textbooks, but geometry readiness is a topic that appears to have been slighted [underline mine]."³ They believe that readiness for learning, as it relates to geometry, is vital, and questions such as those below need to be given consideration:

1. Has adequate attention been given to the factors of readiness in planning for geometric concepts in a course of study?
2. Are current textbooks being designed to accommodate factors of readiness at the respective grade levels for which the content is prepared?

¹H. Stewart Moredock, "Geometry and Measurement," Mathematics Education, National Society for the Study of Education, Sixty-ninth Yearbook, 1970, p. 167.

²Millie Amy, Edward Chittenden, and Paul Miller, Young Children's Thinking (New York: Teachers College Press, 1966), p. 126.

³Thomas G. Gibney and William W. Houle, "Geometry Readiness in the Primary Grades," The Arithmetic Teacher, October 1967, p. 570.

3. Is the preparation of teachers adequate for them to understand geometric concepts well enough to teach readiness for this content?
4. How can teachers interpret readiness factors and present geometric concepts in accordance with the need of the class?
5. What can be done to improve teacher effectiveness in establishing readiness for geometric concepts?
6. Will neglect of readiness in the presentation of geometry destroy the possible benefits that might have been gained at previous or subsequent grade levels?
7. Are geometric concepts placed at appropriate levels in courses of study?¹

Certainly geometric concepts have a place in the primary grades. But regardless of how worthy the content may be, the endeavor to help children develop intellectually will be unsuccessful if they are not ready to understand the concepts. Lack of readiness can render the best instructional situation ineffective. Hence, educators involved in the development of the mathematics curriculum must consider factors regarding readiness to understand geometric concepts.

Need for the Investigation

Jean Piaget, a Swiss psychologist, and his associates² performed a series of experiments concerned with the development of an awareness in children of various properties of length and area

¹Ibid.

²Jean Piaget and his associates are sometimes referred to as the Geneva group.

measurement (Piaget's work will be detailed in Chapter II).¹ Following his study of the intellectual development of length and area measurement, Piaget stated that "The development of conservation and measurement runs exactly parallel whether the objects are lengths or whether they are areas and the level at which they are finally grasped is the same for both."²

Copeland states that:

There is a readiness stage that the child must reach before logical concepts such as those involved in measurement can be . . . learned.³

The necessary concepts . . . to measurement do not appear for many children until age seven to eight or until sometime during the second or third grade of school. Yet many teachers attempt teaching measurement before this time.⁴

This study [Piaget's work in measurement] indicates then that if systematic measurement is to be "taught" it should not be presented before the latter part of what is usually the third grade. Even then, for most children it will have to be an experimental or trial-and-error readiness-type experience . . . The necessary concepts [for measurement] will develop (1) When the child is old enough (eight to eight and a half, according to Piaget) . . .⁵

¹Jean Piaget, Barbel Inhelder, and Alina Szeminska, The Child's Conception of Geometry, trans. E.A. Lunzer (New York: Harper and Row, 1970).

²Ibid., p. 300.

³Richard W. Copeland, How Children Learn Mathematics-Teaching Implications of Piaget's Research (New York: The Macmillan Company, 1970), p. 23.

⁴Ibid., p. 198.

⁵Ibid., p. 209.

Often measurement in one dimension is taught before the child is at the operational or readiness level to understand it, and yet two-dimensional or area measurement is deferred several years past the age at which children can understand it. Children at age nine in general are ready for measurement in two dimensions using the method of superposition of a unit square.

Children in the age range seven to nine should be tested first for an understanding of conservation of area [and length] . . . when they are at the conservation or operational level, they are ready to begin measurement using a unit [of measure] . . . and counting the number of times it is contained in the [object] . . . being measured.¹

Thus, Piaget's research has promoted criticism by Copeland of the present manner in which length and area measurement are being taught in American elementary schools. It appears to be Copeland's belief (as indicated in the previous quotes) that we presently begin teaching length measurement approximately one to two years too soon and begin to teach area measurement several years later than it could be taught without any loss of effectiveness.

Beilin and Franklin conducted a study regarding the intellectual development of length and area measurement on a comparative basis (This study will be discussed in detail in Chapter II).² The subjects were New York City school children from the first and third grades. Contrary to the finding of Piaget, Beilin and Franklin's results indicate that the majority of the children studied achieved

¹Ibid., p. 238.

²The discussion of this study is based on Harry Beilin and Irene Franklin, "Logical Operations In Area and Length Measurement: Age and Training Effects," Child Development, 33, 1962, pp. 607-618.

length measurement prior to area measurement. This finding is not consistent with the aspect of Piaget's developmental sequence of mathematical concepts in which length and area measurement are said to be finally achieved at the same level, and, hence, at the same age. It suggests, rather, that "length and area measurement . . . are achieved in that order" and that "the component operations [congruence, conservation, additivity, unit measure, etc.] are applied most easily first to a single dimension then to two dimensions . . ."1

Thus, there are two contrasting points of view regarding the development of and final achievement of length and area measurement: (1) Piaget states that there is a parallel development of and simultaneous achievement of length and area measurement. Based on this finding Copeland expresses criticism of the present manner in which these two measurement concepts are taught. (2) Beilin and Franklin state that length measurement is learned prior to area measurement and in fact the component operations of the two measurements are first learned in one dimension, then in two dimensions. This belief is consistent with the order of appearance of length and area measurement in elementary school textbooks.

Piaget and his associates rarely described the samples used in their studies, except for the age factor. It is assumed that extensive work by this group involving American children is absent. This assumption is based upon Hunt's description of Piaget's work:

¹Ibid., p. 617.

In these early studies, Piaget's empirical data came almost completely from the language behavior of pairs of children observed in preschool situations at the Rousseau Institute in Geneva.

The second period began with his observations of the origins of intelligence and reality constructions in his own three infant children.¹

Rosenbloom, commenting on the importance of Piaget's work and the lack of American experimentation, stated that:

The implications of Piaget's theories for mathematics education have not yet been realized. Studies by competent researchers involving American children are badly needed. New curricular materials, based on sound psychological evidence should be written. And, in teacher education, more work involving Piaget's theories and their implications would serve as landmarks in improving instruction in the elementary school.²

Lovell states that "although there are a number of points on which I find myself in disagreement with the Geneva school, I strongly urge readers to study the books written by Piaget and Inhelder and to repeat for themselves some of the experiments described."³

¹J. Mcv. Hunt, "The Impact and Limitations of the Giant of Developmental Psychology," David Elkind and John Flavell, Studies In Cognitive Development (New York: Oxford University Press, 1969), pp. 4-5.

²Paul C. Rosenbloom, "Implications of Piaget for Mathematics Curriculum," Improving Mathematics Education, Conference sponsored by The Science and Mathematics Teaching Center, Michigan State University and The National Science Foundation, 1967, ed. by Robert Houston, p. 49.

³Kenneth Lovell, The Growth of Basic Mathematical and Scientific Concepts in Children (London: University of London Press LTD, 1961), p. 7.

Purpose of the Investigation

The purpose of this study is to investigate the cognitive development of four significant properties (congruence, conservation, additivity, and unit measure) of length and area measurement relative to the factor of age. The intent is to lend support to one of the two contrasting points of view identified earlier concerning the attainment of length and area measurement: (1) That there is no difference between the ages at which a child attains corresponding levels of understanding relative to length and area measurement and that both of these concepts are attained at approximately the same age. This view is shared by Jean Piaget as indicated in his statement, "The development of conservation and measurement runs exactly parallel whether the objects are lengths or whether they are areas and the level at which they are finally grasped is the same for both."¹ (2) That there is a difference between the ages at which a child attains corresponding levels of understanding relative to length and area measurement and that the child attains length measurement prior to area measurement. This view is shared by Beilin and Franklin as indicated in their statement: "length and area measurement . . . are achieved in that order" and that "the component operations [congruence, conservation, additivity, unit measure, etc.] are applied most easily first to a single dimension then to two dimensions."²

¹Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 300.

²Beilin and Franklin, "Logical Operations In Area and Length Measurement." p. 617.

The Research Problem and the Hypotheses

In a speech at New York University in March of 1967, Piaget comments as follows:

A few years ago Jerome Bruner made a claim which has always astounded me; namely that you can teach anything in an intellectually honest way to any child at any age if you go about it in the right way. Well, I don't know if he still believes that . . . it's probably possible to accelerate but maximum acceleration is not desirable. There seems to be an optimum time. What this optimum time is will surely depend on each individual and on the subject matter.¹

The question of optimum time to introduce a child to a mathematical concept is of utmost importance to anyone who is responsible for the intellectual development in children. As Gibney and Houle have indicated earlier in this paper (pp. 2-3) in the form of questions, the readiness level of the child and the grade placement of the concept are major factors involved in the success or failure of the concept to be learned.

Piaget claims that a parallel development of conservation and of length and area measurement exists and that the level at which they are finally grasped is the same for both [noted earlier, p. 9]. Based on Piaget's research, Copeland claims that systematic measurement should not be presented before the latter part of what is usually the third grade and that often measurement in one dimension is taught before the child is at the readiness level to understand it, and yet two-dimensional or area measurement is deferred several years past the age at which children can understand it.

¹Frank Jennings, "Jean Piaget, Notes on Learning," Saturday Review (May 20, 1967), p.82.

If Piaget's theory regarding the parallel development of and the simultaneous achievement of conservation and measurement regardless whether the objects are lengths or areas is accepted, then the implications described by Copeland become quite prominent since the ideas are contrary to prevailing modes of thought. That is, many mathematics textbooks present length measurement prior to area measurement. As an example, the textbook series used in the elementary school from which the subjects of this study came introduces length measurement two years prior to area measurement.¹

Research Hypotheses

This study is an attempt to investigate hypotheses regarding length and area measurement and their common component properties of congruence, conservation, additivity, and unit measure (terms to be defined in the next section). The specific procedure of study, the test instruments used, and the tests employed to analyze the data are explained in Chapter III. The operational hypotheses and the statistical alternatives that were developed from the research hypotheses listed below are described more completely in Chapter III.

- I. The cognitive development of length measurement is simultaneous to the cognitive development of area measurement relative to the properties of congruence, conservation, additivity, and unit measure.

¹Joseph N. Payne, et al., Elementary Mathematics Concepts and Topics from Readiness Through Grade 6 (New York: Harcourt, Brace, and World, 1965).

This hypothesis asks the questions:

1. Does a child understand the congruence of length and the congruence of area at the same age?
2. Does a child understand conservation of length and conservation of area at the same age?
3. Does a child understand the additivity of length and the additivity of area at the same age?
4. Does a child understand the use of a unit of length measure and the use of a unit of area measure at the same age ?

II. The understanding of length and area measurement are attained simultaneously.

This hypothesis asks the questions:

1. Has a child who has attained (failed to attain) an understanding of length measurement also attained (failed to attain) an understanding of area measurement?
2. Has a child who has attained (failed to attain) an understanding of area measurement also attained (failed to attain) an understanding of length measurement?

Each of the two hypotheses is suggested as a result of Piaget's investigation into the cognitive development of length and area measurement. The first hypothesis deals with Piaget's proposal

regarding a parallel development of the two concepts of length and area measurement. The second hypothesis deals with Piaget's proposal regarding a simultaneous attainment of length and area measurement.

Mathematical Considerations and Definition of Terms

Some of the newer high school geometry textbooks contain axioms concerning the measurement of the length of line segments and the measurement of the area of polygonal regions.¹ The axioms focus on significant properties of measurement including the properties of congruence, conservation, additivity, and unit measure. These four significant properties of measurement, relative to both length and area, are investigated in this study.

In mathematics the word "measurement" refers both to a process (the method or way measurements are performed) and to the end result of the process if the end result is reported using a numeral and a unit of measure such as an inch or a square inch.² The positive real number that is used to denote the measurement of an object is called the "measure" of the object.³ Considering only whole

¹"A polygonal region is a plane figure which can be expressed as the union of finite number of triangular regions, in such a way that if two of the triangular regions intersect, their intersection is an edge or a vertex of each of them." Edwin E. Moise, Elementary Geometry from an Advanced Standpoint (Reading, Mass.: Addison-Wesley, 1963). p. 153.

²James R. Smart and John L. Marks, "Mathematics of Measurement," The Arithmetic Teacher, April, 1966, p. 283.

³Ibid.

number measures, the measure of a line segment is the number of times the unit segment can be laid end to end along the segment being measured from one endpoint to the other. In the remainder of this text, the term 'measurement' refers to the process of finding the measure of an object.

Congruence Property

The property of congruence is the mathematical basis for the theory of measurement.¹ In the sense of developing spatial perceptions, it is clear that concepts relative to measurement begin well below the school level. The child will begin early to distinguish between such things as a round object and a square or triangular one. An individual who correctly selects the piece to fit into a given space in a jigsaw puzzle is exercising his perception of this extremely important geometric property called congruence. The importance of this measurement property called congruence is illustrated by the devotion of an entire workbook regarding the property of congruence by the University of Illinois Committee on School Mathematics.²

In general, two geometric plane figures are congruent if they have the same size and shape, or, in other words, if one can be moved so as to coincide with the other.³ At the elementary level,

¹Ibid., p. 285.

²Jo McKeeby Phillips and Russell E. Zwoyer, Book 2: Congruence, University of Illinois Committee on School Mathematics (New York: Harper and Row, 1969).

³Moise, Elementary Geometry, p. 58.

congruence is given an operational definition: two segments or plane figures are congruent if a copy of one may be made to fit exactly on the other. The tasks used to test for an understanding of the congruence property (i.e., does a child understand the operational definition of congruence) are presented in Chapter III.

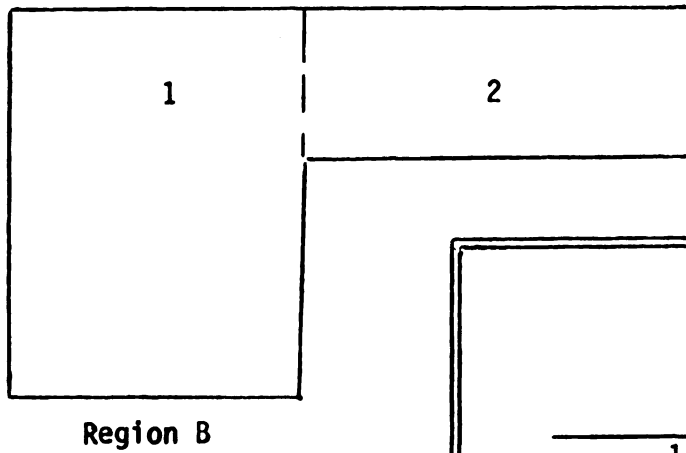
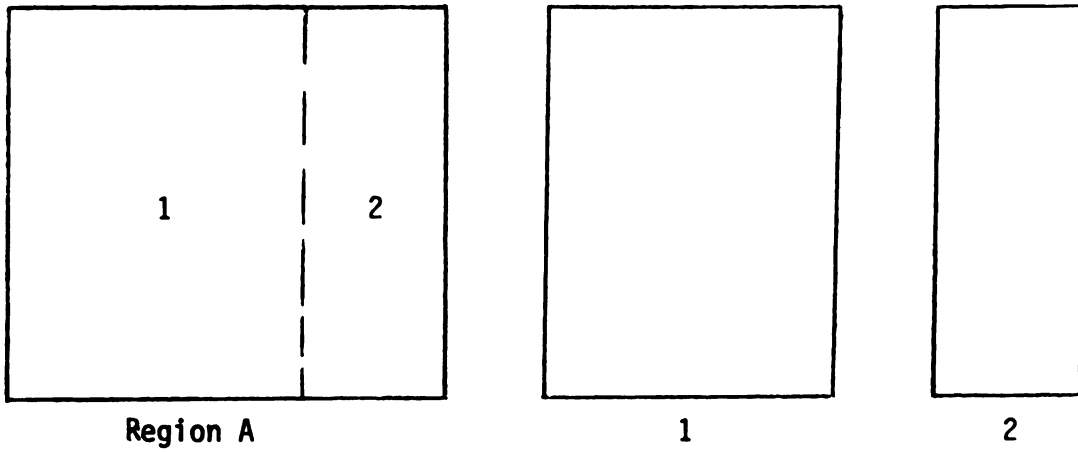
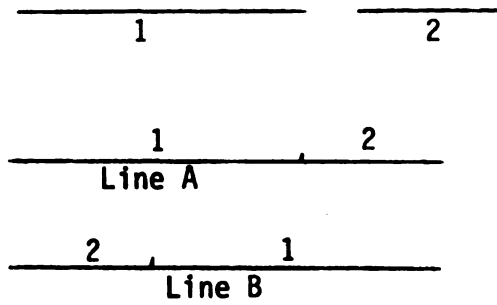
Conservation Property

"Underlying all measurement is the notion that an object remains constant in size throughout any change in position."¹ The property that the length of a line segment or the area of a plane region is unaltered under certain transformations is referred to as conservation. The measure axioms presuppose the concept of conservation of length and of area. For example, the measure axiom regarding the addition of areas states that if a region is the union of two subregions (such that the subregions intersect only in edges or vertices), then the area of the region is the sum of the area of the two subregions.² However, no restriction is placed on how the subregions are combined. Therefore, since they may be combined in more than one way by changing the positions of the two subregions, several regions of various shapes may have the same area (see Figure 1-1, AREA: The area measure of region A is equal to the area measure of region B). In order to make a realistic attempt to solve

¹Jean Piaget, Barbel Inhelder, and Alina Szeminska, The Child's Conception of Geometry, trans. E.A. Lunzer (New York: Harper and Row, 1964), p. 90.

²Moise, Elementary Geometry, p. 154.

FIGURE 1-1

CONSERVATION PROPERTYAREALENGTH

Scale: 1" = 2"

a task requiring the application of this axiom one must have achieved conservation of area. A parallel discussion could apply to lengths as well (see Figure 1-1 LENGTH: The length of line segment A is equal to the length of line segment B). The tasks used to test for conservation (i.e., is the child cognizant of the invariance of length and area under certain transformations) are presented in Chapter III.

Additivity Property

The measure axiom regarding the addition of areas states:

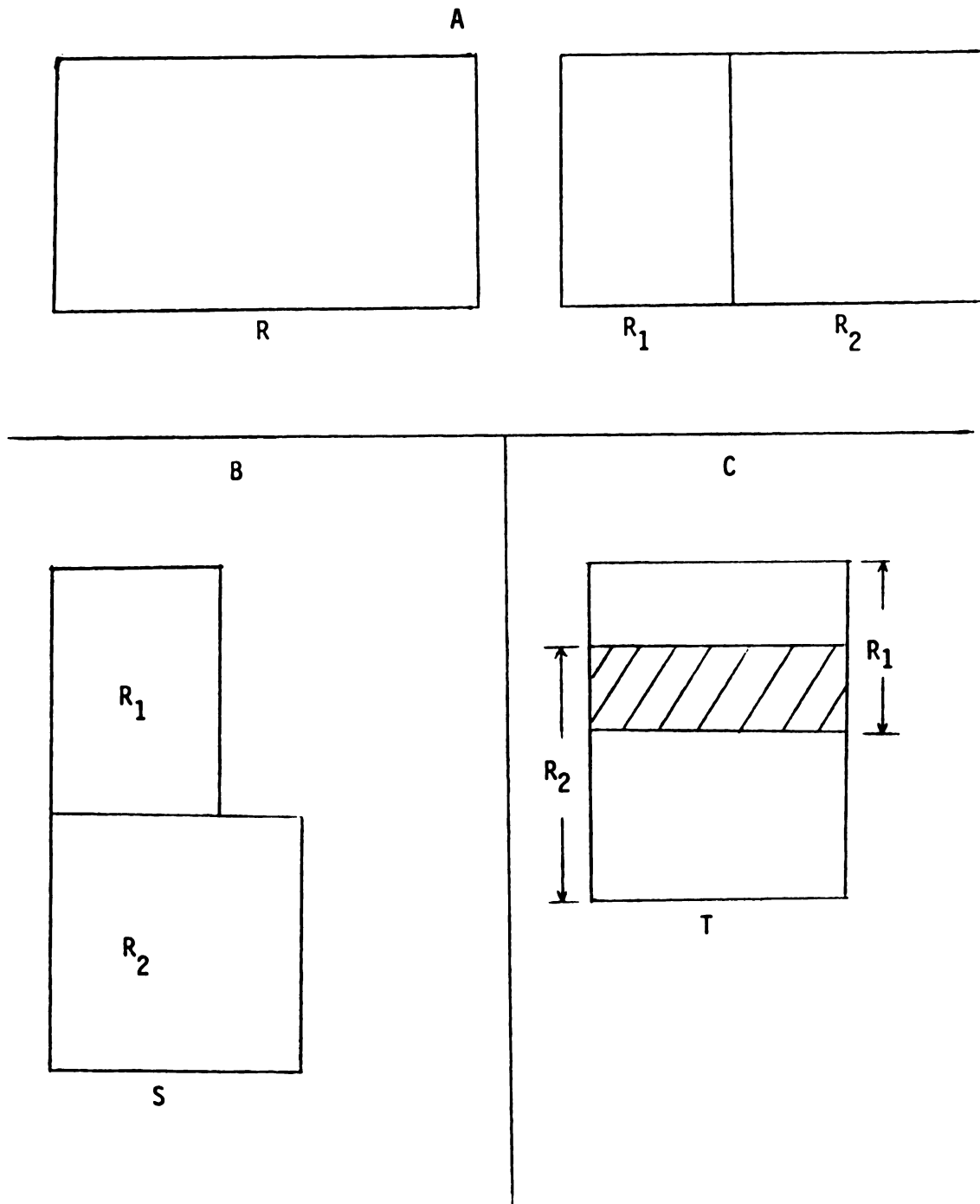
"Suppose that the polygonal region R is the union of two polygonal regions R_1 and R_2 such that the intersections of R_1 and R_2 are contained in a union of a finite number of segments. Then relative to a given unit of area, the area of R is the sum of the areas of R_1 and R_2 ."¹

Suppose we are given a five-inch by three-inch rectangular region denoted by R (see Figure 1-2 A) and are told that R is the union of R_1 , a two-inch by three-inch rectangular region, and R_2 , a three-inch square region. Let a one-inch square be the given unit area. Using the area axiom which states that "if R is any given polygonal region, there is a correspondence which associates to each polygonal region in space a unique positive number such that the number assigned to the given polygonal region R is one"², we have a correspondence which assigns the positive number six to R_1 and the positive number nine to

¹School Mathematics Study Group, Geometry with Coordinates, Part II (New Haven: Yale University Press, 1963), p. 989.

²Ibid.

FIGURE 1-2

ADDITIVITY PROPERTY - AREA

Scale: 1" = 2"

R_2 ; that is, six copies of the given unit area are required to cover R_1 exactly and nine copies to cover R_2 exactly. The intersection of the subregions R_1 and R_2 is the single line segment which is a common side of the two regions. Therefore, using the additivity axiom, the area measure of the region R is the sum of the area measures of R_1 (six) and R_2 (nine), or fifteen.

Suppose a second polygonal region S which is irregular in shape (see Figure 1-2 B) is the union of R_1 and R_2 , such that the intersection of R_1 and R_2 is contained in a single line segment. Then the area measure of S is the sum of the area measures of R_1 and R_2 , or fifteen. Therefore, although regions R and S differ in shape, they have equal area measures relative to a common measuring unit.

A three-inch by four-inch region T is also the union of R_1 and R_2 (see Figure 1-2 C). However, in this case the intersection of R_1 and R_2 is a one-inch by three-inch plane region (see shaded region) which cannot be covered by a finite number of line segments. Therefore, the additivity axiom cannot be used to calculate the area measure of region T . A similar discussion involving lengths would illustrate the use of the additive property of lengths. The tasks used to test for an understanding of the additive property (i.e., is the child aware of the fact that the whole is equal to the sum of its nonoverlapping parts) are presented in Chapter III.

Unit of Measure

The understanding of the notion of a unit of measure and the importance of its size is necessary for proper measurement to take place. For example, consider two congruent rectangles with

dimensions of two inches by four inches (see Figure 1-3, AREA). The one-inch square and the isosceles right triangle whose legs are one inch long are to be used as measuring units. The area measure of rectangle A is found using the unit square and the area measure of rectangle B is found using the isosceles right triangle as measuring units. A child who understands the notion of a unit of measure and the importance of its size would determine that the area measure of rectangle A is eight and the area measure of rectangle B is sixteen relative to their respective units of measure. In addition, the child would also state that the two rectangles, A and B, are congruent since the square measuring unit is exactly twice the size of the triangular measuring unit. Similar statements can be made regarding lengths (see Figure 1-3, LENGTH). The tasks used to test for an understanding of the use of a unit of measure (i.e., does a child consider the number and the size of the units used in the measuring process) are presented in Chapter III.

Organization of the Study

This thesis consists of five chapters.

CHAPTER I. INTRODUCTION

Introduction to and statement of the problem, need and purpose for the study, statement and explanation of the hypotheses, mathematical consideration and definition of terms.

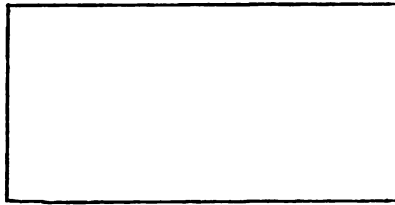
CHAPTER II. REVIEW OF RELATED RESEARCH AND LITERATURE

An over-all view of Piaget's theory of intellectual development, Piaget's description of the development of length and area

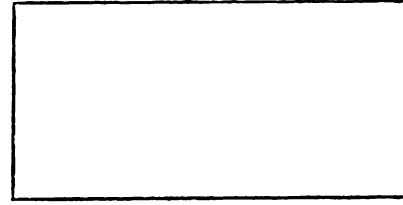
FIGURE 1-3

UNIT MEASURE

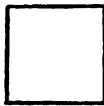
AREA



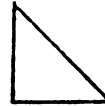
A



B



Square Unit



Triangular Unit

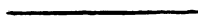
LENGTH



A



B



Unit 1



Unit 2

Scale: 1" = 2"

measurement, other research related to Piaget's investigation of length and area measurement.

CHAPTER III. THE RESEARCH PROCEDURE

Demographic information regarding sample, description of tasks and criteria used in evaluation, the research design, operational hypotheses, and a description of the statistical instruments and the analysis process.

CHAPTER IV. ANALYSIS OF DATA AND FINDINGS

Presentation of results, restatement of hypotheses, conclusions regarding acceptance of hypotheses, correlation analysis, and statistical tests.

CHAPTER V. SUMMARY, CONCLUSIONS AND IMPLICATIONS

Major findings, synopsis of the problem, conclusions, and implications.

Summary

This study examines the development of length and area measurement on a comparative basis relative to the factor of age. It also investigates the final attainment of length and area measurement relative to age. Two questions are central to the study:

1. Is there a parallel development of the significant properties (congruence, conservation, additivity, and unit measure) of length and area measurement?
2. Are length and area measurement finally achieved at approximately the same age?

CHAPTER II

REVIEW OF RESEARCH AND RELATED LITERATURE

Theoretical Background

Perhaps more than any other single person, Jean Piaget [the Swiss psychologist] ranks as the giant of contemporary research into the way in which children think.¹ His work is concerned with investigating the sequential development of intelligence. In effect Piaget has theorized an ordered sequence of stages of intellectual development, and he and his colleagues have amassed a vast amount of research in support of this developmental sequence.

