INVARIANT MANIFOLD THEORY AND ITS APPLICATIONS TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

By

Jiayin Jin

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ABSTRACT

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The theory of invariant manifolds and foliations provides indispensable tools for the study of dynamics of nonlinear systems in finite or infinite dimensional space. As is the case here, invariant manifolds can be used to capture complex dynamics and the long term behavior of solutions and to reduce high dimensional problems to the analysis of lower dimensional structures. Invariant manifolds with invariant foliations provide a coordinate system in which systems of differential equations may be decoupled and normal forms derived. These play an important role in the study of structural stability of dynamical systems or, when a degeneracy occurs, in understanding the nature of bifurcations. This thesis is devoted to the study of the construction of invariant manifolds of solutions with certain spatial structures to some nonlinear parabolic partial differential equations. I approach these problems in two steps: the first step is to construct a manifold of states that is approximately invariant, the second step is to show the existence of a truly invariant manifold of these states near the approximately invariant one, and to determine the dynamics on this manifold. Since this approach may be applied to many different systems, I also develop it in an abstract or general way, extending earlier results of [19].

My thesis consists of two projects, in the first project, we consider the two-dimensional mass-conserving Allen-Cahn Equation,

$$\begin{cases} \phi_t(x,t) = \varepsilon^2 \Delta\phi(x,t) - f(\phi(x,t)) + \fint_\Omega f(\phi(\cdot,t)), & x \in \Omega,\ t > 0, \\ \partial_n \phi(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{0.0.1}$$

where $\Omega \subset \mathbb{R}^2$ is a fixed bounded domain with smooth boundary $\partial\Omega$, ∂_n is the exterior normal derivative to $\partial\Omega$, and $⨍_\Omega = \frac{1}{|\Omega|}\int_\Omega$ means the average over Ω. Here f is the derivative of a double well potential W. We assume the following conditions for f:

$$f(\pm 1) = 0,\ f'(\pm 1) > 0,\ \int_{-1}^{s} f = \int_{1}^{s} f > 0 \text{ for all } s \in (-1, 1). \tag{0.0.2}$$

We establish the existence of a global invariant manifold of bubble states for this equation and give the dynamics for the center of the bubble.

In the second project, we consider the existence, in forward and backward time, of dynamical interior multi-spike states driven by the nonlinear Cahn-Hilliard equation:

$$\begin{cases} u_t = -\Delta(\varepsilon^2 \Delta u - f(u)) & \text{in } \ \Omega \times (0, \infty), \\ \partial_n \Delta u = \partial_n u = 0 & \text{on } \ \partial\Omega \times (0, \infty), \end{cases} \tag{0.0.3}$$

where $\Omega \subset \mathbb{R}^n$ is a fixed bounded domain with smooth boundary $\partial\Omega$ and f is the derivative of a double well potential W, that is , $xf(x) > 0$ for $|x|$ large enough and f has two zeros a and b such that $f'(a), f'(b) > 0$. We construct invariant manifolds of interior multi-spike states for the nonlinear Cahn-Hilliard equation and then investigate the dynamics on it. An equation for the motion of the spikes is also derived. It turns out that the dynamics of interior spikes has a global character and each spike interacts with all the others and with the boundary. Moreover, we show that the speed of the interior spikes is super slow, which indicates the long time existence of dynamical multi-spike solutions in both positive and negative time.

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Chapter 1

Introduction

In dynamical systems, an invariant manifold is a manifold that is invariant under a map or a flow (or semiflow). For instance, a fixed point or periodic orbit of an ordinary differential equation is an invariant manifold for the flow generated by that ODE. The theory of invariant manifolds for discrete and continuous dynamical systems has a long and rich history. Numerous applications can be found where answers to the following questions are needed: 1) Assuming that a dynamical system has an invariant manifold, does a perturbation of this system also have an invariant manifold? 2)When a dynamical system has an invariant manifold, how does one construct locally invariant structures such as the center-stable, center-unstable manifold, and center manifold of the original invariant manifold and invariant foliations of these, which essentially decouple the dynamics.

For the case of the invariant manifold consisting of a single fixed point, Hadamard [49] constructed the unstable manifold of a hyperbolic fixed point of a diffeomorphism of the plane by iterating the mapping applied to a curve in the plane, therefore obtaining a convergent sequence of curves. The limit of the sequence of curves gives the unstable manifold. People now call this geometric approach Hadamard's graph transform. Lyapunov [65] and Perron [78, 79, 80] constructed the unstable manifold of an equilibrium point by formulating the problem as an integral equation. This method is analytic rather than geometric and now is called Lyapunov-Perron method. There is an extensive literature on the stable, unstable, center, center-stable, and center-unstable manifolds of equilibrium points for both finite and infinite dimensional dynamical systems. The general theory for finite dimensional dynamical systems may be found in

[26, 29, 54, 52, 56, 58, 60, 67, 81, 83, 84, 86, 88]. For infinite dimensional dynamical systems we refer the reader to [10, 16, 35, 51, 68, 87, 93]. Most of these works use the approach of Lyapunov-Perron. A good treatment of center manifold theory for ODE's using the Lyapunov-Perron method can be found in the monograph by Carr [29], where several applications are also set forth. Certain infinite dimensional settings are also treated. Vanderbauwhede and Van Gils [88] also use the Lyapunov-Perron method to obtain smooth center manifolds but with some important differences in technique. Ball [10] used the Lyapunov-Perron approach to obtain local stable, unstable and center manifolds for equilibrium points of dynamical systems in Banach space, with application to the beam equation. Henry [51] developed the theory for semilinear parabolic equations. Later, Chow and Lu [35] used this approach to prove the existence of smooth center-unstable manifolds with application to the damped wave equation. For more on center manifold theory in the infinite dimensional setting, using the Lyapunov-Perron method, see [87]. The theory of invariant manifolds for an equilibrium point of finite dimensional dynamical systems using Hadamard's approach may be found in [54]. For infinite dimensional dynamical systems, we refer to [16], where applications are given demonstrating the stability of a pulse solution to the FitzHugh- Nagumo equations and the instability of stationary solutions to the nonlinear Klein-Gordon equation.

Chow, Liu and Yi [34] constructed center manifolds for smooth invariant manifolds for smooth flows in finite dimensional spaces by using the method of Hadamard's graph transform. Krylov and Bogoliubov [24] studied time-periodic ordinary differential equations arising from the study of nonlinear oscillations. Under the assumption that the averaged equation has an asymptotically stable equilibrium point, they proved the existence of periodic integral manifolds, which gives the existence of asymptotically stable periodic orbits for a class of equations. An integral manifold is an invariant manifold in the product space of time and phase space. The above result and many generalizations and related work in collected in the monograph of Bigoliubov and Mitropolsky

[25].

Levinson [62] studied periodic perturbations of an autonomous ordinary differential equation possessing an asymptotically stable periodic orbit. He proved that if the perturbation was sufficiently small, then the perturbed equation has a periodic integral manifold, which may be viewed as a two-dimensional torus. Levinson's results were extended to periodic surfaces by Diliberto [40], Hufford [55], and Kyner [61]. Hale [50] established a general theory of integral manifolds for nonautonomous ordinary differential equations and obtained more general results than those just mentioned above. An extension of Hale's integral manifold theory to a larger class of nonautonomous ordinary differential equations was obtained in [100].

The persistence under perturbation of a compact normally hyperbolic invariant manifolds for a finite dimensional dynamical system was independently obtained by Hirsch, Pugh and Shub [52, 53] and Fenichel [41, 42, 43]. They proved the persistence of normally hyperbolic invariant manifolds, and the existence of the center-stable and center-unstable manifolds and their invariant foliations. Pliss and Sell [82] studied the persistence of hyperbolic attractors for ordinary differential equations. Bates, Lu and Zeng in [11] proved the persistence of compact normally hyperbolic invariant manifolds, and the existence of the center-stable and center-unstable manifolds and their invariant foliations for semiflows in infinite-dimensional spaces and then extended their results without assuming compactness in [17, 18].

Mañé [66] proved that normal hyperbolicity defined in [52] is a necessary condition for the persistence of an invariant manifold under perturbation of a finite dimensional dynamical system.

Henry [51] extended Hale's theory of integral manifolds to general nonautonomous abstract semilinear equations whose linear part generates an analytic semigroup. Henry also studied compact normally hyperbolic invariant manifolds with trivial normal bundle for semilinear parabolic equations and obtained a coordinate transformation which lead to the setting of integral manifolds and persistence results.

In many singular perturbation problems for evolutionary partial differential equations, people are interested in solutions which have certain qualitative features, such as interior or boundary layers or localized spikes, and motions of these layers and spikes, including the location of stationary layers. The canonical shape of such solutions, in a neighborhood of the abrupt spatial disturbance (layer or spike), can usually be determined by a rescaling or blow-up procedure. Thus, a reasonable approximation to the shape of a solution is found quite easily by considering the equation on the whole space and approximately invariant manifolds made up of these approximate solutions have been constructed by many authors. The approach, involving the construction of an approximately invariant manifold of states having a certain spatial structure, was pioneered more than thirty years ago in papers of G. Fusco and J. Hale in [45] and by J. Carr and R. Pego in [31]. In those papers the authors were interested in the slow dynamics of interfaces in solutions to the one-dimensional Allen-Cahn Equation. The same approach was also taken to obtain similar results for the one-dimensional Cahn-Hilliard equation in [4], [21] and [22], and to rigorously establish the slow motion of ”bubble”-like solutions [6, 7] and multipeaked stationary solutions to the Cahn-Hilliard equation [14, 96] in multi-dimensional domains. The approach was also used to produce spike-like stationary solutions to the shadow Gierer-Meinhardt system of biological pattern formation [59]. In most of these papers, the qualitative shape of stationary solutions was the point of interest and so a true invariant manifold was not shown to exist, although that was done in a subsequent paper by Carr and Pego in [32] and also in [22].

It is natural to ask how to deduce the existence of a true invariant manifold in a small neighborhood of the approximately invariant manifold constructed by hand, as described above. In [19], the authors established a systematic way to find a true invariant manifold assuming the approximate one is good enough and they found an invariant manifold of boundary spike states for a class of parabolic equations. In this thesis, we first apply the abstract results in [19] to construct an invariant manifold of boundary droplets for the 2-D mass-conserving Allen-Cahn equation. Then we extend the abstract results in [19] to manifolds with boundary, consisting of approximately stationary states and construct invariant manifolds of dynamic interior-spike states for the Cahn-Hilliard equation in higher space dimensions.

Chapter 2

Preliminaries: Approximately invariant manifolds

Here, we state some results from [19], which provide a tool for obtaining an invariant manifold when a good approximation is available. Roughly speaking, if an immersed manifold $\psi(M)$ is approximately inflowing invariant under map T, and if $\psi(M)$ is approximately normally hyperbolic, then one can find a truly locally invariant manifold W^{cs}, its center-stable manifold, under map T. Furthermore if $\psi(M)$ is approximately overflowing invariant under map T and approximately normally hyperbolic, then we can find a center-unstable manifold W^{cu} under map T. If W^{cs} and W^{cu} intersect transversally, then their intersection is the truly invariant manifold we seek as a graph over $\psi(M)$. Now, we give the precise definitions and statements of the theorems.

Let X be a Banach space and $T \in C^J(X, X)$, $J \geq 1$ with M a connected C^1 Banach manifold and $\psi : M \to X$ an immersion.

Definition 2.0.1. $\psi(M)$ is said to be approximately inflowing invariant under T if the following conditions hold

(1) There exists $\eta > 0$ and $u \in C^0(M, M)$ such that $|T(\psi(m)) - \psi(u(m))| < \eta$, for all $m \in M$;

(2) There exists $r_0 \in (0, 1)$ such that $\psi(\overline{B_c(m_0, r_0)})$ is closed in X for any $m_0 \in u(M)$, where $B_c(m_0, r_0)$ is the connected component of $\psi^{-1}(B(\psi(m_0), r_0))$ containing m_0.

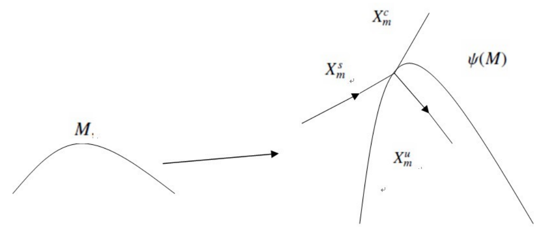

Figure 2.1: Approximately normally hyperbolic invariant manifolds

Condition (1) means that $\psi(M)$ is approximately invariant under T and u on M is an approximation of T on $\psi(M)$. Condition (2) essentially states that the 'distance' between the projection of $T(\psi(M))$ into $\psi(M)$ and the boundary of $\psi(M)$ is bounded from below.

Definition 2.0.2. We say that an approximately inflowing invariant manifold $\psi(M)$ is approximately normally hyperbolic, if conditions (H1)-(H3) hold:

(H1) For each $m \in M$, there is a decomposition $X = X^c_m \oplus X^s_m \oplus X^u_m$ of closed subspaces with projections $\Pi^c_m, \Pi^s_m, \Pi^u_m$.

(H2) For any $m \in M$, Π^c_m is an isomorphism from $D\psi(m)T_mM$ to X^c_m. Furthermore there exist constants B, L, and $\chi \in (0, 1/2)$, such that for any $m_0 \in M$ and $m_1, m_2 \in B_c(m_0, r_0)$, with $m_1 \neq m_2$, for $\alpha = c, u, s$,

$$\begin{cases} \|\Pi^\alpha_{m_0}\| \leq B, \|\Pi^\alpha_{m_1} - \Pi^\alpha_{m_2}\| \leq L|\psi(m_1) - \psi(m_2)|, \\ \\ \dfrac{|\psi(m_1) - \psi(m_2) - \Pi^c_{m_0}(\psi(m_1) - \psi(m_2))|}{|\psi(m_1) - \psi(m_2)|} \leq \chi. \end{cases} \tag{2.0.1}$$

(H3) There exist $\sigma, \lambda \in (0, 1)$ such that, for any $m_0 \in M$, if $m_1 = u(m_0)$, and $\alpha \in \{c, s\}, \beta \in \{c, s, u\}$, with $\alpha \neq \beta$, then

(a) $\|\Pi^{\beta}_{m_1} DT(\psi(m_0))|_{X^{\alpha}_{m_0}}\| \leq \sigma$,

(b) $\|\Pi^{s}_{m_1} DT(\psi(m_0))|_{X^{s}_{m_0}}\| \leq \lambda$,

(c) $\lambda\|(\Pi^{u}_{m_1} DT(\psi(m_0))|_{X^{u}_{m_0}})^{-1}\|^{-1} > \max\{1, \|\Pi^{c}_{m_1} DT(\psi(m_0))|_{X^{c}_{m_0}}\|^{J}\}$.

(H4) There exists B_1, such that $\|D^j T|_{B(\psi(M),r_0)}\| \leq B_1$ for $1 \leq j \leq J$. When $J = 1$, we will need the following function $\mathcal{A}(\delta) = \sup\{\|DT(x_1) - DT(x_2)\| : x_1, x_2 \in B(\psi(M), \delta), |x_1 - x_2| < \delta\}$, and require it to be small enough.

Hypothesis (H3) specifies the different growth rates of DT, the linearization of T, in different directions. Condition (a) represents the approximate invariance of the bundles X^c and X^s under DT. Different rates in the unstable and center-stable directions are assumed in (b) and (c). Hypothesis (H4) is a technical assumption on T, which holds automatically if $\psi(M)$ is precompact.

For $\alpha = c, u, s$, let $X^{\alpha}_m(\varepsilon) = \{x \in X^{\alpha}_m : |x| < \varepsilon\}$ and $X^{\alpha}(\varepsilon) = \{(m, x) : m \in M, x \in X^{\alpha}_m(\varepsilon)\}$.

Theorem 2.0.3. *Assume that (H1)-(H4) hold. Depending on r_0, B, B_1, λ, L, when $\eta, \chi, \sigma, \inf \mathcal{A}(\delta)$ are sufficiently small, there exists a C^J positively invariant manifold W^{cs}, which is given as the image of a map*

$$h : X^s(\delta_0) \to X,$$

for some $\delta_0 > 0$. The mapping h also satisfies

$$h(m, x^s) - \psi(m) - x^s \in X^u_m(\delta_0),$$

and so can be viewed as a graph over the bundle $X^s(\delta_0$.

Furthermore, it holds that, for any $m_0 \in M$, there exists $\tilde{h} : X^c_{m_0}(\delta_0) \oplus X^s_{m_0}(\delta_0) \to X^u_{m_0}(\delta_0)$, so *that*

$$
\begin{aligned}
&\{h(m,x^s) : m \in B_c(m_0,r_0) \cap \psi^{-1}(B(\psi(m_0),\frac{\delta_0}{4})), x^s \in X^s_m(\frac{\delta_0}{4})\} \\
&\subset \psi(m_0) + graph\Big(\tilde{h}|_{X^c_{m_0}(\frac{\delta_0}{2})\oplus X^s_{m_0}(\frac{\delta_0}{2})}\Big) \\
&\subset \{h(m,x^s) : m \in B_c(m_0,r_0) \cap \psi^{-1}(B(\psi(m_0),\delta_0)), x^s \in X^s_m(\delta_0)\}.
\end{aligned}
$$

Remark 2.0.4. Theorem 2.0.3 is a brief statement of the result in [19]. In applications, we will use Theorem 4.2 in [19], which is a precise and rigorous version.

If we want to extend Theorem 2.0.3 to the case of a semiflow T^t, then we need further assumptions [19]:

(H5)

(1) Conditions (H1)-(H4) hold for $\psi(M)$ and T^{t_0} for some t_0,

(2) There exists an integer $k \geq 0$, such that for any $\mu > 0$, there exists $\zeta > 0$, such that for any $x \in B(\psi(M), r)$ and $t \in [kt_0, kt_0 + \zeta]$, we have $|T^t(x) - T^{kt_0}(x)| < \mu$.

The next concept is that of approximately normally hyperbolic overflowing invariant manifold. The results are basically parallel to the case of approximately inflowing invariant manifolds.

Definition 2.0.5. An immersed manifold $\psi(M)$ is said to be approximately overflowing invariant under T if the following conditions hold:

1 There exists a relatively open subset $M_1 \subset M$, a homeomorphism $v : M \to M_1$, and $\eta > 0$ such that $|T(\psi(v(m))) - \psi(m)| < \eta$, for all $m \in M$;

2 There exists $r_0 \in (0,1)$ such that $\psi(\overline{B_c(m_0,r_0)})$ is closed in X for any $m_0 \in v(M)$, where $B_c(m_0,r_0)$ is the connected component of $\psi^{-1}(B(\psi(m_0),r_0))$ containing m_0.

In addition to (H1) and (H2), instead of (H3), we assume the following approximate normal hyperbolicity conditions.

(C3) There exist $a, \lambda \in (0, 1)$ such that, for any $m_1 \in M$, if $m_0 = v(m_1)$, and $\alpha \in \{c, u\}, \beta \in \{c, s, u\}$, with $\alpha \neq \beta$, then

1. $\|\Pi^{\beta}_{m_1} DT(\psi(m_0))|_{X^{\alpha}_{m_0}}\| \leq \sigma, \|(\Pi^c_{m_1} DT(\psi(m_0))|_{X^c_{m_0}})^{-1}\|^{-1} > a$
2. $\lambda\|(\Pi^u_{m_1} DT(\psi(m_0))|_{X^u_{m_0}})^{-1}\|^{-1} > 1,$
3. $\|\Pi^s_{m_1} DT(\psi(m_0))|_{X^s_{m_0}}\| < \lambda \min\{1, \|(\Pi^c_{m_1} DT(\psi(m_0))|_{X^c_{m_0}})^{-1}\|^{-J}\}.$

Theorem 2.0.6. *Assume that (H1), (H2), (C3), and (H4) hold. Depending on r_0, B, B_1, λ, L, when $\eta, \chi, \sigma, \inf \mathcal{A}(\delta)$ are sufficiently small, there exists a C^J negatively invariant manifold W^{cu}, which is given as the image of a map*

$$h : X^u(\delta_0) \to X,$$

for some $\delta_0 > 0$. The mapping h also satisfies

$$h(m, x^u) - \psi(m) - x^u \in X^s_m(\delta_0).$$

Chapter 3

Global invariant manifolds of boundary droplets for the 2-D mass-conserving Allen-Cahn equation

3.1 Introduction

We consider the two-dimensional mass conserving Allen-Cahn equation,

$$\begin{cases} \phi_t^{\hat{\varepsilon}}(y,t) = \hat{\varepsilon}^2 \Delta_y \phi^{\hat{\varepsilon}}(y,t) - f(\phi^{\hat{\varepsilon}}(y,t)) + \fint_\Omega f(\phi^{\hat{\varepsilon}}(\cdot,t)), & y \in \Omega,\ t > 0, \\ \partial_n \phi^{\hat{\varepsilon}}(y,t) = 0, & y \in \partial\Omega,\ t > 0, \end{cases} \tag{3.1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a fixed bounded domain with smooth boundary $\partial\Omega$, ∂_n is the exterior normal derivative to $\partial\Omega$, Δ_y represents the Laplacian with respect to y, and $\fint_\Omega = \frac{1}{|\Omega|}\int_\Omega$ means the average over Ω. Here f is the derivative of a double well potential W. We assume the following conditions for $f \in C^\infty(\mathbb{R})$:

$$f(\pm 1) = 0,\ f'(\pm 1) > 0,\ \int_{-1}^{s} f = \int_{1}^{s} f > 0 \text{ for all } s \in (-1, 1). \tag{3.1.2}$$

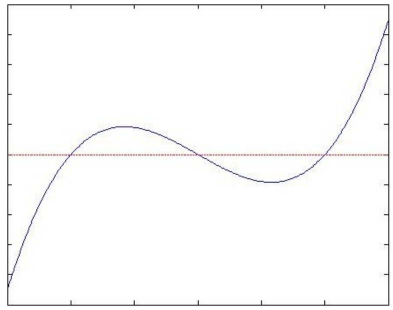

Figure 3.1: Graph of the function f

(3.1.1) can be considered as the associated L^2 gradient flow of the functional

$$J_{\hat{\varepsilon}}(u) = \int_\Omega (\frac{\hat{\varepsilon}^2}{2}|\nabla u|^2 + W(u))dx, \ \ u \in \{v \in H^1(\Omega) : \fint_\Omega v dx = m\}. \tag{3.1.3}$$

This functional has been investigated by several authors, for example, [5, 9, 27, 28, 30, 33, 38, 69, 70, 77, 85].

In [5], N.D. Alikakos et al. constructed an approximately invariant manifold for (3.1.1) using a carefully devised asymptotic expansion. Each element of the manifold is a so-called droplet, or bubble, that is, a state having a roughly semicircular interface attached to the boundary of the domain, the interface separating regions where the solution takes on two different almost constant values. These droplets move slowly towards the increasingly curved region, while maintaining their shape. The motion of the center of the bubble can be determined by the following ODE,

$$\begin{cases} \dfrac{d\hat{\xi}}{dt} = -\dfrac{4\hat{\xi}^2\delta}{3\pi}\mathcal{K}'_\Omega(\hat{\xi}(t)) + O(\hat{\xi}^2\delta^2), \\ \hat{\xi}(0) = \hat{\xi}_0, \end{cases} \tag{3.1.4}$$

where $\hat{\xi}$ is the arc-length parameter of $\partial\Omega$ which represents the center of the bubble. More details can be found in Section 3.2. Moreover, they proved that the bubble shape is stable, that is, if we start from a small H^1-neighborhood of size $O(\varepsilon)$ of the bubble solution, then the flow will stay, for positive time, in a small neighborhood of the manifold of bubble states, in the H^1 sense. Because of the difficulty in handling the contact with the boundary of the domain in higher dimensions, in [5], the authors considered only the two-dimensional case. In higher dimension, there are some results for the interior bubbles, see[6, 7]. In those papers, N.Alikakos and G.Fusco considered bubble solutions for Cahn-Hilliard equation (the mass-conserving Allen-Cahn equation will produce similar dynamics). Roughly speaking, the interface of the bubble has constant curvature and it moves towards the boundary at a exponentially small speed, retaining its shape until it gets close to the boundary. Once near the boundary, it is conjectured that the bubble quickly adhere to the boundary, its energy roughly dropping by half, and then following the dynamics discussed here.

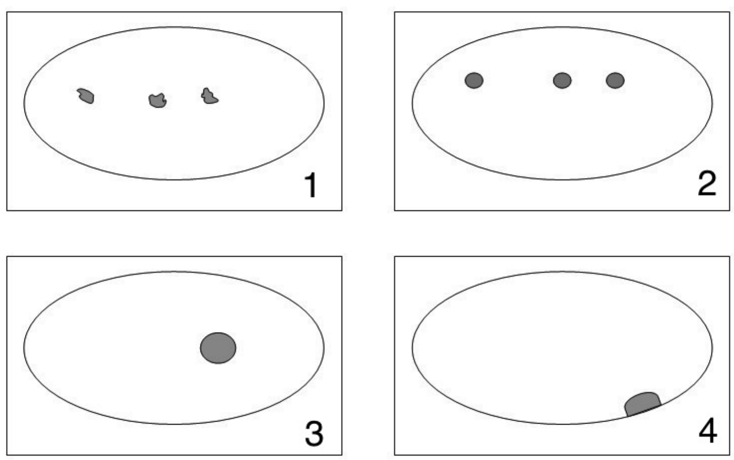

Figure 3.2: Four stages in the evolution for a two-dimensional domain Ω. The last stage is the object of study in this paper.

In this project, we use the framework of [19] to construct a true invariant manifold for (3.1.1), which is close to the approximately invariant manifold given in [5]. The definition of approximate invariancy in [5] is different from the definition in [19]. In [5], a manifold is approximately invariant if the equation, evaluated at a point (i.e., function) of the manifold, is satisfied up to a small error. In [19], approximate invariancy means the manifold is approximately invariant under the solution map for a fixed time (see Section 2.1 for more details). Hence, our main task is to prove that the approximately invariant manifold constructed in [5] satisfies the conditions in the definition given in [19]. Note that once we have obtained the global invariant manifold, bubble solutions on it exist globally in time, forward and backward, being either stationary or connecting stable and unstable equilibria.

This chapter is organized as follows, in Section 3.2 we give some background on the construction of the approximate bubble solution. In Section 3.3, we prove the existence of a true global invariant manifold of bubble states for (3.1.1). Finally, we will discuss the dynamics of the bubble, which includes the motion of the bubble in forward and backward time, and the location of equilibrium bubble states.

3.2 Approximate bubble solution for the mass-conserving Allen-Cahn equation

In this section, we will introduce the approximate bubble solutions of (3.1.1), which were constructed by N.D. Alikakos et al. in [5]. Roughly speaking, each has a semicircular interface Γ, which is the zero level set, with small radius δ. The solution is almost -1 inside the interface, and almost $+1$ outside. This state then moves along the boundary of the domain according to a one-dimensional dynamical system. Now we give a more detailed description. First, we introduce a change of variables that fixes the size of the bubble while δ is varied.

$$y = \delta x,\ \hat{\varepsilon} = \varepsilon\delta,\ u^{\varepsilon}(x,t) = \phi^{\hat{\varepsilon}}(y,t),\ \Omega_{\delta} = \delta^{-1}\Omega := \{x;\ \delta x \in \Omega\}. \tag{3.2.1}$$

Then we can write (3.1.1) as

$$
\begin{cases}
u^{\varepsilon}_t(x,t) = \varepsilon^2 \Delta u^{\varepsilon}(x,t) - f(u^{\varepsilon}(x,t)) + \fint_{\Omega_\delta} f(u^{\varepsilon}(\cdot,t)), & x \in \Omega_\delta,\ t > 0, \\
\partial_n u^{\varepsilon}(x,t) = 0, & x \in \partial\Omega_\delta, t > 0.
\end{cases}
\tag{3.2.2}
$$

We parameterize $\partial\Omega_\delta$ by $z^\delta(\xi)$, where ξ is the arc-length of $\partial\Omega_\delta$ measured from some fixed point of $\partial\Omega_\delta$. We are seeking an invariant manifold $\tilde{\mathcal{M}}$ consisting of bubble-like functions $u(\cdot,\xi,\varepsilon)$, parameterized by ξ, which is the center of the approximately semicircular interface. Obviously, $\tilde{\mathcal{M}}$ is one-dimensional. The invariance means that the vector field is tangent to the manifold, so for $u \in \tilde{\mathcal{M}}$ we can write (3.2.2) analytically in the form:

$$
\begin{cases}
-\varepsilon^2 \Delta u + f(u) + \varepsilon^2 c u_\xi + \varepsilon\sigma = 0, & x \in \Omega_\delta,\ t > 0, \xi \in \mathbb{R}^1 \\
\partial_n u(x,\xi,\varepsilon) = 0, & x \in \partial\Omega_\delta, t > 0, \\
\int_{\Omega_\delta} u(\cdot,\xi,\varepsilon) = |\Omega_\delta| - \pi.
\end{cases}
\tag{3.2.3}
$$

Here $\sigma = \sigma(\xi,\varepsilon)$ and $c = c(\xi,\varepsilon)$ are constants in x and following [5] we have multiplied by powers of ε in anticipation of their size. We call c the speed of the droplet, and $\varepsilon\sigma = \fint_{\Omega_\delta} f(u(\cdot,\xi,\varepsilon))dx$ adjusts for the mass constraint. The motion of the bubble can be represented by the motion of the center ξ, which satisfies

$$
\frac{d\xi(t,\varepsilon)}{dt} = \varepsilon^2 c(\xi,\varepsilon). \tag{3.2.4}
$$

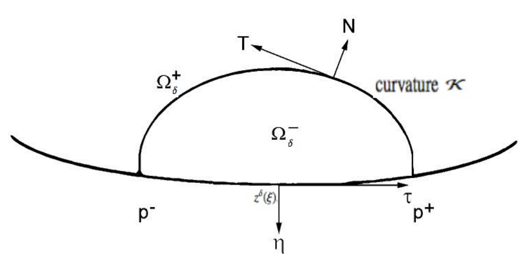

Figure 3.3: Geometry of the bubble

Considering the fact that our equation is the mass-conserving gradient flow of the energy functional (3.1.3), heuristically we can see that asymptotically, solutions should be almost constant taking on values ± 1 everywhere except for an efficient transition between those values, dictated by the predetermined average value. To make the transition efficient, it should take place along a minimal curve enclosing a given area at $\partial\Omega_\delta$, that is, a circular arc intersecting the domain boundary orthogonally. Furthermore, the transition should have width $O(\varepsilon)$ so that the gradient and bulk parts of (3.1.3) are almost equal. So that the dynamics of the bubble state are determined locally, we require the bubble to have small radius $\delta << 1$ in original coordinateds and 1 in our expanded coordinates, thus we fix our mass to be $|\Omega_\delta| - \pi$. In order to rigorously perform the asymptotic analysis in ε, one needs $0 < \varepsilon << \delta$.

