CHOICE OF PERFORMANCE INDEX FOR THE OPTIMAL CONTROL OF MACROECONOMIC SYSTEMS

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ABSTRACT

CHOICE OF PERFORMANCE INDEX FOR THE OPTIMAL CONTROL OF MACROECONOMIC SYSTEMS

By

Jong-goo Park

The specification of a performance criterion is one of the basic requirements for the formal analysis of quantitative decision problems. In macroeconomic policy problems, in particular, in order to analyze the quantitative impact of a given set of policy decisions, the procedure should be based on the policy maker's preference regarding the developments in his economy. Such preferences are specified by a performance index in the analysis.

The purpose of this thesis is to analyze the selection of the performance index by solving the inverse optimal control problem. When a closed-loop system with a known control policy is given, the inverse optimal control problem is to determine performance indices for which the given control rule is optimum.

A sufficient condition is developed for the solution of the inverse problem for a linear discrete-time multi-control regulatory process (the main concern of the regulator problem is to keep the state variables near a fixed target zero), with a quadratic performance index. An explicit solution is obtained for a special case where the performance functional does not contain the control variables, and the result is further extended to a linear tracking

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problem (an important feature of the tracking problem is to make the state variables track the changing targets), which can be easily adapted to the macroeconomic regulation problems.

It is shown that the solution of the inverse problem is not unique in general, which implies that the optimal control policies are fairly robust against different performance indices under the conditions stated above. Also, it is found that the solution of the inverse problem for a linear regulator holds true without any modification for the linear tracking problem.

Finally, the developed techniques for the selection of the performance criteria are illustrated through a macroeconomic stabilization policy model with a quadratic performance functional whose arguments consist of the deviations of the policy goal variables from their target values during the discrete time period of a finite planning horizon. The illustration highlights three main points: the determination of the steady-state values of the trajectories of goal variables which are under control, the computation of the feedback control policy, and the construction of a performance index.

It is shown that the number of the policy goal variables that can be made to attain the prescribed steady-state values is equal to the number of the control variables in the model. It is also demonstrated that the relative weights given to the competing policy goals in the performance index can be quantitatively determined by the method of the inverse optimal control problem for a dynamic policy model.

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By

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INTRODUCTION

Recent trends in control theory deal with the fundamental understanding of large scale systems and decentralized decision making. Economic processes, which are characterized by complex, and often unknown relationships among their constituent components, fall in this class of problems.

The research tools of optimal control theory can be used to study the dynamic responses of the economic system and to evaluate its performances. Most of the economic decisions consist of examining various actions together with their associated consequences, and choosing the particular action which would generate the most desirable outcome. Such elementary decision making could be improved by using econometric models for the purpose of examining various decision rules and their associated results in terms of the trajectories of the variables generated. The decision making process could be further improved by specifying some performance functional to generate certain optimal decision rules because the ad hoc decision rules may not be optimal for certain reasonable performance criterion and some better rules might be discovered by the optimal control approach which would otherwise remain unnoticed by the elementary method.

The performance criterion, in essence, enables one to specify a desired response toward which the system is optimized.

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In order that a performance index be generally applicable, it not only must reveal the performance characteristics of the optimal system, but also must enable the decision maker to choose what the desired characteristics of the optimum system should be.

In practice, however, the actual process of selecting which performance measure is to be used to "measure the distance" between a desired trajectory and its approximation is a major difficulty. That is, any mathematical criterion in practical control problems is rarely explicit enough to define the optimum system uniquely, and consequently even when a realistic performance index can be defined, it is often found that the basic concept of a performance index is too restrictive.

In this case, it is hardly expected that a certain performance index can be generally agreed upon. In fact, it is frequently argued that the choice of the performance functional to be optimized is subjective and arbitrary. This poses the so-called "inverse optimal control problem" -- instead of seeking a control rule corresponding to a given performance index, one can try to determine all performance criteria, if any, for which a given control rule is optimal.

In macroeconomic regulation problems where the performance indices are, in general, of the quadratic form in the deviations of policy goal variables from their target values, the inverse problem is to find, given economic policies, the relative "welfare" weights to be given to the competing policy goals and "costs" associated with the control policies.