Piaget used the "clinical method" as his experimental procedure, which is similar to that used by psychiatrists as a means of diagnosis. This technique involves a single child and an experimenter who interacts with the child by posing questions or presenting the child with a task concerning a particular phenomenon. Piaget believes that this type of exchange between child and investigator is necessary in order that the child's beliefs may be realized. He has remarked about the pitfalls, methods, and advantages of the clinical method:

The good experimenter must, in fact, unite two often incompatible qualities; he must know how to observe, this is to say, to let the child talk freely, without ever checking or side-tracking his utterance, and at the same time he must constantly be alert for

¹David Elkind, "The Continuing Influence of Jean Piaget," Grade Teacher (May/June, 1971), p. 7.

something definitive, at every moment he must have some working hypothesis, some theory, true or false, which he is seeking to check.¹

Using the clinical method, Piaget and his colleagues have collected data that has led to a formulation of a theory of human intellectual development.² According to this theory, the development of the intellect can be outlined in four stages: sensori-motor, pre-operational, concrete operational, and formal operational. The theory holds that the order of the four stages as listed above is invariant and that each stage or substage is a necessary prerequisite for the development of each subsequent stage or substage. Piaget has designated approximate chronological ages for each of the four major stages, but repeatedly emphasizes that these are approximate and are not to be construed as limits or bounds.

The following brief discussion of Piaget's four stages of intellectual development does not pretend to be a comprehensive examination of these developmental stages. The theory upon which the stages are based is elaborate and somewhat complex. Each stage is composed of substages which are interwoven into a highly detailed theoretical structure. For detailed discussions of the four stages of intellectual development the reader is referred to Flavell, 1963,³ and Phillips, 1969⁴.

¹Jean Piaget, The Child's Conception of the World (Patterson, N.J.: Littlefield, Adams, 1963), p. 9.

²John L. Phillips, Jr., The Origins of Intellect: Piaget's Theory (San Francisco, Calif.: W.H. Freeman and Company, 1969).

³John H. Flavell, The Developmental Psychology of Jean Piaget, (Princeton: D. Van Nostrand Co., Inc. 1963).

⁴Phillips, Origins of Intellect.

The first of Piaget's stages of intellectual development, the sensori-motor stage, begins at birth and lasts until approximately two years of age.¹ It is during this stage that the child learns to coordinate and organize perceptual and motor functions and develops simple behavior patterns for dealing with the external world. He learns that objects do not cease to exist when outside his perceptual field and becomes capable of elementary symbolic behavior.

The second stage, the pre-operational stage, begins with the advent of organized symbolic behavior, language in particular, and lasts until around seven years of age.² The essential difference between a child in the sensori-motor stage and one in the pre-operational stage is that the former is restricted to direct interactions with the environment, whereas the latter is capable of representing the environment with symbols (language)³. In the pre-operational stage the child is capable of representational thought, but in a limited sense. He is continually victimized by his perceptual field and thinks in terms of beginning and final configurations when confronted with transformations.

One of the most significant indicators of the pre-operational stage is the child's failure to understand that certain physical properties such as length, number, area, weight, amount (mass), and volume are conserved under certain transformations in the shape or

¹Ibid., p. 11.

²Ibid., p. 51.

³Phillips, Origins of Intellectual Development, p. 54.

configuration of objects. To exhibit these concepts of conservation a child must hold invariant in his mind a given physical property throughout observed changes of state. The child makes perceptual judgments based upon the appearance of the object following the transformation and disregards the invariant qualities of the object.

The third stage of Piaget's developmental scheme, the concrete operational stage, begins about seven years of age and lasts until about eleven years of age.¹ The rules of mathematics and logic are used by Piaget as models of the mental functioning of children in this stage. Piaget believes that the rules of logic have developed out of the interaction of humans with the demands of living in a lawful universe.² The actions that were originally overt, and then internalized, now begin to form tightly organized systems of actions. Piaget refers to any internal act that forms an integral part of one of these systems (such as combining, separating, placing in order, or substituting) as an "operation."³ The development of the "operations" characterizes this stage of intellectual development.⁴

Since birth, the dominant mental activities of the child have changed from overt actions (in the sensori-motor stage) to perceptions (in the pre-operational stage) to the intellectual operations (in

¹Phillips, Origins of Intellect, p. 51.

²Ibid., p. 68.

³Ibid.

⁴Flavell, Developmental Psychology, p. 166.

the concrete operations stage). These operations occur within a framework of class relations that make possible what Piaget calls mobility of thinking - reversibility, decentering, taking the view of the other, etc.¹ As a result, the concrete operations child conserves quantity and number, constructs the time and space that he will live with as an adult, and establishes the foundations of the logical thinking that is the identifying feature of the next and final stage of his development.

Piaget's final stage of intellectual development, formal operations, begins about the age of eleven.² As a child grows older and gains more experience, his construction of reality becomes more precise and extended, and that makes him aware of gaps in his understanding that had been masked by the vagueness of his previous constructions. He fills those gaps with hypotheses, and he is able to formulate, and often even to test, hypotheses without actually manipulating concrete objects. For the first time the child is able to think in terms of all possible combinations when confronted with a problematic situation.

For convenience in identifying the four stages of intellectual development proposed by Piaget, sensori-motor, pre-operational, concrete operational, and formal operational, the remainder of this writing will refer to them as stages I, II, III, and IV, respectively.

¹Phillips, Origins of Intellect, p. 90.

²Ibid., p. 91.

Substages will be identified by capital letters, e.g. I A, I B, II A, II B, III A, III B, IV A, and IV B.

Piaget's Description of the
Cognitive Development of Length Measurement

Piaget's view of the inception of length measurement is described in his tower experiment:¹

He invited children to build with blocks a tower equal in height to a tower already built by the experimenter. This tower, however, was on a table which stood higher than the table on which the subject was to build his tower and some distance from it. Sticks longer, shorter, and equal to the height of the model were available to the subjects.

Children at stage I have an exaggerated confidence in visual comparison; their measuring may be summed up with the words: "I look and I see". That faith is undermined when they come to notice a difference in base levels of the towers. As a result, the two perceptual fields are brought together by manual transfer (substage II A). When the child is required to compare the towers without moving them, they go through the motions of manual transfer. They accommodate their hand movements to the size of the towers, imitating their height. Through body transfer (substage II B) they reach the idea of a common measure. Because body transfer is inaccurate, sooner or later they reject it. A third object is sought as a measuring instrument. This instrument is a common measuring object independent of the subject's own body. Transitivity ($A = B$ and $B = C$, implies $A = C$) at a qualitative level is now present (substage III A). When transitivity is extended to include relations between separate parts of an overall length, the evolution of a metrical system consisting of the use of a unit measure is completed (substage III B).²

¹Piaget, Inhelder, and Szeminska, Conception of Geometry, pp. 30-66.

²Ibid., p. 65.

Piaget and his associates have conducted other single task investigations regarding the cognitive development of length measurement. A description of the intellectual development of the child through the various developmental stages defined by Piaget (similar to the description of the tower experiment) is noted with each investigation.

One of these investigations concerns conservation of length and the extremities of the lines:¹

The subject was presented with a straight wooden rod of length 5 cm. and a longer undulating thread of plasticine shaped like a snake. The objects were placed side by side a few millimeters apart, with their endpoints in exact alignment, and the child was asked to compare the lengths of the two objects. If he said that they were equal, he was made to run his finger along the two lines and the question was repeated. Next, he was shown what happened when the plasticine was straightened, and the question was repeated. Finally, the plasticine was twisted back to its original shape and the original question was asked again.

Of approximately a hundred children who were given these questions, only 15 per cent of those aged four years, six months and younger correctly recognized the inequality of the two lengths. Of those children over the age of five years, six months, 90 per cent gave correct replies.² [This is one of the few times in which any statistics are presented in the descriptions of Piaget's investigations.]

The children in stage I compare the lengths of the lines by focusing on the endpoints. Judgment is modified by movement of fingers for the children in substage II A. The children in substage II B make correct judgments on this task which implies that they are aware of the intervals that lie between the endpoints.

¹Ibid., pp. 91-94.

²Ibid., p. 92.

Another of Piaget's conservation of length investigations relates a comparison of lengths and a change in position of the line:¹

The experiment consisted of showing the subject two straight wood sticks identical in length and with their extremities facing each other; one of the sticks was then moved forward 1 or 2 cm. (the sticks being approximately 5 cm. long), and the subject was asked to say once again which of the two was longer or whether they were the same length. At all levels, the sticks were judged equal before staggering. After that change of position, subjects at the first stage maintain that the stick which has been moved forward is longer, thinking only in terms of the further extremities and ignoring the nearer extremities. This response lasts into substage II A. Between levels II A and II B we find a series of transitional responses, beginning with perceptual regulations and passing from intuitive regulations to operations, when conservation of length is assured (stage III).

An experiment used by Piaget to describe the intellectual development of the child regarding length measurement is noted below:²

The subject is asked to judge between strips of paper in a variety of linear arrangements, involving right-angles, acute angles, etc., but these are pasted on cardboard sheets. When he has given his replies, saying they are equal or that one is longer than the other, he is shown a number of movable strips and asked to verify his judgment. . . . he is given short strips of card 3 cm., 6 cm., sometimes 9 cm. long (these lengths corresponding with those of segments on the mounted strips).

At levels I and II A, subjects had no notion of conservation and consequently they failed to understand the concept of a middle term and that of a unit. At substage II B conservation is dimly perceived, and children at this level also begin to understand transitivity. At substage III A measurement is conducted with reliance on the transitive property but without a metric unit while at substage III B the child now

¹Ibid., pp. 95-103.

²Ibid., pp. 117-127.

uses a metric unit in the iterative process of measurement.

Piaget investigated subdividing a straight line with the following experiment:¹

Two wires, AC and DF which are equal in length are placed parallel to one another with their ends in alignment. The child was told that a bead on the wire was a train traveling along a railway line. The experimenter moved his bead from A to B, and the child was asked to move his bead to do a journey of the same length. Subjects were provided with a ruler, string, strips of card of varying length which they were invited, but not shown how, to use.

The experimenter commenced by moving his bead from A, the child being invited to move his bead from D so that the segment AB equaled the segment DE. This procedure was repeated with the subject having to move his bead from the other end F so that $AB = FE$.

Next D was moved 4 inches to the left of A, so that F was 4 inches to the left of C. The subject was again asked to move his bead to E on DF, starting from F, and making $FE = AB$. Keeping DF in the same position relative to AB, the experimenter then moved his bead 15 inches from A -- a distance longer than any of the measuring instruments provided. The child was again asked to locate E so that $AB = FE$.

Finally the wire DF was replaced by a wire GI which was shorter than AB. The wires were still parallel but GI was displaced 4 inches to the right of AC. The experimenter moved his bead 6 inches from A and the subject was asked to move his bead 6 inches from I.

During stages I and II A, the length of travel is determined solely by the point of arrival so the problem is solved only when the points of departure are in alignment. In substage II B a given length can be reproduced with reasonable accuracy by visual estimate. Measurement is possible in substage III A if the measuring rod provided is equal to, or longer than, the distance to be measured. During substage III B, subjects apply a short ruler by iterate stepwise movements, thus illustrating the use of a unit of length.

¹Ibid., pp. 129-149.

Related Research - Length Measurement

The accuracy of Piaget's account of the cognitive development of length measurement has been investigated by Lovell, Healey, and Rowland.¹ This study contains four replications of the Piagetian studies described in the previous section. The sample for the study consisted of seventy Primary School children and fifty Educationally Subnormal Special School children. The following discussion will pertain only to the seventy Primary School children. The general procedure and the criteria for the evaluation at the various stages were kept as close as possible to those aspects of Piaget's investigation. Only the results of Lovell's study will be noted.

Regarding the first investigation described in the previous section concerning the conservation of length and the endpoints of the lines: Kendall's tau coefficient ($\tau = .26$, significant at the .01 level) indicates a positive correlation between chronological age and measurement stage.² As the age of the subjects increased, so did the measurement level. This finding coincided with that of Piagets. Approximately 65 per cent³ of the subjects six years old and older were aware of the intervals which lie between the endpoints

¹The discussion of this study is based upon K. Lovell, D. Healy, and A.D. Rowland, "Growth of Some Geometrical Concepts," in Logical Thinking In Children, ed. I.E. Sigel and F.H. Hooper (New York: Holt, Rinehart, and Winston, Inc., 1968), pp. 140-157.

²Ibid., p. 144.

³Ibid.

(rated at substage II B) as compared to 90 per cent¹ of Piaget's subjects who gave correct responses.

The second replication of a Piagetian study concerns conservation of length and a change of position of the line: Kendall's tau coefficient ($\tau = .42$, significant at the .01 level) indicates a positive correlation between chronological age and measurement stage.² Increase in age implied a higher measurement level. Approximately 60 per cent of the eight- and nine-year old groups were rated at stage III while a considerably lesser percentage (15 per cent) of those children in the younger groups were rated at this stage.³ This finding agrees with that of Piaget: conservation of length is achieved at a mean age of seven and one-half years.⁴

The third replication of one of Piaget's investigations concerns length measurement with the use of independent objects to be used as units: Kendall's tau coefficient ($\tau = .55$, significant at the .01 level) indicates a positive correlation between chronological age and measurement stage.⁵ The older children were rated at a higher stage than the younger children. Lovell's results

¹Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 92.

²Lovell, "Growth of Some Geometrical Concepts," p. 145.

³Ibid.

⁴Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 126.

⁵Lovell, "Growth of Some Geometrical Concepts," p. 146.

indicate that 70 per cent of the eight- and nine-year old groups were rated at the highest level (substage III A) while only 30 per cent of the younger subjects were rated at this stage.¹ This finding is in agreement with that of Piaget : length measurement is achieved at approximately eight or eight and one-half years.²

The fourth replication of one of Piaget's studies concerns subdividing a straight line: Kendall's tau coefficient ($\tau = .30$, significant at the .01 level) indicates a positive correlation between chronological age and measurement stage.³ The data indicates a slight increase in measurement understanding as the subjects increase in age. Only 13 per cent of the seventy Primary School children studied were scored at measurement stage III regarding subdivision of a line.⁴ There were no statistics presented in Piaget's study to use for comparison purposes, but the subdivision task was noted as Piaget's most difficult length measurement task.

Piaget's Description of the Cognitive Development of Area Measurement

Piaget and his associates have conducted a sequence of tasks to gain information regarding the cognitive development of area measurement. As with length measurement, the stages of intellectual

¹Ibid.

²Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 126.

³Lovell, "Growth of Some Geometrical Concepts," p. 147.

⁴Ibid.

development defined by Piaget are associated with the various levels of attainment regarding the understanding of area measurement.

One of these investigations concerns subtracting smaller congruent areas from larger congruent areas:¹

The child was shown two identical sheets of cardboard painted green, each 20 cm. by 30 cm. These represent meadows. He is asked to compare the meadows and agree that there was the same amount of grass on each. Following this, he was shown a toy cow and asked if it had the same amount of grass to eat in each of the fields. The experimenter then places small wooden houses (1 cm. by 2 cm.) one at a time in each of the fields. In one field the houses were placed end to end, while in the other field the houses were spread about. To begin with, one house was placed in each field, then two, three, etc. After each increase in the number of houses the child was asked to compare the amounts of grass left in each field for the cow to eat.

The child in stage I had difficulty understanding what was being asked. At substage II A, equality of areas was recognized only when there was one house in each field. Children in substage II B determined equality of remaining areas up to a certain number of houses but this varied with the child. Conservation of area was present at substage III A, i.e. equality was determined regardless of the number of houses placed in the meadows.

Another of Piaget's area measurement investigations concerned unit iteration:²

The child was shown a number of shapes which are equal in area but which differ markedly in shape. One is a square which can be composed out of nine smaller squares. The others are irregular figures made up of the same number of small squares. The child was given a choice of three counters to measure the figures. One is a square which is a quarter of one of the

¹Piaget, Inhelder, and Szeminska, Conception of Geometry, pp. 262-273.

²Ibid., pp. 296-301.

figures to be measured. The second is a rectangle that can be composed of two unit squares. The third is a triangle equal to a square cut diagonally in half. The child is asked to compare the sizes of the regions of various shapes.

Children at substage II A make judgments of size by reference of the perceptual appearance of the figure. At substage II B children make correct judgments if the regions being compared can be composed of all squares or all triangles. Squares and triangles are regarded as equivalent units. At substage III A comparison of areas is made by transferring parts of one figure to vacant sites of another. Children at substage III B measure the figures by unit iteration. The area of the unit square can be expressed in terms of the area of the unit triangle, etc.

Piaget investigated subdividing areas with the following experiment:¹

Each of the children (whose ages ranged from four to around seven) was shown a circular slab of modelling clay. He is told that the clay is a cake. His first task was to cut the cake into two pieces so that each piece has the same amount. Next he is asked to cut a similar circular slab of clay into three equal parts. Division into fourths, fifths, and sixths follow using the same procedure. After each request to cut the clay, the child was asked whether the sum of the pieces equaled the whole.

The children in stage I could not divide the clay equally. During substage II A, dividing into halves and quarters is possible but not trisection. Children in substage II B begin to conserve the whole (whole is equal to the sum of its parts) and trisection is accomplished by trial error. During substage III A, trisection is possible and the whole is conserved.

¹Ibid., pp. 302-325.

Related Research - Area Measurement

The accuracy of Piaget's account of the cognitive development of area measurement has been investigated by Lovell, Healey, and Rowland.¹ This study contains three replications of Piagetian investigations described in the previous section. The sample consisted of seventy Primary School children. General procedure and the criteria for evaluation at the various stages were similar to those aspects of Piaget's investigation. Only the results of Lovell's study will be noted.

Regarding the first investigation concerning subtracting small congruent areas from larger congruent areas: Kendall's tau coefficient ($\tau = .29$, significant at the .01 level) indicates a positive correlation between chronological age and measurement stage.² As the age of the subjects increased, so did the measurement level. Lovell's data indicates that 77 per cent³ of the sample completed this task successfully, i.e. were rated at stage III. This is in agreement with Piaget's finding: At stage III (usually at seven and one-half but sometimes as early as six and one-half years) children recognize that remainders are always equal.⁴

¹Ibid., "Growth of Some Geometrical Concepts, pp. 140-157.

²Ibid., p. 152.

³Ibid.

⁴Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 264.

The second replication of a Piagetian study concerns unit iteration:¹ Kendall's tau coefficient ($\tau = .47$, significant at the .01 level) indicates a positive correlation between chronological age and measurement stage.² Only 22 per cent³ of those children whose ages are seven to nine years are rated as being in stage III. This finding is contrary to that of Piaget ; stage III usually begins at the age of seven. (It must be remembered that the ages assigned to the various stages of intellectual development are only approximations.) In agreement with Piaget's findings is the fact that only 8 per cent of those children seven years old and younger are rated as being at stage III.

The third replication of a Piagetian study concerns subdivision of areas:⁴ Kendall's tau coefficient ($\tau = .59$, significant at the .01 level) indicates a positive correlation between chronological age and measurement stage.⁵ Approximately 93 per cent⁶ of the six- and seven-year olds are rated as being in substage II B or higher. This finding is in agreement with that of Piaget : in general, substage II B occurs between six and seven years of age. No comparison can be made using Lovell's eight- and nine-year olds since Piaget's sample for this task included children whose ages ranged from four to seven.

¹Lovell, "Growth of Some Geometrical Concepts," p. 153.

²Ibid. ³Ibid.

⁴Ibid., p. 154.

⁵Ibid. ⁶Ibid.

In summary, the main stages in the cognitive development of length and area measurement proposed by Piaget have been confirmed among English school children by Lovell and his associates.¹ The protocols were classified into the stages enumerated by Piaget and a few intermediate substages such as substage II B - III A. However, the number of children at the various stages were not always what one would expect from Piaget's results. For example, in Lovell's conservation of length task relative to a change of position, only 27 per cent of the seven year old children were rated at stage III.² Piaget claims that "the third stage is reached about the age of seven".³ Also, the data indicates that considerable variability in achievement of an operation may exist at a particular age level. Thus, chronological age is not a very good guide to the stage of cognitive development of some children.

Comparative Study of the Cognitive

Development of Length and Area Measurement

Beilin and Franklin conducted an investigation concerning length and area measurement. The study was conducted on a comparative basis to investigate whether the abilities to solve related problems of length and area measurement are acquired simultaneously, and whether there are age associated limits upon the acquisition of

¹Ibid., pp. 142-157.

²Ibid., p. 145.

³Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 72.

measurement operations when a deliberate training effort is made.¹

The discussion that follows will pertain to the first of the two stated purposes.

The subjects were New York City school children from the first and third grades of a public elementary school in a predominantly middle class area. The two groups were indicated to be above average (by I.Q. scores). The mean age of the twenty-seven first graders is six years, six months (range: six years, zero months to seven years, three months).² The mean age of the thirty-three third graders is eight years, eleven months (range: eight years, one month to nine years, four months).³

Piaget's unit measure tasks for length⁴ and area⁵ measurement were used for the tasks of this investigation. Figure 2-1 illustrates the length and area measurement testing materials.⁶ The area materials were made of white cardboard and the length materials consisted of strips of colored paper pasted on white cardboard. Lengths numbered six to ten were movable strips of white cardboard. The

¹The account of this experiment is taken from Harry Beilin and Irene C. Franklin, "Logical Operations in Area and Length Measurement: Age and Training Effects," Child Development, 33 (September, 1962), pp. 607-618.

²Ibid., p. 609.

³Ibid.

⁴Piaget, Inhelder, and Szeminska, Conception of Geometry, pp. 116-127.

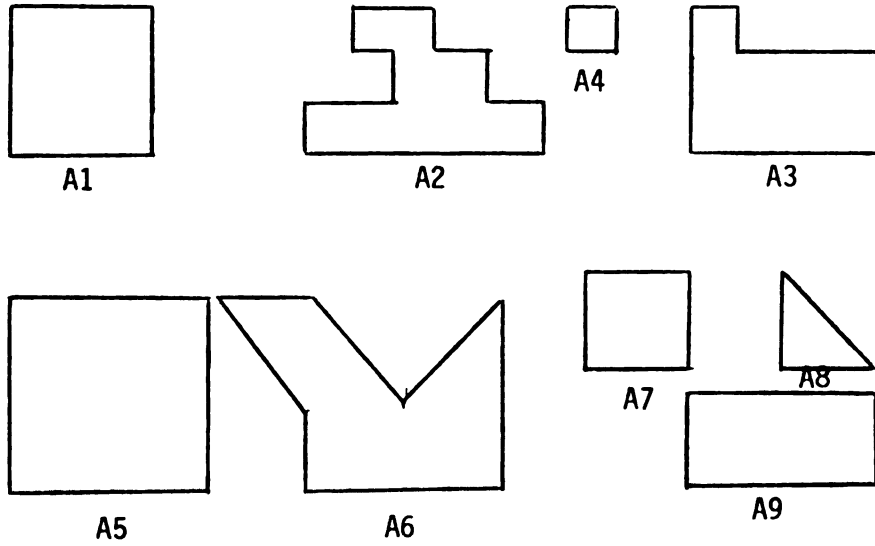
⁵Ibid., pp. 296-301.

⁶Beilin and Franklin, "Logical Operations in Area and Length Measurement," p. 610.

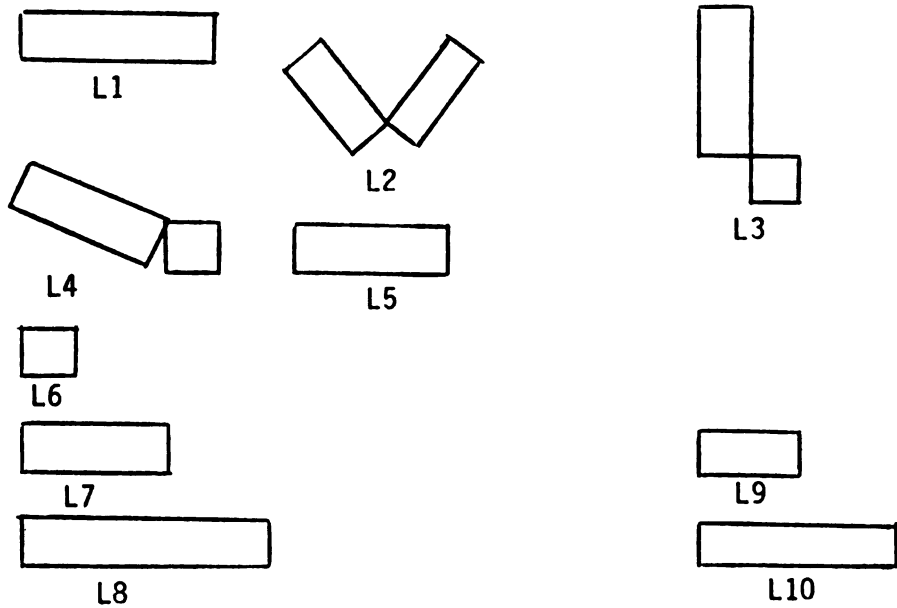
FIGURE 2-1

AREA AND LENGTH MEASUREMENT

AREA



LENGTH



Scale: 1" = 4"

materials were so devised that the measurement of both equalities and inequalities was tested. The shapes were so constructed that a conflict is generated between the perceptual properties of the objects and their logical relations. Shapes equal in area were made to appear unequal.

The procedures used in this study first required testing the subjects with the area materials. The intent was to determine whether the subject could measure the areas without aid from the experimenter. If the child did not answer correctly, the methods of superposition and unit iteration were demonstrated to him. Criteria used in evaluation is similar to the stage descriptions presented in Piaget's The Child's Conception of Geometry. Each child was classified as to the level of measurement he achieved.

The following is the order of area measurement task presentations:¹

Step 1. The subject was given the three-inch square, A1, and the irregular shaped figure, A2, of the same area measure (nine square inches). He was asked whether the space in them was the same and to give a supportive reason for his response. He was permitted to manipulate the figures.

Step 2. Figure A3 was substituted for figure A2 and a comparison asked for. These figures had the same area measure but not the same shape.

Step 3. The subject was given A1, A2, A3, and A4 together. He was told to verify whether his judgments were correct by using the one-inch square, A4. If necessary the experimenter demonstrated superposition and unit iteration processes.

¹Ibid., pp. 611-12.

Step 4. The child was then given figures A5 and A6 which are unequal in area. The procedure of steps 1 to 3 were repeated.

Step 5. Subject is given measuring instruments A7, A8, and A9 to verify his answers. A demonstration of measurement was given if necessary.

Regarding the order of length measurement task presentation:¹

Step 1. The subject was given L1, L2, L3, L4, and L5 and asked which of the lengths were equal and which were unequal. ($L1 = L2$, $L3 = L4 \neq L5$).

Step 2. The subject was given three movable strips one, three, and five inches long (L6, L7, and L8) to be used as measuring units.

Step 3. If measurement was not successfully achieved with L6 to L8, then L9 and L10 were given, which together provided the subject with measuring units that corresponded to all the strips mounted on the card. The experimenter demonstrated unit iteration if necessary.

The results of Beilin and Franklin's investigation support Lovell's observation² that considerable variability in achievement of an operation may exist at a particular age level. Also, the data indicates that first graders differ from third graders in their ability to utilize measuring concepts. The following table³ consists of the numbers of first and third graders who have achieved length and area measurement:

¹Ibid., pp. 611-12.

²Kenneth Lovell, "A Follow Up Study of Some Aspects of the Work of Piaget and Inhelder on the Child's Conception of Space," British Journal of Education Psychology, 1959, p. 104.

³Beilin and Franklin, "Logical Operations in Area and Length Measurement". This is a portion of Table 1, p. 614. Per cents are in parentheses.

	<u>First Grade</u>	<u>Third Grade</u>
Length Measurement	3 (11)	27 (82)
Area Measurement	0 (0)	9 (27)

As indicated by the data, a large proportion of third graders have achieved length measurement but not area measurement. A similar phenomenon exists with the first grade group, but to a lesser degree.