By performing an outer expansion, an inner expansion, and a corner expansion (where the interface meets the boundary of the domain), and patching these together, in [5] the authors constructed an approximate solution to system (3.2.3) having bubble-like structure. This solution is parameterized by c, σ, the length of the interface, $|\Gamma|$, the curvature of the interface, $\mathcal{K}$, and the arc-length from the center, $z^\delta(\xi)$, to $\{p^\pm\}$ the intersection of Γ and $\partial\Omega_\delta$ (see Figure 4.). Invariance of the family with respect to the nonlocal parabolic equation and the mass constraint up to a specified order dictated certain solvable equations for these geometric parameters. Thus they found an approximate solution for (3.2.3) given by the following:

Theorem 3.2.1. *[5] Assume that δ and ε are small parameters satisfying $\varepsilon \leq \frac{1}{2}C_1^*\delta^2$, where C_1^* is a constant defined by (2.65) in [5]. Then for any positive interger k, if ε is sufficiently small, there* *exist* $u = u(x, \xi, \varepsilon),\ \sigma = \sigma(\xi, \varepsilon),\ c = c(\xi, \varepsilon)$ *such that*

$$\begin{cases} \varepsilon^2 \Delta u - f(u) + \varepsilon\sigma = \varepsilon^2 c u_\xi + O(\varepsilon^k) & \text{in } \Omega_\delta, \\ \partial_n u = 0 & \text{on } \partial\Omega_\delta, \\ \int_{\Omega_\delta} u(\cdot, \xi, \varepsilon) = |\Omega_\delta| - \pi, \\ \varepsilon\sigma = \fint_{\Omega_\delta} f(u(\cdot, \xi, \varepsilon) dx. \end{cases} \tag{3.2.5}$$

Remark 3.2.2. 1. In [5], Theorem 3.2.1 requires that $\varepsilon \geq \delta^m$ for some $m \geq 2$. By carefully checking the proof, we found that this condition is not necessary.

2. In fact when δ is small, there are two quasi-steady states, one being the droplet with an interface separating regions where it is approximately $+1$ and -1, the other being a spike state. The former shape is stable and the later is unstable. As δ becomes $O(\varepsilon)$, the droplet and spike merge and cause to exist for smaller δ.

3. For the inner expansion, near the interface, the authors use the coordinates (r, s), where r is the signed distance from the interface, which is positive outside the bubble and negative inside, and s is the arc-length along the interface. The leading order of the interior expansion is the heteroclinic solution to

$$\ddot{U} - f(U) = 0,\ U(\pm\infty) = \pm 1,\ \int_{-\infty}^{\infty} R\dot{U}^2(R) dR = 0, \tag{3.2.6}$$

in the stretched variable $R = \frac{r}{\epsilon}$.

4. The leading term of the outer expansion is ± 1, and the corner expansion is $O(\varepsilon)$ and is exponentially decaying away from the interface.

5. The leading term of $|\Gamma|$ is π, and the leading term of the curvature, $\mathcal{K}$, of Γ is 1, which implies that the interface is approximately a semicircle.

Now, we state some results of the spectral analysis for the operator obtained by linearizing at one of these bubble states. Consider the following eigenvalue problem:

$$\begin{cases} L\bar{\phi} := \varepsilon^2 \Delta\bar{\phi} - f'(u)\bar{\phi} + \fint_{\Omega_\delta} f'(u)\bar{\phi} = \bar{\lambda}\bar{\phi} & \text{in } \Omega_\delta, \\ \partial_n \bar{\phi} = 0 & \text{on } \partial\Omega_\delta, \\ \int_{\Omega_\delta} \bar{\phi} = 0, \end{cases} \tag{3.2.7}$$

where u is the solution to (3.2.5). The largest eigenvalue is of the order $\delta\varepsilon^2$, which is very close to zero since both factors are small, and the corresponding eigenfunction is close to u_ξ. More precisely, the largest eigenvalue is $\frac{4\varepsilon^2}{3\pi}\frac{d^2\mathcal{K}_{\Omega_\delta}}{d\xi^2}(\xi) + O(\varepsilon^2\delta^4)$, where $\mathcal{K}_{\Omega_\delta}$ is the curvature of the boundary $\partial\Omega_\delta$. The rest of the spectrum is negative but is only $O(\epsilon^2)$ away from zero. The precise estimate is given by the following theorem from [5].

Theorem 3.2.3. *[5] Let $u = u(x, \xi, \varepsilon)$ be the solution to (3.2.5). If for large constant C^*, $\delta^2 > C^*\varepsilon$ holds, then for any $v \in H^1(\Omega_\delta)$ satisfying*

$$\int_{\Omega_\delta} v = 0, \int_{\Omega_\delta} v u_\xi = 0, \tag{3.2.8}$$

we have

$$\langle Lv, v\rangle \leq -\frac{2\varepsilon^2\pi^2}{|\Gamma|^2}\int_{\Omega_\delta} v^2. \tag{3.2.9}$$

Thus, there is a gap between the largest eigenvalue and the others because the largest eigenvalue is $\frac{4\varepsilon^2}{3\pi}\frac{d^2\mathcal{K}_{\Omega_\delta}}{d\xi^2}(\xi) + O(\varepsilon^2\delta^4)$, where $\frac{d^2\mathcal{K}_{\Omega_\delta}}{d\xi^2}(\xi)$ is of order δ, and δ is very small.

3.3 Invariant manifolds of boundary droplets

3.3.1 Construction of $\mathcal{M}_\varepsilon$

We choose the space X as $H^1(\Omega_\delta)$ with norm given by $|u|_X^2 = |\varepsilon\nabla u|_{L^2}^2 + |u|_{L^2}^2$. First, we modify the function f to make sure that the evolution defines a semiflow globally in time. Thus, we consider

$$u_t^\varepsilon(x,t) = \varepsilon^2\Delta u^\varepsilon(x,t) - \tilde{f}(u^\varepsilon(x,t)) + \fint_{\Omega_\delta} \tilde{f}(u^\varepsilon(\cdot,t)), \tag{3.3.1}$$

where $\tilde{f}(u) = \eta(u)f(u)$. Here, $\eta(s) \geq 0$ is a C^∞ bump function satisfying

$$\eta(s) = 1,\ \ |s| \leq 2;\ \ \eta(s) = 0, |s| \geq 4. \tag{3.3.2}$$

Note that $|\tilde{f}|_{C^m(\mathbb{R})} < \infty$. This modification does not affect the bubble solution we seek because that solution has its range in $[-1,1]$. For convenience, we keep the notation f, instead of $\tilde{f}$.

Let $W(x,\xi,\varepsilon)$ be the second order approximation of the solution to (3.2.5) given by Theorem 3.2.1, which means that W satisfies

$$\begin{cases} \varepsilon^2\Delta W - f(W) + \varepsilon\sigma = \varepsilon^2 cW_\xi + O(\varepsilon^3) & \text{in } \ \Omega_\delta, \\ \partial_n W = 0 & \text{on } \ \partial\Omega_\delta, \\ \int_{\Omega_\delta} W(\cdot,\xi,\varepsilon) = |\Omega_\delta| - \pi, \\ \varepsilon\sigma = \fint_{\Omega_\delta} W(\cdot,\xi,\varepsilon). \end{cases} \tag{3.3.3}$$

Define $\psi_\varepsilon : \partial\Omega_\delta \to X$ as $\psi_\varepsilon(\xi) = W(x,\xi,\varepsilon)$. Let $\mathcal{M}_\varepsilon = \psi_\varepsilon(\partial\Omega_\delta)$. We will prove that $\mathcal{M}_\varepsilon$ is an approximately normally hyperbolic invariant manifold, so that we can apply the Theorems 2.0.3

and 2.0.6.

3.3.2 $\mathcal{M}_\varepsilon$ is approximately invariant

One may expect that the flow of (3.2.2) starting from $W(x, \xi, \varepsilon)$ will stay close to $W(x, \xi(t), \varepsilon)$, where $\xi(t)$ satisfies (3.2.4). In fact, we have

Lemma 3.3.1. *There exists $C > 0$ such that, for any small ε and $z^\delta(\xi) \in \partial\Omega$, the solution $u(t, x, \varepsilon)$ of (3.3.1) with initial data $u(0, x, \varepsilon) = \psi_\varepsilon(\xi)$ satisfies*

$$|u(t, x, \varepsilon) - \psi_\varepsilon(\xi(t))|_X \leq C\varepsilon^3 e^{Ct}. \tag{3.3.4}$$

Proof. Let $v = u - \psi_\varepsilon$, then v satisfies

$$\begin{cases} v_t = \varepsilon^2 \Delta v - (f(\psi_\varepsilon + v) - f(\psi_\varepsilon)) + \fint_{\Omega_\delta} [f(\psi_\varepsilon + v) - f(\psi_\varepsilon)]dx + O(\varepsilon^3), \\ v(0, \cdot) = 0. \end{cases} \tag{3.3.5}$$

Let $g(v) \equiv f(\psi_\varepsilon + v) - f(\psi_\varepsilon)$. Rewrite (3.3.5) as

$$\begin{cases} v_t = L_\varepsilon v - g(v) + \fint_{\Omega_\delta} g(v)dx + O(\varepsilon^3), \\ v(0, \cdot) = 0 \end{cases} \tag{3.3.6}$$

where $L_\varepsilon = \varepsilon^2 \Delta$, with homogeneous Neumann boundary condition.

Using the variation of constants formula, we have

$$v = \int_0^t e^{L_\varepsilon(t-s)}\big(- g(v) + \fint_{\Omega_\delta} g(v)dx + O(\varepsilon^3)\big)ds. \tag{3.3.7}$$

It is well known that L_ε generates a contraction semigroup, and recall that f has been modified, so we have $|g(v)| \le C|v|$ and $|\nabla g(v)| = |g'(v)\nabla v| \le C|\nabla v|$, which implies

$$|v(\cdot, t)|_X \le \int_0^t (C|v|_X + O(\varepsilon^3))ds. \tag{3.3.8}$$

Applying Gronwall's inequality to $|v|_X$, we have

$$|v|_X \le C\varepsilon^3 e^{Ct}. \tag{3.3.9}$$

□

Lemma 3.3.1 implies that $\mathcal{M}_\varepsilon$ is an approximately overflowing invariant manifold for T being the time t_0 solution operator, by taking the function v in Definition 2.0.5 as $\xi(-t_0, \cdot)$.

3.3.3 Splitting along the manifold $\mathcal{M}_\varepsilon$

From the spectral analysis mentioned in Section 2.2, we can split the space X along $\mathcal{M}_\varepsilon$ naturally. For any $z^\delta(\xi) \in \partial\Omega_\delta$, let

$$\begin{aligned} X^c_{\varepsilon,\xi} &= T_{\psi_\varepsilon(\xi)}\mathcal{M}_\varepsilon = span\{W_\xi(x,\xi,\varepsilon)\}, \\ X^s_{\varepsilon,\xi} &= \{v \in X : \int_{\Omega_\delta} v\tilde{v} = 0, \ \text{for all } \tilde{v} \in X^c_{\varepsilon,\xi}\}. \end{aligned} \tag{3.3.10}$$

Let $\Pi^\alpha_{\varepsilon,\xi}, \alpha = c, s$ be the projections associated with this splitting.

Lemma 3.3.2. $\|\Pi^{\alpha}_{\varepsilon,\xi}\|$ *is uniformly bounded and smoothly depends on* ξ.

Proof. For any $x \in X$, we can write $x = x^c + x^s$, where $x^\alpha \in X^\alpha_{\varepsilon,\xi}$. Since $x^c \perp_{L^2} x^s$, $|x|^2_{L^2} = |x^c|^2_{L^2} + |x^s|^2_{L^2}$. Since $X^c_{\varepsilon,\xi}$ is finite dimensional, we have

$$\begin{aligned} &C|x^c|_X \leq |x^c|_{L^2} \leq |x|_{L^2} \leq |x|_X, \text{ and} \\ &|x|_X \geq |x^s|_X - |x^c|_X \geq |x^s|_X - \frac{1}{C}|x|_X \end{aligned} \tag{3.3.11}$$

for some constant C independent of ε, because of our choice of norm on X.

By carefully checking the construction of the approximate bubble solutions in [5], we find that every term in the asymptotic expansion comes from solving a certain elliptic equation which gives the solution high regularity. Likewise, σ and c are also smooth functions of ξ (see section 2.2 and 2.3 of [5] and the appendix of [8]). This implies that $\psi_\varepsilon(\partial\Omega)$ is at least a C^2 smooth manifold. Then the uniform boundedness follows from the usual compactness argument and the smooth dependence follows from the smoothness of $\psi_\varepsilon(\xi)$.

□

The third inequality in (H2) is satisfied automatically for compact manifolds (the proof can be found in [11]). So far (H1) and (H2) in Definition 2.0.2 have been established. We need to prove that $\mathcal{M}_\varepsilon$ is approximately normally hyperbolic as an approximately overflowing invariant manifold, that is, (C3) holds.

3.3.4 $\mathcal{M}_\varepsilon$ is approximately normally hyperbolic

From the splitting of the space, we can see that the whole space X is the center-stable manifold W^{cs}, because there is no unstable subspace. Hence, we only need to find the center-unstable manifold, which is actually just the center manifold. Since we need to study the linearized flow, we consider the linearized system.

Let $\overline{L}_\varepsilon v = \varepsilon^2 \Delta v - f'(u)v + ⨍_{\Omega_\delta} f'(u)v dx$, where u is the solution to (3.2.2) with initial data $W(x, \xi, \varepsilon)$ from (3.3.3) for some fixed ξ.

Let $\overline{W}(t, \cdot)$ be the solution to

$$\begin{cases} \overline{W}_t = \overline{L}_\varepsilon \overline{W}, \\ \overline{W}(0, \cdot) = W(\cdot, \xi, \varepsilon). \end{cases} \tag{3.3.12}$$

Let $\widetilde{L}_\varepsilon v = \varepsilon^2 \Delta v - f'(\psi_\varepsilon(\xi(t)))v + ⨍_{\Omega_\delta} f'(\psi_\varepsilon(\xi))v dx$, and $\widetilde{W}(t, \cdot)$ be the solution to

$$\begin{cases} \widetilde{W}_t = \widetilde{L}_\varepsilon \widetilde{W}, \\ \widetilde{W}(0, \cdot) = W(\cdot, \xi, \varepsilon). \end{cases} \tag{3.3.13}$$

Lemma 3.3.3. *$|\overline{W}(t, \cdot) - \widetilde{W}(t, \cdot)|_X \leq C\varepsilon^3 |W(\cdot, \xi, \varepsilon)|_X e^{Ct}$, for some constant C*

Proof. The difference $v = \overline{W} - \widetilde{W}$ satisfies

$$\begin{cases} v_t = \varepsilon^2 \Delta v - f'(\psi_\varepsilon(\xi(t))v + g(u - \psi_\varepsilon(\xi(t))\overline{W} + ⨍_{\Omega_\delta} (f'(\psi_\varepsilon(\xi))v - g(u - \psi_\varepsilon(\xi(t))\overline{W})dx, \\ v(0, \cdot) = 0, \end{cases} \tag{3.3.14}$$

where $g = f'(u) - f'(\psi_\varepsilon(\xi(t)))$.

By an argument similar to the proof of Lemma 3.3.1, we get

$$|\overline{W}(t,\cdot)|_X \le |W(\cdot,\xi,\varepsilon)|_X e^{Ct}. \tag{3.3.15}$$

Using the variation of constants formula, we have

$$v = \int_0^t e^{L\varepsilon(t-s)}[-f'(\psi_\varepsilon(\xi(t))v + g(u-\psi_\varepsilon(\xi(t))\overline{W} + \fint_{\Omega_\delta}(f'(\psi_\varepsilon(\xi))v - g(u-\psi_\varepsilon(\xi(t))\overline{W})dx]ds. \tag{3.3.16}$$

Using Lemma 3.3.1 and (3.3.15), we have

$$|v(\cdot,t)|_X \le \int_0^t (C|v(\cdot,s)|_X + C\varepsilon^3|W(\cdot,\xi,\varepsilon)|_X e^{Cs})ds. \tag{3.3.17}$$

Applying Gronwall's inequality gives the desired result. $\square$

Next we study the behavior of $\widetilde{W}(t,\cdot)$ in center and stable directions. Write $W(\cdot,\xi,\varepsilon)$ as $W(\cdot,\xi,\varepsilon) = a(0)W_\xi(x,\xi,\varepsilon) + W^s(x,\xi,\varepsilon)$ and similarly, $\widetilde{W}(t,\cdot) = a(t)W_\xi(x,\xi(t),\varepsilon) + W^s(x,\xi(t),\varepsilon)$, where $W^s(x,\xi(t),\varepsilon) \in X^s_{\varepsilon,\xi(t)}$.

Lemma 3.3.4. *If $a(0) = 0$, i.e., $W(\cdot,\xi,\varepsilon) \in X^s_{\varepsilon,\xi}$, then*

$$|a(t)||W_\xi(x,\xi(t),\varepsilon)|_X \le C\varepsilon^{1/2}e^{Ct}|W(\cdot,\xi,\varepsilon)|_X. \tag{3.3.18}$$

Proof. By differentiating with respect to ξ in (3.3.3), we have

$$\widetilde{L}_\varepsilon W_\xi(x,\xi(t),\varepsilon) = \varepsilon^2 c W_{\xi\xi}(x,\xi(t),\varepsilon) + \varepsilon^2 c_\xi W_\xi(x,\xi(t),\varepsilon) + O(\varepsilon^2). \tag{3.3.19}$$

One finds that the residual $O(\varepsilon^3)$ in (3.3.3) becomes $O(\varepsilon^2)$, because taking a derivative near the interface generates a factor of $\frac{1}{\varepsilon}$. It is easy to see that near the interface W_ξ is of order $\frac{1}{\varepsilon}$ and $W_{\xi\xi}$ is of order $\frac{1}{\varepsilon^2}$. Also note that the width of the layer is $O(\varepsilon)$, so $|W_\xi|_{L^2} \leq C\varepsilon^{-1/2}$ and $|W_{\xi\xi}|_{L^2} \leq C\varepsilon^{-3/2}$, which implies that

$$|\widetilde{L}_\varepsilon W_\xi(x, \xi(t), \varepsilon)|_{L^2} \leq C\varepsilon^{1/2}. \tag{3.3.20}$$

For a detailed proof of these facts, use the expression for u_ξ from p.294 of [5] and the derivative bounds in [5], Sections 2.2 and 2.3. By an argument similar to the proof of Lemma 3.3.1, we have $|\widetilde{W}|_{L^2} \leq Ce^{Ct}|W(\cdot, \xi, \varepsilon)|_{L^2}$ for some constant C, which implies that

$$|a(t)||W_\xi|_{L^2}, \ \ |W^s(x, \xi(t), \varepsilon)|_{L^2} \leq Ce^{Ct}|W(\cdot, \xi, \varepsilon)|_{L^2}. \tag{3.3.21}$$

Since $\widetilde{L}_\varepsilon$ is self-adjoint and $\widetilde{L}_\varepsilon \widetilde{W} = \widetilde{W}_t$, we have

$$\begin{aligned}
\langle \widetilde{W}(t, \cdot), \widetilde{L}_\varepsilon a(t) W_\xi(x, \xi(t), \varepsilon)\rangle =& \langle \widetilde{W}_t(t, \cdot), a(t)W_\xi(x, \xi(t), \varepsilon)\rangle \\
=& \langle a'(t)W_\xi(x, \xi(t), \varepsilon) + a(t)W_{\xi\xi}(x, \xi(t), \varepsilon)\dot{\xi}, a(t)W_\xi\rangle + \\
& \langle W_t^s, a(t)W_\xi\rangle \\
=& a(t)a'(t)\langle W_\xi, W_\xi\rangle + \frac{1}{2}a^2(t)\frac{d}{dt}\langle W_\xi, W_\xi\rangle - \\
& a(t)\langle W^s(x, \xi(t), \varepsilon), W_{\xi\xi} \cdot \dot{\xi}\rangle \\
=& \frac{d}{dt}(\frac{a^2(t)}{2}|W_\xi|^2_{L^2}) - a(t)\langle W^s(x, \xi(t), \varepsilon), W_{\xi\xi} \cdot \dot{\xi}\rangle
\end{aligned} \tag{3.3.22}$$

For the third identity, we use the fact that $\langle W_\xi(x, \xi(t), \varepsilon), W^s(x, \xi(t), \varepsilon)\rangle = 0$. Note that, from p.294 of [5], $|W_\xi|_{L^2} \geq C^{-1}\varepsilon^{-1/2}$, which combined with (3.3.21), gives

$$|a(t)| \leq C\varepsilon^{1/2}e^{Ct}|W(\cdot, \xi, \varepsilon)|_{L^2}. \tag{3.3.23}$$

Combine (3.3.23), (3.3.22), (3.3.21), (3.3.20) and use $\dot{\xi} = \varepsilon^2 c$ to obtain

$$\frac{d}{dt}(a(t)^2 |W_\xi|^2_{L^2}) \leq 4C^2 \varepsilon e^{2Ct} |W(\cdot, \xi, \varepsilon)|^2_{L^2}, \tag{3.3.24}$$

which implies that

$$a^2(t)|W_\xi(x, \xi(t), \varepsilon)|^2_{L^2} - a^2(0)|W_\xi(x, \xi(0), \varepsilon)|^2_{L^2} \leq 2C\varepsilon(e^{2Ct} - 1)|W(\cdot, \xi, \varepsilon)|^2_{L^2}. \tag{3.3.25}$$

Thus, if $a(0) = 0$, i.e., $W(\cdot, \xi, \varepsilon) \in X^s_{\varepsilon,\xi}$, then we have

$$|a(t)||W_\xi(x, \xi(t), \varepsilon)|_{L^2} \leq C\varepsilon^{1/2} e^{Ct} |W(\cdot, \xi, \varepsilon)|_{L^2}. \tag{3.3.26}$$

Since $X^c_{\varepsilon,\xi}$ is finite dimensional, we have

$$|a(t)||W_\xi(x, \xi(t), \varepsilon)|_X \leq C\varepsilon^{1/2} e^{Ct} |W(\cdot, \xi, \varepsilon)|_X. \tag{3.3.27}$$

□

Note that $\xi(t)$ is given by an ODE, so we can consider $\widetilde{W}(-t, \cdot) = a(-t)W_\xi(x, \xi(-t), \varepsilon) + W^s(x, \xi(-t), \varepsilon)$. Following an analagous argument to that in Lemma 3.3.4, we have if $W^s(x, \xi, \varepsilon) = 0$, then

$$a^2(-t)|W_\xi(x, \xi(-t), \varepsilon)|^2_{L^2} - a^2(0)|W_\xi(x, \xi, \varepsilon)|^2_{L^2} \leq 2C\varepsilon e^{Ct} a^2(0)|W_\xi(x, \xi, \varepsilon)|^2_{L^2}, \tag{3.3.28}$$

which implies that

$$|a(t)||W_\xi(x,\xi(t),\varepsilon)|_X \geq \left(\frac{C}{1+C\varepsilon e^{Ct}}\right)^{1/2}|a(0)||W_\xi(x,\xi,\varepsilon)|_X. \tag{3.3.29}$$

We also have the estimate:

Lemma 3.3.5. *If* $a(0)=0$*, i.e.,* $W(\cdot,\xi,\varepsilon)=W^s(x,\xi,\varepsilon)\in X^s_{\varepsilon,\xi}$*, then*

$$|W^s(x,\xi(t),\varepsilon)|_{L^2} \leq (e^{-bt}+C\varepsilon e^{Ct})|W^s(x,\xi,\varepsilon)|_{L^2}, \tag{3.3.30}$$

and

$$|\varepsilon\nabla W^s|_{L^2} \leq C(e^{-bt}+\varepsilon e^{Ct}+\varepsilon^{\frac{3}{2}}e^{Ct}(e^{-bt}+C\varepsilon e^{Ct})^{\frac{1}{2}})|W^s(x,\xi,\varepsilon)|_{L^2}, \tag{3.3.31}$$

where $b=\dfrac{2\pi^2}{|\Gamma|^2}$*, coming from (3.2.9).*

If $W^s(0,\cdot)=0$*, i.e.,* $W(\cdot,\xi,\varepsilon)=a(0)W_\xi(x,\xi,\varepsilon)\in X^c_{\varepsilon,\xi}$*, then*

$$|W^s(x,\xi(t),\varepsilon)|_{L^2} \leq C\varepsilon e^{Ct}|a(0)||W_\xi(x,\xi,\varepsilon)|_{L^2}, \tag{3.3.32}$$

and

$$|\varepsilon\nabla W^s|_{L^2} \leq C\varepsilon e^{Ct}|a(0)||W_\xi(x,\xi,\varepsilon)|_{L^2}. \tag{3.3.33}$$

Proof. With the decomposition of $\widetilde{W}$ given prior to Lemma 3.3.4, we write (3.3.13) as

$$a'(t)W_\xi(x,\xi(t),\varepsilon)+a(t)W_{\xi\xi}(x,\xi(t),\varepsilon)\dot{\xi}+W^s_t(x,\xi(t),\varepsilon)=\widetilde{L}_\varepsilon(a(t)W_\xi(x,\xi(t),\varepsilon)+W^s(x,\xi(t),\varepsilon)). \tag{3.3.34}$$

Using (3.3.19), we get

$$a'(t)W_\xi(x,\xi(t),\varepsilon) + W^s_t = \widetilde{L}_\varepsilon(W^s(t)) + \varepsilon^2 a(t)c_\xi W_\xi(x,\xi(t),\varepsilon) + a(t)O(\varepsilon^2). \tag{3.3.35}$$

Taking the inner product with W^s and using $W^s_t = W^s_\xi \dot{\xi}$, we have

$$\frac{d}{d\xi}(|W^s(t)|^2_{L^2})\varepsilon^2 c = 2\langle \widetilde{L}_\varepsilon W^s, W^s\rangle + a(t)\langle O(\varepsilon^2), W^s\rangle, \tag{3.3.36}$$

Note that from the calculations given in [5], the $O(\varepsilon^2)$ term is of order $O(\varepsilon^2)$ near the interface, but of order $O(\varepsilon^3)$ away from the interface, so its L^2 norm is $O(\varepsilon^{\frac{5}{2}})$. If $a(0) = 0$, we may use Theorem 3.2.3, Lemma 3.3.4, and (3.3.21) to obtain that for some positive constants b and C (which may change from line to line),

$$\frac{d}{d\xi}(|W^s(x,\xi(t),\varepsilon)|^2_{L^2}) \le -b|W^s(x,\xi(t),\varepsilon)|^2_{L^2} + C\varepsilon e^{Ct}|W^s(x,\xi,\varepsilon)|_{L^2}|W^s(x,\xi(t),\varepsilon)|_{L^2}, \tag{3.3.37}$$

which implies that

$$\begin{aligned} |W^s(x,\xi(t),\varepsilon)|_{L^2} &\le (e^{-bt} + \frac{C\varepsilon e^{(b+C)t}}{b+C})|W^s(x,\xi,\varepsilon)|_{L^2}, \\ &\le (e^{-bt} + C\varepsilon e^{Ct})|W^s(x,\xi,\varepsilon)|_{L^2}. \end{aligned} \tag{3.3.38}$$