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By solving the inverse problems, one may discover general properties shared by all optimal control policies. Further, if it can be discovered that there exist many performance indices for which a single control policy is optimal, then the preceding criticism about the choice of performance index is irrelevant since the important aspects of optimality will hold independently of the specific choice of a performance index.

Also, as the inverse optimal problem is the opposite to the optimal control problem, the solution of the inverse problem must distinguish between control policies which are optimal and those which are not, and perhaps disclose practical advantages of using specific control policies in combination with specific performance indices.

As the inverse optimal problem is a relatively new addition to the theoretical and methodological repertories of the systems science and is more so for the social sciences, no general solution of the inverse problem is yet found. In particular, the explicit solution of the inverse problem is not known for the discrete-time, multi-control regulatory processes.

Considering the fact that the market framework of the economic system is working through the myriad individual decisions, and the economic problems are generally formulated in terms of discrete-time difference equations, it is desirable for the economists that the solution to the inverse problem be found for the discrete-time, multi-control systems.

The purpose of this thesis is to solve this subclass of the inverse problem and to demonstrate the applicability of the solution

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solution to macroeconomic regulation models, and thereby to derive new insights for the economic policy manipulations.

The outline of this study is as follows. Chapter One includes some examples of the applications of optimal control theory to the economic field. The main concern of the chapter is to show that an economic regulation process can be viewed as a multi-stage dynamic optimization process. Chapter Two contains a brief discussion on control system design and the significance of the inverse problem. Chapter Three presents a literature survey of the inverse problems. In Chapter Four, the solution of the inverse problem for linear discrete-time multi-control systems is developed. Chapter Five illustrates the applications of the results in Chapter Four to some simple macroeconomic regulation problems. Chapter Six involves a summary and conclusions and suggests some recommendations for further study.

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CHAPTER ONE

ECONOMIC APPLICATIONS OF OPTIMAL CONTROL THEORY

Recently optimal control theory has found substantial applications in economics, because much of modern economic theory is concerned with optimal behavior of economic decision units over time.

Principal applications of control theory in macroeconomic and microeconomic problems have been:

(1) growth models for analysis of expansion of economies over long periods of time;

(2) planning models for sectoral allocation of resources over periods of time;

(3) short-run economic models for the study of short-run effects of economic policies on macroeconomic goals;

(4) consumer choice problems over the household life cycle;

(5) dynamic models of investment and pricing by firms;

(6) multiperiod portfolio analysis models; and

(7) dynamic models of a number of sectors including water resources and banking.

1.1 Optimal Growth Models

The central problem of these models is the division of output over time between consumption and investment so that some measure

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measure of social welfare is maximized. A simplified version of such a model is as follows.

Suppose that in a simple economy, the stream of output $F(K)$ attainable from the services of a given stock K of capital equipment are allocated into consumption flow C and gross investment I:

$$F(K) = C + I \ . \tag{1.1}$$

In turn, after deduction of provision for replacement of capital equipment at a rate of δK investment expenditure leads to further accumulation of capital stock:

$$\dot{K} = I - \delta K \ , \ (\cdot = \frac{d}{dt}) \ . \tag{1.2}$$

Also, aggregate saving rate $s(t)$ describes the composition of output at each moment:

$$I(t) = s(t) \ F[K(t)] \ . \tag{1.3}$$

Physical considerations impose the constraints:

$$C \geq 0 \ , \quad I \geq 0 \ , \quad 0 \leq s \leq 1 \ . \tag{1.4}$$

Equations (1.1) - (1.4) constitute, for given functions F and s and given initial condition $K(0) = K_0$, a differential equation describing the economic system.

Let

$$J = \int_0^N U[C(t)] \exp(-a\ t)dt \tag{1.5}$$

be the social welfare criterion accurately reflecting the desires of the community, where $U(\cdot)$ is a specified smooth, concave, positive

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positive welfare function and a, a non-negative constant, is the social rate of time preference. If the society by choosing proper s(t) wants to attain the maximum value of J, subject to the constraints, (1.1) - (1.4) and at the same time maintain the terminal capital stock equal to a prescribed value K_N, then the solution of a standard problem in the optimal control theory is required (Stoleru, 1965; Lele, et.al., 1971; Shell, 1967; Sakakibara, 1970; Burmeister and Dobell, 1972).