In summarizing their investigation, Beilin and Franklin concluded that:

On the basis of the data of this study . . . , we would suggest that length and area . . . measurement are achieved in that order. Also the constituent operations to measurement (i.e. transitivity, subdivision, change of position, etc.) are applied more easily first to a single dimension, then to two dimensions, The order of achievement is a function of added dimensions Although our data deny the Piaget view of the simultaneous achievement of area and length measurement, we do not feel that this, of necessity, does violence to the unitary or structural interpretation of development It seems likely that within the limits of a particular level (e.g. stage III) tasks which are ordered in difficulty because of complexity (e.g. added dimensions) and which require no different operations for their solution will be achieved in order of such complexity. Certainly more evidence is needed before this important issue is resolved [underlined mine].¹

Summary

A synopsis of Piaget's theory of intellectual development has identified four major stages: (1) sensori-motor, (2) pre-operational, (3) concrete operational, and (4) formal operational. These stages were related to various levels of cognitive development regarding length and area measurement. Piaget concluded from his

¹Ibid., p. 617.

study concerning the cognitive development of length and area measurement that "The development of . . . measurement runs exactly parallel whether the objects are lengths or whether they are areas and the level at which they are finally grasped is the same for both."¹

Lovell, Healey, and Rowland confirmed many of Piaget's findings through replications of his investigations. Seven such investigations have been described.

Beilin and Franklin conducted a comparative study concerning the ability to solve related problems of length and area measurement. On the basis of the data of this study, Beilin and Franklin concluded that "length and area measurement are achieved in that order" and that "constituent operations to measurement . . . are applied more easily first to a single dimension, then to two dimensions."²

The present study will attempt to lend support to one of the two stated contrasting viewpoints (i.e. that of Piaget and that of Beilin and Franklin). Measurement axioms found in modern geometry textbooks have been used to identify four common properties of length and area measurement: (1) congruence, (2) conservation, (3) additivity, and (4) unit measure. This study will investigate the cognitive development of the four common properties to obtain information regarding the two questions:

¹Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 300.

²Beilin and Franklin, "Logical Operations in Area and Length Measurement," p. 617.

1. Is there a simultaneous cognitive development of length and area measurement?
2. Does the final attainment of length measurement occur at the same age as the final attainment of area measurement?

Involved in this study will be four pairs of tasks (one regarding length measurement, the other area measurement) each testing one of the four common properties of measurement. This procedure is unlike the investigations reviewed that consisted of single task studies. It is believed that with this procedure a more accurate assessment of the development of measurement can be made.

Criteria for evaluation will be similar to that used in related studies so that a comparison of results can be accomplished.

Demographic data will be used to describe the sample. The data will be presented in tabular form and be subject to statistical analysis suggested by Bentler.¹ Similar procedures are absent from Piaget's work, a situation that has produced a fair amount of criticism.

¹Peter N. Bentler, "Monotonicity Analysis: An Alternative to Linear Factor and Test Analysis," Measurement and Piaget, ed. by Donald Green, et al., (New York: McGraw-Hill Book Company, 1971), pp. 220-27.

CHAPTER III

THE RESEARCH METHODOLOGY

Pilot Study

Prior to the actual study, a pilot study was conducted to determine the age groups to be used and to refine the tasks based on the length and area axioms. Fourteen children from ages five to eleven comprised the subjects for the pilot study. Twelve of the subjects were students in the same public elementary school in which the actual study was conducted. The two five-year olds were children who would enroll in the kindergarten of the same elementary school the following year.

Chronological age was chosen as the population variable since many of the studies reviewed by the investigator, including those of the Geneva group, relate the results to the ages of the subjects. Since this study relied heavily on the results of the Geneva group for its theoretical basis, the ages of the children were used as a means of grouping the subjects. Almy, Chittenden, and Miller also state that the best predictor of ability to conserve is chronological age.¹

The pilot study five- and six-year olds had difficulty performing the operations required in the tasks. In addition, it could

¹Almy, Chittenden, and Miller, Young Children's Thinking, p. 77.

be expected that a majority of five- and six-year olds would fail the length conservation task since Piaget and his associates have found that children attain conservation of length at a mean age of seven and a half.¹ Seven was taken as the study's base age. Regarding maximum age, the Geneva group has found that children aged eleven to twelve performed successfully in the doubling area task, their most difficult area task, hence, eleven was the pilot study maximum age.² The eleven-year age group remained the maximum age group of the actual study.

Specific modifications of the investigating procedure made as a result of the pilot study are discussed with the respective tasks. The responses obtained in the pilot study were used in connection with the Geneva group's results to determine the stages for each task. A recording sheet based upon this determination was developed and used in the actual study (see Appendix B).

Sample

The population for the study was the student body of a public elementary school in a northern Michigan city serving a middle class neighborhood. In the winter of 1971 the names of nearly all the children in the school in the seven- through eleven-year age

¹Piaget, Inhelder and Szeminska, Conception of Geometry, p. 126.

²Ibid., p. 337.

groups as of February were obtained.¹ There were 59 names in the seven-year old group; 55 in the eight-year old group; 33 in the nine-year old group; 33 in the ten-year old group; and 34 in the eleven-year old group.

Following the procedure outlined by Walker and Lev² the investigator alphabetized and consecutively numbered each of the five sets of names. Then he reordered each group of names using a table³ of random numbers. The preliminary sample consisted of children whose names were among the first twenty in each group. Children who did not pass the criterion for inclusion in the sample (the vocabulary task regarding measurement terms) were not included in the final sample of twenty children in each age group. The child whose name was next on the list was then added to the preliminary sample.

In order to obtain twenty children in each age group for the final sample, one nine-year old and one ten-year old were replaced on the original preliminary sample.⁴ Demographic data for the final

¹Seven years is operationally defined as six years, seven months to seven years, six months; eight as seven years, seven months to eight years, six months; nine as eight years, seven months to nine years, six months; ten as nine years, seven months to ten years, six months; eleven as ten years, seven months to eleven years, six months. Children in the Special Education class were not included.

²Helen M. Walker and Joseph Lev, Elementary Statistical Methods (New York: Holt, Rinehart and Winston, 1958), pp. 202-212.

³Ibid., pp. 280-281.

⁴The nine-year old was replaced due to her absence during the administration of the length and area vocabulary tasks. The ten-year old was replaced because of failure of the length and area vocabulary tasks. It was later learned that the ten-year old should have been placed in the Special Education class.

sample is given in Table 2-1. For notational purposes ages were recorded as years; months (e.g. eight-years, seven-months was recorded as 8;7). The mean age for the seven-year old group is 7 years, 1.2 months; for the eight-year old group is 8 years, 1.3 months; for the nine-year old group is 9 years, 1.0 months; for the ten-year old group is 10 years, 1.9 months; for the eleven-year old group is 11 years, 2.2 months.

Table 2.2 and Table 2.3 give reading comprehension scores and arithmetic concept scores, respectively, in terms of grade level for the age groups of eight through eleven. These scores are the results of the Stanford Achievement Tests published by Harcourt, Brace and World, Inc., administered in March of 1971.¹ Table 2.2 also gives the reading level of the children in the seven-year old group. The evaluation instrument used for the seven-year olds was the Basic Reading Test.² The scores entered in Table 2.2 for the seven-year old group are percentile scores.

The children in the sample came from two first-grade, two second-grade, one third-grade, one fourth-grade, and one fifth-grade classrooms. The mathematics textbooks used by grades are:

1. Grade one - One By One Elementary Mathematics, Joseph Payne, et al., Harcourt, Brace and World, Inc., 1965.

¹For those in Grade 2, Primary 2 Battery of Test W was used; for Grade 3, Primary 2 Battery of Test X was used; for Grade 4, Intermediate 1 of Test W was used; for Grade 5, Intermediate 2 of Test W was used.

²The children in the seven-year old group were tested for reading readiness. The Basic Reading Test, Sixties Edition, copyright 1963 by Scott Foresman and Company was the evaluation instrument used.

TABLE 3-1

DEMOGRAPHY OF SAMPLEAge Distributions (By Year and Month)

<u>7 Years</u>	<u>8 Years</u>	<u>9 Years</u>	<u>10 Years</u>	<u>11 Years</u>
6;7 3	7;7 1	8;7 3	9;8 4	10;10 3
6;10 2	7;9 3	8;9 1	9;11 1	11;0 2
6;11 1	7;10 2	8;10 1	10;0 2	11;1 3
7;0 3	7;11 3	9;0 3	10;1 2	11;2 5
7;1 2	8;1 2	9;1 3	10;2 1	11;3 1
7;3 3	8;3 2	9;2 2	10;3 2	11;5 3
7;4 1	8;4 1	9;3 1	10;4 2	11;6 3
7;5 1	8;5 2	9;4 2	10;6 6	
7;6 4	8;6 4	9;5 2		
		9;6 2		
Total 20	Total 20	Total 20	Total 20	Total 20
Mean 7;1.2	Mean 8;1.3	Mean 9;1	Mean 10;1.9	Mean 11;2.2
<u>SEX DISTRIBUTIONS</u>				
Boys 12	9	10	10	13
Girls 8	11	10	10	7
<u>GRADE DISTRIBUTIONS</u>				
Grade 1 20	Grade 1 19	Grade 2 01	Grade 4 17	Grade 4 03
	Grade 3 01	Grade 3 19	Grade 5 03	Grade 5 17

TABLE 3-2

READING SCORE DISTRIBUTION *(By Grade Level)

7 Years	8 Years	9 Years	10 Years	11 Years
12	1.8	2	2.5	1
15	2.0	1	2.6	2
38	2.1	2	3.0	1
50	2.2	1	3.1	1
53	2.3	1	3.7	1
55	2.4	1	3.8	1
70	2.6	1	4.0	1
75	2.9	1	4.3	1
80	3.0	1	4.4	2
85	3.3	1	5.0	1
92	3.5	1	5.7	2
95	3.6	1	6.4	3
99	3.9	1	6.9	1
	4.2	2	7.0	1
	4.4	1	8.2	1
	4.6	1	8.4	1
	5.5	1	9.5	1
				10.9
Range 12-99	Range 1.8-5.5	Range 2.5-6.9	Range 2.9-9.5	Range 2.8-10.9
Mean 65.8	Mean 3.1	Mean 4.6	Mean 6.0	Mean 6.0
	National Norm 2.7	National Norm 3.7	National Norm 4.7	National Norm 5.7

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*The scores indicated for the seven-year old group are percentile scores. The percentile scores for four subjects in the seven-year old group were unavailable. Table entries for the eight- through eleven-year old group represent grades in school (e.g. 3.4 represents third grade, fourth month).

TABLE 3-3
ARITHMETIC CONCEPT SCORE DISTRIBUTION
 (By Grade Level)¹

<u>8 Years</u>	<u>9 Years</u>	<u>10 Years</u>	<u>11 Years</u>
1.4 1	2.3 1	3.0 1	3.3 1
2.3 3	2.7 1	4.0 2	5.2 2
2.6 2	3.3 2	5.7 1	5.6 1
2.8 1	3.8 2	5.8 2	5.9 1
3.0 1	4.0 1	5.9 3	6.3 1
3.1 1	4.1 1	6.3 2	6.5 3
3.2 2	4.2 1	6.5 1	6.6 3
3.4 1	4.3 1	6.6 1	7.0 1
3.6 1	4.5 2	6.8 1	7.3 4
3.8 1	5.3 1	7.6 2	7.6 1
4.0 1	5.5 4	7.8 1	8.0 1
4.3 1	6.3 2	8.0 2	8.5 1
4.7 1	7.1 1	8.5 1	
5.3 1			
5.6 2			
Range 1.4-5.6	Range 2.3-7.1	Range 3.0-8.5	Range 3.3-8.5
Mean 3.5	Mean 4.6	Mean 6.3	Mean 6.6
National Norm 2.7	National 3.7	National Norm 4.7	National Norm 5.7

¹Arithmetic concept scores were unavailable for the seven-year old group.

2. Grade two - Two By Two, Elementary Mathematics, Joseph Payne, et al., Harcourt, Brace and World, Inc., 1965.
3. Grade three - 3 Elementary Mathematics, Joseph Payne, et al., Harcourt, Brace and World, Inc., 1966.
4. Grade four - 4 Elementary Mathematics, Joseph Payne, et al., Harcourt, Brace and World, Inc., 1966.
5. Grade five - 5 Elementary Mathematics, Joseph Payne, et al., Harcourt, Brace and World, Inc., 1966.

With the consistency present in the textbook series used at this elementary school, it was assumed that all the children in the study had been introduced to similar mathematical topics.

General Procedures

The interview technique, as used by Piaget and others, was employed to determine the level of measurement understanding of each child. The tasks used in this study were administered in a fixed order (vocabulary, congruence, conservation, additivity, and unit measure tasks) during a single interview with each child. The interview was tape recorded to be used later as a means of verifying the evaluation of measurement ability completed during the interview. Before beginning any interview, the investigator visited the classrooms of all the subjects in the study so that he became known to the students prior to the interviews. During this visit, an explanation was given to the children pertaining to the types of activities in which they would be engaged. No mention was made of mathematics. There was enthusiasm on the part of the students as noted by the comment of one child that: "I can play these games at home with my brother." The atmosphere of the interviews was relaxed and friendly.

No time limit was placed on the interviews. They ranged in length from thirty minutes to fifty-six minutes with most interviews lasting from thirty-five to forty-five minutes.

A semi-standardized method of questioning similar to that of the Geneva group was employed. Standardized questions were asked, and those that didn't seem to be understood were rephrased. Some of the more common difficulties were identified during the pilot study, and rephrased questions were then used in the actual study.

Mermelstein and Shulman state that "the employment of many rephrased questions may help to reduce the ambiguity of a particular question . . . , the standard questioning approach, because of its inflexibility, may not reduce the possibility of confusion of events."¹ Dodwell is of the opinion that "real insight into the cognitive processes of the child can best be obtained by presenting the child with a relatively fluid situation and seeing what he makes of it. In this way, . . . one gets closer to the nature and quality of the child's thinking."² These and other similar findings represent the basis of the decision to employ a semi-standardized type of questioning.

Concrete materials such as wire and paper of different colors were used as manipulative devices in the tasks prepared for this

¹Egon Mermelstein and Lee A. Shulman, "Lack of Formal Schooling and the Acquisition of Conservation," Child Development, 38 (1967), p. 51.

²P.C. Dodwell, "Children's Understanding of Number Concepts: Characteristics of an Individual and of a Group Test," Canadian Journal of Psychology, 15 (1961), p. 35.

study. This procedure is in agreement with Piaget's finding that "for a complete understanding of the genesis of intellectual operations, manipulation and experience with objects had first to be considered."¹

Every response was reacted to with a remark of "good" accompanied by smiling and nodding affirmatively in order to increase the subject's confidence.² There was no intentional attempt to elicit a response that was not naturally present with the child.

The first task to be administered in each interview was the vocabulary task which tested the correct use of the vocabulary employed in the remaining tasks. The vocabulary task was the instrument used to determine inclusion of the subject in the final sample. Children who did not pass this task were not included in the final sample.

The remaining sections of this chapter discuss each measurement property relative to length and area and its related task. The property is defined; the procedure for the related task is briefly described; and the method of evaluating the measurement abilities and relating these abilities to Piaget's stages of intellectual development is presented. Appendix A contains the precise interview procedures employed. The properties are discussed in the order in which the related tasks were administered in the interview.

¹Jean Piaget, "Autobiography," A History of Psychology in Autobiography, ed. Edwin G. Boring et al. (Worcester, Mass: Clark University Press, 1952) IV, p. 247.

²This procedure is based upon Jan Smedslund, "Development of Concrete Transitivity of Length in Children," Child Development, 34 (1963) p. 393 and p. 400.

Criterion for Inclusion in the Sample

Children had to demonstrate an understanding of the vocabulary used in the tasks of the study in order for their responses to be of practical use. The decision to develop and administer a vocabulary task to be used for a criterion for inclusion of the subject in the final sample is based upon a study which was conducted by Lovell and Ogilvie in which they report that there exists considerable verbal confusion in children up to about nine years of age.¹ It was reported that children frequently confuse such terms as longer, fatter, shorter, bigger, thicker, and smaller. Consistent with this finding, results of the pilot study indicated that the five- and six-year olds gave inconsistent responses to questions asked during the interviews. This was possibly due to the terms used during the interview, the "play" mannerisms demonstrated by the children, etc. The investigator felt that the responses given by the five- and six-year olds of the pilot study were not of practical use for this investigation, consequently five- and six-year olds were not used in the actual study.

Vocabulary Task

The purpose of the vocabulary task was to discover whether the subject could respond correctly to the standard questions used in the remaining tasks when the response was based only on a perceptual

¹K. Lovell and E. Ogilvie, "A Study of the Conservation of Substance in the Junior School Child, " British Journal of Psychology, 53, pp. 175-188.

discrimination. Measurement terms regarding both length and area were involved in the vocabulary task. The results of a study conducted by Beilin indicated that 88 per cent of the middle class kindergarten children in his sample made correct perceptual judgments regarding unequal areas and 69 per cent of them made correct judgments regarding equal areas.¹ Older children in Beilin's study reached near perfect levels for both equality and inequality judgments. Hence, the investigator assumed that the children of age seven and above in this study possessed the ability to make the required perceptual discrimination relative to both length and area.

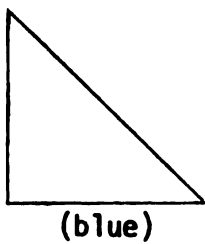
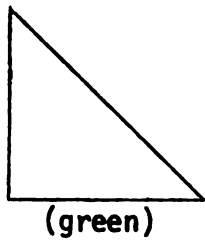
The task was used to elicit an indication of the child's understanding of the vocabulary used in later tasks. If the subject had initial difficulties with the task, the questioning was restructured so that acceptable substitutes for troublesome terms could be found or so the subject could discover how the investigator was using them. The terms used as substitutes were recorded on the Interview Recording sheet (see Appendix B). The substitute terms were used as a variation when needed in the administration of remaining tasks.

The vocabulary task involved comparisons of polygonal regions, both equal and unequal in area measure, and comparisons of line segments, both equal and unequal in length measure, see Figure 3-1. The polygonal regions compared in the vocabulary task were either congruent or had ratios of area measure of at least two to one and had shapes that permitted one to see upon superimposing them that they

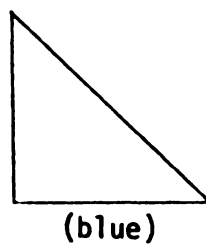
¹Harry Beilin, "Perceptual-Cognitive Conflict in the Development of an Invariant Area Concept," Journal of Experimental Child Psychology, I (1964), p. 217.

FIGURE 3-1

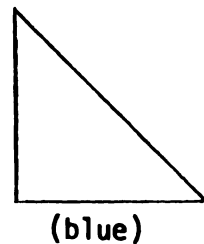
VOCABULARY TASK - AREA



A

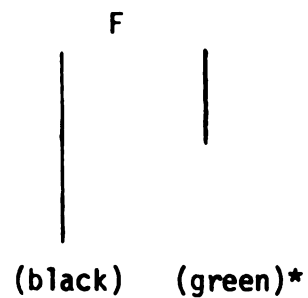
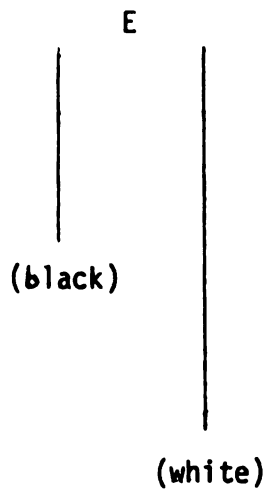


B



C

VOCABULARY TASK-LENGTH



*All wires are 1/8 inch in diameter except the green wire which is 1/4 inch diameter.

Scale: 1" = 8"

were the same size or different size. The line segments compared were either congruent or had ratios of length measure of at least two to one so that decisions regarding length could be made by the superimposing method.

The older children in the pilot study (from seven years on) had no trouble understanding the question: "Which is longer?" Some difficulty occurred with the question, "Which has more space?". Some children attempted to relate the amount of space to the shape or to the number of sides of the polygonal region. To minimize the possibility of a similar situation occurring during the actual study, the vocabulary test was refined. The vocabulary test regarding polygonal regions tested the child's understanding of the term 'area' or 'space' by having the child make a variety of comparisons. In one situation, the child had to compare two polygonal regions constructed so that the region with the greater number of sides had the smaller area. The vocabulary test regarding length tested the child's understanding of the term "length" by having the child make a variety of comparisons. In one situation, the child had to compare two wires, the shorter wire having a larger diameter. This situation was incorporated to test if size judgments were being made on lengths alone and not some other characteristic.

The subject passed the task if he responded correctly to the question either with the original vocabulary ("space," "larger," "smaller," "more," "less") or with substitutes such as "room" for "space" and "bigger" for "larger," otherwise, he failed it. One

child (nine years, eight months) failed the vocabulary task and hence was not included in the final sample.¹ This child had difficulty understanding the terms "more space" relative to the two polygonal regions in Figure 3-1 A, B, C.

WEL (9;8). (The questions on length are answered satisfactorily using terms "longer" and "shorter.") "Look at these two pieces of paper, the blue and the green. (see Figure 3-1 A.) "Do they have the same amount of space or different amounts?" "What do you mean." "Which one has more room?" (pause) "This one." (Child points to the green paper.) "Why?" "It looks bigger." "Why?" "It just looks bigger." "Which has more room the blue or the red?" (See Figure 3-1 B.) "The blue." "How do you know that?" "It looks bigger." "Which has more room, the blue or the yellow paper?" (See Figure 3-1 C.) (pause) "That's easy, the yellow one." "How do you know that?" "It has more sides." (At no time did the child attempt to super-impose the colored pieces of paper.)

The response of WEL on the comparison of the plane regions in Figure 3-1 C indicates that she believes the number of sides of a plane region determines the area of that region. The investigator determined that the terms "area," "space," or "room" were not adequately understood for this child to be included in the sample.

Measurement Tasks

The eight tasks regarding significant properties of length and area measurement (congruence, conservation, additivity, and unit measure) are discussed in the same order in which they were presented during the interviews with each child. In the discussion of each task a definition of the tested property is given, the task is

¹It was learned that this child should have been placed in the Special Education class rather than the regular third grade.

described, criteria for evaluation are noted and examples of various measurement levels are illustrated relative to the property being tested. The evaluation of each child relative to each task is recorded in Appendix C.

Congruence Task -- Length

Two straight line segments are said to be congruent if and only if their lengths are equal. That is, two straight line segments are congruent if they can be positioned in a parallel manner such that their ends coincide. The importance of the property of congruence to measurement is noted by Smart and Marks:

The concept of congruence is the mathematical basis for the theory of measurement . . . the measure of a line segment is the number of times the unit segment can be laid end to end along the segment from one end point to the other. The concept of congruence . . . makes it possible to provide an answer to the rather subtle question of how a number can be applied to a segment . . .¹

In the administration of the task, the term "congruent" was not used. The subject was given three wires (same thickness) one colored red, one black, and the other white. The red and the black wires were the same length (four inches) and the white wire was longer (eight inches long). The child was asked which of the wires have the same length. The child was allowed to manipulate the wires. Those children who could determine that the red and black wires were the

¹James R. Smart and John L. Marks, "Mathematics of Measurement," The Arithmetic Teacher, April, 1966, p. 285.

same length and could give an adequate reason for their answer passed the congruence of length task. An adequate reason would be demonstrating that the ends of the two wires coincide when placed parallel to each other. EMM and HAH are examples of children who passed this task:

EMM (7;6). (The red, black, and white wires are placed on the desk in front of the child in no organized manner.) "Which of the wires have the same size?" (The child arranges the three wires on the desk so that the ends were butted up against the tape recorder case.) "These two." (The child points at the red and the black.) "Why?" "Because their ends match and this one (points to the white wire) doesn't."

HAH (7;0). (The child manipulates the four-inch red and black wires and the eight-inch white wire. He places them in a vertical position with their ends on the desk top.) "The red and black are the same." "Why?" (He places his hand over the top ends of the red and black wires.) "Their ends are the same."

A child who could not determine that the red and black wires were congruent and give an adequate reason for his answer based on the ends of the wires failed the congruence of length task.

Congruence Task -- Area

"Two geometric figures are congruent if they have exactly the same size and shape."¹ In geometry, the child experiences congruence of area through observation and measurement. Sometimes they assist their observations by placing objects in positions relative to each other which will help to see any difference in size or shape the objects may have. This procedure of superimposing is basic to the

¹School Mathematics Study Group, Geometry Part I (New Haven: Yale University Press, 1960), p. 97.

decision of congruency between polygonal regions. The remarks noted earlier by Smart and Marks relative to the importance of the concept of congruence in the measuring process also apply to area as well as length.

Again, the term "congruent" was not used in the administration of this task. The terms "the same in size and shape" were used instead. The subject was given three isosceles right triangles. The blue and the green had four-inch legs and the white triangle had six-inch legs. The child was asked which of the triangles had the same size and shape. Those children who could determine (usually by superimposing) that the blue and green triangles were congruent and could give adequate reasons for their answer passed the task. A reason was judged adequate if it involved the recognition of equality in size and shape. JOH is an example of a child who passed the congruence of area task.

JOH (7;10). (The three triangles are laid on the table in front of the child.) "Which of these triangles have the same size and shape?" (The blue and green triangles are picked up and superimposed by the child.) "These two." "How do you know that?" "They fit together with nothing left over." . . .

When a child hesitated to respond to the question regarding which triangles were congruent, he was told that he could move the triangles if he wished. This freedom of movement usually led to superimposing of the triangles.

Failure to determine that the blue and green triangles were congruent or a mere guess of the same with no supporting reason or manipulative response led to failure of the congruence of area task.