If $W^s(0,\cdot) = 0$, we may use Theorem 3.2.3, (3.3.25) and (3.3.21) to obtain that for some constant b and C,

$$\frac{d}{d\xi}(|W^s(x,\xi(t),\varepsilon)|^2_{L^2}) \le -b|W^s(x,\xi(t),\varepsilon)|^2_{L^2} + C\varepsilon e^{Ct}|a(0)||W_\xi(x,\xi,\varepsilon)|_{L^2}|W^s(x,\xi(t),\varepsilon)|_{L^2}, \tag{3.3.39}$$

which implies that

$$\begin{aligned} |W^s(x,\xi(t),\varepsilon)|_{L^2} &\leq \frac{C\varepsilon e^{Ct}}{b+C}|a(0)||W_\xi(x,\xi,\varepsilon)|_{L^2}, \\ &\leq C\varepsilon e^{Ct}|a(0)||W_\xi(x,\xi,\varepsilon)|_{L^2}. \end{aligned} \tag{3.3.40}$$

If $a(0) = 0$, note that $\int_{\Omega_\delta} W^s dx = 0$, (3.3.36) gives

$$\begin{aligned} 2\langle \widetilde{L}_\varepsilon W^s, W^s\rangle &= -\frac{d}{dt}(|W^s|^2_{L^2}) + a(t)\langle O(\varepsilon^2), W^s\rangle, \\ -2\int_{\Omega_\delta} \varepsilon^2|\nabla W^s|^2 + f'(W)|W^s|^2 dx &= -\frac{d}{dt}(|W^s|^2_{L^2}) + a(t)\langle O(\varepsilon^2), W^s\rangle. \end{aligned} \tag{3.3.41}$$

Using (3.3.37) and (3.3.38) in (3.3.41) gives

$$\begin{aligned} \varepsilon^2|\nabla W^s|^2_{L^2} &\leq \int_{\Omega_\delta} |f'(W)||W^s|^2 dx + b\varepsilon^2|W^s|^2_{L^2} + C\varepsilon^3 e^{Ct}|W^s(x,\xi,\varepsilon)|_{L^2}|W^s(x,\xi(t),\varepsilon)|_{L^2}, \\ &\leq ((C+b\varepsilon^2)(e^{-bt} + C\varepsilon e^{Ct})^2 + C\varepsilon^3 e^{Ct}(e^{-bt} + C\varepsilon e^{Ct}))|W^s(x,\xi,\varepsilon)|^2_{L^2}, \end{aligned} \tag{3.3.42}$$

which implies that

$$|\varepsilon\nabla W^s|_{L^2} \leq C(e^{-bt} + \varepsilon e^{Ct} + \varepsilon^{\frac{3}{2}} e^{Ct}(e^{-bt} + C\varepsilon e^{Ct})^{\frac{1}{2}})|W^s(x,\xi,\varepsilon)|_{L^2}. \tag{3.3.43}$$

If $W^s(x,\xi,\varepsilon) = 0$, we may combine (3.3.36), (3.3.39) and (3.3.40) to obtain

$$\varepsilon^2|\nabla W^s|^2_{L^2} \le \int_{\Omega_\delta} |f'(W)||W^s|^2 dx + b\varepsilon^2|W^s|^2_{L^2} + C\varepsilon^3 e^{Ct}|a(0)||W_\xi(x,\xi,\varepsilon)|_{L^2}|W^s|_{L^2},$$
$$\le (C + b\varepsilon^2)(C\varepsilon e^{Ct})^2|a(0)|^2|W_\xi(x,\xi,\varepsilon)|^2_{L^2} + C\varepsilon^3 e^{Ct} C\varepsilon e^{Ct}|a(0)|^2|W_\xi(x,\xi,\varepsilon)|^2_{L^2}, \tag{3.3.44}$$

which implies that

$$|\varepsilon\nabla W^s|_{L^2} \le C\varepsilon e^{Ct}|a(0)||W_\xi(x,\xi,\varepsilon)|_{L^2}. \tag{3.3.45}$$

□

Now combining Lemma 3.3.2, Lemma 3.3.3, Lemma 3.3.4, (3.3.29), and Lemma 3.3.5, gives (C3) in the definition of normal hyperbolicity for $T_\varepsilon^{t_0}$, the time-t_0 solution map of (3.3.1) provided t_0 is large and with ε chosen small enough. Precisely, $\eta = C\varepsilon^3 e^{Ct_0}$, $\sigma = C\varepsilon^{\frac{1}{2}} e^{Ct_0}$, $a = (\frac{C}{1+C\varepsilon e^{Ct_0}})^{1/2}$, and $\lambda = C(e^{-bt_0} + \varepsilon e^{Ct_0} + \varepsilon^{\frac{3}{2}} e^{Ct_0}(e^{-bt_0} + C\varepsilon e^{Ct_0})^{\frac{1}{2}})$. For instance, we choose t_0 such that $Ce^{-bt_0} \le \frac{1}{2}$, and choose any $\varepsilon \le \varepsilon(t_0)$ to satisfy all the conditions $(C3)$.

3.4 Dynamical bubble solution

So far we have constructed the approximately normally hyperbolic invariant manifold $\mathcal{M}_\varepsilon$. Using the splitting

$$X = H^1(\Omega_\delta) = X^c_{\varepsilon,\xi} \oplus X^s_{\varepsilon,\xi} \tag{3.4.1}$$

and their related estimates we have established approximate normal hyperbolicity. Hence, we may apply Theorem 2.0.6 for sufficiently small ε, to the time-t_0 map $T_\varepsilon^{t_0}$ of the semiflow defined by (3.3.1) for some t_0 large enough. We have the following:

1. For the map $T_\varepsilon^{t_0}$, there exists a unique C^2 normally hyperbolic invariant manifold $\widetilde{\mathcal{M}}_\varepsilon = \Psi_\varepsilon(\partial\Omega_\delta) \subset X$, where Ψ_ε satisfies $\Psi_\varepsilon(\xi) - \psi_\varepsilon(\xi) \in X^s_{\varepsilon,\xi}$.

2. From Lemma 3.3.1 and Theorem 2.0.6, we have that $|\Psi_\varepsilon(\xi) - \psi_\varepsilon(\xi)|_{C^0(\partial\Omega_\delta, X)} \to 0$ as $\varepsilon \to 0$.

To see that $\widetilde{\mathcal{M}}_\varepsilon$ is invariant under the semiflow T_ε^t generated by (3.3.1), we just need to verify condition (H5) stated in Section 2.1: There exists an integer $k \geq 0$, such that for any $\mu > 0$, there exists $\zeta > 0$, such that for any $x \in B(\psi(M), r)$ and $t \in [kt_0, kt_0 + \zeta]$, we have $|T^t(x) - T^{kt_0}(x)| < \mu$. Actually, we can easily prove this by letting $k = 1$ and using the variation of constants formula. We have

$$\begin{aligned}
&T_\varepsilon^t(x) - T_\varepsilon^{t_0}(x) \\
&= e^{L_\varepsilon t}x - e^{L_\varepsilon t_0}x + \int_0^t e^{L_\varepsilon(t-s)} r(T_\varepsilon^s(x))ds - \int_0^{t_0} e^{L_\varepsilon(t_0-s)} r(T_\varepsilon^s(x))ds \\
&= \int_{t_0}^t e^{L_\varepsilon(t-s)} L_\varepsilon e^{L_\varepsilon t_0} x ds + \int_{t_0}^t e^{L_\varepsilon(t-s)} r(T_\varepsilon^s(x))ds + \int_0^{t_0} (e^{L_\varepsilon(t-s)} - e^{L_\varepsilon(t_0-s)}) r(T_\varepsilon^s(x))ds \\
&= \int_{t_0}^t e^{L_\varepsilon(t-s)} L_\varepsilon e^{L_\varepsilon t_0} x ds + \int_{t_0}^t e^{L_\varepsilon(t-s)} L_\varepsilon \int_0^{t_0} e^{L_\varepsilon(t_0-\tau)} r(T_\varepsilon^\tau(x)) d\tau ds + \\
&\quad \int_{t_0}^t e^{L_{\varepsilon,P}(t-s)} r(T_\varepsilon^s(x))ds \\
&= \int_{t_0}^t e^{L_\varepsilon(t-s)} [L_\varepsilon T_\varepsilon^{t_0}(x) + r(T_\varepsilon^s(x))]ds,
\end{aligned} \tag{3.4.2}$$

where

$$r(u) = -f(u) + \fint_{\Omega_\delta} f(u). \tag{3.4.3}$$

Recall that the function f has been cut off, so condition (H5) follows from the smoothing effect of the semigroup operator. Therefore, the manifold $\widetilde{\mathcal{M}}_\varepsilon$ is locally invariant under (3.3.1). Furthermore, since $\widetilde{\mathcal{M}}_\varepsilon$ is in an $O(\varepsilon)$ neighborhood of $\mathcal{M}_\varepsilon$ in $H^1(\Omega_\delta)$, by a regularity argument and

Sobolev inequality, we can actually follow the same proof to get the same result for the space $X = W^{1,q}$ for any large q. Also the invariant manifold is independent of q. Therefore, we have the following theorem.

Theorem 3.4.1. *For every sufficiently small ε, there exists a globally invariant manifold for (3.2.2), $\widetilde{\mathcal{M}}_\varepsilon$, in an $O(\varepsilon)$ neighborhood of $\mathcal{M}_\varepsilon$ in $L^\infty \cap H^1$ and being a graph over $\mathcal{M}_\varepsilon$.*

Qualitatively, $\widetilde{\mathcal{M}}_\varepsilon$ consists of functions each of which has a roughly semicircular interface structure attached to the boundary of Ω_δ. In the next section, we will give the dynamics on $\widetilde{\mathcal{M}}_\varepsilon$.

3.4.1 Motion on $\widetilde{\mathcal{M}}_\varepsilon$

Fix ξ_0, let $\Psi_\varepsilon(\xi(\tau(t)))$ be the solution starting from $\Psi_\varepsilon(\xi_0)$. Here $\xi(\cdot)$ is the motion of the approximate bubble solution, which satisfies (3.2.4), i.e.,

$$\frac{d\xi(t,\varepsilon)}{dt} = \varepsilon^2 c(\xi,\varepsilon). \tag{3.4.4}$$

Note that $c(\xi,\varepsilon)$ is determined by the geometric problems, so it is a known function. The function $\tau(t)$ describes the motion on $\widetilde{\mathcal{M}}_\varepsilon$.

Theorem 3.4.2. $\tau(t)$ *satisfies the equation*

$$\tau' = \frac{O(\varepsilon) + c}{(1 + O(\varepsilon))c}, \tag{3.4.5}$$

which implies that the leading order of $\tau(t)$ is t.

Proof. Since $\Psi_\varepsilon(\xi) - \psi_\varepsilon(\xi) \in X^s_{\varepsilon,\xi}$, we write $\Psi_\varepsilon(\xi(\tau(t))) = W(x, \xi(\tau(t)), \varepsilon) + V(t)$. By the invariance of $\Psi_\varepsilon(\xi(\tau(t)))$ under (3.2.2), we have that

$$\begin{aligned} & W_\xi(x, \xi(\tau(t)), \varepsilon)\xi'(\tau(t))\tau'(t) + V'(t) \\ = & \varepsilon^2 \Delta W - f(W) + ⨍ f(W)dx + \varepsilon^2 \Delta V + N(W, V), \end{aligned} \tag{3.4.6}$$

where $N(W, V) = f(W) - f(W + V) + ⨍ f(W + V) - f(W)dx$, which is at least quadratic in V. Plugging (3.3.3) and (3.2.4) into (3.4.6), we have that

$$\varepsilon^2 c(\xi(\tau(t)), \varepsilon)(\tau' - 1)W_\xi(x, \xi(\tau(t)), \varepsilon) + V'(t) = \varepsilon^2 \Delta V + N(W, V) + O(\varepsilon^3). \tag{3.4.7}$$

Taking the inner product with $W_\xi(x, \xi(\tau(t)), \varepsilon)$, we get

$$\varepsilon^2 c(\tau' - 1)|W_\xi|^2_{L^2} + \langle V'(t), W_\xi\rangle = \langle N(W, V), W_\xi\rangle + \langle O(\varepsilon^3), W_\xi\rangle. \tag{3.4.8}$$

By taking the derivative with respect to t in $\langle V(t), W_\xi\rangle = 0$, we get $\langle V'(t), W_\xi\rangle = \langle V(t), \varepsilon^2 c\tau' W_{\xi\xi}\rangle$. Then we have

$$\varepsilon^2 c(\tau' - 1)|W_\xi|^2_{L^2} + \varepsilon^2 c\tau'\langle V(t), W_{\xi\xi}\rangle = \langle N(W, V), W_\xi\rangle + \langle O(\varepsilon^3), W_\xi\rangle. \tag{3.4.9}$$

Note that $\langle V(t), W_{\xi\xi}\rangle = O(1) = O(\varepsilon)|W_\xi|^2_{L^2}$. Furthermore, since $N(W, V)$ is at least quadratic in V, we have that $\langle N(W, V), W_\xi\rangle = O(\varepsilon^2)|W_\xi|_{L^1} = O(\varepsilon^3)|W_\xi|^2_{L^2}$ and $\langle O(\varepsilon^3), W_\xi\rangle = O(\varepsilon^4)|W_\xi|^2_{L^2}$, since $|W_\xi|_{L^1} = O(1)$. Using these in (3.4.9), we obtain

$$\varepsilon^2 c(\tau' - 1) + O(\varepsilon^3)c\tau' = O(\varepsilon^3) + O(\varepsilon^4), \tag{3.4.10}$$

which implies

$$c(\tau' - 1) + O(\varepsilon)c\tau' = O(\varepsilon), \tag{3.4.11}$$

giving the desired result. □

3.4.2 Equilibria

Theorem 3.4.3. *Let $z^\delta(\xi_0)$ be a point on $\partial\Omega_\delta$ where the curvature of $\partial\Omega_\delta$ experiences a strict extreme; namely:*

$$\mathcal{K}'_{\Omega_\delta} = 0, \quad \mathcal{K}''_{\Omega_\delta} \neq 0. \tag{3.4.12}$$

Then there exists ξ^* in a δ neighborhood of ξ_0 such that $\Psi_\varepsilon(\xi^*)$ is an equilibrium of (3.2.2). If in addition, $\mathcal{K}''_{\Omega_\delta}(\xi_0) > 0$, i.e., the curvature archives a local minimum, then the equilibrium is unstable. If $\mathcal{K}''_{\Omega_\delta}(\xi_0) < 0$, i.e., the curvature archives a local maximum, then the equilibrium is stable.

Proof. We let $\tilde{\xi}(t) = \xi(\tau(t))$, which describes the motion on the invariant manifold $\tilde{\mathcal{M}}_\varepsilon$. From Theorem 3.4.2, we have

$$\begin{aligned}
\tilde{\xi}'(t) = \xi'(\tau(t))\tau' &= \varepsilon^2(1 + O(\varepsilon))c(\xi(\tau(t)), \varepsilon) \\
&= \varepsilon^2(1 + O(\varepsilon))c(\xi(t + O(\varepsilon)), \varepsilon) \\
&= \varepsilon^2(1 + O(\varepsilon))(c(\xi(t), \varepsilon) + O(\varepsilon^3)).
\end{aligned} \tag{3.4.13}$$

Let $\tilde{\xi}(t, \tilde{\xi}_0)$ be the flow of the ODE

$$\begin{cases}
\tilde{\xi}'(t) = \varepsilon^2(1 + O(\varepsilon))(c(\xi(t), \varepsilon) + O(\varepsilon^3)) \\
\tilde{\xi}(0) = \tilde{\xi}_0.
\end{cases} \tag{3.4.14}$$

In [5] the authors show that $c = -\dfrac{4\delta^2}{3\pi}\mathcal{K}'_{\Omega}(\delta\xi) + O(\delta^3)$. We now assume that $\mathcal{K}''_{\Omega_\delta}(\xi_0) > 0$, then there exist $\xi_1 < \xi_0 < \xi_2$, with $|\xi_i - \xi_0| = O(\delta)$ such that

$$(c(\xi_1, \varepsilon) + O(\varepsilon^3)) < 0 < (c(\xi_2, \varepsilon) + O(\varepsilon^3)). \tag{3.4.15}$$

The Intermediate Value Theorem gives the existence of a stationary solution and the positivity of the derivative gives the instability.

$\square$

Chapter 4

Invariant manifolds of interior multi-spike states for the Cahn-Hilliard equation in higher space dimensions

4.1 Introduction

This chapter is concerned with the existence, in forward and backward time, of dynamic interior multi-spike states (see Figure 4.1 for an illustration of an interior multi-spike state) driven by the nonlinear Cahn-Hilliard equation:

$$\begin{cases} u_t = -\Delta(\varepsilon^2 \Delta u - f(u)) & \text{in } \ \Omega \times (0, \infty), \\ \dfrac{\partial \Delta u}{\partial n} = \dfrac{\partial u}{\partial n} = 0 & \text{on } \ \partial\Omega \times (0, \infty), \end{cases} \tag{4.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $\frac{\partial}{\partial n}$ is the exterior normal derivative to $\partial\Omega$, $0 < \varepsilon << 1$ is a small parameter and $f : \mathbb{R} \to \mathbb{R}$ is assumed to be smooth and supports a nondegenerate ground state w with asymptotic value 0 for the equation $\Delta v - g(v) = 0$ in $\mathbb{R}^n$, $g(v) = f(\bar{m} + v) - f(\bar{m})$ for fixed $\bar{m}$ with $f'(\bar{m}) > 0$, that is for $\bar{m}$ in the metastable region. A typical example is $f(u) = u^p - u$ with $1 < p < \frac{n+2}{n-2}$, if $n \geq 3$. The usual choice for the Cahn-Hilliard equation is $f(u) = u^3 - u$.

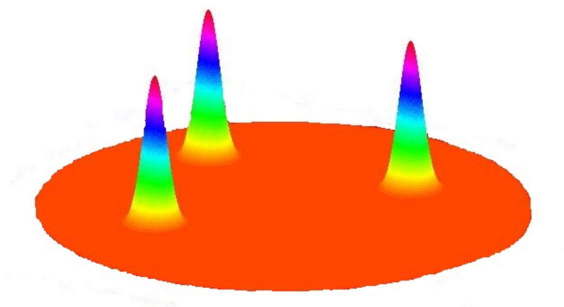

Figure 4.1: Interior multi-spike state

We first prove an abstract result on the existence of an invariant manifold with boundary for a map when one only has a family of approximately invariant manifolds that are approximately normally hyperbolic, each consisting of almost stationary states. The abstract result is an extension of results in [19] to tackle the case of manifolds with boundary. Our abstract result is instrumental in proving the existence of locally invariant manifolds of multi-spike states for the Cahn-Hilliard equation. Though we do not set up a general framework for semiflows, the proof in the current paper is quite general and should be widely applicable.

The Cahn- Hilliard equation (where $f(u) = u^3 - u$) is a widely accepted model for the complicated patterning of the local concentrations in a binary alloy contained in a vessel Ω, as it is rapidly quenched below the curve of miscibility. Above that curve, the alloy is in a homogeneous phase corresponding to thermodynamic equilibrium. Below the curve, the thermodynamic equilibrium corresponds to two separated phases. The separation phenomena that originate after the rapid quenching include nucleation, spinodal decomposition and the formation and dynamics of fronts. We refer the readers to [28] and the references therein for the physical background. For a discussion of the stationary problem for this equation, we refer readers to [14].

The multi-spike equilibria of (4.1.1) has been studied by many authors. In [14], the authors proved the existence of stationary interior multi-spike solutions to (4.1.1) by using an invariant manifold approach. They constructed quasi-invariant manifolds of interior multi-spike states and estimated the motion of each spike, then proved the existence of stationary interior multi-spike states. Similar results were reported independently in [96, 97, 99], where the authors used a Lyapunov-Schmidt reduction technique. In [12] multiple boundary spike solutions were found.

The rich collection of solutions to the stationary problem for the Allen-Cahn equation

$$\begin{cases} u_t = \varepsilon^2 \Delta u - u + f(u), & x \in \Omega \subset\subset \mathbb{R}^n \\ \frac{\partial u}{\partial N} = 0, & x \in \partial\Omega. \end{cases} \tag{4.1.2}$$

has also been studied by many authors, especially for the case where $f(u) = u^p$ with superlinear but subcritical growth. In [71], the authors investigated the Gierer-Meinhardt system in the asymptotic limit as the diffusivity of the inhibitor becomes unbounded. In that limit, one is lead to (4.1.2), referred to as the 'shadow equation'. They showed that no non-constant positive stationary solutions exist when ε is large. For (4.1.2), it was shown in [64] that positive solutions must have peaks with exponentially decaying tails as $\varepsilon \downarrow 0$. The paper [72] studies (4.1.2) with $f(u) = u^p$. The authors obtained a positive solution that has a single peak, the so-called least energy solution, by using a mountain pass argument. They further showed that this peak must actually locate on $\partial\Omega$ and the profile of the solution is a modification of the ground state on $\mathbb{R}^n$, translated to $\partial\Omega$ and rescaled by ε. Later, a topological lower bound on the number of such solutions was given by Z-Q. Wang in [90]. In further work W-M Ni and I. Takagi, in [73], investigated the location of the peaks, and they proved that the peak location tended, as $\varepsilon \to 0$, to the point of $\partial\Omega$ where the mean curvature achieved its maximum. Other papers followed, providing for solutions with spikes at any collection of non-degenerate (in some cases only topologically nontrivial) critical points of the mean curvature, and even multiple spikes accumulating at local minimal points of the mean curvature, or solutions to other singularly perturbed equations and systems (see, e.g., [44], [89], [94], [39], [75], [76], [63], [20], [37], [48], [46], [47] and [98]).

Likewise, the Dirichlet problem has also attracted some attention, with results providing detailed information about the existence and location of a stationary peak (see, e.g., [57], [74], and [36]).

The case of critical growth is quite different, due to a scale invariance and related lack of compactness. Still there are some results and we refer the readers to [91], [92], [2], [1], and [3], for example.

For dynamical spike solutions, there are not many results. In [19], the authors found an invariant manifold of boundary spike solutions to equations of the form

$$\begin{cases} u_t = \varepsilon^2 \Delta u - u + g(u), & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{4.1.3}$$

and showed rigorously that the motion of the boundary spike is driven by the mean curvature of the boundary of the domain. A related dynamical problem was considered in [15], where the authors constructed an invariant manifold of boundary droplet solutions to the 2-d mass-conserving Allen-Cahn equation

$$\begin{cases} u_t(x,t) = \varepsilon^2 \Delta u(x,t) - f(u(x,t)) + \fint_\Omega f(u(\cdot,t)), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{4.1.4}$$

and also showed that the motion of the center of the droplet is driven by the curvature of the boundary of the domain.

In this chapter, we show the existence of dynamical interior multi-spike solutions to (4.1.1) by constructing invariant manifolds of interior multi-spike states. Roughly speaking, we use k ground states, translated to k points of Ω being δ-away from each other and $\frac{\delta}{2}$ away from the boundary and scaled with ε to comprise a manifold $\mathcal{M}_\varepsilon$ parametrized by the location of the centers of the spikes. Thus $\mathcal{M}_\varepsilon$ has dimension nk and has boundary due to the non-proximity constraint. In [19], the authors established a general theorem which states that in a small neighborhood of an approximately invariant and approximately normally hyperbolic manifold, there exists a true invariant manifold being a smooth graph over the former manifold. In their setting, they require that the approximately invariant manifold is inflowing or overflowing at an $O(1)$ rate. However, due to the super slow motion of the interior spikes, the results in [19] can not be applied to our problem directly, so the first thing we do is to extend the results in [19] to make them applicable to our case. This involves a "blow up" technique. Another technical difficulty in our problem is that the linearized operator obtained by linearizing the Cahn-Hilliard equation at a multi-spike state is not self-adjoint, so we cannot split the space according to the spectrum of that linearized operator to prove the normal hyperbolicity. To overcome this difficulty, we turn to deal with the corresponding "integrated" equation. That is, we transform the Cahn-Hilliard equation using $(-\Delta)^{-\frac{1}{2}}$, and by so doing, the corresponding linearized operator becomes self-adjoint. More details about how to make this transformation well-defined can be found in Section 4.3.5. Then we construct a locally invariant manifold of transformed (by $(-\Delta)^{-\frac{1}{2}}$) interior multi-spike states for the transformed semi-flow, and after transforming everything back, we obtain a locally invariant manifold $\widetilde{\mathcal{M}_\varepsilon}$ of interior multi-spike states for the original equation. Note that, once we obtain such an invariant manifold, the solution of (4.1.1) starting from any point in that manifold exists for both positive and negative time. By reducing (4.1.1) on $\widetilde{\mathcal{M}_\varepsilon}$, we derive an equation which determines the velocity of each spike analytically. It turns out that the dynamics of interior spikes for the Cahn-Hilliard equation has a global character where not only the closest spikes interact but each spike interacts with all the others and with the boundary. Furthermore, the speed of the interior spikes is estimated to be exponentially small, which indicates that the multi-spike states exist for a very long time, forwards and backwards.

This chapter is organized as follows: in Section 4.2, we extend the results in [19] to slow manifolds with boundary. In Section 4.3, we construct approximately invariant manifolds of interior multi-spike states for (4.1.1) and then prove the existence of truly invariant manifolds of interior multi-spike states nearby. In Section 4.4, we investigate the dynamics of multi-spike states and give an estimate of the speed of the centers of the spikes.

4.2 Approximately stationary invariant manifolds with boundary

In Definition 2.0.1, a technical assumption that the distance from $u(M)$ to the boundary of the manifold M is bounded below is made. This guarantees that the image under T of a graph over $\psi(M)$ is still a graph over $\psi(M)$. A similar assumption is also made in Definition 2.0.5 for approximately overflowing invariant manifolds. However, sometimes the approximately invariant manifold is approximately stationary, so that the map T is approximated by the identity map, and therefore that assumption can not be satisfied. This happens when we are seeking an invariant manifold of states with super slow motion . In this section, we establish a framework to obtain invariant manifolds near approximately stationary and approximately normally hyperbolic manifolds. The idea is to modify the map T only near the boundary of $\psi(M)$ to get a new map $\widetilde{T}$ such that $\psi(M)$ is approximately inflowing (overflowing) invariant under $\widetilde{T}$ and $\widetilde{T} = T$ when applied to the points away from the boundary. More specifically, if the "projection" of $T(x)$ to $\psi(M)$ is near the boundary of $\psi(M)$, then we move the "projection" inside (outside) $\psi(M)$ and away from the boundary of $\psi(M)$.

Moreover, the movement should be very slight for the purpose of keeping the normal hyperbolicity. More precise statement may be found below.

Let X, Y be two Banach spaces and $T \in C^J(X, X)$ with $J > 1$ and $M \subset Y$ be a C^J finite-dimensional closed manifold with smooth boundary ∂M. Let $\psi \in C^J(M, X)$ be an embedding [1] satisfying $\|D^i\psi\| < B_2$ for $1 \leq i \leq J$ and $\|D\psi^{-1}\| < B_3$ for all $m \in M$. We denote the metric on M by $d(\cdot,\cdot)$. Furthermore, we make the following assumption on the boundary of M.

With $B(\partial M, r_0) = \{m \in M : d(m, \partial M) \leq r_0\}$, we assume that there exists $r > 0, \gamma_1, \gamma_2 > 0$ and $\phi : B(\partial M, r) \to \mathbb{R}^n_+$ such that $\phi(\partial M) \subset \partial\mathbb{R}^n_+$, where $\mathbb{R}^n_+$ is the upper half space of $\mathbb{R}^n$, and $\gamma_1 \leq \|D\phi^{-1}\| \leq \gamma_2$, $\|D\phi\| \leq \gamma_3, \|D^2\phi^{-1}\| \leq \gamma_4$. Here ϕ^{-1} means taking the inverse for a local chart. Since M is assumed to be compact, it is possible to construct such a map ϕ by using a partition of unity.

Definition 4.2.1. $\psi(M)$ is said to be approximately stationary invariant under T if there exists small $\eta > 0$ such that $|T(\psi(m)) - \psi(m)| < \eta$, for all $m \in M$.

Definition 4.2.2. We say that an approximately stationary invariant manifold $\psi(M)$ is approximately normally hyperbolic, if the following conditions hold:

1. For each $m \in M$, there is a decomposition $X = X^c_m \oplus X^s_m \oplus X^u_m$ of closed subspaces with projections $\Pi^c_m, \Pi^s_m, \Pi^u_m$ varying in C^J way with respect to m.