1.2 Development Planning Models

The development planning models are also concerned with the choice between consumption and investment over time, but the primary focus is on the sectoral allocation of investment over the planning period.

Since the development programs, in general, emphasize the overall economic growth as well as the intersectoral consistency of the projects, the planning models are usually formulated in complex nonlinear constraints including sectoral production functions and capital accumulations and the overall resource constraints to each sector. Also, as the development plan involves the changes in the composition of aggregate supply and demand in the economy over the planning period, the performance indices optimized by the planning models are customarily specified to be additive over both various consumption goods and time.

Here the numerical control theory methods[1)] are employed to facilitate the solution of the complex dynamic models (Arrow,

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et.al., 1970; Little, 1969; Levhari, 1969; Simon, 1956; Kendrick and Taylor, 1970).

1.3 Short-Run Economic Policy Models

These models are concerned with the choice of economic policy to best regulate and stabilize the economy. It is assumed that policy maker has a model of the economy that he believes is an acceptable representation of the structure of the economy and on which he will base his policy decisions. With difference equation macroeconomic models in general, focus of the study is on the use of, for instance, fiscal and monetary policies to control unemployment, price fluctuations, and balance of payments.

The main idea of such a study is to steer the policy goal variables close to the targets by choosing appropriate policy instruments.

As an illustration (Turnovsky, 1973), consider a problem of determining government expenditures for regulating a standard multiplier-accelerator model.

The system comprises the aggregate demand equation:

$$Z = cY + I + G \tag{1.6}$$

(aggregate demand Z is broken down into consumption cY, investment I, and government expenditures G); the flexible accelerator:

$$I = \alpha\dot{Y} - k\dot{I} \tag{1.7}$$

(investment is an exponentially declining weighted average of past income changes); and the output adjustment equation for the aggregate excess demand:

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$$\dot{Y} = r(Z - Y) \ , \tag{1.8}$$

where c, α, k, r are positive constants with $0 \le c \le 1$.

From (1.6) - (1.8), the evolution of national income Y over time given values of G and $\dot{G}$ is derived as follows.

$$\ddot{Y} + b_1\dot{Y} + b_2Y - b_3G - r\,\dot{G} = 0 \ , \tag{1.9}$$

where b_1, b_2, b_3 are constants expressed in terms of c, α, k, r.

Now the policy maker is assumed to control government expenditures so as to minimize some objective functional,

$$J = f_1(Y_N - \bar{Y})^2 + f_2(G_N - \bar{G})^2 + \int_0^N \{q_1(Y - \bar{Y})^2 + q_2(G - \bar{G})^2 + n(\dot{G})^2\}dt$$

where $\bar{Y}$ and $\bar{G}$ are some desired level of income and associated level of government expenditures respectively. The terms, $q_1(Y - \bar{Y})^2$ and $q_2(G - \bar{G})^2$ are losses incurred by being away from these optima. The fact that these are quadratic implies that positive and negative deviations from desired levels are weighted equally and are increasingly costly. The term $n(\dot{G})^2$ denotes the cost of changing fiscal policy and can be described as adjustment costs. $f_1(Y_N - \bar{Y})^2$ and $f_2(G_N - \bar{G})^2$ measure the terminal costs of being away from the targets.

In the above problem, it is possible that the government may be subjected to limits on the amount of its expenditures (Theil, 1964; Phillips, 1954, 1957; Pindyck, 1972; Erickson, et.al., 1970; Norman, 1971; Livesey, 1970; Chow, 1970, 1972; Fischer and Cooper, 1971; Turnovsky, 1973).

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1.4 Consumer Choice Problems

The consumer choice problems characterize the individual as choosing a time path of his available time (or energy) to be divided between earning (by renting human capital) and investment in human capital so as to maximize the present value of lifetime earnings.

Following Haley (1973), then the individual's disposible income in any period is the difference between his earnings $rE(t)$ and the cost of input used for the investment in human capital $rK(t)$ during that period, where $E(t)$ is total stock of human capital possessed by the individual at time t, $K(t)$ the human capital stock used as input to the human capital investment at t, and r the constant rental rate per unit of time on the human capital stock (capital markets are assumed to be perfect).