Conservation of Length Task

Conservation of length refers to the concept that the length of a line segment is unaltered under certain transformations; for example, changes in the perceptual aspects of a line segment such as shape and position do not change its length. "Change of position" and "subdivision" are two such transformations that do not change the length of a line. "Change of position" refers to the rearrangement of the line relative to its surroundings, (see Figure 3-2, A-F) which does not change the length of the line. "Subdivision" refers to the dividing of a line into component non-intersecting parts (except for possible endpoints). The rearrangement of the position of these parts constitutes a change of shape of the line (see Figure 3-2, G and H), which again does not change the length of the line. The importance of this property (conservation of length) is indicated in Piaget's statement that: "Conservation of length is the fundamental prerequisite of all measuring."¹

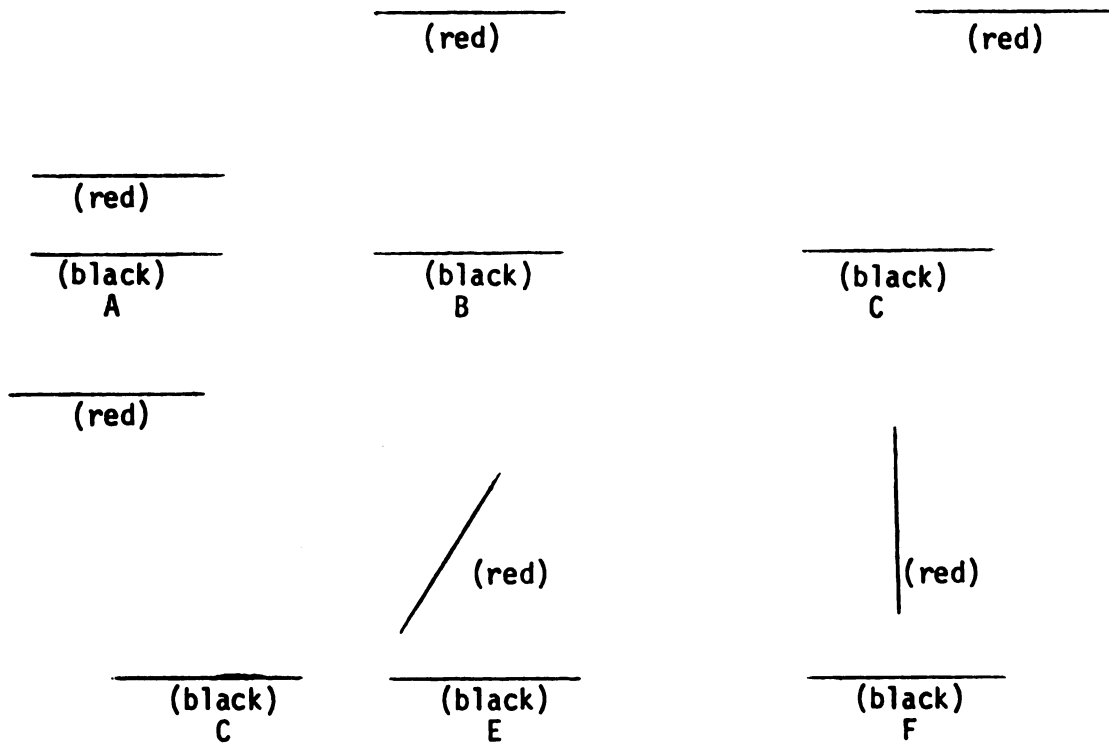
The task used to test for conservation of length is similar to a task used by Piaget for the same purpose.² The distinction between the Geneva group's task and the task used in this study is in the materials used by the child to arrive at answers to questions regarding conservation of length. The manipulative devices used in the Geneva task were "a straight stick made of wood or clay and an undulating thread made of plasticine . . ."³ It was believed by this

¹Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 66.

²Ibid., pp. 91-103.

³Ibid., p. 91.

FIGURE 3-2
 CONSERVATION OF LENGTH
 CHANGE OF POSITION



SUBDIVISION



Scale: 1" = 3"

investigator that the different materials (thread and wood) together with the probable difference in thickness of these materials may interfere with the attempt of the child to focus specifically on the property of length. Hence, electrical wires of the same thickness and composition were used as "lines" in this study (see Conservation of Length task, Appendix A). The task was administered in two parts: (1) a change of position test and (2) a subdivision test. First, to test for the conservation of length relative to a change of position (movement of the objects to be compared) the subject is given two thin wires of the same thickness that have the same length (three inches). One is black, the other red. The child is allowed to move the two wires and to superimpose them to determine that they have the same length. The child is assisted by the experimenter, if necessary, to arrive at the conclusion of equal lengths. The assistance by the experimenter involved placing the wires in a parallel position so that their ends coincided. If a child was still not convinced that the two wires were congruent, the experimenter cut off a portion of the wire thought to be longer by the child. This cutting procedure and realignment of the wires in a parallel manner with the ends coinciding was a convincing practice; the child then would accept the fact that the wires were the same length. Once the child had agreed that the lines were the same length, a precise order of arrangements of the two wires were made (see Figure 3-2, A-F). After each arrangement, the child was asked if the two wires were the same

length or was one longer. He was asked for the reason for his answer only if he answered correctly to all the different arrangements of the wires or after he made the first incorrect response.

There are two categories of response for the conservation of length relative to the change of position test. A child passed this portion of the conservation of length task if he responded correctly to the questions asked after each alteration of positions of the black and red wires and supported his decisions with the reason: changing the position of the wires does not change their size. The following protocol is an example of a child who passed the change of position test.

HAM (8;6). (The subject stated that the two wires are the same length when placed in a parallel manner with the ends coinciding. The positions of the wires are changed in a precise order; see Figure 3-2, A-F). "Which is longer now?" "They are the same." "And now?" "Same." "What about now?" "Same." "Which is longer now?" "Neither, they are the same." "Why?" "Because all you did was move them." "And?" "That doesn't make them longer or shorter."

A child failed the conservation of length task relative to change of position if he answered that the two wires were not the same length after any one of the various arrangements. Some of the younger children gave responses indicating failure similar to the responses of ASI, LOV, and AND.

ASI (6;10). (The child stated that the two wires were the same length when they were in a parallel position with the ends coinciding. The two wires were separated approximately three feet but still remained parallel; see Figure 3-2 B). "Which one is longer or are they the same?" "This one is longer." (The one farthest away from the subject.) "Why?" "Because it is farther away."

LOV (7;6). (Wires are parallel with the ends coinciding). "Which wire is longer or are they the same?" "Same." (Wires separated approximately three feet but still parallel with ends coinciding.) "Which is longer now or are they the same?" "Same." (The wire farthest away is moved to the right so that the ends no longer coincide; see Figure 3-2 C). "Which is longer now or are they the same?" "That one is longer." (Child points to the wire farthest to the right.) "Why?" "Because it is more that way." (Subject points to the right.)

AND (7;6). . . . (Wires were placed in a parallel manner with ends coinciding approximately three feet apart.) "Which wire is longer or are they the same?" "Same." (Wires moved so that ends do not coincide but still remain parallel.) "Which one is longer or are they the same?" "Same." (Wires arranged so that the black one is horizontal and the red one is vertical, see Figure 3-2 F.) "Which is longer or are they the same?" "That one is longer." "Why?" (Child points to red wire.) "Because it is pointing up."

In each of the above examples of subjects who failed the conservation of length task relative to change of position, the subjects' decisions were based on perceptual judgments rather than on any logical property of length. Piaget states that: "The first stage . . . is one in which perceptual comparison is the only basis of comparison . . ."¹ "After a change of position, subjects at the first stage maintain that the stick which has been moved forward is longer, thinking only in terms of the further extremities and ignoring the nearer extremities. This response lasts into substage II A."² Based on Piaget's findings, the subjects who failed the conservation of length test relative to change of position would be at substage II A or lower in Piaget's scheme of intellectual development.

¹Ibid., p. 31.

²Ibid., p. 95.

The second part of the conservation of length task involves subdivision, i.e. the altering of the shape of one of two wires known to be the same length. The experimenter assisted the child, if necessary, to verify that the red and black wires were the same length. Then, in the child's view, the experimenter shaped the red wire in a wavy manner (see Figure 3-2 H). The child was asked which of the two wires was longer or were they the same length. Some of the children who passed the change of position portion of the conservation of length task failed the subdivision portion. JOH and COR are two children who failed this portion of the conservation of length task.

JOH (7;10). (The straight wires are placed in a parallel manner such that their ends coincide.) "Which wire is longer or are they the same length?" "Same." "Why?" "They match." (The red wire is bent so that the ends no longer coincide; see Figure 3-2 H.) "Which wire is longer now or are they the same length?" "The black one is longer." "Why?" "The red wire is bent and that makes it shorter."

COR (9;2). (The child states that the parallel wires are the same length. The red one is bent; see Figure 3-2 H.) "Which wire is longer or are they the same length?" "The black one is longer." "Why?" "Because it is straight."

An example of a child who passed this portion of the length conservation task is QUI.

QUI (7;10). . . . (The red wire is bent, see Figure 3-2 H.) "Which wire is longer or are they the same length?" "Same length." "Why?" "You just bent the red one." "And?" "You didn't change its size and it was as long as the black one before you bent it."

A child passed the conservation of length task if he passed both the change of position and the subdivision portions. A child who possesses the conservation of length would be rated at substage

III A in Piaget's intellectual developmental scheme.¹ A child failed the conservation of length task if he failed either of the two portions. Failure of both portions of the conservation of length task (change of position and subdivision) indicated that the child is at substage II A or lower.²

The evaluation of each child relative to the conservation of length task was recorded as:

P: indicating the child passed both the change of position and the subdivision portions.

Fc: indicating the child failed the change of position portion only.

Fs: indicating the child failed the subdivision portion only.

F: indicating the child failed both portions.

Based on the above evaluations, the child was placed in the following stages of intellectual development relative to conservation of length:

A. attainment (evaluated with score of P).

B. transitional (evaluated with score of Fc or Fs).

C. none (evaluated with score of F).

The conservation stages (attainment, transitional, and none) were recorded in Appendix C as A, B, and C, respectively.

Conservation of Area Task

"Conservation of area" refers to the concept that the area of a polygonal region is unaltered under certain transformations; for

¹Ibid., p. 126.

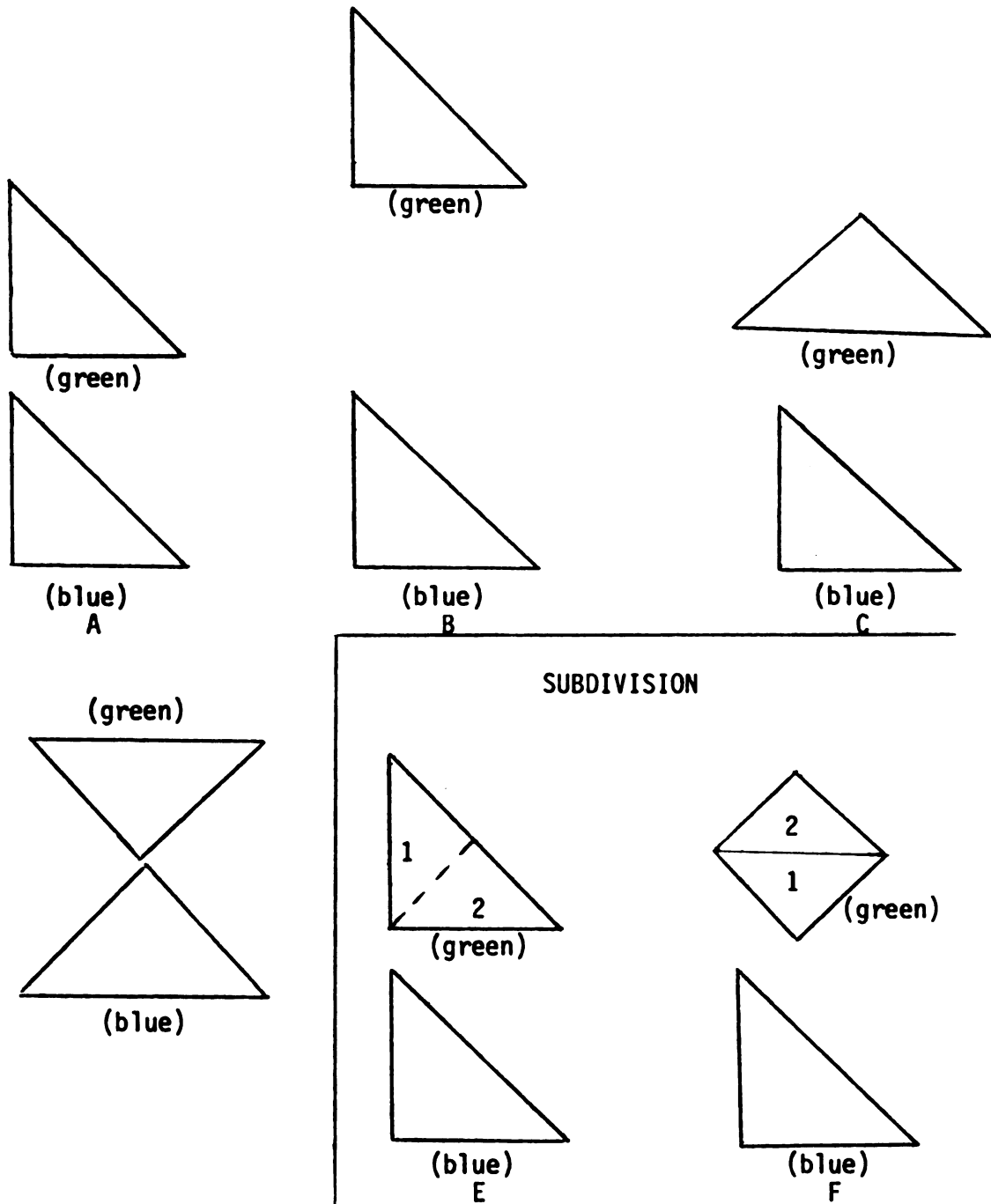
²Ibid., p. 95.

example, changes in the perceptual aspects of the polygonal region such as shape and position do not modify its area. "Change of position" and "subdivision" relative to polygonal regions are two such transformations that do not alter area. Change of position refers to the rearrangement of the polygonal region relative to its surroundings (see Figure 3-3, A-D) which does not change the area of the region. "Subdivision" refers to the dividing of a polygonal region into component non-intersecting parts (except for a point or a line). The rearrangement of the positions of these parts constitutes a change of shape of the region (see Figure 3-3, E and F), which again does not change the area of the polygonal region. Conservation of area is presupposed in the axioms of area: For example, the additive property of area implies that if a polygonal region is the union of two subregions (such that the subregions intersect only in edges or vertices), then the area measure of the polygonal region is the sum of the area measures of the two subregions, regardless how the two subregions are combined. Hence, several regions of various shapes may have the same area measure.

The task used to test for conservation of area is a variation of the task used by Piaget for the same purpose.¹ The distinction between the Geneva group's task and the task used in this study is the shape of the polygonal region used prior to subdividing and rearranging of its parts. In the Geneva task the child was shown two

¹Ibid., p. 274.

FIGURE 3-3
 CONSERVATION OF AREA
 CHANGE OF POSITION



Scale: 1" = 8"

rectangles recognized as congruent and the experimenter then cut a portion off one and moved it to another part of the same figure (the experimenter cut the rectangle diagonally and put the two sections together in the shape of a triangle). The polygonal regions used in the conservation of area task in this study were two isosceles right triangles having eight-inch legs. A cut was made on one triangle along the line that bisected the right angle, thus forming two forty-five degree right triangles (see Figure 3-3, E and F). The investigator chose to begin this portion of the task with isosceles triangles rather than with rectangles (as in the Geneva task) since the vocabulary task of this study involved comparisons of triangular regions. The task was administered in two parts: (1) a change of position test and (2) a subdivision test. First, to test for conservation of area relative to change of position (movement of the plane regions to be compared) the subject was given two isosceles right triangles with eight-inch legs. The triangles were made of paper, one blue, the other green. The child was allowed to manipulate the two triangles and superimpose them to determine that they are the same size and shape (congruent). The child was assisted by the experimenter, if it was necessary, to arrive at a conclusion of equal area. The assistance by the experimenter involved superimposing the triangles and trimming with a scissors whenever the child thought it necessary to make the two triangles congruent. Once the child agreed that the two triangles had the same amount of area ("space" or "room") the triangles were placed in an ordered sequence of arrangements relative to each other (see Figure 3-3, A-D). After each arrangement,

the child was asked which region had more area ("space" or "room") or were they the same. The child was asked for a reason supporting his answer when he stated that one triangle had more space than the other (an incorrect response) or after he had answered correctly after all the different arrangements of the triangles.

There are two categories of responses for the conservation of area task relative to the change of position portion. A child passed this portion of the conservation of area task if he responded correctly to the question asked after each alteration of positions of blue and green triangles and supported his decisions with the reason: changing the position of the triangle does not change its size. The following protocol is an example of a child who passed the conservation of area task relative to change of position.

BOW (7;11). (The child stated that the two triangles were the same size by the superimposing method. After each rearrangement of the triangles -see Figure 3-3, A-D- he stated that they were the same size.) "Why?" "Because moving them does not make them get bigger or smaller." . . .

A child failed the conservation of area task relative to change of position if he answered that the two triangles were not the same size after any one of the various arrangements. Some of the younger children gave responses indicating failure similar to the responses of BUF, BUR and MCS.

BUF (7;1). (The child determined that the triangles were the same size by superimposing them. A correct response was given after arrangement A. Then the two triangles were separated as in arrangement B; see Figure 3-3 B). "Which triangle has more space, the blue or the green, or are they the same?" "The green one." "Why?" "Because it is farther away."

BUR (8;1). . . . (The triangles are arranged as in arrangement B; see Figure 3-3 B.) "Which triangle has more room or do they have the same amount of room?" (For this child the term "room" had to be used instead of "area" or "space" as indicated in the vocabulary task.) "The green one has more room." "Why?" "Because it is up higher."

MCS (7;3). . . . (The triangles are arranged as in arrangement C; see Figure 3-3 C). "Which triangle has more space or do they have the same?" "The green one." "Why?" "It is pointing up."

Again, as in the conservation of length task, incorrect responses were made on the basis of perceptual judgments. As Piaget has indicated, the child beginning to learn the measuring process places a lot of faith in visual transfer at the earliest stages of measurement, thus answering on the basis of "it looks as . . ." rather than on properties of logic. A child that relies heavily on perceptual judgments would be judged at no higher level than substage II A.¹

The second portion of the conservation of area task involves subdivision, i.e. the altering of the shape of one of the two triangles known to be congruent. The experimenter assisted the child, if it was necessary, to verify that the blue and green triangles are congruent. Then, in the child's view, the experimenter cut the green triangle as illustrated in Figure 3-3 E. The cut portions are then joined to form a plane region of a different shape (square). The child is asked which of the two plane regions has the most space or are they the same size.

¹Ibid., p. 274.

This portion of the conservation of area task was passed with responses similar to that of CAR's.

CAR (9;4). . . . (The green triangle was cut and its parts rearranged to form a square, see Figure 3-3 F). "Which of the two spaces, the blue or the green, is larger or are they the same size?" "Same." "Why?" "You don't change the size when you cut them if you put the parts back together."

Failing responses were given by LOV and CLO.

LOV (7;6). . . . (After cut and rearrangement; see figure 3-3 F). "The green one is smaller". "Why?" "Because it was cut."

CLO (6;7). . . . (After cut and rearrangement of green parts; see Figure 3-3 F.) "The green one is smaller." "Why?" "Because it is a square."

A child passed the Conservation of Area task if he passed both the change of position portion and the subdivision portion. He is then at substage III A according to Piaget, i.e. he possesses conservation of area.¹ Failing only one part of this task would place the child at substage II B, i.e. not all of his responses are based upon visual transfer.² Failing both parts of the conservation of area task would imply the child is no more advanced than substage II A where children confine themselves to perceptual judgments and the areas are not conserved when their appearances are modified.³

The evaluation of each child relative to the conservation of area task was recorded as:

P: indicating the child passed both the change of position

¹Ibid., p. 275.

²Ibid., p. 274.

³Ibid.

and the subdivision portion.

Fc: indicating the child failed the change of position portion only.

Fs: indicating the child failed the subdivision portion only.

F: indicating the child failed both portions.

Based on the above evaluations, the child was placed in one of the following stages of intellectual development relative to conservation of area:

- A. attainment (evaluated with score of P).
- B. transitional (evaluated with score of Fc or Fs).
- C. none (evaluated with the score of F).

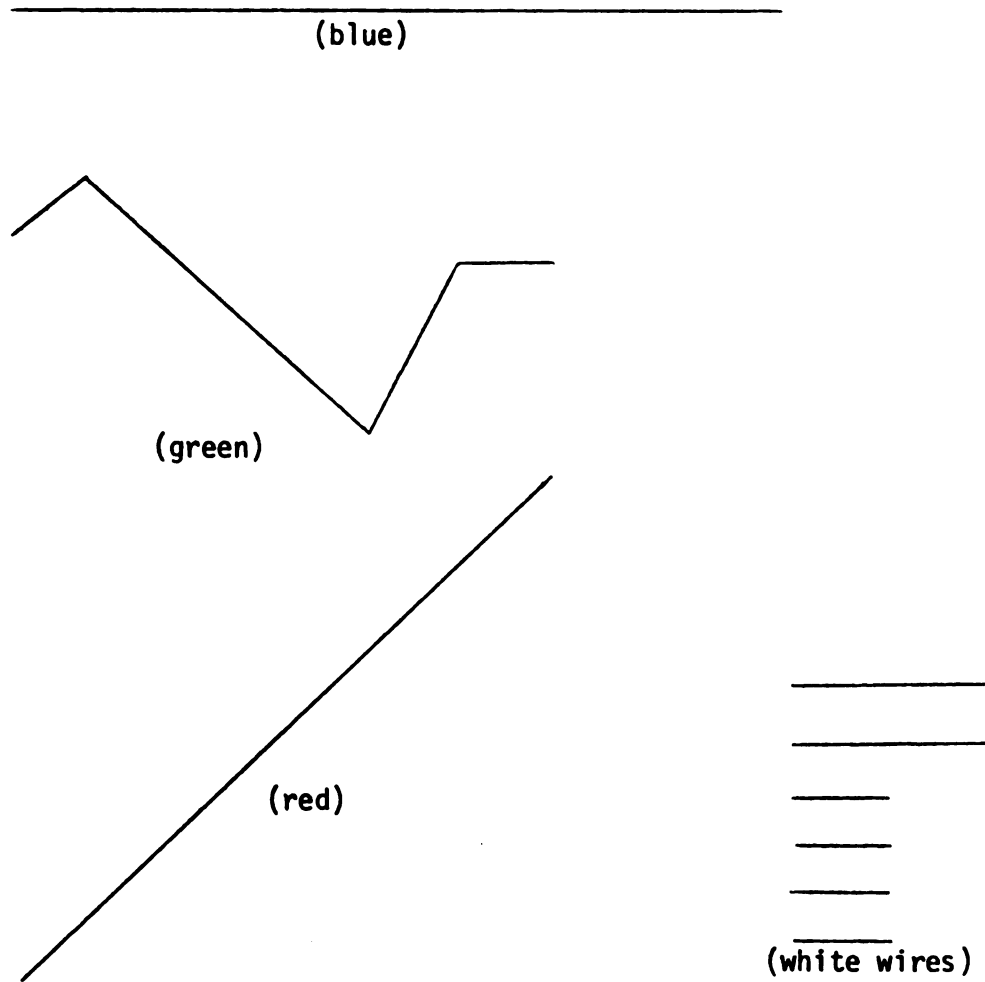
The conservation stages (attainment, transitional, and none) were recorded in Appendix C as A, B, and C, respectively.

Additivity of Length Task

The additivity property regarding length measurement states that if line segment P is the union of two (or more) line segments P_1 and P_2 such that the intersection of P_1 and P_2 is at most the end-point of each line, then relative to a given unit length, the length measure of P is the sum of the length measures of P_1 and P_2 . If a second line segment T is composed of these segments P_1 and P_2 arranged in a different way, it has the same length measure as the line segment P .

The task used to test for the understanding of the additive property of length involved three colored lines on a piece of white paper (see Figure 3-4). The blue line is straight and horizontal (sixteen inches long), the green line is broken into segments of two

FIGURE 3-4
ADDITIVITY OF LENGTH



Scale: 1" = 4"

inches, eight inches, four inches, and two inches and the red line is straight and oblique (fifteen inches long).

The child was given six pieces of wire of the same thickness (two pieces were four inches long and four pieces were two inches long.) The child was asked to cover each of the three colored lines with the pieces of wire, one color at a time, by placing the wires end to end on the lines. If the child was unable to perform this function, he was given assistance by the investigator. The question to be answered in this task was which of the colored lines have the same length and why.

The original task used in the pilot study to test for the understanding of the additive property of lengths was developed involving a similar procedure and objective (relative to length) as Wagman's task¹ used to test for the understanding of the additive property of areas. (Wagman's additive property of area task was used in this study and is discussed in the next section). The additive property of length task that was used in the actual study is a variation of the one used in the pilot study. The broken green line used in the pilot study consisted of a first segment of eight inches followed by two four-inch segments. The investigator observed that this arrangement did not cause the child to rearrange the wires when he placed them on the green line from the blue line. That is, he used exactly the same arrangement for both the blue and the green lines. Since it was desired to have different arrangements of wires for the two

¹Wagman, Conception of Area, pp. 59-60.

lines, and almost invariably a child used a four-inch wire for the first portion of the blue line, the first portion of the green line was made to be two inches long for the additivity task used in the actual study.

The question to be answered in this task was which of the colored lines have the same length. A reason for the child's response was requested. The correct response by the child is that the blue line segment (sixteen inches long) and the broken green line segment have the same length. The supporting reason is that each of these two colored line segments requires the same amount of wire to cover them, but in different arrangements. The different arrangement of the joined wires does not vary their total length. This type of response requires an understanding of conservation and transitivity ($A=B$ and $B=C$ imply $A=C$). The child makes the union of wires congruent to the blue line by butting the ends of the wires together. Thus, the blue line and the joined wires are of the same length. Due to conservation, the equality of length is unaltered when he arranges the wires and makes their union congruent to the green line. Hence, as a result of transitivity, the blue and green lines have equal lengths.

Piaget states that "conservation, and . . . transitivity, are achieved at a mean age of 7 1/2 years, . . ."¹ A study reported by Smedslund shows that the average age at which children acquire

¹Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 126.

transitivity of length relations lies somewhere between the ages of seven and eight years.¹ Thus, successful completion of the additive property of length task (which involves conservation and transitive properties) is possible for children at substage III A (which Piaget claims to begin at approximately age seven)² or higher in Piaget's scheme of intellectual development.

The following protocol is an illustration of a correct response to this task:

MCN (10;6). "Can you completely cover the blue line with the wires by placing the wires end to end on the blue line?" (She does so using the two four-inch pieces first.) "Yes." "Now, can you cover the green line with the wires?" (Completes task using two-inch wire first.) "There." "Which of the two lines is longer, the blue or the green?" "Or are they the same?" "They are the same." "Why?" "They both took the same amount of wire to cover them." "Now cover the red line with the wire." (Attempts to do so but discovers the last wire extends beyond the end of the red line.) "It doesn't match exactly." "Which line is longer, the blue or the red?" "Blue." "Why?" "There is wire left over when I tried to cover the red."

A type of response that represents failure of the task is one in which the child could not make accurate judgments about the lengths of the blue, green, and red lines. If the child could not support his decision with reasoning similar to that stated earlier, he did not pass the task. A child who failed the additivity of length task would be at a level below substage III A, according to Piaget's scheme of intellectual development.

¹Jan Smedslund, 'Development of Concrete Transitivity of Length in Children,' Child Development 34 (1963), p. 393 and p. 400.

²Phillips, Origins of Intellect, p. 75.

PUR and OLE are illustrations of children who had difficulty with the additivity of length task:

PUR (7;7). "Can you cover the blue line with the wires?" "Sure." (He does so,) "Now, can you cover the green line with the wires?" (Hesitates, then completes task.) "Which of the two lines is longer, the blue or the green?" "The blue." (Incorrect answer.) "Why?" "It is straight and sticks out this much more on this side." (He held his fingers apart the approximate distance that the blue line extended to the right beyond that of the green line.) "Do the wires help?" "No, I can see which one is longer."

PUR was making a judgment based upon the ends of the line. He did not consider applying the transitive property to determine that the blue and green lines were the same length. A child who relies on perception rather than on the transitive property is placed at stage II of Piaget's intellectual developmental scheme.¹

OLE (7;9). (He responds correctly to the comparison of the blue and green lines and gives adequate reason for his decision.) "Now, can you cover the red line with the wires?" "I think so." (He does so, but to make the wires fit on the green line, he overlaps the last two wires by one inch.) "Which line is longer, the blue or the red?" "Or are they the same length?" "The same." "Why?" "They both took the same wires to cover them up."

OLE failed the additivity property of length task because he did not take into account the overlapped wires.

Additivity of Area Task

The additivity property relative to area implies that: "Suppose the polygonal region R is the union of two polygonal regions R_1 and R_2 such that the intersections of R_1 and R_2 are contained in a

¹Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 65.

union of a finite number of line segments. Then relative to a given unit area, the area of R is the sum of the areas R_1 and R_2 ."¹ If a second polygonal region S is composed of these same subregions R_1 and R_2 arranged in a different way, it has the same area as region R .