2. For any $m \in M$, Π^c_m is an isomorphism from $D\psi(m)T_mM$ to X^c_m and $T_mM = X^c_m + \Lambda_m(X^c_m)$, where $\Lambda_m \in L(X^c_m, X^s_m \oplus X^u_m)$. Furthermore there exist B, L, and $\chi \in (0, 1/2)$, such that for

[1] This theory could be developed for immersed Banach manifolds, but in our opinion the increase in generality is not worth the loss of clarity and increase of pages.

any $m_0 \in M$ and $m_1, m_2 \in B_c(m_0, r_0)$, with $m_1 \neq m_2$, for $\alpha = c, u, s$,

$$\begin{cases} \|\Pi^\alpha_{m_0}\| \le B, \|\Pi^\alpha_{m_1} - \Pi^\alpha_{m_2}\| \le L|\psi(m_1) - \psi(m_2)|, \\ \\ \dfrac{|\psi(m_1) - \psi(m_2) - \Pi^c_{m_0}(\psi(m_1) - \psi(m_2))|}{|\psi(m_1) - \psi(m_2)|} \le \chi. \end{cases} \tag{4.2.1}$$

3. There exist $\sigma, \lambda \in (0,1)$ and a such that, for any $m_0 \in M$, and $\alpha \in \{c, s, u\}, \beta \in \{c, s, u\}$, with $\alpha \neq \beta$, then

   (a) $\|\Pi^\beta_{m_0} DT(\psi(m_0))|_{X^\alpha_{m_0}}\| \le \sigma,\ \|(\Pi^c_{m_0} DT(\psi(m_0))|_{X^c_{m_0}})^{-1}\|^{-1} > a$,

   (b) $\|\Pi^s_{m_0} DT(\psi(m_0))|_{X^s_{m_0}}\| \le \lambda,\ \lambda\|(\Pi^u_{m_0} DT(\psi(m_0))|_{X^u_{m_0}})^{-1}\|^{-1} > 1$,

   (c) $\lambda\|(\Pi^u_{m_0} DT(\psi(m_0))|_{X^u_{m_0}})^{-1}\|^{-1} > \max\{1, \|\Pi^c_{m_0} DT(\psi(m_0))|_{X^c_{m_0}}\|^J\}$,
   $\|\Pi^s_{m_0} DT(\psi(m_0))|_{X^s_{m_0}}\| < \lambda \min\{1, \|(\Pi^c_{m_0} DT(\psi(m_0))|_{X^c_{m_0}})^{-1}\|^{-J}\}$.

4. There exists B_1, such that $\|D^j T|_{B(\psi(M), r_0)}\| \le B_1$ for $1 \le j \le J$.

Remark 4.2.3. (4.2.1) implies that $\|\Lambda_m\| \le B\chi$, for any $m \in M$

Define

$$X^\alpha_m(\delta) := \{x^\alpha \in X^\alpha_m : |X^\alpha_m| \le \delta\},\ \alpha = s, u,$$

and

$$N(M, \varepsilon) := \{\psi(m) + x^s + x^u : x^s \in X^s_m(\varepsilon), x^u \in X^u_m(\varepsilon)\}.$$

Lemma 4.2.4. *If $\varepsilon < \min\{\frac{L}{8}, \frac{BL}{8}\}$, then $N(M, \varepsilon)$ is a tubular neighborhood of M satisfying*

1. *For any two points $\psi(m_i) + x^s_i + x^u_i \in N(M, \varepsilon)$, $i = 1, 2$, if*

$$\psi(m_1) + x^s_1 + x^u_1 = \psi(m_2) + x^s_2 + x^u_2,$$

then

$$m_1 = m_2,\ x_1^s = x_2^s,\ x_1^u = x_2^u.$$

2. *There exists small θ such that if $|x-(\psi(m)+x^s+x^u)| \le \theta$, where $(\psi(m)+x^s+x^u) \in N(M,\varepsilon)$, then*

$$x = \psi(m_*) + x_*^s + x_*^u,$$

where $|x_*^s| \le \varepsilon$ and $|x_*^u| \le \varepsilon$.

Proof. We refer to the proof of Lemma 3.6 in [19]. □

Remark 4.2.5. As a consequence of the proof, we have that $m(x), x^s(x), x^u(x)$ are all smooth in x for $x \in N(M,\varepsilon)$.

Now we start to construct a center-stable manifold for T. We first define several functions that will be used later in this section. Write $x \in \mathbb{R}^n$ as (x', x_n), and define $S_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ as

$$S_\alpha(x', x_n) = (x', x_n + \alpha), \tag{4.2.2}$$

and define $e : \mathbb{R}^n \to \mathbb{R}$ as

$$e(x', x_n) = x_n. \tag{4.2.3}$$

Let $b(z)$ be a smooth monotone function satisfying

$$b(z)\begin{cases} = 1, & z \le 0, \\ \in (0,1), & z \in (0,1), \\ = 0, & z \ge 1, \end{cases} \tag{4.2.4}$$

then for any $m \in M$, we define

$$\widetilde{m} = \phi^{-1}(S_{\alpha(m)} \circ \phi(m)), \tag{4.2.5}$$

where

$$\alpha(m) = b\Big(\frac{e(\phi(m)) - d}{d}\Big) \cdot l, \tag{4.2.6}$$

where $d > 0$ is a constant satisfying $d < \frac{\gamma_1 r}{2}$ and $l > 0$ is a small constant to be determined.

Remark 4.2.6. Note that $\|D\phi^{-1}\| \leq \gamma_2$, so $\|D\phi\| \geq \frac{1}{\gamma_2}$. It follows immediately that for any m satisfying $2\gamma_2 d < d(m, \partial M) < r$, we have $e(\phi(m)) > 2d$, which implies that $\alpha(m) = 0$. So $\widetilde{m} = m$ if $2\gamma_2 d < d(m, \partial M) < r$.

For $x \in N(m, \varepsilon)$, by Lemma 4.2.4, we can write x as $x = \psi(m(x)) + x^s(x) + x^u(x)$. We construct a new map $\widetilde{T}$ as

$$\widetilde{T}(x) = \begin{cases} T(x), & m(x) \in M \backslash B(\partial M, r), \\ \\ T(x) + \psi(\widetilde{m(x)}) - \psi(m(x)), & m(x) \in B(\partial M, r)). \end{cases} \tag{4.2.7}$$

By Remark 4.2.6, we have $\widetilde{T} \in C^J(X, X)$.

Let

$$u(m) = \begin{cases} m, & \text{for } \ m \in M \backslash B(\partial M, r), \\ \\ \widetilde{m}, & \text{for } \ m \in B(\partial M, r). \end{cases} \tag{4.2.8}$$

By Remark 4.2.6, one can see that u is continuous.

Lemma 4.2.7. *For any $m \in M$, $d(u(m), \partial M) \geq \gamma_1 l$.*

Proof. We write $\phi(m) = (x', 0)$ and for any $\bar{m} \in \partial M$, we write $\bar{m} = (\bar{x}', 0)$. Then one can check that

$$|u(m) - \bar{m}| = |\phi^{-1}(x', t) - \phi^{-1}(\bar{x}', 0)| \geq \gamma_1 |(x', t) - (\bar{x}', 0)| \geq \gamma_1 t.$$

Thus $d(u(m), \partial M) = \inf_{\bar{m} \in \partial M} d(u(m), \bar{m}) \geq \gamma_1 t$. □

Lemma 4.2.8. $|\widetilde{T}(\psi(m)) - \psi(u(m))| < \eta$, *for any* $m \in M$.

Proof. By direct computation, we have

$$\widetilde{T}(\psi(m)) - \psi(u(m)) = T(\psi(m)) - \psi(m),$$

which implies the desired result. □

Combining Lemma 4.2.7 and Lemma 4.2.8, it is clear that $\psi(M)$ is an approximately inflowing invariant manifold for $\widetilde{T}$. The map $\widetilde{T}$ is almost the same as T, except that it shifts the "base points" of x on $\psi(M)$ that are near the boundary of $\psi(M)$ a little. Intuitively, one may expect that if l is small enough, $\psi(M)$ is also approximately normally hyperbolic for $\widetilde{T}$, so that we can apply Theorem 2.0.3 to conclude the existence of a center-stable manifold. However, one may notice that the distance from $u(M)$ to the boundary of M depends on l and l is also involved in the trichotomy property of $D\widetilde{T}$. As we apply Theorem 2.0.3, the distance from $u(M)$ to the boundary of M needs to be fixed first, then we make the other parameters small. Therefore, Theorem 2.0.3 can not be applied directly. To overcome this problem, we perform the following blow-up analysis.

Let

$$
\begin{aligned}
&M^l = \frac{M}{l} = \{m^l = \frac{m}{l} : m \in M\},\\
&\psi^l(m^l) = \frac{1}{l}\psi(lm^l),\\
&T^l(x) = \frac{1}{l}T(lx), \widetilde{T}^l(x) = \frac{1}{l}\widetilde{T}(lx),\\
&u^l(m^l) = \frac{1}{l}u(lm^l),\\
&\bar{X}^\alpha_{m^l} = X^\alpha_{lm^l}, \bar{\Pi}^\alpha_{m^l} = \Pi^\alpha_{lm^l}, \alpha = c, s, u.
\end{aligned}
\tag{4.2.9}
$$

First, it is easy to check that

$$
|\widetilde{T}^l(\psi^l(m^l)) - \psi^l(u^l(m^l))| < \frac{\eta}{l} \tag{4.2.10}
$$

for any $m^l \in M^l$, and

$$
d(u(M^l), \partial m^l) \geq \gamma_1. \tag{4.2.11}
$$

Thus, $\psi^l(M^l)$ is an approximately inflowing invariant manifold for $\widetilde{T}^l_\varepsilon$.

Clearly, for each $m^l \in M^l$, $X = \bar{X}^c_{m^l} \oplus \bar{X}^s_{m^l} \oplus \bar{X}^u_{m^l}$ of closed subspaces with projections $\bar{\Pi}^c_{m^l}, \bar{\Pi}^s_{m^l}, \bar{\Pi}^u_{m^l}$, and for any $m^l_0 \in M^l$ and $m^l_1, m^l_2 \in B_c(m^l_0, \frac{r_0}{l})$, with $m^l_1 \neq m^l_2$, for $\alpha = c, u, s$,

$$
\begin{cases}
\|\bar{\Pi}^\alpha_{m^l_0}\| \leq B, \|\bar{\Pi}^\alpha_{m^l_1} - \bar{\Pi}^\alpha_{m^l_2}\| \leq lL|\psi^l(m^l_1) - \psi^l(m^l_2)|,\\
\\
\dfrac{|\psi^l(m^l_1) - \psi^l(m^l_2) - \bar{\Pi}^c_{m^l_0}(\psi^l(m^l_1) - \psi^l(m^l_2))|}{\psi^l(m^l_1) - \psi^l(m^l_2)} \leq \chi.
\end{cases}
\tag{4.2.12}
$$

Note that

$$\bar{\Pi}^{\alpha}_{u^l(m_l)} D\widetilde{T}^l(m_l)|_{\bar{X}^{\beta}_{m_l}} = \Pi^{\alpha}_{u(m)} D\widetilde{T}(m)|_{X^{\beta}_m},$$

thus in order to show the trichotomy property for $\widetilde{T}^l$, we just need to consider $\Pi^{\alpha}_{u(m)} D\widetilde{T}(m)|_{X^{\beta}_m}$. Furthermore, for $m \in M\backslash B(\partial M, r)$, we have $\widetilde{T} = T$ and $u(m) = m$, so we just need to deal with $m \in B(\partial M, r)$.

Since $DS_{\alpha(m)} = I + D\alpha(m) \cdot e_n$, where $e_n = (0, \cdots, 1)$, and $D\phi^{-1}(\phi(x))D\phi(x) = I$, we have

$$\begin{aligned}
& D\psi(\widetilde{m(x)}) - D\psi(m(x)) \\
& = D\psi(\widetilde{m(x)})D\widetilde{m}(x) - D\psi(m(x))Dm(x) \\
& = D\psi(\widetilde{m(x)})D(\phi^{-1})(S_{\alpha(m)}\phi(m))(I + D\alpha(m(x)) \cdot e_n)D\phi(m(x))Dm(x) - D\psi(m(x))Dm(x) \\
& = (D\psi(\widetilde{m(x)}) - D\psi(m(x)))D(\phi^{-1})(S_{\alpha(m)}\phi(m))(I + D\alpha(m(x)) \cdot e_n)D\phi(m(x))Dm(x) + \\
& \quad D\psi(m(x))D(\phi^{-1})(S_{\alpha(m)}\phi(m)) - D(\phi^{-1})(\phi(m))(I + D\alpha(m(x)) \cdot e_n)D\phi(m(x))Dm(x) + \\
& \quad D\psi(m(x))D(\phi^{-1})(\phi(m))(D\alpha(m(x)) \cdot e_n)D\phi(m(x))Dm(x).
\end{aligned} \tag{4.2.13}$$

Using the fact that $\|D^i\psi\| \leq B_2$, $\|D\phi^{-1}\| < \gamma_2$, $\|D\phi\| < \gamma_3$, $\|D\alpha(m(x))\| = O(l)$, $\|Dm(x)\|$ is uniformly bounded, $|\widetilde{m} - m| \leq \gamma_2 t$ and $\|D\phi^{-1}(S_{\alpha(m)}\phi(m)) - D\phi^{-1}(\phi(m))\| = O(l)$, we have

$$\|D\psi(\widetilde{m(x)}) - D\psi(m(x))\| \leq Cl, \tag{4.2.14}$$

for some constant C being independent of x.

Thus we have

$$
\begin{aligned}
\|\Pi^{\alpha}_{u(m)} D\widetilde{T}(m)|_{X^{\beta}_{m}}\| &\le \|\Pi^{\alpha}_{u(m)} - \Pi^{\alpha}_{m}\|\|D\widetilde{T}(m)|_{X^{\beta}_{m}}\| + \|\Pi^{\alpha}_{m} D\widetilde{T}(m)|_{X^{\beta}_{m}}\| \\
&\le \|\Pi^{\alpha}_{u(m)} - \Pi^{\alpha}_{m}\|(\|DT(m)|_{X^{\beta}_{m}}\| + \|D\psi(\widetilde{m(x)}) - D\psi(m(x))\|)+ \\
&\quad \|\Pi^{\alpha}_{m} DT(m)|_{X^{\beta}_{m}}\| + \|\Pi^{\alpha}_{m}\|\|D\psi(\widetilde{m(x)}) - D\psi(m(x))\| \\
&\le \|\Pi^{\alpha}_{m} DT(m)|_{X^{\beta}_{m}}\| + Cl.
\end{aligned}
\tag{4.2.15}
$$

Similarly, one can prove

$$
\|\Pi^{\alpha}_{u(m)} D\widetilde{T}(m)|_{X^{\beta}_{m}}\| \ge \|\Pi^{\alpha}_{m} DT(m)|_{X^{\beta}_{m}}\| - Cl. \tag{4.2.16}
$$

Therefore, when l is small enough, the trichotomy properties are satisfied, so that Theorem 2.0.3 can be applied to $\widetilde{T}^{l}$ and M^{l} to obtain a center-stable manifold $\bar{W}^{cs}(l)$ for $\widetilde{T}^{l}$. Note that if $\bar{W}^{cs}(l)$ is invariant under $\widetilde{T}^{l}$, then $W^{cs}(l) = l\bar{W}^{cs}$ is invariant under $\widetilde{T}$. Since $\widetilde{T}(x) = T(x)$ when $e(\phi(m(x))) > 2d$ or $d(m(x), \partial M) > r$, it is clear that $W^{cs}(l)$ is locally invariant under T. So we have the following theorem:

Theorem 4.2.9. *Depending on r_0, B, B_1, λ, L, when η, χ and σ are sufficiently small, for every l sufficiently small, there exists a C^{J} positively locally invariant manifold $W^{cs}(l)$ for T, which is given as the image of a map*

$$
h : \{(m, x^{s}) : m \in M, x^{s} \in X^{s}_{m}(\delta_0)\} \to X,
$$

for some $\delta_0 > 0$. Also for any $x \in W^{cs}(l)$ with $e(\phi(m(x))) > 2d$ or $d(m(x), \partial M) > r$, we have $T(x) \in W^{cs}(l)$.

Remark 4.2.10. Theorem 4.2.9 essentially says that $W^{cs}(l)$ is invariant under T if the "base point" on $\psi(M)$ is away from the boundary of M.

Similarly, we can construct center-unstable manifolds for T. Let

$$\widetilde{m} = \phi^{-1}(S_{-\alpha(m)} \circ \phi(m)), \tag{4.2.17}$$

then we construct $\widetilde{T}$ and v respectively as

$$\widetilde{T}(x) = \begin{cases} T(x), & m(x) \in M \backslash B(\partial M, r), \\ \\ T(x) + \psi(\widetilde{m(x)}) - \psi(m(x)), & m(x) \in B(\partial M, r)), \end{cases} \tag{4.2.18}$$

and

$$v(m) = \begin{cases} m, & \text{for } \; M \in M \backslash B(\partial M, r), \\ \\ \phi^{-1}(S_{\alpha(m)} \circ \phi(m)), & \text{for } \; M \in B(\partial M, r). \end{cases} \tag{4.2.19}$$

By the construction of v, one can easily see that v is a homeomorphism. Following the same argument as above, we obtain the follow theorem:

Theorem 4.2.11. *Depending on r_0, B, B_1, λ, L, when η, χ and σ are sufficiently small, for every t sufficiently small, there exists a C^J negatively locally invariant manifold $W^{cu}(l)$ for T, which is given as the image of a map*

$$h : \{(m, x^u) : m \in M, x^u \in X^u_m(\delta_0)\} \to X,$$

for some $\delta_0 > 0$. Also for any $x \in W^{cu}(l)$ with $e(\phi(m(x))) > 2d$ or $d(m(x), \partial M) > r$, we have $T(x) \in W^{cu}(l)$.

4.3 Invariant manifolds of interior multi-spike states

4.3.1 Construction of the manifold $\mathcal{M}_\varepsilon$

We start from the following equation,

$$-\varepsilon^2 \Delta u + f(u) = \sigma, \tag{4.3.1}$$

whose solutions are stationary solutions to (4.1.1).

Let $\sigma = f(\bar{m})$, where $\bar{m}$ is in a metastable region, which means that $f'(\bar{m}) > 0$. We make the following transformation,

$$u = v + \bar{m},\ \ g(v) = \sigma - f(v + \bar{m}). \tag{4.3.2}$$

Obviously $g(0) = 0$. Let $g'(0) = -\mu$, then we can write $g(v) = -\mu v + h(v)$ with h satisfying $h(0) = h'(0) = 0$. Then (4.1.1) becomes

$$\begin{cases} v_t = -\Delta(\varepsilon^2 \Delta v + g(v)) & \text{in } \ \Omega \times (0, \infty), \\ \dfrac{\partial \Delta v}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \ \partial\Omega \times (0, \infty), \end{cases} \tag{4.3.3}$$

or

$$\begin{cases} v_t = -\Delta(\varepsilon^2 \Delta v - \mu v + h(v)) & \text{in } \ \Omega \times (0, \infty), \\ \dfrac{\partial \Delta v}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \ \partial\Omega \times (0, \infty). \end{cases} \tag{4.3.4}$$

And (4.3.1) becomes

$$\varepsilon^2 \Delta v + g(v) = 0, \tag{4.3.5}$$

or

$$\varepsilon^2 \Delta v - \mu v + h(v) = 0. \tag{4.3.6}$$

By the main theorem of [23], there exists a ground state w satisfying

$$\begin{cases} \Delta w + g(w) = 0 & \text{in } \mathbb{R}^n, \\ w(0) = \max w(x), & w > 0, \\ w(x) = w(|x|), & \\ w(x) \to 0, & |x| \to \infty. \end{cases} \tag{4.3.7}$$

Also, $w(x)$ satisfies that $|\partial_r^k w(x)| \leq Ce^{-\nu|x|}$ for any $x \in \mathbb{R}^n$ and $k \geq 0$, where ∂_r is the derivative in the radial direction. Note that g is assumed to be smooth, so w is smooth. Furthermore, we assume the ground state w is non-degenerate, that is, the operator obtained by linearizing at w has 0 as an eigenvalue of multiplicity n (when $f = u^3 - u$, the ground state is non-degenerate).

We will construct an approximately invariant manifold of interior multi-spike states parametrized by the locations of the spikes for (4.1.1). Roughly speaking, we patch k translated and ε-rescaled ground states together, requiring that k center points of the spikes are at least $\delta > 0$ away from each other and $\frac{\delta}{2}$ away from the boundary. Here the constant δ is not necessarily $O(1)$, but must satisfy $\frac{\delta}{\varepsilon} \to \infty$ as $\varepsilon \to 0$, so δ could be $O(|\varepsilon ln\varepsilon|)$ or $O(\sqrt{\varepsilon})$, for example.

Let

$$\widetilde{\Omega}^k = \{P = (p_1, p_2, \cdots, p_k) : \quad p_i \in \Omega, \ \ |p_i - p_j| \geq \delta, \ \ d(p_i, \partial\Omega) \geq \frac{\delta}{2}\}, \tag{4.3.8}$$

obviously, $\widetilde{\Omega}^k$ is a closed submanifold of $\underbrace{\Omega \times \cdots \times \Omega}_{k-\text{copies}}$ with boundary.

Then we let

$$W_{\varepsilon,P}(x) = \sum_{i=1}^{k} w_{\varepsilon,p_i}, \tag{4.3.9}$$

where $P = (p_1, p_2, \cdots, p_k) \in \tilde{\Omega}^k$ and $w_{\varepsilon,p_i} = w(\frac{x-p_i}{\varepsilon})$.

From this point on, to simplify the notation, we have replaced $\frac{\nu}{2}$ by ν in the exponentially small terms.

Lemma 4.3.1. *$W_{\varepsilon,P}$ satisfies*

$$\varepsilon^2 \Delta W_{\varepsilon,P} - \mu W_{\varepsilon,P} + h(W_{\varepsilon,P}) = R_{\varepsilon,P}(x), \tag{4.3.10}$$

where $|R_{\varepsilon,P}(x)|_{C^m(\Omega)} = O(e^{-\frac{\nu\delta}{\varepsilon}})$ for any m.

Proof. First, note that $W_{\varepsilon,P}$ and h are smooth, so $R_{\varepsilon,P}$ is smooth. For any $x \in \{x \in \Omega : d(x, p_i) < \frac{\delta}{2}\}$, we have $|w(\frac{x-p_l}{\varepsilon})| = O(e^{-\frac{\nu\delta}{\varepsilon}})$ for $l \neq i$. Using the fact that $h(0) = 0$, we have

$$\begin{aligned}|R_{\varepsilon,P}(x)| &= |h(W_{\varepsilon,P}) - h(w(\frac{x-p_i}{\varepsilon})) - \sum_{l\neq i} h(w(\frac{x-p_l}{\varepsilon}))| \\ &\leq C \sum_{l\neq i} |w(\frac{x-p_l}{\varepsilon})| \\ &= O(e^{-\frac{\nu\delta}{\varepsilon}}).\end{aligned}$$

For any $x \in \{x \in \Omega : d(x, p_i) > \frac{\delta}{2}, i = 1, \cdots, k\}$, we have $|w(\frac{x-p_i}{\varepsilon})| = O(e^{-\frac{\nu\delta}{\varepsilon}})$ for $i = 1, \cdots, k$. Thus

$$|R_{\varepsilon,P}(x)| = |h(W_{\varepsilon,P}) - \sum_{i=1}^{k} h(w(\frac{x-p_l}{\varepsilon}))| = O(e^{-\frac{\nu\delta}{\varepsilon}}).$$

Therefore, $|R_{\varepsilon,P}(x)|_{C^0(\Omega)} = O(e^{-\frac{\nu\delta}{\varepsilon}})$. One can follow the same argument to show that for any m,

$$|R_{\varepsilon,P}(x)|_{C^m(\Omega)} = O(e^{-\frac{\nu\delta}{\varepsilon}}).$$

Note that taking a derivative of $R_{\varepsilon,P}$ generates $\frac{1}{\varepsilon}$ which is absorbed by $e^{-\frac{\nu\delta}{\varepsilon}}$ by changing ν slightly.

□

Now, we see that $W_{\varepsilon,P}(x)$ approximately satisfies (4.3.6). However it does not satisfy the

Neumann boundary conditions in (4.1.1), so we need to modify it slightly.

Let $\mathcal{H}(\rho)$ be the solution to

$$\begin{cases} \varepsilon^2 \Delta v - \mu v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial n} = \frac{\partial \rho}{\partial n} & \text{on } \partial\Omega, \end{cases} \tag{4.3.11}$$

and let $\overline{\mathcal{H}}(\rho)$ be the solution to

$$\begin{cases} \varepsilon^2 \Delta v - \mu v = \varepsilon^2 \chi \Delta \rho - \varepsilon^2 \Delta \mathcal{H}(\rho) & \text{in } \Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3.12}$$

where $\chi(x)$ is a smooth cut-off function satisfying

$$\begin{cases} \chi(x) = 0, & x \in \{x : d(x, \partial\Omega) \geq \frac{\delta}{4}\}, \\ \chi(x) = 1, & x \in \{x : d(x, \partial\Omega) \leq \frac{\delta}{8}\}. \end{cases} \tag{4.3.13}$$

One can easily check that $\mathcal{H}(W_{\varepsilon,P}) = \sum_{1 \leq i \leq k} \mathcal{H}(w_{\varepsilon,p_i})$ and $\overline{\mathcal{H}}(W_{\varepsilon,P}) = \sum_{1 \leq i \leq k} \overline{\mathcal{H}}(w_{\varepsilon,p_i})$. To estimate $\overline{\mathcal{H}}(W_{\varepsilon,P})$ and $\mathcal{H}(W_{\varepsilon,P})$, we first prove the following lemma. It may have been proved elsewhere, but for the completeness, we give a proof here.

