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ABSTRACT

THE MATHEMATICAL BEHAVIORS DERIVABLE FROM THE PROGRAM OF UNIFIED SCIENCE AND MATHEMATICS FOR ELEMENTARY SCHOOLS

By

Sompop Krairojananan

This Case-Study is a comprehensive account of intensive continual observation of four 'USMES' classes at Lansing, Michigan, during the nine weeks from late September to early December, 1972. USMES (Unified Science and Mathematics for Elementary Schools) is an activity-oriented and integrated course in science, mathematics and social science. Many units, designed to promote the problem-solving skills of students in their attempt to answer some major 'challenges', have been developed, and four such units (Soft Drink Design; Dice Design; Designing for Human Proportion; and Burglar Alarm Design) were selected for observation in this Case-Study with emphasis on mathematical behaviors which arose naturally from these courses of activities. The mathematical topics observed were then categorized into seven areas: Arithmetic, Algebra, Graph and Tabulation, Geometry, Application and Practical Mathematics, Statistics or Physics, and Foundation of Mathematics. The mathematical behaviors arising from these four USMES

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units were found to be distributed fairly evenly in all seven areas, and thus it could be deduced that these USMES units gave rise to a well-balanced mathematics program.

Since each unit aimed at providing a partial solution to a long-range challenge, the mathematical problems that arose were thus highly relevant and did not contain artificial data. These problems were systematically tackled by techniques as near to the 'scientific method' as possible. Children were observed to have learned just as much about scientific process as mathematics. One outstanding feature was that students had to verify the correctness of their results by checking with the practical outcomes, and not by recourse to teachers.

Uniformity was not to be expected in an activity-oriented almost-realistic program like USMES. In fact, it could be observed that diversity of students' achievements, rather than uniformity, was encouraged in these classes. Despite divergence in activities, all students in these four units eventually learned at least the following mathematical topics, which the 1963-Cambridge Conference had termed "the bed-rock foundation of elementary school mathematics," namely, counting and fractions, symmetry and invariant properties of geometric figures, real experience in collecting data, use of graphs and other visual displays of data, and the vocabulary of elementary logic.

THE MATHEMATICAL BEHAVIORS DERIVABLE FROM THE PROGRAM OF UNIFIED SCIENCE AND MATHEMATICS FOR ELEMENTARY SCHOOLS

By

Sompop Krairojananan

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Finally, I would like to thank the Mathematics Department, Michigan State University, for giving me partial financial support, and the Thai Government for giving me the 3-year study leave.

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CHAPTER 1

INTRODUCTION

This research is a case-study type of work which involves intensive continual observations of four USMES classes in Lansing, Michigan, from late September to early December 1972. The variable singled out for this observation was the mathematical behavior, pure or applied, exhibited by students while participating in USMES activities.

What is USMES?

USMES (Unified Science and Mathematics for Elementary Schools) is a working project of Education Development Center, funded by the National Science Foundation, to develop interdisciplinary units focusing on problems whose partial solution implies some realistically useful achievement and a substantial amount of learning in science and mathematics. The USMES project grew out of the 1967 Cambridge Conference on the Correlation of Science and Mathematics in the Schools. This Conference Report[1] stated the primary aim of drawing maximum benefits in scientific education from an integrated program in elementary schools.

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[1]Cambridge Conference, "Goals for the Correlation of Elementary Science and Mathematics," Boston: Houghton Mifflin, 1968.

In the implementing phase of the project, this integrated program makes use of both activity learning and expository instructions (frequently in the form of "How to" cards). Its underlying philosophical principle has been 'macro-openness and micro-structuring.' USMES has selected special types of practical or 'real' problems which meet the following criteria: attainable results within the time and resources available (in elementary schools), inter-disciplinary nature with emphasis on problem-solving, and definite academic achievement in elementary science and mathematics appropriate to the age level.

The USMES challenges are hypothesized to have four important advantages as follows:

(a) Motivation, based on appeal and direct application to the students' own environment,

(b) Accuracy of learning, to be checked against the practical outcomes,

(c) Opportunities for problem defining,

(d) Experience in total problem-solving, in contrast to isolated problem-solving in most programs. (Total problem-solving resembles the working of a real scientist. The students will be actively involved in observation, quantification, formulation, trial of successive models, and, above all, critical thinking.)

Statement of the Research Problem:

A case-study is to be undertaken to ascertain the mathematical behaviors, or potential mathematical topics, derivable from the USMES Program.

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Justification of the Method Used:

Why should the case-study method involving intensive observation of a few units have the preference over the other methods involving a larger sample size? There are two reasons. First, the nature of a program like USMES will permit a high degree of flexibility. There will be considerable divergence between any two USMES units even though they may bear the same title. For example, one Dice Design Unit may concentrate on the construction of polyhedra (and hence learn a great deal of Geometry), while another Dice Design Unit may collect data on coin-tossing and dice-rolling (and hence learn many topics in Elementary Probability). The 'Human Proportion' Unit may branch out into designing garments, or designing furniture and fixtures for an elementary school, or measuring a human body to study graphic art, or even going to the extent of studying the bone-structure, muscles, and other anatomical topics. Secondly, J. Piaget,[1] K. Lovell,[2] Z. P. Dienes,[3] the Soviet Academy

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[1]See, e.g., Piaget, Jean, "The Child's Conception of Number," London: Routledge and Paul, 1952.

[2]Lovell, Kenneth, "The Growth of Basic Mathematical and Scientific Concepts in Children," London: London University Press, 1961.

[3]Dienes, Zoltan P., "An Experimental Study of Mathematics Learning," London: Hutchinson, 1963.

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of Pedagogical Science[1] and other clinical psychologists, have already demonstrated the immense gain in the knowledge about mathematics learning when case-studies based on intensive continuous observations were carried out.

Limitation of the Study:

This study is not intended for generalizing the results and conclusions about the observed populations to those of all USMES populations. Rather, it is meant to provide a paradigm case showing what amount of mathematical learning can be derived from part of the USMES program.

In a way, the content of this Case-Study is very much like that of the teachers' Handbooks[2] in the new approach of teaching mathematics in British primary schools at present.

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[1]Kilpatrick, Jeremy and Wirsup, Izaak (ed.), "Soviet Studies in the Psychology of Learning and Teaching Mathematics," Chicago: The University Press, 1969.

[2]Schools Council, "Mathematics in Primary Schools," Curriculum Bulletin No. 1, London: H. M. Stationery Office, 1965.

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1. <u>The literature related to activity-oriented curricula</u>:

The idea of an activity-oriented curriculum is not new: Pestalozzi[1] and Herbart[2] emphasized real experience and the use of real objects for their pupils' learning ever since mid-nineteenth century. Also in the last century, Froebel[3] stressed the value of play in learning. Later, Montessori[4] advocated the idea of allowing children freedom of movements in class, and other reasonable liberties to promote 'more natural' learning. Finally, with John Dewey's publications:[5] "Democracy and Education" (1916), "Experience and Nature" (1925), "Experience and Education" (1938), the child-centered activity-oriented classroom did begin to take shape and lingered on even after the zenith of the 'progressive education' movement. More recently, the Plowden Report[6] recommended a completely open classroom in British

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[1]Heafford, Michael, "Pestalozzi, His Thought and Its Relevance Today," London: Methuen, 1967.

[2]Dunkel, Harold B., "Herbart and Education," New York: Random House, 1969.

[3]Priestman, Barbara, "Froebel Education Today," London University Press, 1960.

[4]Montessori, Maria, "The Montessori Method," translated by Anne E. George, New York: Schoecher Books, Inc., 1964.

[5]Dewey, John, "Democracy and Education"; "Experience and Nature"; "Experience and Education," New York: Macmillan, 1916, 1925 and 1938 respectively.

[6]Central Advisory Council for Education (England), "Children and Their Primary Schools," London: H. M. Stationery Office, 1967.

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[2]*Ibid*.,

primary schools. For activity-oriented or experience-oriented curricula in mathematics, it began with John Perry's address[1] to the British Association For the Advancement of Science in 1901, followed closely by E. H. Moore's presidential address[2] to the American Mathematical Society in 1902. Both men strongly urged reforms in the mathematics curriculum and mathematics teaching. A few quotations from Moore's speech might help to sum up the ideas here:

> "A grade teacher must make wiser use of the foundations furnished by the kindergarten. The drawing and paper folding must lead on directly to...intuitive geometry, the construction of models, the elements of mechanical drawing, (and) simple exercises in geometrical reasoning....This program of reforms calls for the development of a thorough-going laboratory system of instruction in mathematics... to develop on the part of <u>every</u> student the true spirit of research, and an appreciation practical as well as theoretic, of the fundamental methods of science."

The above quotation is most relevant to the proper conduct of at least two USMES classes observed in this Case-Study.

The activity curricula seemed to be quite popular during the 20's and the 30's, maybe due to Dewey's influence. Franklin Bobbitt (1934, 1936) wrote two articles, "The Trend of the Activity Curriculum," and "What is the Activity School?" giving indication of the kind of learning that took place in some elementary schools. G. W. Diemer (1925)

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[1]Bidwell, James K. and Clason, Robert G. (ed.), "Readings in the History of Mathematics Education," Washington: N.C.T.M., 1970, pp. 221-245.

[2]<u>Ibid</u>., pp. 246-255.

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earlier described a half-way progressive school--the Platoon School, whose schedule in the afternoon was devoted entirely to activities. C. M. Reinoehl (1934) did a comparative study on the relative efficiency of platoon and non-platoon schools with regard to pupils' progress and achievement in common school subjects, and his finding was that platoon schools, on the average, did somewhat better. The concept that children could learn 'homeroom' subjects (arithmetic, reading, spelling, etc.) from activities was put into practice even earlier. R. Beatley (1921) reported that his students at Horace Mann School for Boys learned 'math' in the following manner:

> "Some students trail a tape measure around the room,...,squint along a rod out of the window, ...,learn to use a pantograph,...or a slide rule, ..."

Alice Stewart (1939) of Lincoln School described how her class learned from a 'community lab' and afterwards an 'urban lab' by actually living in those settings over a period--a variation from the usual 'camping' activities. Her children's logs revealed a great deal of mathematical behaviors learned from such experiences. Frank Freeman (1935) summed up the advantages of an activity curriculum as follows:

> "Activities furnish the acquaintance with the world of things and persons, which is the necessary basis for ideas and thinking; the traditional school has done violence to the children's nature by attempting to curb their natural disposition to be active."

The 'activity' trend was not confined to pupils alone; good teachers, too, initiated and conducted 'activityorient

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oriented' experiments which would finally result in better teaching. Claire Zyre (1927) experimented with 31 third-graders by giving them free 'conversation periods,' and she recorded the subjects of the children's conversation. Out of the 47,575 items of conversation recorded over a period, 12,983 were about play at home; 7,048 about animals; 6,051 about auto-trips and picnics and only 49 (i.e. 0.1%) about arithmetic. The writer of this Case-Study feels that, if this experiment were replicated today in a traditional school, the percentage pertaining to arithmetic or mathematics might not have altered very much.

The Activity Movement then captured so much attention that the National Society for the Study of Education (1934) issued a yearbook devoted entirely to the historical development, pros and cons of the activity curriculum. Forty-two leading personalities in education contributed to the writing of this yearbook. They defined the functions[1] of Activity in the Learning Process as encompassing some or all of the following: to be the basis of learning (replacing the former basis of passive reception); to be a natural mode of living (not just studying); to explore, experiment and interact with the environment, to develop body, mind and spirit, and, as a result, to make school life a happier phase of child life; to promote further growth and critical thinking.

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[1]N.S.S.E., "The Activity Movement," 33rd Yearbook, Part II, pp. 48-50.

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[1]Ibid.,

As expected, there was heated arguement over the evaluation in an activity program. By means of a nation-wide questionnaire sent to a cross-section of school personnel, the yearbook committee found that the following points[1] were dominant in the responses: "The important consideration is not how much factual information the child has gained, but how able he is to use it in new situations. ...Tests (for the Activity programs) should be designed to determine how learning takes place, whether a child can do things, how much progress he has made in the growth of independence, purposefulness, tolerant understanding, social habits, and social attitudes in general."

The 'activity' concept of teaching children has always had implication for the teaching of mathematics. W. F. Downey (1928) published an article: "New Mathematics as a part of New Education" in the Mathematics Teacher. The underlying principle of his 1928-essay was strikingly similar to the 1970-guideline of USMES. According to Downey, the process of education through self-activity requires three conditions:

(a) The pupils should be given opportunity to be problem-finders as well as problem-solvers,

(b) Whatever activity is undertaken: academic studies or mechanical, practical, and fine arts, or athletics, the pupils should feel that it is worth mastering,

(c) Before considering any problem as completed, the pupils should feel sure, through checks and other means

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[1] Ibid., pp. 153-156.

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that his work is correct. Then they would reach 'assured success' which would bring encouragement for the next undertaking.

H. Rugg (1924) who sat in the National Committee for revising mathematics curriculum suggested three guiding principles for the future, most of which sounded progressive enough. He wrote:

> "First, what are the abilities and interests of the pupils? Secondly, what mathematics materials do they need, both as growing youths in schools and as prospective citizens in a highly industrialized society? Thirdly, in what grades should these materials be <u>utilized</u>?" (Notice the word 'utilized', not 'verbally taught').

Charles H. Judd (1925), who was a prominent educational psychologist at the time, designed a laboratory study investigating the early stages of <u>number</u> <u>consciousness</u>. By means of elaborate apparatuses, he noted the difference between counting in the abstract and counting objects, sounds and flashes of light. He concluded that concrete counting of objects, sounds, light-flashes, etc. aroused number consciousness more rapidly. Judd (1925) also advocated more emphasis on the informational aspect of arithmetic as opposed to the cut-and-dry computational aspect. In fact, the mathematical behaviors observed in the Soft Drink Design Unit (See Chapter 4) utilized more informational aspect than computational aspect of cardinal and ordinal numbers, as well as fractions.

Although the 29th Yearbook of the National Society for the Study of Education (1930) came out at a time when progressive education was popular, the Yearbook Committee (i.e. the

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[1] N.S.S.
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[2] Ibid.,

(i.e. the Committee on Arithmetic) made it plain that they would not allow children's interest and 'felt needs' to degenerate into whims (often requiring less effort) and thereby neglect all the other dynamics of learning. A child's curriculum had to include materials which would become fundamentals for competent and useful adult living, argued the Committee. "Let the child eat, drink and be merry, but (forget not) that tomorrow he will be an adult."[1] Despite these harsh warnings, the curriculum proposed in the Yearbook was relatively liberal, even suggesting the gradual movement from 'mental discipline' to 'activity learning'. Examples of the suggestions[2] were:

--The place of object-teaching: more concrete work, less abstract drill;

--Larger use of the pupils' environment and increased provision for self-activity: counting toys, marbles, objects in a picture, candles for birthday cakes, scores in games, etc.;

--Motivating abstract calculations through games and other devices.

Some people questioned the actual mathematical learning obtained from an activity curriculum. Many research studies have been, and are being carried out to measure the mathematical outcomes (if any) of activity-oriented programs. These included studies done by MacLatchy (1930), Woody (1931), Hizer and Harap (1932), Thiele (1938), Tompkins and

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[1]N.S.S.E., "Report of the Society's Committee on Arithmetic," 29th Yearbook, Bloomington: Public School Pub. Co., 1930, pp. 4-5.

[2]<u>Ibid</u>., pp. 85, 89, 95.

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Stokes (1940), Lowry and Inez (1944) and Williams (1949). Most mathematics and science programs in the 50's and 60's, activity-oriented or otherwise, included research studies evaluating their own outcomes. It is relevant to note the findings of the first two studies cited above, since they were carried out during the 'progressive' era. Both MacLatchy and Woody investigated the number ideas of young children and the types of arithmetical information they possessed. Both studies concluded that young children acquired incidentally through their informal activities outside schools a much larger arithmetical background than has been believed possible.

During the Second World War, the traditional method of instruction seemed to take priority over activity classes. But many teachers, for example Catherine Stern,[1] still conducted their mathematics classes in a healthy manner by allowing the students to manipulate concrete materials before verbalizing abstract rules. In a way, Stern's book was the spearhead of another new wave of activity-oriented mathematics instruction to appear later on. Also in the period the National Society for the Study of Education (1951) issued another yearbook on the Teaching of Arithmetic. This yearbook still reflected the strong position of the 'manipulative instructional aids' and the 'laboratory method', and this time it also discussed the

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[1]Stern, Catherine, "Children Discover Arithmetic," New York: Harper, 1949.

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[1] N.S.S.
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'audio-visual instructional aids' which was becoming more and more popular. It contained a whole section on visual and manipulative materials with an almost exhausting list of 98 suitable devices.[1] It further recommended that manipulative objects and certain types of pictorial materials should be made in the classroom, since there was great educational value in having the pupils make these materials. This was, in fact, parallel to USMES's principle of encouraging students to make things in their 'Design Lab'. The National Council of Teachers of Mathematics also made available such information on manipulative materials to teachers in two of their yearbooks:

(a) the 16th Yearbook, "Arithmetic in General Education," 1941 (See the Section on 'Utilizing supplementary devices and materials);

(b) the 18th Yearbook, "Multisensory Aids in the Teaching of Mathematics," 1945.

Needless to mention that in this period many articles could still be found in professional journals describing various 'activity methods' of teaching mathematics. Typical among these were articles by Steiss and Baxter (1943), F. M. Burns (1944), R. Morris (1947) and M. B. Wilson (1955).

To learn mathematics through activities and concrete manipulations received another impetus in the late 50's and early 60's, as a result of the 'New Math Era', in which the mathematics content in the curriculum was drastically

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[1] N.S.S.E., "The Teaching of Arithmetic," 50th Yearbook, 1951, pp. 172-185.

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revised. Unlike the learning-through-activity programs of the 30's which was very flexible, the mathematics-through-activity programs of the 60's were more specific, quite often with stated objectives and corresponding manipulative devices for sale. The important ones were: the Cuisenaire rods, the Madison Project's "shoe boxes", the geo-board, and Z. P. Dienes' multibase arithmetical blocks. Numerous articles appeared in professional journals to describe these devices and the 'Mathematics Laboratory' in general. Typical were the articles by Clara Davidson (1962) on Cuisenaire Rods, Z. P. Dienes (1962), R. B. Davis (1960), W. Liedtke (1969), Patricia Davidson and A. W. Fair (1970).

Although these activities related to Mathematics Laboratory are mostly short-ranged, their usefulness in promoting the students' understanding of mathematics has been demonstrated beyond any doubt. T. E. Kieren (1969) reviewed all research works in this connection for the period 1964-1968 and reported that most of the findings were in favor of the concrete manipulation or activity type of learning.

Many recent comparative studies were carried out to evaluate the mathematical learning obtained from abstract/verbal vs. concrete/games classes. Several games or activities described were similar to those observed in USMES classes. The control group in all cases were the abstract/verbal type. F. L. Coltharp (1969) and D. Ross (1970)

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dealt with the teaching of integers to 6th graders. The 'concrete counting' of their experimental groups was similar to that observed in the Dice Design Unit. R. M. Bisio (1971) did a comparative study on fraction at Berkley, California, and he concluded that even passive use of manipulative materials was better than no use at all. This conclusion then supported the USMES position, since the knowledge on fraction which the children learned from the Soft Drink Design Unit, Human Proportion Unit, etc. was all derived from concrete situation. J. J. Bowen (1970) used games as instructional media (to teach logic), just like the games involving logical circuits, which some children played in the Burglar Alarm Design Unit. W. P. Palow (1970) researched on the ability of children (Grades 3-12) to visualize perspectives of solid figures, and his finding that such visualization was impossible for children under twelve should have direct bearing on the placement of that USMES unit which involves polyhedra construction. Several USMES units observed (in Lansing) could provide excellent opportunity for introducing elementary topological ideas to children, and E. T. Esty's study (1970), carried out at Harvard, could throw some light on the subject. He carefully distinguished between the children's recognition of 'topologically drawn' and 'geometrically drawn' figures.

The review of literature on learning mathematics via activity classes would not be complete if the present

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revolution in English primary schools is not mentioned. The 'revolution' started slowly and quietly during the 60's when Piaget's research became more well-known, and the Schools Council[1] published a paradigm account of organic growth of mathematical ideas in children, once the environment became apt. Many teachers took initiative on their own, and they were officially supported by the Plowden Report, and practically by the publication of the Nuffield Mathematics Project and the Association of Teachers of Mathematics.[2] Instead of fully relying on textbooks, chalk and talk like in the past, the British teachers are now using the following guiding principles[3] for their new approach:

(1) Children learn mathematical concepts more slowly than adults realized. They learn by their own activities.

(2) Children pass through certain stages depending on their chronological and mental ages and their experience.

(3) Their learning can be accelerated by providing suitable experiences and suitable language development simultaneously.

(4) Practice is necessary to fix a concept once it has been understood. Practice should follow, not precede, discovery.

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[1]Schools Council, "Mathematics in Primary Schools," Curriculum Bulletin No. 1, London: H. M. Stationery Office, 1965.

[2]A. T. M., "Notes on Mathematics in Primary Schools," London: Cambridge University Press, 1969.

[3]Quoted from the Plowden Report, p. 237.

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Joseph Featherstone,[1] who visited British primary schools as an impartial non-educationist observer, noted:

> "Mathematics is now an important catalyst for schools making a transition from formal to less formal teaching....Teachers provide experience and introduce children to various means of expression (communication). This takes time and a flexible schedule, so that children can see a piece of continuous work through to their satisfaction (parallel to USMES strategy). It implies an environment stocked with materials to see, listen and touch; and it suggests the necessity for varied activities to suit different tastes, talents and skills...."

In the U.K., The Nuffield Mathematics Project, since 1967, has published (via John Wiley, New York) a series of books: I Do and I Understand, Computation and Structure, Shape and Size, Graphs Leading to Algebra, Environmental Geometry, Probability and Statistics,..., to act as "guides" to the teachers and pupils alike. These are not textbooks in the traditional sense: the stress is on how to learn (organically), not on what to teach.[2]

Edith Biggs, who has been in the forefront since the start of this 'revolution', has also published a number of books in this direction. Rather than telling the teachers what to do, she usually fills her books[3] with children's actual reports, drawn graphs, number patterns discovered by children, etc.

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[1]Featherstone, Joseph, "Informal Schools in Britain Today--An Introduction," New York: Citation Press, 1971, pp. 29-30.

[2]Quoted from "I Do and I Understand," p. (i).

[3]See, (i) Biggs, Edith, "Mathematics for Younger/Older Children," (2 volumes), New York: Citation Press, 1971; (ii) Biggs, E. and McLean, J. R., "Freedom to Learn," Reading, Mass.: Addison-Wesley, 1969.

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2. <u>The literature on the Integration of Mathematics with Science, Social Science and Other Courses</u>:

William F. Russell, who was the Dean of Columbia's Teachers College during the 30's, produced the most appropriate definition of "integration" as follows:[1]

> "Integration is not just another educational slogan. It constitutes a direct answer to the profound and wide-spread disintegrations that now exist in all areas of human experience. It is the integration of experiences that educators should aim at. The end to be sought is a unified and related pattern of experience in each child. All subject-matter must become integral and truly functional in the student's growth. There must be an integrating curriculum: the study of all aspects of culture--on all its fronts, and in all its interrelationships."

Integration of mathematics with other subjects did take place extensively during this 'progressive' era. In order to convey the true spirit of teaching according to an integrating curriculum in that period, <u>details</u> of a <u>few</u> paradigm cases will be presented in this review instead of an exhaustive summarized list of articles.

Arretta L. Watts (1928) reported the following lively scene she observed in several arithmetic classes:

> "Arithmetic today is an extremely dynamic and flexible affair. Students do their calculations on automobile and airplane distances, baseball records, weather reports, railroad timetables, averages, and various scientific and technical reports, all being worked out in decimals. They also work out a formula for getting around the city, figuring out a saving account, the number of calories afforded in a plate of corned beef, or

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[1]Quoted from Russell, William F., "Education and Divergent Philosophies," <u>Teachers College Record</u>, Vol. 39, December 1937, pp. 186-188.

> making a graph showing the increase during the last five years of summer tourists to Europe. By motion pictures, the students learn that the movement of a baseball pitcher in pitching his ball included more than a hundred distinct motions. In the same way, the modern psychologist has shown us (teachers) how to chart the mental motions the mind goes through in each of the processes of arithmetic. It is not the aim in modern mathematics to stump the youth with some unworkable or trick problems, but to give him the satisfaction that he can do things successfully and the message that mathematics is useful."

Dwight S. Davis (1923) called for 'live problems' in teaching mathematics. In a long article, he suggested problems arising from many real-life sources: agriculture, mechanical arts, domestic science, banking, athletics, radio and electricity, local bus routes, building and construction, and the post office. For his own 'Algebra via Agriculture' class (which included field work), he listed the following mathematical behaviors observed: formulae in area; distribution over areas; workable formulae for sprays, feeds and fertilizers, statistical graphs on yearly production, profits, and weather; the laws of levers; formulae in Steam and Gasoline Engines; regular linear equations drawn from catalogues; percentage; friction; machine efficiency.

H. E. Slaught (1928) compared mathematics with sunshine in the sense that, without either, civilization would gradually come to a halt. Students could learn a great deal of science and social studies, as well as mathematics, if they set out, as a team, to investigate the truth of the statement "Mathematics underlies present-day civilization in much the same far-reaching manner as sunshine underlies all forms of life."

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Alfred North Whitehead, in two different books published during this period, also hinted at teaching interdisciplinary courses from such themes as "rhythm" and "order of nature." To learn the abstract (mathematical) concept of 'periodicity', he advised students to examine 'rhythms': to time heart-beats, breathing, wave-phenonmena, music, etc. He concluded:

> "Apart from recurrence, knowledge would be impossible; for nothing could be referred to our past experience. Also, apart from some regularity of recurrence, measurement would be impossible. In our experience, as we gain the idea of exactness, recurrence is fundamental."[1]

In another book, Whitehead wrote:

> "The thought organization of experience is made possible only by the conviction that there is 'order of nature', and by observing the interconnectedness of all events. This possibility of disentangling the most complex evanescent circumstances into various permanent laws (like periodicity) is the controlling idea of modern thought."[2]

E. V. Sadley (1926) described his 'pleasant approach to demonstrative geometry' as follows: Given a simple lesson in central, axial, and plane symmetry and equipped with a background of knowledge of geometric forms gained from the students' own experience, the children could, with the help of picture magazines and field trips, come up with many interesting themes for 'geometry with social science':

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[1]Whitehead, Alfred North, "Science and the Modern World," New York: MacMillan Co., 1935, p. 47.

[2]Whitehead, A. N., "An Introduction to Mathematics," New York: Henry Holt and Co., 1911, p. 11.

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--symmetry of a colonial house and a Greek temple;
--geometric forms in a Gothic Cathedral, beautiful bridges, furniture, and snow-flakes;
--symmetry in plants, flowers and trees;
--whether the primitive people knew any geometry.

One girl, who visited Europe during the summer chose the topic 'The Geometry of Sulgrave Manor,' and took the class (and the teacher) on a personally conducted tour of the ancestral home of the Washingtons to observe many examples of geometric forms and symmetry in the architecture, furniture, and landscapes of the gardens.

W. V. Lovitt (1924) used the easily observed growth and decay of natural and man-made objects to express the mathematical principle of continuity. David Eugene Smith (1928) pointed out the examples of a dependent variable (i.e., a mathematical function) in everyday life usage. "The cost of a piece of cloth depends on its length, width and quality," wrote Smith, "Similarly, all costs, expenses, weights, volumes, success, joy, etc. depend on something else." He did not elaborate how to quantify 'success' and 'joy', which the USMES children (of the Soft Drink Design Unit) have done.

The National Council of Teachers of Mathematics issued two yearbooks,[1] in 1931 and 1936, to suggest ways in which mathematics teachers could bring their subject closer to real life and other disciplines. In the 1931 Yearbook,

[1]N.C.T.M., "Mathematics in Modern Life," 6th Yearbook, 1931, Chap. 1-2, 5-10, and "Mathematics in Modern Education," 11th Yearbook, 1936, Chap. 1, 4-8.

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the chapters on the application of mathematics to social sicence, biology, and investment gave a survey of literature and techniques used in those days. Unlike the high-powered mathematics (Linear Algebra, Monte Carlo Method, etc.) which is applied to these three fields today, the methods and calculations in those days were simple and easy to follow, but are, and will always be, suitable for teaching children at the more elementary level. The possible hang-up might be what M. J. Lighthill[1] called "the language problem." The children hardly knew enough language (vocabulary, concepts, facts, etc.) in social science, biology, or investment. And it would be useless to impose verbal teaching before appropriate real-life experience. In the same Yearbook, the chapter on the application of mathematics to physics pointed out clearly the lack of the teaching of modelling or problem-formulation, as opposed to problem-solving. Again, this was due to the fact that most mathematics teachers at all levels did not know enough about the language of physics. This problem still persists even in 1973.

In the 1936 Yearbook, a whole chapter on "Mathematics in General Education" called for the teaching of mathematics as a means to facilitate language (communication), to develop the sense of consecutiveness (sequencing or

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[1]Lighthill, Sir James, "The Art of Teaching the Art of Applying Mathematics," Presidential Address to the Mathematical Association, London, Mathematical Gazette, Vol. 55, June 1971, pp. 249-270.

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ordering), responsibility (accuracy) and the ability to combine ways of thinking (application). All these were to be implemented by "relating mathematics to other fields of knowledge" which was fully explained in the following Chapter in that yearbook. Several sections of the material in this Chapter had direct bearing on the USMES units of polyhedron-construction and designing for human proportion. For instance,[1] the models for the crystals of magnetic iron, zinc, lime and salt are octahedra, tetrahedron, rhombohedron and cube respectively; any sea-shell of the ammonite type has the spiral as their outer contour; Michelangelo (1475-1564), following the ancient Roman artist Vitruvius, established the following 'Canon' (law) for drawing a human body: "If a = the total height of man, from top of hair line to sole of a foot, then, top of the head to chest = a/4, chest to upper part of thigh = a/4, upper part of thigh to knee = a/4, knee to sole of a foot = a/4, lower neck bone to chin = a/16, foot of a man = a/6, etc."

The concern for having too many cut-and-dry 'algebraic' problems in the teaching of physics was already voiced in the 1931 yearbook (of N.C.T.M.). James P. Davis (1937) spelled it out in more details and added a new concern that most physics problems were not 'real' enough. He wrote:

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[1]All examples are quoted from the N.C.T.M.'s 11th Yearbook, "Mathematics in Modern Education," 1936, pp. 218-219, 239-242.

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"One of the most important goals of physics instruction is the development of scientific thinking. First, the students should be given opportunities for discussing the meaning of the problem, reading its constituent data accurately, using symbols and known relationships of the parts, and discovering new relationships. Secondly, the problem should be in the pupil's world, not the 'brain-teaser' type which has no place in the student's life. Thirdly, it should be a real problem, a felt need, and its solution produces some satisfaction."

After a lull due to the compartmentalized traditional teaching during the War, the late forties' literature was once again teeming with activity-oriented integrating suggestions. Carrie Lou Early (1947) was a typical healthy example of the period: her students learned a great deal of mathematics from her Nutrition class. The following is part of her log, which, in several respects, was similar to Piaget's account:

"I placed a quart of milk, several large-size glasses and a pitcher on the table. Three children took turns to pour a full glass of milk, and a fourth child poured the remaining milk into a jug.

--'How many full glasses do we have?' (Three).
--'How many meals do you eat every day?' (Three).
--'Will a quart of milk give you a full glass for each meal?' (Yes).
--'Will that take the whole quart of milk?' (No, we have some left in the jug for breakfast cereal).
--'Now, let's pour the milk out for our party.'

One child counted the number of paper-cups, another estimated the number of tables required to put the cups on, others helped to pour the milk...."

Her unit also included the actual making of butter and vanilla-milkshakes in class, and a trip to the dairy barn. Apart from milk, she also utilized eggs, cereals, potatoes, green vegetables for other 'Nutrition' units.

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The National Council of Teachers of Mathematics, in 1944, appointed a commission to set up principles for building a stronger mathematics program for the post-war era. The Second Report of this commission was widely publicized. The important recommendations for Grades 1-6 were:[1] to conceive arithmetic as having both mathematical and social aims, (arithmetic is not a mere tool subject); to recognize that readiness for learning arithmetical ideas and skills is primarily the product of relevant experience; and to supplement paper-and-pencil tests with other procedures such as interviews, observation, examination of work products and use of arithmetical ideas in ordinary happenings. Such recommendations clearly pointed towards the concept of integrating arithmetic with other phases of a child's school life.

The theoretical groundwork for the integration of all educational experiences came forward in 1958 with the publication of the 57th Yearbook of the National Society for the Study of Education. Benjamin S. Bloom listed three essential means that could be used as "integrative threads":[2]

(a) Major ideas or methods, e.g., the scientific process which is common to all branches of science and social science,

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[1]Quoted from Bidwell, J. K., and Clason, R. G. (ed.), "Readings in the History of Mathematics Education," Washington: N.C.T.M., 1970, pp. 624-29.

[2]N.S.S.E., "The Integration of Educational Experiences," 57th Yearbook, 1958, pp. 97-101.

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(b) Major problems whose desired solutions, whether short-ranged or long-ranged, demand a careful study of several disciplines and their interrelationships,

(c) A connected theory, or encyclopedic organization to unify, often by a higher level of abstractness, several seemingly divergent topics. (This has, traditionally, been an accepted form of applied mathematics, where scattered facts observed or measured from various topics in physical science or economics were accounted for by an elaborate deductive system stemming from only a few well-chosen postulates.)

By late 50's and during the 60's, several integrated projects appeared, relying mainly on the integrative threads (1) and (2) above. The important projects, which were long-ranged, were:

--Man: A Course of Study,[1]

--The Minnemast Project,[2]

--The Man-Made World,[3]

--this USMES Project,

--The 'Environmental Studies' Project[4]

and, more recently, the People and Technology Project.[5]

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[1]Initiated by Bruner, J. S., in "Toward a Theory of Instruction," Cambridge: Harvard University Press, 1967, pp. 73-101.

[2]Minnesota School Mathematics and Science Teaching Project, 1963.

[3]Engineering Concept Curriculum Project, Polytechnic Institute of Brooklyn, New York, 1968.

[4]Initiated by the American Geological Institute at Boulder, Colorado; typical activities were described by its director, Samples, Robert E., "Environmental Studies," Science Teacher, Vol. 38, October 1971, pp. 36-37.

[5]Directed by Peter B. Dow, Education Development Center, 1972.

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In fact, teaching a more or less integrated course has always been an appealing concept for many teachers, but the above projects have represented, for the first time, a massive organized effort instead of the usual piecemeal one-man crusade. Examples of these individual efforts were:

--A. G. Doherty and L. A. Shaw, (1943) who integrated mathematics with music,

--Earl Murray (1944) who assigned students to groups to analyze the assumptions, intermediate deductions, and conclusions of statements drawn from advertising, economics, and governments under the long-ranged headings: personal living, economic living, and social-civic living,

--Gertrude D. Hillman (1958)--who taught the technique of data-gathering, adding, counting, percentage, and graphs from a school-wide polling on "What is your favorite dessert?" This was very similar to the 'Soft Drink' Unit of USMES,

--James H. Humphrey, (1967 and 1972), who used Physical Education as a learning medium in the development of first mathematical, then science concepts with 2nd and 5th graders,

--Dorothy Amsden and Edward Szado (1958), who combined arithmetic with the unit 'Fish' in biology using real specimens in an aquarium. "How many fish will the aquarium accommodate?"--"Well, first let Mary measure the dimensions of the aquarium while I measure those of the fishes...." Students also drew feed charts, graphs of the temperatures of the tank, the room and outside. They also learned a great deal of biology of fishes.

--J. Richard Suchman (1964) who regarded elementary science primarily as an 'inquiry training.' His program at Illinois emphasized both on the focusing points of children's attention and on the freedom to branch out their own investigations. This writer was famous for setting up apparent 'discrepency' in science classes, and asking the children why.

--John C. Archbold (1967), who integrated graphs with geography. The class made cylinders whose heights were proportional to some geographical data (population, mineral resources, etc.) and placed the corresponding cylinders at the correct places on a map.

--Henry Lulli (1972) who integrated art with geometry by guiding the students to construct polyhedra. This was very similar to some classes of the USMES 'Dice Design' Unit.

Although most of these integrated projects, organized or individual, specified the appropriate grade level to try out, closer examinations of the materials and methodology have strongly suggested that a continuous non-graded approach might be more suitable. As a matter of fact, this non-graded approach is even independent of readability and mathematical skill, because a genuinely interesting

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project will always be a strong motivation to acquire the needed skills in reading and mathematics.

The 'new' science education in the 60's moved considerably away from scientific facts and information to the process of scientific enquiry, which, in effect, meant coming nearer to mathematics, because scientific enquiry involves inventing a hypothesis, collecting data, experimenting, graphing results, drawing inferences, etc. which demand considerable mathematical skill. The New Science is also more akin to social science now as the latter begins to focus on the process of inquiry rather than just the finished product.