The human capital stock grows at a rate governed by

$$\dot{E}(t) = F[K(t)] - \delta E(t) , \qquad (1.10)$$

where $F(K)$ is the flow of investment at time t attainable from the only input of human capital stock and δ, a constant rate of human capital depreciation.

Assume that the individual wants to maximize the present value of his disposible earnings over the life cycle:

$$J = \int_0^N r[E(t) - K(t)] \exp(-at)dt , \qquad (1.11)$$

where N is the end of the earning life cycle and a, the constant individual time preference rate.

Then the general form of the individual optimization problem is to maximize the objective functional (1.11) subject to the

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the constraint (1.10) as well as such physical constraints as

$$K(t) \geq 0, \ E(t) > 0,$$

$$E(t) > K(t) \ ,$$

and at the same time to keep the positive human capital stock at the end of the earning period (Hakansson, 1969; Haley, 1973; Stafford and Stephen, 1972; Yaari, 1964; Ben-Porath, 1967).

1.5 Theory of Firms

Various dynamic models for the theory of firm have been developed. Let us consider a firm which seeks to maximize the integral of the discounted profit flows over the planning period by choosing optimal level of capital stock in each period (Leland, 1972).

If growth permits better achievement of the firm's goal in the future, capital stock K is one of the key decision variables for the firm. Following most dynamic formulation, it is assumed that the rate of change of capital stock, $\dot{K}$, is a function of current profits P. This formulation describes a self-financing firm which reinvests all or a positive fraction $(0 < \alpha \leq 1)$ of its profits.

Then, the problem is given by the following. In a dynamic environment, the firm will maximize

$$J = \int_0^N e^{-\delta t} \cdot P(K,L)\,dt$$

subject to

$$\dot{K} = \alpha P(K,L) , \qquad 0 < \alpha \leq 1;$$

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where ι is a constant discount rate, L is labor input for the production of the firm, and N is a fixed terminal time. (Arrow, et.al., 1958; Hakansson, 1970; Lucas, 1967; Thompson, et.al., 1971; Zabel, 1967).

1.6 Portfolio Selection

The individual and/or firm is faced with the problem of selecting a portfolio of bonds, stocks, and cash that will maximize his utility function defined over the probability distributions of returns generated by the various portfolios. For instance, Samuelson (1969) considers an individual's portfolio selection problem, postulating the existence of a risky asset that makes \$1 invested in, at time t, return \$1 Z_t after one period, where Z_t is a random variable subject to the probability distribution, Prob $\{Z_t \le z\} = p(z)$, $z \ge 0$, along with the safe asset that makes \$1 invested in it at time t return \$1(1 + r) at the end of the period. Yields at different time are assumed to be independent so that $p(z_0, z_1, \ldots, z_N) = p(z_0)p(z_1)\ldots p(z_N)$.

The problem is to find the optimal fraction of total wealth, a_t, that should be put into the risky asset, with $1 - a_t$ going into the safe assets, at each instant of time. Specifically,

$$\underset{\{C_t, a_t\}}{\text{Max}} \ J = E \sum_{t=0}^{N} (1 + \rho)^{-t} U(C_t)$$

subject to the wealth constraint,

$$C_t = W_t - W_{t+1} [(1 - a_t)(1 + r) + a_t Z_t]^{-1},$$

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$W_{N+1} = 0$ (no bequeating of wealth at death),

where $U(C_t)$ is the individual utility function depending on consumption C at time t, ρ a constant discount rate, W_t the individual's wealth at t, and E an "expectation" operator.

This problem is solved simultaneously for optimal saving-consumption and portfolio-selection decisions over time using a dynamic programming method. (Hakansson, 1969; Merton, 1969).

1.7 Pollution Control Problems

Haurie, et.al. (1972) analyze optimal policies of consumption and of pollution control in an economy using a two-sector macroeconomic planning model.

It is assumed that one sector (Sector 1) of the economy produces good Y to be consumed or invested and another sector (Sector 2) produces good Z used exclusively to purify the environment.