Lemma 4.3.2. *If $G \in L^q(\Omega)$ and $H \in L^q(\partial\Omega)$ for some $q \geq 2$, and if v is the solution to*

$$\begin{cases} \Delta v - \mu v = G & \textit{in } \Omega, \\ \frac{\partial v}{\partial n} = H & \textit{on } \partial\Omega, \end{cases} \tag{4.3.14}$$

then

$$|v|_{L^{\frac{nq}{n-2}}(\Omega)} \leq C(|G|_{L^q(\Omega)} + |H|_{L^q(\partial\Omega)})$$

Proof.

$$\int_\Omega \Delta v \cdot v - \mu v^2 dx = \int_\Omega G v dx.$$

By Green's formula, we have

$$-\int_\Omega |\nabla v|^2 dx + \int_{\partial\Omega} vH ds - \int_\Omega \mu v^2 dx = \int_\Omega G v dx.$$

Then, we have

$$\begin{aligned}
&\int_\Omega |\nabla v|^2 dx + \int_\Omega \mu v^2 dx \\
&\leq \int_{\partial\Omega} |vH| ds + \int_\Omega |Gv| dx \\
&\leq \lambda \int_{\partial\Omega} |v|^2 ds + \frac{1}{4\lambda}\int_{\partial\Omega} |H|^2 ds + \lambda \int_\Omega |v|^2 dx + \frac{1}{4\lambda}\int_\Omega |G|^2 dx \\
&\leq \lambda C(\int_\Omega |v|^2 ds + \int_\Omega |\nabla v|^2 dx) + \frac{1}{4\lambda}\int_{\partial\Omega} |H|^2 ds + \lambda \int_\Omega |v|^2 dx + \frac{1}{4\lambda}\int_\Omega |G|^2 dx,
\end{aligned}$$

which implies that

$$(1-\lambda C)\int_\Omega |\nabla v|^2 dx + (\mu - \lambda C - \lambda)\int_\Omega |v|^2 dx \leq \frac{1}{4\lambda}\int_\Omega |G|^2 dx + \frac{1}{4\lambda}\int_{\partial\Omega} |H|^2 ds.$$

Choosing λ small enough, we have

$$|v|_{W^{1,2}} \leq C(\int_\Omega |G|^2 dx + \int_{\partial\Omega} |H|^2 ds).$$

For $q > 2$, and with $v_+ = \max\{v, 0\}$,

$$\int_\Omega \Delta v \cdot v_+^{q-1} dx - \mu \int_\Omega v_+^q dx = \int_\Omega G v_+^{q-1} dx.$$

By Green's formula, we get

$$-(q-1) \int_\Omega v_+^{q-2} |\nabla v_+|^2 dx + \int_{\partial\Omega} v_+^{q-1} H ds - \mu \int_\Omega v_+^q dx = \int_\Omega G v_+^{q-1} dx.$$

Then

$$(q-1) \int_\Omega v_+^{q-2} |\nabla v_+|^2 dx + \mu \int_\Omega v_+^q dx \le \int_{\partial\Omega} v_+^{q-1} |H| ds + \int_\Omega |G| v_+^{q-1} dx,$$
$$\frac{4(q-1)}{q^2} \int_\Omega |\nabla (v_+^{\frac{q}{2}})|^2 dx + \mu \int_\Omega v_+^q dx \le \lambda \int_{\partial\Omega} v_+^q ds + \kappa(\lambda) \int_{\partial\Omega} |H|^q ds + \lambda \int_\Omega v_+^q dx + \int_\Omega |G|^q dx,$$

which indicates that

$$|v_+^{\frac{q}{2}}|_{W^{1,2}(\Omega)} \le C(|G|_{L^q(\Omega)} + |H|_{L^q(\partial\Omega)}).$$

So

$$|v_+|_{L^{\frac{nq}{n-2}}(\Omega)} \le C(|G|_{L^q(\Omega)} + |H|_{L^q(\partial\Omega)}).$$

One can prove the same inequality for v_-, which completes the proof. $\square$

Using Lemma 4.3.2 and routine bootstrap argument, we have that for any m, $|\mathcal{H}(W_{\varepsilon,P})|_{C^m(\Omega)}$ and $|\overline{\mathcal{H}}(W_{\varepsilon,P})|_{C^m(\Omega)}$ are both $O(e^{-\frac{\nu\delta}{\varepsilon}})$. Let

$$\widetilde{\mathcal{H}}(W_{\varepsilon,P}) = \mathcal{H}(W_{\varepsilon,P}) + \overline{\mathcal{H}}(W_{\varepsilon,P}) + K_{\varepsilon,P}, \tag{4.3.15}$$

then let

$$\widetilde{W}_{\varepsilon,P} = W_{\varepsilon,P} - \widetilde{\mathcal{H}}(W_{\varepsilon,P}), \tag{4.3.16}$$

where $K_{\varepsilon,P}$ is a constant for fixed P with $K_{\varepsilon,P} = 0$ for some $P = P^*$, such that the mass $\int_\Omega \widetilde{W}_{\varepsilon,P} dx$ remains constant as we vary $P \in \widetilde{\Omega}^k$. One can easily check that

$$\frac{\partial \widetilde{W}_{\varepsilon,P}}{\partial n} = \frac{\partial \Delta \widetilde{W}_{\varepsilon,P}}{\partial n} = 0.$$

By the exponentially decaying property of ground states and the fact that $\mathcal{H}(W_{\varepsilon,P}), \overline{\mathcal{H}}(W_{\varepsilon,P}) = O(e^{-\frac{\nu\delta}{\varepsilon}})$, one can easily see that $K_{\varepsilon,P} = O(e^{-\frac{\nu\delta}{\varepsilon}})$. By using the symmetry of the ground states, one also can check that $D_P K_{\varepsilon,P} = O(e^{-\frac{\nu\delta}{\varepsilon}})$. More precisely, by direct computation, one has $D_P K_{\varepsilon,P} = \frac{1}{|\Omega|}\int_\Omega D_P W_{\varepsilon,P} - (D_P \mathcal{H}(W_{\varepsilon,P}) + D_P \overline{\mathcal{H}}(W_{\varepsilon,P}))dx$. First, it is easy to verify that $D_P \mathcal{H}(W_{\varepsilon,P}) = -\nabla_x \sum_i \mathcal{H}(W_{\varepsilon,p_i}) = O(e^{-\frac{\nu\delta}{\varepsilon}})$, similarly we have $D_P \overline{\mathcal{H}}(W_{\varepsilon,P}) = O(e^{-\frac{\nu\delta}{\varepsilon}})$. It remains to estimate $\int_\Omega D_P W_{\varepsilon,P} dx$. Since $D_P W_{\varepsilon,P} = -\sum_i \nabla_x w_{\varepsilon,p_i}$, it is sufficient to show that $\int_\Omega \nabla_{x_l} w_{\varepsilon,p_i} dx = O(e^{-\frac{\nu\delta}{\varepsilon}})$. Note that $\nabla_{x_l} w$ is odd in x_l and $B(p_i, \delta/2) \subset \Omega$, so we have $\int_{B(p_i,\delta/2)} \nabla_{x_l} w_{\varepsilon,p_i} dx = 0$. By the exponentially decaying property of the ground state, it is also clear that $\int_{\Omega \setminus B(p_i,\delta/2)} \nabla_{x_l} w_{\varepsilon,p_i} dx = O(e^{-\frac{\nu\delta}{\varepsilon}})$. It follows immediately that $D_P K_{\varepsilon,P} = O(e^{-\frac{\nu\delta}{\varepsilon}})$. Moreover, since $K_{\varepsilon,P}$ is a constant in x, one finds that for any m,

$$|\widetilde{\mathcal{H}}(W_{\varepsilon,P})|_{C^m(\Omega)} = O(e^{-\frac{\nu\delta}{\varepsilon}}). \tag{4.3.17}$$

Define

$$\psi_\varepsilon(P) = \widetilde{W}_{\varepsilon,P}. \tag{4.3.18}$$

Then the approximately invariant we construct is

$$\mathcal{M}_\varepsilon = \psi_\varepsilon(\widetilde{\Omega}^k). \tag{4.3.19}$$

Remark 4.3.3. Every point on the manifold $\mathcal{M}_\varepsilon$ has the same mass. Since we are only building an approximately invariant manifold, this may seem unnecessary. However, as we will see in Section 4.3.5, this is important for the transformed equation.

4.3.2 Spectral analysis of the linearized Allen- Cahn operator

Let

$$\begin{aligned} L_0 v &:= \Delta v - \mu v + h'(w)v, \qquad && H^2(R^n) \to L^2(R^n), \\ \widetilde{L}_0^\varepsilon v &:= \varepsilon^2 \Delta v - \mu v + h'(\widetilde{W_{\varepsilon,P}})v, \qquad && H^2(\Omega) \to L^2(\Omega), \end{aligned} \tag{4.3.20}$$

and

$$B_u(v,v) = \int_\Omega (\varepsilon^2 |\nabla v|^2 + \mu v^2 - h'(u)v^2)dx \tag{4.3.21}$$

Recall that the ground state w is assumed to be non-degenerate, which means that there exists $b > 0, \lambda_1 > 0$, such that $\sigma(L_0) \cap (-b, \infty) = \{0, \lambda_1\}$ with 0 having multiplicity n. In fact λ_1 is simple, and the corresponding eigenfunction V is radially symmetric and exponentially decaying with the same rate as that of w. Also, the corresponding eigenspace with respect to 0 is $span\{\frac{\partial w}{\partial x_i}, i = 1, \cdots, n\}$. We will use the spectrum of L_0 to estimate the spectrum of $\widetilde{L}_0^\varepsilon$. Now we consider the eigenvalue problem

$$\begin{cases} \widetilde{L}_0^\varepsilon \phi = -\lambda^\varepsilon \phi, & \text{in } \Omega, \\ \frac{\partial \phi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3.22}$$

First we let

$$\begin{aligned} W_\varepsilon^{ij} &= \chi_i w_{x_j}\left(\frac{x - p_i}{\varepsilon}\right), \\ V^i &= \chi_i V\left(\frac{x - p_i}{\varepsilon}\right), \end{aligned} \tag{4.3.23}$$

where χ_i is a smooth cut-off function satisfying

$$\chi_i(x) = \begin{cases} 1, x \in \Omega_i := \{x : d(x, p_i) \le \frac{1}{2}\delta, d(x, \partial\Omega) \ge \frac{1}{4}\delta\}, \\ 0, x \in \tilde{\Omega}_i := \{x : d(x, p_i) \ge \frac{3}{4}\delta, d(x, \partial\Omega) \le \frac{1}{8}\delta\}. \end{cases} \tag{4.3.24}$$

Lemma 4.3.4.

$$\begin{aligned} \frac{B_{\widetilde{W}_{\varepsilon,P}}(W_\varepsilon^{ij}, W_\varepsilon^{ij})}{\langle W_\varepsilon^{ij}, W_\varepsilon^{ij}\rangle} &= O(e^{-\frac{\nu\delta}{\varepsilon}}), \\ \frac{B_{\widetilde{W}_{\varepsilon,P}}(V^i, V^i)}{\langle V^i, V^i\rangle} &= -\lambda_1 + O(e^{-\frac{\nu\delta}{\varepsilon}}), \end{aligned} \tag{4.3.25}$$

Proof. Here, we just prove the first statement, the proof of the other one is similar.

$$\begin{aligned} & B_{\widetilde{W}_{\varepsilon,P}}(W_\varepsilon^{ij}, W_\varepsilon^{ij}) \\ = & B_{w_{\varepsilon,p_i}}(W_\varepsilon^{ij}, W_\varepsilon^{ij}) + \int_\Omega (h'(w_{\varepsilon,p_i}) - h'(\widetilde{W}_{\varepsilon,P}))(W_\varepsilon^{ij})^2 dx \\ = & \int_{\Omega_i} (\varepsilon^2(\nabla W_\varepsilon^{ij})^2 + \mu(W_\varepsilon^{ij})^2 - h'(w_{\varepsilon,p_i})(W_\varepsilon^{ij})^2) dx + \int_\Omega (h'(w_{\varepsilon,p_i}) - h'(\widetilde{W}_{\varepsilon,P}))(W_\varepsilon^{ij})^2 dx \\ & + \int_{\Omega\backslash(\Omega_i\cup\tilde{\Omega}_i)} (\varepsilon^2(\nabla W_\varepsilon^{ij})^2 + \mu(W_\varepsilon^{ij})^2 - h'(w_{\varepsilon,p_i})(W_\varepsilon^{ij})^2) dx \\ = & I + II + III. \end{aligned} \tag{4.3.26}$$

Obviouly, $I = O(e^{-\frac{\nu\delta}{\varepsilon}})$. By direct computation, we have

$$\begin{aligned} |II| &\le \int_\Omega C|\widetilde{W}_{\varepsilon,P} - w_{\varepsilon,p_i}|(W_\varepsilon^{ij})^2 dx \\ &\le \int_{\Omega\backslash\tilde{\Omega}_i} (C|\sum_{l\neq i} w_{\varepsilon,p_l}| + O(e^{-\frac{\nu\delta}{\varepsilon}}))(W_\varepsilon^{ij})^2 dx \le Ce^{-\frac{\nu\delta}{\varepsilon}} \int_\Omega (W_\varepsilon^{ij})^2 dx. \end{aligned} \tag{4.3.27}$$

We are finish the proof by estimating III. First it is easy to check that $W_\varepsilon^{ij}, \varepsilon\nabla W_\varepsilon^{ij} = O(e^{\frac{-\nu\delta}{\varepsilon}})$ in $\Omega\backslash(\Omega_i \cup \tilde{\Omega}_i)$ and $|\varepsilon\nabla\chi_i|_{L^2} = o(1)$. Combining with the fact that $h'(0) = 0$, we have $|III| \leq O(e^{-\frac{\nu\delta}{\varepsilon}})\langle W_\varepsilon^{ij}, W_\varepsilon^{ij}\rangle$.

$\square$

So far we see that $\widetilde{L_0^\varepsilon}$ has k positive eigenvalues and nk eigenvalues near 0. In [20], the authors used a scaling and limiting process to show that there is an $O(1)$ spectral gap between the negative spectrum and the spectrum near zero for the linearized Allen-Cahn operator obtained by linearizing at a boundary multi-spike state. In [95], the authors used a similar technique to show such spectral gap for the operator obtained by linearizing at a single interior-spike state. The same argument adapts here. Performing a change of variable $y = \frac{x-p_i}{\varepsilon}$, where p_i is the center of one of the spikes, extending the eigenfunction for (4.3.22) to $\mathbb{R}^n$, then letting ε tend to zero (see [20, 72, 73] for more details about this process), one finds that the eigenvalue problem (4.3.22) converges along a sequence weakly in $H^1(\mathbb{R}^n)$ to the limiting eigenvalue problem

$$\Delta\phi_\infty - \mu\phi_\infty + h'(w)\phi_\infty = \lambda_\infty\phi_\infty \quad \text{in } \mathbb{R}^n. \tag{4.3.28}$$

If we let $y = \frac{x-z}{\varepsilon}$ for $z \in \Omega$ and $z \notin \{p_i\}_{i=1}^k$ and then perform the same limiting process, we find the limiting eigenvalue problem

$$\Delta\phi_\infty - \mu\phi_\infty = \lambda_\infty\phi_\infty \quad \text{in } \mathbb{R}^n. \tag{4.3.29}$$

In both (4.3.28) and (4.3.29), the negative spectrum is bounded away from 0. Therefore, for ε sufficiently small the rest of the spectrum of $\widetilde{L_0^\varepsilon}$ lies in $(-\infty, -C]$ for some constant $C > 0$.

4.3.3 Spectral analysis of the linearized Cahn-Hilliard operator

Consider the eigenvalue problem:

$$\begin{cases} -\Delta(\varepsilon^2\Delta\phi - \mu\phi + h'(\widetilde{W}_{\varepsilon,P})\phi) = \lambda^\varepsilon\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial n} = \dfrac{\partial\Delta\phi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3.30}$$

First, we recall the variational characterization of the eigenvalues of Cahn-Hilliard equation ([13] Theorem 5):

$$-\lambda_n^\varepsilon = \min{}_n \max{}_n \frac{B_{\widetilde{W}_{\varepsilon,P}}(v,v)}{\langle(-\Delta)^{-1})v, v\rangle}, \tag{4.3.31}$$

where $\max_n$ is over all $v \in T_n$, and $\min_n$ is over all n-dimensional subspaces T_n of $\hat{H}^1(\Omega) := \{v : v \in H^1(\Omega), \int_\Omega v dx = 0\}$. Also, $(-\Delta)^{-1}$ is defined on $\hat{H}^1(\Omega)$ by $(-\Delta)^{-1}v = \eta$ for $v \in \hat{H}^1(\Omega)$ if and only if $\eta \in \hat{H}^1(\Omega)$ is the solution to

$$\begin{cases} -\Delta\eta = v & \text{in } \Omega, \\ \frac{\partial\eta}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, we construct a test function for (4.3.31) using the unstable eigenfunction V. Let θ be a function satisfying

$$supp(\theta) \subset \{x : \frac{3}{8}\delta \le |x| \le \frac{1}{2}\delta\}, \quad \text{and} \quad \int_{\mathbb{R}^n} \theta(x) = \int_{\mathbb{R}^n} V(x),$$

then let

$$\theta_\beta(x) = \beta^n\theta(\beta x). \tag{4.3.32}$$

It is easy to verify that

$$|\theta_\beta(x)|^2_{L^2(\mathbb{R}^n)} = \beta^n|\theta(x)|^2_{L^2(\mathbb{R}^n)}, \quad |\nabla\theta_\beta(x)|^2_{L^2(\mathbb{R}^n)} = \beta^{n+2}|\nabla\theta(x)|^2_{L^2(\mathbb{R}^n)}.$$

We let

$$\overline{V}^i = \alpha_i(x)V(\frac{x-p_i}{\varepsilon}) - \theta^i_\beta(x). \tag{4.3.33}$$

Here, $\alpha_i(x)$ is an smooth cut-off function satisfying

$$\alpha_i(x) = \begin{cases} 1, x \in \{x \in \Omega : d(x,p_i) \le \frac{1}{4}\delta\}, \\ \\ 0, x \in \{x \in \Omega : d(x,p_i) \ge \frac{3}{8}\delta\}, \end{cases} \tag{4.3.34}$$

and

$$\theta^i_\beta(x) = \tau_i\theta_\beta(\frac{x-p_i}{\varepsilon}), \tag{4.3.35}$$

where τ_i is a constant such that $\int_\Omega \overline{V}^i dx = 0$. Note that $\tau_i = O(1)$ and

$$supp(\theta^i_\beta) \subset \{x : \frac{3\varepsilon\delta}{8\beta} \le d(x,p_i) \le \frac{\varepsilon\delta}{2\beta}\}.$$

So if we choose β such that $1 < \frac{\varepsilon}{\beta} < 1+\varepsilon$, then $supp(\theta^i_\beta)$ and $supp(\alpha_i)$ are disjoint, and furthermore $supp(\theta^i_\beta)$ is contained in Ω for ε small enough. Then, one has

$$\begin{aligned} B_{\widetilde{W}_{\varepsilon,P}}(\overline{V}^i,\overline{V}^i) &= B_{\widetilde{W}_{\varepsilon,P}}(\alpha_i(x)V(\frac{x-p_i}{\varepsilon}),\alpha_i(x)V(\frac{x-p_i}{\varepsilon})) + B_{\widetilde{W}_{\varepsilon,P}}(\theta^i_\beta(x),\theta^i_\beta(x)) \\ &= (-\lambda_1 + O(e^{-\frac{\nu\delta}{\varepsilon}}))\langle\alpha_i(x)V(\frac{x-p_i}{\varepsilon}),\alpha_i(x)V(\frac{x-p_i}{\varepsilon})\rangle + O(\beta^n). \end{aligned} \tag{4.3.36}$$

It remains to estimate $\langle(-\Delta)^{-1}\overline{V}^i,\overline{V}^i\rangle$, which is the point of the following lemma.

Lemma 4.3.5. $0 < \langle(-\Delta)^{-1}v, v\rangle \leq C\langle v, v\rangle$, *for any* $v \in \hat{H}^1(\Omega)$, $v \neq 0$ *for some constant* $C > 0$.

Proof. $(-\Delta)$ as a operator acting on functions with mean-value zero and homogenous Neumann boundary condition is a positive operator, so $\langle(-\Delta)^{-1}v, v\rangle > 0$. Let η be the mean value zero solution of the equation

$$\begin{cases} (-\Delta)\eta(x) = v(x) & \text{in } \Omega, \\ \dfrac{\partial \eta}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3.37}$$

By direct calculation, we have

$$\langle(-\Delta)^{-1}v, v\rangle = \int_\Omega v\eta dx = \int_\Omega |\nabla\eta|^2 dx. \tag{4.3.38}$$

For any $\lambda > 0$,

$$\begin{aligned} \int_\Omega v\eta dx &\leq \frac{1}{4\lambda}\int_\Omega v^2 dx + \lambda\int_\Omega \eta^2 dx \\ &\leq \frac{1}{4\lambda}\int_\Omega v^2 dx + \lambda C\int_\Omega |\nabla\eta|^2 dx. \end{aligned} \tag{4.3.39}$$

By choosing λ small enough, it follows immediately that

$$\int_\Omega v\eta dx \leq C\int_\Omega v^2 dx.$$

□

Since $\overline{V}^i$ is constructed by scaling V and θ_β by ε, and since V decays exponentially, one has $< (-\Delta)^{-1}\overline{V}^i, \overline{V}^i >= O(\varepsilon^2)$. One may find a rigorous proof of this fact in the proof of Lemma 4.4.3 in the current paper. Combining (4.3.31), (4.3.36) and Lemma 4.3.5, we have that there are k positive eigenvalues $\lambda_\varepsilon^i (1 \leq i \leq k)$ greater than $\frac{C\lambda_1}{\varepsilon^2}$ with corresponding eigenfunctions denoted by

V_ε^i for some constant C. Similarly, one can prove that there are nk eigenvalues which are $O(e^{-\frac{\nu\delta}{\varepsilon}})$. If there are other positive eigenvalues away from zero, then by (4.3.31) one can see that there will be extra positive eigenvalues away from 0 for the corresponding linearized Allen-Cahn operator, which is a contradiction. Therefore, there are exactly k positive eigenvalues bounded away from 0 for (4.3.30). Moreover, if there is a negative eigenvalue for (4.3.30) of order $o(\frac{1}{\varepsilon^2})$, then by (4.3.31) and Lemma 4.3.5, one sees that there will be a negative eigenvalue approaching zero as $\varepsilon \to 0$ for the corresponding linearized Allen-Cahn operator, which is again a contradiction. Thus, the rest of the spectrum is in $(-\infty, -\frac{\bar{b}}{\varepsilon^2})$ for some $\bar{b} > 0$. For convenience, we denote the size of this spectral gap by $b_\varepsilon = \frac{b}{\varepsilon^2}$.

Note that

$$-\Delta(\varepsilon^2 \Delta \widetilde{W}_\varepsilon^{ij} - \mu \widetilde{W}_\varepsilon^{ij} + h'(\widetilde{W}_{\varepsilon,P})\widetilde{W}_\varepsilon^{ij}) = O(e^{-\frac{\nu\delta}{\varepsilon}}), \tag{4.3.40}$$

where

$$\widetilde{W}_\varepsilon^{ij} = D_j(w_{\varepsilon,p_i} - \widetilde{\mathcal{H}}(w_{\varepsilon,p_i})), \tag{4.3.41}$$

and $\widetilde{W}_\varepsilon^{ij}$ almost satisfies the boundary conditions. Thus, we may use $span\{\widetilde{W}_\varepsilon^{ij}, i = 1, \cdots, k, j = 1, \cdots, n\}$ to approximate the center space for $L_{\varepsilon,P}$, where $L_{\varepsilon,P}v := -\Delta(\varepsilon^2\Delta v - \mu v + h'(\widetilde{W}_{\varepsilon,P})v)$ with the domain

$$\{v : v \in W_\varepsilon^{4,q}(\Omega), \frac{\partial v}{\partial n} = \frac{\partial \Delta v}{\partial n} = 0\}.$$

4.3.4 $\mathcal{M}_\varepsilon$ is approximately stationary

First of all, we define a rescaled Sobolev space $W_\varepsilon^{k,q}$ as the family of $k-th$ differentiable functions (in distribution sense) endowed with the rescaled norm

$$|\cdot|_{W_\varepsilon^{k,q}} = \sum_{0\le|\alpha|\le 2} |\varepsilon^{|\alpha|} D^\alpha(\cdot)|_{L^q(\Omega,\varepsilon^{-n}d\mu)}.$$

Then we choose our phase space $X := \{u \in W_\varepsilon^{2,q} : \frac{\partial \Delta u}{\partial n} = \frac{\partial u}{\partial n} = 0\}$ with large q. Recall that we define $L_{\varepsilon,P}$ as

$$L_{\varepsilon,P}v := -\Delta(\varepsilon^2\Delta v - \mu v + h'(\widetilde{W}_{\varepsilon,P})v) \tag{4.3.42}$$

with the domain

$$\{v : v \in W_\varepsilon^{4,q}(\Omega), \frac{\partial v}{\partial n} = \frac{\partial \Delta v}{\partial n} = 0\}.$$

It is clear that $L_{\varepsilon,P}$ generates an analytic semigroup $e^{tL_{\varepsilon,P}} : L_\varepsilon^q \to L_\varepsilon^q$, where $L_\varepsilon^q = W_\varepsilon^{0,q}$. By using the spectral property of $L_{\varepsilon,P}$ and the embedding theorem of fractional power spaces (see [51]), we have

$$\begin{aligned} |e^{tL_{\varepsilon,P}}|_{(L_\varepsilon^q, L_\varepsilon^q)} &\le Ce^{\frac{Ct}{\varepsilon^2}}, \\ |e^{tL_{\varepsilon,P}}|_{(L_\varepsilon^q, W_\varepsilon^{2,q})} &\le C(1 + (\frac{t}{\varepsilon^2})^{-\frac{1}{2}})e^{\frac{Ct}{\varepsilon^2}}. \end{aligned} \tag{4.3.43}$$

We first modify h to make sure (4.3.4) generates a semiflow globally in time. Let $\tilde{h}(u) = \eta(u)h(u)$, where $\eta(s)$ is a smooth cut-off function satisfying

$$\eta(s) = \begin{cases} 1, & |s| < \max w + 1, \\ 0, & |s| > \max w + 2. \end{cases}$$

We now consider the equation

$$u_t = -\Delta(\varepsilon^2 \Delta u - \mu u + \tilde{h}(u)) \tag{4.3.44}$$

with the same Neumann boundary conditions. For convenience, we still keep the notation h instead of $\tilde{h}$.

For any fixed $P \in \widetilde{\Omega}^k$, we consider the initial value problem

$$\begin{cases} u_t = -\Delta(\varepsilon^2 \Delta u - \mu u + h(u)) & \text{in } \ \Omega \times (0, \infty), \\ \dfrac{\partial \Delta u}{\partial n} = \dfrac{\partial u}{\partial n} = 0 & \text{on } \ \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = \widetilde{W}_{\varepsilon,P} & \text{in } \ \Omega. \end{cases} \tag{4.3.45}$$

Lemma 4.3.6. *Let u be the solution to (4.3.45). Then u satisfies*

$$|u - \widetilde{W}_{\varepsilon,P}|_X \leq C(t + \varepsilon t^{\frac{1}{2}}) e^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}}. \tag{4.3.46}$$

Proof. Let v be the difference $u - \widetilde{W}_{\varepsilon,P}$, then v satisfies

$$\begin{cases} v_t = -\Delta(\varepsilon^2 \Delta v - \mu v + h'(\widetilde{W}_{\varepsilon,P})v) + \Delta(\varepsilon^2 \Delta \widetilde{\mathcal{H}} - \mu \widetilde{\mathcal{H}} + h(\widetilde{W}_{\varepsilon,P} + \widetilde{\mathcal{H}}) - h(\widetilde{W}_{\varepsilon,P} + v) \\ \qquad + h'(\widetilde{W}_{\varepsilon,P})v - R_{\varepsilon,P}) \\ \quad = L_{\varepsilon,P} v + r(v), \\ \dfrac{\partial \Delta v}{\partial n} = \dfrac{\partial v}{\partial n} = 0, \\ v(\cdot, 0) = 0. \end{cases} \tag{4.3.47}$$

Using Lemma 4.3.1, (4.3.17) and the fact that h has been modified, we get

$$|r(v)|_{L^q_\varepsilon} \leq C(|v|_X + e^{-\frac{\nu\delta}{\varepsilon}}). \tag{4.3.48}$$

Using the variation of constants formula, we write the solution v as

$$v = \int_0^t e^{L_{\varepsilon,P}(t-s)} r(v) ds. \tag{4.3.49}$$

Applying (4.3.43) to (4.3.49), we have

$$|v|_X \leq \int_0^t C(1 + (\frac{t-s}{\varepsilon^2})^{-\frac{1}{2}}) e^{\frac{C(t-s)}{\varepsilon^2}} (|v|_X + e^{-\frac{\nu\delta}{\varepsilon}}) ds.$$

Then it follows directly by Gronwall's inequality that

$$|v|_X \leq C(t + \varepsilon t^{\frac{1}{2}}) e^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}}. \tag{4.3.50}$$

$\square$

4.3.5 Transformation of the phase space

Observe that the linearized Cahn-Hilliard operator $L_{\varepsilon,P}$ is not self-adjoint in L^2, so it is hard to show the normal hyperbolicity which leads to the existence of an truly invariant manifold directly. To handle this, we introduce the following transformation. For any $\phi \in X$ satisfying $\int_\Omega \phi dx = 0$, let ψ be the solution to the equation

$$\begin{cases} -\Delta\psi = \phi & \text{in } \Omega, \\ \frac{\partial\psi}{\partial n} = 0 & \text{on } \Omega, \\ \int_\Omega \psi dx = 0. \end{cases} \tag{4.3.51}$$

Define $A : \hat{W}_\varepsilon^{4,q} \cap \{u \in W_\varepsilon^{4,q} : \frac{\partial u}{\partial n} = 0 \text{ on } \Omega\} \to \hat{W}_\varepsilon^{2,q}$ by $A\psi = \phi$, where $\hat{W}_\varepsilon^{k,q} = W_\varepsilon^{k,q} \cap \{u : \int_\Omega u dx = 0\}$. For any $\phi \in \hat{W}_\varepsilon^{2,q}$, let $\phi^\sharp \in \hat{W}_\varepsilon^{3,q}$ satisfy $A^{\frac{1}{2}}\phi^\sharp = \phi$, i.e. $\phi^\sharp = A^{-\frac{1}{2}}\phi$. One can check that the spectrum of A lies in $(0, \infty)$ (A is a positive operator), so $A^{-\frac{1}{2}}$ is well-defined ([51]). Let $u(t, \cdot)$ be the solution to (4.3.44). Since the Cahn-Hilliard equation conserves the mass, one can define $u^\sharp = A^{-\frac{1}{2}}(u - q(u))$, where $q(u) = \frac{1}{|\Omega|}\int_\Omega u dx$ is a constant in t and x. It is clear that $u^\sharp$ satisfies

$$\begin{cases} ((-\Delta)^{\frac{1}{2}}u^\sharp)_t = -\Delta(\varepsilon^2\Delta((-\Delta)^{\frac{1}{2}}u^\sharp) - \mu((-\Delta)^{\frac{1}{2}}u^\sharp) + h((-\Delta)^{\frac{1}{2}}u^\sharp + q(u))) & \text{in } \Omega, \\ \frac{\partial(-\Delta)^{\frac{1}{2}}u^\sharp}{\partial n} = \frac{\partial(-\Delta)^{\frac{3}{2}}u^\sharp}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3.52}$$

Note that all points on the approximately invariant manifold $\mathcal{M}_\varepsilon$ have the same mean value, which we denote by q_ε. Let

$$\mathcal{M}_\varepsilon^\sharp = A^{-\frac{1}{2}}(\mathcal{M}_\varepsilon - q_\varepsilon) := \{A^{-\frac{1}{2}}(\psi_\varepsilon(P) - q_\varepsilon) : P \in \widetilde{\Omega}^k\}. \tag{4.3.53}$$