All the three successful programs in science education: Elementary Science Study, Science Curriculum Improvement Study, and Science--a Process Approach, have science topics thickly intertwined with mathematics components. The ESS has units like tangrams, attribute games and patterns, mirror cards, etc. which are essential mathematics. The SCIS has units like 'position and motion', 'relativity', 'subsystems and variables' which are mathematical in character. The SaPA[1] of AAAS overtly includes the following units of mathematics: using numbers, measuring, using space-time relationship, communicating (by charts, graphs, etc.), interpreting data and infering.

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[1]For a full discussion of the mathematics component of SaPA, see Mayor, John R., "Science and Mathematics in Elementary School," Arithmetic Teacher, Vol. 14, December, 1967, pp. 630-632.

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The 'new' mathematics program of the English primary schools is interdisciplinary in nature. Mathematics is no longer taught as an isolated subject, but it is always mixed with art, dancing, music, recipes, needlework, stories, etc.[1] The Nuffield Mathematics Project's first and foremost aim has been to integrate the children's mathematical experience with other aspects of school life. For example, geometry is to be learned from having visual and tactile contact with "Shape and Size," graphs and probability are to be studied as ways of communications in social studies and science. Many more examples of integrative experiences were described in the book written by Leonard Marsh.[2]

Several research studies were carried out to evaluate integrated or partial integrated programs. Only the more recent and typical ones are included here. W. J. Coffia (1971) did a detailed study on the mathematics outcomes of SCIS vs. regular science teaching program. For five consecutive years, the 'SCIS' students scored higher on mathematics application, but the 'regular' students did better on concepts and reasoning. W. L. Gray (1970) reported a significantly positive effect of an integrated program on the acquisition and retention of mathematics

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[1]See, e.g., Biggs, Edith, "Mathematics for Younger Children," New York: Citation Press, 1971, pp. 22-28, 34-37.

[2]Marsh, Leonard, "Alongside the Child: Experiences in the English Primary School," New York: Praeger, 1970.

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and science behaviors for 5th graders. Betty Willmon (1971) discovered a total of 473 technical words of mathematics hidden within 24 non-mathematical textbooks of grades 1-3.

As for the integrated activities whose main component was elementary science, and not mathematics, Mary B. Rowe (1965), did a large-scale experimental study with an elaborate design on selected task-oriental science programs. Her finding was that there was some indication that the accumulation of experiences in various contexts tended to increase the probability that a cognitive network would form.

In 1967, a colloquium on "How to teach Mathematics so as to be useful" was held in Utrecht, Netherland, bringing together applied mathematics educators from all over Europe, U.S.S.R. and U.S.A. In the opening address, Hans Freudenthal[1] said, in part,

> "What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematicizing reality....Take, for example, fraction. In its traditional teaching, the concrete context is no more than a ceremony which is hurried through, and then the abstract theory (of rational number) is applied. The result, intermediate or final, is not connected back to any previous experience on a level which fractions have been introduced."

He, in another part of his speech, commented on the narrowness of classical applied mathematics. His remark was:

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[1]Freudenthal, Hans, "Why to teach mathematics so as to be useful," <u>Educational Studies in Mathematics</u>, Vol. 1, May, 1968, pp. 6-7.

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[1] Polla
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"It is a fact that biologists, economists, sociologists are better prepared to apply modern mathematics than physicists who carry the burden of a longer tradition (of using calculus alone)."

This remark was in full agreement with Henry O. Pollak, who wrote:[1]

"The traditional picture of the user of mathematics as a man who looks up a formula in a handbook and turns the crank is very much out of date. When one applies mathematics in practice, the problems which arise are not exactly like the problems in the book. ...Applications typically begin with an ill-defined situation outside mathematics--in physics, engineering, biology, economics, or almost any field of human activity. The job is to understand this situation as well as possible, and make a mathematical model. One has had to know his mathematics previous to this point, and, more important, to know the conditions for his success. One can then examine what is required to make it fit the new situation (model)."

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[1]Pollak, H. O., "Application of Mathematics," N.S.S.E. 69th Yearbook: Mathematics Education, 1970, pp. 328-329.

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3. The literature on the philosophical and psychological aspects of an activity-oriented and integrated curriculum like USMES:

This type of curriculum is based on two philosophical points of view: Naturalism and Pragmatism. The first view holds that all knowledge boils down to nature study. Homo sapiens is part of nature. Man must give up his longing for assistance from supernatural sources, and must solve his own problems--drought and flood, disease and hunger, etc., by 'scientific' method: making hypothesis, experimentation, verification, and inferences. Naturalism also stresses the importance of an 'open' person (one who welcomes new knowledge and challenges) as opposed to a 'closed' person (one who looks askance at new knowledge as a threat) because knowledge, like nature, grows and changes. Pragmatism views knowledge as an awareness of continually changing relationships not only between man and his physical environment, but also between man and society. Changes bring new truths. Knowledge is "true" when it solves "practical" problems in society and in the physical world. Seeking new knowledge and integration of existing knowledge are never-ending tasks.

These two philosophical points of view were put forward forcefully early this century by people like Santayana,[1] Russel, Poincare and Dewey. But, going back

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[1]Santayana, George, "Reason and Common Sense," New York: Charles Scribners and Sons, 1905.

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in history one would find that earlier philosophers like Hobbes, Locke, Hume and Kant, expressed essentially the same ideas. For knowledge of any kind, Hobbes placed emphasis on the purely physical character of sensation and the action of the brain. Locke asserted that there were no innate ideas in men; experience, through the sense organs, produced simple ideas (figures, numbers, etc.) in the mind, which then united simple ideas into complex ones. Knowledge concerned the connection of ideas, such as, their agreement or their inconsistency when combined. Hume perceived impressions (sensations) and ideas (images and memories) as knowledge, but he added that ideas were really effects of impression. Kant was the only one who distinguished between knowledge obtained directly from experience and knowledge somehow obtained by the mind independently of experience. These two kinds of knowledge have eventually developed into inductive and deductive methods of learning.[1]

So one could easily conclude that philosophers, past and present, have all favored experience and sense impressions as a means to gain knowledge. Verbalized statements without appropriate experiences are, in fact, not knowledge. The existentialists go even one step further: they demand

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[1]For a more expounded account of this paragraph, see, e.g., Horne, Herman H., "Philosophy of Education," London: McMillan, 1924.

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a completely open and free experience. R. G. Olson[1] put it this way:

> "An education which intensifies awareness is what an existentialist would strive for. It refers to the awareness of that subjectivity--known publicly as Johnny sitting presently in this classroom, but existing pour soi in another world in which he wants to live his own life. It refers to his awareness of his precarious role as a baseless chooser who cannot escape choosing, and therefore who must create his own answers to all questions and problems in or out of his classroom...."

The USMES's insistence on long-range projects is very much in line with Whitehead's thinking:

> "Progress in the right direction is the result of a slow, gradual process of continual comparison of ideas with facts. The important criterion is that we should be able to formulate empirical laws connecting various parts into conceivable interrelated whole...."[2]

"Try it and see if it works" is a familiar expression heard so often in an USMES class. John Dewey always advocated such a philosophy of education, as seen from the following quotation:

> "To prove a thing means primarily to try, to test it. Not until a thing has been tried out do we know its true worth. Till then it may be a pretense. The thing coming out victorious in a test carries credentials with it. So it is with inferences, too. Every inference shall be tested. We shall discriminate between beliefs that rest on tested evidence and those that do not...."[3]

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[1]Olson, Robert G., "An Introduction to Existentialism," New York: Dover Publication, 1962, pp. 17-18.

[2]Whitehead, Alfred North, "The Aims of Education," New York: McMillan Co., 1929, pp. 156-157.

[3]Dewey, John, "How We Think," New York, D. C. Heath and Co., 1910, p. 27.

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The psychological base of such a program as USMES will now be reviewed. Gestalt psychology tended to fit in with this approach more than other schools of psychology. It explains learning in a large reference frame in terms of goal, insight, and maturation.[1] The 'goal' corresponds to the interesting realistic challenge or sub-challenge of USMES. The learning takes place, not through exercises, but through insight, and the neurological changes are not modifications in definite synaptic connections (Thorndike's word) but are differentiations of maturation patterns conceived as energy patterns within the total psychological organism. The progress is made through the constant interpretation of the individual with his environment field. The learning, dynamically considered, manifests itself in organized thinking and not in isolated bond.

John Dewey seemed to supply USMES not only with a philosophical base(with his emphasis on experience, activities, child-centered instruction, etc.), but he together with McLellan, also supplied a psychological base for the mathematical aspect of an USMES class. They wrote:[2]

> "The two methods: Symbols and Things. The first method is teaching numbers merely as a set of symbols, leaves out the objects entirely, or at least makes no reflective use of them. It lays the emphasis on symbols, never showing clearly what they symbolize, but leaving it to chances of future

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[1]Dickey, John W., "Arithmetic and Gestalt Psychology," Elementary School Journal, Vol. 39, September 1938, pp. 47-50.

[2]N.C.T.M., "Reading in the History of Mathematics Education," 1970, pp. 158-159. (The article by Dewey and McLellan).

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experience to put some meaning into these empty abstractions. The second method treats numbers as tangible properties of objects, and nothing more. It neglects the mental activity which uses them. It also makes numbers meaningless.

Both methods are vitated by the same fundamental psychological error: they do not take account of the fact that number arises in and through the activity of the mind in dealing with objects."

What Dewey and McLellan meant was, in fact, what is now called "learning-through-discovery," which is one of the most important principles for a program like USMES. Lee S. Shulman explained the psychological aspect of discovery learning as follows:[1]

"The child finds in his manipulation of the materials regularities that correspond with intuitive regularities he has already come to understand. It is rarely something outside the learner that is discovered, but the discovery involves an internal re-organization of the previously known ideas in order to establish a better fit between those and the regularities of an encounter to which the learner has had to accommodate.

This is precisely the pedagogical psychology attributed to Socrates.[2] Socrates maintained that he was not teaching the student anything new, but he was merely helping the boy to reorganize his thought and bring to fore what he had always known.

More recently, Tallmadge, Kasten and Shearer (1971) did a comparative study on the inductive vs. deductive

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[1]N.S.S.E., "Mathematics Education," 69th Yearbook, Chicago: University of Chicago Press, 1970, p. 28. (The article by L. S. Shulman).

[2]See Jones, Phillip S., "Discovery Teaching-From Socrates to Modernity," The Arithmetic Teacher, Vol. 17, October 1970, pp. 503-510.

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methods of instruction for two kinds of subject-matter contents: Content I based on mathematical and logical rules (the 'Transportation Technique' Unit) and Content II on arbitrary rules (the 'Aircraft Recognition' Unit). The subjects were Navy enlisted men. Their preliminary findings were that the inductive method was significantly better than the deductive method for Content I ($p < .01$) and that there was no statistically significant difference for Content II. Other learners characteristics (attitude toward such technical units, anxiety level, being introverts or extroverts, etc.) were controlled in this preliminary experiment.

The battle "inductive vs. deductive teaching" has always been (and will remain) in the pedagogical world. Probably it would be best to go back and seek some wisdom from an old, experienced and highly competent teacher, J. W. A. Young. His success was really based on a mixture of both inductive and deductive technique. He listed his pedagogical principles[1] as follows:

1. Mathematics should grow.
2. Experimental (concrete) origins or preliminaries are necessary.
3. Pupils must abstract their own mathematics.
4. A moderate amount of drills, wisely administered, should help.

The word 'drill' used by Young in those days would correspond to the term 'simulation' of today. The following

[1]Young, J. W. A., "The Teaching of Mathematics," first published in 1924, reprinted in The Mathematics Teacher, Vol. 61, March 1968, pp. 287-295.

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example of Young's drill-exercise shows clearly that it is, in fact, learning by simulation:

> "On quadratic equations, the pupils should be led from the solution of
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> $$x^2 + 2x + 1 = 16,$$
>
> via that of $x^2 + 8x + 16 = 2,$
>
> and that of $x^2 + 2ax + a^2 = b,$
>
> to the solution of $x^2 + 2ax = b,$
>
> and hence that of any equation of this type."

The psychological base for an integrated curriculum like USMES was essentially the principle of 'organizing experiences.' The 57th Yearbook of the National Society for the Study of Education contained the following deliberations in David R. Krathwohl's article:[1]

> "Integration can be defined as 'organizing experience,' that takes place in the learner's mind. It is an important factor in learning. Without such organization, learning would only be an unrelated sequence of perceptions analogous to the snapshots recorded on camera films. Application or retention would be impossible."

Integration of several subject-matter fields have been psychologically analyzed by George A. Miller[2] in a most interesting way. He analyzed the cognitive operations in terms of Information Theory, which is a mathematical model for the process of communication. Among other things studied in this model is the abstraction of 'essence' or

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[1]N.S.S.E., "The Integration of Educational Experiences," 57th Yearbook, Chicago: The University Press, 1958, p. 45.

[2]Miller, George A., "Information and Memory," Scientific American, Vol. 195, 1956, pp. 42-57.

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'information' contained in a certain message. For example, the multiple-choice question

"Pilot: parachute as boiler: ----
(1) safety valve, (2) fire, (3) pressure gauge, (4) steam whistle" implies the same relationship between appropriate pairs, rather than aiming at the meanings of individual words instructed, students could learn from such a test many topics from many disciplines in a truely integrated manner.

The British are now teaching integrated mathematics, science, art and reading in 'open' classes, somewhat similar to the USMES classes observed. They often refer to Z. P. Dienes when they discuss the psychological base of their present work. It would be appropriate to examine Dienes' thinking about an activity-oriented curriculum. The following excerpt shows how Dienes connected play with mathematical learning:[1]

> "Much has been written about play and its function in a child's life. We are concerned with the function of play only as it affects the cognitive process of mathematics learning. A child usually begins with exploratory-manipulative type of play. He enjoys the feel of objects, running his fingers down the surfaces, putting things together and taking them apart. He then gradually becomes aware of the observed (and abstract) properties of the materials, asks questions about them, and generally moves up along an awareness time curve. So it is unlikely that the play will remain purely manipulative for long. He will proceed on to the two next stages: representational play, and search for regularities.

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[1]Dienes, Z. P., "An Experimental Study of Mathematics Learning," London: Hutchinson, 1963, pp. 21-23.

Representational play adds another component into the activity: imagination. There may be some attempt to re-shape the objects so that the result resembles the thing imagined but often it is pure representation where objects stand for other things by definition or assignment. In another direction, manipulative play may quite imperceptibly move over to a search for regularities or even the abstract rules governing them. Children are delighted in regularities. The formulation of a rule-structure is a kind of 'closure' which ties up all the past experience. Once the regularities and the rule-structure have been familiarized, they become part of the play. In the next stage, the play opens with the rules themselves acting as objects of manipulation, and it is now at a higher (intellectual) level."

A brief review will now be given about the literature on the case-study type of research conducted in the field of mathematics education. Jean Piaget was one of the first men to do this type of research and his work has been so widely reviewed that it is almost unnecessary to repeat here. The depth of his case-study results could be seen from the following paragraph:[1]

"The last stage of development of intelligence is the stage of formal operations. The child becomes capable of reasoning not only on the basis of objects, but also on the basis of hypotheses or of propositions. Here a class of novel operations appears which is superimposed on the operations of logical class and numbers. The first novelty is a combinative structure; somewhat like a mathematical structure. It is superimposed on the structure of simple classifications. A second novelty is the appearance of a structure which constitutes a group of transformations: reversibility by inversion, reversibility by reciprocity, (e.g. $A > B \Rightarrow B < A$), and a combination of both into a single but larger structure."

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[1]Piaget, Jean, "The Stages of the Intellectual Development of the Child," in Kramer, Klaas, Problems in the Teaching of Elementary Mathematics, Boston, Mass.: Allyn and Bacon, Inc., 1970, pp. 57-65.

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K. Lovell and his associates carried out almost 10,000 Piaget-type experiments. Lovell's book[1] testified the value of this type of research. Barbel Inhelder wrote the following in the Foreward of this book:

> "Dr. Lovell does not accept any of our (Piaget's and hers) findings or take over anything from our work without the most scrupulous checking. We were a little worried when we waited for the results of such verifications, because they dealt with researches that were carried out in a rather different school setting. It is reassuring to see that Dr. Lovell's results and our own agree perfectly."

Lastly, the reports of the case-study type researches carried out by the Soviet Academy of Pedagogical Science during the last 25 years points to the important role of this kind of work. Commenting on their success, Jeremy Kilpatrick and Izaak Wirszup[2] wrote:

> "A major difference between the Soviet and American educational research is that the former has used qualitative rather than quantitative methods. American readers may find the Soviet papers do not comply exactly to the U.S. standard of design and statistical analysis. But, by using qualitative method, the Soviet psychologists have been able to penetrate into the child's thought and to analyze his mental processes which the quantitative method seldom achieves."

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[1] Lovell, K., "The Growth of Basic Mathematical and Scientific Concepts in Children," London: The University Press, 1965, p. 5-6.

[2] Kilpatrick, Jeremy and Wirszup, Izaak, (ed.), "Soviet Studies in the Psychology of Learning and Teaching Mathematics," Chicago: University of Chicago, 1969, pp. iv-v.

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CHAPTER 3

CATEGORIZING THE MATHEMATICAL BEHAVIORS DERIVABLE FROM FOUR 'USMES' UNITS

Many topics in the usual mathematical curriculum arise naturally in the course of carrying out the USMES activities. This case-study is the first attempt to record and classify all the mathematical behaviors observed in the USMES units of "Soft Drink Design, Dice Design, Designing for Human Proportion and Burglar Alarm Design." During the course of nine weeks' intensive continual observation (September-December, 1972), the writer of this Case-Study realized that many potential mathematical topics could have been fruitfully discussed if the teacher had had an appropriate check-list in her class, and thus branched out into activities or discussions involving those sophisticated mathematical ideas at the most opportune moment. These potential mathematical topics will be marked ** in this Chapter. No doubt another observer might have noticed some other potential mathematical topics and suggested a different list. The main purpose of this Chapter is to provide teachers with paradigm cases and a suggestive list demonstrating how to teach mathematics from a seemingly non-mathematical unit like Soft Drink Design. It is not the purpose of the writer, nor anybody connected with USMES, to expect uniformity in all USMES classes.

Details of the mathematical behaviors observed in the above-mentioned USMES classes will be given in Chapter 4. In this Chapter, the mathematical behaviors of each Unit, together with its <u>potential</u> mathematical lessons marked ** will be listed under the following headings:

(1) Arithmetic,
(2) Algebra,
(3) Graph and Tabulation,
(4) Geometry, Topology, and Trigonometry,
(5) Application and Practical Mathematics,
(6) Statistics and Probability,
(7) Foundation of Mathematics.

THE SOFT DRINK DESIGN UNIT

(1) <u>Arithmetic</u>: Calculations (addition through division) involving whole numbers, decimals and fractions which arose from the measured quantity of sugar, Kool-Aid powder, soda-water, etc. Detailed study of the 'Long Division' process, e.g. $33\overline{)3225}$. Multiplicative inverses, e.g., to put in $\frac{1}{3}$ spoonful of sugar three times is equivalent to 1 spoonful). Units of measurement (spoonful, standard cup, fl. oz., etc.) Place value** (9 cups = 1 pitcher, etc.) Percentage (e.g. to calculate the percentage of sugar present in a sweetened packet of Kool-Aid.) Percentage-increase** (when the price of a can of soda-pop increases from 10 to 15¢.) Estimation (of sugar, soft-drink powder and liquid.)

(2) Algebra: A variable amount (of sugar, water, etc.) vs. a fixed amount. Independent and dependent variables (Popularity of a drink depends on taste, color, etc.) Composite functions (e.g. popularity depends on taste, and taste depends on the amount of sugar). An arbitrary system of assigning scores or marks to one's favorite drinks. Weighted scores. Matrices (e.g., defined by raters/drinks rated, or drinks/1st, 2nd, 3rd ...choices). Reduction of the size of a matrix (by combining several similar flavors into one flavor). Frequency of being voted 1st, 2nd, 3rd... choices when the drinks were polled. Algebraic problems in one unknown (e.g., to find the amount of Kool-Aid powder in a sweetened packet). The Number Line** (showing the scores obtained by the 1st, 2nd, 3rd...drinks). An algebraic method** showing that change of grading systems did not affect the relative ranking of the drinks.

(3) Graphs and Tabulations: Tabulation of experimental results in mixing drinks, and the results of students' voting for their favorite drinks. Tabulation of the blindfold tests' responses.

(4) Geometry and Topology: Circular right cylinders (the soda-pop can). Curved surfaces** of various bottles. The circle, diameter, circumference, and the number π . (Measuring the circumference and diameter of a cylindrical can). Simple closed curves** found on various patterns in bottle-designs. Between-ness** (exhibited by various scores' points

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on the Number line). Change of grading systems which corresponded to translation** and shrinkage of scales.** Invariance produced by such transformations** (i.e. translation and shrinkage of scales did not affect 'between-ness' or relative ranking of the drinks.)

(5) Application and Practical Mathematics: Experience in measuring sugar, soft-drink powder, water, etc. Rate** of flow of liquid and measuring time** with a stop-watch. Conjectures. Proposing hypotheses and testing them, (e.g. one boy proposed: $\frac{1}{2} + \frac{1}{3} \simeq \frac{2}{5}$ but later found out that $\frac{1}{2}$ cup of water + $\frac{1}{3}$ cup of water $\neq \frac{2}{5}$ cup of water). Using the definition to derive something else, (e.g. using the definition of "soft-drink" to design a questionnaire for a school-wide polling of soft-drinks).

(6) Statistics: Polling. To quantify the "popularity" of a drink by counting votes or total scores assigned by people. Representative samples. Random selection** of sample population. Stratified sampling. To control some variables (e.g. to control the independent variable 'look' by blindfold tests). Confounding variables, (e.g. the independent variable "color of a drink" is confounding with the independent variable "amount of water added"). Irrelevant variables (e.g. the color of the measuring spoon). Voting: arrangement of scores in ascending or descending order. Mean scores.** Discussion of whether the experimental results (mixing drinks, polling, etc.) were partly due to chances.

(7) Foundation of Mathematics: Ordinal and cardinal numbers (1st, 2nd, 3rd...choices are ordinal numbers, while the scores are cardinal numbers). Reversibility of orderings, (e.g. 1st, 2nd, 3rd...choices become the scores of 7, 6, 5,...points). Two mutually exclusive sets (e.g. soft-drinks vs. non-soft-drinks, to guess right vs. wrong in the blindfold tests), and the introduction of the binary digits (0, 1). Discussion of 'equality' in mathematics, (e.g. drink 'a' = drink 'b' as far as popularity was concerned, but their other attributes might be different). Logic: negation (non-soft-drink), implication (more Kool-Aid powder ⇒ deepened color). The continuum in a real model, (the color of lime soda changed continuously from emerald green to light green while water was slowly added.) Sets and their elements: defining the set of non-soft-drink by enumerating its elements, and then abstracting the properties of this set afterwards. Intersection of two sets (e.g. the common properties belonging to both orange soda and strawberry soda).

THE DICE DESIGN UNIT

(1) Arithmetic: Various methods of counting and tallying the scores of coin-tossing games, e.g., recording in groups of 5 strokes, or marking each square of a graph-paper to mean 5 or 10 points. Place-value (A boy used △ for 5, and □ for 10 strokes). Practices on multiplication when counting up the groups of 5 strokes each. The important concept that multiplying by an integer implies repeated

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additions, (e.g. $5 \times 3 = 5 + 5 + 5$). Addition for a long column of numbers. Commutative Law (adding backwards). Closure Law (The sum of 2 integers is still an integer). Partial sums or sub-totals. Mental arithmetic involving division by 2 (One half of 1482 is 741). Approximation, (e.g. $\frac{745}{1482} \simeq \frac{1}{2}$). Relative Error** (e.g. $\frac{741}{1482}$ and $\frac{745}{1482}$ differ only by 1 part in 371). Distributive Law, e.g. $19 \times 10 = (10 \times 10) + (9 \times 10)$ when counting tallies arranged in groups of ten.

(2) Algebra: Ideas of increasing or decreasing expressed by integers (e.g. the points gradually won or lost in any game). Negative numbers** for counting points lost or penalties. A set of consecutive integers. Arithmetical Progression,** e.g. 7, 13, 19, 25... read from the concentric circular track of a game (See Figure 4.9). The arithmetical mean** and the subsequent derivation of the algorithm $\bar{a} = \frac{1}{n}(a_1 + a_2 + a_3 + \ldots + a_n)$ for any set $\{a_i\}$. The combination ${}^{8}C_2$, ${}^{n}C_2$ (finally generalized to ${}^{n}C_r$**) when discussing a tournament for 8 or n competitors. (This last formula may be more suitable for the High School level.)

(3) Graphs and Tabulations: Tabulation of H-scores and T-scores obtained by the whole class. Bar-graphs showing each pair (H,T) as well as the pooled H vs. the pooled T. Mean values read from a set of bar-graphs put together. A histogram drawn from the actual data recorded from exactly 50 throws of coin-tossing. The histogram as

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an approximation to the Normal Curve.** Introducing co-ordinate axes,** abscissa, ordinate, and ordered pairs. A skew Normal Curve obtained by plotting 'frequency of H' against H itself in the above data. Locating the maximum frequency from this (skew) Normal Curve.**

(4) Geometry:

(a) Observed from the activities of children who played a horserace game: (See Fig. 4.9) Concentric circles (their circular 'track'), radii, sectors, annuli, chords, the inscribed hexagon consisting of 6 equilateral triangles. The internal and external angles of a hexagon.** The congruence** of two triangles (by paper-folding). The theorem: $\frac{\text{Arc length}}{\text{radius}}$ = constant, and hence defining "radian" as a unit for measuring angles.**

(b) Observed from the activities of children who constructed polyhedra from pre-cut polygon during the August Workshop: Symmetry. The pre-cut polygons: equilateral triangles, squares, rhombuses, pentagons, hexagons, and octagons, and their simple geometrical properties obtained by tessellation, or direct measurements. Convex polyhedra: the cube, tetrahedron, duodecahedron, icosahedron, and octahedron. Intersection of two planes (an 'edge' of a polyhedron), and the angle** between them. Three or more planes defining a point** (the 'corner' of a polyhedron).

(5) Application and Practical Mathematics: Use of an abacus** for tallying as well as adding. Keeping two records to show "points won" as well as "points lost" for

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checking the correctness of record-keeping similar to the 'double-entry' book-keeping. Principle of book-keeping.** Word-problems about budgeting.** Counterclockwise (circular) motion and complete revolutions (360°) studied via the experience on the circular track (Figure 4.9).

(6) Statistics and Probability: Theoretical probabilities of the coin-game (1/2 for H, 1/2 for T), and of the lot-game consisting of 3 straws (1/3 for the 'short' straw, and 2/3 for a 'long' straw). Theoretical and experimental results compared, and discrepency computed in terms of relative error.** Equally likely events. $P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1)P(A_2)P(A_3)P(A_4)$ for the independent events of throwing 'heads' for 4 consecutive times. Normal Distribution inferred from the constructed histogram. Probabilistic model exemplified by the histogram. Discussion of the Deterministic model.

(7) Foundation of Mathematics: Two mutually exclusively sets (H vs. T). Complement of a set (e.g. complement of element H in the set $\{H,T\}$ is T). Generalization from "get--don't get", "won--lost" etc. into $\{H,T\}$. Ordinal numbers utilized in keeping tallies, and cardinal numbers in totalling. Idea of equality viewed from the fact that points won by one side equals points lost by the other side (Reflexive Law). A counter-example of the Transitive Law: in a certain game, A beats B, and B beats C; does A necessarily beat C?--(NO!) Symmetry as an example of a necessary, but not sufficient, condition for a fair die.

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as a means to study A.P.**

THE 'HUMAN PROPORTION' UNIT

(1) <u>Arithmetic</u>: Units of measurement (first, the egocentric units: hand, elbow-length, the child's own body; then the scientific units: inch, foot, meter, etc.) Accuracy of measurements (correct to the nearest 1/16 in., or nearest millimeter). Approximation (e.g., height of a drinking fountain ≈ 2/3 of a child's height, etc.) Estimation (e.g. estimating the extra heights due to wearing shoes). Many opportunities to work on long-division or fractions, e.g.,

25)76'197½" when calculating the mean-height of 25 students;

$1\frac{3}{4} - 1\frac{11}{16}$ when calculating the error due to measurement;

$(7/16)(1\frac{1}{2})$ when enlarging a photo $1\frac{1}{2}$ times, etc.

Also, calculations involving the concept of <u>ratio</u> or <u>proportionality</u> when building a scaled model of the school on a piece of tri-wall board.

(2) <u>Algebra</u>: A linear function $y = \frac{3}{2}x$, when x referred to any length measured in the smaller photo, and y the <u>corresponding</u> length in the enlarged photo. Also, a non-linear function exemplified by the verified fact that human height is not a linear function of age. (A 10-year-old child is not twice as tall as a 5-year-old.) Introducing arrays of data in the form of a matrix.

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(3) Graphs and Tabulations: Twenty-five kindergarteners' heights neatly tabulated as a 5x5 matrix, by pre-arranging the children into 5 groups; each group consisting of 5 children. Identifying each child by an ordered pair (a,b) where 'a' referred to group-membership, and 'b' individual number within that group, i.e., the child (1,2) was not the same as the child (2,1). Experience in drawing pictorial graphs and bar-graphs from such data as: the 25 kindergarteners' heights, the waist-sizes of 22 fifth-graders, etc. Introducing the Cartesian Co-ordinate grid to facilitate the drawing of pictures from rough drafts or given photographs. Playing the game of tic-tac-toe.**

(4) Geometry and Trigonometry: The concept of an orthogonally projected height (i.e. the length between two points obtained by orthogonally projecting the highest point of the hair and the lowest point of the heel on to a vertical plane.) Such projections carried out by placing two horizontal planes (card-boards) touching the said highest and lowest points, and being perpendicular to the vertical scale. 90° rotation of the two horizontal planes and the vertical scale in order to measure projected width of a human face. Measuring a curve-length (e.g. waist size, by a tape or a straight ruler rotating as a variable tangent to the curved waist-line). Idea of an envelop or line-conic.** Difficulty involved in projecting sets of points

on a curved surface (e.g. the sets corresponding to the organs in a human face) on to a plane (a flat sheet of drawing paper.) Sterographic projection.** Study of similar figures** from two photographs, one being the enlarged copy of the other, and a ratio of 3:2 in linear measurements ⇒ 9:4 in area measurements. Experience in drawing parallel and perpendicular lines for the Cartesian Co-ordinate grids. Simple trigonometry for calculating the height of classroom ceiling. Using the slopes of right-angled triangles' hypotenuses to indicate the relative 'fatness'. Introducing P $\tan^{-1}(y/x)$ as another way to measure angles,** when P refers to the principal value.

(5) Statistics: Collecting data about the kindergarteners' heights, the lengths and widths of 5th graders' faces, and various measurements of a 5th grader's body. Calculating the mean, median, mode and range from these data, and comparing the mean values of their leg-lengths, arm-lengths, trunk-lengths, etc. with the measurements of classroom furniture and fixtures to find out why some furniture and fixtures caused inconvenience to persons of children's size. Writing up a physical profile of a 5th grader.**

(6) Application and Practical Mathematics: Measuring by means of a foot-rule, yard-stick or a tape measure. Building a scaled model of a school (not completed). Learning to make allowance when measuring a human body for the purpose of making a garment (e.g., collar size should be slightly bigger than neck-size.) Curve-stitching**

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(e.g., making a hyperbola, a parabola, or other line-conics.) Making simple furniture from wood and tri-wall to fit a 5th grader's size.** Verifying (by means of a steel tape measure) the ceiling height previously calculated from trigonometry tables.

(7) Foundation of Mathematics: An example of upper limits: "The library shelf should be at most 5 ft. high." Discussion of some lower limits** such as: the speed of 50 m.p.h. on a freeway, 18 years old for being eligible to vote, etc. As mentioned in (2) above, the use of ordered pairs (a,b) with $1 \leqslant a \leqslant 5$, $1 \leqslant b \leqslant 5$ to identify 25 children. Many-to-one correspondence when mapping these ordered pairs to the real numbers representing the heights of these children.

THE BURGLAR ALARM DESIGN UNIT

(1) Arithmetic: Counting the number of turns when making an electromagnet. Practice on fractions and decimals:

(Ex. To wind 20 ft. of (insulated) electric wire around a nail of diameter 1/4 in., it required $\frac{20 \times 12}{(3.14)(1/4)} = 305.7$, or roughly 306 turns. Further, if these 306 turns of winding produced a solenoid of 5", then the diameter of this electric wire $= \frac{5}{306} = .01634$", whose order of magnitude could not be measured by a ruler.) The powerful method of arithmetical deduction from indirect measurements, as demonstrated above. Scientific notation, e.g. 1.634×10^{-2}. Positive and negative exponents.** Discussion of possible error, (

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error, (e.g., in the above example, there was a discrepancy of .0004" when checked against a standard Table which gives d = .01594",) and ways to improve the accuracy of such work. Informational aspect of integers, e.g. #26 wire; 0 means 'off' and 1 means 'on' in the logical circuit. Estimation, e.g., obtaining a wooden block of comparable size to that of a battery, when constructing a battery-holder.

(2) Algebra: Algebraic equation of one unknown, e.g. $(3.1416)d = \frac{3 \times 12}{92}$. (This work included both the formation and the solution of the equation.) Inverse proportion, e.g. current $\propto \frac{1}{\text{resistance}}$. The Inverse Square Law. Permutation of 5 elements in a circuit diagram: + pole, - pole, chemical solution (of the battery), bulb and switch. Combination ${}^{n}C_2$ arising from all possible connections, e.g.,

bottom of bulb connected to bottom of battery,
bottom of bulb connected to curved side of battery,
threaded part of bulb connected to top of battery,
etc.

(3) Graphs and Tabulations: Tabulation showing how the magnetic force depends on the number of turns in a solenoid. Study of First Difference to derive the growth-rate and decay, e.g.

No. of turns	No. of clips suspended	First Difference
50	9	
100	17	8
150	25	8
200	33	8
250	40	7
300	45	5
350	39	-6

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Ordered pairs (8,72); (9,81), etc. giving the straight-line graph $y = 9x$. Graph of the experimental results about the Inversed Square Law.**

(4) Geometry and Topology: Circular cylinder, (e.g. the battery, the empty coffee can.) The circle: its center, diameter, circumference $= 2\pi r$. Cylindrical surface of the coffee can flattened out into a rectangle. Parallel strips cut from this rectangular sheet. Spherical surface (a bulb). Mid-point of a line, (where the mechanical 'swinging' switch was pivoted.) Discussion of 'geometrical contact' vs. 'electrical contact'.

Relative positions on a circuit-diagram: connection and disconnection. A topological line (any wire connecting the + and - terminals.) Simple closed curve (a complete circuit). Simple discontinuities. Separation and distance between 2 planes. (The 'springy' door-mat and the floor).

(5) Application and Practical Mathematics: Measuring long distances for electric wiring. Cutting empty coffee cans into thin straight strips. Soldering intersections. Winding electromagnets. Gluing springs and electrical contact (tin-foil) to a door-mat. Manipulating logical circuits.** Use of a micrometer** to measure small distances.

(6) Physics: Dry cells. Conductor (wire) and insulator. Electric current, shown by the lighting of the bulb or the heating of the wire. A complete or closed circuit. Switch. Open circuit. Short-circuit phenomenon. 'On' and 'Off'.

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Positive and negative poles of a battery. Batteries in series. Electromagnets. A solenoid. Magnetic force $\propto$ number of turns in the solenoid. External resistance and Ohm's Law. Resistance of a wire $\propto$ its length. Resistors in series. Potential Difference. Internal resistance of a battery. Voltmeters. Ammeters. Magnetic Induction. The Inversed Square Law. Electric terminals. Domestic electric wiring.** Logical circuits.

Gravitational pull. Spring constant and potential energy of a compressed spring. Equilibrium or balancing. Limiting friction.

(7) Foundation of Mathematics: Abstraction (e.g. noting relevant elements of an electrical set-up and abstracting them into a set called circuit-diagram.) To operate formally (i.e. to permute, or combine symbolic elements of a circuit diagram, to do calculations by referring to the diagram alone.) To gain a deep impression of the cardinal numbers '1' and '2', e.g.

--"ONE battery doesn't work; TWO work beautifully."

--"ONE metal-strip (in a battery holder) leads to the short-circuit phenomenon; TWO separate metal-strips are quite safe."

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CHAPTER 4

DETAILS OF THE FOUR CASE-STUDIES

The full text of the four Case-Studies will now be presented. They were written from the chronological records of the writer's own observation, supplemented by the teachers' logs in order to clarify some of the teachers' input during these activity-oriented classes. The four USMES units observed are: the 'Soft Drink Design' Unit, the 'Dice Design' Unit, the 'Designing for Human Proportion' Unit, and the 'Burglar Alarm Design' Unit.

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The CASE-STUDY of the "Soft Drink Design" Unit,
Observed at Wexford School, Lansing, Michigan
September 26 to November 28, 1972

I. The behaviors to be observed: Mathematical content and mathematical process, which arose from the unit's activities, class discussions, teacher's advice, students' responses to questions, and other remarks.