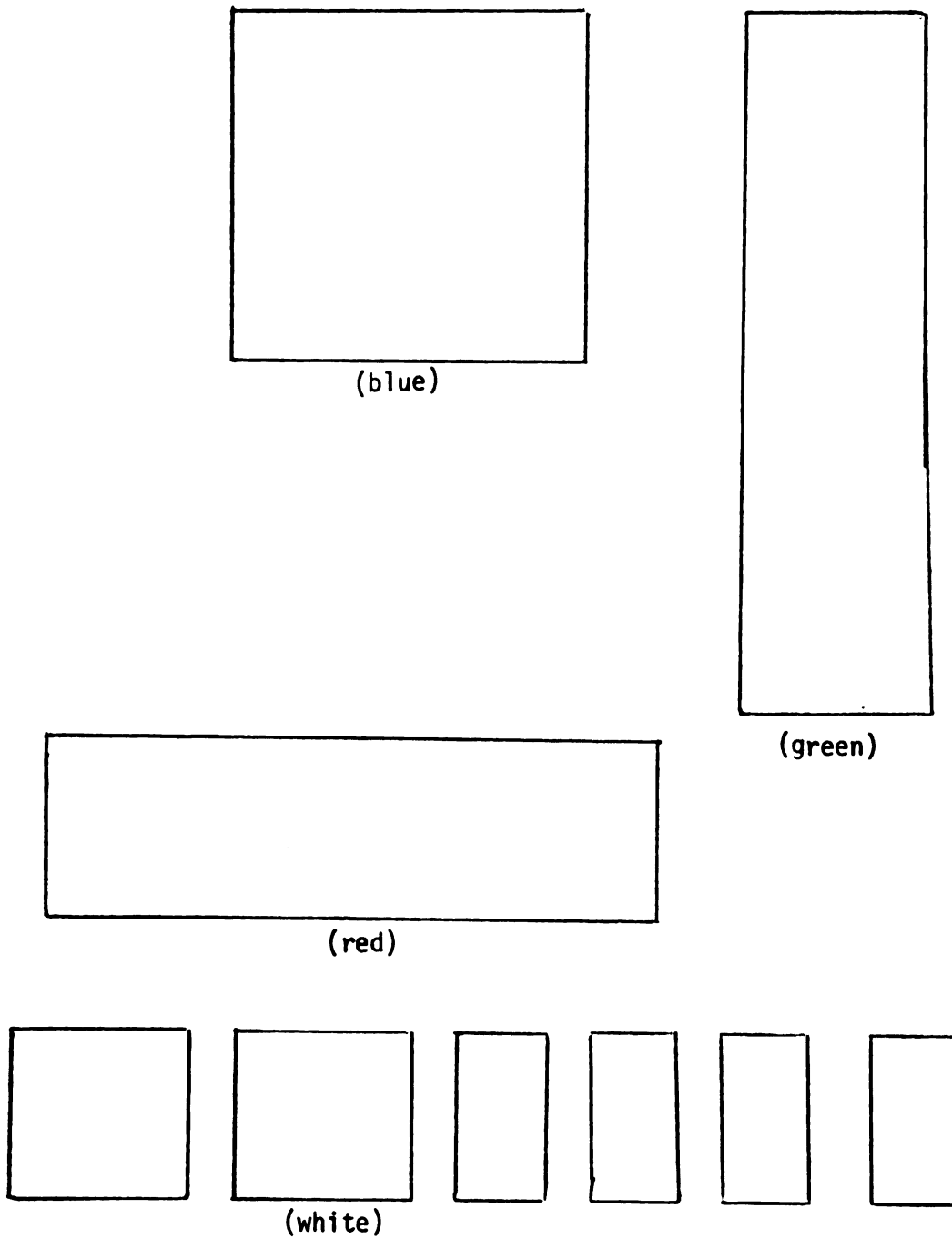
The task used to test for the understanding of the additive property of area was suggested by the task used in Wagman's study for the same purpose.² The task used in the present study involved three polygonal regions: a four-inch square, a two- by eight-inch rectangle, and a two- by seven-inch rectangle made of blue, green, and red paper, respectively (see Figure 3-5). These regions were pasted to a piece of white paper so that they could not be manipulated. The child was given two square pieces of white paper with two-inch sides and four rectangular pieces of white paper with one- and two-inch sides. The child was asked to cover each of the three colored regions with the white pieces of paper, one color at a time. If the child was unable to perform this function, he received help from the investigator. The question to be answered in this task is which of the colored pieces of paper had the same area. A reason for the child's response was requested.

The correct response by the child is that the blue and the green pieces of paper have the same area. The supporting reason being that each of these two colored papers requires the same amount of white paper to cover them, but in different arrangements. The

¹School Mathematics study Group, Geometry With Coordinates Part II (New Haven: Yale University Press, 1963), p. 989.

²Wagman, Concept of Area, pp. 146-47.

FIGURE 3-5
ADDITIVITY OF AREA



Scale: 1" = 2"

different arrangements of the white papers do not vary their total area. This type of response requires an understanding of conservation and transitivity ($A=B$ and $B=C$ imply $A=C$). The subject makes the union of the subregions congruent to the blue polygonal region and therefore equal in area to that region. Due to conservation, the equality of area is unaltered when he rearranges the subregions and makes their union congruent to the green region. Hence, as a result of transitivity, the blue and green regions have equal areas. Since conservation of area and application of the transitive property are characteristic of the first part of the concrete operational stage, substage III A, the correct response described above belongs to substage III A or a higher level.¹

The following protocol is an example of this type of response:

LAR (7;4). "Cover the blue with the white pieces." (The subject does so.) "Can you cover the green exactly with the white pieces?" "I'll try." (He does so.) "Now, which has more space, the blue or the green?" "Or do they have the same amount of space." "Same." "How do you know that?" "The same pieces of white paper fit exactly on both the blue and the green." "Now, fit the white pieces on the red paper." (Child attempts to do so.) "I have one piece left over." (It is a one- by two-inch piece.) "Which has more space, the blue or the red, or are they the same?" "The blue has more space." "Why?" "Because I had one piece left over when I covered the red."

A type of response that represents failure of the task is a response in which the child could not make accurate judgments about the area of the blue, green, and red papers. If the child could not

¹Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 275.

support his decision with reasoning similar to that stated earlier, his responses were placed in this failure category. A child who failed the additivity of area task would be at a level below substage III A.¹

The following protocols are examples of children who failed the additivity of area task:

AND (7;6). "Could you cover the blue piece of paper with these white pieces?" "I think so." "Could you now cover the green paper with the white ones?" "I'll try." (Again, he does so.) "Which space is bigger; Why do you say that?" "It is longer, it looks bigger." (The two- by eight-inch rectangle looks as if it has more space than the four-inch square to the child.) "Could these white pieces help you decide which has more space?" "No, I can tell without them."

BUR (8;1). (The questions pertaining to comparing the blue and green papers are answered correctly.) "Can you cover the red paper with the white ones?" "I'll try." "Yes, I can." "Which paper has more space, the blue or the red paper?" "They are the same." "Why?" "I covered them both with the white papers." "What about this piece of white paper?" (The one- by two-inch piece that wasn't needed to cover the red paper but was needed to cover the blue.) "Oh, I didn't need that to cover the red paper."

In the cases of AND and BUR, the transitive property was not considered in comparing the areas of the colored pieces of paper. Perceptual judgments were made instead. This is characteristic of children below substage III A.

Unit Length Task

Piaget indicated what is meant by the measuring process when he stated: "To measure is to take out of a whole one element, taken

¹Ibid.

as a unit, and to transpose this unit on the remainder of the whole; . . ." ¹ Thus, the measure of an object is dependent upon the size of the unit of measure chosen. In general, for each unit of length measure, there is a correspondence that assigns a unique positive real number to each line segment and one is assigned to the given unit of length measure.

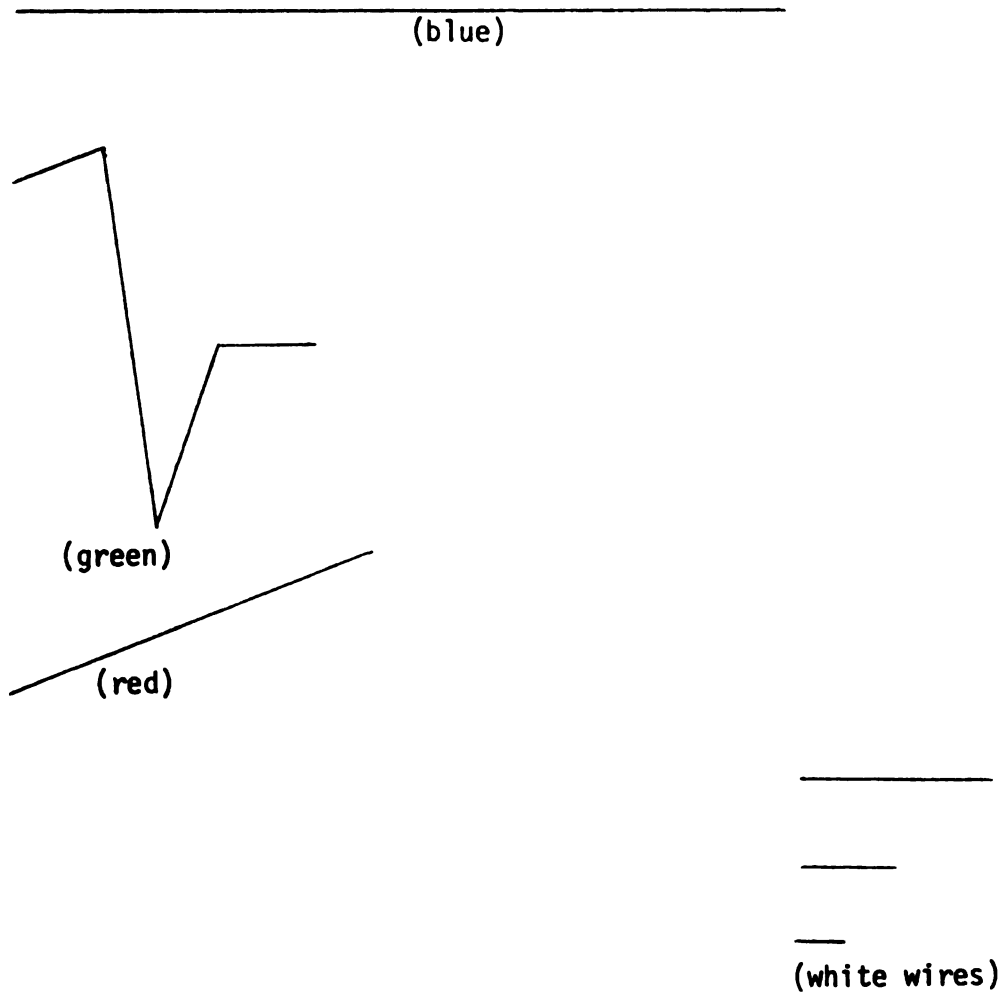
The task used in this study to test for the understanding of the unit of length measure was developed involving the same procedure and objective (relative to length measure) as Wagman's task ² used to test for the understanding of the unit of area measure (Wagman's unit of area measure task was used in this study and is discussed in the next section). The task developed to test for the understanding of the unit of measure, i.e. to test to see if the subject is cognizant of the importance given to the size of the unit of length measure in the measuring process, involves three colored lines drawn on white paper. The blue line is straight and sixteen inches long, the green line is broken into segments of two inches, eight inches, four inches, and two inches, respectively and the red line is straight, oblique, and eight inches long (see Figure 3-6).

The child is given the paper with the three colored lines drawn on it. Also, he is given six wires that are four inches long, ten wires that are two inches long, and ten wires that are one inch long. No mention is made of the length measures of the lines. All

¹Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 3.

²Wagman, Concept of Area, pp. 51-52.

FIGURE 3-6
UNIT OF LENGTH MEASURE



Scale: 1" = 4"

wires are of the same thickness, one-eighth inch in diameter. The child is asked to cover the blue line with the long pieces (four inches) of wire. Then he is asked to cover the green line with the two-inch wires and the red line with the small (one-inch) wires. If the child seems uncertain as to what is being asked, the investigator assists the child in the covering process. The assistance given by the investigator involves demonstrating how to place the wires end to end on the colored lines. This completed, he has four four-inch wires on the blue line, eight two-inch wires on the green line, and eight one-inch wires on the red line. The wires are left on the colored lines for the next part of the task. This is consistent with Smedlund's requirement that the subject's remembering the relevant information be insured.¹

In the first part of the task the child is asked to compare the lengths of the blue and the green line (a different number of pieces of wire of different size were used to cover the blue and green line). He is allowed to manipulate the wires that were not used in the covering process (two four-inch wires, two two-inch wires, two one-inch wires). Next, he is asked to compare the green and the red lines in terms of length (it requires eight wires of different lengths, two- and one-inch lengths, to cover the green and red lines, respectively). This question is asked to see if the subject determines length erroneously by comparing the number of wires used in the covering and disregarding the length of the wires. The second part of the

¹Jan Smedslund, Concrete Reasoning: A Study of Intellectual Development ("Monograph of the Society for Research in Child Development", 29, No. 2, Serial no. 93), 1964, p. 4.

task requires the child to determine how many of the one-inch wires are needed to cover the green line. The child may manipulate the extra wires that are not covering the colored lines if desired. A reason to support his answer is requested after each response from the child.

The first type of response includes those that are correct and are based on comparing the lengths and the numbers of the different measuring units. In a similar experiment, the Geneva group found that "almost unhesitatingly, they [children at substage III B] compare unit lengths, and discover both that the small unit is a third or a half of the other, . . ." ¹ Thus, the correct response that is based upon recognition of the different size units of measure belongs to substage III B. This type of response indicates the understanding of the unit length property of measurement in a concrete situation, and hence, is rated as passing of the unit length task. The following protocol is an example of this type of response:

MYE (9;0). (The subject placed four four-inch wires, eight two-inch wires, and eight one-inch wires on the blue, green and red lines, respectively.)
 "Which line is longer, the blue or the green, or are they the same?" "Same." "Why?" "Because each of these wires (Child points to a wire that is four-inches long) makes two of those." (Child points to the two-inch wires on the green line.) "Is the green or red line longer or are they the same?" "The green line is longer." "How do you know that?" "Both lines take eight pieces to cover them, but the pieces covering the green line are bigger (assumed she meant longer) than those covering the red line." "Could you tell how many of these (one-inch wires) are needed to cover the green line without placing the wires on the line?" "I think so." (The subject is allowed to manipulate the extra wires. After comparing the two-inch wire with the one-inch wire the subject responds.) "It

takes sixteen." "How do you know that?" "Each of these (child points to the two-inch wire) makes two of those (child points to the one-inch wires)."

The next type of response is transitional to substage III B. The child would answer all questions correctly with adequate reasons except for determining how many one-inch wires are needed to cover the green line without placing the wires on the line. The child felt it necessary to lay one-inch wires next to the two-inch wires on the green line. This type of response was judged as passing the unit length task, although the investigator believes that the child would have continued to lay the one-inch pieces of wire on the green line if the two-inch pieces of wire had not been there for comparison of lengths. SAU is an example of this type of response:

SAU (9;3). (Up to this point all questions are answered properly.) "Can you tell how many of these (one-inch) wires are needed to cover the green line without placing them on the green line?" "I don't think so." "You may use these wires if you wish." (He begins to place the remaining one-inch wires along the green line. Then he borrows the one-inch wires from the red line and continues to cover the green line. After laying six one-inch wires along the green line he responds to the question.) "It takes sixteen." "How do you know that?" "These (the two-inch) wires are twice as big as these (one-inch) wires."

The final type of response belongs to children who are below the stage of concrete operations, stage III. This type of response indicates poor understanding of the unit length measure property and, consequently, failure of the task. WES and REH are examples of this type of response:

WES (9;1). (The subject placed the four four-inch wires on the blue line, eight two-inch wires on the green line, and eight one-inch wires on the red line.) "Which line is longer, the blue or the green or are

they the same?" "The blue one is longer." "How do you know that?" "Because it has longer pieces in it." (Child points to the four-inch wires.)

REH (10;1). (Wires are placed on the colored lines properly.) "Which line is longer, the blue or the green, or are they the same?" "The blue line." "How do you know that?" "It looks like it?" "What do you mean?" "This one (the blue line) is straight." "Which is longer the green line or the red line?" "The green line." "How do you know that?" "Because it has more wires." (Since this was a correct answer with an invalid reason, the investigator decided to ask the subject to compare the blue and red lines in terms of length.) "Which is longer the red line or the blue line?" "The red line." "How do you know that?" "Because it takes more (eight one-inch) wires to cover the red line than it takes to cover the blue line (four four-inch wires)." (This subject based his decision of length upon the number of units used and disregarded the length of those units.)

As indicated, both WES and REH were rated as failure on the unit length task. WES based his response on the size of the units while disregarding the number of units used. Thus, he gave incorrect responses. REH based his responses on the number of units and disregarded the size of the units. Hence, again failure of the task.

Unit Area Task

The unit area property implies that "for each given unit polygonal region, there is a correspondence that assigns a unique positive real number to each polygonal region; one is assigned to the given unit polygonal region."¹ The importance of the relationship between the unique area measure of the polygonal region and the

¹Myron F. Rosskopf, Harry Sitomer, and George Lenchner, Modern Mathematics: Geometry (Morristown, N.J.: Silver Burdett Co., 1966), p. 431.

standard unit used is stressed in modern geometry textbooks.¹ According to Piaget: "Unit measurement may occur . . . if he [the subject] counts the measuring units and uses them according to their respective sizes."² The task, as suggested by Piaget, requires the subject to demonstrate the use of the unit area regarding the number of units used and the size of each unit.

The task used in this study to test for the understanding of the unit of area measure is the same task used in Wagman's study except for the shape of one of the polygonal regions whose areas are to be compared.³ An "L" shaped polygonal region was replaced by a rectangular polygonal region (see Figure 3-7 C). The rectangular region was used since its length dimension gave it an appearance of having an area that was greater than that of the other two polygonal regions (it, in fact, had a smaller area). With the adaptation, the investigator believes that those children who judge the amount of area solely on one of the dimensions of the region would be identified. The three polygonal regions were made of colored paper, blue, green, and red (see Figure 3-7) and pasted on a sheet of white paper. The blue and green region have the same area and the red region has a smaller area.

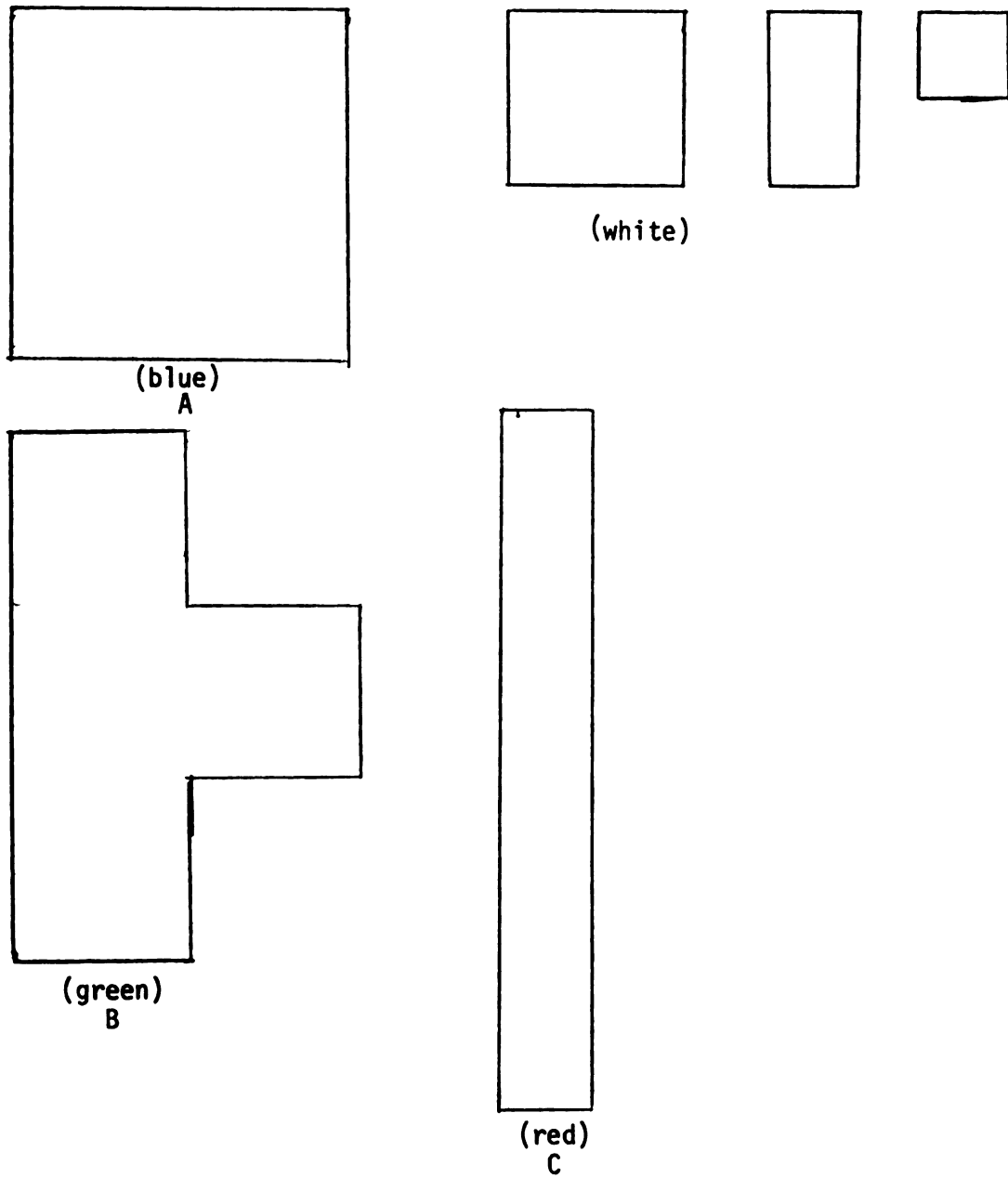
¹School Mathematics Study Group, Geometry with Coordinates, Part II (New Haven: Yale University Press, 1963), p. 744.

²Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 293.

³Wagman, Concept of Area, p. 51.

FIGURE 3-7

UNIT OF AREA MEASURE



Scale: 1" = 2"

The investigator gives the child six squares with two-inch sides, ten rectangles with one- and two-inch sides, and ten squares with one-inch sides made of white paper. These small pieces of white paper are to be used as measuring units. There are more pieces of white paper than is necessary to completely cover all three colored regions.

The child is asked to cover the blue region (a square with four-inch sides) with the square pieces of paper having two-inch sides. Next, the child is asked to cover the green paper (see Figure 3-8 B) with the rectangular pieces of white paper. Then, he is asked to cover the red paper with the small squares (one-inch sides). If the child has difficulty with the covering process, he is given assistance by the investigator. The pieces of white paper are left on the colored pieces of paper for the next part of the task. This is consistent with Smedlund's requirement that the subject's remembering the relevant information be insured.¹

In the first part of the task, the child is asked to compare the amount of space on the blue paper with that of the green paper. A different number of white pieces of paper with different areas were used to cover the blue and green regions. This is done to see if the child is cognizant of the different sizes of units used. He is allowed to manipulate the pieces of white paper that were not used in the covering process (two two-inch squares, two one- by two-inch rectangles, and two one-inch squares). Next, the subject is asked to

¹Smedslund, Concrete Reasoning, p. 4.

compare the amount of space on the green paper with that of the red paper. This time, the same number of pieces of white paper (but different sizes) were used to cover the green and red regions. This question is asked to see if the subject determines area erroneously by comparing the number of white pieces used in the covering and disregarding the size of the pieces used. The last question asked requires the child to determine how many of the one-inch squares are needed to cover the green region without placing them on the green region. He may manipulate the extra white pieces of paper. A reason to support his answer is requested after each response from the child.

The first type of response includes those that are correct and are based on comparing the areas and numbers of the different measuring units. The Geneva group found that "at level III B children understand the notion of a unit and . . . take the size of the measuring elements into account."¹ Thus, this type of response belongs to substage III B since it is evidence of an understanding of the relative size of the units of measure. This type of response indicates the understanding of the use of the unit area property in a concrete situation, and thus, is rated as passing of the unit area task. The following protocol is an example of this type of response:

KOS (9;11). (The child has placed the four two-inch squares on the blue region, the eight one- by two-inch rectangles on the green region, and the eight one-inch squares on the red region.) "Which space is larger, the blue or the green? Or are they the same?" "Same."
 "How do you know that?" "Each of these squares makes two rectangles and there are four squares covering the

¹Piaget, Inhelder, and Szeminska, Concept of Geometry, p. 296.

blue and eight rectangles covering the green, so they are the same." "Which space is larger, the green or the red?" "The green." "Why?" "It takes eight pieces of paper to cover the green and eight for the red, but the pieces on the green paper are larger." "How many of these pieces (one-inch squares) would be necessary to completely cover the green paper?" (The child places a one-inch square that was not used in the covering on top of a one- by two-inch rectangle.) "There are two of these small pieces (one-inch squares) for each of these one- by two-inch rectangles so it takes sixteen of the small ones to cover the green paper."

The next type of response is transitional to substage III B.

All answers were correct with adequate reasons given by the child.

The distinction between this type of response and KOS's response is that the former had to lay the one-inch squares on the green paper to answer the last question. This type of response was judged as

passing the unit area task. MAN is an example of this type of response:

MAN (10;4). (Correct responses and adequate supporting reasons are given to this point.) "Can you tell how many of these (one-inch squares) pieces of paper are needed to cover the green paper without placing them on the green paper?" (Hesitates.) "Would you like to use these extra pieces of paper?" "Yes." (He picks up only the two extra one-inch pieces and places them on a white rectangle (one- by two-inch) that is on the green paper. He then realizes the relationship between the one-inch square and the one- by two-inch rectangle.) "I know it would take sixteen."

The distinction between the responses of KOS and MAN is that KOS immediately recognized that two one-inch squares are equivalent to one one- by two-inch rectangle while MAN observed this only after attempting to place the one-inch squares on the green region. It is assumed by the investigator that MAN would have continued placing the

one-inch squares on the green region if the one- by two-inch rectangles were not on the green region. Both types of responses were judged as passing responses on the unit area task.

The final type of response belongs to children who are below stage III, the stage of concrete operations. This type of response indicates poor understanding of the unit area measurement property and, consequently, failure of the task. CAR, BIS and LEI are examples of this type of response:

CAR (9;4). (The investigator assists in placing the one- by two-inch rectangle on the green region; the child placed the squares on the blue and red regions properly.) "Which has more space, the blue or the green? Or are they the same?" "The green." "How do you know that?" "Because I had the most trouble doing that one." . . .

BIS (7;11). (The child covers all three colored regions properly.) "Which of the papers has more space, the blue or the green? Or are they the same?" "The green." "How do you know that?" "It takes more pieces of white paper to cover it." "Which has more space, the green or the red?" (Pause, while she counts the white pieces of paper.) "They are the same." "How do you know that?" "They both take eight pieces to cover them."

LEI (9;2). (Covers the colored regions properly.) "Which piece of paper has more space, the blue or the green? Or are they the same?" "The blue paper." "How do you know that?" "Because it takes bigger pieces to cover it."

As noted earlier, this type of response indicates failure of the unit area task. CAR regarded neither the size nor the amount of units used to cover the colored regions. He focused his attention on only the degree of difficulty in covering the regions. BIS believed that the area of a region is dependent upon the number of

units used in the covering of the region and disregarded the size of those units. LEI did take into consideration the size of the measuring units, but disregarded the number of those units.