Since $D_P \int_\Omega \psi_\varepsilon(P)dx = \int_\Omega D_P\psi_\varepsilon(P)dx = 0$, the tangent space of $\mathcal{M}_\varepsilon^\sharp$ is also well-defined as $A^{-\frac{1}{2}}T\mathcal{M}_\varepsilon$. In order to obtain a truly invariant manifold near $\mathcal{M}_\varepsilon$ for the semiflow $u(t, \cdot)$ generated by (4.3.44), we first find a truly invariant manifold near $\mathcal{M}_\varepsilon^\sharp$ with zero mass for $u^\sharp(t, \cdot)$, then by the injectivity of $A^{-\frac{1}{2}}$, one obtains a truly invariant manifold for $u(t,\cdot)$ near $\mathcal{M}_\varepsilon$. Naturally, we choose the phase space $X^\sharp$ for $u^\sharp$ to be $\hat{W}^{3,q}_\varepsilon$. First of all, one can easily check that $\mathcal{M}^\sharp_\varepsilon$ is approximately stationary for $u^\sharp(t,\cdot)$. In fact, by Lemma 4.3.6, one has

$$\begin{aligned}
|u^\sharp(t, A^{-\frac{1}{2}}(\widetilde{W}_{\varepsilon,P} - q_\varepsilon)) - A^{-\frac{1}{2}}(\widetilde{W}_{\varepsilon,P} - q_\varepsilon)|_{W^{3,q}_\varepsilon} &= |A^{-\frac{1}{2}}(u(t, \widetilde{W}_{\varepsilon,P}) - \widetilde{W}_{\varepsilon,P})|_{W^{3,q}_\varepsilon} \\
&\leq C|u(t, \widetilde{W}_{\varepsilon,P}) - \widetilde{W}_{\varepsilon,P}|_X \\
&\leq C(t + \varepsilon t^{\frac{1}{2}})e^{-\frac{\nu\delta}{\varepsilon}}e^{\frac{Ct}{\varepsilon^2}}
\end{aligned} \tag{4.3.54}$$

Linearizing equation (4.3.52) at $A^{-\frac{1}{2}}(\widetilde{W}_{\varepsilon,P} - q_\varepsilon)$, we obtain

$$\delta u^\sharp_t = (-\Delta)^{\frac{1}{2}}(\varepsilon^2\Delta(-\Delta)^{\frac{1}{2}}\delta u^\sharp - \mu(-\Delta)^{\frac{1}{2}}\delta u^\sharp + h'(\widetilde{W}_{\varepsilon,P})(-\Delta)^{\frac{1}{2}}\delta u^\sharp) = L^\sharp_{\varepsilon,P}\delta u^\sharp, \tag{4.3.55}$$

where

$$L^\sharp_{\varepsilon,P} := (-\Delta)^{\frac{1}{2}} \circ (\varepsilon^2\Delta - \mu + h'(\widetilde{W}_{\varepsilon,P})) \circ (-\Delta)^{\frac{1}{2}}, \tag{4.3.56}$$

with the domain

$$\{v : v \in W^{4,q}_\varepsilon,\ \frac{\partial(-\Delta)^{\frac{1}{2}}v}{\partial n} = \frac{\partial(-\Delta)^{\frac{3}{2}}v}{\partial n} = 0\}.$$

By the main theorems of [13], we know that $L^\sharp_{\varepsilon,P}$ is self-adjoint in L^2_ε (after taking the self-adjoint extension) and the eigenvalue problem for $L^\sharp_{\varepsilon,P}$ is equivalent to the eigenvalue problem for $L_{\varepsilon,P}$. Let

$$\begin{aligned}
\widetilde{W}^{ij,\sharp}_\varepsilon &= A^{-\frac{1}{2}}D_{p_{ij}}\widetilde{W}^{ij}_\varepsilon, \\
V^{i,\sharp}_\varepsilon &= A^{-\frac{1}{2}}V^i_\varepsilon.
\end{aligned} \tag{4.3.57}$$

These are well-defined since both $D_{p_{ij}}\widetilde{W}^{ij}_{\varepsilon}$ and V^{i}_{ε} have mean-value zero. Also, one can check

$$L^{\sharp}_{\varepsilon,P}\widetilde{W}^{ij,\sharp}_{\varepsilon} = O(e^{-\frac{\nu\delta}{\varepsilon}}), \tag{4.3.58}$$

and

$$L^{\sharp}_{\varepsilon,P}V^{i,\sharp}_{\varepsilon} = \lambda^{i}_{\varepsilon}V^{i,\sharp}_{\varepsilon}. \tag{4.3.59}$$

Without loss of generality, we assume that $|\widetilde{W}^{ij,\sharp}_{\varepsilon}|_{L^2_{\varepsilon}} = |V^{i,\sharp}_{\varepsilon}|_{L^2_{\varepsilon}} = 1$. Furthermore, it is clear that $L^{\sharp}_{\varepsilon,P}$ generates an analytic semigroup $e^{L^{\sharp}_{\varepsilon,P}t}$ with similar properties to $e^{L_{\varepsilon,P}t}$.

4.3.6 Splitting space $X^{\sharp}$ along the manifold $\mathcal{M}^{\sharp}_{\varepsilon}$

Spectral analysis of $L^{\sharp}_{\varepsilon,P}$ yields that the approximate unstable space of $L^{\sharp}_{\varepsilon,P}$ is $span\{V^{i,\sharp}_{\varepsilon}\}$ and the approximate center space of $L^{\sharp}_{\varepsilon,P}$ is $span\{\widetilde{W}^{ij,\sharp}_{\varepsilon} : 1 \le i \le k, 1 \le j \le n\} \sim T_{\psi^{\sharp}_{\varepsilon}(P)}\mathcal{M}^{\sharp}_{\varepsilon}$. We will use this to construct center-stable and center-unstable manifolds for $\mathcal{M}^{\sharp}_{\varepsilon}$.

Remark 4.3.7. $\widetilde{W}^{ij,\sharp}_{\varepsilon}$ does not satisfy the boundary conditions, but we can modify it slightly with the same way that we modified $W_{\varepsilon,P}$ in Section 4.3.1. Note that the modification will be $O(e^{-\frac{\nu\delta}{\varepsilon}})$. For convenience, we will keep the notation $\widetilde{W}^{ij,\sharp}_{\varepsilon}$ for the modified function.

Since $L^{\sharp}_{\varepsilon,p}$ is self-adjoint, we have

$$\lambda^{l}_{\varepsilon}\langle\widetilde{W}^{ij,\sharp}_{\varepsilon}, V^{l,\sharp}_{\varepsilon}\rangle = \langle\widetilde{W}^{ij,\sharp}_{\varepsilon}, L^{\sharp}_{\varepsilon,p}V^{l,\sharp}_{\varepsilon}\rangle = \langle L^{\sharp}_{\varepsilon,p}\widetilde{W}^{ij,\sharp}_{\varepsilon}, V^{l,\sharp}_{\varepsilon}\rangle = O(e^{-\frac{\nu\delta}{\varepsilon}}),$$

which implies that

$$\langle\widetilde{W}^{ij,\sharp}_{\varepsilon}, V^{l,\sharp}_{\varepsilon}\rangle = O(e^{-\frac{\nu\delta}{\varepsilon}}), \tag{4.3.60}$$

where $\langle,\rangle$ denotes the L^2_{ε} inner product, i.e., $\langle f,g\rangle = \int_{\Omega} fg\varepsilon^{-n}dx$. Since $\langle\widetilde{W}^{ij,\sharp}_{\varepsilon}, \widetilde{W}^{lm,\sharp}_{\varepsilon}\rangle = \langle(-\Delta)^{-1}\widetilde{W}^{ij}_{\varepsilon}, \widetilde{W}^{lm}_{\varepsilon}\rangle$, we have the following lemma

Lemma 4.3.8.

$$\langle \widetilde{W}_{\varepsilon}^{uv,\sharp}, \widetilde{W}_{\varepsilon}^{lm,\sharp} \rangle = o(\varepsilon) \quad \textit{if } (l,m) \neq (u,v). \tag{4.3.61}$$

Proof. We just state this lemma here, the proof is similar to the proof of Lemma 4.4.3 in the last section of the current paper. □

Let $\bar{V}_{\varepsilon}^{i,\sharp}$ be the component of $V_{\varepsilon}^{i,\sharp}$ which is orthogonal (in the sense of L_{ε}^{2}) to $span\{\widetilde{W}_{\varepsilon}^{ij,\sharp}\}$. Clearly $|\bar{V}_{\varepsilon}^{i,\sharp} - V_{\varepsilon}^{i,\sharp}|_{X^{\sharp}} = O(e^{-\frac{\nu\delta}{\varepsilon}})$. Then we decompose $X^{\sharp}$ as $X^{\sharp} = X_{\varepsilon,P}^{c} \oplus X_{\varepsilon,P}^{s} \oplus X_{\varepsilon,P}^{u}$, where

$$\begin{aligned}
X_{\varepsilon,P}^{c} &= span\{\widetilde{W}_{\varepsilon}^{ij,\sharp}, i = 1, \cdots, k, j = 1, \cdots, n\}, \\
X_{\varepsilon,P}^{u} &= span\{\bar{V}_{\varepsilon}^{i,\sharp}, i = 1, \cdots, k\}, \\
X_{\varepsilon,P}^{s} &= \Big\{v : \int_{\Omega} v\tilde{v}dx = 0, \text{for any } \ \tilde{v} \in X_{\varepsilon,P}^{c} \oplus X_{\varepsilon,P}^{u}\Big\}.
\end{aligned} \tag{4.3.62}$$

Denote the associated projection maps by $\Pi_{\varepsilon,P}^{\alpha}$, $\alpha = c, u, s$. By the smoothness of $\widetilde{W}_{\varepsilon}^{ij,\sharp}$ and $V_{\varepsilon}^{i,\sharp}$, we have that $\Pi_{\varepsilon,P}^{\alpha}$, $\alpha = c, s, u$, are smooth in P. Using the L_{ε}^{2}-orthogonality and finite dimensionality of center and unstable spaces, we have the boundedness of these projections. Following from the compactness of $\mathcal{M}_{\varepsilon}$, we immediately have that $\Pi_{\varepsilon,P}^{\alpha}$ are uniformly bounded and uniformly Lipschitz in P. Clearly all the bounds are independent of ε for ε small enough. Furthermore, we have that for small enough ε, there exists $B > 0$, independent of ε, such that

$$|\Pi_{\varepsilon,P}^{\alpha}|_{C^{m}(\widetilde{\Omega}^{k}, L(x))} \leq B, \quad \text{for any positive integer } \ m. \tag{4.3.63}$$

4.3.7 Trichotomy

To investigate the linearized flow at a solution $u^\sharp$ to (4.3.52) with initial condition $A^{-\frac{1}{2}}\widetilde{W}_{\varepsilon,P}$, we consider

$$\bar{\delta u}_t^\sharp = (-\Delta)^{\frac{1}{2}} \circ (\varepsilon^2\Delta - \mu + h'((-\Delta)^{\frac{1}{2}}u^\sharp)) \circ (-\Delta)^{\frac{1}{2}}\bar{\delta u}^\sharp. \tag{4.3.64}$$

To get estimates for $\bar{\delta u}^\sharp$, we also consider

$$\delta u_t^\sharp = L^\sharp_{\varepsilon,P}\delta u^\sharp = (-\Delta)^{\frac{1}{2}} \circ (\varepsilon^2\Delta - \mu + h'(\widetilde{W_{\varepsilon,P}})) \circ (-\Delta)^{\frac{1}{2}}\delta u^\sharp. \tag{4.3.65}$$

Lemma 4.3.9. *If $\delta^\sharp u(0) = \bar{\delta u}^\sharp(0)$, then $|\bar{\delta u}^\sharp - \delta u^\sharp|_{X^\sharp} \le C(t + \varepsilon t^{\frac{1}{2}})(t + \varepsilon^{\frac{3}{2}}t^{\frac{1}{4}})e^{-\frac{\nu\delta}{\varepsilon}}e^{\frac{Ct}{\varepsilon^2}}|\bar{\delta u}^\sharp(0)|_{X^\sharp}$.*

Proof. First it is easy to prove that

$$|\bar{\delta u}^\sharp|_{X^\sharp},\ |\delta u^\sharp|_{X^\sharp} \le Ce^{\frac{Ct}{\varepsilon^2}}|\bar{\delta u}^\sharp(0)|_{X^\sharp}. \tag{4.3.66}$$

Let $v = \bar{\delta u}^\sharp - \delta u^\sharp$. Then v satisfies

$$\begin{cases} v_t = L^\sharp_{\varepsilon,P}v - (-\Delta)^{\frac{1}{2}}[(h'(\widetilde{W}_{\varepsilon,P}) - h'(u))(-\Delta)^{\frac{1}{2}}\bar{\delta u}^\sharp], \\ v(0) = 0. \end{cases}$$

Using the variation of constants formula, we get

$$v = \int_0^t e^{L^\sharp_{\varepsilon,P}(t-s)}(-\Delta)^{\frac{1}{2}}[-(h'(\widetilde{W}_{\varepsilon,P}) - h'(u))\Delta^{\frac{1}{2}}\bar{\delta u}^\sharp]ds.$$

Note that $|\widetilde{W}_{\varepsilon,P} - u|_{W^{2,q}_\varepsilon} \le C(t + \varepsilon t^{\frac{1}{2}})e^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}}$, so for q large enough, we have

$$|\widetilde{W}_{\varepsilon,P} - u|_{C^1} \le C(t + \varepsilon t^{\frac{1}{2}})e^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}},$$

which implies that

$$|(-\Delta)^{\frac{1}{2}}[(h'(\widetilde{W}_{\varepsilon,P}) - h'(u))(-\Delta)^{\frac{1}{2}}\delta\bar{u}^\sharp]|_{L^q_\varepsilon} \le C(t + \varepsilon t^{\frac{1}{2}})e^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}} |\delta\bar{u}^\sharp|_{W^{2,q}_\varepsilon}. \tag{4.3.67}$$

Clearly,

$$|e^{L^\sharp_{\varepsilon,P}t}|_{L(L^q_\varepsilon, X^\sharp)} \le C(1 + (\frac{t}{\varepsilon^2})^{-\frac{3}{4}})e^{\frac{Ct}{\varepsilon^2}}.$$

Then it follows immediately that

$$|v|_{X^\sharp} \le C(t + \frac{t^{\frac{1}{2}}}{\varepsilon})(t + \varepsilon^{\frac{3}{2}} t^{\frac{1}{4}})e^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}} |\delta\bar{u}^\sharp(0)|_{X^\sharp}.$$

□

Decompose $\delta u^\sharp$ as

$$\delta u^\sharp = \varepsilon D\widetilde{W}^\sharp_{\varepsilon,P} \cdot a(t) + \sum b_i(t)\bar{V}^{i,\sharp}_\varepsilon + W^s(t), \tag{4.3.68}$$

where D means the gradient with respect to P. It is easy to check that $|\delta u^\sharp|_{X^\sharp} \le Ce^{\frac{Ct}{\varepsilon^2}}|\delta u^\sharp(0)|_{X^\sharp}$ and $|\delta u^\sharp|_{L^q_\varepsilon} \le Ce^{\frac{Ct}{\varepsilon^2}}|\delta u^\sharp(0)|_{L^q_\varepsilon}$. Then by the boundedness of the projection maps, we have

$$|a(t)| \le Ce^{\frac{Ct}{\varepsilon^2}}|\delta u^\sharp(0)|_{L^q_\varepsilon}, \quad |b_i(t)| \le Ce^{\frac{Ct}{\varepsilon^2}}|\delta u^\sharp(0)|_{L^q_\varepsilon}, \quad |W^s(t)|_{L^q_\varepsilon} \le Ce^{\frac{Ct}{\varepsilon^2}}|\delta u^\sharp(0)|_{L^q_\varepsilon}, \tag{4.3.69}$$

and

$$|W^s(t)|_{X^\sharp} \le Ce^{\frac{Ct}{\varepsilon^2}} |\delta u^\sharp(0)|_{X^\sharp}. \tag{4.3.70}$$

Taking the inner product of (4.3.68) with $L^\sharp_{\varepsilon,P} \varepsilon D\widetilde{W}^\sharp_{\varepsilon,P} \cdot a(t)$, we have

$$\begin{aligned}\langle \delta u^\sharp, L^\sharp_{\varepsilon,P} \varepsilon D\widetilde{W}^\sharp_{\varepsilon,P} \cdot a(t)\rangle &= \langle L^\sharp_{\varepsilon,P} \delta u^\sharp, \varepsilon D\widetilde{W}^\sharp_{\varepsilon,P} \cdot a(t)\rangle \\ &= \langle \delta u^\sharp_t, \varepsilon D\widetilde{W}^\sharp_{\varepsilon,P} \cdot a(t)\rangle \\ &= \langle \varepsilon D\widetilde{W}^\sharp_{\varepsilon,P} \cdot a'(t), \varepsilon D\widetilde{W}^\sharp_{\varepsilon,P} \cdot a(t)\rangle + \langle \sum_i b_i'(t) \bar{V}^{i,\sharp}_\varepsilon, \varepsilon D\widetilde{W}^\sharp_{\varepsilon,P} \cdot a(t)\rangle.\end{aligned} \tag{4.3.71}$$

Note that $|\delta u^\sharp|_{L^q_\varepsilon} \le Ce^{\frac{Ct}{\varepsilon^2}} |\delta u^\sharp(0)|_{L^q_\varepsilon}$ and $|L_{\varepsilon,P} \varepsilon D\widetilde{W}^\sharp_{\varepsilon,P} \cdot a(t)|_{L^\infty} \le Ce^{-\frac{\nu\delta}{\varepsilon}} |a(t)| \le Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}} |\delta u^\sharp(0)|_{L^q_\varepsilon}$. Combining with (4.3.69) and Lemma 4.3.8, we obtain

$$|a(t) \cdot a'(t)| \le Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}} (|\delta u^\sharp(0)|^2_{L^q_\varepsilon} + |\delta u^\sharp(0)|_{L^q_\varepsilon} |b_i'(t)|). \tag{4.3.72}$$

Since $\frac{d}{dt}|a(t)|^2 = 2a(t) \cdot a'(t)$, by integrating both sides of (4.3.72) we get

$$\Big| |a(t)|^2 - |a(0)|^2 \Big| \le Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}} |\delta u^\sharp(0)|^2_{L^q_\varepsilon}. \tag{4.3.73}$$

Similarly, taking the inner product of (4.3.68) with $L^\sharp_{\varepsilon,P} \bar{V}^{i,\sharp}_\varepsilon$, we have

$$|b_i(t) - e^{\lambda^i_\varepsilon t} b_i(0)| \le Ce^{\frac{Ct}{\varepsilon^2}} e^{-\frac{\nu\delta}{\varepsilon}} |\delta u^\sharp(0)|_{L^q_\varepsilon}. \tag{4.3.74}$$

Now we consider the stable direction. Since $L^{\sharp}_{\varepsilon,P}$ is self-adjoint, one has

$$\begin{aligned} &|\langle L^{\sharp}_{\varepsilon,P} W^s, \varepsilon D\widetilde{W}_{\varepsilon,P} \cdot \tau\rangle| = |\langle W^s, L^{\sharp}_{\varepsilon,P}\varepsilon D\widetilde{W}_{\varepsilon,P} \cdot \tau\rangle| \leq Ce^{-\frac{\nu\delta}{\varepsilon}}|\tau||W^s|_{L^2_\varepsilon}, \\ \text{and } &|\langle L^{\sharp}_{\varepsilon,P} W^s, \bar{V}^{i,\sharp}_\varepsilon\rangle| = |\langle W^s, L^{\sharp}_{\varepsilon,P}\bar{V}^{i,\sharp}_\varepsilon\rangle \leq Ce^{-\frac{\nu\delta}{\varepsilon}}|W^s|_{L^2_\varepsilon}, \end{aligned} \tag{4.3.75}$$

which implies that

$$|\Pi^c_{\varepsilon,P} L^{\sharp}_{\varepsilon,P} W^s|_{L^2_\varepsilon} + |\Pi^s_{\varepsilon,P} L^{\sharp}_{\varepsilon,P} W^s|_{L^2_\varepsilon} \leq Ce^{-\frac{\nu\delta}{\varepsilon}}|W^s|_{L^2_\varepsilon}. \tag{4.3.76}$$

Using the fact that center and unstable bundles are finite dimensional, we have

$$|\Pi^c_{\varepsilon,P} L^{\sharp}_{\varepsilon,P} W^s|_{X^\sharp} + |\Pi^s_{\varepsilon,P} L^{\sharp}_{\varepsilon,P} W^s|_{X^\sharp} \leq Ce^{-\frac{\nu\delta}{\varepsilon}}|W^s|_{X^\sharp}. \tag{4.3.77}$$

Write (4.3.65) as

$$\varepsilon D\widetilde{W}_{\varepsilon,P}\cdot a'(t) + \sum b'_i(t)\bar{V}^{i,\sharp}_\varepsilon + \frac{d}{dt}W^s(t) = L^{\sharp}_{\varepsilon,P}\varepsilon D\widetilde{W}_{\varepsilon,P}\cdot a(t) + \sum b_i(t) L^{\sharp}_{\varepsilon,P}\bar{V}^{i,\sharp}_\varepsilon + L^{\sharp}_{\varepsilon,P}W^s(t). \tag{4.3.78}$$

Applying $\Pi^s_{\varepsilon,P}$ to both sides, we obtain

$$\frac{d}{dt}W^s(t) = \Pi^s_{\varepsilon,P} L^{\sharp}_{\varepsilon,P} W^s + \Pi^s_{\varepsilon,P}(L^{\sharp}_{\varepsilon,P}\varepsilon D\widetilde{W}_{\varepsilon,P}\cdot a(t) + \sum b_i(t) L^{\sharp}_{\varepsilon,P}\bar{V}^{i,\sharp}_\varepsilon). \tag{4.3.79}$$

By (4.3.77), we see that $\Pi^s_{\varepsilon,P} L^{\sharp}_{\varepsilon,P} : X^s_{\varepsilon,P} \to X^s_{\varepsilon,P}$ is a small perturbation of $L^{\sharp}_{\varepsilon,P}$, so $\Pi^s_{\varepsilon,P} L^{\sharp}_{\varepsilon,P}$ generates a semigroup. Then by the spectral gap for $L^{\sharp}_{\varepsilon,P}$ and perturbation theory, we have

$$|W^s(t)|_X \leq e^{-b\varepsilon t}|W^s(0)|_X + Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}}|\delta u(0)|_{L^q}. \tag{4.3.80}$$

Denote the semiflow generated by (4.3.52) by $T_\varepsilon^{\sharp t}$. Combining (4.3.73), (4.3.74), (4.3.80) and Lemma 4.3.9, we obtain the following trichotomy properties:

Lemma 4.3.10. *If $t = O(\varepsilon^2)$ and ε is small enough, for $\alpha, \beta = c, u, s$ and $\alpha \neq \beta$, we have*

$$\begin{aligned}
&\|\Pi^{\beta}_{\varepsilon,P} DT_\varepsilon^{\sharp t}|_{X^{\alpha}_{\varepsilon,P}}\| \le Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}},\\
&\|\Pi^{s}_{\varepsilon,P} DT_\varepsilon^{\sharp t}|_{X^{s}_{\varepsilon,P}}\| \le e^{-b\varepsilon t} + Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}},\\
&1 - Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}} \le \|(\Pi^{c}_{\varepsilon,P} DT_\varepsilon^{\sharp t}|_{X^{c}_{\varepsilon,P}})^{-1}\|^{-1} \le \|\Pi^{c}_{\varepsilon,P} DT_\varepsilon^{\sharp t}|_{X^{c}_{\varepsilon,P}}\| \le 1 + Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}},\\
&\|(\Pi^{u}_{\varepsilon,P} DT_\varepsilon^{\sharp t}|_{X^{u}_{\varepsilon,P}})^{-1}\|^{-1} \ge e^{\tilde{\lambda}\varepsilon t} - Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}},
\end{aligned} \tag{4.3.81}$$

where $\tilde{\lambda}_\varepsilon$ is defined in section 4.3.2.

Proof. If $\Pi^{c}_{\varepsilon,P}\delta u^{\sharp}(0) = 0$, i.e., $a(0) = 0$, by (4.3.73), we have

$$|a(t)| \le Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}} |\delta u^{\sharp}(0)|_{L^q_\varepsilon},$$

which implies

$$|\Pi^{c}_{\varepsilon,P}\delta u^{\sharp}|_{L^q_\varepsilon} \le Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}} |\delta u^{\sharp}(0)|_{L^q_\varepsilon} \le Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}} |\delta u(0)|_{X}.$$

Since $X^{c}_{\varepsilon,P}$ is finite dimensional, we have

$$|\Pi^{c}_{\varepsilon,P}\delta u^{\sharp}|_{X^{\sharp}} \le Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}} |\delta u^{\sharp}(0)|_{X^{\sharp}},$$

which implies $\|\Pi^{c}_{\varepsilon,P} DT_\varepsilon^{\sharp t}|_{X^{\alpha}_{\varepsilon,P}}\| \le Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}}$, for $\alpha \neq c$.

If $\delta u^{\sharp}(0) \in X^{c}_{\varepsilon,P}$, i.e., $\delta u^{\sharp}(0) = \varepsilon D\widetilde{W}^{\sharp}_{\varepsilon,P} \cdot a(0)$, by (4.3.73), we have

$$|a(t)| \le (1 + Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}})|a(0)|, \tag{4.3.82}$$

which implies $|\Pi^c_{\varepsilon,P}\delta u^\sharp|_{L^q_\varepsilon} \leq (1 + Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}})|\delta u^\sharp(0)|_{L^q_\varepsilon}$. Again we use the finite dimensionality of the center space to obtain

$$\|\Pi^c_{\varepsilon,P} DT^{\sharp t}_\varepsilon|_{X^c_{\varepsilon,P}}\| \leq 1 + Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}}. \tag{4.3.83}$$

Observe that when $t = O(\varepsilon^2)$ and ε is small enough, (4.3.72) combined with Lemma 4.3.8 implies that $\Pi^c_{\varepsilon,P} DT^{\sharp t}_\varepsilon|_{X^c_{\varepsilon,P}}$ is a small perturbation of the identity map, so $\Pi^c_{\varepsilon,P} DT^t_\varepsilon|_{X^c_{\varepsilon,P}}$ is also an isomorphism. One can follow a similar argument to obtain

$$1 - Ce^{-\frac{\nu\delta}{\varepsilon}} e^{\frac{Ct}{\varepsilon^2}} \leq \|(\Pi^c_{\varepsilon,P} DT^{\sharp t}_\varepsilon|_{X^c_{\varepsilon,P}})^{-1}\|^{-1}. \tag{4.3.84}$$

The other inequalities can be proved similarly, this being left to the readers. The only thing we want to point out is that by (4.3.74), $\Pi^u_{\varepsilon,P} DT^{\sharp t}_\varepsilon|_{X^u_{\varepsilon,P}}$ is a small perturbation of an isomorphism if $t = O(\varepsilon^2)$ and ε is small enough, which indicates that $\Pi^u_{\varepsilon,P} DT^{\sharp t}_\varepsilon|_{X^u_{\varepsilon,P}}$ is an isomorphism. □

Recall that $b_\varepsilon, \tilde{\lambda}_\varepsilon = O(\frac{1}{\varepsilon^2})$. For fixed δ, we first choose $t_o = \varepsilon^2 K$ with K large enough to make $e^{-b_\varepsilon t_0}$ small enough and $e^{\tilde{\lambda}_\varepsilon t_0}$ large enough, then we let ε be sufficiently small, so that $\mathcal{M}_\varepsilon$ is an approximately stationary invariant and approximately normally hyperbolic manifold for $T^{\sharp t_0}_\varepsilon$. Then by the general theorem we established in Section 4.2, we have a locally truly invariant manifold near $\mathcal{M}_\varepsilon$ for $T^{\sharp t_0}_\varepsilon$ by intersecting the center-stable manifold and the center-unstable manifold of $\mathcal{M}_\varepsilon$. However, due to the non-uniqueness of center manifolds (different modifications give different center manifolds), we cannot conclude that the center manifold for $T^{\sharp t_0}_\varepsilon$ is invariant under the semiflow $T^{\sharp t}_\varepsilon$. The correct way should be to modify the vector field to generate a modified semiflow, then prove the existence of a truly invariant manifold for the modified semiflow.

4.3.8 Modification of vector field

In this section, we take $t_0 = \varepsilon^2 K$ such that $\mathcal{M}_\varepsilon$ is an approximately stationary invariant and approximately normally hyperbolic manifold for $T_\varepsilon^{\sharp t_0}$. We also assume that the boundary of Ω satisfies the conditions listed in Section 4.2. Following the set up in Section 4.2, we define $\tilde{P} \in \widetilde{\Omega}^k$ the same way as $\widetilde{m}$. For any $x \in N(\mathcal{M}_\varepsilon^\sharp, r)$, where $N(\mathcal{M}_\varepsilon^\sharp, r)$ is a tubular neighborhood of $\mathcal{M}_\varepsilon^\sharp$, let $m(x)$ be the projection of x into $\mathcal{M}_\varepsilon^\sharp$ and let $P(x) \in \widetilde{\Omega}^k$ be that point such that $\psi_\varepsilon^\sharp(P(x)) = m(x)$, then we write x as $x = \psi_\varepsilon^\sharp(P(x)) + x^s(x) + x^u(x)$.