II. Brief Description of the "Soft Drink" Group: Seven students (four girls and three boys) were selected, on a voluntary basis from a combined Grade 5-6 class of about 90 children under a team-teaching situation which involved three teachers. The teacher of this Unit was one of the three, and she had actively participated in the Soft Drink Unit of the USMES Summer Workshop here in August. One girl student had also taken part in that Workshop.

While doing USMES, they had a room of their own with a kitchen sink, a drinking fountain, a long table. The Kool-Aid powder, sugar, spoons, pitchers, paper-cups and paper-towels were provided. The time allotted to the unit was twice a week (Monday, Wednesday 1:30-2:15 p.m.).

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Six students were at Grade 6 level in mathematics and science and only one student was at Grade 5 level in these subjects. No serious reading nor communication problem was observed.

Activity 1: Mixing drinks and tasting these drinks among themselves until each student came up with his or her own 'brand' of soft-drink.

The children were experiencing a unique classroom environment which involved a great deal of laughter, manipulative skills in using household utensils, but not without mathematical learning. They mixed different kinds of Kool-Aids with various measured amount* of sugar and water, tasted the new inventions, looked at each other's colorful tongues, (the writer hoped that food-coloring was relatively harmless), and sometimes left the room hurriedly to avoid the near-nauseating feeling caused by the grossness of certain inventions. Still, all was done in good faith, and, after much experimenting with the variable amount* of Kool-Aids that should be used for a fixed amount* of water and sugar, every student did come up with an acceptable formula of his own. They learned that the success (popularity) of one's drink depended on two factors: the taste and the look of the drink. (One girl remarked that a drink

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*Throughout this Case-Study, the words marked * refer to the mathematical behaviors observed, and those marked ** refer to the potential mathematical topics which could have been discussed in an USMES class, but were not included in the observed unit.

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which looked like vampire blood would not sell.) Hence they were introduced to the concept of independent variables* (taste, look) and a dependent variable* (popularity). The latter was easily quantifiable*, and, as one boy suggested, they were going to poll* the whole school to measure the relative popularity of the drinks. The independent variable was brought up for discussion, and, to the surprise of the two adults in the room, the children discovered four (not two) independent variables, as follows:

(1) the kinds of Kool-Aids (This is qualitative in nature: the variable being strawberry, cherry, orange, lime, cola flavors, etc.).

(2) the volume of the Kool-Aids used (This is quantitative, measured by a 'standard' teaspoon).

(3) the volume of sugar used (quantitative, again measured by a teaspoon). Here the children did not realize that there were many kinds of sugar.

(4) the volume of water used (quantitative, measured by bottles of 16 fl. oz. capacity, or 1-pint pitchers).

The majority of the children seemed to possess the idea that "the color of the drink" was an independent variable which directly affected the popularity of the drinks. But, after some experimentation, they were satisfied that the variables (1), (2), and (4) above did take care of this extra variable. At this point, the teacher could have introduced the concept of a composite

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composite function**, which the students would meet later on. (The color of a drink depends on (1), (2) and (4), and the popularity of the drink partially depends on color, and hence the popularity, in turn, depends on (1), (2) and (4), and perhaps more variables. Alternatively, the teacher could have explained this in terms of confounding variables** in Statistics, a concept which would be useful later on in such USMES units as Consumer Research and Pedestrian Crossing. (If one says that the popularity of a drink depends on its color, then one must not introduce variables (1), (2) or (4), since these are confounding variables to 'color'). In fact, there was some advantage of treating 'color' as an independent variable, as suggested by the children; in that way, they could see the concept of a continuum**: e.g. in mixing drinks of a lime flavor, the whole continuum from emerald green to light green could be seen as more and more water was added to the solution.

Another concept the children learned from this activity was that of irrelevant variables*. (All USMES units which involve graphing will demand the identification of relevant and irrelevant variables as a prerequisite.) When the writer (of this Case-Study) asked a student what the irrelevant factors were, he replied, after a momentary reflection, "the color of the measuring spoon; the kinds of containers used in testing." "But will the kind of container be a relevant factor in selling the soft drink?" asked his teacher. "Yes," replied the student, "because it would change the popularity of the drink."

One interesting behavior observed was the way the children quantified* the ingredients used. They did not use a weighing pan, nor a graduated cup to judge the amount of Kool-Aids and sugar; they just used ordinary teaspoons. This might sound unscientific, but they had good reasons. "I have always kept this same spoon every-time I measured the Kool-Aids and sugar," said one girl, "and one teaspoon means this full." (She demonstrated the amount.) So she got the basic idea of a unit of measure*, (which is, usually, arbitrary).

"But what happens if you want to communicate to other people who have never seen your spoon?" asked the teacher.

"Then I will say one-quarter, one-half or a whole packet of Kool-Aid" replied the girl. So she viewed the subject of fraction* from the utilitarian's vantage point. This was precisely how fractions had been discovered and used in antiquity: 1/2, 1/4, 1/8, ...

During the same period, another student proudly told his teacher that he discovered a practical way to 'prove' (meaning 'verify') that 1/2 + 1/4 = 3/4, because he first used one-half, and then one-quarter packet of Kool-Aid, and he noticed there was a quarter left over, and so he used up three-quarters of the packet. Earlier that morning, he had been taught that $1/2 = 2/4$ and $2/4 + 1/4 = 3/4$ in a traditional 'math' period.

An USMES class could provide many opportunities for practical verifications* of this type. The writer once advised a boy (who ran out of Kool-Aid) to 'play' with sugar. The boy was asked to divide a certain amount of sugar on a large plate into 5 heaps, and to divide the same amount of sugar on a second plate into 7 heaps. When the student added the larger heap of sugar to the smaller heap, he could tell, just by looking, that the sum was approximately a third of the original amount. This appeared to be more meaningful to the child than the traditional presentation $1/5 + 1/7 = 7/35 + 5/35 = 12/35$, which had been shown to him a week before. Incidentally, such 'playing' (with concrete objects) would leave a strong impression in the mind of the child --an impression saying $1/5 + 1/7 \neq 2/12$, or generally,

$$\frac{a}{b} + \frac{c}{d} \neq \frac{a + c}{b + d}$$

Also, during this 'mixing drinks' activity, while the students were measuring water from the drinking fountain, the teacher could have discussed the rate of flow** of a liquid--a very important topic both for mathematics (rate of change) and for science (property of fluids). Both a graduated pitcher and a stop-watch were available in that school, so the (volume) rate of flow of the water could easily be calculated, and children would enjoy the experience of using a stop-watch. This is also a very effective way to study the concept of time.**

Activity 2: Evaluating each other's drink:

Seven children produced seven 'brands' of soft-drinks, and they decided that only the top three drinks should be used in the general polling. A way had to be found to choose the top three drinks from seven. After some discussions, two types of evaluations were proposed and adopted. (The evaluations would be carried out by the seven students, plus the teacher and the writer of this case-study.) In essence, the two methods of evaluation were:

(i) Counting the frequencies*, by which each drink was named first choice, second choice, third choice,... and seventh choice;

(ii) Assigning a common maximum score to the drinks which each rater liked best, and then descending scores* at equal interval to the next preference, until he assigned a minimum score to the least-liked drink. The ranking* of the drinks would then be decided by the total score* each drink received.

They proceeded with the first method (counting frequencies). At first, the seven drinks were named Tom's drink, Kathy's drink, etc., but, for the sake of modesty as well as impartiality, the sample drinks were

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re-named 'a, b, c, d, e, f, g', and only the teacher knew the key to these code-names. A <u>tabulation</u>* had to be set up to keep track of everyone's rating. To do this, a student suggested that each rater should write the names of his most preferred drink at the top of a column, and proceed downwards till he wrote the name of his last choice (seventh choice) at the bottom of the column. It was the concensus of the class to adopt this method of recording votes; and so the evaluation began. All nine people were busy tasting the seven sample drinks, trying to remember their pleasing flavors (or the opposite) before walking over to the chalkboard to fill up his own column of rating. Nearly every rater had to taste some, if not all, drinks for the second or even third time before he could make up his mind about the relative merits of the drinks. Some people might object that such a rating was highly subjective, but it should be remembered that the main purpose of this activity was not to rate home-made soft-drinks, but to learn some mathematics from the experience. First, the children here had the opportunity to see <u>a matrix</u>* <u>in the making</u> (the 7 x 9 matrix in Table 4.1 below), an opportunity missed by most college students taking a course on Matrix Theory. Secondly, they learned from first-hand experience the contrast of <u>ordinal numbers</u>* (used in defining the rows of the matrix below) and card

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When the nine raters filled up their respective columns, a matrix was obtained:

Table 4.1: A Matrix relating the raters to their preference in drinks

Choices \ Raters	Tom	Eddie	Kathy	Shari	Sheri	Allen	Debbie	The Teacher	Sompop
1st	c	a	a	c	b	g	a	a	c
2nd	a	e	c	a	c	a	c	c	b
3rd	d	c	b	f	a	f	f	g	a
4th	e	g	d	d	e	e	b	b	g
5th	g	f	e	g	f	d	d	f	e
6th	f	b	g	e	d	c	g	e	f
7th	b	d	f	b	g	b	e	d	d

The next job was to spot winners, especially winners of 1st, 2nd and 3rd places. The children read from this Table the <u>frequencies</u>* by which some drinks were voted 1st choice and 2nd choice. Drink 'a' was 4 times voted to be 1st, and 3 times 2nd, while drink 'c' was 3 times 1st and 4 times 2nd. There seemed to be little doubt that 'a' should be declared the winner, and 'c' the runner-up. But the teacher quickly pointed out that 'a' was still a shaky winner, since only 4 people out of 9 (less than 50%) voted for it as their first choice. "How many % does 4/9 correspond to?" asked a student, and so they went on to have a mini-session on percentage.* Of all the drinks voted to be 1st choice, 'a' accounted for 44% of the votes, 'c' 33%, 'b' and 'g' 11% each.

A big problem arose when they tried to determine the 3rd place winner. Looking at the 3rd row of Table 4.1, the children found that 'f' appeared 3 times, 'a' twice, and 'b, c, d, g' once each. It was tempting to take 'f' as the 3rd place winner. But, fortunately, a student in the class sensed that it was wrong to ignore 'b' and 'g', because both 'b' and 'g' were voted top at least once, while 'f' was never chosen top, and, for that matter, never even for 2nd place. How could 'f' be ranked before 'b'? Again, from the percentage point of view, 'f' was put into 3rd place only by 33% of the voters (3 people out of 9), and yet 3 other persons had made it quite clear that they preferred 'b' to 'f'. (See the first 3 rows of Table 4.1 regarding the positions of 'b'). But the f-supporters argued that 'b' was voted to be bottom of the list 3 times, while 'f' suffered this fate only once.

After some (heated) discussion, it was resolved that the arguement 'f' vs. 'b' should be settled by the second method of evaluation, i.e., by assigning appropriate scores to each of the drinks according to the raters' likes or dislikes. For example, 1 point could be given to the drinks in the last row (7th choice), 2 points for the 6th choice, 3 points for the 5th choice, and so on, till finally 7 points for the 1st choice on the top row. It was decided by the class that, rather than tasting the sample drinks again, they would make use of the data* in Table 4.1. The mere conversion of the data in Table 4.1 into another representation involving scores was a good exercise in mathematics: it taught the children one-one correspondence* and Piaget's "reversibility of ordering"*:

1st choice ⟶ 7 points
2nd choice ⟶ 6 points
3rd choice ⟶ 5 points
. . .
7th choice ⟶ 1 point

It also taught the usual arithmetical skills* in multiplication and addition. In the next period, the children were busy working out the scores obtained by drinks 'a, b, c, d, e, f, g'. For examples,

Drink 'a':	7 points, 4 times	28
	6 points, 3 times	18
	5 points, twice	10
		56 pts.
Drink 'c':	7 points, 3 times	21
	6 points, 4 times	24
	5 pts. & 2 pts, once each	7
		52 pts.

<u>Drink 'f'</u>:	5 points, 3 times	15
	3 points, 3 times	9
	2 points, twice	4
	1 point, once	1
		29 pts.

<u>Drink 'b'</u>:	7 points, 6 points, once each	13
	5 points, 2 points, once each	7
	4 points, twice	8
	1 point, 3 times	3
		31 pts.

At this point, the 'b'-supporters clapped; so 'b' was proved to be better than 'f' and hence 'b' won the 3rd place after 'a' and 'c'. (It was purely coincident that the three top drinks were labelled like the first three alphabets.) It could not escape the notice of any observer that the children seemed to show more enthusiasm to do this sort of arithmetic than they would in a traditional 'math' class. When all this arithmetical work was done, the children recorded all the scores in a Table like Table 4.2:

Table 4.2: Scores assigned to the seven drinks

Drinks	a	b	c	d	e	f	g
Scores	56	31	52	24	29	29	31

Although the dispute 'b' vs 'f' was settled, 'b' still could not solely claim the 3rd place, because 'g' came up with an equal score of 31. Also, unexpectedly, 'e' and 'f' shared the 4th place, with an equal score of 29. These two instances left a strong impression on the children about equality* in mathematics: things that looked and tasted so differently (such as drinks 'b' and 'g') could be said to be equal when we considered another attribute (score) possessed by both. This subtle meaning of 'equality' should always be empnasized in applied mathematics. The writer has met high school graduates who did not appreciate the equality expressed in "Force = Mass x Acceleration."

Something else seemed to bother the children of this group.

--"How did you know that the top drink should receive 7 points, and not, say, 10 points?"

--"What made you decide that the last three choices should deserve any point at all, since they would probably not sell very well?"

The children's doubts were quite legitimate, and the teacher should enlighten the children, once for all, that all grading systems** (methods of assigning scores) were, to a certain extent, arbitrary. A different grading system, say,

1st choice	⟶	10 points,
2nd choice	⟶	9½ points,
3rd choice	⟶	9 points,
	. . .	
7th choice	⟶	7 points.

might very well be applied to Table 4.1, and data quite different from Table 4.2 would be obtained. This second grading system would certainly narrow the gap* between the best-liked drink, and the least-liked drink, but would the ranking (ordering)** among a, c, b, g, f, e, d be any different? It would be an interesting exercise on multiplying and adding fractions** to find this out. (See results in Table 4.3).

Table 4.3: Scores obtained from another grading system.

Drinks	a	b	c	d	e	f	g
Scores	86½	74	84½	70½	73	73	74

So, the children could have been given the opportunity to discover that changing from one equal-interval grading system into another did not affect the ranking or ordering of the drinks. It would have been even more interesting if the children were led to discover why. This would be the children's first opportunity to meet an invariance under a transformation,** the key to all modern mathematics.

[Think of the 7 scores in Table 4.2 as 7 geometric points** on the number-line**. The changing from one equal-interval grading system into another corresponds to the translation** of the 7 points, together with the shrinking** of the scale. These two transformations do not affect between-ness** of points on the number-line, and hence the ordering remains the same.]

The above geometrical** explanation can easily be demonstrated in the Design Laboratory, using a fairly thick elastic band with 7 points marked off when the elastic is stretched. The band is then bodily shifted and allowed to shrink. None of the 7 points swap places.

Without changing the theme of this sub-challenge, the data in Table 4.1 could also be used as a spring-board to introduce 'reasoning' by means of algebra**. That is to say, an algebraic method could be used to show why the ordering among 'a, c, b, g, f, e, d' did not change when different grading systems were assigned, provided equal intervals were maintained between pairs of scores given to two consecutive choices. [Let X_1, X_2, X_3, X_4, X_5, X_6, X_7 be the scores assigned to the 1st, 2nd, 3rd, . . ., 7th choices respectively, with the condition

$$X_1 > X_2 > X_3 > X_4 > X_5 > X_6 > X_7 > 0.$$

This condition should be 'equal interval', i.e., select $d > 0$ such that

$$\begin{aligned} X_1 - X_2 &= d \qquad & X_1 &= X_2 + d \\ X_2 - X_3 &= d \qquad & X_2 &= X_3 + d \\ &\ldots \qquad \text{or} & &\ldots \\ X_6 - X_7 &= d \qquad & X_6 &= X_7 + d \end{aligned}$$

Score of drink 'a' = $4X_1 + 3X_2 + 2X_3$

Score of drink 'c' = $3X_1 + 4X_2 + X_3 + X_6$

Therefore, 'a' - 'c' = $X_1 - X_2 + X_3 - X_6$

Since $X_1 > X_2$, and $X_3 > X_6$, 'a' - 'c' > 0

Hence, the score of drink 'a' is always above that of 'c', regardless of the numerical scores of X_1, X_2, X_3, X_6, provided the above condition is satisfied.

<u>The 'b' vs 'f' dispute</u>: Score of 'b' = $X_1 + X_2 + X_3 + 2X_4 + X_6 + 3X_7$

Now $X_6 = X_7 + d$

$X_5 = X_6 + d = (X_7 + d) + d = X_7 + 2d$

and so $X_4 = X_5 + d = (X_7 + 2d) + d = X_7 + 3d$

$X_3 = X_4 + d = (X_7 + 3d) + d = X_7 + 4d$

$X_2 = X_3 + d = (X_7 + 4d) + d = X_7 + 5d$

$X_1 = X_2 + d = (X_7 + 5d) + d = X_7 + 6d$

Hence Score of 'b' = $(X_7 + 6d) + (X_7 + 5d) + (X_7 + 4d) + 2(X_7 + 3d) + X_7 + d + 3X_7$

$= 9X_7 + 22d$

Score of 'f' = $3X_3 + 3X_5 + 2X_6 + X_7$

$= 3(X_7+4d) + 3(X_7+2d) + 2(X_7+d) + X_7$

$= 9X_7 + 20d$

'b' - 'f' = $2d > 0$. Thus 'b' > 'f']

The above algebra may be a little too difficult for Sixth graders, but, in view of the fact that USMES will be extended into Grades 7 and 8 soon, this algebraic work may be introduced with many subsequent benefits.

Activity 3: Controlling one of the independent variables:

Now that the 'best' drink had been chosen by the group, mass-production, suggested a boy, should be started immediately so that it could be sold to everybody in the school. A girl was not too happy about the suggestion, because drink 'a' had been voted 1st choice only by 44% of a very small group which was not representative of the whole school (She meant 'target population'). Besides, she felt that the first two choices (drinks 'a' and 'c') tasted equally good. Drink 'a' had won because it had an attractive color, and 'c' might have won if it were judged on the basis of taste alone. The teacher said that this statement (conjecture)* could be tested out scientifically by controlling* one independent variable (look) and leaving the other independent variable to function alone. One girl who had taken part in the USMES Summer Workshop in Lansing recalled the blindfold method to eliminate the variable 'look'. She suggested to use the method, and the class then proceeded to rate 'a' and 'c' once again, but this time each rater was blindfolded when he tasted the two drinks.

The grading system was the same as before: 7 points for 1st choice and 6 points for 2nd choice. The results were tabulated in Table 4.4.

Table 4.4: Comparing drinks 'a' with 'c'.

Raters / Drinks	Tom	Eddie	Kathy	Shari	Sheri	Allen	Debbie	The Teacher	Sompop	Total
Drink 'a'	6	6	7	6	6	7	7	7	6	58
Drink 'c'	7	7	6	7	7	6	6	6	7	59

The class decided that this result was inconclusive, because

(i) it was a close race; the <u>result might be due to chance</u>*. (Afterall, the rating was subjective).

(ii) When Table 4.1 was compared with Table 4.4, it could be seen that there was only one person changing the ordering of 'a' and 'c'. It was this person's voting that gave 'c' the lead this time. The majority still voted as before.

Although the result was inconclusive, the time was well worth spent. It gave the children some idea how to <u>eliminate</u>* or control a variable. It was also the

first time that the children experienced the formation of a conjecture or hypothesis*, conducting an experiment to test* the hypothesis, and deciding whether the results were conclusive or not.

The teacher also pointed out that it did not really matter even if this test was inconclusive. In a real business situation, the store would sell both brands in any case. Consumers are usually ready to switch to another brand of the same product, if the brand of their choice is sold out.

Activity 4: Preparing a large quantity of the two kinds of drinks for all their peers (Sixth graders) to taste and comment:

Apart from the purpose of communicating the progress of the Soft-drink Unit to their peers who were working on either Consumer Research or Designing for Human Proportions, the Soft-drink group also wanted to survey their peers' opinion on the drinks they had produced. The teacher advised them to make the two top drinks ('a' and 'c') in large pitchers, so that their peers might taste them and make comments. Thirty-two fifth and sixth graders (those who were in the same home-room) were expected to take part. From past experience, it was estimated* that 1/4 of a paper cup of each drink should be allowed for each person to taste and re-taste. "How many cups of each drink would be required for 32 people?" asked the teachers. This was a problem on proportion*, and the whole class helped out to find its answer:

"1/4 of a cup for one person means 1 cup for 4 persons. 32 = 8 x 4, so we need 8 times as much, i.e., 8 cups."

"How much drink will be needed if the teachers of the other two home-rooms want to take part too?" asked the writer.

"There would be ample drink for two more persons," replied a girl, "because 1/4 cup for each person is already a lot."

"What Mr. Sompop meant was how to figure it out mathematically if there were 34 persons instead of 32?" said the teacher.

"Let me see," replied the girl, "Two more persons would require 2/4 of a cup extra,"

"What is 2/4?" asked the teacher.

"Two-quarters means one-half" was the final reply.

So the class proceeded to make the drinks by measuring out 8 times everything specified in the previous recipies. First, it meant the pitcher should be big enough to hold 8 cups of water. A boy suspected that the pitcher was not large enough, because, as he put it, "the (linear) measurements*, i. e., the width of the mouth and the height, of the pitcher are only slightly more than double those of the cup, and they are

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not 8 times as much." The teacher asked him to check it out by measuring 8 cups of water and pour them, one by one, into the pitcher. To his surprise, he found that the pitcher could hold slightly more than 8 cups of water.

At this point, an informal discussion on the relationship of linear measurements and volume/capacity measures** could have been introduced. The most appropriate experiences seemed to be:

(1) Starting from a certain cube, say, a Dienes block whose dimensions are 3 units each, if the child is asked to double the linear dimensions of the cube in every way (i.e. to make a new cube whose dimensions are 6 units each from several smaller ones), the child will find out that he needs eight smaller blocks, and not two, and we write 8 as 2^3 or 2 X 2 X 2.

(2) A pop can or beer can should be introduced as a cylinder**, and an empty (open) coffee can as a larger cylinder. If the diameter** and height** of the coffee can are roughly twice those of the pop can, then it will take 8 pop cans of water, not two, to fill the coffee can. Again, it is a sort of cubic relationship** between a linear measurement and the volume measure. This second experience is necessary to convince the child that such a cubic relationship exists not only for similar rectilinear objects, but for 'round' objects as well.

The recipe for making one cup of drink 'a' was as follows: 3 teaspoons of sugar, 1/4 spoon of orange Kool-Aid, 1/4 spoon of cherry Kool-Aid, and 1/3 + 1/4 spoons of an 'undisclosed' flavor. (The students insisted that each winning entry should contain an undisclosed ingredient). As for 1/3 + 1/4, they wrote it this way not because they did not know 1/3 + 1/4 = 7/12, but they thought 1/3 + 1/4 was an easier fraction to work with: they put in 1/3 spoon, and then 1/4 spoon of the same powder. It would be more difficult to estimate 7/12 of a spoon.

When the writer asked the students how they would deal with 8 times the quantity (1/3 + 1/4) spoons, one student explained:

"Four times 1/4 spoons will give one full spoon, and so eight times (twice as much) will give 2 spoons. Three times 1/3 spoons give 1 spoon, and six times give 2 spoons; but we want eight times, so we put in 1/3 spoons for two more times."

Apparently, this student grasped the intuitive idea of a <u>multiplicative inverse</u>*, and she also knew how to use the <u>distributive law</u>* <u>of multiplication over addition</u>, since she worked with 8 × 1/4 and 8 × 1/3, and then added.

During this period of Activity 4, two more mathematical problems arose naturally. Unfortunately, these two problems were not pursued in details, and no solutions to the problems were given in class. The first problem was:

"Each packet of Kool-Aid (of the unsweetened type) weighed 0.15 oz. and dissolved in 2 quarts of water (sugar to be added afterwards). The teacher just brought the class another type of Kool-Aid with sugar already added: this packet (3.30 oz.), when dissolved in 1 quart of water, was ready to drink. The teacher asked what percentage of sugar was contained in the sweetened type of Kool-Aid."

The second problem was:

"A boy who was mixing drink 'c' in a pitcher found that the drink produced according to the recorded recipe turned out to be too "gross" (thick). So he poured out 1/10 of the content (of the pitcher) into a glass, and filled the pitcher up with water. He repeated this for 3 times, and the drink then had the clear and attractive look he had hoped for. He wanted to know what fraction of the original Kool-Aid remained in the pitcher, and what amount of sugar must be added to maintain the same level of sweetness as before."

The writer of this Case-study was of the opinion that the solutions of these two problems should be discussed and made available to the students while the level of enthusiasm was high. Grade 6 is the proper time to introduce <u>algebraic</u> <u>techniques</u>.** The solutions of these two problems, as given below, would have provided the best motivation for learning algebra.

[<u>Solution to Problem 1</u>**: Let X (oz.) be the amount of sugar in each packet of the sweetened type, and thus the amount of pure Kool-Aid in the packet = 3.30 - X oz.

**It should be pointed out to the children that the symbol X used in Problem 1 connotes a different meaning from that in Problem 2. In Problem 1, the equation is true only for ONE value of X, while in Problem 2, the calculations will remain true for <u>all</u> values of X. This was a point which SMSG decided to emphasize, and it was one of the first deliberations of SMSG during its first Writing Session.

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Twice this amount would function just like 0.15 oz. of the unsweetened type of Kool-Aid, therefore

$$0.15 = 2\ (3.30 - X)$$

$$0.15 = 6.60 - 2X$$

$$2X = 6.45$$

$$X = 3.225 \text{ oz.}$$

Thus the percentage of sugar in the packet

$$= \frac{3.225}{3.300} \times 100 = \frac{3225}{3300} \times 100$$

$$= \frac{3225}{33} = 97.7$$

Besides algebra, this problem would give the children good practices on decimals** and long-division**. It also reminded the consumers that, when they bought the sweetened type, they were paying mostly for sugar (97.7%), and very little (0.03%) for Kool-Aid.

Solution to Problem 2*: Let X (oz.) be the original amount of Kool-Aid in the pitcher. Every time the drink went through the diluting process, 1/10 of the amount of Kool-Aid present was poured out, and so, after the 1st dilution, $X - \frac{1}{10}X = \frac{9}{10}X$ oz. of Kool-Aid were left in the pitcher.

After the 2nd dilution, the amount left

$$= \frac{9}{10}X - \frac{1}{10}\left(\frac{9}{10}X\right)$$

$$= \frac{9}{10}X - \frac{9}{100}X = \frac{81}{100}X \qquad \text{oz.}$$

After the 3rd dilution, the amount left

$$= \frac{81}{100} X - \frac{1}{10} \left(\frac{81}{100} X\right)$$

$$= \frac{81}{100} X - \frac{81}{1000} X = \frac{729}{1000} X \quad \text{oz.}$$

Hence the new 'clear' drink contained only $\frac{729}{1000}$ or roughly 3/4 of the original Kool-Aid put in.
The same fraction of sugar was poured out, i.e. $\frac{81}{1000}$ of the original amount. Originally, he put in 8 times 3, or 24 teaspoons of sugar, and so he had to add $(\frac{81}{1000})(24)$ or $\frac{1944}{1000}$, or roughly 2 more spoons.

The same mathematics can be used to discuss some problems in science, e.g. the problem of pumping out air from an enclosed space.]

Going back to the activity of preparing drinks 'a' and 'c' in large pitchers for 32 sixth graders to taste and comment, it could be observed that both the preparation and the tasting went on fairly well. But the result of this polling was again inconclusive. So the class decided to give up polling on 'a' and 'c', and, instead they would ask <u>representative samples</u>* from every grade of the whole school to state their favorite soft-drinks or pops. This immediately led to the tightening-up of the definition of 'soft-drink' and using this concise definition to design a questionnaire for the polling.

Activity 5: Defining 'Soft Drink' by examining the properties of two mutually exclusive sets: soft drinks and non-soft drinks:

The teacher led the discussion and pointed out to the children that before they seriously defined "soft-drink", it would help to list items which were definitely not soft-drinks. In effect, the children learned of two mutually exclusive sets* (although the teacher need not say so). A beverage could belong to one of the sets but it would never belong to both sets simultaneously. Another concept which arose from the discussion of this dichotomy was negation*, and also the principle that the unary operation of 'negation', applied twice, would produce the original statement again:

$$\sim \sim p \Longrightarrow p$$

The statement $\sim \sim p$ took place when the class decided against a student's suggestion that cider was a non-soft drink. The student was somewhat sure that cider contained alcohol. Although he was not totally forbidden to drink cider, he was under the impression that it was not good for children. Many argued that cider was apple-juice, and although some brands of cider might be slightly alcoholic, nobody got drunk with cider yet. So the class decided that cider was not a non-soft drink, and without any difficulty on logic, they concluded that it was a soft drink.

The list of non-soft drinks was long, original, and really interesting. (This might be slightly different from an adult's list). According to these children, soft-drinks or pops did not include the following: pure water, whisky, brandy, beer, cooking sherry, milk, tea, coffee, chocolate, ovaltine, fresh orange juice, fresh fruit juice, soup, vinegar, alker-selzer, ... (This is one very good way of describing a set: listing its elements*).

Then came the first abstraction*: "Soft-drinks or pops are thirst-quenching liquids which (i) do not contain alcohol, (ii) are not fully natural."

The condition (ii) above precluded milk, soups, tomato juices, fruit-juices, etc., for being soft-drinks or pops. The children also discovered one subtle implication* derived from condition (ii): "Soft-drinks or pops must contain artificial flavors and sugar." It was this condition that differentiated between fresh orange juice and orangeade soda. Also by this implication, tea and coffee and the like were precluded from being called 'pops'. When the properties of the Non-soft drinks were fully examined, it was comparatively easy for the children to formulate the definition of soft-drink, or pop:

"A soft-drink or pop is a beverage which contains soda-water or water (i.e. which is fizzy or not fizzy), sugar, and artificial flavor(s)."

The (organic) formulation of a definition by the children themselves after a great deal of pertinent experience (i.e. mixing and tasting soft-drinks) was another instance of the healthy method of teaching mathematics and science in this program. Usually, in a traditional 'math' or science class, definitions were thrown at the children, who were then ordered to memorize as many as they could.

Activity 6: Using the definition of "soft-drink" to design a questionnaire for general polling:

In learning mathematics and science, it is very important to realize that the mere formulation and memorization of some definitions are utterly useless, unless the definitions* are subsequently used to derive something else. Here, the definition of "soft-drink" could be put to use immediately in designing a questionnaire for polling sample population of the school. Since 'fizziness' and 'artificial flavor' were the two main features of the "soft-drink" defined by the children, the first two items in the questionnaire were:

(1) Do you like carbonated (fizzy, bubbly) drinks?

Check: ( ) Yes ( ) No

(2) Read the following list, and write '1' for your best-liked flavor, '2' for your next favorite, '3' for the next one still, and all the way down till you write '15' for your least-liked flavor.

___ Cherry	___ Lemon	___ Raspberry
___ Grape	___ Root-beer	___ Cream-soda
___ Pepsi Cola	___ Orange	___ Ginger-ale
___ Coca Cola	___ Blackcherry	___ 7-up
___ Royal Crown Cola	___ Strawberry	___ Lime
		___ Other flavors

At first the children phrased item #2 this way:

(2) Please fill in the blank space below the flavor of the pop or soft-drink you like best: ______________.

Soon it was apparent that, if they wanted to construct a preference/flavor matrix (like that of Table 4.1), the sheer <u>size</u>* of this <u>matrix</u> might be prohibitive to carry this work any further. The <u>sample-population</u>* was about 30 in number, and this matrix would be of the size 30 x 30 if everyone happened to have a different 'favorite' flavor. By presenting the sample-population with the above list, they could limit the size of this matrix down to 16 x 16. (In the next Activity, they discovered that even this size was still far too big, and, in the last Activity of this Unit, they used the blindfold test to convince themselves that many of the flavors above were, in fact, the same, and the size of this matrix could further be reduced to 10 x 10).

The questionnaire also contained two more items, both of which could lead to some meaningful mathematical exercises. These items were:

(3) What is the convenient size for a bottle or a can of pop?

( ) 8 fl. oz., ( ) 12 fl. oz.

(4) How much are you willing to pay for a 12 fl. oz. bottle or can of soft-drink?

( ) 10¢, ( ) 15¢, ( ) 20¢

Item (3) should have furnished the most opportune moment to answer the question "Just how much is 1 fl. oz.?" which must have bothered many children's (and adult's) minds for some length of time. First, many children still think that this is a unit of weight, which is not true. Secondly, all too many children, by rote learning, verbalize '16 fl. oz. = 1 pint' without any appropriate experience in dealing with the liquid content of 1 pint, or 2 pints, or 8 fl. oz., etc. The writer introduced a one-pint Coca Cola bottle into an USMES laboratory one day. He asked a boy to fill half the bottle with water, and said '8 oz.' He then asked the boy to half the content twice, and the boy said, on his own, '4 oz., 2 oz.' respectively. 'Now, you have to use your imagination a bit,' said the writer, 'Half this amount is what we call 1 fl. oz.' 'Oh, I see now,' remarked the boy. Alternatively, any teacher could have introduced perfume bottles of 1 fl. oz., so that children might see the actual content of this unit measure.*

The terms '8 fl. oz.', '12 fl. oz.' as introduced by the questionnaire should also be called one-half pint and two-thirds pints in order to have some practice on fraction. For the boy mentioned above, '2/3 pints' had a deeper significance than '12 fl. oz.', because '2/3 pints' produced in his mind an image of some liquid in a one-pint Coca Cola bottle which was two-thirds full.

During the discussion on pop bottles, mathematico-artistic creativity** could have taken place with regard to the design of new kinds of attractive bottles, and the approximate measurements of such designs. All kinds of bottles could be brought into the classroom for the children to examine qualitatively the curved surfaces** of some bottles, and to study quantitatively their diameters, heights, and capacities.

Another kind of container of soft-drink, the can, could also be used in many ways as an apparatus for teaching elementary mathematics. The prototype cans are circular right cylinders.** It would be easier to wrap a piece of metal wire round a cylinder than to measure the circumference** of a two-dimensional circle by using a piece of thread.

Hence, by using metal wires and a soda-pop can, the attempt to discover the number π** would be less painful than the usual method of using a thread and a 'flat' circle on paper.

Finally, the last item in the questionnaire about the prices 10, 15, or 20¢ for a bottle of soft-drink could be used to generate a discussion on inflation which is usually quoted in terms of percentage increase.* The middle-class children may not be too thoughtful about an increase of 5¢, but it represents an increase of 50% (from 10¢ to 15¢) and 33⅓% (from 15¢ to 20¢) respectively. In Thailand, the inflation has more or less forced the coins of 1, 5, and 10 cents out of circulation (1 U.S. $ = 20.80 Bahts = 2,080 Thai cents). The businessmen use the dirty trick of increasing the retail prices in steps of 25 cents. It is not healthy to accustom children to thinking of increases in steps of 5¢ while only an increase of 1¢ is justifiable.

Activity 7: An attempt to poll the entire school with the questionnaires:

The original plan was to poll the entire school, but as soon as the discussion about this ambitious project started, they realized their limitations: the quantity* of ditto paper required, the insurmountable task of counting* and analysing* the returns, and the time* it would take to carry out a large-scale* survey. In any case, what was the use of acquiring information pertinent to one school only? Two children hit on the correct approach of getting the opinion of a cross-section* of this school population from K to 6, and hoped that the cross-sections of other elementary school populations were similar. Although the teacher did not specificly give a formal lesson on stratified sampling*, the children did learn the essence of this topic from this experience. As events turned out later, the responses to the questionnaires did come from a fairly even cross-section of the school population with respect to grade levels and age-groups. It would have been even better if these stratified samples were randomly selected**. The writer sincerely believed that the excellent co-operation between the office and the class-room teachers at Wexford would have permitted random selections (pulling names out of pre-arranged bags) to take place.

The survey group came across one difficulty when polling the samples from kindergarten, grades 1 and 2: the primary-grade children could not read most of the writing in the questionnaires. This difficulty was overcome by the service of three volunteers from the Soft-drink group who patiently talked to the younger children (K-2) about fizziness, the flavors of the drinks, and the price, and these volunteers tried to record their verbal responses as accurately as possible. One girl remarked afterwards that it was delightful to talk in 'kindergarten' language once again, (e.g. 'bubbling' for carbonation). But she was seriously doubtful whether the younger children really knew some particular flavor she was talking about, unless they had tasted the real thing before, and in such cases it would be meaningless to ask the question whether they liked that flavor or not. So, even a sixth-grade student can deduce that experience reigns supreme in education! It was unfortunate for the overall result of this polling that, when the younger students did not know any flavor, they marked them as 15th choice (least liked) instead of 'no comment,' and hence the 15th choice here did not necessarily reflect the quality of the drink. (See the 15th column of Table 4.5).