Since in this problem there are two conflicting goals -- maximization of consumption flow and minimization of the accumulation of pollution generated in the production processes -- over the planning period, the policy maker faces the allocation of investments between two sectors in each period such that the objective function including the weighted sum of social benefits derived from consumption and the social cost of pollution accumulation is to be maximized.

Emphasis is given to the following. By taking a sufficiently long interval of time, an optimum path may contain an arc along which the pollution and per capita consumption are maintained constan

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constant. If such an arc is uniquely defined for each performance criterion which gives different relative weights $(\lambda, 1 - \lambda)$ to per capita consumption and cost of pollution, it is possible for policy maker to choose a particular weighting based on a long-term cost effectiveness analysis.

Formally, the problem is to maximize

$$J = \lambda \int_0^N e^{-\delta t} c \, dt + (1 - \lambda) \int_0^N e^{-\delta t} \psi(x_3) dt \ , \ 0 \leq \lambda \leq 1,$$

subject to the constraints of capital accumulations in two sectors, pollution accumulation and the per capita consumption equations:

$$\dot{x}_1 = u_1 y - (d + n) x_1$$
$$\dot{x}_2 = u_2 y - (d + n) x_2$$
$$\dot{x}_3 = G(Ly, Lz) - r \, x_3$$
$$c = (1 - u_1 - u_2) u_3 f_1\left(\frac{x_1}{u_3}\right), \ 0 \leq u_i \leq 1, \ i = 1,2,3;$$
$$u_1 + u_2 \leq 1,$$

where

x_i $(i = 1,2)$ is the capital stock in sector i (measured in terms of total labor (L) unit),

x_3 is quantity of pollution agents accumulated,

u_i $(i = 1,2)$ is the gross investment in sector i (measured in terms of good Y),

u_3 is the proportion of labor engaged in Sector One,

y is production of Y (measured in total labor unit),

z is production of Z (measured in total labor unit),

c is per capita consumption,

n is a constant rate of labor force increase, d

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d is a constant rate of capital depreciation in both sectors,

$f(\cdot)$ is production function of Sector One (homogeneous of degree one),

$G(\cdot)$ is pollution generating function with $\frac{\partial G}{\partial Y} > 0$, $\frac{\partial G}{\partial Z} < 0$ for all $Y \geq 0$ and $Z \geq 0$,

r is a constant rate of natural elimination of pollution,

$\psi(x_3)$ is the social cost of pollution.

The main concern of this chapter was to show that an economic regulation process can be viewed as a multi-stage dynamic optimization process. This permits one to discuss meaningfully the inverse optimum control problem in the economic field, because the inverse problem starts with a given control rule which is assumed to have already been obtained in the optimization process.

Before examining the inverse problem formally, we need to discuss some aspects of the control system design in the next chapter (Chapter Two).

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CHAPTER TWO

DESIGN OF CONTROL SYSTEMS AND THE INVERSE OPTIMAL CONTROL PROBLEM

2.1 Performance Index

In modern control system design, it is emphasized that an admissible control must have, in addition to the stability property, an optimizing property in some sense, for example, minimizing the error of the system under control or satisfying certain specifications of accuracy and speed of performance of the system.

In the design of control systems, the starting point is the system specification. This includes a description of the input to the system and the desired response. Also included is a statement of the basis on which the system performance will be judged. This statement is in the form of a performance index. That is, the performance index enables one to specify a desired response towards which the system is optimized.

In order that a performance index be generally applicable, it not only must reveal the performance characteristics of the optimal system, but also must enable the designer to choose what the desired characteristics of the optimum system should be.

The actual process of selecting which performance measure is to be used to "measure the distance" between a desired output time-function and an approximation to the desired function is the major

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major difficulty. If a realistic performance index is defined that represents most of the design requirements of a problem, then the solution for the optimal control function can usually be obtained only by numerical methods which yield solutions to only a particular problem. On the other hand, if it is desired to obtain a closed-form solution for the control and, thereby, to solve more than numerical problem, simple performance indices must be used which often do not specify many of the design requirements. Thus the choice of a performance index is generally a compromise between a realistic criterion and one that is mathematically tractable.

Even when a realistic criterion can be defined, it is often found that the basic concept of a performance index is too restrictive. In practice, any mathematical criterion is rarely explicit enough to define the optimum system uniquely. It is here that a certain amount of personal opinion is found.