Rewrite (4.3.52) as

$$u_t^\sharp = L_{\varepsilon,P}^\sharp u^\sharp + r_P(u^\sharp), \tag{4.3.85}$$

where $r_P(u^\sharp) = (-\Delta)^{\frac{1}{2}}(h((-\Delta)^{\frac{1}{2}}u^\sharp) - h'(\psi_\varepsilon(P))(-\Delta)^{\frac{1}{2}}u^\sharp)$.

Using the variation of constants formula, we write (4.3.85) as

$$u^\sharp = e^{L_{\varepsilon,P}^\sharp t}u(0) + \int_0^t e^{L_{\varepsilon,P}^\sharp (t-s)} r_P(u^\sharp)ds. \tag{4.3.86}$$

By Lemma 4.3.6, we have that for ε small enough,

$$\psi_\varepsilon^\sharp(P) = e^{L_{\varepsilon,P}^\sharp t}\psi_\varepsilon^\sharp(P) + \int_0^t e^{L_{\varepsilon,P}^\sharp (t-s)} r_P(\psi_\varepsilon^\sharp(P))ds + O(e^{-\frac{\nu\delta}{\varepsilon}})e^{\frac{Ct}{\varepsilon^2}}, \tag{4.3.87}$$

where $O(e^{-\frac{\nu\delta}{\varepsilon}})$ is in the $X^\sharp$ sense.

Now we modify the vector field to obtain a new equation:

$$\begin{cases} \tilde{u}_t^\sharp = L_{\varepsilon,P}^\sharp \tilde{u}^\sharp + r_P(\tilde{u}^\sharp) + \frac{1}{t_0}e^{L_{\varepsilon,P}^\sharp t}\psi_\varepsilon^\sharp(\tilde{P}) + r_P(\psi_\varepsilon^\sharp(\tilde{P})) - [\frac{1}{t_0}e^{L_{\varepsilon,P}^\sharp t}\psi_\varepsilon^\sharp(P) + r_P(\psi_\varepsilon^\sharp(P))], \\ \tilde{u}^\sharp(0) = x. \end{cases} \tag{4.3.88}$$

Using the variation of constants formula, we write (4.3.88) as

$$\begin{aligned}\tilde{u}^\sharp &= e^{L^\sharp_{\varepsilon,P}t}x + \int_0^t e^{L^\sharp_{\varepsilon,P}(t-s)}[r_P(\tilde{u}^\sharp) + \frac{1}{t_0}e^{L^\sharp_{\varepsilon,P}s}\psi^\sharp_\varepsilon(\tilde{P}) + r_P(\psi^\sharp_\varepsilon(\tilde{P})) - (\frac{1}{t_0}e^{L^\sharp_{\varepsilon,P}s}\psi^\sharp_\varepsilon(P) + r_P(\psi^\sharp_\varepsilon(P)))]ds \\ &= e^{L^\sharp_{\varepsilon,P}t}x + \int_0^t e^{L^\sharp_{\varepsilon,P}(t-s)}r_P(\tilde{u}^\sharp)ds + \frac{t}{t_0}e^{L^\sharp_{\varepsilon,P}t}\psi^\sharp_\varepsilon(\tilde{P}) + \int_0^t e^{L^\sharp_{\varepsilon,P}(t-s)}r_P(\psi^\sharp_\varepsilon(\tilde{P}))ds - (\frac{t}{t_0}e^{L^\sharp_{\varepsilon,\tilde{P}}t}\psi^\sharp_\varepsilon(P) \\ &\quad + \int_0^t e^{L^\sharp_{\varepsilon,P}(t-s)}r_P(\psi^\sharp_\varepsilon(P))ds).\end{aligned} \tag{4.3.89}$$

Denote the semiflow generated by (4.3.88) by $\widetilde{T}^{\sharp t}_\varepsilon$. Recall that $T^{\sharp t}_\varepsilon(x)$ is the semflow defined by (4.3.85). Then $T^{\sharp t}_\varepsilon(x) = e^{L^\sharp_{\varepsilon,P}t}x + \int_0^t e^{L^\sharp_{\varepsilon,P}(t-s)}r_P(T^{\sharp s}_\varepsilon(x))ds$. It follows from (4.3.89) that

$$\begin{aligned}\widetilde{T}^{\sharp t}_\varepsilon(x) = T^{\sharp t}_\varepsilon(x) + \int_0^t e^{L^\sharp_{\varepsilon,P}(t-s)}(r_P(\widetilde{T}^{\sharp s}_\varepsilon(x)) - r_P(T^{\sharp s}_\varepsilon(x)))ds + \frac{t}{t_0}e^{L^\sharp_{\varepsilon,P}t}\psi^\sharp_\varepsilon(\tilde{P}) + \\ \int_0^t e^{L^\sharp_{\varepsilon,P}(t-s)}r_P(\psi^\sharp_\varepsilon(\tilde{P}))ds - (\frac{t}{t_0}e^{L^\sharp_{\varepsilon,P}t}\psi^\sharp_\varepsilon(P) + \int_0^t e^{L^\sharp_{\varepsilon,P}(t-s)}r_P(\psi^\sharp_\varepsilon(P))ds).\end{aligned} \tag{4.3.90}$$

Recall that in Section 4.2, we use l to denote how much we shift the base point. Note that $|\tilde{P} - P| \le Cl$ for any P, one can easily prove by using Gronwall's inequality that

$$\begin{aligned}&|\widetilde{T}^{\sharp t}_\varepsilon(x) - T^{\sharp t}_\varepsilon(x)| \le C(t + \varepsilon t^{\frac{1}{2}})e^{\frac{Ct}{\varepsilon^2}}l, \\ &\text{and} \\ &\|D\widetilde{T}^{\sharp t}_\varepsilon(x) - DT^{\sharp t}_\varepsilon(x)\| \le C(t + \varepsilon t^{\frac{1}{2}})e^{\frac{Ct}{\varepsilon^2}}l.\end{aligned} \tag{4.3.91}$$

For $x \in W^{3,q}_\varepsilon$, let $\bar{x} = (-\Delta)^{\frac{1}{2}}x$ and $(-\Delta)^{\frac{1}{2}}\widetilde{T}^{\sharp t}_\varepsilon(x) = \widetilde{T}^t_\varepsilon(\bar{x})$. Then by direct computation, we have

$$r_P(\widetilde{T}^{\sharp t}_\varepsilon(x)) - r_P(T^{\sharp t}_\varepsilon(x)) = -(-\Delta)^{\frac{1}{2}}\Big(h'(\psi_\varepsilon(P))(\widetilde{T}^t_\varepsilon(\bar{x}) - T^t_\varepsilon(\bar{x})) - \big(h(\widetilde{T}^t_\varepsilon(\bar{x})) - h(T^t_\varepsilon(\bar{x}))\big) \tag{4.3.92}$$

We have

$$h(\widetilde{T}^t_\varepsilon(\bar{x})) - h(T^t_\varepsilon(\bar{x})) = h'(T^t_\varepsilon(\bar{x}))(\widetilde{T}^t_\varepsilon(\bar{x}) - T^t_\varepsilon(\bar{x})) + O(|\widetilde{T}^t_\varepsilon(\bar{x}) - T^t_\varepsilon(\bar{x})|^2). \tag{4.3.93}$$

Recall that h has been modified, by using Lemma 4.3.6, one can check that for $x = \psi^\sharp_\varepsilon(P)$, we have

$$|r_P(\widetilde{T}^{\sharp t}_\varepsilon(x)) - r_P(T^{\sharp t}_\varepsilon(x))|_{W^{1,q}_\varepsilon} \leq C\Big((t + \varepsilon t^{\frac{1}{2}})e^{\frac{Ct}{\varepsilon^2}} e^{-\frac{\nu\delta}{\varepsilon}} l + ((t + \varepsilon t^{\frac{1}{2}})e^{\frac{Ct}{\varepsilon^2}} l)^2\Big). \tag{4.3.94}$$

Therefore, combining with (4.3.87), (4.3.90), (4.3.91) and (4.3.94), we have that for any $x = \psi^\sharp_\varepsilon(P)$,

$$\begin{aligned}
&\widetilde{T}^{\sharp t_0}_\varepsilon(x) = T^{\sharp t_0}_\varepsilon(x) + \psi^\sharp_\varepsilon(\widetilde{P(x)}) - \psi^\sharp_\varepsilon(P(x)) + O(e^{-\frac{\nu\delta}{\varepsilon}} l + l^2),\\
&D\widetilde{T}^{\sharp t_0}_\varepsilon(x) = DT^{\sharp t_0}_\varepsilon(x) + D\psi^\sharp_\varepsilon(\widetilde{P(x)}) - D\psi^\sharp_\varepsilon(P(x)) + O(e^{-\frac{\nu\delta}{\varepsilon}} l + l^2).
\end{aligned} \tag{4.3.95}$$

Following the estimates in Section 4.2.1, now one can apply Theorem 2.0.3 to obtain a C^m (m depends only on the smoothness of the nonlinearity and the boundary $\partial\Omega$) center-stable manifold $W^{cs\sharp}_\varepsilon$ near $\mathcal{M}^\sharp_\varepsilon$ for $\widetilde{T}^{\sharp t_0}_\varepsilon$. To obtain the invariance for the semiflow $\widetilde{T}^{\sharp t}_\varepsilon$, we need to verify the weak uniform continuity in (H5). In fact, we will prove

- For any $\mu > 0$, there exists $\zeta > 0$, such that for any $x \in B(\mathcal{M}^\sharp_\varepsilon, r)$ and $t \in [t_0, t_0 + \varepsilon^2\zeta]$, we have $|\widetilde{T}^{\sharp t}_\varepsilon(x) - \widetilde{T}^{\sharp t_0}_\varepsilon(x)| < \mu$.

An important point here is that we want ζ and μ to be independent of ε. First we write (4.3.88) as

$$\tilde{u}^\sharp_t = L^\sharp_{\varepsilon,P}\tilde{u}^\sharp + \tilde{r}_P(\tilde{u}^\sharp), \tag{4.3.96}$$

where for $x \in X^\sharp$, $\tilde{r}_P(x,t) = r_P(x) + \frac{1}{t_0} e^{L^\sharp_{\varepsilon,P} t} \psi^\sharp_\varepsilon(\tilde{P}) + r_P(\psi^\sharp_\varepsilon(\tilde{P})) - [\frac{1}{t_0} e^{L^\sharp_{\varepsilon,P} t} \psi^\sharp_\varepsilon(P) + r_P(\psi^\sharp_\varepsilon(P))]$.

Then by direct calculation, we have

$$
\begin{aligned}
&\widetilde{T}^{\sharp t}_\varepsilon(x) - \widetilde{T}^{\sharp t_0}_\varepsilon(x) \\
&= e^{L^\sharp_{\varepsilon,P} t} x - e^{L^\sharp_{\varepsilon,P} t_0} x + \int_0^t e^{L^\sharp_{\varepsilon,P}(t-s)} \tilde{r}(\widetilde{T}^{\sharp s}_\varepsilon(x)) ds - \int_0^{t_0} e^{L^\sharp_{\varepsilon,P}(t_0-s)} \tilde{r}(\widetilde{T}^{\sharp s}_\varepsilon(x)) ds \\
&= \int_{t_0}^t e^{L^\sharp_{\varepsilon,P}(t-s)} L^\sharp_{\varepsilon,P} e^{L^\sharp_{\varepsilon,P} t_0} x ds + \int_{t_0}^t e^{L^\sharp_{\varepsilon,P}(t-s)} \tilde{r}(\widetilde{T}^{\sharp s}_\varepsilon(x)) ds \\
&\quad + \int_0^{t_0} (e^{L^\sharp_{\varepsilon,P}(t-s)} - e^{L^\sharp_{\varepsilon,P}(t_0-s)}) \tilde{r}(\widetilde{T}^{\sharp s}_\varepsilon(x)) ds \qquad (4.3.97) \\
&= \int_{t_0}^t e^{L^\sharp_{\varepsilon,P}(t-s)} L^\sharp_{\varepsilon,P} e^{L^\sharp_{\varepsilon,P} t_0} x ds + \int_{t_0}^t e^{L^\sharp_{\varepsilon,P}(t-s)} L^\sharp_{\varepsilon,P} \int_0^{t_0} e^{L^\sharp_{\varepsilon,P}(t_0-\tau)} \tilde{r}(\widetilde{T}^{\sharp \tau}_\varepsilon(x)) d\tau ds + \\
&\quad \int_{t_0}^t e^{L^\sharp_{\varepsilon,P}(t-s)} \tilde{r}(\widetilde{T}^{\sharp s}_\varepsilon(x)) ds \\
&= \int_{t_0}^t e^{L^\sharp_{\varepsilon,P}(t-s)} [L^\sharp_{\varepsilon,P} \widetilde{T}^{\sharp t_0}_\varepsilon(x) + \tilde{r}(\widetilde{T}^{\sharp s}_\varepsilon(x))] ds.
\end{aligned}
$$

First note that $e^{L^\sharp_{\varepsilon,P} t}$ is an analytic semigroup, so

$$
\|(-L^\sharp_{\varepsilon,P})^{\frac{1}{2}} e^{L^\sharp_{\varepsilon,P} t_0}\| \le C(\frac{t_0}{\varepsilon^2})^{-\frac{1}{2}} e^{\frac{Ct_0}{\varepsilon^2}} = CK^{-\frac{1}{2}} e^{CK}.
$$

Observe that $|\tilde{r}_P(\cdot,t)|_{W^{1,q}_\varepsilon}$ is uniformly bounded on any bounded set in $X^\sharp$ for any time period, therefore $(-L^\sharp_{\varepsilon,P})^{\frac{1}{2}} \widetilde{T}^{\sharp t_0}_\varepsilon(\Gamma)$ is bounded for any bounded set $\Gamma \subset X^\sharp$. Again we use the fact that $\|(-L^\sharp_{\varepsilon,P})^{\frac{1}{2}} e^{L^\sharp_{\varepsilon,P}(t-s)}\| \le C(\frac{t-s}{\varepsilon^2})^{-\frac{1}{2}} e^{\frac{C(t-s)}{\varepsilon^2}}$ to obtain

$$
|\widetilde{T}^{\sharp t}_\varepsilon(x) - \widetilde{T}^{\sharp t_0}_\varepsilon(x)| \le C\zeta^{\frac{1}{2}} e^{C\zeta}, \qquad (4.3.98)
$$

which implies the weak uniform continuity required in (H5).

Therefore $W_\varepsilon^{cs\sharp}$ is locally invariant under $\widetilde{T}_\varepsilon^{\sharp t}$. Note that $T_\varepsilon^{\sharp t}(x) = \widetilde{T}_\varepsilon^{\sharp t}(x)$ if the base point $P(x)$ is l-away from the boundary of $\widetilde{\Omega}^k$, which implies that $W_\varepsilon^{cs\sharp}$ is locally invariant under $T_\varepsilon^{\sharp t}$. Similarly, we can construct a center-unstable manifold $W_\varepsilon^{cu\sharp}$ for $\widetilde{T}_\varepsilon^{\sharp t}$. The manifolds $W_\varepsilon^{cs\sharp}$ and $W_\varepsilon^{cu\sharp}$ are graphs over the stable bundle and unstable bundle, respectively, of $\mathcal{M}_\varepsilon^\sharp$, both having small Lipschitz constant. The intersection of these gives $\widetilde{\mathcal{M}}_\varepsilon^\sharp$, a locally invariant manifold for $T_\varepsilon^{\sharp t}$ in forward and backward time. Also, since $\widetilde{\mathcal{M}}_\varepsilon^\sharp$ is a graph over $\mathcal{M}_\varepsilon^\sharp$, it may be written as $\Psi_\varepsilon^\sharp(\widetilde{\Omega}^k)$ and $\Psi_\varepsilon^\sharp(P) - \psi_\varepsilon^\sharp(P) \in X_{\varepsilon,P}^s \oplus X_{\varepsilon,P}^u$. Furthermore, since the original measure of non-invariance, η, is of order $O(e^{-\frac{\nu\delta}{\varepsilon}})$, $\widetilde{\mathcal{M}}_\varepsilon^\sharp$ is in an $O(e^{-\frac{\nu\delta}{\varepsilon}})$ neighborhood of $\mathcal{M}_\varepsilon^\sharp$ in the $X^\sharp(W_\varepsilon^{3,q})$ topology. Then, by the injectivity of $A^{\frac{1}{2}}$, there exists a locally invariant manifold $\widetilde{\mathcal{M}}_\varepsilon = \Psi_\varepsilon(\widetilde{\Omega}^k)$ for T_ε^t, and $\widetilde{\mathcal{M}}_\varepsilon$ is in an $O(e^{-\frac{\nu\delta}{\varepsilon}})$ neighborhood of $\mathcal{M}_\varepsilon$ in the $W_\varepsilon^{2,q}$ sense. Also, from the fact that $A^{\frac{1}{2}}$ acting on any function in its domain gives a function with mean-value zero, the mass of each state in $\Psi_\varepsilon(P)$ is q_ε for any $P \in \widetilde{\Omega}^k$. However, T_ε^t is the semiflow generated by (4.3.44), where we modified the nonlinearity h. Notice that when q is large, $W_\varepsilon^{2,q}$ is embedded into $C^{0,\alpha}$, which indicates that $\widetilde{\mathcal{M}}_\varepsilon$ is in an $O(e^{-\frac{\nu\delta}{\varepsilon}})$ neighborhood of $\mathcal{M}_\varepsilon$ in the L^∞ sense. Therefore, $\widetilde{\mathcal{M}}_\varepsilon$ is locally invariant under the semiflow generated by the original equation (4.3.4). In summary, we have the main result:

Theorem 4.3.11. *For fixed δ, if ε is sufficiently small, then there exists a $C^m(\widetilde{\Omega}^k, X)$(m only depends on the smoothness of the nonlinearity and the boundary $\partial\Omega$) manifold $\widetilde{\mathcal{M}}_\varepsilon = \Psi_\varepsilon(\widetilde{\Omega}^k)$ which is locally invariant under the semiflow generated by (4.3.4). Furthermore, $\widetilde{\mathcal{M}}_\varepsilon$ lies in a $O(e^{-\frac{\nu\delta}{\varepsilon}})$ neighborhood of $\mathcal{M}_\varepsilon$ in the $L^\infty \cap W_\varepsilon^{2,q}$ sense.*

Remark 4.3.12. 1. The proof of Theorem 4.2 and the parallel version of the center-unstable manifolds in [19] indicates that $\Psi_\varepsilon - \psi_\varepsilon \in C^m(\widetilde{\Omega}^k, X)$ with bounded C^m norm, furthermore

$$|\Psi_\varepsilon - \psi_\varepsilon|_{C^0(\widetilde{\Omega}^k,X)} \le Ce^{-\frac{\nu\delta}{\varepsilon}}, \qquad \lim_{\varepsilon\to 0} |\Psi_\varepsilon - \psi_\varepsilon|_{C^1(\widetilde{\Omega}^k,X)} \to 0. \tag{4.3.99}$$

2. Roughly speaking, Theorem 4.3.11 yields that multi-spike states exist until the spikes attach to the boundary of the domain or they collide with each other . We will show in the next section that the motion of each spike is exponentially slow, so multi-spike states exist for very long positive and negative time.

4.4 Long time dynamics on $\widetilde{\mathcal{M}}_\varepsilon$

So far, we constructed a locally invariant manifold $\widetilde{\mathcal{M}}_\varepsilon$ of interior multi-spike states for (4.3.4), such that an interior multi-spike state maintains until a spike attaches to the boundary of the domain or a collision between spikes occurs. Now we investigate the dynamics on $\widetilde{\mathcal{M}}_\varepsilon$. By the invariance of $\widetilde{\mathcal{M}}_\varepsilon$, we have that for any $P \in \widetilde{\Omega}^k$, there exists $\tau_\varepsilon(P)$ such that

$$D\Psi_\varepsilon(P) \cdot (\varepsilon\tau_\varepsilon(P)) = -\Delta(\varepsilon^2\Delta\Psi_\varepsilon(P) - \mu\Psi_\varepsilon(P) + h(\Psi_\varepsilon(P))), \tag{4.4.1}$$

where D means the derivative with respect to P and $\varepsilon\tau_\varepsilon(P)$ is the velocity vector of all spikes. Here we include a factor of ε with τ_ε to eliminate the $\frac{1}{\varepsilon}$ in our calculations generated by differentiating Ψ_ε.

Let

$$\tilde{h}_\varepsilon(P) = \Psi_\varepsilon(P) - \psi_\varepsilon(P). \tag{4.4.2}$$

Note that

$$|\tilde{h}_\varepsilon(P)|_{C^0(\widetilde{\Omega}^k, X)} \leq Ce^{-\frac{\nu\delta}{\varepsilon}}, \text{ and } \lim_{\varepsilon\to 0} |\tilde{h}_\varepsilon(P)|_{C^1(\widetilde{\Omega}^k, X)} \to 0. \tag{4.4.3}$$

Recall that $\psi_\varepsilon(P) = W_{\varepsilon,P} - \widetilde{\mathcal{H}}(W_{\varepsilon,P})$, and rewrite (4.4.1) as

$$D\Psi_\varepsilon(P) \cdot \Big(\varepsilon\tau_\varepsilon(P)\Big) = \Big(D\psi_\varepsilon(P) + D\tilde{h}_\varepsilon(P)\Big) \cdot \Big(\varepsilon\tau_\varepsilon(P)\Big) = L_{\varepsilon,P}\tilde{h}_\varepsilon + \gamma_{\varepsilon,P}(x), \tag{4.4.4}$$

where

$$\gamma_{\varepsilon,P} = \Delta\Big(R_{\varepsilon,P}+\varepsilon^2\Delta\widetilde{\mathcal{H}}(W_{\varepsilon,P})-\mu\widetilde{\mathcal{H}}(W_{\varepsilon,P})-(h(W_{\varepsilon,P}-\widetilde{\mathcal{H}}(W_{\varepsilon,P})+\tilde{h}_\varepsilon(P))-h(W_{\varepsilon,P})-h'(\widetilde{W}_{\varepsilon,P})\tilde{h}_\varepsilon(P))\Big). \tag{4.4.5}$$

It is clear that

$$|\gamma_{\varepsilon,P}|_{L^q_\varepsilon(\Omega)} \le \frac{C}{\varepsilon^2}e^{-\frac{\nu\delta}{\varepsilon}}. \tag{4.4.6}$$

So far, we only know $\lim\limits_{\varepsilon\to 0}|D\tilde{h}_\varepsilon(P)|_X \to 0$. In order to get a refined estimate for $|D\tilde{h}_\varepsilon(P)|_X$, an expression for $A^{-\frac{1}{2}}\tilde{h}_\varepsilon(P)$ should be obtained from (4.3.52) by using the normal hyperbolicity. Reducing the equation (4.3.52) to the invariant manifold $\widetilde{\mathcal{M}}^\sharp_\varepsilon$, we get

$$D\Psi^\sharp_\varepsilon(P)\cdot\Big(\varepsilon\tau_\varepsilon(P)\Big) = L^\sharp_{\varepsilon,P}\tilde{h}_\varepsilon(P) + \gamma^\sharp_{\varepsilon,P}(x). \tag{4.4.7}$$

Clearly,

$$|\gamma^\sharp_{\varepsilon,P}(x)|_{L^q_\varepsilon(\Omega)} \le \frac{C}{\varepsilon}e^{-\frac{\nu\delta}{\varepsilon}}.$$

Our purpose is to get an expression for $\tilde{h}^\sharp_\varepsilon(P)$ without involving $D\Psi^\sharp_\varepsilon(P)\cdot(\varepsilon\tau_\varepsilon(P))$, so we consider the following decomposition

$$L^q_\varepsilon = X^\perp_{\varepsilon,P} \oplus T_{\Psi^\sharp_\varepsilon(P)}\widetilde{\mathcal{M}}^\sharp_\varepsilon,$$

where $T_{\Psi^\sharp_\varepsilon(P)}\widetilde{\mathcal{M}}^\sharp_\varepsilon = \{D\Psi^\sharp_\varepsilon(P)\cdot\tau : \tau \in \mathbb{R}^{nk}\}$ and $X^\perp_{\varepsilon,P} = \{v \in L^q_\varepsilon(\Omega) : \int_\Omega vD\Psi^\sharp_\varepsilon(P)\cdot\tau dx = 0 \text{ for any } \tau\in\mathbb{R}^{nk}\}$. The corresponding projection map $\Pi^\perp_{\varepsilon,P}$ for $X^\perp_{\varepsilon,P}$ is

$$\begin{aligned}\Pi^\perp_{\varepsilon,P} &= I - D\Psi^\sharp_\varepsilon(P)\Big(D\psi^\sharp_\varepsilon(P) + \Pi^c_{\varepsilon,P}D\tilde{h}^\sharp_\varepsilon(P)\Big)^{-1}\Pi^c_{\varepsilon,P} \\ &= \Big(I-\Pi^c_{\varepsilon,P}\Big)\Big(I - D\tilde{h}^\sharp_\varepsilon(P)\Big(D\psi^\sharp_\varepsilon(P) + \Pi^c_{\varepsilon,P}D\tilde{h}^\sharp_\varepsilon(P)\Big)^{-1}\Pi^c_{\varepsilon,P}\Big).\end{aligned} \tag{4.4.8}$$

By (4.4.3), one can verify that $\Pi^{\perp}_{\varepsilon,P}$ is well-defined and uniformly bounded for any P and small enough ε. Since $L^{\sharp}_{\varepsilon,p}$ almost leaves each subspace $X^{\alpha}_{\varepsilon,P}$, $\alpha = c, s, u$ invariant, and $T_{\Psi^{\sharp}_{\varepsilon}(P)}\widetilde{\mathcal{M}}^{\sharp}_{\varepsilon}$ is very close to $X^{c}_{\varepsilon,P}$, it is clear that there exists $C > 0$, independent of P and ε, such that

$$|(\Pi^{\perp}_{\varepsilon,P}L^{\sharp}_{\varepsilon,p}|_{X^{s}_{\varepsilon,P}\oplus X^{u}_{\varepsilon,P}\cap D(L^{\sharp}_{\varepsilon,p})})^{-1}|_{L(X^{\perp}_{\varepsilon,P},W^{4,q}_{\varepsilon}(\Omega)\cap D(L^{\sharp}_{\varepsilon,P}))} \leq C\varepsilon^{2}. \tag{4.4.9}$$

Applying $\Pi^{\perp}_{\varepsilon,P}$ to (4.4.7) and using the fact that $\tilde{h}^{\sharp}_{\varepsilon}(P) \in X^{s}_{\varepsilon,P} \oplus X^{u}_{\varepsilon,P}$, we have

$$\tilde{h}^{\sharp}_{\varepsilon}(P) = (\Pi^{\perp}_{\varepsilon,P}L^{\sharp}_{\varepsilon,p}|_{X^{s}_{\varepsilon,P}\oplus X^{u}_{\varepsilon,P}\cap D(L^{\sharp}_{\varepsilon,p})})^{-1}\Pi^{\perp}_{\varepsilon,P}(\gamma^{\sharp}_{\varepsilon,P}). \tag{4.4.10}$$

By direct computation, one can check that for some C independent of P and ε,

$$|D\Pi^{\perp}_{\varepsilon,P}|_{L(L^{q}_{\varepsilon})} \leq C, \quad |D\gamma^{\sharp}_{\varepsilon,P}|_{L^{q}_{\varepsilon}} \leq \frac{C}{\varepsilon}e^{-\frac{\nu\delta}{\varepsilon}}.$$

Therefore, it follows immediately from (4.4.10) that

$$|D\tilde{h}^{\sharp}_{\varepsilon}(P)|_{W^{4,q}_{\varepsilon}} \leq C\varepsilon e^{-\frac{\nu\delta}{\varepsilon}}, \tag{4.4.11}$$

which implies that

$$|D\tilde{h}_{\varepsilon}(P)|_{W^{3,q}_{\varepsilon}} \leq C\varepsilon e^{-\frac{\nu\delta}{\varepsilon}}. \tag{4.4.12}$$

We now derive an equation for $\tau_{\varepsilon,P}$. Since $\widetilde{\mathcal{M}}_{\varepsilon}$ is constructed with all points having equal mass, one has $\int_{\Omega} D\Psi_{\varepsilon}(P)dx = 0$. Then (4.4.1) can be rewritten as

$$\begin{aligned}\Big((-\Delta)^{-1}(D\psi_{\varepsilon}(P) + D\tilde{h}_{\varepsilon}(P))\Big)\cdot(\varepsilon\tau_{\varepsilon}(P)) &= \varepsilon^{2}\Delta\Psi_{\varepsilon}(P) - \mu\Psi_{\varepsilon}(P) + h(\Psi_{\varepsilon}(P)) + \rho_{\varepsilon} \\ &= \Lambda_{\varepsilon,P}\tilde{h}_{\varepsilon}(P) + \bar{\gamma}_{\varepsilon,P},\end{aligned} \tag{4.4.13}$$

where $\rho_\varepsilon = -\frac{1}{|\Omega|}\int_\Omega[\varepsilon^2\Delta\Psi_\varepsilon(P) - \mu\Psi_\varepsilon(P) + h(\Psi_\varepsilon(P))]dx$ is added to make the right hand side of mean value zero as required by our definition of $(-\Delta)^{-1}$ and $\Lambda_{\varepsilon,P}v := \varepsilon^2\Delta v - \mu v + h'(\psi_\varepsilon(P))v$ and $\bar{\gamma}_{\varepsilon,P} = -(R_{\varepsilon,P} + \varepsilon^2\Delta\widetilde{\mathcal{H}}(W_{\varepsilon,P}) - \mu\widetilde{\mathcal{H}}(W_{\varepsilon,P}) - (h(W_{\varepsilon,P} - \widetilde{\mathcal{H}}(W_{\varepsilon,P}) + \tilde{h}_\varepsilon(P)) - h(W_{\varepsilon,P}) - h'(\widetilde{W}_{\varepsilon,P})\tilde{h}_\varepsilon(P)) + \rho_\varepsilon$. As in Section 4.3.5, $(-\Delta)^{-1}$ acting on a mean-value zero function v is defined to be the mean-value zero solution of $-\Delta\eta = v$ with the homogeneous Neumann boundary condition. Note that $\varepsilon^2\Delta\Psi_\varepsilon(P) - \mu\Psi_\varepsilon(P) + h(\Psi_\varepsilon(P)) = O(e^{-\frac{\nu\delta}{\varepsilon}})$, so $\rho_\varepsilon = O(e^{-\frac{\nu\delta}{\varepsilon}})$.