The sample population* was 32, but only 26 of the returns (if counting both verbal and written returns) were analysable. The class decided to use the Frequency Method (See Activity 2) to analyse the returns with regard to the popularity of different flavors. The result is tabulated as follows: (See Table 4.5 on following page).

It took the children five class-periods to establish Table 4.5, and they were somewhat tired at the end. This feeling was not altogether bad, because it forced them to take another good look at the 15 x 16 matrix they had constructed, and to find an effective way to reduce* the size of this matrix. (See the next Activity).

Table 4.5: Frequency by which each flavor was voted 1st, 2nd,...15th choices. The last column (total* votes) was for checking whether the frequencies were counted properly.

Choices / Flavors	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th	12th	13th	14th	15th	No Comment	Total Votes
Cherry	1	2	7	1	1	3	3	1	0	2	1	2	0	0	0	2	26
Grape	4	1	0	2	2	3	5	2	1	1	1	0	0	0	1	3	26
Pepsi	3	4	2	2	3	3	0	0	3	1	1	1	1	1	0	1	26
Coca-Cola	3	5	2	5	3	0	1	0	0	2	1	0	3	1	0	0	26
Royal Crown	0	0	4	3	3	1	1	2	1	2	4	2	2	1	0	0	26
Lemon	0	2	0	1	0	0	1	2	2	1	3	3	3	3	3	1	26
Root beer	2	4	3	3	2	3	0	4	1	1	0	1	0	2	0	0	26
Orange	3	2	0	0	1	3	5	3	1	1	4	3	1	0	0	0	26
Blackberry	1	3	1	1	0	1	3	3	2	0	2	1	4	0	1	3	26
Strawberry	1	1	3	2	3	2	1	3	2	3	0	3	2	0	0	0	26
Raspberry	1	0	2	1	1	1	1	1	1	2	4	3	1	5	0	2	26
Cream-Soda	4	1	0	2	3	2	2	0	1	3	0	1	0	1	6	0	26
Ginger-ale	1	0	0	1	2	0	2	1	3	1	0	2	4	1	6	2	26
7-up	3	1	2	3	1	3	1	1	4	1	0	2	2	2	0	0	26
Lime	0	0	0	0	1	1	1	2	2	3	3	0	2	6	3	2	26

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With regard to the other two items in the questionnaires, it was reported that

(i) 20 had voted for carbonated drinks, and 6 for non-carbonated drinks.

(ii) 14 had voted for the price of 10¢ per bottle and 12 for 15¢ per bottle.

The way the children interpreted Table 4.5 was interesting. On the criterion of using maximum* 1st choice frequency, they decided that Grape and Cream-Soda (1st choice frequency = 4) might be the two most popular flavors. But they were not too sure because Pepsi and Coca-Cola ranked 1st choice with frequency = 3 and 2nd choice with very high frequencies compared with the 2nd choice frequencies of Grape and Cream-Soda. Orange and 7-up were definitely runners-up with 1st choice frequency = 3. Ginger-ale seemed to be least liked by this sample-population because it ranked 15th choice with frequencies = 6. Perhaps the majority of this population had never tasted this flavor before. The case of Cream-Soda seemed to confirm this hypothesis.* While Cream-Soda ranked 1st choice with frequency 4, it simultaneously ranked bottom-choice with frequency 6. It meant that some children who had not tasted Cream-Soda before wrote down 15th choice instead of 'no comment'. The same reasoning* could be applied to the cases of Coca Cola and Pepsi. Every child in the sample group seemed to have had experience with these flavors before. The

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result was that Coca-Cola and Pepsi enjoyed high frequencies for 1st and 2nd choices and zero frequency for bottom choice. The teacher pointed out to the class the tremendous effect of advertising, which at least helped to introduce certain products to the public.

Just like Activity 2, the next step was to assign some scores corresponding to the 1st, 2nd, ... choices. This time the children knew that the scores could be assigned arbitrarily*, so long as they were consistent* in their calculation. They seemed to be proud of the fact that they could assign scores freely, and they decided to give 5 points for the 1st choice, 3 points for the 2nd choice, 1 point for the 3rd choice and none for the rest. The teacher called this system a weighted score*, because it was tilted in favor of the top choices. The children then proceeded to calculate the total score for each drink. The list of top drinks, with their respective scores, is shown in Table 4.6.

As expected, Coca Cola and Pepsi turned out to be the top two, having scores much higher than those of Grape and Cream Soda. The scores for Coca Cola and Pepsi would have been even higher if the 4th choice were given some points as well. In fact these scores could be regarded as a measure* of how well-known a certain drink was among the population sampled here.

Table 4.6: The 8 most popular flavors, arranged according to their scores.

Flavors	Scores
Coca Cola	32
Pepsi	29
Root-beer	25
Grape	23
Cream Soda	23
Orange	21
7-up	20
Cherry	18

The children then discussed the common features of these two top flavors. They all agreed that Coca Cola and Pepsi were very similar to each other: both were carbonated, sweet and both had a tainty 'Cola' taste. The girl who participated in the Summer Workshop immediately suggested the blindfold test to prove that nobody could tell the difference between Coca Cola, Pepsi Cola, Royal Crown Cola, and indeed <u>any</u> Cola (such as Faygo Cola, Shasta Cola, etc.) The rest of the class did not believe her, and so the blindfold test was going to be the next and last activity of the Soft-Drink Group before the Christmas vacation.

Activity 8: The blindfold test to determine the equivalence of all Colas:

This activity was one way to introduce the concept that two things are equivalent* if one cannot be distinguished from the other. It is an important concept in geometry**, e.g., since a circle and a square cannot be differentiated from each other if we consider only the property of simple closed curves,** then a circle and a square are said to be topologically** equivalent, and the properties associated with simple closed curves are called topological properties. By the same line of reasoning, since it could be established by the blindfold test that all Colas could not be distinguished from one another if only the variable 'taste' was considered, all Colas should then be equivalent as far as taste was concerned.

All students and an invited teacher from another class took part in this experiment. Each took turns to be blindfolded and then given 3 kinds of Colas (Pepsi, Royal Crown, and Coca Cola) in 3 different cups, and asked to supply the correct names. Re-tasting was allowed and everyone was urged to think seriously of all the previous experiences one had with Colas, so that guesswork or playful answers were cut down to the minimum. The experiment, whose result was tabulated in Table 4.7, revealed that the majority of this group could not give even a single correct response, and everyone gave at least two incorrect responses.

Table 4.7: Results of the blindfold test: ✓ referred to correct responses, otherwise the names of the incorrect responses were shown.

Correct Responses / Experimenters	Coca Cola	Pepsi	R.C.Cola
Allan	R.C.	Coke	Pepsi
Tom	R.C.	Coke	Pepsi
Eddie	✓	R.C.	Pepsi
Kathy	Pepsi	R.C.	Coke
Shari	R.C.	✓	Coke
Sheri	Pepsi	R.C.	Coke
Debbie	R.C.	Coke	Pepsi
The Teacher	R.C.	✓	Coke

In fact, Table 4.7 gave more details than required. (It showed which drink was indistinguishable from which one.) A simplified matrix, using 0 for incorrect responses and 1 for correct responses, would serve the same purpose, (Table 4.8), and it would immediately lead to the discussion of <u>binary</u>** <u>digits</u> (0, 1) which the children might have heard before in connection with computers. The <u>binary</u> <u>system</u> could easily be followed by playing with <u>Dienes</u>' <u>Multibase</u> <u>blocks</u>,** and studying <u>place</u> <u>value</u>.**

Table 4.8: Scores given for correct identifications of the drinks

Experimenters \ Drinks	Coca Cola	Pepsi	R.C. Cola
Allan	0	0	0
Tom	0	0	0
Eddie	1	0	0
Kathy	0	0	0
Shari	0	1	0
Sheri	0	0	0
Debbie	0	0	0
The Teacher	0	1	0

The teacher pointed out to the class that, although no guesswork was implied, the small percentage* of correct responses might well be due to chance. The writer of this Case-Study informed the class that, in Montery, California, the Soft Drink group had done the same blindfold test on 4 Colas (Coca Cola, Pepsi, Royal Crown, and Shasta Cola), and they, too, had obtained only a small percentage of correct responses. Only one student (from a class of 23) gave all 4 identifications correctly; the majority (10 of them) got the names all wrong, 8 of them made three mistakes, and 4 made two mistakes.

The children were consequently satisfied that all Colas were equivalent as far as taste was concerned.

One boy further remarked: "Since you can't tell one Cola from another, and since the TV commercial says one of them is the real thing, all of them would be real things then." The writer would like to add: "Another way to look at it is that none of them could be the real thing, since they all are artificial, when referred back to the definition."

Now, the children could start working on the intended objective: to reduce the size of the matrix in Table 4.5 by cutting down repetitious items in the questionnaire. First, Pepsi, Royal Crown, and Coca-Cola could be grouped into one: the Cola flavor. The children further grouped 7-up, Lemon, and Lime into one flavor, and also cherry and blackcherry into just one flavor. There were now only ten flavors left, ranking from 1 to 10, and hence there were eleven choices including the 'no comment' column. The size of the corresponding matrix* was thus reduced to 10 x 11. A student suggested further that all the other sweet fruit flavors: grape, strawberry or even raspberry could be grouped with the cherry set, since the taste-difference among these fruit flavorings was minimal. Only the colorings differed somewhat. In fact many firms sold these products as one item: the mixed fruit flavor, or Hawaiian punch. Although many in the class agreed with this idea which would result in reducing the matrix even further, it was almost the concensus of the class that

orange should not be combined with the cherry/strawberry group, because a good orange drink should have a distinctive 'orange' taste and a remarkable 'orange' smell.

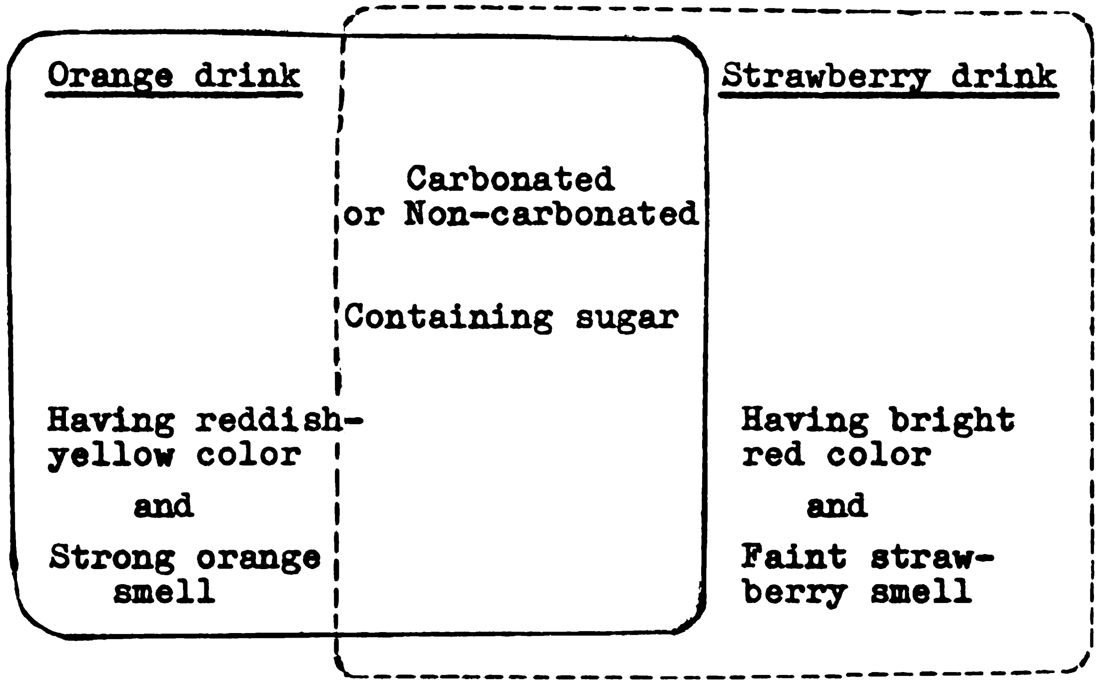
The discussion on the different characteristics of the orange and strawberry drinks unexpectedly led to the mathematical topic of '<u>intersection of two sets</u>'.* The children listed the characteristics of the two drinks as follows:

<u>Orange drink</u>	<u>Strawberry drink</u>
May be carbonated or non-carbonated	May be carbonated or non-carbonated
Containing sugar	Containing sugar
Having reddish-yellow color	Having bright red color
Having strong orange smell	Having faint strawberry smell

"You can, in fact, write the common properties only once instead of writing twice," said the teacher. So the children wrote the two common properties above in the middle between the two columns:

<u>Orange drink</u>	<u>Strawberry drink</u>
Carbonated or non-carbonated	
Containing sugar	
Having reddish-yellow color	Having bright red color
Having strong orange smell	Having faint strawberry smell

The first line of this revised presentation tended to promote the misunderstanding that the orange drink was carbonated and the strawberry drink was not. To make things clearer, they circled each side with a different color-chalk, and obtained a perfect Venn-diagram:



The teacher told the class that the common part in the middle is called the intersection of the two sets.

This concluded the major types of activities of the Soft-Drink Unit during the observed period (September 26 - November 28, 1972).

Conclusion: It seemed hardly possible that some forty items of mathematical behaviors plus some twenty potential mathematics lessons should have arisen from such a non-mathematical theme like 'Soft Drink Design.' But a reader who actually counted up the mathematical behaviors marked * and the potential mathematics lessons marked ** could verify that this vast amount of mathematical learning did take place. Moreover, a reader who thoroughly read Activities 1-8 would easily see why this mathematical learning took place. The whole approach was organic and evolutionary,[1] but care was taken so that the approach was not revolutionary[1] against the traditional content of mathematics. Many of the topics marked * and ** were 'traditional' mathematics in every sense of the word, and, hopefully they were conventional enough to make it possible for a conscientious student to pass the usual Stanford Achievement Test or any State-wide Assessment Program. Needless to mention that a conscientious student would acquire more practice on his own with such important topics as fraction and percentage. The Soft Drink Unit activities were arranged to motivate the truth-searching spirit, and to relate mathematics to one of the child's most enjoyable habits (drinking soda-pops), but not to replace the hard work demanded by any course in mathematics, even at the elementary level. But hard work plus enjoyment is a surer way toward success than hard work alone.

Hard work does not mean regimentation. As a matter of fact, the USMES philosophy allows divergence within the same unit. If this Case-Study is compared

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[1]The words 'Evolution, not Revolution' were used by E. H. Moore in his 1902 Presidential Address to the American Mathematical Society, reprinted in The Mathematics Teacher, April, 1967, pp. 360-374.

with the reports of other Soft Drink groups, say, the one in Monterey, California, one striking difference is that the Monterey group did a great deal of work on graphs, while the Lansing group did not have time to do any graphing in the Fall term, because they did considerably more on tabulations and calculations. The Lansing children would hopefully graph most of their results in the following term.

The "Soft Drink Design" is a good choice to motivate the children. The British also include the polling of favorite soft-drinks in their 'presentation of mathematical challenge to Primary School children,' usually done on a series of work cards. The following, quoted in full, is one such card.

> "Mathematics Workshop (Card) No. 178
>
> Make up a list of ten flavours, such as cocoa, strawberry, lemon. Write them on a large sheet of paper and give every child in the school one vote for his or her favourite flavour.
>
> When you have counted the votes, arrange them in order with the favourite first. Now illustrate the choice of children by means of a graph."[2]

The content of this card is very similar to the description of part of an USMES Soft-drink unit, but the British work card seemed to be more directed.

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[2]Pethen, R. A. J. "The Workshop Approach to Mathematics," Toronto: McMillan Co., 1968, p. 8.

The CASE-STUDY of the "Dice Design" Unit,
Observed at Wexford School, Lansing, Michigan,
September 25 to November 27, 1972

I. The behaviors to be observed would include children's remarks while tossing coins, drawing long-short straws, choosing left-right hands, and playing various horserace games; the ingenious ways of recording (tallying) results in Experimental Probability; the children's methods of graphing, particularly their special method of building up a histogram; and, time permitting, the children's manipulation of various polyhedra to be used as dice in this Unit.

Any mathematical topics arising from class discussions and students' responses to questions put by the teacher or the writer would also be recorded.

II. Brief Description of the 'Dice Design' Group:

Twenty-six 3rd graders from an entire wing of 'Team 2' (i.e. Grade 3-4) took part. The time allotted was Monday, Wednesday, Friday, 1:50-2:30 p.m., but cancellation of classes due to Parent-Teacher Conferences, concerts, etc. effectively reduced USMES activities down to twice a week on the average. The teacher had taken part in the USMES Summer Workshop in Lansing, but, as it turned out, she conducted the class by using her own originality (about coin-games) rather than repeating the techniques and materials (about polyhedra) she had picked up from the Workshop.

This was not an all-volunteer class, and so a few children's overt and covert boredom with the coin-games was to be expected, but even these reluctant students did learn at least addition, multiplication and graphs from this Unit. There was another drawback: two children were observed to have reading and communication problems, but they did try to learn something from the coin-games.

<u>Activity 1</u>: <u>The games of (i) drawing straws, (ii) choosing hands, and (iii) flipping coins; and the various tallying methods devised by the children</u>:

The initial discussion on how to choose just one person from two equally 'good' (meaning 'qualified') people led to the suggestion of a fair method of selection by means of a fair game. After eliminating a few unfair practices such as

- fighting,
- reaching for the highest shelf in the library,

(although this was a favorite game for third graders), the children gradually converged on fairer types of games such as drawing the shorter straw out of <u>two</u> partially concealed straws; choosing correctly the hand that held a tiny object (while the other hand was empty); and flipping a coin. It was delightful to hear that they did not say outright these were perfectly fair games. They only <u>conjectured</u>* that these might be fair, and they set out to <u>verify</u>* their conjectures. The class then split into three groups, fetched the necessary instruments (long and short straws, tiny counters taken from 'Monopoly' sets, pennies and tallying sheets), and proceeded with these three games. The writer asked one pair of children (who were about to toss a penny) whether a thumbtack might be

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*Through the Case-Study, the underlined words which are also marked * refer to the mathematical behaviors observed, while those marked ** refer to <u>potential</u> mathematical topics that could have been discussed in class.

used instead of a coin. "No," replied the children firmly, "because a thumbtack is not flat on both sides; it has something sticking out on one side." "It is not balanced," said another child. Hence the children learned that symmetry* or a balanced shape* is a necessary condition* for fairness.

In another group who were drawing straws, one pair of children wanted to be different from the rest: they used 3 straws (two of which were 'long') instead of the usual set of two straws (1 long, 1 short). It did not take them long to realize that the game was not fair. The holder of the straws had a 'better chance' of winning than the drawer. Theoretically,** the probability* of winning = 2/3 for the holder and 1/3 for the drawer, and, as a matter of fact, the result obtained from this game came remarkably close to the theoretical values, i.e., in 30 trials, the drawer succeeded to get the shorter straw 9 times.

Although most couples in these games kept only one tallying sheet, a few pairs hit upon the ingenious idea of checking the accuracy* of the scores by keeping two tallying sheets. Each student simultaneously recorded his own winning score and the other side's losing score, or vice versa. At the end of the game, the points lost by one should equal* the points won by the other unless one of them cheated. This was an instance in which equality* occurred in a non-trivial manner. This method of checking debit/credit items (double entry book-keeping*) had, in fact, been invented by Simon Stevin around 1600 A.D., and has gradually developed into the modern accounting. During this activity, the elementary principle of book-keeping** could have been introduced, by considering on one side the teacher's (real or hypothetical) incomes from the usual salary and summer jobs, and, on the other side, the grocery bills, the mortgage payment, the insurance premiums, etc. Alternatively, the teacher could have introduced budgeting** for such activities like camping which, again, involved incomes and expenses. (The children sold apples to realize some income for the 'Camping' group.)

Now, back to the games. First, it was interesting to observe different tallying methods. Most of them seemed to know intuitively that the results to be tallied were divided into two mutually exclusive sets* (heads-tails, or winning-losing). Only a few students recorded in terms of mixed data like HHHTTHTTTHHTH... These children would find it hard to count accurately afterwards. Some children drew "smiling faces" for 'heads' and "fish-tails" for 'tails'. But some tried to abstract* the acts of winning and losing by writing one X for a point won, and one O for a point lost, and then counted the X's and O's afterwards. Negative integers** could have been introduced to these students by writing -1 for the first point lost, -2 for the next point lost, then -3, -4, -5 and so on. One girl, Lori, did use the set of integers* for recording her results of flipping a coin, but she used it to record the total number of throws rather than separating the throws into 2 sets: positive integers for heads and negative integers for tails. Had she used positive and negative integers, she would not have to count the numbers of heads and tails afterwards. But Lori's method had the advantage that, if she wanted 50 throws exactly, she would know when to stop.

Since no student in this group used positive and negative integers to record the scores of the two sides, they all had to undertake the next task of counting* the two sets of tallies to find the (total) scores for 'heads' and 'tails', or points 'won' and 'lost' -- as the case might be. Most of them wrote the tallies in each set into groups of 5*, as the following examples:

Example 1. Throwing a coin:

	Heads	Tails
Group 1	/////	/////
Group 2	/////	/////
Group 3	/////	/////
Group 4	/////	/////
Group 5	/////	/////
Extra	//	////

The student counted: 5, 10, 15, 20, 25 and then 26, 27 points for 'heads' and 28, 29 points for 'tails'. This presentation showed that the two children here knew the Multiplication Table* for 5.

Example 2. Choosing hand:

Points won		Points lost	
X X X X X	5	0 0 0 0 0	5
X X X X X	10	0 0 0 0 0	10
X X X X X	15	0 0 0 0 0	15
X X X X	19	0 0	17

This pair of children did not just verbalize 5, 10, 15, ..., they actually wrote down what they said at the end of each line. This was one of the best ways to prevent a miscount.* It also shows very clearly that multiplication of integers* is successive addition. (Here 5 X 3 = 5 + 5 + 5).

The activity of counting up the tallies turned out to be the best motivation for learning multiplication. Many surprising results followed, for example, the use of graph-papers* in conjunction with counting in groups of 5. This method of putting 5 tallies in each square of the graph-paper and considering 10 squares at a time was most efficient, and it enabled some students in the 'coin-tossing' group to record scores as high as 196. The method is demonstrated as follows, with part of the graph-paper shown:

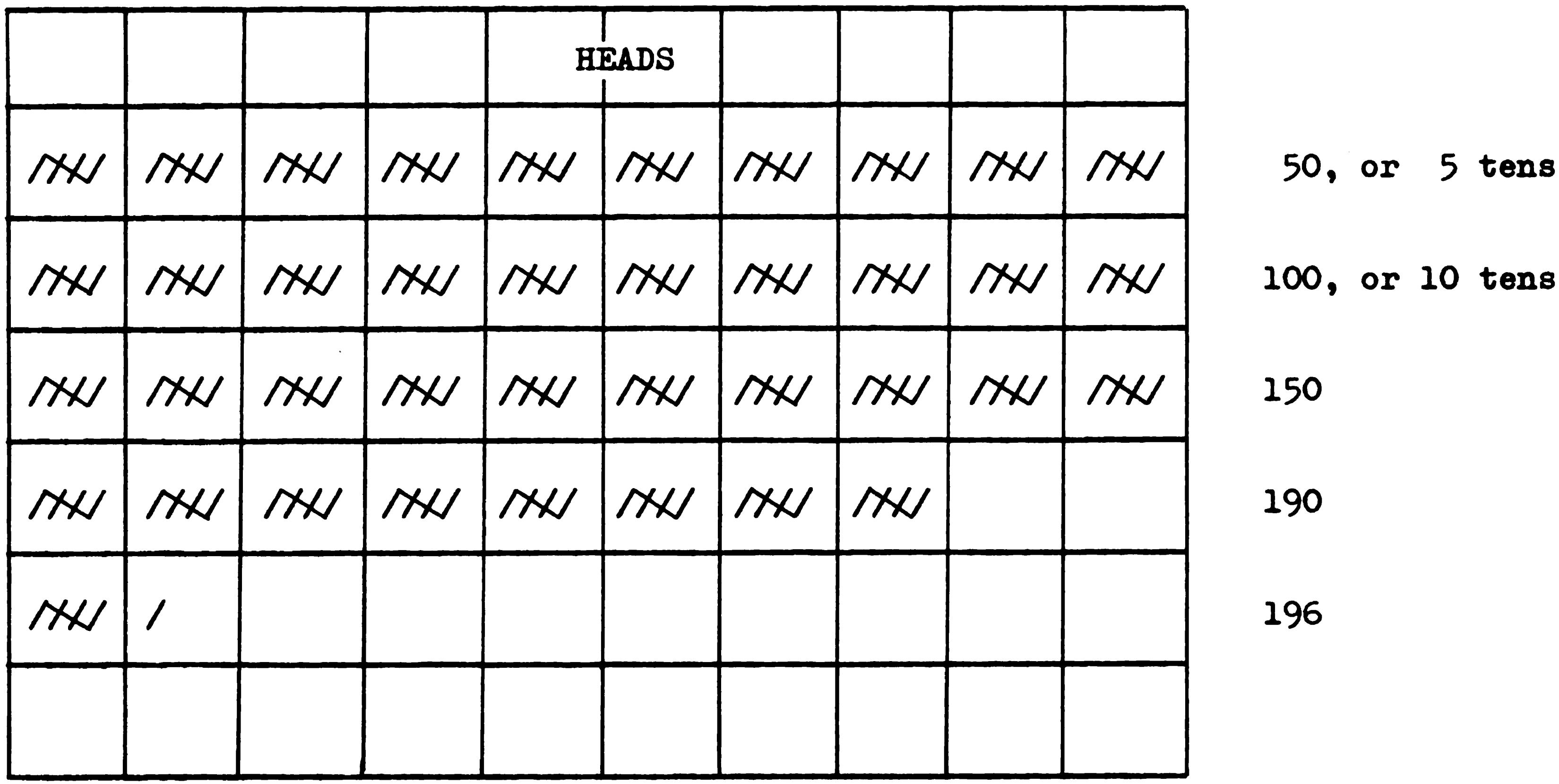

Figure 4.1. An efficient method of tallying.

The brute-force method of counting would certainly have led to some mistakes at some stage, while this systematic method* of counting could be done accurately up to several hundreds.* The subject of place-value** could have been discussed here. In conjunction with this idea, a boy at North School, Lansing, did even better: he drew a small rectangle to represent 5, and a small triangle to represent 50. He kept on tallying until he got ten rectangles before he changed them into a triangle. This is how he finally wrote 65 heads and 64 tails:

△ ▭ ▭ ▭ heads, △ ▭ ▭ //// tails.

The exchange of ten rectangles into a triangle should have prompted the discussion of different denominations** of the U.S. currency. The discussion, focused on the one-dollar, ten-dollar, and a hundred-dollar bills, would in turn have led to very vivid explanations of the old favorite subjects of 'carrying' and 'borrowing'. ('Exchange'** is probably a better word for both these terms.)

Activity 2: Abstracting some concepts in Probability from the results of playing the three types of games:

All the three types of games mentioned in Activity 1 provided a common experience to the children: they all learned the vocabulary of a dichotomous situation constituted by two mutually exclusive sets,* and this situation was the realistic counterpart of the theoretical probability* of value 1/2. In the coin-tossing group, at first it was a certain James vs. a certain Stacey, but eventually they realized that it was simply a matter of heads vs. tails. Moreover, James and Stacey could easily switch sides (because the scores* for 'heads' and 'tails' were nearly the same), and thus there was no advantage in monopolizing 'heads' by one person or the other. For the other two groups, the teacher carefully explained that the game was not really about one child betting against another, but was more concerned with the chance* (probability) of choosing correctly when somebody had to choose one thing from a set of two, (either two hands or two straws). The children apparently understood the implication of the games, and they by themselves changed the egocentric names in their tally-sheets from Kim vs. Lana, Peter vs. Dee, etc. into more abstract names such as

Get	--	Don't get,
Right	--	Wrong,
Yes	--	No,
In	--	Out.

The final abstraction* occurred when a child wrote H vs. T, and the whole class agreed that from then on the two scores on their tally sheets would be re-named H and T respectively. This was an example of generalization*. Here the scores generalized H = 19 and generalized T = 17, etc. in Table 4.9 below need not refer to heads and tails, but they might refer to the scores obtained by the two sides playing the games of choosing hands or drawing straws. It could be seen from this Table that the scores obtained by these children, especially the pooled scores,* were not too far off from theoretical values.** Another USMES class at North School, Lansing, did coin-tossing only, and their results were also close to the theoretical values.

Table 4.9 The actual data recorded from the H/T types of games.

Wexford School		North School	
H-points	T-points	Heads	Tails
27	29	65	64
19	17	7	9
141	119	24	25
176	196	8	8
196	196	35	39
19	23	29	26
21	19	12	16
10	8	27	29
27	21	15	20
64	66	38	40
45	43	30	30
		18	21
745	737	308	327

To find the sums* of the two columns of H and T became a formidable task for the third-graders at Wexford School but it was no problem at all for the sixth-graders at North School. At Wexford, several methods were attempted by the children: adding two numbers* at a time, using a graph-paper* to regulate the place-values* while column addition was carried out, and there might be more ideas. But no student got the correct answers, and finally the correct sums had to be supplied by the teacher. But they did learn something during various attempts, for example, "adding two numbers at a time" was a real-life instance of experiencing the closure law* in algebra. Column-addition was a happier event at North School: the teacher reported that the children merrily 'chorused the addition' as they produced partial sums* along the column (for each place-value). But she did not report whether they checked the answers by adding backwards.** If they did, that would be an example of using the commutative law** for addition.

The mathematical learning derived from Table 4.9 was considerable. The teacher (at Wexford) asked the children to find the total number of 'throws' (generalized throws) of the pooled results. Most students could do this without difficulty: they just added up the two sub-totals* (745 and 737) to get the grand total* of 1482. In an attempt to introduce the statements:

Probability of getting H = 1/2,
Probability of getting T = 1/2,

the teacher asked the class how much was one-half of 1482. Surprisingly, quite a few students gave the correct answer of 741. It showed that they knew the Multiplication Table* for 2, and the process of division by 2*. Then the

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teacher explained: "If you split the grand total into 2 halves, about one-half will go to H, and also about one-half will go to T." The children did not quarrel with this small discrepancy, but, at this point, the subject of <u>relative error</u>** could have been discussed.

[<u>On Relative Error</u>: Score for H = 745

One-half of the grand total = <u>741</u>

Error = 4

Thus the relative error is 4 parts in 741 parts, or, approximately, 1 part in 185 parts, which is a small error.]

This activity also gave the students a first-hand experience about the well-known fact that a small relative error is to be expected whenever <u>theoretical and experimental results are compared</u>,** like the present case: $\frac{745}{1482}$ against the theoretical value of $\frac{1}{2}$. In any case, such <u>approximations</u>** as $\frac{745}{1482} \simeq \frac{1}{2}$ and $\frac{4}{741} \simeq \frac{1}{200}$ are valuable facts to note. The writer often recalls such answers to center-of-gravity problems as "$\frac{745}{1482}$ ft. above the base," which should, in reality, be approximated by $\frac{1}{2}$ ft. Strange enough, the elementary students in these USMES classes did not find it difficult to accept $\frac{745}{1482} \simeq \frac{1}{2}$ or $\frac{308}{635} \simeq \frac{1}{2}$, although they knew very little about fractions. While playing coin-tossing or similar games, they were keenly aware of the fact that this activity was a means to figure out the <u>chance</u>* of getting one ou

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one out of two possibilities. Since they obtained 745 'heads' out of 1482 throws, or 308 heads out of 635 throws, these two probabilities had to be approximately equal to "one out of two".

During this activity, another concept could have been discussed: that of equally likely events.** The three games of choosing hands, drawing straws and flipping coins will each generate two events (called generalized H and T), both of which are equally likely. Children might be asked to suggest many more opposite-pairs, such as: to be healthy vs. to be ill; rich vs. poor; genius vs. idiot, etc., which are not pairs of equally likely events, and the probabilities** associated with these pairs are certainly not $(\frac{1}{2}, \frac{1}{2})$. Take for example the dichotomy 'Rich vs. Poor'. In many countries, the poverty line is clearly defined and evidently visible, the probability of being born in a poor family can be calculated. If only 10% of the population live above the subsistant level, then the probability of being born 'rich' is only 1/10, while the probability of being born 'poor' is as high as 9/10.

The dichotomous situation also lends itself to the discussion of complement** of a universal set. In the coin-tossing game, the complement of 'head' is 'tail'. In the set of human beings, the complement of 'men' is the subset of 'women'; the complement of 'the rich' is the subset of poor people. This last example illustrates that

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that the complement of a set may have a larger or smaller cardinal number** than the original set.

One last point about Activities 1-2: an abacus** could have been used both for tallying and for adding integers, especially those arranged in a column.

Activity 3. Graphing the data obtained from the games:

So far the children in this class had been using graph-paper for keeping tallies, facilitating the process of counting, and other purposes, but not for graphing. It was apparent that such a rich collection of H and T data could be graphed* in a meaningful way. The teacher at the point motivated the class to graph their results. First, she asked what was the purpose of spending so many USMES periods playing the H/T types of games, and the children replied: "To prove (verify) that tossing a coin, or playing other games like it, is a fair means to choose one person from two equally good people." The teacher then asked the class how to communicate this important message to other people by means of graphs. The following graphs were suggested by some children:

(1) A pair of bar-graphs* for each couple of (H, T) data: the two columns were to be placed side by side but colored differently, say red for H and green for T. Nearly every red column had approximately the same height* as the adjoining green column. People could then tell

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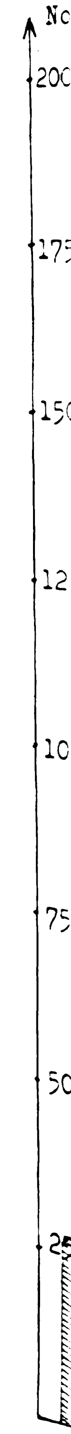
at a glance that, in most cases, the chance* of getting H is approximately the same as getting T. (See Figure 4.2).

(2) Putting all the (red) H-columns on one graph-paper and all the (green) T-columns on another sheet: the first graph-paper then contained a patch of red area, and the second a patch of green area. Using scissors to cut out the two areas* and lay one on top of the other could demonstrate that the overall chances* of getting H and T were approximately the same (provided all the columns had the same width.*)

The writer would like to add that, in (2) above, the sets* of H-columns (already cut) could have been arranged in ascending order** according to heights, and thus the median** of H could have been spotted directly from this arrangement. Similarly, the median of T could have been found. (See Figure 4.3).



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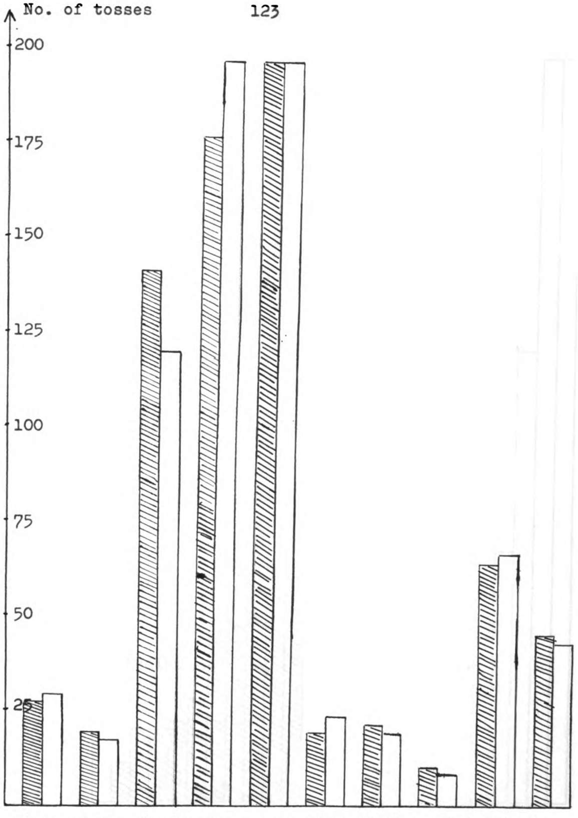

Figure 4.2: A visual presentation to show that the chance of getting H is, in most cases, nearly equal to that of getting T.
(Data were taken from Table 4.9.
Scale 1 cm = 10 tosses).

Heads Tails

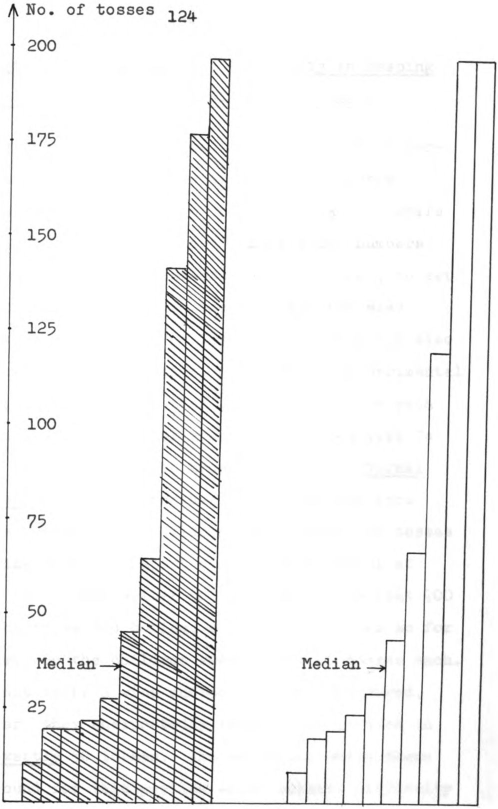

Figure 4.3: A visual presentation to show that, for a prolonged game, the overall chance for H ≃ that of T.