The Research Design

This investigation has attempted to bring empirical evidence to bear on the theoretical assumptions regarding the association of length and area measurement. The two research hypotheses assumed an association between the development of the measurement of length and the development of the measurement of area. The study focused on the following two questions: (1) Is there a parallel development of the significant properties (congruence, conservation, additivity, and unit measure) of length and area measurement? (2) Is the understanding of length measurement attained at the same age as the understanding of area measurement?

Analysis

Twenty-five operational hypotheses were developed from the two research hypotheses noted in Chapter I. Statistical alternative hypotheses were formed relative to each operational hypothesis. A null hypothesis was formed for each statistical alternative hypothesis. Pearson's Chi-square Test of Independence and the Phi-coefficient were used in testing the null hypotheses for acceptance.¹ If a null hypothesis was rejected, then the Phi-coefficient was used to indicate the strength of association between the understanding of length and

¹William Hays, Statistics For Psychologists, (New York: Holt, Rinehart, and Winston, 1963), pp. 589-597.

area measurement relative to the four common properties of congruence, conservation, additivity, and unit measure. The use of the Phi-coefficient and the fourfold contingency table as a statistical means of testing results relative to Piaget's theories is suggested by Bentler.¹ A seven step statistical procedure for testing the hypotheses is explained following the statements of the hypotheses.

Research Hypothesis I

I. The cognitive development of length measurement is simultaneous to the cognitive development of area measurement relative to the properties of congruence, conservation, additivity, and unit measure.

Operational Hypotheses of Research Hypothesis I

- IO₁ The seven-year old subject's score on the Congruence of Length Task will correlate positively with the subject's score on the Congruence of Area Task.
- IO₂ The eight-year old subject's score on the Congruence of Length Task will correlate positively with the subject's score on the Congruence of Area Task.
- IO₃ The nine-year old subject's score on the Congruence of Length Task will correlate positively with the subject's score on the Congruence of Area Task.

¹Peter N. Bentler, "Monotonicity Analysis: An Alternative to Linear Factor and Test Analysis," Measurement and Piaget, ed. by Donald Green, et al. (New York: McGraw-Hill Book Company, 1971), pp. 220-27.

- 10_4 The ten-year old subject's score on the Congruence of Length Task will correlate positively with the subject's score on the Congruence of Area Task.
- 10_5 The eleven-year old subject's score on the Congruence of Length Task will correlate positively with the subject's score on the Congruence of Area Task.
- 10_6 The seven-year old subject's score on the Conservation of Length Task will correlate positively with the subject's score on the Conservation of Area Task.
- 10_7 The eight-year old subject's score on the Conservation of Length Task will correlate positively with the subject's score on the Conservation of Area Task.
- 10_8 The nine-year old subject's score on the Conservation of Length Task will correlate positively with the subject's score on the Conservation of Area Task.
- 10_9 The ten-year old subject's score on the Conservation of Length Task will correlate positively with the subject's score on the Conservation of Area Task.
- 10_{10} The eleven-year old subject's score on the Conservation of Length Task will correlate positively with the subject's score on the Conservation of Area Task.
- 10_{11} The seven-year old subject's score on the Additivity of Length Task will correlate positively with the subject's score on the Additivity of Area Task.
- 10_{12} The eight-year old subject's score on the Additivity of Length Task will correlate positively with the subject's score on the Additivity of Area Task.

- 10_{13} The nine-year old subject's score on the Additivity of Length Task will correlate positively with the subject's score on the Additivity of Area Task.
- 10_{14} The ten-year old subject's score on the Additivity of Length Task will correlate positively with the subject's score on the Additivity of Area Task.
- 10_{15} The eleven-year old subject's score on the Additivity of Length Task will correlate positively with the subject's score on the Additivity of Area Task.
- 10_{16} The seven-year old subject's score on the Unit of Length Measure Task will correlate positively with the subject's score on the Unit of Area Measure Task.
- 10_{17} The eight-year old subject's score on the Unit of Length Measure Task will correlate positively with the subject's score on the Unit of Area Measure Task.
- 10_{18} The nine-year old subject's score on the Unit of Length Measure Task will correlate positively with the subject's score on the Unit of Area Measure Task.
- 10_{19} The ten-year old subject's score on the Unit of Length Measure Task will correlate positively with the subject's score on the Unit of Area Measure Task.
- 10_{20} The eleven-year old subject's score on the Unit of Length Measure Task will correlate positively with the subject's score on the Unit of Area Measure Task.

Statistical Hypotheses of Operational Hypotheses

Each of the twenty operational hypotheses was cast into an appropriate form of a statistical alternative hypothesis. For example: in the case of operational hypothesis IO_1 , the "statistical alternative" became $H_1: \phi_{xy} > 0$. Where the symbol (H_1) denotes the statistical alternative hypothesis and form ϕ_{xy} indicates the population correlation coefficient.

The statistical alternative hypothesis was employed to derive a corresponding null hypothesis (H_0), in this case $H_0: \phi_{xy} \leq 0$; this null hypothesis was submitted to test by means of the following seven step process:

1. $H_0: \phi_{xy} \leq 0$
2. Statistical tests employed:
 - a) Chi-Square Test of Independence:

$$\chi^2 = \sum_j \sum_k \frac{(fojk - fejk)^2}{fejk} \quad \text{d.f.} = 1$$

where $fojk$ denotes observed frequency and $fejk$ denotes expected frequency in a fourfold contingency table is used to test for independence of (x) length and (y) area relative to the four measurement properties studied: i.e. $\phi_{xy} = 0$.¹

¹Hays, Statistics For Psychologists, pp. 589-606.

b) Phi-coefficient:

$$\phi = \frac{bc - ad}{\sqrt{(a+b)(c+d)(a+c)(b+d)}} \quad d.f. = 1$$

used to test for a negative correlation between

length and area measurement: i.e. $\phi_{xy} < 0$.¹

3. Level of significance is identified in a Chi-square (χ^2) Table.
4. A critical region is identified.
5. Values of " χ^2 " and " ϕ " are calculated.
6. Decision is made whether the observed value of " χ^2 " is greater than its critical value. If the value of " χ^2 " is greater than its critical value or $\phi > 0$, then the null hypothesis (H_0) is rejected and the statistical alternative hypothesis (H_1) is accepted indicating non-independence of the variables tested.
7. The value of " ϕ " indicates the strength of association between variables.²

The seven step statistical process was applied to each of the twenty operational hypotheses developed from Research Hypothesis I. The statistical analysis procedure used in this study was suggested by the Research Consultation Center, Erickson Hall, Michigan State University.

¹Ibid., pp. 604-06.

²Ibid.

Research Hypothesis II

II The understanding of length and area measurement are attained simultaneously.

**Operational Hypotheses of
Research Hypothesis II**

- II0₁ The seven-year old subjects who have attained (failed to attain) an understanding of either length or area measurement have attained (failed to attain) an understanding of both length and area measurement.
- II0₂ The eight-year old subjects who have attained (failed to attain) an understanding of either length or area measurement have attained (failed to attain) an understanding of both length and area measurement.
- II0₃ The nine-year old subjects who have attained (failed to attain) an understanding of either length or area measurement have attained (failed to attain) an understanding of both length and area measurement.
- II0₄ The ten-year old subjects who have attained (failed to attain) an understanding of either length or area measurement have attained (failed to attain) an understanding of both length and area measurement.
- II0₅ The eleven-year old subjects who have attained (failed to attain) an understanding of either length or area measurement have attained (failed to attain) an understanding of both length and area measurement.

The same statistical procedure including the seven step testing procedure that was used on Research Hypothesis I was used on Research Hypothesis II.

Collection of Data

The following investigative procedures were used. Nine tasks were administered to each subject involving (1) vocabulary of measurement terms and (2) four properties common to both length and area measurement: (a) congruence (b) conservation (c) additivity and (d) unit measure. The tasks are discussed in Chapter III and detailed descriptions are contained in Appendix A. An interview recording sheet was used during the interview (see Appendix B). The criteria used in the evaluation process is defined with each task discussion in Chapter III and the recording of the evaluations for each subject is contained in Appendix C. In addition to evaluations on nine tasks, Appendix C also contains the following data:

1. Subject identification.¹
2. Age.
3. Grade.
4. Reading level.
5. Arithmetic concept level.
6. Conservation stage (attainment, transitional, none).
7. Measurement stage (II A, II B, III A, III B).

¹Subject is identified by three letters and his age.

Summary

In this chapter the pilot study was described and demographic data given to define the sample. General procedure of the study was followed by detailed descriptions of each task including: (1) a definition of the measurement property being tested, (2) a description of the materials and procedure, (3) criteria used for evaluation, and (4) sample interviews illustrating various levels of achievement.

The research design involved a statement of research hypotheses which were transformed into statistical hypotheses that will be tested using Pearson's Chi-square test for independence. The Phi-coefficient will be used as an indication of the strength of association between the length and area measurement properties studied.

CHAPTER IV

ANALYSIS OF DATA AND FINDINGS

Introduction

The statistical findings regarding the measurement properties of congruence, conservation, additivity, and unit measure and the final attainment of length and area measurement are reported in this chapter. The first section concerns itself with sorting and classifying the data. Each of the twenty-five operational hypotheses are restated followed by: (1) supporting data in tabular form (fourfold contingency table), (2) values of the Chi-square and Phi-coefficient tests statistics, (3) the alpha level relative to the Chi-square statistic, (4) a statement of rejection or acceptance of the null hypothesis, and (5) a restatement of the Phi-coefficient to indicate the strength of association between the compared variables. The summary of this chapter concerns a table which summarizes the tested hypotheses, the significance levels, and statements of rejection or acceptance.

Evaluation of the Subjects

All of the subjects in the sample were interviewed by a single investigator. The results for each subject were kept on a recording sheet (see Appendix B), and all of the interviews were tape recorded. The tapes were used where needed to clarify and substantiate

comments written on the recording sheets during the interviews. Following the criteria for evaluation (noted with the discussion of each task in Chapter III) each subject was scored for each task using the information on the recording sheet. This was done by the investigator who conducted the interviews.

The interviews were numbered consecutively corresponding to the order in which they were conducted (grade one through grade five). Every fifth interview was selected and scored directly from the recording sheets by a second investigator who was familiar with the study. There was 93 per cent agreement between the scorings of the recording sheets by the two investigators.¹

Results for Criterion for Inclusion in Sample

The final sample for the study was composed of twenty subjects in each of the five age groups who passed the criterion for inclusion in the sample, the vocabulary task. In order to have this requisite number of subjects, the vocabulary task was administered to 101 children. One child in the ten-year old group had difficulty using terms regarding size ("more", "less", "larger", etc.) and thus failed this task.² The remaining 100 children all passed the

¹There was 100 per cent agreement for the vocabulary and the congruence of length and area tasks, 95 per cent for the conservation of length and area tasks and the conservation stages relative to length and area, 90 per cent for the additivity of length and area tasks and the unit of measure tasks relative to length and area, and 85 per cent for the measurement stages relative to length and area.

²It was learned that this child should have been placed in the Special Education class.

vocabulary task. These results are consistent with the finding of Beilin referred to in Chapter III that post-kindergarten children reached near perfect levels of performance on a similar task.¹

Research Hypothesis I

I. The cognitive development of length measurement is simultaneous to the cognitive development of area measurement relative to the properties of congruence, conservation, additivity, and unit measure.

The following twenty operational hypotheses were developed to test specifically this research hypothesis by converting them to statistical alternative hypotheses (i.e. $\phi_{xy} > 0$). A null hypothesis ($\phi_{xy} \leq 0$) was formed from each statistical alternative hypothesis and was tested using the Chi-square and Phi-coefficient test statistics. Tabular data and a decision concerning the rejection or acceptance of the null hypotheses accompany each operational hypothesis.

The operational hypotheses are grouped according to the measurement properties in the order of congruence, conservation, additivity, and unit measure. Each measurement property is considered relative to the factor of age (seven through eleven years). Then an operational hypothesis (HT_i) is formed concerning each measurement property and all subjects in the sample regardless of

¹Beilin, "Perceptual-Cognitive Conflict in the Development of an Invariant Area Concept," p. 217.

age. The seven step statistical procedure outlined earlier is used in determining the acceptance or rejection of this summarizing hypothesis across all ages. All results are summarized in Table 4-31.

Congruence Task

$I0_1$ The seven-year old subject's score on the Congruence of Length task will correlate positively with the subject's score on the Congruence of Area Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between congruence of length and congruence of area is $\phi = .63$

TABLE 4-1. Congruence task for the seven-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	18	0	$\chi^2 = 9.47$	$\alpha = .05$
	Fail	1	1		
				$\phi = .63$	

The decisions of acceptance or rejection of the statistical alternative hypotheses for the operational hypotheses I_{02} , I_{03} , I_{04} , and I_{05} are not reported relative to an χ^2 value. Due to the fact that expected cell frequencies (fe_{jk}) of zero occur the Chi-square test for independence is not applicable (zero denominators are present.):

$$\chi^2 = \sum_j \sum_k \frac{(fo_{jk} - fe_{jk})^2}{fe_{jk}} .$$

The Phi-coefficient is not applicable to this type of situation (expected cell frequencies of zero) since it can be expressed in terms of χ^2 :

$$\phi = \sqrt{\frac{\chi^2}{N}} .$$

There are two solutions to this problem: (1) make no decision regarding the acceptance of any of the operational hypotheses I_{02} , I_{03} , I_{04} , or I_{05} ; or (2) if the observed cell frequencies indicate an extreme direction regarding either a pass-pass, fail-fail situation or a pass-fail, fail-pass situation make a decision of acceptance or rejection, respectively, relative to the operational hypothesis based upon inspection of the fourfold contingency table. Based upon the observed cell frequencies of the contingency tables for the operational hypotheses I_{02} , I_{03} , I_{04} , and I_{05} , the investigator chose the latter of the two solutions.

- IO_2 The eight-year old subject's score on the Congruence of Length Task will correlate positively with the subject's score on the Congruence of Area Task. The cell frequencies in the following contingency table indicate that 90 per cent of the eight-year old subjects passed both the Congruence of Length and the Congruence of Area tasks. Based on the above statistic this operational hypothesis is accepted.

TABLE 4-2. Congruence task for the eight-year old group.

		<u>Area</u>	
		<u>Pass</u>	<u>Fail</u>
Length	Pass	18	2
	Fail	0	0

- IO_3 The nine-year old subject's score on the Congruence of Length Task will correlate positively with the subject's score on the Congruence of Area Task. The cell frequencies in the following contingency table indicate that 100 per cent of the nine-year old subjects passed both the Congruence of Length and the Congruence of Area Tasks. Based on the above statistic this operational hypothesis is accepted.

TABLE 4-3. Congruence task for the nine-year old group.

		<u>Area</u>	
		<u>Pass</u>	<u>Fail</u>
Length	Pass	20	0
	Fail	0	0

IO₄ The ten-year old subject's score on the Congruence of Length Task will correlate positively with the subjects score on the Congruence of Area Task. The cell frequencies in the following contingency table indicate that 100 per cent of the ten-year old subjects passed both the Congruence of Length and the Congruence of Area Tasks. Based on the above statistic this operational hypothesis is accepted.

TABLE 4-4. Congruence task for the ten-year old group.

		<u>Area</u>	
		<u>Pass</u>	<u>Fail</u>
Length	Pass	20	0
	Fail	0	0

IO₅ The eleven-year old subject's score on the Congruence of Length Task will correlate positively with the subject's score on the Congruence of Area Task. The cell frequencies in the following contingency table indicate that 100 per cent of the eleven-year old subjects

passed both the Congruence of Length and the Congruence of Area Tasks. Based on the above statistic this operational hypothesis is accepted.

TABLE 4-5. Congruence task for the eleven-year old group.

		<u>Area</u>	
		<u>Pass</u>	<u>Fail</u>
Length	Pass	20	0
	Fail	0	0

In summarizing the property of congruence relative to length and area measurement, all of the subjects in the sample are considered regardless of age. The following operational hypothesis is considered for acceptance.

HT₁ The subject's score on the Congruence of Length Task will correlate positively with the subject's score on the Congruence of Area Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between congruence of length and congruence of area is $\phi = .88$.

TABLE 4-6. Congruence task for all subjects in the sample.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	96	2	$\chi^2 = 15.49$	$\alpha = .001$
	Fail	1	1		
				$\phi = .88$	

Conservation Task

IO₆ The seven-year old subject's score on the Conservation of Length Task will correlate positively the subject's score on the Conservation of Area Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between conservation of length and conservation of area is $\phi = .88$.

TABLE 4-7. Conservation task for the seven-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	5	0	$\chi^2 = 15.56$	$\alpha = .001$
	Fail	1	14	$\phi = .88$	

IO₇ The eight-year old subject's score on the Conservation of Length Task will correlate positively with the subject's score on the Conservation of Area Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between conservation of length and conservation of area is $\phi = .50$.

TABLE 4-8. Conservation task for the eight-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	7	2	$\chi^2 = 5.05$	$\alpha = .025$
	Fail	3	8	$\phi = .50$	

IO₈ The nine-year old subject's score on the Conservation of Length Task will correlate positively with the subject's score on the Conservation of Area Task. Since the null hypothesis submitted to test was not rejected, the statistical alternative could not be accepted. Hence this operational hypothesis cannot be accepted. The correlation between conservation of length and conservation of area is $\phi = .39$.

TABLE 4-9. Conservation task for the nine-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	6	1	$\chi^2 = 2.97$	$\alpha = .05$
	Fail	6	7	$\phi = .39$	

IO₉ The ten-year old subject's score on the Conservation of Length Task will correlate positively with the subject's score on the Conservation of Area Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational

hypothesis is accepted. The correlation between conservation of length and conservation of area is $\phi = .44$.

TABLE 4-10. Conservation task for the ten-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	17	1	$\chi^2 = 3.95$	$\alpha = .05$
	Fail	1	1	$\phi = .44$	

IO_{10} The eleven-year old subject's score on the Conservation of Length Task will correlate positively with the subject's score on the Conservation of Area Task. Since the null hypothesis submitted to test was not rejected, the statistical alternative could not be accepted. Hence this operational hypothesis cannot be accepted. The correlation between conservation of length and conservation of area is $\phi = .19$.

TABLE 4-11. Conservation task for the eleven-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	14	4	$\chi^2 = .74$	$\alpha = .05$
	Fail	1	1	$\phi = .19$	

In summarizing the property of conservation relative to length and area measurement, all of the subjects in the sample are

considered regardless of age. The following operational hypothesis is considered for acceptance.

HT₂ The subject's score on the Conservation of Length Task will correlate positively with the subject's score on the Conservation of Area Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between conservation of length and conservation of area is $\phi = .59$.

TABLE 4-12. Conservation task for all subjects in the sample.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	49	8	$\chi^2 = 34.73$	$\alpha = .001$
	Fail	12	31	$\phi = .59$	

Additivity Task

IO₁₁ The seven-year old subject's score on the Additivity of Length Task will correlate positively with the subject's score on the Additivity of Area Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between additivity of length and additivity of area is $\phi = .73$.

TABLE 4-13. Additivity task for the seven-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	3	2	$\chi^2 = 10.5881$	$\alpha = .005$
	Fail	0	15	$\phi = .73$	

¹⁰₁₂ The eight-year old subject's score on the Additivity of Length Task will correlate positively with the subject's score on the Additivity of Area Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between additivity of length and additivity of area is $\phi = .80$.

TABLE 4-14. Additivity task for the eight-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	8	1	$\chi^2 = 12.74$	$\alpha = .001$
	Fail	1	10	$\phi = .80$	

¹⁰₁₃ The nine-year old subject's score on the Additivity of Length Task will correlate positively with the subject's score on the Additivity of Area Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational

hypothesis is accepted. The correlation between additivity of length and additivity of area is $\phi = .65$.

TABLE 4-15. Additivity task for the nine-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	6	4	$\chi^2 = 8.57$	$\alpha = .005$
	Fail	0	10	$\phi = .65$	

IO_{14} The ten-year old subject's score on the Additivity of Length Task will correlate positively with the subject's score on the Additivity of Area Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between additivity of length and additivity of area is $\phi .49$.

TABLE 4-16. Additivity task for the ten-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	15	1	$\chi^2 = 4.80$	$\alpha = .05$
	Fail	2	2	$\phi = .49$	

IO_{15} The eleven-year old subject's score on the Additivity of Length Task will correlate positively with the subject's score on the Additivity of Area Task. Since the

null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between additivity of length and additivity of area is $\phi = .79$.

TABLE 4-17. Additivity task for the eleven-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	17	1	$\chi^2 = 12.59$	$\alpha = .001$
	Fail	0	2	$\phi = .79$	

In summarizing the property of additivity relative to length and area measurement, all of the subjects in the sample are considered regardless of age. The following operational hypothesis is considered for acceptance.

HT₃ The subject's score on the Additivity of Length Task will correlate positively with the subject's score on the Additivity of Area Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between additivity of length and additivity of area is $\phi = .76$.

TABLE 4-18. Additivity task for all subjects in the sample.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	49	9	$\chi^2 = 58.38$	$\alpha = .001$
	Fail	3	39	$\phi = .76$	

Unit of Measure Task

IO₁₆ The seven-year old subject's score on the Unit of Length Measure Task will correlate positively with the subject's score on the Unit of Area Measure Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between the unit of length measure and the unit of area measure is $\phi = .46$.

TABLE 4-19. Unit of measure task for the seven-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	1	3	$\chi^2 = 4.21$	$\alpha = .05$
	Fail	0	16	$\phi = .46$	

IO₁₇ The eight-year old subject's score on the Unit of Length Measure Task will correlate positively with the subject's score on the Unit of Area Measure Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this

operational hypothesis is accepted. The correlation between the unit of length measure and the unit of area measure is $\phi = .88$.

TABLE 4-20. Unit of measure task for the eight-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	5	0	$\chi^2 = 15.55$	$\alpha = .001$
	Fail	1	14	$\phi = .88$	

IO₁₈ The nine-year old subject's score on the Unit of Length Measure Task will correlate positively with the subject's score on the Unit of Area Measure Task. Since the null hypothesis submitted to test was not rejected, the statistical alternative could not be accepted. Hence this operational hypothesis cannot be accepted. The correlation between the unit of length measure and the unit of area measure is $\phi = .38$.

TABLE 4-21. Unit of measure task for the nine-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	2	2	$\chi^2 = 2.81$	$\alpha = .05$
	Fail	2	14	$\phi = .38$	

IO₁₉ The ten-year old subject's score on the Unit of Length Measure Task will correlate positively with the subject's score on the Unit of Area Measure Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between the unit of length measure and the unit of area measure is $\phi = .56$.

TABLE 4-22. Unit of measure task for the ten-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	11	2	$\chi^2 = 6.28$	$\alpha = .025$
	Fail	2	5		
				$\phi = .56$	

IO₂₀ The eleven-year old subject's score on the Unit of Length Measure task will correlate positively with the subject's score on the Unit of Area Measure Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between the unit of length measure and the unit of area measure is $\phi = .49$.

TABLE 4-23. Unit of measure task for the eleven-year old group.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	9	3	$\chi^2 = 4.85$	$\alpha = .05$
	Fail	2	6		
				$\phi = .49$	

In summarizing the property of a unit of measure relative to length and area measurement, all of the subjects in the sample are considered regardless of age. The following operational hypothesis is considered for acceptance.

HT₄ The subject's score on the Unit of Length Measure Task will correlate positively with the subject's score on the Unit of Area Measure Task. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between the unit of length measure and the unit of area measure is $\phi = .64$.

TABLE 4-24. Unit of measure task for all of the subjects.

		<u>Area</u>			
		<u>Pass</u>	<u>Fail</u>		
Length	Pass	28	10	$\chi^2 = 40.59$	$\alpha = .001$
	Fail	7	55		
				$\phi = .64$	

Research Hypothesis II

II. The understanding of length and area measurement are attained simultaneously.

The following five operational hypotheses were developed to test specifically this research hypothesis by converting them to statistical alternative hypotheses (i.e. $\phi_{xy} > 0$). A null hypothesis ($\phi_{xy} \leq 0$) was formed from each statistical alternative hypothesis and was tested using the Chi-square and Phi-coefficient test statistics. Tabular data and a decision concerning the rejection or acceptance of the null hypotheses accompany each operational hypothesis.

The operational hypotheses are grouped according to the age of the subjects. Each of the hypotheses concerns the final attainment of both length and area measurement, i.e. is the subject at substage III B or is he at a substage lower than III B. An operational hypothesis is formed concerning the final attainment of length and area measurement and all of the subjects in the sample regardless of age. The seven step statistical procedure outlined in Chapter III is used in determining the acceptance or rejection of this summarizing hypothesis across all ages. All results are summarized in Table 4-32.

II0₁ The seven-year old subjects who have attained (failed to attain) an understanding of either length or area measurement have attained (failed to attain) an understanding of both length and area measurement. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this

operational hypothesis is accepted. The correlation between the final attainment of length measurement and the final attainment of area measurement is $\phi = .46$.

TABLE 4-25. Final attainment of length and area measurement (sub-stage III B) for the seven-year old group.

		<u>Area</u>			
		<u>III B</u>	<u>< III B</u>		
Length	III B	1	3	$\chi^2 = 4.21$	$\alpha = .05$
	< III B	0	16		
				$\phi = .46$	

II0₂ The eight-year old subjects who have attained (failed to attain) an understanding of either length or area measurement have attained (failed to attain) an understanding of both length and area measurement. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between the final attainment of length measurement and the final attainment of area measurement is $\phi = .88$.

TABLE 4-26. Final attainment of length and area measurement (sub-stage III B) for the eight-year old group.

		<u>Area</u>			
		<u>III B</u>	<u>< III B</u>		
Length	III B	5	0	$\chi^2 = 15.56$	$\alpha = .001$
	< III B	1	14		
				$\phi = .88$	

II0₃ The nine-year old subjects who have attained (failed to attain) an understanding of either length or area measurement have attained (failed to attain) an understanding of both length and area measurement. Since the null hypothesis submitted to test was not rejected, the statistical alternative could not be accepted. Hence this operational hypothesis cannot be accepted. The correlation between the final attainment of length measurement and the final attainment of area measurement is $\phi = .14$.

TABLE 4-27. Final attainment of length and area measurement (sub-stage III B) for the nine-year old group.

		<u>Area</u>			
		<u>III B</u>	<u>< III B</u>		
Length	III B	1	2	$\chi^2 = .39$	$\alpha = .05$
	< III B	3	14		
				$\phi = .14$	

II0₄ The ten-year old subjects who have attained (failed to attain) an understanding of either length or area measurement have attained (failed to attain) an understanding of both length and area measurement. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between the

final attainment of length measurement and the final attainment of area measurement is $\phi = .56$.

TABLE 4-28. Final attainment of length and area measurement (sub-stage III B) for the ten-year old group.

		<u>Area</u>		
		<u>III B</u>	<u>III B</u>	
Length	III B	11	2	$\chi^2 = 6.28 \quad \alpha = .025$
	< III B	2	5	$\phi = .56$

II0₅ The eleven-year old subjects who have attained (failed to attain) an understanding of either length or area measurement have attained (failed to attain) an understanding of both length and area measurement. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between the final attainment of length measurement and the final attainment of area measurement is $\phi = .45$.

TABLE 4-29. Final attainment of length and area measurement (sub-stage III B) for the eleven-year old group.

		<u>Area</u>		
		<u>III B</u>	<u>III B</u>	
Length	III B	1	2	$\chi^2 = 4.01 \quad \alpha = .05$
	< III B	3	14	$\phi = .45$

In summarizing the consideration of the final attainment of length and area measurement, all of the subjects in the sample are considered regardless of age. The following operational hypothesis is considered for acceptance.