Recall that $\psi_\varepsilon(P) = \sum_{1\le i\le k} w_{\varepsilon,p_i} - \widetilde{\mathcal{H}}$, so direct computation yields

$$D\psi_\varepsilon(P) = (-\nabla w_{\varepsilon,p_1}, \cdots, -\nabla w_{\varepsilon,p_k}) - D\widetilde{\mathcal{H}},$$

where ∇ means the derivatives with respect to x. Write $P = (p_1, \cdots, p_k)$ and $p_i = (p_{i1}, \cdots, p_{in})$, then let

$$\begin{aligned}\Theta_{ij}^{lm} &= \langle(-\Delta)^{-1}\varepsilon D_{p_{ij}}\Psi_{\varepsilon,P}, -\varepsilon\nabla_{x_m}w_{\varepsilon,p_l}\rangle \\ &= \langle(-\Delta)^{-1}(\varepsilon\nabla_{x_j}w_{\varepsilon,p_i} - q_\varepsilon^{ij}), \varepsilon\nabla_{x_m}w_{\varepsilon,p_l}\rangle + \langle(-\Delta)^{-1}(q_\varepsilon^{ij} + \varepsilon D_{p_{ij}}\widetilde{\mathcal{H}} - \varepsilon D_{p_{ij}}\tilde{h}_\varepsilon), \varepsilon\nabla_{x_m}w_{\varepsilon,p_l}\rangle,\end{aligned} \tag{4.4.14}$$

where $q_\varepsilon^{ij} = \frac{1}{|\Omega|}\int_\Omega \varepsilon\nabla_{x_j}w_{\varepsilon,p_i}dx = -\frac{1}{|\Omega|}\int_\Omega \varepsilon D_{p_{ij}}\widetilde{\mathcal{H}} - \varepsilon D_{p_{ij}}\tilde{h}_\varepsilon dx$.

Clearly, $q_\varepsilon^{ij} = O(e^{-\frac{\nu\delta}{\varepsilon}})$ since $\int_\Omega \varepsilon D_{ij}\widetilde{\mathcal{H}} - \varepsilon D_{ij}\tilde{h}_\varepsilon dx = O(e^{-\frac{\nu\delta}{\varepsilon}})$. Before we estimate Θ_{ij}^{lm}, we prove the following lemma. Here we want to remind the readers that $\langle\cdot,\cdot\rangle$ is still the L^2_ε inner product defined as $\langle f, g\rangle = \int_\Omega fg\varepsilon^{-n}dx$.

Lemma 4.4.1. *If $f(x) : \mathbb{R}^n \to \mathbb{R}$ satisfies $|f(x)| + |D_if(x)| \le Ke^{-k|x|}$ for any $x \in \mathbb{R}^n$, then $|S(D_if)(x)| := |c_n\int_{\mathbb{R}^n}\frac{(D_if)(y)}{|x-y|^{n-2}}dy| \le \frac{C}{1+|x|^{n-1}}$ for any $x \in \mathbb{R}^n$, where $c_n = \frac{1}{n(n-2)\omega_n}$ with ω_n the volume of the unit ball in $\mathbb{R}^n$.*

Proof. Let B_r be the ball with radius $r > 0$ centered at origin. We write

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{(D_i f)(y)}{|x-y|^{n-2}} dy &= \int_{B_R} \frac{(D_i f)(y)}{|x-y|^{n-2}} dy + \int_{\mathbb{R}^n \backslash B_R} \frac{(D_i f)(y)}{|x-y|^{n-2}} dy \\ &= I_R + J_R. \end{aligned} \tag{4.4.15}$$

For fixed x and any $y \in \mathbb{R}^n \backslash B_R$ with R large enough, we have

$$|y| \geq |x-y| - |x|,$$

$$|x-y| \geq R - |x|.$$

It follows that

$$|J_R| \leq K e^{k|x|} \int_{\mathbb{R}^n \backslash B_R} \frac{e^{-|x-y|}}{|x-y|^{n-2}} dy \leq K e^{k|x|} \int_{R-|x|}^{\infty} e^{-kr} r dr = \frac{K}{k} e^{2k|x|} e^{-2kR} (\frac{1}{k} + R - |x|), \tag{4.4.16}$$

which indicates that

$$\lim_{R \to \infty} J_R = 0. \tag{4.4.17}$$

Write $y = \eta + y_i e_i$ where $\eta = \sum_{j \neq i} y_j e_j$ and e_j is the canonical basis of $\mathbb{R}^n$. Then, with an integration by parts, we have

$$\begin{aligned} I_R &= \int_{|\eta|<R} \int_{-\sqrt{R^2-|\eta|^2}}^{\sqrt{R^2-|\eta|^2}} \frac{(D_i f)(\eta + y_i e_i)}{|\eta + y_i e_i - x|^{n-2}} dy_i d\eta \\ &= \int_{|\eta|<R} \frac{f(\eta + \sqrt{R^2-|\eta|^2} e_i)}{|\eta + \sqrt{R^2-|\eta|^2} e_i - x|^{n-2}} - \frac{f(\eta - \sqrt{R^2-|\eta|^2} e_i)}{|\eta - \sqrt{R^2-|\eta|^2} e_i - x|^{n-2}} d\eta \\ &\quad + (n-2) \int_{|\eta|<R} \int_{-\sqrt{R^2-|\eta|^2}}^{\sqrt{R^2-|\eta|^2}} \frac{f(\eta + y_i e_i)(\eta + y_i e_i - x) \cdot e_i}{|\eta + y_i e_i - x|^n} dy_i \\ &= I_R^a + I_R^b. \end{aligned} \tag{4.4.18}$$

Since $|f(\eta \pm y_i e_i)| \leq Ke^{-kR}$ and $|\eta \pm y_i e_i - x| \geq |R| - |x|$, one has

$$I_R^a \leq 2K \frac{e^{-kR}}{(R - |x|)^{n-2}} \omega_n R^{n-1}, \tag{4.4.19}$$

and so

$$\lim_{R \to \infty} I_R^a = 0. \tag{4.4.20}$$

By direct computation, we have

$$\begin{aligned} |I_R^b| &= (n-2) \Big| \int_{B_R} \frac{f(y)(y-x) \cdot e_i}{|y-x|^n} dy \Big| \\ &\leq (n-2) \int_{B_R} \frac{|f(y)|}{|y-x|^{n-1}} dy \leq (n-2) \int_{\mathbb{R}^n} \frac{|f(y)|}{|y-x|^{n-1}} dy. \end{aligned} \tag{4.4.21}$$

Combining (4.4.17), (4.4.20) and (4.4.21), we have

$$\Big| \int_{\mathbb{R}^n} \frac{(D_i f)(y)}{|x-y|^{n-2}} dy \Big| \leq (n-2) \int_{\mathbb{R}^n} \frac{|f(y)|}{|y-x|^{n-1}} dy. \tag{4.4.22}$$

We now estimate $\int_{\mathbb{R}^n} \frac{|f(y)|}{|y-x|^{n-1}} dy$. We first consider the case $|x| \leq 1$. Since $|y| \geq |x-y| - |x|$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}} dy &\leq K \int_{\mathbb{R}^n} \frac{e^{-k|y|}}{|x-y|^{n-1}} dy \\ &\leq Ke^{k|x|} \int_{\mathbb{R}^n} \frac{e^{-k|x-y|}}{|x-y|^{n-1}} dy \\ &\leq Ke^k \int_{\mathbb{R}^n} \frac{e^{-k|x-y|}}{|x-y|^{n-1}} dy \\ &= C_0, \end{aligned} \tag{4.4.23}$$

where $C_0 = Ke^k \int_{\mathbb{R}^n} \frac{e^{-k|y|}}{|y|^{n-1}} dy$. Assume now $|x| > 1$, let $B_r(x)$ be the ball centered at x with radius

r, then write

$$\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}} dy = \int_{B_{|x|/2}(x)} \frac{|f(y)|}{|x-y|^{n-1}} dy + \int_{\mathbb{R}^n \setminus B_{|x|/2}(x)} \frac{|f(y)|}{|x-y|^{n-1}} dy = I + II. \tag{4.4.24}$$

We estimate

$$\begin{aligned} I &\leq K e^{-k\frac{|x|}{2}} \int_{B_{|x|/2}(x)} \frac{1}{|x-y|^{n-1}} dy \\ &= \omega_n K e^{-k\frac{|x|}{2}} \frac{|x|}{2} = \frac{1}{2} \omega_n K |x|^n e^{-k\frac{|x|}{2}} |x|^{-(n-1)} \\ &\leq C|x|^{-(n-1)}, \end{aligned} \tag{4.4.25}$$

and

$$\begin{aligned} II &\leq (\frac{|x|}{2})^{-(n-1)} \int_{\mathbb{R}^n \setminus B_{|x|/2}(x)} |f(y)| dy \\ &\leq (\frac{|x|}{2})^{-(n-1)} \int_{\mathbb{R}^n} |f(y)| dy \\ &\leq C|x|^{-(n-1)}. \end{aligned} \tag{4.4.26}$$

Then, the desired result follows directly. □

Lemma 4.4.2. *Assume* $\Omega \subset \mathbb{R}^n$ *is a smooth bounded connected open set. Let* $\hat{L}^2(\Omega)$ *be the subset of* $L^2(\Omega)$ *of the functions with mean-value zero. Then, for each* $f \in \hat{L}^2$*, there is a unique solution* $\bar{u}$ *with mean-value zero of the problem*

$$\begin{cases} -\Delta u = f(x), & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases}$$

Moreover, there exits a Neumann function (sometimes called second type of Green's function)

$N : \Omega\times\Omega\backslash\{(x,x) : x\in\Omega\}\to\mathbb{R}$ such that

$$\bar{u}(x) = ((-\Delta)^{-1}f)(x) = \int_\Omega N(x,y)f(y)dy. \tag{4.4.27}$$

For $n\geq 3$, the Neumann function is of the form

$$N(x,y) = G(x,y) + \phi(x,y), \tag{4.4.28}$$

where $G(x,y) = \frac{|x-y|^{2-n}}{n(n-2)\omega_n}$ and $\phi(x,y) : \Omega\times\Omega\to\mathbb{R}$ is smooth.

Proof. One can find the proof of this Lemma in [85], where they construct an Neumann function for $n = 3$. The same argument is valid in general.

□

Lemma 4.4.3.

$$\begin{aligned}
\Theta_{ij}^{lm} &= C^*\varepsilon^2 + O(\varepsilon^{n+1}), \quad \textit{if } i = l, j = m,\\
\Theta_{ij}^{lm} &= O(\varepsilon^{n+1}), \quad \textit{if } i = l, j\neq m,\\
\Theta_{ij}^{lm} &= O(\frac{\varepsilon^{n+1}}{\delta^{n-1}}), \quad \textit{if } i\neq l,
\end{aligned} \tag{4.4.29}$$

where C^* is a positive constant defined below.

Proof. Since $N(x,y)\sim|x-y|^{2-n}$, we have $\int_\Omega\int_\Omega|N(x,y)|dydx\leq C$. Combining this with the fact that $q_\varepsilon^{ij}, \varepsilon D_{p_{ij}}\widetilde{\mathcal{H}}, \varepsilon D_{p_{ij}}\tilde{h}_\varepsilon) = O(e^{-\frac{\gamma\delta}{\varepsilon}})$, it follows that we can ignore the contribution of these terms. Therefore it is sufficient to estimate

$$\int_\Omega N(x,y)\varepsilon\nabla_{x_j}w_{\varepsilon,p_i}(y)dy\varepsilon\nabla_{x_m}w_{\varepsilon,p_l}(x)dx. \tag{4.4.30}$$

Let $\nabla_i w(x) = \frac{\partial w}{\partial x_i}$, then one has $\varepsilon \nabla_{x_j} w_{\varepsilon,p_i}(y)dy = \nabla_j w(\frac{x-p_i}{\varepsilon})$. By Lemma 4.4.2, we have

$$\begin{aligned}\int_\Omega N(x,y)\varepsilon\nabla_{x_j} w_{\varepsilon,p_i}(y)dy &= \int_\Omega N(x,y)\nabla_j w(\frac{y-p_i}{\varepsilon})dy \\ &= c_n \int_\Omega \frac{1}{|x-y|^{n-2}}\nabla_j w(\frac{y-p_i}{\varepsilon})dy + \int_\Omega \phi(x,y)\nabla_j w(\frac{y-p_i}{\varepsilon})dy \\ &= I_{ij} + J_{ij}.\end{aligned} \tag{4.4.31}$$

Changing variables by setting $y = \varepsilon z + p_i$, we have

$$\begin{aligned}I_{ij} &= c_n\varepsilon^2 \int_{\Omega_\varepsilon^i} \frac{\nabla_j w(z)}{|\frac{x-p_i}{\varepsilon} - z|^{n-2}} dz \\ &= c_n\varepsilon^2 \int_{\mathbb{R}^n} \frac{\nabla_j w(z)}{|\frac{x-p_i}{\varepsilon} - z|^{n-2}} dz - c_n\varepsilon^2 \int_{\mathbb{R}^n\setminus\Omega_\varepsilon^i} \frac{\nabla_j w(z)}{|\frac{x-p_i}{\varepsilon} - z|^{n-2}} dz \\ &= I_{ij}^a + I_{ij}^b,\end{aligned} \tag{4.4.32}$$

where $\Omega_\varepsilon^i = \{z : p_i + \varepsilon z \in \Omega\}$. We estimate

$$\begin{aligned}|I_{ij}^b| &\le c_n\varepsilon^2 \int_{\mathbb{R}^n\setminus\Omega_\varepsilon^i \cap\{|\frac{x-p_i}{\varepsilon}-z|\le 1\}} \frac{|\nabla_j w(z)|}{|\frac{x-p_i}{\varepsilon} - z|^{n-2}} dz + c_n\varepsilon^2 \int_{\mathbb{R}^n\setminus\Omega_\varepsilon^i \cap\{|\frac{x-p_i}{\varepsilon}-z|>1\}} \frac{|\nabla_j w(z)|}{|\frac{x-p_i}{\varepsilon} - z|^{n-2}} dz \\ &\le C\varepsilon^2 e^{-\frac{\nu\delta}{\varepsilon}} \int_{|\xi|\le 1} \frac{1}{|\xi|^{n-2}} d\xi + C_n\varepsilon^2 \int_{|z|>\frac{\delta}{2\varepsilon}} |\nabla_j w(z)| dz \\ &= O(e^{-\frac{\nu\delta}{\varepsilon}}).\end{aligned} \tag{4.4.33}$$

To estimate J_{ij}, we write

$$\begin{aligned} J_{ij} &= \int_{B_{\frac{\delta}{2}}(p_i)} \phi(x,y)\nabla_j w(\frac{y-p_i}{\varepsilon})dy + \int_{\Omega\setminus B_{\frac{\delta}{2}}(p_i)} \phi(x,y)\nabla_j w(\frac{y-p_i}{\varepsilon})dy \\ &= J_{ij}^a + J_{ij}^b. \end{aligned} \tag{4.4.34}$$

Since $\nabla_j w(x)$ is odd in x_j, we have $\int_{B_{\frac{\delta}{2}}(p_i)} \nabla_j w(\frac{y-p_i}{\varepsilon})dy = 0$. It follows that

$$\begin{aligned} |J_{ij}^a| &= |\int_{B_{\frac{\delta}{2}}(0)} (\phi(x,z+p_i) - \phi(x,p_i))\nabla_j w(\frac{z}{\varepsilon})dz| \\ &\leq C\int_{B_{\frac{\delta}{2}}(0)} |z|e^{-\frac{C|z|}{\varepsilon}}dz \\ &= Cnw_n\varepsilon^{n+1}\int_0^{\frac{\delta}{2\varepsilon}} s^n e^{-Cs}ds \\ &\leq C_\delta\varepsilon^{n+1}. \end{aligned} \tag{4.4.35}$$

Also, one can estimate

$$|J_{ij}^b| \leq Ce^{-\frac{\nu\delta}{\varepsilon}}\int_\Omega |\phi(x,y)|dy = O(e^{-\frac{\nu\delta}{\varepsilon}}). \tag{4.4.36}$$

Let $N_{ij}(x) = \int_\Omega N(x,y)\varepsilon\nabla_{x_j} w_{\varepsilon,p_i}(y)dy = \int_\Omega N(x,y)\nabla_j w(\frac{y-p_i}{\varepsilon})dy$. For $i \neq l$, recall that $|p_i - p_l| > \delta$ and $d(p_l,\partial\Omega) \geq \frac{\delta}{2}$, therefore, for any $x \in B_{\frac{\delta}{2}}(p_l)$, we have $|x - p_i| \geq \frac{\delta}{2}$. From this and using Lemma 4.4.1, we have

$$
\begin{aligned}
&|\int_{\Omega} N_{ij}(x)\nabla_m w(\frac{x-p_l}{\varepsilon})dx| \\
&\le |\int_{B_{\frac{\delta}{2}}(p_l)} N_{ij}(x)\nabla_m w(\frac{x-p_l}{\varepsilon})dx| + |\int_{\Omega\backslash B_{\frac{\delta}{2}}(p_l)} N_{ij}(x)\nabla_m w(\frac{x-p_l}{\varepsilon})dx| \\
&\le C\varepsilon^2(\frac{\varepsilon}{\delta})^{n-1}\int_{B_{\frac{\delta}{2}}(p_l)} |\nabla_m w(\frac{x-p_l}{\varepsilon})|dx + Ce^{-\frac{\nu\delta}{\varepsilon}}\int_{\Omega\backslash B_{\frac{\delta}{2}}(p_l)} |N_{ij}(x)|dx \\
&\le \frac{C}{\delta^{n-1}}\varepsilon^{2n+1} + Ce^{-\frac{\nu\delta}{\varepsilon}}(\int_{B_{\frac{\delta}{2}}(p_i)} |N_{ij}(x)|dx + \int_{\Omega\backslash(B_{\frac{\delta}{2}}(p_l)\cup B_{\frac{\delta}{2}}(p_i)} |N_{ij}(x)|dx \\
&\le \frac{C}{\delta^{n-1}}\varepsilon^{2n+1} + Ce^{-\frac{\nu\delta}{\varepsilon}}\varepsilon^{n+1} + \frac{C}{\delta^{n-1}}e^{-\frac{\nu\delta}{\varepsilon}}\varepsilon^{n+1} \\
&= O(\frac{\varepsilon^{2n+1}}{\delta^{n-1}}).
\end{aligned} \tag{4.4.37}
$$

For $i = l$, $j \neq m$, we start with

$$
\begin{aligned}
|\int_{\Omega} N_{ij}(x)\nabla_m w(\frac{x-p_l}{\varepsilon})dx| \le |\int_{\Omega} I^a_{ij}\nabla_m w(\frac{x-p_l}{\varepsilon})dx| + |\int_{\Omega}(I^b_{ij}+J^b_{ij})\nabla_m w(\frac{x-p_l}{\varepsilon})dx| \\
+ |\int_{\Omega} J^a_{ij}\nabla_m w(\frac{x-p_l}{\varepsilon})dx|.
\end{aligned}
$$

Since $I^a_{ij}(x+p_i)$ is odd in x_j and $\nabla_m w$ is odd in x_m, we have

$$
\begin{aligned}
|\int_{\Omega} I^a_{ij}\nabla_m w(\frac{x-p_l}{\varepsilon})dx| &= |\int_{B_{\frac{\delta}{2}}(p_i)} I^a_{ij}\nabla_m w(\frac{x-p_l}{\varepsilon})dx + \int_{\Omega\backslash B_{\frac{\delta}{2}}(p_i)} I^a_{ij}\nabla_m w(\frac{x-p_l}{\varepsilon})dx| \\
&= |\int_{\Omega\backslash B_{\frac{\delta}{2}}(p_i)} I^a_{ij}\nabla_m w(\frac{x-p_l}{\varepsilon})dx| \\
&\le C\frac{\varepsilon^{n+1}}{\delta^{n-1}}e^{-\frac{\nu\delta}{\varepsilon}}.
\end{aligned} \tag{4.4.38}
$$

One can also easily check that

$$
\begin{aligned}
&|\int_\Omega (I^b_{ij} + J^b_{ij})\nabla_m w(\frac{x-p_l}{\varepsilon})dx| \le Ce^{-\frac{\nu\delta}{\varepsilon}}\varepsilon^n,\\
&|\int_\Omega J^a_{ij}\nabla_m w(\frac{x-p_l}{\varepsilon})dx| \le C\varepsilon^{2n+1}.
\end{aligned}
\tag{4.4.39}
$$

Therefore, we have for $i = l$, $j \neq m$,

$$
\int_\Omega N_{ij}(x)\nabla_m w(\frac{x-p_l}{\varepsilon})dx = O(\varepsilon^{2n+1}). \tag{4.4.40}
$$

Finally, we consider the case $i = l$, $j = m$. We write

$$
\begin{aligned}
\int_\Omega N_{ij}(x)\nabla_j w(\frac{x-p_i}{\varepsilon})dx = &\int_\Omega I^a_{ij}\nabla_j w(\frac{x-p_i}{\varepsilon})dx + \int_\Omega (I^b_{ij} + J^b_{ij})\nabla_j w(\frac{x-p_i}{\varepsilon})dx\\
&+ \int_\Omega J^a_{ij}\nabla_j w(\frac{x-p_i}{\varepsilon})dx.
\end{aligned}
\tag{4.4.41}
$$

We compute

$$
\int_\Omega I^a_{ij}\nabla_j w(\frac{x-p_i}{\varepsilon})dx = \int_{B_{\frac{\delta}{2}}(p_i)} I^a_{ij}\nabla_j w(\frac{x-p_i}{\varepsilon})dx + \int_{\Omega\setminus B_{\frac{\delta}{2}}(p_i)} I^a_{ij}\nabla_j w(\frac{x-p_i}{\varepsilon})dx. \tag{4.4.42}
$$

One can check by direct computation that

$$
\begin{aligned}
\int_{B_{\frac{\delta}{2}}(p_i)} I^a_{ij}\nabla_j w(\frac{x-p_i}{\varepsilon})dx &= c_n\varepsilon^2 \int_{B_{\frac{\delta}{2}}(p_i)}\int_{\mathbb{R}^n} \frac{\nabla_j w(z)}{|\frac{x-p_i}{\varepsilon} - z|^{n-2}}dz\nabla_j w(\frac{x-p_i}{\varepsilon})dx\\
&= c_n\varepsilon^{n+2}\int_{B_{\frac{\delta}{2\varepsilon}}(0)}\int_{\mathbb{R}^n}\frac{\nabla_j w(z)}{|y-z|^{n-2}}dz\nabla_j w(y)dy\\
&= C^*\varepsilon^{n+2},
\end{aligned}
\tag{4.4.43}
$$

where $C^* = c_n \int_{B_{\frac{\delta}{2\varepsilon}}(0)} \int_{\mathbb{R}^n} \frac{\nabla_j w(z)}{|y-z|^{n-2}} dz \nabla_j w(y) dy$. Also we have

$$
\begin{aligned}
&|\int_\Omega (I_{ij}^b + J_{ij}^b) \nabla_j w(\frac{x-p_i}{\varepsilon}) dx| \leq C e^{-\frac{\nu\delta}{\varepsilon}} \varepsilon^n, \\
&|\int_\Omega J_{ij}^a \nabla_j w(\frac{x-p_i}{\varepsilon}) dx| \leq C \varepsilon^{2n+1}.
\end{aligned}
\tag{4.4.44}
$$

Then the desired result follows directly.

□

Taking the inner product of (4.4.13) with $-\nabla_{x_j} w_{\varepsilon,p_i}$ for $i = 1, \cdots, k$ and $j = 1, \cdots, n$, we get a linear system consisting of nk equations which can be written as

$$
\Theta_{\varepsilon,P} \cdot \tau_{\varepsilon,P} = \Gamma_{\varepsilon,P}, \tag{4.4.45}
$$

where

$$
\Theta_{\varepsilon,P} = \begin{bmatrix}
\Theta_{11}^{11} & \cdots & \Theta_{1n}^{11} & \cdots & \cdots & \Theta_{k1}^{11} & \cdots & \Theta_{kn}^{11} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Theta_{11}^{1n} & \cdots & \Theta_{1n}^{1n} & \cdots & \cdots & \Theta_{k1}^{1n} & \cdots & \Theta_{kn}^{1n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Theta_{11}^{k1} & \cdots & \Theta_{1n}^{k1} & \cdots & \cdots & \Theta_{k1}^{k1} & \cdots & \Theta_{kn}^{k1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Theta_{11}^{kn} & \cdots & \Theta_{1n}^{kn} & \cdots & \cdots & \Theta_{k1}^{kn} & \cdots & \Theta_{kn}^{kn}
\end{bmatrix},
$$

and

$$\Gamma_{\varepsilon,P} = \begin{bmatrix} \langle \Lambda_{\varepsilon,P}\tilde{h}_{\varepsilon}(P) + \bar{\gamma}_{\varepsilon,P}, -\varepsilon\nabla_{x_1} w_{\varepsilon,p_1}\rangle \\ \vdots \\ \langle \Lambda_{\varepsilon,P}\tilde{h}_{\varepsilon}(P) + \bar{\gamma}_{\varepsilon,P}, -\varepsilon\nabla_{x_n} w_{\varepsilon,p_1}\rangle \\ \vdots \\ \vdots \\ \langle \Lambda_{\varepsilon,P}\tilde{h}_{\varepsilon}(P) + \bar{\gamma}_{\varepsilon,P}, -\varepsilon\nabla_{x_1} w_{\varepsilon,p_k}\rangle \\ \vdots \\ \langle \Lambda_{\varepsilon,P}\tilde{h}_{\varepsilon}(P) + \bar{\gamma}_{\varepsilon,P}, -\varepsilon\nabla_{x_n} w_{\varepsilon,p_k}\rangle \end{bmatrix}.$$

Since $\Lambda_{\varepsilon,P}$ is self-adjoint and $\bar{\gamma}_{\varepsilon,P} = O(e^{-\frac{\nu\delta}{\varepsilon}})$, one immediately has

$$\Gamma_{\varepsilon,P} = O(e^{-\frac{\nu\delta}{\varepsilon}}).$$

By Lemma (4.4.3), one finds that $\Theta_{\varepsilon,P}$ is a diagonally dominant matrix, which implies that $\Theta_{\varepsilon,P}$ is invertible. Thus $\tau_{\varepsilon,P}$ can be expressed as

$$\tau_{\varepsilon,P} = (\Theta_{\varepsilon,P})^{-1}\Gamma_{\varepsilon,P}, \tag{4.4.46}$$

and so

$$|\tau_{\varepsilon}(P)| = O(\varepsilon^{-2}e^{-\frac{\nu\delta}{\varepsilon}}). \tag{4.4.47}$$

Remark 4.4.4. Once the matrix $\Theta_{\varepsilon,P}$ has been computed, equation (4.4.46) determines the complete dynamics of the spikes on the invariant manifold. Although entries of $\Gamma_{\varepsilon,P}$ may have different exponential rates, each entry of the matrix $\Theta_{\varepsilon,P}$ is of order ε to some power, therefore $\Theta_{\varepsilon,P}$ mixes all the entries of $\Gamma_{\varepsilon,P}$ together, which yields that the interior multi-spike dynamics of the Cahn-

Hilliard equation has a global character where not only the closest spikes interact but each spike interacts with all the others and with the boundary.

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