Median for H = 36, Median for T = 36.

Activity 4. Discussing the difficulty in keeping the tallies for tossing a coin EXACTLY 50 times:

The teacher then introduced a special kind of bar-graph which would show that relatively few players obtained an excessive number of heads (or tails), while the majority of players obtained almost equal numbers of heads and tails. The students were told that, to get such a graph, everyone had to use exactly* the same number of tosses (no more, no less). The class was also informed that, at Eaton Rapids, Michigan, an experimental USMES group had tried this idea with 20 tosses by each student, and got a beautiful graph that looked like "a tower with several layers of broader bases." (Normal Distribution*). But the teacher added that the more tosses, the better. The children volunteered 100 tosses each. Having read the report of the "Dice" group at Lexington, Mass., the writer informed the class that 100 tosses might prove too boring, or at least it was so for some people. So the children settled for 50 tosses each. It turned out to be a wise choice: nobody got bored, and moreover, they unexpectedly faced a new problem in tallying exactly 50 tosses. The solutions which these children found for this problem would enhance difficulty in counting H and T separately, if the number of tosses was too big.

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The new difficulty arose from the fact that the older method of tallying (as used in Activity 1) did not work any more. Many children started by the old method: they tallied in two separate columns, H and T, grouped the tallies in sets of 5, and meant to count them up afterwards. But, by the time they did the actual counting, most of them discovered that they had done more than 50* throws, and, since no history of the tally-marks were kept, it was impossible to figure out which marks were the first 50 throws.* Hence many had to start all over again. Quite a few made the same mistake again in the second attempt, and even in third attempt. Obviously there were better and more systematic ways of recording the tallies that would take care of the total number* of tosses as well as the numbers of H and T. About one quarter of the class were lucky (or bright) enough to come up with a system that served the above triple purposes. Examples of such systems were:

(1) Lori's Method: This girl came up with so ingenious a method that her teacher and the observers decided to xerox her work and send a copy to the USMES headquarters in Boston, Mass. Her secret was to record not only the number* of tosses, but the history* of all the 50 throws. In this way, there was no question that she might go over 50 throws, because her method could stop at any n, when n = some natural number.* The record of her work is fully reproduced here:

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NAME: Lori

H	T
8	1
9	2
10	3
13	4
14	5
15	6
16	7
20	11
21	12
24	17
25	18
26	19
27	22
29	23
30	28
32	31
33	34
37	35
39	36
41	38
43	40
45	42
46	44
47	
48	
49	
50	

Figure 4.4. Lori's Method of Tallying Exactly 50 Tosses.

This is one of the most delightful ways of using natural numbers.* She utilized the ordinal* aspect of natural numbers rather than the usual cardinal* aspect, which is implied in the 'word-problems' found in most arithmetic texts. Her thinking was very much in line with the interpretation of real numbers** in Classical Physics: in a classical problem, it is usually required to determine the position and/or velocity of a particle at time $t = 3$, say, what they really mean is to follow the spatial behavior of the particle at the end of the 3rd second (or some other time-unit) which is an ordinal number.

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The marvelous thing about Lori was that nobody taught her Classical Physics before, nor any usage of ordinal numbers. This is another instance to substantiate the claim that the interplay between the ordinal and cardinal aspects of natural numbers is innate within the human race. Hence in the older days (before Peano, Cantor and other pedantic logicians), such interplay between the ordinal and cardinal interpretations used to be the foundation** on which addition** was defined. In the opinion of this writer, this is still the best way to define the addition of two natural numbers, for example: 17 + 25 should imply the result of counting** until you reach the twenty-fifth natural number after 17, and you will unmistakenly land on the natural number 42. The writer once had the experience of teaching 'addition' to adults (by using the above definition) for the Program of Illiteracy Eradication, implemented in certain remote parts of rural Thailand, and the result seemed to be encouraging. (No comparative studies were done to support this claim yet, but one would be carried out in the near future.)

Going back to Lori's work: in counting up the number of H, she used the 'grouping in 5' method, i.e. she underlined every fifth integer, and counted 5, 10, 15, 20, 25, 26 and 27 for H. Similarly it was 5, 10, 15, 20, 21, 22 and 23 for T.

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(2) A girl wrote down 1, 2, 3, ..., 50 first, and then placed either H or T (as the case might be) underneath each natural number, and she carefully counted up the H's and T's afterwards. This method was inefficient in the sense that a miscount could have easily occurred, and it was almost impossible for a large number of tosses,* say 500.

(3) A student kept the tallies for H and T in the usual manner, i.e., tallying in two columns, and wrote the tally-marks in groups of 5. But he kept a second sheet to record the total number* of throws all along. This method had the snag that it was quite easy to forget tallying the second sheet. This boy had to do the experiment twice, because, in his first trial, he found that the two sub-totals for H and T gave the sum of 52 instead of the required 50.

(4) Just like (3) above, minus the second sheet. Using the fact that 8 times 5 = 40, a student could have given himself the first 'warning shot' when he got 8 groups of tallies, say 5 groups of H and 3 groups of T, or vice versa. (Such deviation** from Normality as 6 or more groups of H is usually rare). Care was to be exercised after the 41st throw until the student completed the 9th group or the set of 45 throws. After that he could have used Method (2) for the last five throws of the game.

It was a pity that nobody in class used Method (4), or at least the writer did not observe any. This is a relatively efficient method, for, with the slight modification, it can be used to record H and T up to the total of 200 throws or more. For an assigned task of 200 throws, one starts with 19 times 10 = 190, and hence one could proceed leisurely for the first 19 groups. (Again, by normal distribution, the scores should appear as 10 groups of H and 9 groups of T, or vice versa.) Then one uses Method (2) for the last ten throws.

Incidentally, this method would clearly exhibit the distributive law**. In the first numerical example mentioned above, 40 could be viewed as either "5 times 8" or "5 times 5 plus 5 times 3". In the second numerical example, 190 could be viewed either as "10 times 19" or "10 times 10 plus 10 times 9."

Activity 5: Building up a histogram by using the data collected from the 50-throw games, and making spontaneous comments about the Probabilistic Model:

Of the 26 children in this class, only two did not succeed to record the tallies of H and T for the particular game involving exactly 50 throws, and so there were 24 sets of data to build up a histogram* on the chalkboard. The teacher began by drawing a horizontal axis* and put 50 dots (spaced at equal interval*) on the axis. She then wrote ordered pairs* of natural numbers underneath these dots:

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0	1	2	3	. . .	47	48	49	50	H
50	49	48	47	. . .	3	2	1	0	T

Her ordered pairs were written in the form $\binom{H}{T}$, where H referred to the number of 'heads' and T the number of 'tails'. Although this was an unconventional notation* for an ordered pair, which is usually written as (H,T), there should be no doubt about the children's internalization of this important concept in mathematics. During this activity, the children could differentiate between $\binom{24}{26}$ and $\binom{26}{24}$.

Each child took a turn to march forward to the chalkboard, proudly announced his scores for H and T (whose total* was checked by the class to make sure that the total = 50), and then the child put a box above the corresponding* ordered pair on the axis. If there was a box there already, he just put another box on top so that his was still above that particular ordered pair. The class nearly went wild when a boy mistakenly put his box above $\binom{24}{26}$ instead of $\binom{26}{24}$ because he had previously announced H = 26, T = 24. (Of course he quickly corrected the mistake.) The teacher should be congratulated for having transmitted the concept of an ordered pair so successfully.

Gradually, the central part of the axis began to look like multi-story towers. Although the highest 'tower' (maximum frequency*) did not fall on the point corresponding to $\binom{25}{25}$ as expected, the data obtained here clearly communicated the concept of normal distribution* to the children. (See the histogram in Figure 4.5). Most 'boxes' clustered around the center, and a few off-center boxes were found on both sides.

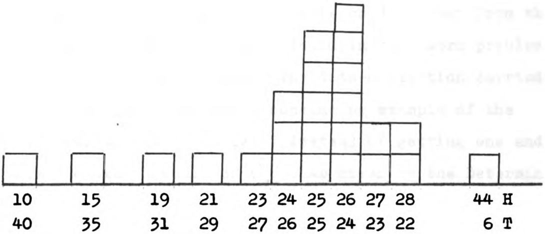

Figure 4.5. The Histogram obtained from the 50-throw games.

Children made several interesting comments about this histogram and one boy even commented about the Probabilistic Model* in general. For example,

-- " $\binom{25}{25}$ means a tie;* five people got a tie because there are five boxes above $\binom{25}{25}$."

-- "Most boxes are around the middle, and so this game (coin-tossing) is fair."

-- "This picture (the histogram) should be called the 'USMES Board' because we got it during the USMES period."

-- "It is an unusually nice bar-graph." (Perhaps the child referred to the pattern of normal distribution*).

Teacher: "We have two boxes $\binom{10}{40}$ and $\binom{44}{6}$ a long way off the center. Are they wrong?"

A student: "No, it just happened that way."

This last sentence, in a naive way, summarized the underlying philosophy of the Probabilistic Model*, or the Mathematics of Uncertainty. It is so different from the Deterministic Model* usually found in the 'word problems' of arithmetic and algebra. The data-collection carried out in this Activity was a convincing example of the Probabilistic Model of 1/2. Instead of getting one and only one exact result of $\binom{25}{25}$, as given by the Deterministic Model, the children now learned to appreciate the Probabilistic way of thinking: one should not expect an exact answer; one could only expect the answers to cluster around some value,* and one even makes allowance for a few off-target values.

Another mathematical behavior could have followed immediately: the continuous curve**. The teacher could have joined the tops of the bar-graphs together by means of a continuous curve (See Figure 4.6) because this histogram is an approximation to the Normal Curve**. Although the curve obtained here was slight skew,** it could have been a good introduction to the subject of Normal Curve, which children would meet later on.

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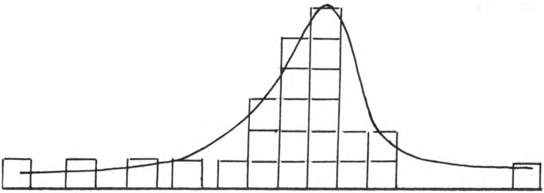

Figure 4.6. The histogram as an approximation to the Normal Curve.

A comment should be made about the unusual way this teacher labelled the horizontal axis of Figure 4.5. Instead of writing $(^{0}_{50})$, $(^{1}_{49})$, $(^{2}_{48})$, etc. for values $(^{H}_{T})$, she could have used the easier abscissas** 0, 1, 2, ... by showing the values of H alone. This should be possible because T = 50 - H, and hence, once H assumed a certain value, T could assume only one corresponding value.

The way she used 'boxes' laid above the abscissas to denote the frequencies* (her'y-co-ordinates') for various $(^{H}_{T})$ was a good method in introducing ordinates* or y-co-ordinates. If she had introduced a vertical axis** at the left of the diagram, and labelled this axis "frequencies", then the whole idea of a co-ordinate plane** could have arisen by plotting 'frequencies' against H.** This plane would consist of the set of points** defined by the ordered pair (H,f), where H referred to the number of 'heads', and f referred to frequencies or the heights of t

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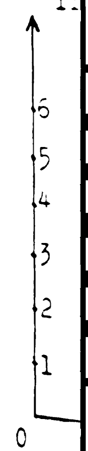

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of the 'towers' in Figure 4.5. Significant points** in this set are shown in Figure 4.7, and they also approximate the Normal Curve.

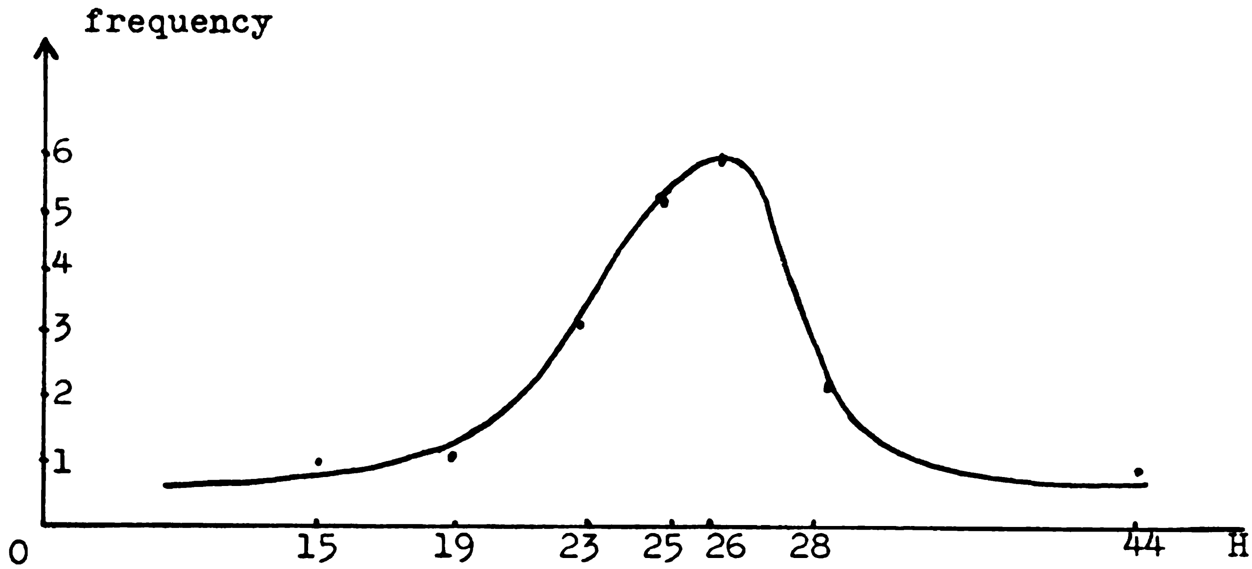

Figure 4.7. The set of ordered pairs (H,f) visually presented.

If one were not happy about the approximate nature of the graph in Figure 4.7, one could have asked the children to graph

$$T = 50 - H$$

and one would have obtained a perfect straight line.* (Figure 4.8). In either case, the following generalization** could be made:

A graph* is a visual presentation of the set of ordered pairs (x, y), once the (horizontal) x-axis, and the (vertical) y-axis have been clearly defined.

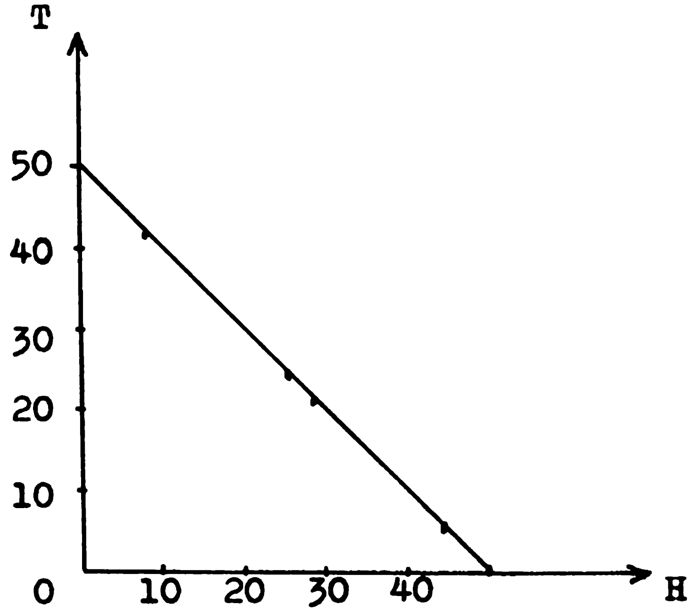

Figure 4.8. The graph of T = 50 - H, drawn from typical ordered-pairs: (0,50), (10,40), (25,25), (28,22) and (44,6)

<u>Activity 6</u>: <u>Devising horse-race types of games to be played in conjunction with coin-tossing</u>:

The children were now satisfied that tossing a coin is a fair means to decide between two persons, and they could conceive of many interesting games whose moves depended on the results of tossing a coin. Obviously the children enjoyed this period which was a welcome change from a very structured Reading Program just before it. Several horserace types of games were devised by the children themselves (with very little help from the three adults in that classroom). By far the most interesting game observed was the game whose 36 moves were constrained to the <u>sectors</u>* and <u>annuli</u>* of six <u>concentric</u> <u>circles</u>;* each circle was divided into six parts by six <u>radii</u>*, making an angle* of 60° to each other at the common center.* (See Figure 4.9). The 36 spaces were numbered 1 through 36 in a counterclockwise* direction. At the start of the game, all the players placed their counters at position* '1', and the main idea was to reach position '36' before the opponents did.

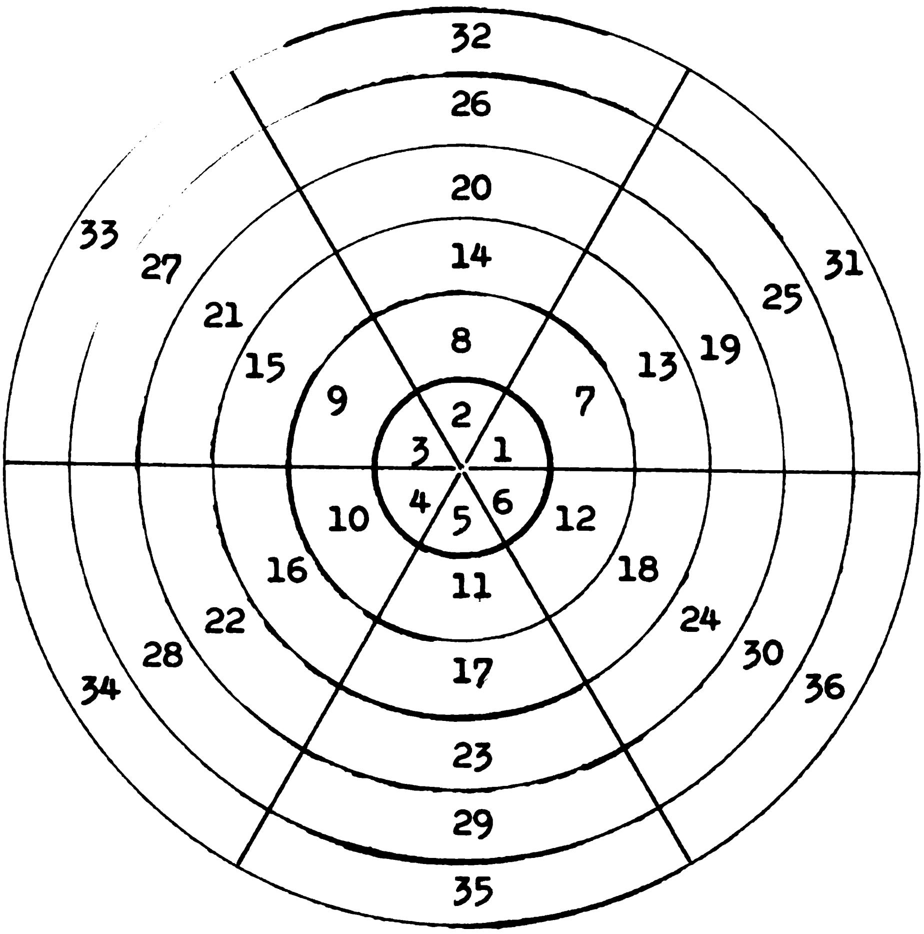

Figure 4.9. The circular tracks in which the counters moved.

Normally each player would 'go' one step at a time by getting a 'head' or a 'tail'. If he got four consecutive* heads, he would be awarded bonus points, and allowed to move his counter radially* outwards, i.e., to gain an increase* of 6 points. If he got four consecutive tails, he would then be permitted to move his counter to 'the other side' of the same circle (to the space diametrically opposite*), i.e., to gain an increase of 3 points. The 5th consecutive head or tail would, however, be awarded one point only, and this should be taken as a warning of the penalty to follow soon. The 6th consecutive head or tail would bring a penalty of -12 points or going back 12 steps. (If any player was less than 12 steps from 'start', he had to move his counter back to 'start'). The children explained that the 6th consecutive head or tail would be more likely effected by cheating than by luck. The probability** of getting 6 consecutive heads could have been discussed at this point. On the whole, this Activity provided a rich experience for mathematical learning. It could act as a catalyst to spark off the following mathematical lessons:

(a) The discussion on the probability of getting 2, 3, 4 ... consecutive heads: Although it would not be difficult to verify by coin-tossing that the probability** of getting 2 consecutive heads $= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, it would be rather tedious to do so, and it was certainly not advisable to experiment with the next cases of 3, 4, 5, 6, ... consecutive heads. It is a big question mark whether the teacher should explain the principle

$$P(A \cap B) = P(A) \cdot P(B)$$

to elementary school children or not. It involves this difficult concept: the <u>joint</u> <u>occurrence</u>** of two events which are <u>independent</u>** of each other. Piaget[1] contended that children before age 11 would not appreciate probability in terms of 'proportion of favorable outcomes,' but Yost[1] disputed this contention on several grounds especially the following:

(a) Piaget's experiment seemed to confound 'preference' with 'expectation'.

(b) Piaget's experiment relied too heavily on the verbal skill of the children -- a doubtful quantity before age 11. Yost re-designed the experiment so that it involved mainly non-verbal responses from the children, and his finding was that children between 7 and 10 <u>could</u>, in fact, learn elementary concepts of probability if the learning environment was appropriate. Yost's conclusion was later supported by two studies, carried out by Davies[2]

[1]See Yost, A., "Non-verbal Probability Judgements by Young Children." <u>Child Development</u>, Vol. 30, December 1962, pp. 769-780.

[2]Davies, C., "Development of the Probability Concept in Children," <u>Child Development</u>, Vol. 36, September 1965, pp. 779-788.

and Goldberg[1] respectively. Davies went even further than Yost by concluding that children above the age of 9 could learn not only the non-verbal tasks in probability but also the verbal ones, if these tasks were properly managed; while Goldberg re-affirmed that the rudimentary concept of probability could be learned even earlier, i.e. by 5-year-old children. It could be inferred from Davies' and Goldberg's studies that the probability of the joint occurrence of independent events should be within the grasp of upper-grade children in elementary schools.

(b) The concept of angle measurement derived directly from Figure 4.9: Both units of angle-measurement, the degree** and the radian**, arose naturally from the activity of designing Figure 4.9. Take the radian first: the children would have a good chance of discovering by practical means that the arc-length increases in the same proportion as the radius, if the vertical angle is kept constant.** For example, using a copper wire or a fuse, the children could verify that the arc-length** separating '7' and '13' is twice the arc-length separating '1' and '7' when the radius of the 2nd circle is twice the radius of the innermost circle. Similarly, the arc-length separating '13' and '19' is 3 times that separating '1'

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[1]Goldberg, S., "Probability Judgements of Preschool Children," Child Development, Vol. 37, March 1966, pp. 157-167.

and '7' when the radius of the 3rd circle is 3 times that of the innermost circle.

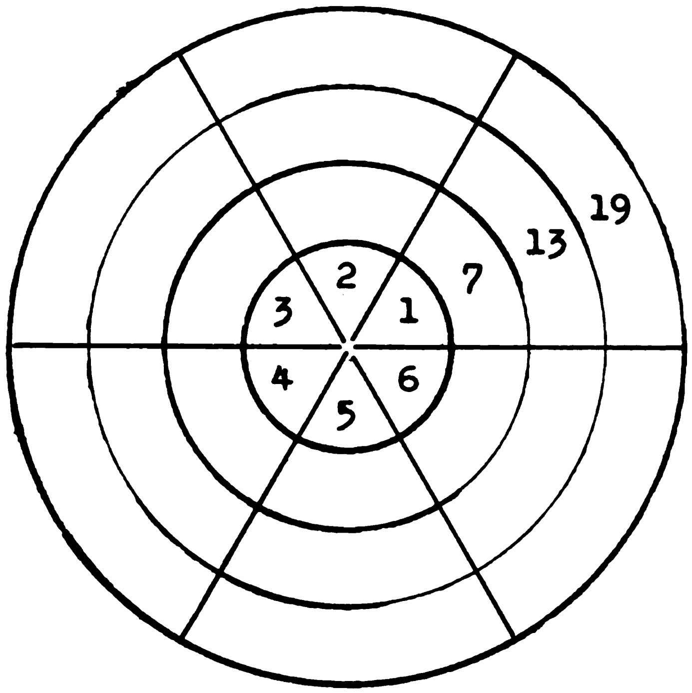

Figure 4.9, partially reproduced.

Hence the children would have the opportunity to associate** a constant angle** with the constant ratio: $\frac{\text{arc-length}}{\text{radius}}$. The teacher may or may not verbalize this association or correspondence.** If she does, the word 'radian' can be introduced. The idea of measuring angles in degrees is somewhat easier and perhaps more suitable for children at this level. A complete revolution* (like moving the counter along positions '1' through '6') is assigned 360° (an arbitrary number). This was divided into 6 equal parts, as shown above, and hence each angle = $\frac{360°}{6}$ = 60°.

(c) Introduction of the equilateral triangle and the hexagon: Six triangles can be formed by joining together the points A,B,C,D,E,F on the innermost circle, as shown in Figure 4.10.

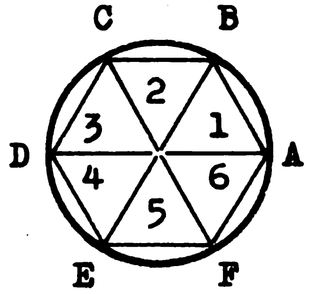

Figure 4.10. The six equilateral triangles and the hexagon formed inside a circle.

By direct measurement,** or by consideration of symmetry,** the six triangles can be said to be 'equal in all respects,'** and moreover, each is equilateral.** Paper-cutting and paper-folding would help tremendously in uncovering the above facts. After cutting out the six equilateral triangles from paper, the children should be encouraged to try to do the same on cardboard, and this time they would be asked to produce a greater number of equilateral triangles. "Why?" may be the children's reaction to having so many equilateral triangles in front of them. At this point, the major challenge of the Dice Design Unit could be put forward: "How can you design some other kinds of dice, besides the cube, which are fair, and easily tested to be so?" The children, playing with the equilateral triangles in front of them, might come up with a tetrahedron.** If several pairs of hands were helping to stick the triangles together into some sizeable 3D objects, they might even discover the icosahedron** and the octahedron,** which are fair dice. Obviously they would also produce a host of other convex polyhedra,** which might not act as fair dice. Holden[1] listed eight possible convex polyhedra which could be constructed from equilateral triangles alone. They are: the tetrahedron (4 faces), the octahedron (8 faces), the icosahedron (20 faces) and the remaining five contain 6, 10, 12, 14 and 16 faces respectively. The experience in constructing polyhedra would help to clarify many concepts in 3D geometry later on, e.g., the angles between pairs of planes,** (children called them the 'corners' of a die), and the intersection of two planes** (an 'edge').

As for 2D geometry, the learning resulting from this Activity would be equally spectacular. When a child lays an equilateral triangle casually on top of another and rotated until the two coincide, he would learn that the acts of lifting, moving and rotating a triangle do not change its sides nor its angles. (Lengths and angles are invariant** under the Euclidean group.**) When a base angle (of the top equilateral triangle) is

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[1]Holden, Alan, "Space Visualisation," USMES Working Paper, Serial No. D3 in the 'Dice Design' Unit, 1969, p. 5.

made to coincide with the vertical angle (of the lower triangle) after some rotation, it carries a message that each angle of either triangle is 60° by the arguement in (b). Hence the sum** of the 3 internal angles = 180° for the case of the equilateral triangle, and the child should then be encouraged to speculate the result (after several direct measurements and addition) for the general case. He would soon find out that the sum of the internal angles of any triangle is 180°.

Figure 4.10 also demonstrated the plane figure of the hexagon.** Various activities could follow the recognition of this figure, e.g., the investigation of the size of its internal angle (each = 60° + 60° = 120°), and the sum of all its external angles (= 180°, and why?), not forgetting the probable construction of polyhedra using both equilateral triangles and hexagons.

(d) Introducing Arithmetical Progression** and Arithmetic Mean:** Each of the six sectors of the largest circle in Figure 4.9 could have provided the best introductory example of an Arithmetical Progression,** (in this case, the common difference** = 6). Children are naturally curious and attracted to patterns, including number patterns**. (The writer still remembers vividly the sight of a 4-year old Thai baby-girl counting the numbers of flowers and octagons appearing in the floral-patterned sarong (skirt) her mother was wearing that day.) Referring back to Figure 4.9 again, one could justly say that the sequence 6,12,18,24,30,36 might not generate too much excitement (because that was merely the Multiplication Table* for 6), but all other sequences,** such as 7,13,19,25,31 would be a challenging pattern to the children if a teacher asked such timely questions as:

-- "Will an arbitrary number, say 85, be included** in the above sequence? How do you check it?"

-- "Now if you call the next number (37) the sixth term,** and one next to that (43) the seventh term, etc. What number is going to be the 100th term?"

Surely there could be found some children who would keep on adding 6 and tally every stage of the addition until they finally reached the 100th term, and meanwhile, if all the partial sums were correct, they would hit on the number '85' somewhere along the line. Although this was a good practice on addition, a more efficient method did exist, and all it needed was the concept of one-one correspondence**:

$$\begin{array}{cccccc} 7, & 13, & 19, & 25, & 31, & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 6, & 12, & 18, & 24, & 30, & \ldots \end{array}$$

The writer asked the 'Soft-Drink' Group (6th graders) to solve the above two problems for the 'Dice Design' Group (3rd graders) and some positive reinforcements (Rare postal stamps from Thailand) were promised to be given to any 6th graders who could communicate the ideas or at least the correct answers to the 3rd graders. All the seven 6th graders got the rewards. After informing the writer of the (correct) answers she obtained, one 6th grade girl said she was going to explain to the 3rd graders in the following way:

"Look at these two lines:

7, 13, 19, 25, 31, ...
6, 12, 18, 24, 30, ...

Do you see that the top line is always ONE more than* the bottom line? - (Yes). Do you also see that the bottom line is just the Multiplication Table* for 6? - (Yes). Now take the last number in the bottom line (30); call it by another name,* (the 6th term) is 36 or 6 x 6. The seventh term is going to be 6 x 7, and you always do that for the bottom line. Its one-hundredth term = 6 x 100 = 600. The top line is always ONE more, so the answer is 601."

"Why don't you try out your teaching method now? The boy concerned is in the Design Lab perfecting his 'circular' track," said the writer.

The meeting was cordial and the boy did pay much attention to her explanation, because it flattered him to think that somebody else should take interest in his circular track (Figure 4.9). But the 6th grader spoke so fast that the writer was not sure if her listener could absorb any part of her explanation except the final answer '601' which he took for granted. In any case, it is difficult for a third-grader to deal with 3 almost simultaneous mathematical operations: treating 100 as an ordinal number,* multiplying* 100 by 6, and adding* 1 to 600.

As for the first question: "Is 85 a member* of the set* 7,13,19,25,31,37,...?" the sixth grader again dealt with that on the basis of the one-one correspondence* with the set 6,12,18,24,30,36,... They multiplied successively all the natural numbers by 6, until they got 14 x 6 = 84. The next product 15 x 6 = 90 was too big, so they went back to '84' and added ONE to it. So they concluded that '85' is in the set 7,13,19,25,30,... Apparently it did not occur to this group of children that the two inversed operations**: subtracting 1 from 85, and dividing by 6, could have provided them with an easier test.

The A.P. 7,13,19,25,31 would give an arithmetic mean** of $\frac{1}{5}$ (7 + 13 + 19 + 25 + 31) or 19 which would carry a deeper significance than the mere addition and division by 5. For an A.P., its arithmetic mean is always a median*: the 'middle' number in the proper sense of the word. For example, the '19' above is mid-way between '7' and '31', between '13' and '25' etc. Unfortunately, the 'mean' has been generalized to include results obtained from applying the algorithm $\frac{1}{n}(a_1 + a_2 + a_3 + \dots + a_n)$ on any set of numbers $\{a_i\}$, not just A.P., and thus the 'mean value' gets considerably distorted from the 'median value'. In this writer's opinion, the best way to introduce mean-values or averages to children is to use examples from Arithmetical Progression, so that the children may see the ideas behind its algorithm.

Watching the children playing this 'circular track' game was enjoyable. Two children, A and B, used two pennies to play the game and they were very excited whenever bonus points or penalties were about to come up. B won when he was 'home' in 35 moves. He was obviously lucky: he got four consecutive heads twice and four consecutive tails once out of 35 throws, while the theoretical probability for getting four consecutive heads or tails is only 1/16. The chronological movements of A's and B's counters are shown in Table 4.10 (The integers in this Table refer to the positions of the counters on the circular track.)

So A lost the game by 36-32 or 4 points. Out of the 35 tosses, B got 18 heads and 17 tails, both of which were remarkably close to the theoretical result of a perfect coin. It should be noticed that both B and A suffered the penalty of (-12) points once in their 35 tosses. The <u>theoretical probability</u>** of getting this penalty is $(1/2)^6$ or 1/64, but neither the number of throws nor the size of the circular chart was big enough to test this theoretical value. So this experience should be taken as a means to introduce and utilize <u>negative numbers</u>* more than anything else. The writer also questioned the children why they gave 6 bonus points for '4 heads' and only 3 bonus points for '4 tails'. They replied that they just wanted two different methods of

Table 4.10. Chronological results of a two-player game using the circular track in Figure 4.9; the integers shown in boxes refer to either bonuses (if increasing) or penalties (if decreasing).

	A		B	
	H	T	H	T
	4	1	1	2
	5	2	3	4
	7	3	5	7
	10	6	6	8
	12	8		9
	13	9		12
	14	11		13
	20		2	1
	21		3	
	9	10	4	
	11		10	11
	12	13		12
	14			13
	15		17	16
	16		18	
	22	23	19	
	25	24	25	26
	26	27	27	
	28	29	28	29
	32	30	31	30
		31	32	
			33	34
			35	36
No. of tosses	20	15	18	17

jumping on the circular track. Although they learned earlier that the chance for getting 4 heads is the same as that of getting 4 tails, the two different bonus systems should bring home the truth that equal chances do not always bring equal rewards in real life.

Another boy in this group was playing by himself on 'championship': he was tossing a coin to find out the semi-finalists, the pair of finalists, and the champion from 8 competitors or 8 teams. His arrangement was simple: the 8 competitors or teams were grouped arbitrarily into four pairs,* and he flipped the coin 4 times to get the 4 winners or winning teams of the first round. These were again grouped arbitrarily into two pairs, and he tossed his coin twice to select the finalists. He then flipped the coin for the last time to choose the champion. The process of successive elimination* was something like the following diagram:

A			
B	A		
C		A	
D	D		
E			A
F	E		
G		G	
H	G		

It should be pointed out to the child that the above method of elimination is unfair, although such method is actually used in many tournaments, and also in selecting candidates for political offices. The transitive law** does not hold when one discusses human ability. There have been numerous counter-examples** taken from tennis or chess tournaments, or any fields of human activities whose records showed that A beat B, B beat C and yet C beat A (if C was allowed to challenge A.) A fairer arrangement is to have each competitor playing against all the rest, one by one. Two points will be awarded for winning, one point for a draw, and none for losing. For 8 competitors, there will be ${}^{8}C_{2} = \frac{8x7}{1x2} = 28$ games altogether in the first round and the maximum points earned = 2(8-1) = 14. There may not be a second round if only one competitor gets the highest total**. (Here the highest total need not be the maximum score of 14, because there might be some draws.) This is another instance of using mathematics to create a fairer situation to all concerned. It also provides the best motivation to learn the Combination formula**

$$ {}^{n}C_{r} = \frac{n(n-1)(n-2) \ldots \text{ for r factors}}{1.2.3 \ldots (r-1)r} $$

The ${}^{n}C_{2}$ method has another advantage: it can be used for any number of competitors,** while the child's method of elimination described above can be used conveniently only for 8,16,32, or generally 2^k competitors.

This discussion brings to a close the major types of mathematical behaviors that arose, or could have arisen, from the Dice Design Unit during the observed period (September 25-November 27, 1972).

Conclusion: Despite the name 'Dice Design Unit,' the activities undertaken by the observed children were far from designing a die, or even playing with the usual 6-sided commercial dice. They spent almost the whole term investigating problems concerning the simplest 'die': the coin. They also extended the game of coin-tossing to include drawing straws and choosing hands, and then generalized the head/tail concept into a pair of equally likely events. Some forty mathematical behaviors arose naturally from the children's determined effort to answer such demanding questions as:

-- how to devise an effective method of tallying H and T so that anyone could quickly and conveniently count them up afterwards,

-- how to graph the results,

-- how to devise an enjoyable horserace type of game to be played in conjunction with coin-tossing.