HT₅ The subjects who have attained(failed to attain) an understanding of either length or area measurement have attained (failed to attain) an understanding of both length and area measurement. Since the null hypothesis submitted to test was rejected, the statistical alternative was accepted. Hence this operational hypothesis is accepted. The correlation between the final attainment of length measurement and the final attainment of area measurement is $\phi = .61$.

TABLE 4-30. Final attainment of length and area measurement (sub-stage III B) for all of the subjects.

		<u>Area</u>		
		<u>III B</u>	<u>< III B</u>	
Length	III B	17	10	$\chi^2 = 37.22 \quad \alpha = .001$
	< III B	8	55	$\phi = .61$

Summary

In this chapter, the statistical results concerning the measurement properties of congruence, conservation, additivity and unit measure have been stated.

Concerning Research Hypothesis I:

I. The cognitive development of length measurement is simultaneous to the cognitive development of area measurement relative to the properties of congruence, conservation, additivity, and unit measure.

Twenty operational hypotheses relating each age level to each of the measurement properties were statistically tested. In addition four hypotheses were formed relating each of the four measurement properties to all the subjects in the sample. These summarizing hypotheses were also submitted to statistical test. Statements of acceptance or rejection were made in addition to noting the correlation coefficient (ϕ) for each operational hypothesis as a means of investigating Research Hypothesis I. These results are summarized in Table 4-31.

Concerning Research Hypothesis II:

II. The understanding of length and area measurement are attained simultaneously.

Five operational hypotheses relating each age level to the final attainment of length and area measurement were statistically tested. In addition, a hypothesis was formed relating the final attainment of length and area measurement to all the subjects in the sample. This summarizing hypothesis was also submitted to statistical test. Statements of acceptance or rejection were made in addition to noting the correlation coefficient (ϕ) for each operational hypothesis as a means of investigating Research Hypothesis II. These results are summarized in Table 4-32.

Table 4-31 and Table 4-32 are the summarizations of all the operational hypotheses tested including: (1) the significance level " α " for the Chi-square test for independence, (2) a statement of rejection or acceptance, and (3) the value of the Phi-coefficient¹ indicating the correlation between the variables tested.

TABLE 4-31. Summary of the operational hypotheses tested for Research Hypothesis I.

<u>Measurement Property</u>	<u>Hypothesis</u>	<u>Alpha level</u>	<u>Accept</u>	<u>Reject</u>	<u>Correlation Coefficient</u>
Congruence	IO ₁	.05	X		.63
	IO ₂	-	X		-
	IO ₃	-	X		-
	IO ₄	-	X		-
	IO ₅	-	X		-
	HT ₁	.001	X		.88
Conservation	IO ₆	.001	X		.38
	IO ₇	.025	X		.50
	IO ₈	.05		X	.39
	IO ₉	.05	X		.44
	IO ₁₀	.05		X	.19
	HT ₂	.001	X		.59

¹The variables tested in Research Hypothesis I are the attainment of each of the four measurement properties relative to length measurement and the attainment of each of the four measurement properties relative to area measurement. The variables tested in Research Hypothesis II are the final attainment of length measurement (substage III B) and the final attainment of area measurement (substage III B) relative to the age levels studied.

<u>Measurement Property</u>	<u>Hypothesis</u>	<u>Alpha level</u>	<u>Accept</u>	<u>Reject</u>	<u>Correlation Coefficient</u>
Additivity	IO ₁₁	.005	X		.73
	IO ₁₂	.001	X		.80
	IO ₁₃	.005	X		.65
	IO ₁₄	.05	X		.49
	IO ₁₅	.001	X		.79
	HT ₃	.001	X		.76
Unit Measure	IO ₁₆	.05	X		.46
	IO ₁₇	.001	X		.88
	IO ₁₈	.05		X	.38
	IO ₁₉	.025	X		.56
	IO ₂₀	.05	X		.49
	HT ₄	.001	X		.64

Note that twenty-one of the twenty-four operational hypotheses tested concerning Research Hypothesis I are accepted.

TABLE 4-32. Summary of the operational hypotheses tested for Research Hypothesis II.

<u>Age Group</u>	<u>Hypothesis</u>	<u>Alpha level</u>	<u>Accept</u>	<u>Reject</u>	<u>Correlation Coefficient</u>
7	II0 ₁	.05	X		.48
8	II0 ₂	.001	X		.88
9	II0 ₃	.05		X	.14
10	II0 ₄	.025	X		.56
11	II0 ₅	.05	X		.45
All	HT ₅	.001	X		.61

Note that five of the six operational hypotheses tested concerning Research Hypothesis II are accepted.

CHAPTER V

SUMMARY OF MAJOR FINDINGS, CONCLUSIONS, AND IMPLICATIONS

Introduction

It was the stated purpose of this study to investigate the cognitive development of length and area measurement relative to four common component properties (congruence, conservation, additivity, and unit measure) and the factor of chronological age. The intent was to conduct a comparative investigation to lend support to one of the two contrasting points of view identified to be:

1. There is no difference between the ages at which a child attains corresponding levels of understanding relative to length and area measurement and that both of these concepts are finally attained at approximately the same age.
2. There is a difference between the ages at which a child attains corresponding levels of understanding relative to length and area measurement and that a child finally attains length measurement prior to area measurement.

The need for this study was based on the conflicting conclusions made from two separate investigations: (1) Piaget, Inhelder, and Szeminska have completed extensive research on the cognitive

development of length and area measurement.¹ Based upon the results of his work, Piaget stated that "The development of . . . measurement runs exactly parallel whether the objects are lengths or whether they are areas and the level at which they are finally grasped is the same for both."² (2) Beilin and Franklin conducted an investigation of length and area measurement on a comparative basis.³ One of the stated purposes of this study was to investigate whether the abilities to solve related problems of length and area measurement are acquired simultaneously. The intent was to test the validity of Piaget's conclusion regarding the simultaneous development of and final attainment of length and area measurement. Based on the results of Beilin and Franklin's study, they concluded that "We would suggest that length and area . . . measurement are achieved in that order. Also the constituent operations to measurement . . . are applied more easily first to a single dimension, then to two dimensions, . . ."⁴

Piaget's research has prompted Copeland to criticize the present manner in which length and area measurement are being taught in American elementary schools. Copeland stated that "Measurement in one dimension is taught before the child is at the operational or readiness level to understand it, and yet two-dimensional or area

¹Piaget, Inhelder, and Szeminska, Conception of Geometry.

²Ibid., p. 300.

³Beilin and Franklin, "Logical Operations in Area and Length Measurement," pp. 607-618.

⁴Ibid., p. 617.

measurement is deferred several years past the age at which children can understand it."¹ If Piaget's research is to be an influence in the development of the mathematics curriculum of the elementary schools, verification of his results is necessary.

Most of the related studies focused on only one of these two measurement concepts, either length or area, and had not investigated them on a comparative basis. In this study, four common component properties of length and area measurement have been identified using measurement axioms contained in modern geometry textbooks. These measurement properties were studied as a means of investigating the cognitive development of length and area measurement on a comparative basis. The study has collected and analyzed data regarding the four measurement properties in research of statistical evidence to test hypotheses central to the cognitive development of length and area measurement.

Within the limitations of this study the following major findings and implications are presented.

Conclusions Concerning the Four Measurement Properties

The investigation of four common component properties of length and area measurement (i.e. congruence, conservation, additivity, and unit measure) has provided a more refined procedure for investigating the cognitive development of length and area measurement than has been previously available. The findings show that it is possible

¹Copeland, How Children Learn Mathematics, p. 238.

to identify stages of cognitive development (as suggested by Piaget) relative to these two measurement concepts by the child's operational understanding of the four component properties common to both length and area measurement, although, in some instances, the number of subjects at certain stages were not what would be expected from Piaget's results. (e.g. The nine-year old group performed at a lower level than one would have expected; they are much nearer to the eight-year olds than the ten-year olds in ability to successfully complete the tasks concerning the conservation, additivity, and unit measure properties.)

Congruence Property

The six operational hypotheses concerning a positive correlation between the subject's score on the Congruence of Length Task and his score on the Congruence of Area Task within and across all ages were accepted. Thus, these results imply that a child understands what is meant by the property of congruence regardless whether the objects are lengths or areas. This conclusion needs to be verified with younger children (e.g. four to seven years) due to the high rate of success of the subjects in this study on the congruence tasks (92 per cent and 97 per cent of the sample, respectively, passed the length and area measurement tasks concerning the congruence property).

Conservation Property

Three of the five operational hypotheses concerning a positive correlation between the subject's score on the Conservation of Length

Task and his score on the Conservation of Area Task within each age group were accepted. The two hypotheses that were not accepted concerned the nine- and eleven-year old subjects, although 65 per cent and 75 per cent of the subjects in the nine- and eleven-year old groups, respectively, were rated the same on both conservation tasks (i.e. relative to length and area measurement). The operational hypothesis concerning a positive correlation between the subject's score on the Conservation of Length Task and his score on the Conservation of Area Task across all ages was accepted. Thus, there is evidence to support the claim that a child understands what is meant by the property of conservation regardless whether the objects are lengths or areas.

Additivity Property

The six operational hypotheses concerning a positive correlation between the subject's score on the Additivity of Length Task and his score on the Additivity of Area Task within and across all ages were accepted. Thus, these results imply that a child understands what is meant by the additivity property regardless whether the objects are lengths or areas.

Unit Measure Property

Four of the five operational hypotheses concerning a positive correlation between the subject's score on the Unit of Length Task and his score on the Unit of Area Task within each age group were accepted. The hypothesis regarding the nine-year old group was not accepted, although 80 per cent of the nine-year olds were scored the

same on the length and area measurement tasks concerning the unit measure property. The operational hypothesis concerning a positive correlation between the subject's score on the Unit of Length Task and his score on the Unit of Area Task across all ages was accepted. Thus, there is evidence to support the claim that a child is cognizant of the size and number of the units used in the measuring process regardless whether the objects are lengths or areas.

The following table indicates the percentage of subjects who have passed each of the measurement property tasks relative to length and area:

	<u>Congruence</u>	<u>Conservation</u>	<u>Additivity</u>	<u>Unit Measure</u>
Length	92	56	58	38
Area	97	61	52	35

Consistent with the findings of Piaget, the findings of this study indicate the measurement properties in order of difficulty are:

(1) congruence, (2) conservation, (3) additivity, and (4) unit measure.

Conclusions Concerning the Research Hypotheses

Two research hypotheses were established in this investigation. The first research hypothesis states, in essence, that there is a simultaneous cognitive development of length and area measurement. Seventeen of the twenty related operational hypotheses within age groups and all four related hypotheses across all ages were accepted in support of this research hypothesis. Based upon the analysis of the data collected in this study, Research Hypothesis I is supported.

Piaget, Inhelder, and Szeminska also indicate the existence of a similar cognitive development of length and area measurement: "The development of . . . measurement runs exactly parallel whether the objects are lengths or whether they are areas . . ." ¹

The second research hypothesis states that length and area measurements are finally understood at approximately the same age. Four of the five operational hypotheses concerning a simultaneous final attainment of length and area measurement within each age group were accepted. The hypothesis concerning the nine-year old group was not accepted, although 75 per cent of the nine-year olds were rated the same (substage III B for final attainment, < III B for lacking final attainment) regarding the final attainment of length and area measurement. The hypothesis concerning a simultaneous final attainment of length and area measurement across all ages was accepted. Based upon the analysis of the data collected in this study, Research Hypothesis II is supported.

Piaget, Inhelder, and Szeminska have made a similar conclusion indicated by their statement: "The level at which they [length and area measurement] are finally grasped is the same for both". ²

Implications

Certain implications, over and beyond the study, warrant mentioning.

¹Piaget, Inhelder, and Szeminska, Conception of Geometry, p. 300.

²Ibid.

1. The elementary school teacher would gain insight regarding the thought processes of a child through replicating some of the tasks presented in this study. The stages of mental growth that would be observed should be taken into account when planning learning experiences.
2. An effective individual interview technique would be a beneficial pedagogical skill for the elementary school teacher to acquire. Incorrect responses from the child could be pursued and misconceptions eliminated effectively through the use of this technique.
3. Before introducing a new concept such as area measurement, the child should be tested with Piagetian type tasks to be sure that he has all the prerequisites for mastering the concept. If he is not yet ready for the concept, the child should be provided with experiences that will help him become ready. The English Nuffield Project is using this procedure to chart the cognitive growth of a child concerning mathematical concepts.
4. Concrete materials should be used wherever possible to provide children with certain experiences which will prepare them to learn a particular mathematical concept. The Mathematics Laboratory makes extensive use of physical objects that can be manipulated by the children.
5. The component properties of measurement (i.e. congruence, conservation, additivity, etc.) should be introduced using both lengths and areas simultaneously. As an

example, when the child is near the stage of understanding conservation (the age will differ with different children but may be found through the use of Piagetian type tasks) the physical objects used in the instructional process should be both lengths and areas.

Implications for Future Research

1. If Beilin and Franklin's conclusion regarding one dimensional concepts being learned earlier than two dimensional concepts because of the complexity caused by the additional dimension is extended, one may assume that concepts of length, area, and volume measurements are developed cognitively in that order; this assumption is contrary to the results of Piaget. Thus, investigation is needed to determine the optimum order of placement regarding length, area, and volume measurements. Further investigation could involve weights, liquid volume, three dimensional surface area, etc.
2. All investigations, including this one, need varification. It is suggested that any one of the four measurement properties (congruence, conservation, additivity, and use of a unit of measure) be used to investigate the cognitive capability relative to measurements of length, area, volume, weights, etc. Tasks employing the use of objects of varying sizes and shapes should be developed to thoroughly test each measurement property.

BIBLIOGRAPHY

BIBLIOGRAPHY

- Abbas, LaVonne. "The Grade Placement of Linear and Area Measurement." Unpublished Master's Thesis, Macalester College, 1962.
- Adler, Irving. Mathematics and Mental Growth. New York: The John Day Company, 1968.
- Almy, Millie, Chittenden, Edward, and Miller, Paula. Young Children's Thinking. New York: Teachers College Press, 1966.
- Ballard, William R. Geometry. Philadelphia: W. B. Saunders Company, 1970.
- Banks, Houston J. "Concepts of Measurement." Enrichment Mathematics for the Grades. Twenty-Seventh Yearbook of the National Council of Teachers of Mathematics. Washington D.C.: The National Council of Teachers of Mathematics, Inc., 1963. pp. 108-126.
- Beard, Ruth M. An Outline of Piaget's Developmental Psychology For Students and Teachers. New York: Basic Books, Inc., Publishers, 1969.
- Beck, Anatole, Bleicher, Michael, and Crowe, Donald. Excursions Into Mathematics. New York: Worth Publishers, Inc., 1969.
- Beilin, Harry. "Perceptual-Cognitive Conflict in the Development of an Invariant Area Concept." Journal of Experimental Child Psychology, I (1964), pp. 208-226.
- Beilin, Harry and Franklin, Irene C. "Logical Operations in Area and Length Measurement: Age and Training Effects." Child Development, 33, 1962. pp. 607-618.
- Bentler, Peter M. "Monotonicity Analysis: An Alternative to Linear Factor and Test Analysis." Measurement and Piaget. Edited by Donald Green, Marquerite Ford, and George B. Flamer. New York: McGraw-Hill Book Company, 1971. pp. 220-244.
- Borg, Walter. Educational Research An Introduction. New York: David McKay Company, Inc., 1963.

- Braine, Martin D. "The Ontogeny of Certain Logical Operations - Piaget Formulation Examined by Nonverbal Methods." Psychological Monographs: General and Applied. 73, 1959. p. 43.
- Brune, Irvin H. "Geometry in the Grades." Enrichment Mathematics for the Grades. Twenty-Seventh Yearbook of the National Council of Teachers of Mathematics. Washington D.C.: The National Council of Teachers of Mathematics, Inc., 1963. pp. 143-147.
- Bumby, Douglas R. "An Analysis of Selected Geometric Concepts in the New Mathematical Programs for the Elementary School and Their Implications for Teacher Education." Unpublished Master's Thesis, Brooklyn College, 1968.
- Cambridge Conference on School Mathematics. Goals for School Mathematics. Boston: Houghton Mifflin Company, 1963.
- Cambridge Conference on Teacher Training. Goals For Mathematics Education of Elementary School Teachers. Boston: Houghton Mifflin Company, 1967.
- Carroll, John B. "Words, Meanings and Concepts." Current Research on Instruction. Edited by Richard Anderson, Gerald Faust, Marianne Roderick, Donald Cunningham, and Thomas Andre. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1969. pp. 221-241.
- Copeland, Richard. How Children Learn Mathematics. New York: The Macmillan Company, 1970.
- D'Augustine, Charles. Factors Relating To Achievement With Selected Topics In Geometry and Topology. When Taught to Fifth, Sixth, and Seventh Grade Pupils Via a Programmed Test. Unpublished Doctoral Dissertation, Florida State University, 1963.
- Davis, Robert B. "Mathematics Teaching With Special Reference to Epistemological Problems." Journal of Research and Development in Education, 1 (Fall, 1967), p. 46.
- Davis, Robert B. The Changing Curriculum: Mathematics. Washington, D.C.: Association for Supervision and Curriculum Development, National Education Association, 1967.
- Dodwell, P.C. "Children's Understanding of Number Concepts: Characteristics of an Individual and of a Group Test." Canadian Journal of Psychology, 15 (1961), pp. 29-36.

- Elkind, David. "The Continuing Influence of Jean Piaget." Grade Teacher, 88 May/June, 1971, p. 7.
- Engleman, Sidfried E. "Does the Piagetian Approach Imply Instruction." Measurement and Piaget. Edited by Donald Green, Marquerite Ford, and George Flamer. New York: McGraw-Hill Book Company, 1971.
- Flavell, John H. The Developmental Psychology of Jean Piaget. Princeton: D. Van Nostrand Co., Inc., 1963.
- Foley, Jack L. Metric Geometry, Concepts of Area Measure. Washington, D.C.: United States Office of Education ERIC. Department of Health, Education, and Welfare. August, 1967.
- Furth, Hans G. Piaget for Teachers. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1970.
- George, Linda Olsen, "Selected Factors Which Affect Young Children's Concepts of Conservation of Length." Dissertation Abstracts. February, 1971. p. 2735-A.
- Gibney, Thomas C. and Houle, William W. "Geometry Readiness in the Primary Grades." The Arithmetic Teacher. October, 1967, pp. 470-472.
- Glennon, Vincent J. and Gallahan, Leroy G. Elementary School Mathematics -- A Guide to Current Research. 3rd edition. Washington, D.C.: Association for Supervision and Curriculum Development, National Education Association, 1968.
- Halmos, Paul R. Measure Theory. Princeton, N.J.: D. Van Nostrand Co., Inc., 1950.
- Hays, William L. Statistics for Psychologists. New York: Holt, Rinehart, and Winston, 1963.
- Hooper, Frank H. "Piagetian Research and Education." Logical Thinking in Children-Research Based on Piaget's Theory. Edited by Irving Sigel and Frank H. Hooper, 1968. pp. 423-434.
- Hunt, J. Mcv. "The Impact and Limitations of the Giant of Developmental Psychology." Studies In Cognitive Development. Edited by David Elkind and John Flavell. New York: Oxford University Press, 1969.
- Jackson, Stanley. "Congruence and Measurement." The Arithmetic Teacher. February, 1967, pp. 94-102.
- Joyce, Bruce R. The Teacher-Innovator: A Program to Prepare Teachers. Washington: United States Government Printing Office, 1968, pp. 415-435.

- Kamps, Kenneth George. "An Investigation of Portions of a Model for Acquisition of Conservation and Measurement of Length Based on Performance of Selected Second Grade Children on Six Piaget-Type Tasks." Dissertation Abstracts. June, 1971. p. 4550-A.
- Lovell, Kenneth. "The Development of Scientific Concepts." Perceptual Development in Children. Edited by Aline H. Kidd and Jeanne L. Rivoire. New York: International Universities Press, Inc., 1966. pp. 407-444.
- Lovell, K., Healey, D., and Rowland, A.D. "Growth of Some Geometrical Concepts." Logical Thinking in Children - Research Based on Piaget's Theory. Edited by Irving Sigel and Frank H. Hooper, 1968. pp. 140-157.
- Lovell, K., Mitchell, B., Everett, I.R. "An Experimental Study of the Growth of Some Logical Structures." Logical Thinking in Children - Research Based on Piaget's Theory. Edited by Irving Sigel and Frank H. Hooper, 1968. pp. 224 - 239.
- Lovell, K. and Ogilvie, E. "A Study of the Conservation of Substance in the Junior School Child." British Journal of Psychology, 30, 1960. pp. 109-118.
- Mermelstein, Egon and Shulman, Lee A. "Lack of Formal Schooling and the Acquisition of Conservation." Child Development, 38 (1967), pp. 39-52.
- Moise, Edwin E. Elementary Geometry from an Advanced Standpoint. Reading, Mass.: Addison-Wesley, 1963.
- National Conference on Curriculum Experimentation. Modern Viewpoints in the Curriculum. Edited by Paul C. Rosenbloom, New York: McGraw-Hill Book Company, 1961.
- National Council of Teachers of Mathematics. Piagetian Cognitive - Development Research and Mathematics Education. Edited by Myron F. Roszkopf, Leslie P. Steffe, and Stanley Taback. Washington, D.C.: National Council of Teachers of Mathematics Inc., 1971.
- National Society for the Study of Education - The Sixty-Ninth Yearbook. Mathematics Education. Edited by Edward G. Begle. Chicago: The University of Chicago Press, 1970.
- Noll, Victor H. Introduction to Educational Measurement. Boston: Houghton Mifflin Company, 1965.
- Phillips, John L., Jr. The Origins of Intellect: Piaget's Theory. San Francisco, Calif.: W.H. Freeman and Company, 1969.

- Phillips, Jo McKeeby and Sqoyer, Russell E. Book 2: Congruence. New York: Harper and Row, 1969.
- Piaget, Jean. "Autobiography." A History of Psychology in Autobiography, Vol IV. Eds. Edwin G. Boring et. al. Worchester Mass.: Clark University Press, 1952, pp. 237-256.
- Piaget, Jean. "Cognitive Development in Children: Piaget." Journal of Research in Science Teaching. Volume 2, 1964. pp. 176-195.
- Piaget, Jean. The Child's Conception of the World. Patterson, N.J.: Littlefield, Adams, 1963.
- Piaget, Jean. "The Theory of Stages in Cognitive Development." Measurement and Piaget. Edited by Donald Green, Marqueritte Ford, and George Flamer. New York: McGraw-Hill Book Company, 1971. pp. 1-11.
- Piaget, Jean and Inhelder, Barbel. The Child's Conception of Space. Translated by F.J. Langdon and J.L. Lunzer. London: Routledge and Kegan Paul Ltd., 1956.
- Piaget, Jean, Inhelder, Barbel, and Szeminska, Alina. The Child's Conception of Geometry. 2nd ed. Translated by E.A. Lunzer. New York: Harper and Row, 1960.
- Ringenber, Lawrence A. College Geometry. New York: John Wiley and Sons, Inc., 1967.
- Rosenbloom, Paul C. "Implications of Piaget for Mathematics Curriculum." Improving Mathematics Education for Elementary School Teachers. Edited by Robert Houston. Conference sponsored by The Science and Mathematics Teaching Center, Michigan State University and The National Science Foundation. pp. 44-49.
- Rosskopf, Myron F., Sitomer, Harry, and Lenchner George. Modern Mathematics: Geometry. Morristown, N.J.: Silver Burdett Co., 1966.
- Royden, H.L. Real Analysis. New York: The Macmillian Company, 1963.
- School Mathematics Study Group. Geometry Part I. New Haven: Yale University Press, 1960.
- School Mathematics Study Group, Geometry with Coordinates, Part II. New Haven: Yale University Press, 1963.

- Shah, Sair Ali. "Selected Geometric Concepts Taught to Children Ages Seven to Eleven." Edited by C. Alan Riedesel and Len Pikaart. The Arithmetic Teacher. February, 1969, pp. 119-128.
- Smart, James R. and Marks, John L. "Mathematics of Measurement." The Arithmetic Teacher, April, 1966, pp. 283-87.
- Smedslund, Jan. Concrete Reasoning: A Study of Intellectual Development. ("Monograph of the Society for Research in Child Development," 29, No. 2, Serial No. 93), pp. 389-405.
- Smedslund, Jan. "Development of Concrete Transitivity of Length in Children". Child Development, 34 (1963), pp. 389-405.
- Steffe, Leslie P. "Thinking About Measurement." Edited by C. Alan Riedesel. The Arithmetic Teacher. May, 1971, pp. 332-338.
- Stoll, Earline Lillian. Geometric Concept Formation In Kindergarten Children. Unpublished Doctoral Dissertation, Stanford University, 1962.
- Taloumis, Thalia. "The Understanding of Area Measurement." The Mathematics Education of the Elementary School Teacher. A Report of the TTT Project in Science and Mathematics Tri-University Project in Elementary Education. Volume II, 1970. pp. 113-124.
- Towler, J.O. Training Effects and Concept Development: A Study of the Conservation of Continuous Quantity in Children. Washington, D.C.: United States Office of Education ERIC. Department of Health Education, and Welfare. 1968.
- Turabian, Kate L. A Manual for Writers. 3rd, edition. Chicago: The University of Chicago Press, 1967.
- Wagman, Harriet. A Study of the Child's Concept of Area. Unpublished Doctoral Dissertation, Columbia University, 1968.
- Walker, Helen M. and Lev, Joseph. Elementary Statistical Methods. New York: Holt, Rinehart, and Winston, 1958.
- Wallace, J.G. Concept Growth and the Education of the Child. New York: New York University Press, 1967.
- Wertheimer, Max. Productive Thinking. Edited by Michael Wertheimer. New York: Harper and Brothers Publishers, 1959.
- Young, Beverly S. Inducing Conservation of Number, Weight, Volume, Area, and Mass in Pre-School Children. Washington, D.C.: United States Office of Education ERIC. Department of Health, Education, and Welfare. February, 1969.

Zimiles, Herbert. "A Note on Piaget's Concept of Conservation."
Logical Thinking in Children - Research Based on Piaget's
Theory. Edited by Irving Sigel and Frank H. Hooper, 1968.
pp. 355-381.

APPENDIX A
DETAILED DESCRIPTION OF TASKS

Preparatory Remarks

In preliminary conversation with each child, the following remarks were made:

1. "This is not a test; I would like to know what you think about some of the materials that I have."
2. "There aren't any right or wrong answers; just tell me what you think."
3. "Don't guess on these tasks. Try to work things out."
4. "I'm going to write down some things and keep this tape recorder on during our meeting. In case I forget something, I can listen to our conversation later."

Each child was asked to state his name, grade, and teacher's name for identifying purposes. The child was allowed to listen to his voice on tape prior to the presentation of the tasks.

Any variation in terms used was indicated on the interview recording sheet (see Appendix B).

Vocabulary Task

Purpose: To determine whether the subject is familiar with the vocabulary to be used in the measurement tasks. Only subjects who are will be included in the sample.

Materials: 8 inch length of $\frac{1}{8}$ inch diameter wire, colored black.
8 inch length of $\frac{1}{8}$ inch diameter wire, colored red.
4 inch length of $\frac{1}{4}$ inch diameter wire, colored green.
16 inch length of $\frac{1}{8}$ inch diameter wire, colored white.
2 isosceles right triangles with 8 inch legs, one colored blue, the other green.
1 isosceles right triangle with 4 inch legs, colored red.
1 hexagon with area measure equal to 16 square inches, colored yellow.