The above implied that a great deal of mathematical learning occurred around the themes of Probability and Graph, both of which had been termed 'basic' or 'unifying' themes by the National Council of Teachers of Mathematics. The Council's 24th Yearbook (1959) presented a list of seven "most basic mathematical themes which should be central to the entirety of a modern mathematical curriculum."[1] The word 'entirety' was later elaborated to mean "recurring and varied contact with these fundamental concepts and processes so that understanding might grow within children throughout their school careers." Two years later, the

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[1]N.C.T.M., "Growth of Mathematical Ideas, Grades K-12," 24th Yearbook, 1959, pp. vi and 2.

N.C.T.M. issued an important pamphlet,[1] stressing ten "unifying themes" for teaching modern mathematics. On both occasions, Graph and Probability were included in the list. By introducing Graph and Probability in an activity-oriented atmosphere, this USMES unit is a positive step towards the idealism deliberated by the N.C.T.M.

Some people may doubt the real value of such intuitive ideas on Probability as introduced by this USMES unit, but William Feller,[2] an experienced teacher in this field, gave the assurance that intuition would serve as background knowledge and guide towards deeper theory and more sophisticated applications.

These USMES children had another advantage over the children who learned Grade 3 mathematics in a traditional class: the former had the first-hand experience in recording an experiment (by tallying) and counting before they performed any piece of the resulting arithmetical work. Such arithmetical problems immediately became more relevant and meaningful: they were not just another sheet of paper, full of numbers, to be handed to the teacher aide who would grade it the next day, and to be cast away soon after that.

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[1]N.C.T.M., "The Revolution in School Mathematics," A report of regional orientation conferences in mathematics, 1961, p. 22.

[2]Feller, William, "An Introduction to Probability Theory and Its Applications," New York, John Wiley and Son, 1968, p. 2.

Professor Lomon and Mrs. Beck[1] ably described four kinds of coin-games most suitable for this Unit. It had been a real concern of this writer that the teacher might have taken the USMES Manual like a cook-book, and just prescribed the 4 games to her class, but fortunately this did not happen. In fact the children of the observed class had ample opportunity to show originality, from creating ingenious tallying methods to constructing circular horserace track on tri-walls, so long as such originality did not involve gambling with money.

The Wexford group concentrated on learning Graphs, Probability and other derivable mathematics from coin-games. The writer was informed that, in the following term, the children would have the opportunity to construct and play initially with the cube, the tetrahedron, and the icosahedron, in response to the challenge: "How would you design dice for 3, 4 and 5 players?" Some 'Dice Design' groups[2] elsewhere did begin the unit by building regular polyhedra from precut polygons. This is also a healthy approach. The children in such classes

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[1]Lomon, Earl and Beck, Betty, "Coin Games," <u>USMES Working Paper</u> Serial No. D10, 1972, p. 1-18.

[2]Take, for example, the 'Dice' group at Champaign, Illinois. The teacher introduced two cubes and a dodecahedron as early as the second class meeting. Scarcely had the third meeting started when the children were on the way to construct various polyhedra and test their 'fairness'.

will have some valuable first-hand experience about the space of 3 dimensions (particularly space perception). They can also get familar with various terminologies and concepts in Solid Geometry such as, the angle between two planes (faces), the intersection between two planes (i.e. an edge of the polyhedron), the altitude of a tetrahedron, etc. The precut polygons themselves offer many valuable theorems in plane geometry.

Throughout the history of mathematics education, the 'Dice Design' unit is probably the first practicable blueprint bringing together two divergent disciplines: 3D Geometry and Probability. But, on reflection, one would see that both disciplines really represent different parts of man's efforts to study Nature as a whole. It would be apt to close the Case-Study of this Unit with the following lines:

> "All Nature is but Art, unknown to thee,
> All Chance, Direction, which thou canst not see..."
>
> Alexander Pope (1688-1744)

The CASE-STUDY of the "Designing for Human Proportion" Unit, Observed at Pleasant View School, Lansing, Michigan October 3 - November 30, 1972

I. The behaviors to be observed would focus on the use of numbers to describe a human body in contrast to the use of such vague terms as fat/skinny, tall/short, etc. Children's involvement with these numbers would, hopefully, not be confined to arithmetical work on "word-problems" about body measurements, but would also include the actual experience of measuring (not excluding the discussion on the accuracy of measurements); the recording and tabulating of data; graphing and the calculation of the mean, median, mode and range; extensive applications to the designing of clothing, furniture and house fixtures; application to Graphic Art, especially the drawing of a human face. A small group of children here posed another branch of challenge: "What is inside a human body?" which would lead to an elementary study on anatomy.

II. Brief Description of the 'Human Proportion' Group:

Fourteen 5th graders left their usual classroom every Tuesday and Thursday (1:50 - 2:45 p.m.) to work on USMES in the School Library. Measuring tapes, strings, yard-sticks, foot-rules were provided for the class. The library had a beautiful carpet and children found it most convenient to work on the floor, because the measuring activities often involve kneeling down, sitting on the floor with their backs against the wall, or even lying flat on the floor. The round tables (seating 5 or 6 children) in the library were most useful, because children, while graphing and drawing pictures, could compare each other's work and get immediate feedback. The library also contained many things worth measuring and comparing with the children's own heights, for examples, book-shelves, coat-racks, drinking fountains, slide-projectors, film-loop projectors, etc. Geographical globes were there to provide a case for measuring on a curved surface. Encyclopedias were available for the small group interested in simple aspects of human anatomy.

All the children were in the traditional Grade 5 level of mathematics and science. No reading nor communication problem was observed.

<u>Activity 1</u>: <u>Using numbers to describe people; Measuring children's bodies with specific purposes such as making garments, and designing furniture and other things for children's use</u>:

The teacher posed the following set of questions to the class: "How can your mother or a tailor make garments to fit you? How can a carpenter make chairs, desks and other furniture to suit children of your size? How can the shoe-factories manage to produce shoes that will somehow fit most children?" The answers to all these questions led the children to the conclusion that <u>measuring</u>* various parts of a human body <u>is</u> a very useful and essential activity of life. They seemed eager to have the experience and practice of measuring people, because they felt that qualitative terms like 'tall/short', 'fat/skinny', etc. were not adequate for many purposes, especially to answer the above questions. Measuring would provide them with <u>quantitative data</u>* to work on. They could then use <u>numbers</u>* to describe people, e.g. "He is a 180 lb. man," instead of "He is fat."

Three or four measuring tapes were then passed along, and the children merrily measured each other in small groups. The teacher warned the class about two things:

Throughout this Case-Study, the words <u>underlined and followed by</u> * refer to the mathematical behaviors observed in this group, while those marked ** refer to the <u>potential</u> mathematical topics that could have been discussed.

first, each group should measure with a specific purpose, and secondly, the accuracy* of each measurement should be checked by at least one of the peers in the group. The children then organized themselves into 3 groups: the first group pretended to be tailors, and shoe-makers, (but one girl did produce a pair of sandals from tri-wall and strings); the second group pretended to be carpenters, and the third group to be builders who planned facilities for a new elementary school, e.g. drinking fountains, bath-room basins, coat-racks, etc.

1. The Garment Group: After some discussion, the group decided to concentrate first on measuring and checking the accuracy of each measurement. It was hoped that they would be provided with needles, thread, and inexpensive material later on to try out their measurements like a real tailor. A girl said she would definitely learn how to use the sewing machine afterwards. Each person in the group then fetched a piece of paper and was responsible for recording* his or her own data*obtained from tape-measurements. These would-be tailors seriously measured the parts of their peers' bodies and checked the data before recording. Most children in the group included the following measurements*: collar size (which is slightly bigger than the measurement round the neck); width of shoulders (distance from left shoulder to right shoulder); arm-length; breast size or chest size; width and length of the back; waist size; hip size; pants length; skirt

length or dress length; length and width of feet; depth of a shoe.

At first they wanted to measure to the nearest 1/8 of an inch,* but this high degree of accuracy only resulted in too many arguements and the standard had to be relaxed down to the nearest half inch.* Little did the children realize that they were measuring curved length* when the curve* itself was only defined vaguely. This was particularly true when measuring hip sizes and waist sizes. Watching these children measuring each other made the writer realize that a tailor's measuring never involved straight-line measurements, and the children here were quite realistic about it. For example, the length of the pants leg has to be longer than the distance between heel and waist, measured by two marks on a vertical wall. The distance between the marks on the wall would give a straight-line measurement, which is, by Euclid's first postulate, shortest,* and hence no allowance would be made for the movement and bending of the knee.

This activity gave the children an opportunity missed by most children in a traditional 'math' class: the measuring of curved distance. Too many 'word-problems' in arithmetic provide measurements which involve only straight lines. It was not surprising to find a girl in this group trying to measure her girl-friend's waist size by means of a straight ruler. Her method was ingenious, though.

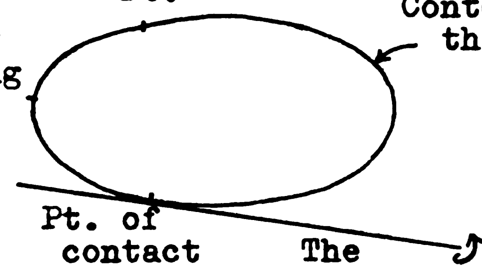

Figure 4.11: Measuring a curve by means of a straight-edge ruler.

She started with one end of the ruler on her friend's left side and gradually swung the ruler round the waist, making sure that the ruler was always a tangent* to the curve. The point of contact* was, in this case, a variable point, varying along the length of the curve as well as the length of the ruler. The last point of contact (the point at the other end of the ruler) became the second starting point* in order to continue the measuring along the curve. She completed her measuring when the ruler finally swung back to the first starting point on the left side. Her result (12 + 4½ inches) came remarkably close to the tape-measured result (17 inches), although she complained that the ruler slipped too easily and too often. This activity should give the students a concrete experience on tangency to a curve, and the "envelope"* outlined by a set of (variable) lines. The activity of curve-stitching (using needlework to envelope line-conics like the parabola, ellipse and hyperbola**) as suggested by both the Schools Council of U. K.[1] and the National Council of Teachers of Mathematics[2] could have been introduced at this point. Figure 4.12 below shows the simple stitches outlining the <u>line-conics</u>** parabola and hyperbola.

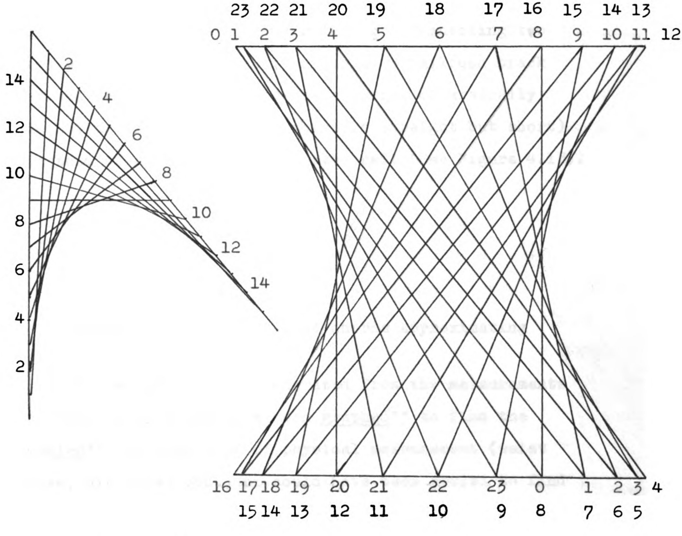

Figure 4.12: The line-conics parabola and hyperbola.

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[1] Schools Council: "Mathematics in Primary Schools," London: H. M. Stationary Office, 1965, p. 85.

[2] N. C. T. M., "Multisensory Aids in the Teaching of Mathematics," 18th Yearbook, 1945, p. 79.

The tape-measuring of any curve-length should be coupled with an important approximation,** which is used so often in practice. It is to approximate the length of any curve by those of a series of short chords.** For examples, in navigation, a set of 'rhumb lines' often replace part of the great-circle arc** connecting two sea-ports on a globe, and, in laying a railroad track around the bend, engineers use a series of gradually deviating tracks, (each of which is straight but short) instead of a geometrically curved rail (See Figure 4.13).

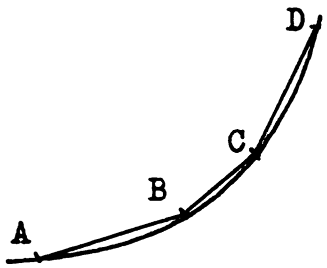

Figure 4.13: A series of chords approximating a curve.

The wealth of data generated from the measurements of this group could have been graphed** to find the median** for each kind of physical measurement (waist size, hip size, etc.) or could have been pooled to find the mean** for each item. The median or mean would have provided a profile of an average 5th grader. The range** for each item could also be calculated, and within this range, several sizes of garments would be recommended for production, with maximum** production on sizes around the mean. The children might even write to the garment manufacturers about their findings because the accuracy of their tape-measurings had been checked several times.

2. The Furniture Group: Each member of this group undertook to design his own desk and chair, temporarily ignoring the measurements of the desks and chairs provided by school. The greatest divergence of opinions seemed to center on the height* of the desk; some wanted the desk-top to be half-way* between eye-level and seat-level, some wanted it lower, and some higher. There was, however, the agreement that it could not be too low, otherwise the drawer might interfere with the knees and thighs. So they learned the existence* of a lower bound* for the vertical height* of the desk-top. The criteria for building one's own chair were easier to agree on: everyone wanted the height to be determined by one's leg length* (from knee to heel), everyone wanted a deeper seat (than that of a school chair), and the depth of the seat was to be determined by one's thigh-length.* Nearly all children in the group complained about the seat-back: it was too short. They wanted the back to be three-fourths* of each one's trunk-length* (from neck to hips). Each student also measured his or her own arm-length,* bent at the elbow at his or her preferred angle,* just in case one might be allowed the luxury of having arm-rests in a classroom chair!

It would be unwise to suggest the calculation of means and medians for the data of this group, because that would result in a set of desks and chairs comfortable to nobody, (either too big or too small). But the children were pointed out that, from the furniture-manufacturers' point of view, it was the mean or median that determined the size of their products.

3. The School Design Group: This group started with an ambitious plan of designing a model for a new elementary school. They wanted to make scaled models* for classrooms, offices, corridors, store-rooms, restrooms, the playground, trees, etc., and their "lot of land" was a 3 ft. x 2½ ft.* tri-wall board. They used yardsticks and a steel-tape measure to determine the length* and width* of typical rooms, corridors, the paths and the parking lot outside the school building. At first they thought of using a scale* of 1:50, but after measuring the length and width of a typical classroom (18 ft. x 16 ft.), they decided to change the scale of the model to 1:100. The height of the model was approximated by that of the ceiling. They looked up the USMES "How to" cards and learned to use simple trigonometry* to find the height* of the ceiling (8½ ft.). This figure was checked by means of a yardstick plus the custodian's ladder. Most of them seemed to enjoy the experience of measuring distances* and working out the corresponding lengths* on their model. This involved a lot of calculations on proportion.* These were the benefits derived from this experience, although the model was never completed. Moreover, this group of children were the only ones that were exposed to the concept of areas* while the other groups (the Garment group and the Furniture group) dealt only with linear measurement.*

When the children's interest changed from building a school model to interior design, they spent the rest of the time measuring* various things inside the school: library shelves, drinking fountains, coat racks, heights of chalkboard and bulletin boards, dimensions of the work-bench in the Design Laboratory, etc. They recorded and made suggestions for improvement. For example, their measurements revealed that the top shelf of the library was too high even for the tallest child in class (4 ft. 11 inches), and the children recommended that the spaces between the 3 shelves could be reduced* by 3 inches at each level, thereby the top shelf could be lowered by 9 inches. They also said the things used by children should be at most 5 feet high. This represents the concept of an upper limit* which the children saw before, e.g., on road signs. In fact, the children could have been asked if they had seen lower limits* written somewhere, and the children should be able to cite several examples: 45 miles per hour on a free-way, 18 years old for voting, $50 for opening a bank account, etc.

They were surprised to find that the height of the drinking fountain (2 ft. 2 in.) was much lower than* they thought. One student said this extra low height might be designed for the sake of kindergarteners. This led the whole class into guessing or estimating* the height of a typical kindergartener. One girl said jokingly: "I am

10 years old, and a kindergartener is 5 years old, or one-half* my age. My height is 4 ft. 10 in., and a kindergartener should be one-half my height, so typically he is 2 ft. 5 in.--just the right height for the drinking fountain." Everyone in the class knew that she was only joking, because the attributes* 'age' and 'height' were different. It could have been pointed out that 'height' is not a linear function** of time (age). However, the class seriously desired to know whether her estimated height of a kindergartener was correct or not. So, in their next Activity, they were going to measure the heights of a sample population* taken from the kindergarten classes.

Activity 2: Measuring the heights of a sample population of kindergarten classes, and graphing the results:

The measuring instrument was a simple device constructed by these 5th graders in the Design Laboratory. It consists of a graduated* vertical* tri-wall board about 4½ ft. high, and a piece of flat* cardboard used as a "height indicator" to be put horizontally* on top of the kindergartener's head at right angles* to the vertical scale. One boy stood by the vertical board and used the 90° corner* of a hard-bound book to check whether the cardboard indicator and the vertical triwall were really perpendicular* to each other. Further it was agreed that the kindergarteners could keep their shoes on while being measured, so that the data gathered were really the natural heights plus* the heights of shoe-heels. (The kindergarteners were measured with their back leaning against the vertical scale.) Three volunteers measured the heights of the shoe-heels as the kindergarteners marched in, five at a time. They found that the heights of the shoe-heels were roughly 1/2 in. in most cases, and so they were going to subtract* 1/2 in. from every item in the collected data. This should be good enough because the overall heights were measured to the nearest 1/2 in.* only. The additional heights due to hair-styles presented a bigger problem. Some hair-styles could easily add an inch to the natural height, and many wanted this, too, to be subtracted from the relevant items* in the data, because, these people argued, the class was primarily interested in designing facilities like drinking fountains, coat-racks, etc. Other children disagreed and they wanted to measure "from hair to heel" whatever type of hair a person might possess, because, they argued, the data should be accurately describing people how they really looked at that moment. The class finally decided to go along with the latter because the additional height due to one's hair was difficult to estimate* in most cases. The children were very thorough in their measuring and recording: they recorded the names as well as the heights of the kindergarteners, so that their data could indeed be used "to describe people," (i.e., to store data for identification purpose) if so desired.

When they presented the data in tabulated form,* they used code-names like A_1, A_2, ... A_5, B_1, B_2 etc. instead of real names, and the students interested in the Describing People Unit held the key to these code-names. The children's recorded data are shown here in Table 4.11.

Table 4.11: The heights measured from a sample population of kindergarteners.

Group A		Group B		Group C		Group D		Group E	
#1	3'7½"	#1	3'8"	#1	3'7½"	#1	3'8"	#1	3'4"
2	3'8½"	2	3'7"	2	3'8"	2	3'8"	2	3'10"
3	3'8"	3	3'11½"	3	3'8"	3	3'8"	3	3'7"
4	3'5"	4	3'11½"	4	3'11"	4	3'6½"	4	3'10"
5	4'0"	5	3'9½"	5	3'9"	5	3'8"	5	3'8"

Many mathematical topics could be learned from this Table. Kindergarteners A_3, B_1, C_2, D_1, D_2, D_3, D_5, E_5 all corresponded to height 3'8", and this was a real-life example of many-to-one correspondence.* The labels A_1, A_2, etc. were, in fact, ordered pairs,* the first label being used for group-identification, and the second label for individual-identification within a group. This notion was brought out very clearly when a boy tabulated the above data as Group 1, Group 2, etc., and named the kindergarteners (1, 1), (1, 2), ... (1, 5), (2, 1), (2, 2) etc. He quickly realized that kindergartener (1, 2) was not the same as kindergartener (2, 1). The class also had a chance to see a matrix in the making*: one student shortened Table 4.11 above into the following matrix from:

Group / Individual	A	B	C	D	E
1	3'7½"	3'8"	3'7½"	3'8"	3'4"
2	3'8½"	3'7"	3'8"	3'8"	3'10"
3	3'8"	3'11½"	3'8"	3'8"	3'7"
4	3'5"	3'11½"	3'11"	3'6½"	3'10"
5	4'0"	3'9½"	3'9"	3'8"	3'8"

The sample population were not randomly selected,** which was a pity. If the procedure of random selection** were explained to the class beforehand, the children would certainly have carried this out and learned from the experience.

It should also be pointed out to the class that the data in Table 4.11 were projected heights**, and not the curved distance measured by means of a tape from skull to heel. Instead, a projected height is the distance between the highest and lowest points** when a human figure is projected* on to a vertical plane.* The British model text on modern mathematics teaching[1] suggested a similar activity: students lying down on a large sheet of paper or

[1]Schools Council (of U.K.), "Mathematics in Primary Schools," London, H. M. Stationary Office, 1965, p. 119.

cardboard, while the peers nearby drew the contours of their bodies from heads to heels and then measured the distance between the two extremities** of each contour on paper. This is also another way of projecting a human figure on to a plane, provided the pencil that draws the contour is always perpendicular** to the plane of the paper. Many measurements, such as widths of the neck, arm, wrist, waist, hips, etc., could be taken from this 2D diagram,** and checked with the earlier tape-measurements. The immediate application** seems to be the designing of a comfortable sleeping-bag for camping, or a comfortable bed.

Next, the teacher asked the children to put the data of Table 4.11 in a graph.* The emphasis was that the children had to think of some methods of graphing by themselves (without any help from the adults in the classroom), but the teacher gave the following hint: "The main idea is to communicate the important aspects about the numbers in Table 4.11 to other people." After working for some time, two students came up with the following graphing methods, which the teacher showed to the whole class:

1. <u>A bar-graph (upside-down) showing the distribution of heights among the sample population</u>:

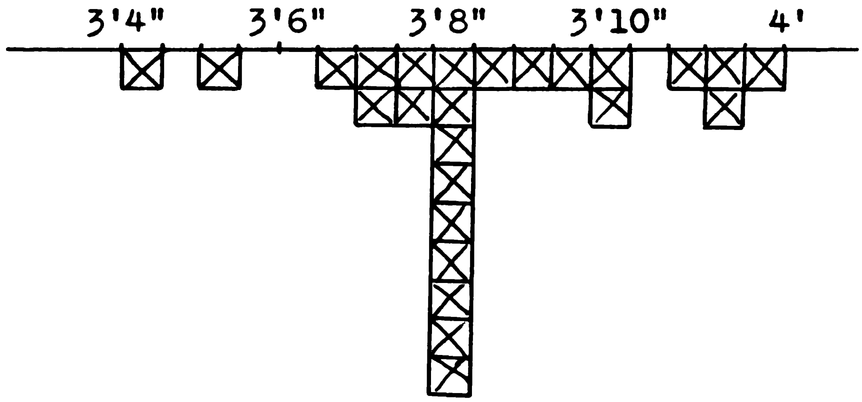

Figure 4.14: Heights shown in a bar-graph.

Apparently this student was interested in tallying the number of people* against each possible height* in Table 4.11. He labeled the various heights in an increasing sequence* (3'4", 3'4½", 3'5", 3'5½", 3'6", etc.) along a horizontal line (axis*). He wrote an x for each person tallied, and put the x's beneath the corresponding* heights of those persons. The teacher suggested the graph would look clearer and better if he put a box around each x. "The resulting picture is called a bar-graph,*" said the teacher, "and the height (3'8") corresponding to the longest column is called the mode.*" In fact, the median** could have been read directly, and the concept of Normal Distribution** could have been discussed. One girl, interested in coat-rack design for kindergarteners, proceeded to calculate the "average" (mean*) of the data in Table 4.11. She did not merely add up the 25 items and divide; instead she used the distributive law* and wrote:

9 persons of height 3'8" :	27'72"
2 persons from each of the groups 3'7", 3'7½", 3'10", 3'11½"; Twice 12'36" ;	24'72"
1 person from each of the rest; 3'4:, 3'5", 3'6½", 3'8½", 3'9", 3'9½", 3'11", 4' :	25'53½"
	76'197½"

She seemed to have some trouble with long division* (the divisor being 25). The writer asked her to think of the exchange of quarters into pennies.* She said 3 quarters = 75 pennies, and so 25 can go into 76 three times* and the remainder* = 1.

```
25 ) 76' 197½"
     3' ...
```

The writer watched carefully to see whether she made the usual mistake in the next stage by forgetting the remainder 1 . But she avoided the mistake by writing

```
       197½
     + 12
25 )  209½
```

At this stage, the writer helped her along by saying

1 dollar = 4 quarters = 100 pennies,

and she echoed: 2 dollars = 8 quarters = 200 pennies.

i.e. 8 x 25 = 25 x 8 = 200

So 25 could go into 209½ eight times, and the remainder = 9½". The writer gave her further encouragement (B. F. Skinner would say reinforcement) by saying that it was correct so far because 9 quarters = 9 x 25 = 225 pennies which is beyond the dividend* of 209½. She nodded in agreement, but then she did not know how to combine such a diversity of results to produce the final answer. In the writer's opinion, adult assistance was essential at this point, otherwise she would be so muddled up that she might never reach her objective and might soon lose all her interest in the problem. The writer reminded her that her original objective was to find the mean* by dividing* 76'197½" by 25. This she accomplished in two stages: the first yielded a partial answer* of 3'; the second a partial answer of 8" with a little "left-over". This left-over was 9½ parts out of 25*, or proportionately* 19 parts out of 50 (2 x 9½ = 19), and the writer introduced the notation of a rational number*, 19/50, to the student. Combining these partial answers as indicated, the student could now write:

3' and 8" and 19/50 inches,

which, finally, became 3' $8\frac{19}{50}$".

2. A Pictorial Graph or a visual presentation of range and differences in heights recorded in Table 4.11:

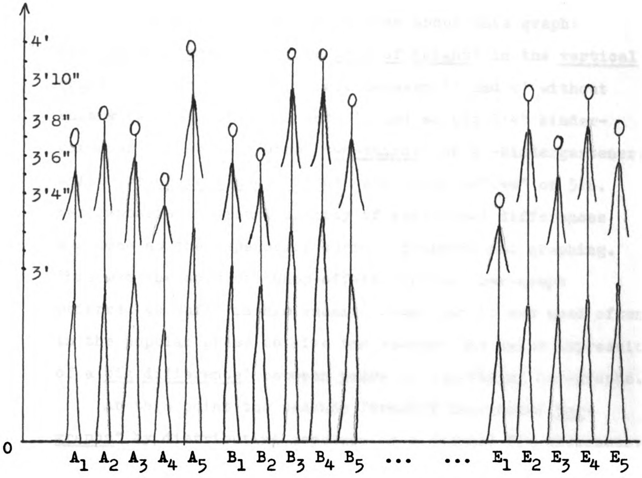

Figure 4.15: A Pictorial Graph

Apparently the girl who invented this graph was interested in the chronological* presentation of these measurements as well as visual clarity to see at a glance the differences* (in the individuals' heights) and the range*. The vertical scale* on the left helped to achieve this visual clarity. Nobody would fail to see that there was a difference in heights between, say, A_2 and B_1. The range, determined by the difference in heights between A_5 and E_1, could be read directly:

Range = 4'0" - 3'4" = 8"

There was one strong criticism about this graph: the inconsistency* over the unit of height* in the vertical axis*. She enlarged the scale between 3' and 4' without enlarging that between 0 and 3', and so the 3'4" kindergartener looked only about two-thirds* of 4'-kindergartener, while the correct proportion* should be 40":48" or 5:6. Her emphasis on visual clarity of individual differences was done at the expense of correct proportional graphing. This was the sort of "Chop-off-the-bottom" bar-graph referred to Huff[1] in his amusing book, and it was used often in the popular press to give the readers the false impression of a big difference* between pairs of individual bar-graphs.

At this point the teacher formally introduced bar-graphs* by distributing reprints of a journal for elementary school children. The journal was Scholastic Young Citizen,[2] Vol. 37, No. 7, October 30, 1972, which contained an article called 'A healthier me'. This article used a colorful bar-graph to show the increments* in life expectation of

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[1]Huff, Darrell, "How to lie with Statistics," N. Y.: W. W. Norton and Co., pp. 62-63.

[2]Scholastic Young Citizen is published weekly, from September through May inclusive, by Scholastic Magazine, Inc., McCall St., Dayton, Ohio 45401.

mankind from prehistoric time till 1970. (See Figure 4.16) The children examined this Figure carefully and then each produced a set of bar-graphs, arranged in <u>ascending order</u>*, from such data as 'the waist sizes of our group', 'the trunk lengths of our group', etc.

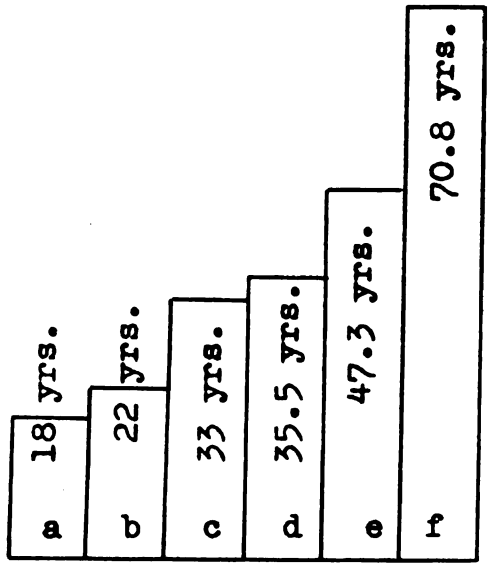

Figure 4.16: The average life span of mankind has considerably increased since ancient times: a) pre-historic time, b) 2,000 years ago, Rome, c) Middle Ages, England, d) 18th century, New England, e) 1900, U.S., f) 1970, U.S.

Activity 3: Measuring the dimensions of a human face and other facial features in order to draw better pictures:

Nearly all the children in this class liked drawing, and they wanted to tie this USMES activity with Art, particularly with the crayon drawing of a human face, which seemed to be a difficult task for them. They thought that the numerical description* of the facial features would perhaps make the job easier. So they started to collect quantitative data* about the faces of the students in class. This was an admirable and enterprising task, especially when it was initiated by such young students as fifth-graders. Great masters in the past (Leonardo di Vinci, Albrecht Durer, and others) had long advocated the quantitative study of human proportion** and projections** as prerequisites to good painting. Durer himself had written four books on Human Proportion, one of which contained the following quotation:[1]

> "First, give the figures a right proportion** according to the canon (laws of physiology and anatomy), arrange them orderly,** lay out the outlines, give the effect of depth by perspective,**...see that every limb be made right in all the smallest as well as the greatest things...."

By means of a tailor's tape, the children measured each other's facial features in considerable details:

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[1]From "The Writings of Albrecht Durer," as quoted by Henrietta Midonick, "The Treasury of Mathematics," N. Y., Philosophical Library, 1965, pp. 345 & 348.

width* and depth* of forehead, distances* between eyes, distance between an ear and the nearer* eye, shape and size of the nose, width of the lips, shape and size of cheeks and chin, etc.

After gathering these data, the children used the scale of 1:2* to put all the facial characteristics on paper and began to draw. Much to their disappointment, they found that even such a carefully planned method of drawing produced only distorted pictures, because the tape-measured distances of various organs on the face were distances on a curved surface*, and it seemed geometrically impossible to project* a curved distance of a fixed "length" on to a plane distance* of the same "length". At this point the teacher could have pointed out to the class that mathematicians had already discovered that distances are not invariant** under projective transformations.** A discussion on Geometry on a curved surface**, or even the elementary aspects of Riemannian Geometry** could have also been initiated, because the children were highly enthusiastic about knowing why they got distorted pictures. One simple but profound activity would be to measure the distance between any two cities, the area** of a certain country, etc. on a spherical** globe, and then compare with the distance, area etc. printed on a 2D map. The more enthusiastic students might be introduced to stereographic projection** which is the usual method of transforming a contour** from a sphere** to a plane**.

An easy-to-handle activity-oriented version of stereographic projection[1] is as follows:

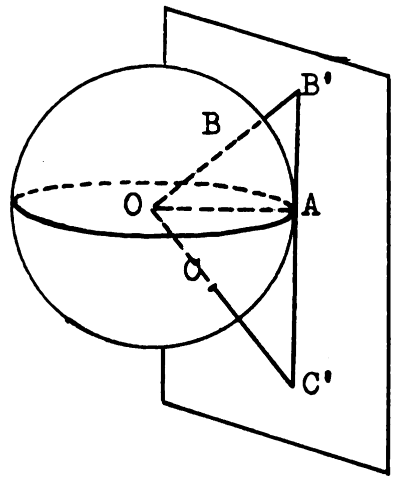

Figure 4.17. Stereographic Projection

A piece of vertical** cardboard is fixed in contact with** a spherical globe at some point A on the equatorial curve.** The center** O is the theoretical point of projection**. B' and C' are images** of B and C respectively. In practice, the theoretical point of projection (inside the globe) is not essential to locate B' and C'. To draw the line OBB', a student can use a toothpick, or a small stick to create a line through B, perpendicular** to the spherical surface. (All lines perpendicular to the sphere will go through center O.) This line, OBB', meets the vertical board at B'.

[1]See, for example, Morris Kline: "Mathematics in Western Culture," N. Y.: Oxford University Press, 1953, p. 152.

Two or three examples of projecting** points like B and C on to their images B' and C' should be sufficient to convince the children that, when two end-points** of a curve (say, the two eye-balls) are transferred from a curve surface** (the human face) to a plane** (the paper), distance is not preserved,** and hence the tape-measured distance between the two eyes could not be used to draw the picture of the two eyes on a flat sheet of paper.

It was then apparent that the tape-measured data collected thus far offered very little help in drawing a better picture of the human face. The teacher then distributed magazine cuttings that bore photographs showing the front view (not side view) of human faces. The children examined them, and, after a long discussion (not without adult input), they realized that the distances between any two organs on a face, as appeared on the photograph, were really vertically projected distances.* (See the definition of "Projected distance" in Activity 2). So the children decided to measure projected distances instead. To start with, they could measure the projected lengths and widths of the faces of students in class. The vertical scale on triwall board, invented for Activity 2, was being brought out once again, but this time the scale was subdivided* further to allow readings to the nearest 1/4 in.* Unlike Activity 2, which measured from ground level,* this new activity would involve the vertical measurement from chin to forehead and horizontal measurement from left ear to right ear, and so two pieces of cardboard indicators would be required this time. One cardboard was placed horizontally at the level* of the chin, and the other at the level of the top of forehead, and both cardboards were placed perpendicular* to the vertical scale. The difference* in heights from ground level gave the projected length* of the face. Both the triwall board (graduated scale) and the two cardboards (indicators) were then turned around 90 degrees* in order to measure horizontal distances. The two indicators were vertically* placed in contact with* the two ears, and the projected width* of the face was read off from the triwall board. (See Figure 4.18).

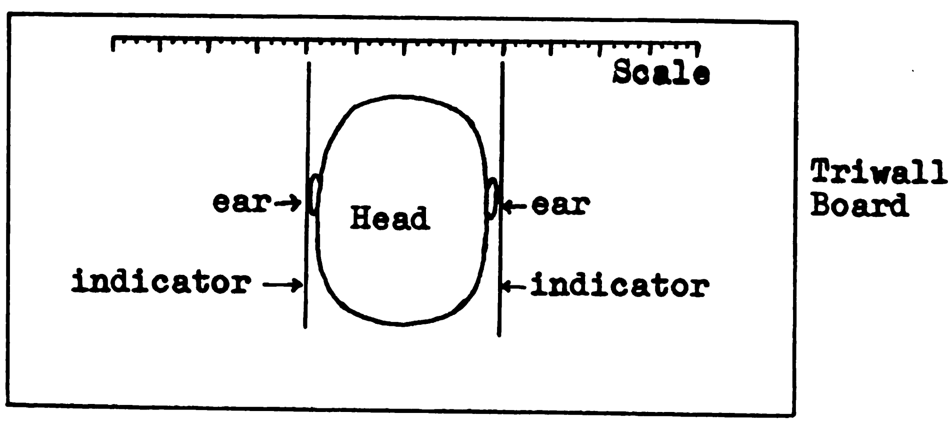

Figure 4.18: How to measure the Projected Width of a human face.

This activity taught the children many valuable concepts in Solid Geometry*: the two cardboard indicators were concrete examples of parallel planes*; each plane is perpendicular to the triwall scale, and there the children could see the concept of the perpendicularity of 2 planes*;

when a cardboard indicator was put in contact with the outermost point* of the curved ear, it demonstrated the concept of tangency* between a curve and a plane; and, finally, when the triwall scale and the cardboard indicators were turned around 90°, the children experienced a concrete example of a finite rotation.*

The data collected in this Activity are shown in Table 4.11a.

Table 4.11a. Projected lengths and widths of 12 fifth-graders' faces.

Length	6	6	7	7	$6\frac{3}{4}$	$6\frac{1}{2}$	$6\frac{1}{2}$	7	6	6	6	$6\frac{1}{2}$
Width	$4\frac{1}{2}$	5	$5\frac{1}{2}$	$4\frac{3}{4}$	$4\frac{1}{4}$	$5\frac{1}{2}$	$4\frac{1}{2}$	$4\frac{1}{2}$	5	5	5	5

Once again the median, mean, mode, and range** could have been calculated from each set** of the above data, giving part of a 5th grader's physical profile. Other interesting profiles might also be suggested: how much soft-drink he consumes a week, how concerned he is about pollution, etc.

The teacher wanted to compare the size of a child's face to that of an adult's. So she herself and the writer of this Case-study took turns to kneel down and let the children measure the projected lengths and widths of their faces. The results were:

Teacher's face: Length 8 in., width $6\frac{1}{4}$ in.,

Writer's face: Length 7 in., width $6\frac{1}{4}$ in.

The measurements taken from the two adults in class gave the students some idea about the slow rate of growth* of the human face once the stage of childhood was over.

The projected length and width of a face would help to fix the boundary lines** of a co-ordinate grid** ruled on rough sketches with a view to improving the likeness of the picture and the subject. The use of a co-ordinate grid to improve a drawing was going to be the main theme of the next Activity.

Activity 4: Using the ideas of proportionate enlargement and co-ordinate grid to draw better pictures of a human being:

While the teacher distributed the magazine cuttings bearing the pictures of human beings, she asked everyone in class to choose a picture one liked best and draw that picture using anything one is capable of: color pencils, crayon, water colors, or even oil paint. The emphasis was, just for once, not on creativity or subjective perception, but on the degree of likeness* between the given picture and the drawing.

The children seemed to enjoy this period of "USMES Art" because they drew only the pictures they really liked. But, despite the intensive effort of everyone, no student in class could produce a drawing anywhere close to the original photographic picture he had chosen. The big problem was that things (objects) in the drawing were mostly out of proportion.* The children themselves noticed this: when all the pairs of original pictures and drawings were placed side by side, the children made the following comments:

--"The girl's face (in one drawing) is too large."

--"The furniture - tables and television sets - look crooked. The lines should be straight."

--"The man's body is not big enough, compared with the pack of Pepsi he was carrying."

--"The pants legs look unreal - too long." (This cutting was an advertisement on fashion.)

--"This (drawn) picture is OK except that the man's shoes look awfully big."

The teacher then explained that photographic pictures looked much more realistic than those drawn by free hand because the camera could capture the correct proportion* of things as they really were except minor shrinkage* of objects further away due to perspectivity.* To clarify the statement about the correct proportion enjoyed by the image of every object seen in a photograph, she showed the children two copies of the same photo: one copy was enlarged* and was $1\frac{1}{2}$ times as big* as the other. It was the photograph of a 10-year-old girl holding a cat, and standing by a table. On the table there were soda-pop bottles and glasses, a flower vase, and a solid state radio set. The teacher pointed out that all the objects appearing in the enlarged photo were $1\frac{1}{2}$ times as big, and she asked the students to verify this. The two photos were then passed from one child to the next, and most children could be observed to measure at least one item (say, the girl's face) from one photo and, after multiplying* the length by $1\frac{1}{2}$, checked the length of the corresponding* item in the other photo. One boy measured the length of the girl's face in the smaller photo and recorded 7/16 in. (His ruler could measure to the nearest 1/16 in.*). He then multiplied* by means of the distributive law*:

$$\begin{aligned}(7/16)(1\tfrac{1}{2}) &= (7/16) + (7/16)(1/2)\\ &= 7/16 + 3\tfrac{1}{2}/16\\ &= 10\tfrac{1}{2}/16 \text{ in.}\end{aligned}$$

He then measured the corresponding length* (i.e., the girl's face) in the larger photo and the result was 10/16 in.

The usual manipulation on fraction**:

$$(7/16)(1\tfrac{1}{2}) = (7/16)(3/2) = 21/32$$

was not observed, and so two topics: (i) the method of converting a mixed number* like $1\frac{1}{2}$ into a single rational number** 3/2, and (ii) the principle of multiplying two rational numbers**

$$(a/b)(c/d) = ac/bd$$

could both have been introduced, because the motivation was there already. But one interesting mathematical behavior was observed here: this boy used a geometrical method* to determine $(7/16)(1/2) = 3\frac{1}{2}/16$. (See Figure 4.19).

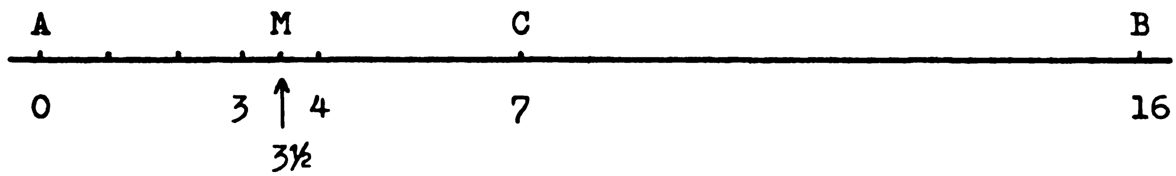

Figure 4.19: How to find (7/16)(1/2) geometrically.

AB was a length, measured 16 units, and AC measured 7 units, so that 7/16 = AC/AB. By the commutative law*,

$$(7/16)(1/2) = (1/2)(7/16) = 1/2 \text{ of } AC/AB$$
$$= AM/AB, \text{ if } M \text{ is mid-point* of } AC$$
$$= 3\tfrac{1}{2}/16, \text{ since } M \text{ is half-way between 3 and 4.}$$

Another girl measured some other things in this pair of photos. She used a ruler, correct to the nearest 1/8 in.*, and measured the side-way length of the table and the height of the flower vase. Measuring from the smaller photo, she reported 2" for the table-length and 1+1/8" for the vase-height. The teacher encouraged her to express the results as $2:\frac{9}{8}$, or 16:9 when both numbers were increased by a factor of 8*. This is a very good way of introducing the subject of ratio*. Next, the girl measured the table-length in the enlarged photo, and she got 3", which was correct, because

$$2 \cdot (1\tfrac{1}{2}) = 2 + (\tfrac{1}{2} \text{ of } 2)$$
$$= 2 + 1 = 3$$

When she measured the vase-height in the larger photo, she obtained $1\frac{3}{4}$", and hastily she expressed the pair of numbers as a ratio. She wrote

$$3 : 1\tfrac{3}{4} = 12 : 4 + 3, \text{ increasing by a factor of 4}$$
$$= 12 : 7$$

She was now perplexed because she was expecting the answer of 16:9, since the teacher had said earlier that, in the larger photo, all things (including the table and the vase) should increase their sizes by the same proportion*, and hence the ratio

table length : vase height

should remain the same. In fact, this student worked out the ratio $3:1\frac{3}{4}$ too hastily before she checked the validity* of her raw datum* of $1\frac{3}{4}$". The vase-height in the smaller photo was $1\frac{1}{8}$" (assumed correct), and the writer showed her that

$$(1\tfrac{1}{8})(1\tfrac{1}{2}) = (9/8)(3/2) = 27/16 = 1\tfrac{11}{16}$$

and so the theoretical model* predicted* a height of $1\frac{11}{16}$ in., not $1\frac{3}{4}$". The student went to fetch a better ruler (which could measure to the nearest 1/16"), and this time, the vase-height, did turn out to be $1\frac{11}{16}$" when she ignored the blurred part in the bottom. So the error* in her previous measurement was

$$1\tfrac{3}{4} - 1\tfrac{11}{16} = (3/4) - (11/16) = (12/16) - (11/16) = 1/16 \text{ in.}$$

The writer comforted her that it was only a small error, because the relative error* = $1/16 : 1\frac{11}{16} = 1/16 : 27/16 = 1 : 27$, or 1 part in 27 parts.

No measurement by rulers will ever be perfectly correct, and most rulers have been graduated to measure up to a certain degree of accuracy,* say, within 1/16 or 1/32 in.

While the children were comparing the enlarged picture against the original copy, it was the most opportune moment

to introduce the subject of Similar Figures** in geometry. Both rectilinear and curvilinear figures** in the two photographs were similar to each other. Their linear measurements** were proportional**, the smaller length being always 2/3 of the larger one (since the enlargement factor $= 1\frac{1}{2}$). The children would have enjoyed the exciting discovery that the area measurements** of any two similar figures above were in the ratio of 4:9**, the smaller surface being always 4/9 of the larger one.

Next, the teacher introduced the idea of using a co-ordinate-grid* to help towards further improvement of the drawing of a human face or any still-life. The teacher asked a boy to rule a co-ordinate grid* over the smaller photo. Each square* in the grid was chosen to be $\frac{1}{2}'' \times \frac{1}{2}''$, and the origin* (starting point) was, in this case, the top-left corner* of the photo. Drawing such a grid was a good exercise on drawing parallel* and perpendicular* lines, and gave the children a feeling that graph-paper* had been invented with a definite purpose. The teacher also asked another student to rule a proportionally* bigger grid over the larger photo. This time the length of each side of a square in the grid was measured $(\frac{1}{2})(1\frac{1}{2})$, or 3/4 in. Care had to be taken to put the origins in the top-left corners in both cases, otherwise a contour* inside one set of squares* in the smaller grid would not be similar to* the contour inside the corresponding set* of squares in the larger grid. When both grids were done, the two ruled photos were passed on to the children who obviously seemed to enjoy tracing similar contours on corresponding sets of squares in the two grids. The children themselves later deduced* that this was a systematic way* of improving the drawing of a human face when a ruled photograph or even a (ruled) rough-sketch of the subject was available.

In the next period, students brought old photographs, magazine cuttings, and rough sketches into the classroom and ruled a co-ordinate grid over the original picture, using the intercept-distance* of either 1/4 in. or 1/2 in. between parallel lines. They then drew enlarged pictures of the original copies on graph-paper* whose squares were $\frac{1}{2}'' \times \frac{1}{2}''$ each. Those who drew from ruled photos did not seem to have good results, because the photo-paper was often bent (curved) while the graph-paper was perfectly flat. This was, once again, the problem arising from the transformation* of a set of points* from a curved surface* onto a plane.*

This last activity was one of the best ways to lead to the formal introduction of the Cartesian Co-ordinate System.** While drawing a contour or a curve which covered many squares in the co-ordinate grid, the children would soon sense the difficulty of keeping track* and counting* the correct number of squares* involved. Sooner or later they would discover that an easier way was to select certain key positions* on the contour or curve in the original copy and to draw on the graph paper by joining* together the points corresponding to those key positions. Once a pair of mutually perpendicular* Co-ordinate Axes* was set up, these key positions would be easily identified by an ordered pair* (x,y), where x and y referred to the perpendicular distances* from the vertical axis* and horizontal axis* respectively. Since any two adjacent edges* of a photograph are perpendicular, they can serve as the Cartesian Co-ordinate Axes. The left vertical edge* and the bottom horizontal edge* are strongly recommended so that all points in the picture are situated in Quadrant I* of the Co-ordinate Plane.

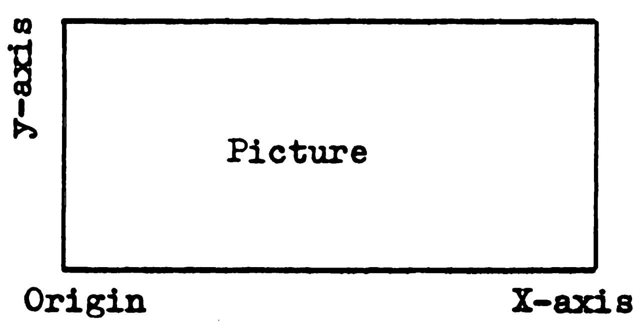

Figure 4.20: Cartesian Axes defined by two edges of a picture.

At this point the game of Tic-tac-toe** could have been introduced. After preliminary playing with 2D Tic-tac-toe, the children should also be encouraged to play with the 3D version of the game. This is one way of introducing the concepts of ordered triplets**, and three mutually perpendicular axes,** as well as consolidating many general concepts about co-ordinate systems.

This concluded the major types of activities of this Unit during the observed period from the beginning of October to the end of November, 1972.

Conclusion: This USMES Unit was originally an off-shoot of the 'Describing People' Unit, and a new branch of challenge has been established: how to design clothing, comfortable furniture, household fixtures and other utensils appropriate to the body size of each age-group. Although this Lansing school started with the same challenge about designing clothes, furniture, etc., to fit the physical proportion of 5th graders, their activities soon branched out into Graphic Art: how to apply this knowledge about human proportion to draw better pictures of human faces and full-size human beings. A small group of children here also embarked on another branching out into the study of elementary aspects of physiology and anatomy. This arose from one boy's inquisitive question: "Since my waist size is different from Anne's, although we are both 10 years old, do we have different things (organs) inside our waists?" After some lengthy search in the library, the children did have a chance to examine photographs and diagrams of the main organs in the vicinity of the human waist. With the writer's help, they managed to draw the following simplified diagram.

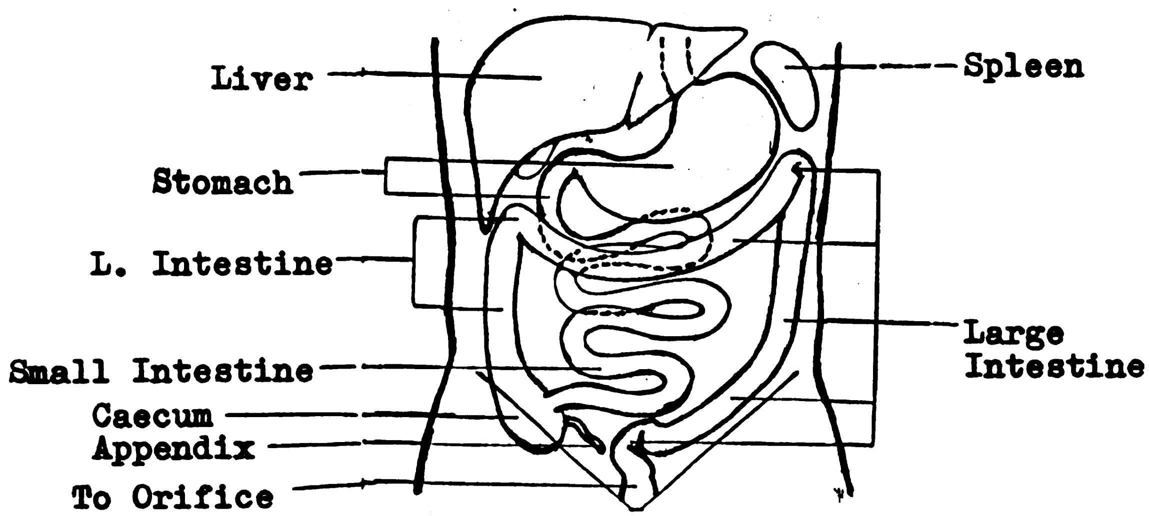

Figure 4.21. The main organs around the front waist-line.

It should be noted that there should be more coils* of intestines than displayed in Figure 4.21. The small and large intestines together measure up to 40 ft., spirally curled up within an internal volume* of 1 x 1 x ½ cubic feet. This experience would be reinforced by such activities as looking at a spiral staircase in some museum, the spring inside a clock, or the magnified picture of the wiring inside a transistor. All of them would represent the idea of saving space whenever a lengthy object is put in spiral forms.*

The names used in Figure 4.21 were all common names because it was agreed that the Latin names appearing in some encyclopedia were too difficult. To the delight of the writer, this small group of children were observed to be interested not only in the positions, but also the different functions of these organs. Given enough time for guided studies in the library, they would have learned and drawn the entire anatomical set-up of man in a somewhat simplified manner: the skeleton, the breathing system, the circulation system, the muscles, etc. It is still an open question whether a 5th grader should be allowed to dissect specimens of animal organs. But a group of 6th graders at Wexford School, Lansing, did dissect some frog legs when they became inquisitive about the variable sizes* of the thigh and calf muscles while moving their legs.

For those children who have learned even the simplified version of man's anatomical structure, they would find it an inspiring example of a well-organized society. The society's work forces on food-production, transportation, commerce, communications, waste-treating, defense, etc., all have their counterparts inside a human body. In many ways, the organs of a man function more efficiently than the corresponding units of a society.

John Dewey once urged people to consider education as growth and growth as education.[1] This USMES unit has grown from "using numbers to describe people," to the designing of clothing and furniture, to Graphic Drawing, to the study of simplified anatomy and its social implication. Every stage of this growth is some tangible motivation for the study of mathematical, biological and social sciences.

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[1]Dewey, John, "Experience and Education," Ontario: Collier-McMillan, 1969, p. 36.

One observation should be noted about this group of 5th graders. They measured things in inches and feet from the start, and hence did not experience the gradual evolution of "units of measurement"* as suggested by Piaget.[1] Piaget advocated the use of a child's body (from shoulder to feet) as his own "measuring unit" first, before the formal introduction of commonly acknowledged (but arbitrary) units like 'foot' and 'meter'. An USMES teacher, Charlotte Hayes,[2] at Eaton Rapids, Michigan, did take up this Piagetian idea and encouraged each child to use his body or parts of the body to be units of measurement. Many delightful equations* were consequently noted, for example:

2 lengths of arms = length from toes to chest,

4 times around the foot = once around the thigh.

etc.

This enterprising teacher also suggested a very good way to introduce the right-angled triangle*, and the slope* of a straight line by using the data collected from human bodies.[3] Each student took measurements of his own height (in inches) and weight (in lbs.). He then drew to scale

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[1]Piaget, Jean, "How Children Form Mathematical Concept," Scientific American, Vol. 189, November, 1953, pp. 81-82.

[2]Hayes, Charlotte, "Teacher's Log," USMES Working Paper on "Designing for Human Proportion," Serial No. C6, 1972, pp. 2-3.

[3]Hayes, Charlotte, "Teacher's Log," USMES Working Paper on "Describing People," Serial No. C3, 1972, p. 6.

a right-angled triangle OA_1B_1, with

length OA_1= the number of lbs. from weight measurement,

length A_1B_1= the number of inches from height measurement.

On the same diagram, other right-angled triangles, such as OA_2B_2, OA_3B_3, etc. were similarly drawn from his peers' measurements. (See Figure 4.22). The slopes of the hypotenuses* obtained would indicate whether a person was relatively fat or skinny.

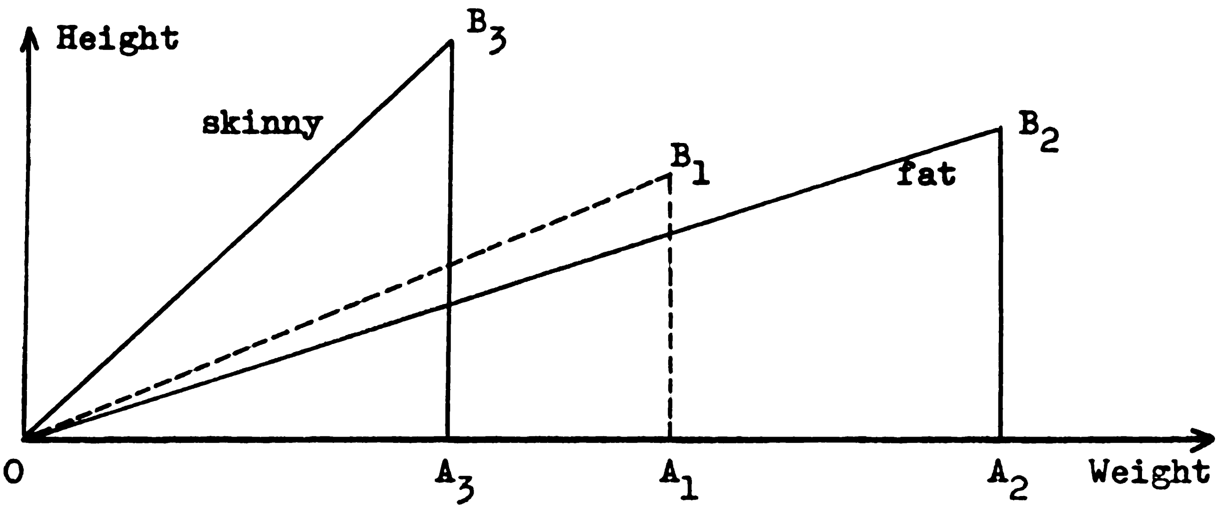

Figure 4.22. The slopes of the hypotenuses of a series of right-angled triangles telling the relative body sizes of people.

Besides learning the interpretation of a graph, the children here were also introduced to a new way of measuring angles:* by means of the ratios B_1A_1/OA_1, B_2A_2/OA_2 etc. It should be emphasized to the children that such angular measures are constantly used in more advanced mathematics, where they are represented by the symbols $P\ \tan^{-1}(B_1A_1/OA_1)$, etc., with P referring to the principal range* of values between $-\pi/2$ and $\pi/2$.

The CASE-Study of the 'Burglar Alarm Design' Unit
Observed at Pleasant View School, Lansing, Michigan
October 3 - December 5, 1972
And also the Summer Workshop's Afternoon Sessions
(with children participating)
from August 25 to September 1, 1972

I. The Behaviors to be observed would be focused mainly on the children's participation in setting up simple direct-current circuits (and thereby learning from scratch the basic concepts of current electricity), in abstracting the relevant physical objects into symbols or elements of a circuit diagram, in making and experimenting with home-made electromagnets (and hence learning about magnetic forces and magnetic induction), and, finally, in hooking up electric alarm-bells to doors; windows, desks, door-mats, etc. This last activity would give the children the experience of measuring large distances, because they wanted to hide the alarm-bell circuits at obscure places well away from the 'bugged' doors and windows.

II. Brief Description of the Classes Observed: The two groups at Pleasant View School were 3rd graders and 5th graders respectively. Both classrooms were very near the Laboratory which contained all the electrical apparatuses as well as measuring devices, e.g. yardsticks, and workshop devices for soldering, metal cutting, wire-bending, etc. On the average, the USMES period was twice a week: Tuesday and Thursday, 12:30 to 1:40 p.m., but sometimes an additional period was arranged for Fridays. The children participating in the Summer Workshop's afternoon sessions were mostly 5th or 6th graders from Wexford School.

Both teachers at Pleasant View School took part in the Summer Workshop. One of them is a male teacher who is quite familiar with all types of electricians' work, and the lady teacher of the other class was quite good at arranging a suitable environment for her 3rd grade students to learn from scratch the concepts of electricity and electromagnetism.

No reading nor communication problem was observed in these groups of children.

Activity 1: Playing with batteries, wires and bulbs; and learning from scratch the fundamental ideas of current electricity.

This activity was observed in the 3rd grade class. The teacher posed the main challenge of this Unit by talking about thieves' breakings into stores and homes, and asking the class what could be done to discourage them. Apparently most children in this class had heard of burglar alarms (perhaps from television movies), because they suggested to install such instruments, but, on closer examination, the teacher realized that, although these children talked about burglar alarms, few had seen a real one, and fewer knew (even vaguely) how it worked. The trouble was that even the few children who had seen a burglar alarm did not have a chance to see the entire set-up: they had been exposed to only part of an alarm system (usually, a bell or a buzzer). So the real challenge of this Unit became: how to find out the 'mysterious' (hidden) parts of any burglar alarm system, or, better still, how to construct an alarm system that would be equally mysterious to outsiders. The teacher gave the children a hint that an easy-to-make, and yet effective, alarm was an electric* one. This hint

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*Throughout this Case Study, the underlined words marked * refer to the significant topics in Mathematics or Physics observed in these classes, while those marked ** refer to the potential lessons in these two fields.

turned out to be a kind of pretest, because the children's responses revealed a high degree of ignorance about electric circuits.* They had heard of electricity*; (one boy even said "lightning* is electricity"); they knew they could get electricity from batteries (dry cells*) as well as from wall sockets, but none of them was aware that it required a complete circuit* for electric current* to exist. The children were thus recommended to go and fetch batteries, copper wires (properly insulated), bulbs, crocodile clips, etc. from the Laboratories, and they were going to play 'Light up your heart' because one girl said that the bulb represented the 'heart' of the system, and a few burned fingers would later testify that the bulb was indeed the heart (regulator) of the system in this case.

It was gratifying to watch children picking up the ideas about an electric circuit from scratch. Most of them only knew that somehow the battery could light up a bulb (e.g. in a torch light). So they began by putting the bottom of the bulb on the top of the battery, but nothing happened. One boy said, "Daddy always talks about 'electric wires' and wiring; so let us try a wire." The addition of a third element (the wire) did not present any permutation* problem, because the children had the intuitive idea that the wire, being a connection*, was to be placed between* the battery and the bulb. The first trial: connecting the bottom of the bulb to the top of the battery (by means of a wire) produced no result again. But the children could see and did try other <u>combinations</u>*, such as:

bottom of bulb connected to bottom of battery,
bottom of bulb connected to curved side of battery,
threaded part of bulb connected to top of battery
etc.

It would be interesting to ask the children <u>how many</u>* distinct combinations they could try. Besides, they learned many <u>geometrical ideas</u>* from this experience: (1) the battery was a real-life <u>circular right cylinder</u> which they could see and manipulate, (2) the top part of a bulb was approximately a <u>hollow sphere</u>,* and (3) the wire was <u>topogically equivalent</u>* to a <u>straight line</u>* in their first set of experiments. In this case the wire was <u>topologically different</u>* from a <u>loop</u>.* As a matter of fact, the children discovered <u>electric current</u>* only when one of them abandoned the straight line and thought of the next element in the topological hierarchy: the <u>simple closed curve</u>,* or loop. Once the idea of bending the wire into a loop set in, all children would in due course experience the sensation of some unwelcome <u>heat</u>* (on the wire) and a pretty <u>light</u>* (on the bulb). But it usually took a long time (nearly an hour for this class) to discover the correct <u>spatial relationship</u>* among the battery, the bulb and the wire. Once the bulb was properly lit up, it meant success as far as learning to establish a <u>simple electric circuit</u>* was concerned. In fact, most students who learned by discovery had established a mischievous, though legitimate, circuit before that: the short-circuit* obtained by connecting the bottom of the battery to the threaded part of the bulb while this threaded part remained in contact* with the top of the battery.

It should be noted that this experience was contrary to all their previous experiences: they had seen Mother using a wire to connect an electric iron or vacuum-cleaner to a wall socket, or Father using a wire to connect a lamp to a switch. But this was the first time something (i.e. the battery) had to be connected to itself in order to bring about the phenomena of heat and light mentioned above. This generated a lively discussion among three students because they by now suspected that there were really two things inside the battery. "The battery is not really connected to itself, but we just connect together the two things inside the battery: one is at the top knob and the other at the bottom," explained one child. So the concept of positive and negative poles* began to take shape. Further, since no heat or light was observed in the events of such ordinary connections as the connection of two things by a string, the connection of a dress to a coat-rack by means of a coat-hanger, etc., this connection between the two poles of a battery was indeed a special event. Something happened to the wire, the bulb and the make-up of the battery itself, and this special "something" was the children's idea of the abstract concept of an electric current.*

Children also tried to connect the two poles by means of an ordinary string, a sticky tape, two pencils, etc., nothing happened. So the class of insulators* began to form. Besides copper wire, they would later discover that other metal wires, tin foil and aluminum foil could be used, giving them the other complementary* set: the set of conductors*.

Activity 2: Making a switch from an empty coffee can, and defining an open or closed circuit by inserting the switch:

One child suggested to solder together the battery, the wire-loop, and the bulb to have a permanent light, but other children objected.

"The battery will soon be run down," said one boy.

"If the light is going to be an alarm," said another student, "we want it 'on'* only when there is a burglary, otherwise the light should be 'off'.* We should work out an automatic switch.*"

This second remark came exceedingly close to the underlying principle* of the Burglar Alarm circuit: "The whole thing is one big switch, whether it is mechanical, optical, or electromagnetic."** The idea of a switch naturally led to the concept of an open* or closed* circuit, and other relevant terminology like breaking* or closing* a circuit.

The teacher suggested that the students should make a switch themselves by using metal strips cut from an empty coffee can, and she explained her rationale: "To make a switch yourself will enable you to see its mechanism* in a constructive manner (in contrast to seeing the internal mechanism of any toy by tearing it into pieces). This is better than just hooking a commercial switch to your circuit, because a commercial switch has everything concealed inside a case."

The writer could spot many mathematical behaviors from the children's activities of cutting coffee cans and drawing up plans for circuit switches. First, the bottom of a can is a circle.* The children used an ordinary can-opener to remove the bottoms of these cans, and they competed to get the most perfect circle. Secondly, the side of a can (where one gets most of the metal) is a cylindrical surface.* The children had to flatten* this curved surface into a plane-figure* in the form of a rectangle* after cutting along the side of the cylinder. A more traditional teacher might have objected to the deafening sound of children's hammering the cut cylindrical surfaces into rectangles. The students then measured* and ruled* the rectangular area into thin strips* and cut accordingly. While they were cutting metal sheets into strips, it should drive home vividly the concept of a straight line,* and parallel lines*. "Hey, you don't cut it straight. Follow the line," said one boy to his peer. The teacher advised the children to fold over the sharp edges (and corners) of each strip, so that it would be safe to work with these strips later on.

The next job was to make a switch from the metal strips. The teacher told the class that they had to think and reason out* how a switch might fit into the complex of the battery, copper wires and the bulb, before deciding on the mechanism of the switch. Working individually at first, the children did not meet with much success, and so the teacher suggested that drawing a diagram* might help to figure out the (spatial) relationship* of the switch to other things in the complex. The diagramatic representation* of a physical situation is indeed mathematical learning: the students had to abstract all important junctions (here, the positive and negative poles, the bottom of the bulb) into geometrical points* (shown as 'boxes' in Figure 4.23) and the connecting wires into topological lines.* They also had to discard irrelevant variables* such as length of the wires, colors of the battery case, shape of the bulb, etc. To sum up, a diagram is a kind of mathematical abstraction,* showing only relevant factors such as relative positions,* connections* and disconnections.*

On their first attempt to draw the circuit diagram, the children learned one more thing: a diagram is not necessarily the photographic copy of a physical phenomenon. A diagram may reveal some internal or subtle connections or relationships which the eye cannot see. For example, it took these children a long time to realize that, in a simple circuit, the 'wire-loop' connecting the negative pole to the bulb and the positive pole was, in fact, not a loop at all, but a topological line. Such a conceptual scheme for relative positions* of the three things implied that, in the diagram to be drawn shortly, the bulb had to lie between* the negative and the positive poles. (See Figure 4.23) Of course, there had to be a loop somehow, otherwise it could not be a complete circuit. The 'missing' link in the diagram was, in fact, the internal connection via the chemical solution, (represented here by the dotted line.)

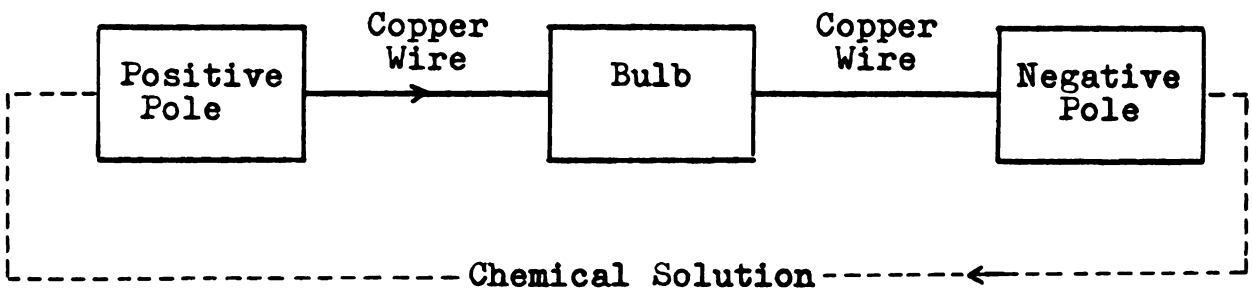


Figure 4.23: An abstract diagram showing a simple circuit without a switch.

It was, at this point, relatively easy to insert a switch into the above circuit because the students could operate formally* on the diagram by means of pencils and erasers. The teacher reminded the class that the idea of a switch was to break or make a complete circuit whenever one wanted. A girl quickly suggested: "We can break the circuit anywhere on the 'copper wire' part of the diagram and insert the switch there." The children by now began to see the significance of the metal strips cut earlier, as one boy remarked: "When two strips touch* each other, the light is on; but when they are moved apart,* the light is off." The teacher reinforced that the remark was correct, and added that the light is on or off, when the circuit is said to be closed* or open*.

Discussions later led to the conclusion that one moveable* metal strip touching two fixed* Fahnestock clips would be preferable to two moveable strips. The children drilled a hole near the mid-point* of the metal strip, put a screw through it to serve as a pivot, and mounted the metal strip and screw on a tri-wall board. Two Fahnestock clips were then fixed to touch the ends of the strip when it looked like a horizontal line* (i.e. parallel to* an edge of the board), and this was 'on'. When the strip was rotated to make an angle* with the horizontal line, it was 'off'. (See Figure 4.24).

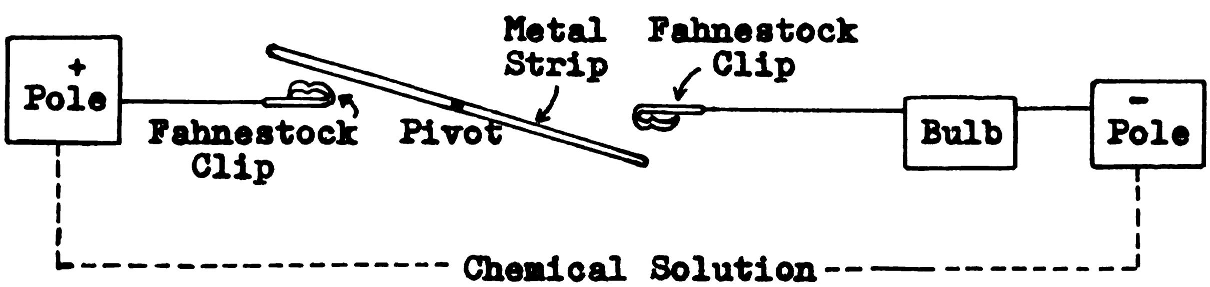


Figure 4.24: A mechanical switch added to the simple circuit.

Activity 3: Making a battery-holder.

Another activity related to cutting empty coffee cans into metal strips was the construction of battery holders. This was to be accomplished in three stages: (i) obtaining a wooden block or a tri-wall board of comparable size* to that of the battery, (ii) nailing a 2" x 1" rectangular* piece of metal strip to the wooden block or triwall board, and bending both ends of the strip upwards into half-a-cylinder* to fit the curved side* of a battery laid horizontally,* (This was enough to prevent it to move sideways*), (iii) fixing something close to the ends of the battery to prevent it from moving lengthwise.*

Stage (i) involved the measuring* of the length* and diameter* of the battery and then the drawing of a rectangle* whose dimensions were numerically*equal to the measurements just obtained. This activity also involved the children in sawing wood or using a saber-saw to cut tri-wall into a rectangle slightly bigger than* the one just drawn.

Stage (ii) involved an estimation* of the middle-portion's length of the battery and the semi-circumference* of the cross-section. The bending of a rectangular metal strip into half-a-cylinder gave the children the unique experience of making a curved surface* in contrast to the traditional experience of looking at one. Many children made this curved surface almost three-fourth* of a cylinder, because it seemed to hold the battery better. These children also experienced the 'springy' reaction* of the curved surface as they pushed the battery down. (See Figure 4.25a).

In stage (iii), most children made the mistake fixing only one piece of metal strip (about 3" x ½") at right angles* to the first strip and then bending the ends upwards to stop lengthwise slipping.* Sooner or later the ends of this lengthwise strip would touch* the positive and negative poles simultaneously,* and a violent short-circuit phenomenon resulted! (See Figure 4.25b). The children learned their lesson quickly, though. One boy remarked: "Two separate* pieces of metal strips are needed, because one continuous* piece does not work." So the children started to nail two L-shape metal pieces to the wooden block, the distance* between the two vertical planes* being approximately the length of the battery. Some children achieved the same objective by removing the middle portion* of the lengthwise metal strip and fixing an extra nail to the now loose metal strip on the left (See Figure 4.25c). They explained: "This is to disconnect* the mischievous 'wire' (lengthwise strip)."

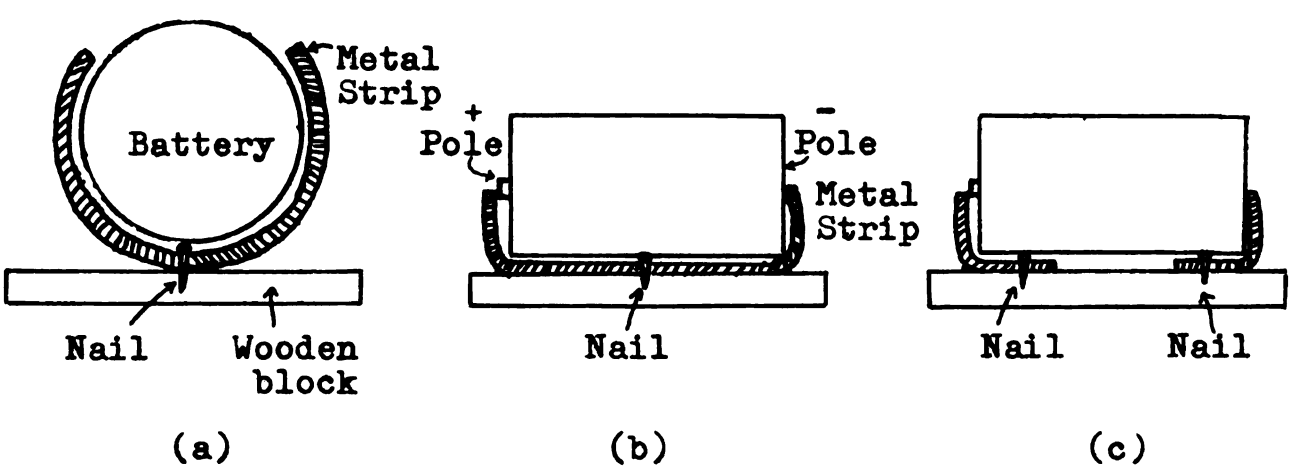


Figure 4.25: A home-made battery-holder;

(a) Viewed from one end,
(b) Side-view, short-circuit phenomenon,
(c) Side-view of the improved model. (Cylindrical strap in the middle is not shown.)