See Figure 3-1 for details of material.

Procedure regarding length: "I'd like you to look at these two pieces of wire, the black and the red. Do they have the same amount of length or different amounts?"

Permit child to handle the wires and superimpose them if necessary.

"Which one has more length? Which one has less length? Which one is longer? Which one is shorter?"

Repeat the above with the black and white wires.

Repeat the above with the black and green wires.

Variation: Child says, "What do you mean by 'length'?" or "I don't know what you mean."

Investigator replies, "Is there the same amount of distance from one end to the other?"

If the child is still puzzled (rare), investigator says, "How are these two pieces of wire different? Are they the same in every way besides color? Is there more white or more black? Why?"

Procedure regarding area: "I'd like you to look at these two pieces of paper, the blue and the green. Do they have the same area? Do they have the same amount of space or different amounts?"

Permit child to handle the triangles and superimpose them if necessary.

"Which one has more space? Which one has less space? Which one is larger? Which one is smaller?"

Repeat the above with the blue and red polygons.

Repeat the above with the blue and yellow polygons.

Variation: Child says, "What do you mean by 'space'?" or "I don't know what you mean."

Investigator replies, "Do they have the same amount of room inside or different amounts?"

If child is still puzzled (rare), investigator says, "How are these two pieces of paper different? Are they the same in every way besides color? Is there more red or more blue, why?"

If the child uses "room" rather than "space", when he indicates the polygon with more room, say, "So this one has more space." Similarly for the polygon with less room. Try using "space" in the questions when repeating the task with the blue and yellow polygons. If the child has trouble using "space", then indicate same on recording sheet and substitute "room" for "space" in the remaining tasks.

Congruence Task-Length

Purpose: To determine whether the child can identify congruent lines (i.e. lines whose endpoints coincide when placed in a parallel manner). To determine whether the child can make correct judgements regarding equal and unequal lengths.

Materials: 2 - 4 inch lengths of wire, 1/8 inch diameter, one made of black, the other of red wire.

1 - 8 inch length of wire, 1/8 inch diameter, made of white wire.

The wires are identical in every way except the stated differences of color and length.

Procedure: The three wires (black, red, and white) are placed on the desk in front of the child in no organized manner. The child is asked to respond to the following questions. Each question is followed by the child's response.

"Are any of the wires the same length?" (response)

"Which of the wires are the same length?" (response)

"How can you tell that?" (response)

"Are any wires of different length?" (response)

"Which wires are of different length?" (response)

"How can you tell that?" (response)

The child is allowed to manipulate the wires and superimpose them if desired.

Congruence Task-Area

Purpose: To determine whether the child can identify congruent polygonal regions (i.e. polygonal regions that have the same size and shape). To determine whether the child can make correct judgments regarding equal and unequal areas.

Materials: 2 isosceles right triangles with 4 inch legs, one made of blue, the other of green paper.

1 isosceles right triangle with 6 inch legs, made of white paper.

Paper of the same composition and thickness was used in the construction of the triangles.

Procedure: The three triangles (blue, green, and white) are placed on the desk in front of the child in no organized manner. The child is asked to respond to the following questions. Each question is followed by the child's response.

"Do any of the pieces of colored paper have the same size and shape?"
(response)

"Which pieces of paper have the same size and shape?" (response)

"How can you tell that?" (response)

"Do any of the pieces of colored paper have different sizes or shapes?" (response)

"How can you tell that?" (response)

The child is allowed to manipulate the triangles and superimpose them if he desires.

Conservation of Length Task

Purpose: To determine whether the subject can conserve length, relative to both a change of position and subdivision.

Materials: 2 - 3 inch long pieces of thin wire (same diameter), one colored black, the other red. See Figure 3-2.

Procedure: This task is administered in two parts:

Part A: Change of position

"I'd like you to look at these two pieces of wire, the black and the red. Do they have the same length? How can you tell?"

The subject is permitted to handle the wires and superimpose them if necessary. Eventually, with the assistance of the experimenter if necessary, the subject concludes that the wires are of equal length.

The wires are placed in front of the subject in a parallel manner so that their ends coincide. "Do the wires have equal length? Why?"

The following arrangements as indicated in Figure 3-1, A-F are formed, each followed by the questions: "Do the wires have the same length? Why?"

Part B: Subdivision

"I'd like you to look at these two pieces of wire, the black and the red. Do they have the same length? Is one

longer than the other? (Which one is longest?) How can you tell?"

The subject is permitted to handle the wires and superimpose them if necessary. Eventually, with the assistance of the experimenter if necessary, the subject concludes that the wires are of equal length.

The black wire is not altered, while the red wire is bent into the shape indicated in Figure 3-2 H. The child is asked: "Do the black and red wires have the same length? Is one longer than the other? (Which is longest?) How can you tell?"

Conservation of Area Task

Purpose: To determine whether the subject can conserve area, relative to both a change of position and subdivision.

Materials: 2 isosceles right triangles with 8 inch legs, one made of blue paper, the other made of green paper. See Figure 3-3.

Procedure: This task is administered in two parts:

Part A: Change of position

"I'd like you to look at these two pieces of paper, the blue and the green. Do they have the same area? Do they have the same amount of space? How can you tell?"

The subject is permitted to handle the triangles and superimpose them if necessary. Eventually, with the assistance of the experimenter if necessary, the subject concludes that the triangles have the same area or same amount of space.

The triangles are placed in front of the subject according to the arrangements indicated in Figure 3-3, A-D. After each arrangement, the subject is asked: "Are the areas the same? Do they have the same amount of space? (Which has more space?) How do you know that?"

Part B: Subdivision

"I'd like you to look at these two pieces of paper, the blue and the green. Do they have the same area? Do they

have the same amount of space or different amounts? (Which one has more space?) How do you know that?"

The subject is permitted to handle the triangles and superimpose them if necessary. Eventually, with the assistance of the experimenter if necessary, the subject concludes that the triangles have the same area or same amount of space.

The blue triangle is not altered, while a segment is cut from the green triangle and placed in the positions indicated in Figure 3-3, E-F. The child is asked "Do the blue and green pieces of paper have the same area (same or different amount of space or room)? Which one has more space? How do you know that?"

Additivity of Length Task

Purpose: To determine whether the subject understands that the whole is equal to the sum of its nonoverlapping parts (except for possible common endpoints) regardless of the arrangement of the parts. The test for additivity of length involves a test for conservation and transitivity.

Materials: The following lines were drawn on white paper (see Figure 3-4).

1 - straight blue line, 16 inches long.

1 - broken green line, with segments of 2, 8, 4, and 2 inches, respectively.

1 - oblique, straight, red line, 15 inches long.

2 - lengths of white wire, 4 inches long.

4 - lengths of white wire, 2 inches long.

The wires were made of the same composition (solder) and were the same thickness ($\frac{1}{8}$ inch in diameter).

Procedure: The blue, green, and red lines are drawn on a piece of paper. The white wires are the only movable pieces in this task. No indication of dimensions is given to the subject.

The child is given all of the white wires and is asked to cover exactly each of the colored lines, one at a time, with the white wires.

The subject is asked, "Can you arrange the white wires so they cover up exactly the blue line by placing them end to end on the blue line?" (Assist the subject if necessary.) "Now, can you do the same with the green line?"

"Let's think about the blue and green lines. Do they have the same length? Is one line longer than the other line? How do you know that? (Which line is longer? How do you know that?)"

"Can you cover up exactly the red line with the white wires? Let's think about the blue, green, and red lines. Does the red line have the same length as the blue line? Is one line longer than the other line? How do you know that? (Which line is the longer line? How do you know that?)"

Additivity of Area Task

Purpose: To determine whether the subject understands that the whole is equal to the sum of its nonoverlapping parts (except for possible common sides) regardless of the arrangement of the parts. The test for additivity of area involves a test for conservation and transitivity.

Materials: The following polygonal regions were pasted on white paper (see Figure 3-5).

1 - 4-inch square made of blue paper.

1 - 2-inch by 8-inch rectangle made of green paper.

1 - 2-inch by 7-inch rectangle made of red paper.

2 - 2-inch squares made of white paper.

2 - 1-inch by 2-inch rectangles made of white paper.

All the paper used was the same composition and same thickness.

Procedure: The blue, green, and red pieces are glued to a large piece of paper. The white segments are the only movable pieces in this task. No indication of dimensions is given to the subject.

The child is given all of the white pieces of paper and is asked to cover exactly each of the colored pieces of paper, one at a time, with the white paper.

The subject is asked, "Can you arrange the white pieces of paper so they exactly cover the blue?" Assist the subject if

necessary. "Now, can you do the same with the green?"

"Let's think about the blue and the green. Do they have the same amount of space or different amounts? How do you know that? (Which has more space? How do you know that?)"

"Can you cover up exactly the red paper with the pieces of white paper? Let's think about the blue, green, and the red. Does the red have the same amount of space or different amounts as the blue or green? How do you know that? (Which has more space? How do you know that?)"

Unit Length Task

Purpose: To determine if the child is aware of the use of a unit of length measure relative to its size and number of units.

Materials: The following lines were drawn on white paper (see Figure 3-6).

1 - straight blue line, 16 inches long.

1 - broken green line, with segments of 2, 8, 4, and 2 inches, respectively.

1 - oblique, straight red line, 8 inches long.

6 - lengths of white wire, 4 inches long.

10 - lengths of white wire, 2 inches long.

10 - lengths of white wire, 1 inch long.

The wires were made of the same composition (solder) and were the same thickness ($1/8$ inch in diameter).

Procedure: The child is given the paper with the blue, green, and red lines drawn on it. No indication of dimensions are given.

"Can you completely cover the blue line with these pieces of wire by placing them end to end on the blue line?" The child is given the six pieces of 4-inch long wires. If the child is not sure of what is being asked, the investigator gives assistance in the form of placing a few of the wires end to end on the blue line.

"Now can you completely cover the green line with these pieces of wire?" The child is given the ten pieces of wire that are 2 inches long. The same type of assistance is given by the investigator if needed.

"Now, can you completely cover the red line with these pieces of wire?"

The child is given the ten pieces of 1-inch wire. Again, similar assistance is given if needed.

The child now has all three colored lines covered with the wires of different sizes.

"Which line is longer, the blue or the green, or are they the same length?" (response) "How do you know that?"

"Which line is longer, the green or the red, or are they the same length?" (response) "How do you know that?"

Throughout this task, the child is permitted to manipulate only those pieces of wire that were not used to cover any of the colored lines (two pieces of 4-inch wire, two pieces of 2-inch wire, and two pieces of 1-inch wire).

"Without placing these small (1-inch) wires on the green line, can you tell me how many of the small wires it would take to completely cover the green line?" (response) "How do you know that?"

Unit Area Task

Purpose: To determine if the child is aware of the use of a unit of area measure relative to its size and number of units.

Materials: The following polygonal regions (blue, green, and red) were pasted on a piece of white paper (see Figure 3-7).

- 1 - square with 4-inch sides made of blue paper.
- 1 - rectangular region, 2 inches by 6 inches, with a 2-inch square adjoined (see Figure 3-6 B) made of green paper.
- 1 - rectangular region, 1 inch by 8 inches, made of red paper.
- 6 - squares, with 2-inch sides, made of white paper.
- 10 - rectangles, with 1-and 2-inch sides, made of white paper.
- 10 - squares, with 1-inch sides, made of white paper.

All stated regions were made of paper that had the same composition and same thickness.

Procedure: The child is given the white paper on which the colored polygonal regions are pasted. No dimensions of sides are indicated. "Can you completely cover the blue paper with these pieces of white paper?" The child is given the six squares with 2-inch sides. If the child is not sure of what is being asked, the investigator gives assistance in the form of placing a few of the white squares on the blue paper.

"Now can you completely cover the green paper with these pieces of white paper?" The child is given the ten rectangular pieces of white paper. The same type of assistance is given by the investigator if needed.

"Now, can you completely cover the red paper with these pieces of white paper?" The child is given the ten squares of white paper with 1-inch sides. Again, similar assistance is given if needed. The child now has all three colored regions covered with the white pieces of paper of different areas.

"Which has more space, the blue or green paper, or are they the same?" (response) "How do you know that?"

Which has more space, the green or the red paper, or are they the same?" (response) "How do you know that?"

Throughout this task, the child is permitted to manipulate only those pieces of white paper that were not used to cover any of the colored regions (two squares with 2-inch sides, two rectangles with 1- and 2-inch sides, and two squares with 1-inch sides).

"Without placing these small (1-inch squares) pieces of white paper on the green paper, can you tell me how many of them it would take to completely cover the green region?" (response) "How do you know that?"

APPENDIX B
INTERVIEW RECORDING SHEET

Name: _____ **Date:** _____
Age: _____ **Birth Date:** _____
Grade: _____ **Reading Score:** _____

I. Summary of Performance:¹

<u>Task</u>	<u>Length</u>	<u>Area</u>	<u>Comments (level)</u>
A. Vocabulary			A.
B. Congruence			B.
C. Conservation			C.
D. Additivity			D.
E. Unit Measure			E.

II. Measurement level: Length _____ Area _____

Reason:

¹Tasks are evaluated on pass (P) or fail (F) basis. Comments include statements regarding the child's responses and measurement level. Comments regarding vocabulary include terms that vary from those used in the task statements for the sake of clarity.

APPENDIX C
RESULTS FOR EACH SUBJECT
(By Age Group)

<u>Column Heading</u>	<u>Contents</u>	<u>Comments</u>
1. Subject	Subject Identification	Subject is identified by three letters and his age. Subject marked with asterisk does not belong in sample due to failure of vocabulary task.
2. Age	Age	Age given in years and months.
3. Grade	Grade	Grades one through five, although all children were in the ninth month of their respective grades.
4. Reading Level	Reading comprehension score	Score given in school years except for seven-year olds (percentile scores).
5. Arithmetic Concepts	Arithmetic concept score	Score given in school years. Not available for seven-year olds.
6. Vocabulary	Vocabulary Task score	P: pass; F: fail
7. Congruence L / A	Congruence Task score for length and area	P: pass; F: fail

<u>Column Heading</u>	<u>Contents</u>	<u>Comments</u>
8. Conservation L / A	Conservation Task score for length and area.	P: pass both change of position and subdivision. Fc: fail change of position; pass subdivision. Fs: fail subdivision; pass change of position. F: fail both change of position and subdivision.
9. Additivity L / A	Additivity Task score for length and area.	P: pass; F: fail
10. Unit Measure L / A	Unit Measure Task score for length and area.	P: pass; F: fail
11. Conservation Stage L / A	Length and Area Conser- vation stage.	Possible entries: A: (attainment stage) B: (transitional stage) C: (none)
12. Measurement stage L / A	Length and Area Measurement stage	Possible entries: Substages II A, II B, III A, III B.

RESULTS FOR EACH SUBJECT
(By Age Group)

<u>Subject</u>	<u>Age</u>	<u>Grade</u>	<u>Reading Level</u>	<u>Arith- metic Concepts</u>	<u>Vocab- ulary</u>	<u>Congru- ence</u> <u>L / A</u>	<u>Conser- vation</u> <u>L / A</u>	<u>Addi- tivity</u> <u>L / A</u>	<u>Unit Measure</u> <u>L / A</u>	<u>Conser- vation Stage</u> <u>L / A</u>	<u>Measure- ment Stage</u> <u>L / A</u>
AND	7;6	1	12	-	P	P	P	F	F	C	B
ASI	6;10	1	99	-	P	P	P	F	F	C	C
BUF	7;1	1	-	-	P	P	P	F	F	C	C
CLO	6;7	1	70	-	P	P	P	F	F	C	C
DEM	7;3	1	85	-	P	P	P	F	F	C	C
EMM	7;6	1	-	-	P	P	P	Fs	F	B	A
GIB	7;0	1	80	-	P	P	P	P	F	A	A
HAH	7;0	1	55	-	P	P	P	F	F	C	C
HAN	7;1	1	75	-	P	P	P	Fs	F	C	B
HAR	7;0	1	80	-	P	P	P	P	P	A	A
JOH	7;3	1	95	-	P	P	P	P	P	A	A
KEL	6;7	1	38	-	P	P	P	Fs	F	B	B
KUE	6;7	1	99	-	P	P	P	F	F	C	C
LAB	7;5	1	-	-	F	P	P	P	P	A	A

<u>Subject</u>	<u>Age</u>	<u>Grade</u>	<u>Reading Level</u>	<u>Arith- metic Concepts</u>	<u>Vocab- ulary</u>	<u>Congru- ence</u> L / A	<u>Conser- vation</u> L / A	<u>Addi- tivity</u> L / A	<u>Unit Measure</u> L / A	<u>Conser- vation Stage</u> L / A	<u>Measure- ment Stage</u> L / A
LAR	7;4	1	92	-	P	P	P	P	P	A	IIIB IIIA
LIS	6;10	1	53	-	P	F	F	F	F	C	IIA IIA
LOV	7;6	1	50	-	F	P	F	F	F	C	IIA IIA
MCS	7;3	1	-	-	P	P	Fs	F	F	B	IIB IIA
MIC	7;6	1	55	-	P	P	Fs	F	F	B	IIB IIA
MUN	6;11	1	-	-	P	P	Fs	F	F	B	IIB IIB
BAU	8;6	3	4.6	4.3	F	P	P	P	F	A	IIIA IIB
BIS	7;11	2	2.3	2.3	P	P	F	F	F	C	IIA IIA
BOW	7;11	2	3.3	3.2	P	P	P	P	F	A	IIIA IIIB
BUR	8;1	2	1.8	1.4	P	P	F	F	F	C	IIA IIA
COO	7;9	2	2.1	2.6	P	P	P	P	P	A	IIIB IIIB
DEM	8;5	2	4.4	4.0	P	P	F	F	F	C	IIA IIA
HAA	8;6	2	3.6	5.3	P	P	P	P	P	A	IIIB IIIB
HAM	8;6	2	4.2	5.6	P	P	P	P	P	A	IIIB IIIB
HAK	8;6	2	3.5	4.7	P	P	Fs	F	F	B	IIB IIIB

<u>Subject</u>	<u>Age</u>	<u>Grade</u>	<u>Reading Level</u>	<u>Arith- metic Concepts</u>	<u>Vocab- ulary</u>	<u>Congru- ence</u>		<u>Conser- vation</u>		<u>Addi- tivity</u>		<u>Unit Measure</u>		<u>Conser- vation Stage</u>		<u>Measure- ment Stage</u>	
						<u>L</u>	<u>A</u>	<u>L</u>	<u>A</u>	<u>L</u>	<u>A</u>	<u>L</u>	<u>A</u>	<u>L</u>	<u>A</u>	<u>L</u>	<u>A</u>
JEN	8;3	2	4.2	3.1	P	P	P	P	P	P	P	F	F	A	A	IIIA	IIIA
JOH	7;10	2	2.2	2.8	P	P	P	F	F	F	F	F	F	C	C	IIA	IIA
MAN	7;9	2	2.1	2.3	P	P	P	F	F	F	F	F	F	C	C	IIA	IIA
MCC	8;5	2	1.8	2.6	P	P	P	F	F	F	F	F	F	C	C	IIA	IIA
MCG	8;1	2	2.0	3.6	P	P	P	F	Fs	F	F	F	F	C	B	IIA	IIB
MUN	7;11	2	3.9	3.8	P	P	P	P	P	P	P	P	P	A	A	IIIB	IIIB
NIE	8;3	2	2.9	3.2	P	P	P	Fs	P	P	P	F	F	B	A	IIB	IIIA
OLE	7;9	2	2.6	3.4	P	P	P	Fs	P	F	F	F	F	B	A	IIB	IIIA
PUR	7;7	2	3.0	3.0	P	P	P	Fs	Fs	F	F	F	F	B	B	IIB	IIB
QUI	7;10	2	2.4	2.3	P	P	P	P	Fs	F	F	F	F	A	B	TTTA	IIB
WIL	8;4	2	5.5	5.6	P	P	P	P	P	P	P	P	P	A	A	IIIB	IIIB
BEE	8;7	3	3.1	4.5	P	P	P	P	P	P	P	F	F	A	A	IIIA	IIIA
CAR	9;4	3	6.9	5.5	P	P	P	Fs	P	P	P	P	P	B	A	IIB	IIIB
COR	9;2	3	2.6	4.5	P	P	P	F	F	F	F	F	F	C	C	IIA	IIA
COT	8;7	3	5.0	6.3	P	P	P	P	Fs	F	F	P	P	A	B	IIIB	IIB

Subject	Age	Grade	Arith-		Vocab-	Congru-		Conser-		Addi-	Unit	Conser-		Measure-
			Heading	Concepts		ence	vation	vation	tivity			Measure	Stage	
			Level		ulary	L / A	L / A	L / A	L / A	L / A	L / A	L / A	L / A	
DAV	8;7	3	3.1	4.5	P	P	P	P	P	P	F	A	IIIA IIIA	
EMM	9;4	3	3.8	5.3	P	P	P	Fs	P	F	F	B	IIB IIIA	
ENT	9;6	3	2.5	2.7	P	P	P	F	F	F	F	C	IIA IIA	
GEH	9;5	3	6.4	5.5	P	P	P	P	P	F	P	A	IIIB IIIA	
HAH	8;9	3	5.0	4.3	P	P	P	Fs	P	P	F	B	IIB IIIB	
JOH	8;10	3	3.8	3.8	P	P	P	F	Fc	F	F	C	IIA IIB	
KEL	9;5	3	5.0	5.5	P	P	P	P	P	P	F	A	IIIA IIIA	
LEI	9;2	3	4.3	4.2	P	P	P	Fs	Fs	F	F	B	IIB IIB	
MAN	9;0	2	2.6	2.3	P	P	P	F	F	F	F	C	IIA IIA	
MCG	9;6	3	4.3	5.5	P	P	P	F	P	F	F	C	IIA IIIA	
MCW	9;1	3	4.0	4.0	P	P	P	Fs	Fs	F	F	B	IIB IIB	
MYE	9;0	3	4.4	6.3	P	P	P	P	P	P	P	A	IIIB IIIB	
ROG	9;0	3	3.0	3.8	P	P	P	Fs	P	F	F	B	IIB IIIA	
SAU	9;3	3	3.7	3.3	P	P	P	Fs	P	F	F	B	IIB IIIA	
SHA	9;1	3	5.7	7.1	P	P	P	P	P	P	F	A	IIIA IIIB	

Subject	Age	Grade	Reading Level		Arith- metic Concepts	Vocab- ulary	Congru- ence		Conser- vation		Addi- tivity		Unit Measure		Conser- vation Stage		Measure- ment Stage
							L / A	L / A	L / A	L / A	L / A	L / A	L / A	L / A	L / A	L / A	
WES	9;1	3	3.8	3.3		P	P	P	Fs	Fs	F	F	F	F	B	B	IIB IIB
AND	10;0	4	5.7	5.8		P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
ASI	10;6	5	8.4	7.8		P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
BEL	10;6	4	9.5	8.0		P	P	P	P	P	P	P	P	P	A	A	IIIB IIIA
COT	10;3	4	4.6	6.5		P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
EBE	10;0	4	9.5	7.6		P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
FRA	10;6	4	7.0	8.0		P	P	P	P	P	P	P	F	P	A	A	IIIA IIIB
GOL	10;3	4	8.2	8.5		P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
HAR	10;6	5	6.7	6.6		P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
HER	10;4	4	2.9	3.0		P	P	P	F	F	F	P	F	P	C	A	IIA IIIB
HOU	9;8	4	3.4	4.0		P	P	P	F	F	F	F	F	F	C	C	IIA IIA
JOH	10;6	4	5.9	7.6		P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
KOS	9;11	4	4.6	6.3		P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
LAR	9;8	4	8.2	5.9		P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
MAN	10;4	4	5.7	5.8		P	P	P	P	P	P	P	P	P	A	A	IIIB IIIA

Subject	Age	Grade	Reading Level	Arith- metic Concepts	Vocab- ulary	Congru- ence		Conser- vation		Addi- tivity		Unit Measure		Conser- vation Stage		Measure- ment Stage
						L / A	L / A	L / A	L / A	L / A	L / A	L / A	L / A	L / A	L / A	L / A
MCN	10;6	5	6.8	6.8	P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
MEY	10;1	4	3.9	5.9	P	P	P	P	P	F	F	F	F	A	A	IIIA IIIA
MCR	10;2	4	4.8	5.7	P	P	P	P	P	P	P	F	F	A	A	IIIA IIIA
FIE	9;8	4	6.0	6.3	P	P	P	P	P	F	F	F	F	A	A	IIIA IIIA
REH	10;1	4	3.4	4.0	P	P	P	P	Fs	P	F	F	F	A	B	IIIA IIB
UTK	9;8	4	4.3	5.9	P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
BAU	11;0	5	5.7	7.3	P	P	P	Fs	P	P	P	F	F	B	A	IIB IIIA
BEE	11;1	5	6.2	7.6	P	P	P	P	P	P	P	P	P	A	A	IIIB IIIA
BIR	11;1	5	6.0	7.0	F	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
BIS	11;6	5	6.0	6.5	P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
CAR	11;0	4	2.8	3.3	P	P	P	P	Fs	F	F	F	F	A	B	IIB IIB
DAV	11;2	5	4.3	6.6	P	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
EMM	11;6	5	4.7	6.6	F	P	P	P	P	P	P	P	P	A	A	IIIB IIIB
GEH	11;2	5	7.8	6.5	P	P	P	P	Fs	P	F	P	F	A	B	IIIB IIIA
HOL	11;6	5	5.0	5.6	F	P	P	P	P	P	P	P	P	A	A	IIIB IIIB

<u>Subject</u>	<u>Age</u>	<u>Grade</u>	<u>Reading Level</u>	<u>Arith- metic Concepts</u>	<u>Vocab- ulary</u>	<u>Congru- ence</u>		<u>Conser- vation</u>		<u>Addi- tivity</u>		<u>Unit Measure</u>		<u>Conser- vation Stage</u>		<u>Measure- ment Stage</u>	
						<u>L</u>	<u>A</u>	<u>L</u>	<u>A</u>	<u>L</u>	<u>A</u>	<u>L</u>	<u>A</u>	<u>L</u>	<u>A</u>	<u>L</u>	<u>A</u>
KEN	11;3	5	6.9	6.5	P	P	P	F	F	F	F	F	F	C	C	IIA	IIA
LAR	11;5	5	10.9	8.0	P	P	P	P	P	P	P	P	P	A	A	IIIB	IIIB
MAS	11;1	5	6.8	8.5	P	P	P	P	P	P	P	F	P	A	A	IIIA	IIIB
MOR	11;5	5	6.7	7.3	P	P	P	P	P	P	P	P	P	A	A	IIIB	IIIB
NIE	11;2	5	6.7	6.6	P	P	P	P	P	P	P	P	F	A	A	IIIB	IIIA
PAT	10;10	4	3.7	5.2	P	P	P	P	P	P	P	F	F	A	A	IIIA	IIIA
REH	11;2	5	3.0	5.9	P	P	P	P	P	P	P	F	F	A	A	IIIA	IIIB
SMI	11;2	5	6.8	7.3	P	P	P	P	P	P	P	P	P	A	A	IIIB	IIIB
STR	10;10	4	3.4	6.3	F	F	P	P	P	P	P	P	P	A	A	IIIB	IIIB
WAL	11;5	5	9.2	7.3	F	P	P	P	Ps	F	P	F	F	A	B	IIIA	IIB
WES	10;10	5	6.8	5.2	P	F	F	P	Ps	P	P	F	F	A	A	IIIA	IIB
WEL*	9;7	3	-	-	F	-	-	-	-	-	-	-	-	-	-	-	-

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