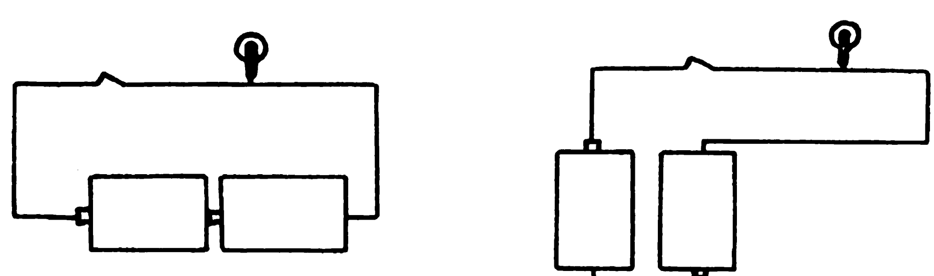
The short circuit mentioned earlier gave the children a very deep impression of the distinction* of the cardinal numbers '1' and '2'. ("ONE metal strip was disastrous, TWO were quite safe," said one child.) All too often students in a traditional methematics class (including College students) just verbalize the numbers 1, 2, 3, 4, ... etc., without having too much impression about the events involving these numbers.* Recently, an Economics professor at Michigan State University spent a greater part of his lecturing time analysing meat prices, and one student remarked afterwards: "It is the first time these weird things called numbers mean anything to me."

This short-circuit phenomenon also made quite an impression on children about the difference between a continuous curve* (metal strip) and two separate* curves (strips) involving simple discontinuity.*

Lastly, the children in this class found out that it was misleading to look at a commercial battery-holder (without careful examination of the parts). In a commercial battery-holder, the two metal strips touching the top and bottom of the battery respectively were separated* by a very thin* rubber strip (an insulator*) which was barely visible. "This gives people the wrong impression that it is one piece of metal there, but in fact they are two pieces," said one child.

Activity 4: Discovering a way to have a stronger current; discussing the anatomy of the electric bell; and winding electromagnets:

A girl in the 5th grade class wrapped a sticky tape round the bottom portion of a battery and continued, with the same tape, to wrap the top part of another battery, so that the two batteries were joined together with the positive pole of the second battery touching* the negative pole of the first. It did not take her long to discover that such a "series" arrangement* of two dry cells was an effective way to provide a stronger current* to any circuit: she connected this 'lengthwise* combination' (as she called it) of two batteries to her previous circuit involving a bulb and a switch, and this arrangement gave her a brighter* light than the single battery alone. She was puzzled about the fact that a commercial battery-holder (which could hold two batteries) had the batteries laid side by side instead of joining them lengthwise. It was indeed a good exercise in topology* as well as in physics* to figure out the equivalence* of the two simple closed curves* (current loops*) in (a) and (b) of Figure 4.26 below:



(a) The girl's arrangement (b) The commercial arrangement

Figure 4.26: Two batteries arranged in series.

She went on to join together 3 and 4 batteries in series* and obtained a brighter and brighter light until the rest of the class gathered round her desk to see what she had done. This discovery turned out to be crucial for the whole class. Before that, several students who had connected commercial alarm-bells to their circuits tried in vain to get them to ring, or at best they rang only faintly. "I have got it!" exclaimed one boy, "we need more batteries! Since two batteries made the bulb light up twice* as bright, they would make the bell ring twice as loud as before." His statement was not altogether correct; two batteries would make the electric bell sound as it should, and certainly not louder than usual. (The commercial electric bells provided were designed for a potential difference* of 3 volts,* while each battery generated about 1.4 volts only.) Besides a bulb, a buzzer might be a better current-indicator* than a bell.

The class now knew how to set off the electric bell and in the next ten minutes the whole room was filled with the ringing noise of electric bells. Soon the children got interested in the anatomy of the electric bell. This interest sprang from the request of a boy who wanted to take apart the different pieces of an alarm bell system, but the teacher instead advised him to study how* an electric bell works and then make a buzzer from coffee-can strips by using the same principle* as an electric bell.

"What are the parts of an electric bell?" the teacher asked the class. I see a spring, hooked to a tiny hammer that strikes the bell," one student responded. "What is the function of the spring?" asked the teacher. A boy was observed to take a careful look at the spring and hammer; he displaced* the hammer slightly with his finger and the hammer swung back,* striking the bell once. (This demonstrated the work* done via the potential energy* of a spring.) The student reported this finding to the class. The teacher remarked further: "All right, the spring causes* the hammer to swing back whenever the latter is displaced, but what caused the displacement* of the hammer in the first place? Surely we are not going to use our fingers every time." This was a timely remark because it made the children look at their electric bells carefully once again. After several wildly speculative and incorrect statements such as

--"the spring works both ways: it pushes as well as pulls."

--"the hammer strikes the bell too hard, it bounces back."

etc.

finally one boy did spot a valuable lead, though the whole set-up was still a mystery to him. He asked the teacher: "Could it be that the thing which looks like a "cotton reel" (the electromagnet**) attracted* the hammer away from the bell?" The teacher responded in a wise manner: he did not say 'yes' right away, but, instead, he advised the whole class to make electromagnets* by first winding an insulated electric wire* round a nail, a bolt or any piece of iron, and then passing an electric current* through that winding wire. "If your home-made electromagnets attract metals," said the teacher, "then the electromagnet looking like 'a cotton reel' in an alarm bell will certainly attract the hammer nearby."

This activity of winding electromagnets offered many relevant mathematical learnings about circular cylinders* and spirals*. First, if the iron bolt was treated as a right circular cylinder, one complete turn* of the copper wire = $2\pi r$ units, or nearly 3 times the diameter* of the core*. The next complete turn represented the beginning of a circular spiral* path, if there was no overlapping* with the first turn. If a fine copper wire (coated with an insulating chemical) was used, the children could wind an incredible length* of this wire round cylindrical bolt or nail of about 5 inches long or even shorter. At Roxbury, Mass., an USMES class managed to wind 20 feet* or more of copper wire around a nail, and to add to the excitement, they measured* up the length of the wire before or after winding. This gave the children two valuable experiences: (i) a clear-cut view of saving space by means of setting up a spiral, (ii) the practice of measuring sizeable distances*. (The writer has to concede that throughout his school and College training he has never had a chance to measure any length greater than 10 feet, nor to play with a wire or string of any sizeable length.) At this point, the use of a trundle wheel** to measure great distances could have been introduced.

This activity could also have been an interesting and highly relevant exercise on <u>fractions</u>** and <u>decimals</u>** as exhibited in the following sample calculation:

By direct measurement, e.g., by a screw-gauge or even a ruler, the student should be able to note that

The diameter of the cylinder = 1/4 in.

The length of 1 turn of the wire = (3.14)(1/4)

= .785 in.

To wind 20 ft. of wire, it requires $\frac{20x12}{.785}$ or 305.7, i.e. roughly 306 times. (The quotient $\frac{20x12}{.785}$ should be worked out fully: $\frac{20x12}{.785} = \frac{20x12x1000}{785} = \frac{240000}{785} = 305.7$)

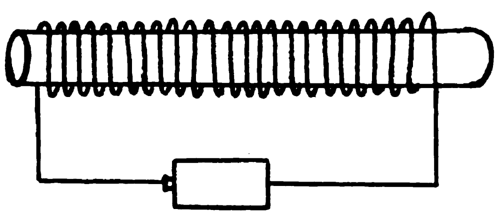


Figure 4.27: Winding an insulated wire round a cylindrical iron bolt.

If the #26 AWG[1] wire is used, and if the turns are really <u>tight</u> and <u>closed</u> <u>to</u> <u>each</u> <u>other</u>, 306 turns will cover approximately 5 in. of the length of the cylinder.

Hence the diameter of this fine electric wire = $\frac{5}{306}$ in., or roughly 1/61 in. This is equivalent to 0.01634 in., or,

[1]A.W.G. stands for 'American Wire Gauge'. AWG Standard copper wires (bare and insulated) range from #0000 (diameter 0.46 in.) to #40 (diameter 0.0031 in.). See, for example, Lister, Eugene C. <u>Electric Circuits and Machines</u>, N. Y., McGraw-Hill Co., 1960, p. 364.

as an engineer would say, 16.34 mils** (1 in.= 1,000 mils). This simple calculation would reveal to the students the powerful method of mathematical deduction** from indirect measurements. Nobody expects a 5th grader to use a micrometer** to measure anything which is in the order** of 1/61 in., but, by indirect measurements and calculations, he can, in fact, determine small magnitudes of that order, or something even smaller. (He can usually measure things correct to 1/16 of an in. by a ruler and never anything which is more refined). After the calculation, he could check the result with standard AWG Tables, which give the diameter of #26 wire as 15.94 mils.

The small discrepancy of 0.01634-0.01594, or 0.0004 in., should not become a source of discouragement for this kind of calculation, but it should encourage the students to investigate further into the reasons for this discrepancy: first, the approximate** nature of the solenoid** length (approximately 5"), the number of turns (approximately 306), and π = 3.14, and secondly, the imperfect winding (i.e., the turns might be less close to each other than appeared to the eye). Such an investigation would have been a good exercise in learning Scientific Process** and systematic improvement for accuracy** in measurement. In any case, the children who went through the above process of calculating the thickness of #26 wire would have learned at least one topic: the order of magnitude**, for 0.01634 in. and 0.01594 in. really belonged to the same order of magnitude, i.e.

1.6×10^{-2} in. (in <u>Scientific Notation</u>**). The concepts of <u>positive</u>** <u>and</u> <u>negative</u> <u>exponents</u>** could have been discussed here in conjunction with topics in which children were normally interested, e.g., the large distances involved in space travels.

An arithmetical problem, similar to the one described in this Activity, arose during the USMES Summer Workshop at Lansing in September 1972. A 5th grade girl wanted to make an electromagnet out of a 2" nail by winding #26 wire round it. At the request of the writer, she counted up the number of turns, and she reported 92 turns when she used up exactly 3 ft. of the #26 wire. The writer then formulated a 'nice' arithmetic problem for her to think about:

> "92 turns of #26 AWG wire wound around a 2 inch nail measures exactly 3 ft. What is the mean diameter of the nail?"

This was a 'nice' problem in several respects:

(i) it contained two irrelevant numbers: #26 and 2 inches; (the girl could spot them immediately, though),

(ii) the word "mean diameter" was used in a practical sense, not just an 'average' calculated from some fixed algorithm,

(iii) it introduced the child to the important fact that 1 turn $= 2\pi r$,

(iv) it introduced a simple first degree algebraic equation, and its solution,

(v) it was a good exercise on decimal fractions,

(vi) the final answer could be verified by means of a ruler.

[Solution: 1 turn = πd = $(3.1416)d = \frac{3x12}{92}$

Hence $d = \frac{36}{92x3.1416} = 0.124"$

Direct measurement gave d = 1/8 in.]

Activity 5: Experimenting with electromagnets:

After most children had made their own electromagnets from nails and #26 wire, they connected the ends of the wire with a battery, and began to play the enjoyable games of testing which articles could or could not be attracted* by these magnets. For many children, the phenomena of magnetism* was not altogether new to them: they had played with bar magnets or horseshoe magnets before. Many toys today also utilize the properties of attraction and repulsion of magnets, although the magnets are often concealed within the toys. But the following could be observed to be new experience on Magnetism for these 5th graders:

(1) Increasing the strength of an electromagnet by winding more turns of wire around it: The children quantified* the strength of their magnet by counting* the number of paper-clips it could pick up and hold them steady in mid-air for some length of time (i.e. by balancing the magnetic force* against the gravitational pull* on the paper-clips). Most children's recorded results were almost the same, plus or minus 1 or 2 paper-clips at most. Table 4.12 below is a typical result for the circuit consisting of only one battery:

Table 4.12 Showing how the magnetic force depends on the number of turns of the winding wire.

No. of turns	No. of clips suspended
50	9
100	17
150	25
200	33
250	40
300	45
350	39

This result, whether it was graphed* or not, showed clearly that as the number of turns increased during the first stage of the winding, the magnetic strength also increased at a constant rate* equivalent to the weight* of 8 paper-clips for every 50 turns of wire. But, after the 200th turn, there was a decline* in the growth-rate* of magnetic strength, and, finally, the magnetic strength itself began to drop after the 300th turn. The children were very puzzled by this unexpected decline both in the growth-rate and in the magnetic strength itself.

At this point, a simple (qualitative) explanation based on Ohm's Law** could have been offered to these children who had already been motivated by the above puzzle. The reason was that, as the electrical resistance** of the wire gradually increased (with the increasing length**), the current gradually decreased.** Since the magnetic strength depended both on** the number of turns and the current, a decline in the growth-rate of the magnetic strength could be observed if the current began to decline. (This was an example of a <u>function of two independent variables</u>**). Finally when the (<u>external</u>) <u>resistance</u>** due to the wire was nearly the same as the <u>internal resistance</u>** of the battery, the current was drastically <u>halved</u>**, and the increasing number of turns could not make up for the loss of magnetic strength due to the <u>sharp decline</u>** in the current, and hence a <u>solenoid of 350 turns</u>* or more would pick up less and less clips than before. (See Table 4.12). The more enthusiastic students could now experiment with <u>ammeters</u>, <u>voltmeters</u>**, etc., and study the quantitative aspect of Ohm's Law:

$$I = \frac{E}{R + r}$$

Problems based on Ohm's Law are always good exercises on <u>fractions</u>** and solving simple algebraic <u>equations</u>** <u>involving one or two unknowns</u>.

(2) <u>A 'paper-clip' experiment demonstrating the essence of the Inversed Square Law</u>:

Several children were playing with their electromagnets in the following way: each moved his electromagnet nearer and nearer to a paper-clip, until suddenly, under the <u>force of attraction</u>,* the clip jumped towards the tip of the electromagnet (with about 20 turns of electric wire in most cases). The writer asked each student to take note of the <u>distance</u>* between the clip and the magnet when the 'jump' took place. "Now, <u>double</u>* that distance," said the writer, "and test to see how many more turns of wire you would have to wind around the nail--in order to make the paperclip jump as before."

"Twice as many*, naturally," remarked one boy.

"Let us all try it," said the writer.

To their disappointment, the children found that after doubling the number of turns, the magnets in all cases were still not strong enough to attract the clips placed twice as far as previously. Some even tripled* the number of turns, but the paper-clip in each case still failed to jump as expected. Most children were on the verge of giving up when the writer suggested the last brute-force attempt: to increase the number of turns by a factor of 4*, and if it still did not work, then nobody would be able to blame them for changing activities. But, fortunately, it did work: all their electromagnets had now enough strength* (magnetic force*) to overcome the limiting frictional force* between the clips and the table, and the clips jumped toward the magnets as expected.

The writer then asked each child to prepare two electromagnets, having either 8 and 72 turns, or 9 and 81 turns, or 10 and 90 turns, and tried out the same experiment with the new condition* that the distance between the clip and the 'stronger' magnet (having more turns) was 3 times* as much as the distance between the clip and the 'weaker' magnet on the other side. "What is the significance of these number-pairs like (8,72), (9,81), etc.?" asked the writer. "The stronger magnet is always having 9 times the strength of the weaker magnet," one boy replied, after a long and deep thought. At this point, the mathematical function*

$$S = 9W$$

(S stands for 'Strong', and W for 'Weak'), or generally $y = 9x$ could have been introduced.

The children carried out the suggested experiment. They found that, when only one magnet was used, there was always a critical stage* for the stronger magnet was always 3 times that of the weaker magnet. When both magnets were simultaneously placed on the two sides of only one clip, and if critical distances were used in both cases, then the clip would not move: it was in equilibrium* under two equal but opposite forces.* Only two boys out of the whole group achieved this spectacular equilibrium, or balancing*, which was a very difficult task to manipulate.

The writer introduced the following two columns to the children:

If you increase the distance between clip and magnet by	then you have to increase the strength of magnet by
2-fold 3-fold	4-fold 9-fold

Unfortunately most children in this group were not accustomed to thinking of 4 as 2^2 and 9 as 3^2, and so no mathematical relationship evolved from this experiment, nor was there any sign of awareness about its implication of a very important law of Physics. The writer did not draw any conclusion from the above two columns, nor did he attempt to verbalize the Inversed Square Law.** Hopefully, one day in the future, when some High School teacher teaches these children Physics verbally, they might recall this earlier experience and learn the Inversed Square Law in a more meaningful way than those who have not been exposed to this experience.

(3) The phenomenon of magnetic induction:

In Experience (1) described earlier, the children had the opportunity to see a great number of paper-clips (See Table 4.12) being suspended by an electromagnet. On closer look, the children realized that most clips did not cling to the magnet directly, but they clung, instead, to other clips which were nearer to the magnet. It could have been pointed out to the children that the inner* clips really became temporary magnets** and this phenomenon was known as magnetic induction**. Several children here discovered that, if these clips remained in contact with an electromagnet for a length of time, they could retain a certain degree of magnetization* even after leaving the electromagnet. Incidentally, an electromagnet with several layers of paper-clips clung to it was an effective instrument to study spatial relationship,** especially such concepts as innermost or outermost elements**, something lying wholly within** something else, volume** and surface area**, etc.

Activity 6: Connecting alarm-bell circuits to doors, windows, desk-drawers, door-mats, etc.

Apparently most children in these two classes knew by now how an alarm circuit could be set up, and so the next job was to connect these alarm-bell circuits to doors, windows, etc. to achieve the original objective of preventing burglary. The children spent a great deal of time on this Activity, but some were quite successful even in their first or second attempt. The more successful ones are described as follows:

(i) The 'positive' and 'negative' terminals* (i.e. the two sides of the circuit switch) were connected to two tin-foils (conductors) separated only by a thin piece of cardboard (insulator). A tight string tied the cardboard to a (closed) door or window. As the door or window was forced open, the string pulled the cardboard away from the tinfoils whose contact closed* the circuit* and set forth the alarm.

(ii) Several children made use of the spring* in a clothespin to ensure the contact of the tin-foils when the insulator (e.g. a cardboard) was pulled away by the door or window. Two bare copper wires could be used in place of tinfoils in this case: the wires were wound round each side of the jaws of the clothespin, and the insulator was inserted in-between. The electrical principle employed here was exactly like (i) above.

(iii) A group of students cut out a 2 ft. x 1 ft. triwall board and labeled it 'Welcome Mat.' Four springs were fixed at the four corners of the mat. A large sheet of tinfoil (about 1-1/2 ft. x 3/4 ft.) was nailed to the bottom side of the mat, and it was secretly connected to the 'positive' terminal of an alarm-bell circuit. When the mat was placed on a floor, the 4 springs held it up (about 1" above ground level). A second tinfoil, connected to the 'negative' terminal of the same alarm-bell circuit, was placed on the floor immediately underneath the mat, and thus was separated from the first tinfoil by <u>less than 1"</u>. When somebody stepped on the mat, the springs were depressed, the mat was lowered, and the two tinfoils came into contact, setting off the alarm.

Activity 6 was, indeed, a sort of climax of the Burglar Alarm Design Unit. It partially answered the Unit's main challenge about designing a burglar alarm which would give effective warning. It also brought together all the physics learned in previous activities, especially Activities 1, 2 and 4.

The children in one class agreed that it was a good idea to have the bell or buzzer hidden well away from the doors, windows and the 'Welcome Mat' which were "bugged." So a <u>considerable length</u>* of wire was to be used. The children tried to do a bit of 'advance planning': they, by a means of a steel tape measure or a yard stick, <u>measured</u>* the distance between the door or window to the obscure corner of the room where the alarm-bell circuit was hidden. Many students forgot to double* the measured distances (for the purpose of wiring* a complete circuit) and so their advance planning was incorrect, e.g., when a boy came forward and asked the teacher for 14 ft. of electric wire to hook up his hidden alarm circuit to a window, he actually needed 28 ft. of wire. This kind of realistic distant wiring* also gave the children a realistic sense of the problem of electrical resistance* which gradually added up* to a sizeable magnitude. The current was considerably reduced* as judged by the faint ringing of the bell. (This was a real-life example of inversed proportion*, i.e. current $\propto \frac{1}{\text{resistance}}$). The children had already learned how to remedy this: adding more voltage* by connecting one or two more batteries (in series*) into the existing circuit. (See Activity 4).

This concluded the major types of activities of the Burglar Alarm Design Unit during the observed period from early October to early December 1972.

Conclusion: It was already implied in the short description of Activity 6 that, although the main challenge of the Burglar Alarm Design Unit provided a kind of focusing point within an otherwise divergent set of activities on basic electricity and magnetism, opportunities for learning some forty different topics in Mathematics and Physics arose mainly from these divergent activities, and fewer opportunities spring from the main challenge itself. The closest analogy was that, although an orchestra conductor provided directions for focusing on one or more groups of instruments at a time, the pretty sound of music came from a divergent set of instruments, and not from the conductor himself.

All USMES Units have encouraged the use of real objects, realistic experiments, real measurements and actual counting, etc., in teaching mathematics and science. The Burglar Alarm Design Unit seems to be more so than other USMES units observed thus far. Teaching children by means of real objects, and arranging children to have some real experience before verbalization of any type, are in fact not new. One and three-quarter centuries ago, Johann Heinrich Pestalozzi published: How Gertrude Teaches Her Children, a classic in the theory of "Natural Education." But it should be emphasized that to learn from real objects does not mean playing aimlessly with these objects; sense impression, observation and intuition should lead naturally to concept-formation, the art of abstraction and classification, etc. which Pestalozzi called "the child's developed powers." The following words[1] were written by Pestalozzi himself:

> "All instruction...is only the Art of helping Nature to develop in her own way; and this Art rests essentially on the relation and harmony between

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[1]Pestalozzi, Johann H., How Gertrude Teaches Her Children, (translated by L. E. Holland & F. C. Turner), Syracuse: Bardeen, 1900, p. 26.

> the impressions received by the child and the exact degree of his developed powers. It is also necessary that the beginning and progress (of these impressions) should keep pace with the beginning and progress of powers to be developed in the child."

Pestalozzi's practice was followed and greatly enriched by Johann Friedrich Herbart who augmented Pestalozzi's Anschauung (intuition due to sense impression and observation) with association, generalization and the "method."[1] Herbart's "method" meant the development of general principles for dealing with generalized systems, as well as the applications of existing principles to new situations. Both Pestalozzi and Herbart voiced regrets at the separation of theory and practice in traditional teaching.

Those teachers who would like to add some "Modern Math" flavor to this Unit could easily do so by branching out into the study of logical circuits involving only a battery, switches and bulbs. This is probably one of the best ways of introducing mathematical logic to elementary school children, because they could check the validity of each logical statement by practical means. Figure 4.28 below shows how to set up simple circuits for starting off this new set of activities.

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[1]Dunkel, Harold B., Herbart and Education, New York: Random House, 1969, pp. 75 and 116.

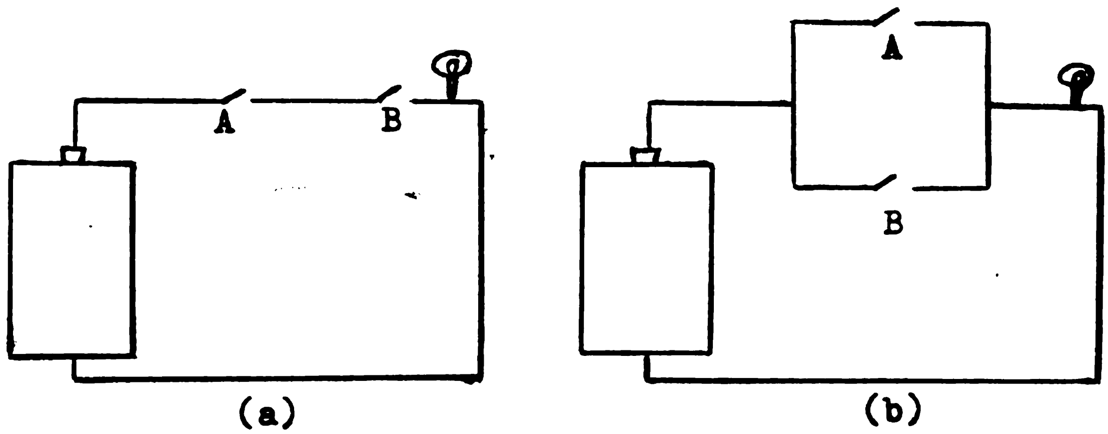

Figure 4.28. The circuits displaying the consequences of the logical operations $p_A \wedge p_B$ and $p_A \vee p_B$.

Statements like "switch A is on" and "switch A is off" are abbreviated to be p_A and $\sim p_A$ respectively. Various joint operations upon switches A and B would result in one of the two possibilities: the light is either <u>on</u> or <u>off</u>, written briefly as L, or $\sim$L. Sooner or later the children will be able to abstract the following basic statements from their practical experience:

(a) $p_A \wedge p_B \Rightarrow L$; $(\sim p_A) \wedge (\sim p_B) \Rightarrow \sim L$;

$(\sim p_A) \wedge p_B \Rightarrow \sim L$; $p_A \wedge (\sim p_B) \Rightarrow \sim L$.

Similarly four other basic statements could be attracted from (b). Usually the children could easily spot the first two possibilities and verbalize "Both 'on' means 'on', and both 'off' means 'off'." The teacher had to step in at this stage and asked them to think of more possibilities until all four were exhausted. Many logical games can then be played, and points are scored by correctly supplying logical symbols for the question-marks in the following complicated circuits, e.g.

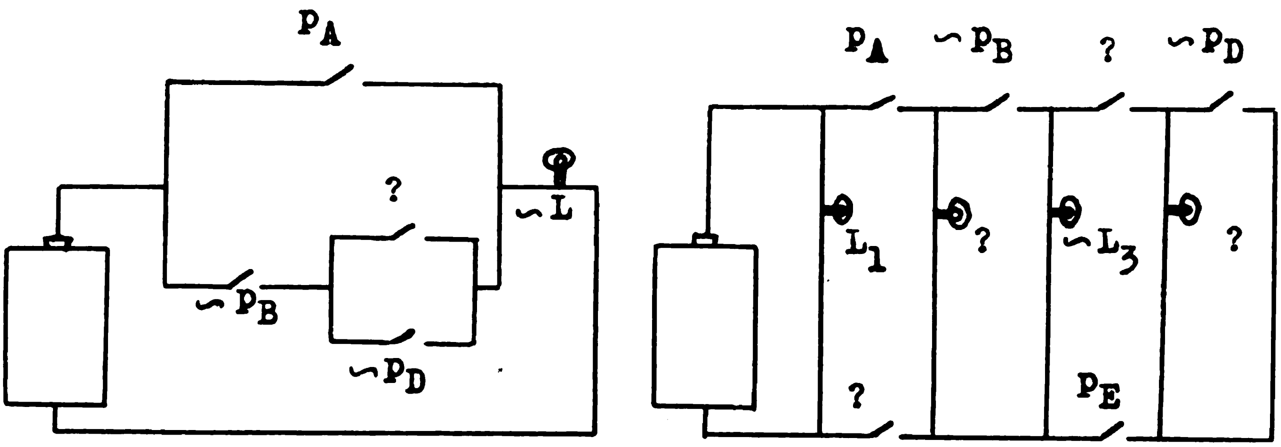

Figure 4.29. Logical circuits for a kind of guessing game.

These games can be made open-ended by allowing children to change freely the logical symbols in the given circuits into opposites, or better still, to devise new logical circuits of their own and play accordingly. The only important rule is that they must not cheat by operating the switches to get the answers; they should figure out the answers first before operating the switches to verify their answers which have been arrived at logically. At this point, the kind of elementary logic advocated by the Cambridge Conference[1] could have been introduced: open sentences and their true/false solutions, equivalence as a 'double implication,' contradictions, indirect proofs, etc. These topics were recommended for classes between Grades 4 and 6, depending on their interests and maturity.

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[1]Cambridge Conference Report: Goals for School Mathematics, Boston: Houghton Mifflin Co., 1963, pp. 31-41.

To sum up, although Symbolic Logic can be introduced here, this USMES Unit is <u>not</u> primarily designed for that purpose. Instead, it should be viewed as an effective means to provide children with first-hand experience about many principles and concepts in mathematics and physics which will, unfortunately, be presented <u>symbolically</u> in the next rung of the educational ladder. They will soon meet with a sea of symbols in Junior High Mathematics: π, sq. rt., m(AB), $.\overline{3}$, etc. plus a wider sea of symbols in Physics:

n for number of turns of a solenoid,
H for magnetic field,
I for current
E for voltage
$|\!\!\cdot$ for a dry-cell
$\sim\!\sim\!\sim$ for a resistance
etc.

Hopefully the experience they gain from this Unit will help to put some vitality into an otherwise boring manipulation of symbols followed by meaningless calculations. The writer showed the script of this Case-Study to a personal friend, an instructor in Psychology. After reading it, he exclaimed: "What a <u>new</u> and <u>revolutionary</u> way of teaching math and physics! It's good for the kids, though. If I were taught this way when I was little, I might have ended up with a Ph.D. in Physics, and not Psychology like the one I've got." The writer reflected afterwards: 'revolutionary' was the wrong word. To put vitality into mathematical symbols has been practiced by good teachers for centuries. The following words are quoted from one of them, Harold Fawcett:[1]

> "Behind every symbol is an idea. It is the idea which is important, and it is familiarity with the idea which puts life in the symbol. It is, therefore, of the greatest importance that the idea be developed before it is symbolized, for, to give a child a symbol for an idea he does not have is to encourage him to take a long tragic step toward intellectual disaster."

The same sentiment was also implied by work of another retired teacher, F. L. Griffin,[2] and it can be concluded that good teaching knows no temporal boundaries.

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[1]Fawcett, Harold P., "Reflection of a Retiring Teacher," The Mathematics Teacher, Vol. 57, November 1964, pp. 450-456.

[2]Griffin, F. L., "Some Teaching Reminiscences," The American Mathematical Monthly, May, 1969, pp. 460-467.

CHAPTER 5

DISCUSSION AND CONCLUSION

The foregoing description of mathematical behaviors arising from the four USMES units shared considerable common ground with the mathematical content for K - 6 proposed by the 1963 Cambridge Conference for School Mathematics. The following points of similarity between the USMES Units and the Cambridge Report[1] are important:

1. The role of physical equipment is emphasized. Whether one thinks in terms of pre-mathematical experience, or in terms of aids to effective communication, or simply as attractive objects that increase motivation, the conclusion is inescapable that the children can study mathematics more satisfactorily when each child has abundant opportunity to manipulate suitable objects.

2. The bed-rock foundation for elementary mathematics consists of counting and physical interpretation of fractions; symmetry and invariant properties of geometrical figures; real experience in collecting data; use of graphs and other visual displays of data; and the vocabulary of elementary logic.

3. Students are encouraged to seek methods for checking the correctness of their solutions or answers without recourse to the teacher.

4. Long-range projects undertaken by individual students or small groups of students are preferred so that students may have experience with the extended aspects of mathematical learning involving many types of modeling and mathematical techniques.

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[1]Cambridge Conference for School Mathematics, "Goals for School Mathematics," Boston: Houghton Mifflin Co., 1963, pp. 31-41.

Nevertheless, some mathematical topics recommended by the Cambridge Conference were not observed in the four USMES units, for examples,

(a) Arithmetic of signed numbers,

(b) Inequalities and absolute values,

(c) Prime numbers and factoring,

(d) Use of straight-edge and compasses to do geometrical construction,

(e) Interpolation.

On the other hand, these USMES children learned many sophisticated concepts in Probability, Topology and Foundation of Mathematics which were conspicuously absent from the Cambridge Report. This Case-Study would, hopefully, help future USMES teachers to locate and utilize the potential mathematical behaviors inherent in these Units' activities. No doubt the list of mathematical behaviors (in Chapter 3) will grow as future groups of USMES teachers and students discover more and more mathematical topics in the course of their work.

The writer of this Case-Study fully realized that his graduate training in Mathematics was a positive asset in doing this research, because it helped him "to see mathematics in action" whenever such an opportunity arose. But he also realized that graduate mathematics alone would not be sufficient for this work. It required another component: he had to be interested in the children's development in the cognitive, affective, and psychomotor domains.

In doing this Case-Study, the focus was on:

1. The integration of several kinds of educational experiences pertinent to mathematics learning, and the integration of activity learning with the usual paper-and-pencil work,

2. The relation of mathematics to some real-life long-range problems, not just the short-range problems involving only symbol manipulation,

3. Students' opportunity to acquire problem-solving ability and to learn the scientific method,

4. The mathematical behaviors arising naturally from these USMES Units or their sub-tasks.

The first three points mentioned above are, in fact, concurrent with Harold Fawcett's thinking.[1] He wrote:

> "To meet the responsibilities of citizenship, the mathematics program should be designed to promote problem-solving process, to define and deal with problems of concern to the society, to gather and organize relevant data, to detect underlying assumptions of the system, and to draw valid conclusions by both inductive and deductive reasoning. Mathematical principles should be the outgrowth of experience rather than the basis of it. Operational skills should serve a recognized and useful purpose."

The USMES activities <u>motivate</u> and <u>introduce</u> a student to certain mathematical ideas, say fraction and percentage, but it is up to the student to practice more on those topics if he wants to achieve such mathematical skills. There are

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[1]Fawcett, Harold P., "Mathematics for Responsible Citizenship," <u>Mathematics Teacher</u>, Vol. 40, November 1947, pp. 199-205.

students who genuinely like expository teaching to guide them along in these sub-tasks. Others continue to explore and discover their own solutions to these sub-challenges. At this point diversity is desirable. (Eventually the results of these sub-tasks will be combined to answer the major challenge.) The most appropriate underlying principle here seems to be: "Teaching should increase the diversity of students' achievement rather than the uniformity."[1]

Conclusion: The following mathematical behaviors were observed to have permeated throughout all the four USMES Units here: counting and measuring which lead to meaningful usage of whole numbers, decimals and fractions; place value; the four fundamental operations of arithmetic; percentage; ratio and proportion; estimation; collection and tabulation of raw data; graphs; proposing hypotheses and testing them; elementary logic; abstracting concrete situations into mathematical relations; sets and their elements; ordinal and cardinal numbers; ordered pairs; simple geometrical shapes like triangles, rectangles, and circular cylinders.

In addition, the following mathematical behaviors were observed in the individual units described below:

The 'Soft Drink Design' Unit: Independent and dependent variables; composite function; multiplicative inverses; matrices and reduction of the size of a matrix; frequency;

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[1]Fitzgerald, William M., "On the Learning of Mathematics by Children," Mathematics Teacher, Vol. 56, November 1963, pp. 517-521.

points on the Number Line and transformations of these points; invariance under a transformation; confounding variables.

The 'Dice Design' Unit: Theoretical probability; approximations; closure, commutative and distributive laws; arithmetical mean and progression; the combination ${}^nC_2$; the histogram; the Normal Curve; plane geometry of the triangle and the hexagon; construction of polyhedra from pre-cut polygons; the 'double-entry' principle of book-keeping; one-one correspondence.

The 'Human Proportion' Unit: Examples of linear and non-linear functions; curved lengths and curved surfaces; orthogonally projected heights; rotation; tangency; simple trigonometry; calculation of mean, median, mode and range; many-to-one correspondence, parallel and perpendicular lines; Cartesian Co-ordinates.

The 'Burglar Alarm Design' Unit: Arithmetical deduction from indirect measurements; scientific notation; inverse proportion; inverse square law; permutation of five elements; first difference; spherical surface, a topological line; simple closed curves; simple discontinuity; separation; distance between two planes; direct proportion; informational aspect of cardinal numbers.

The following potential mathematical topics could have been included: algebraic problems involving one or two unknowns (Soft Drink and Burglar Alarm Units); sets which are topologically equivalent (Soft Drink, Human Proportion and Burglar Alarm Units); random selection; stratified sampling (Soft Drink Unit); equality (Soft Drink, Dice Design Units); relative error (Dice Design, Human Proportion, Burglar Alarm Units); intersection of two planes and the angle between them (Dice Design Unit); envelop and curve-stitching (Human Proportion Unit); various measures of angles: degree, radian, and $\tan^{-1}(y/x)$ when referring to principal values (Dice Design, Human Proportion Units); positive and negative exponents (Burglar Alarm Unit) and logical circuits (Burglar Alarm Unit.)

This Case-Study could serve as a guideline or recommended procedure for teaching mathematics in an USMES. The list of potential mathematical topics would help teachers to initiate mathematical sub-tasks whenever an opportunity arises, so that these USMES Units might be utilized to their full potential in generating activities and discussions which are pertinent to mathematical learning.

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