Very often, the quadratic performance index is considered as a generalized criterion for designing linear multivariable systems. The advantages of using this particular quadratic index are: (1) it results in a closed-form solution for the control and, therefore, the properties of the control as well as the optimal system can be determined; (2) under a reasonable set of restrictions, it always produces a stable system; (3) once the numerical elements of the performance index are specified, the optimal feedback gains can be determined by a straightforward computer solution (Tyler and Tuteur, 1966); (4) when the dynamic systems are under the random shifts, it enables one to disregard the uncertainty elements for obtaining the control rule. (Simon, 1956; Theil, 1957).

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The quadratic functional which is often times called a quadratic social disutility function in economic problems on optimum resource allocation over time and on macroeconomic stabilization policies, has been used quite extensively (Sengupta, 1970; Chow, 1972; Simon, 1956; Theil, 1965). Also, it has been frequently used as a generalized measure of the system performance in the control theory (Kalman, 1964; Kuo, 1970); information theory (Adorno, 1962); production, employment and inventory scheduling (Holt, et.al., 1956), and statistical quality control theory (Barnard, 1959).

The quadratic index contains weighting matrices whose elements can be specified to satisfy the requirements of specific control problems -- in the short-run macroeconomic regulation problems, the elements of the weighting matrix are designed to measure the social disutility associated with the deviations of the economic variables from their specified targets -- and, in effect, these elements become the design parameters of the optimal system.

2.2 Feedback Control

A feedback control system is a combination of elements which automatically cooperate to maintain a physical quantity or process in accordance with a given command. It has three predominant features; (1) it is a closed-loop system in which the control is actuated by a quantity that is affected by the result of the control operation; (2) it can establish control throughout a wide range of command that may vary in a random manner, and (3) it permits the control of high-power operations at a remote point by low-power operations at a local point.

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Some of the major reasons for employing feedback control are; (1) the process or actuator which supplies the output may have signal transmission characteristics that make accurate open-loop operation (open-loop operation yields an entire sequence of controls to be followed from initial conditions) very difficult, (2) with feedback, the precision of control can be made to depend largely on the equipment used to measure the system output and to compare it with its "ideal" value. This fact may enable accurate control to be achieved in spite of inaccuracies and variable characteristics in the process. That is, the feedback will reduce the sensitivity of the system characteristics to changes in parameters, (3) the effects of disturbances on the output may be suppressed by employing feedback, thereby eliminating the need for the elaborate disturbance compensators that would be needed with open-loop control.

In the economic stabilization models, a linear feedback control policy (the control policy is set to respond linearly on the policy goal variables) is generally chosen in such a way that the system under control (closed-loop system) will have a small weighted sum of variances.

It can be seen that feedback is used to overcome limitations of the physical components and is introduced to effect specific changes in the characteristics of the system (Ku, 1962).

In connection with the latter application of feedback, there exists a problem of arbitrary pole (eigenvalue of the system matrix) assignment (Willner, et.al., 1972). Given a controllable, linear, time-invariant system,

$$\dot{x}(t) = Ax(t) + Bu(t) \ , \qquad (2.1)$$

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where $x(t)$ is an n-vector of state variables, $u(t)$ is an m-vector of control variables, and A, B, are constant matrices, find a feedback gain matrix $G(m \times n)$ such that the feedback control

$$u(t) = Gx(t) \tag{2.2}$$

yields the closed-loop system

$$\dot{x}(t) = (A + BG)x(t) \tag{2.3}$$

whose eigenvalues can be assigned arbitrary.

For a single-control system (B in (2.1) becomes an n-vector), the problem of the pole assignment is solved as follows.

A nonsingular matrix T is found such that

$$TFT^{-1} = A + Bh'T^{-1} = A + BG \ , \tag{2.4}$$

where F is the Jordan canonical matrix of desired closed-loop eigenvalues (λ_i, $i = 1,2,\ldots,n$), with only a single Jordan block associated with each multiple eigenvalue, and

$$G = h'T^{-1} \ , \tag{2.5}$$

where $h' = (1, 1,\ldots, 1)$, an n-vector.

For T, we solve $TF - AT = Bh'$ by solving a sequence of problems of the form

$$(\lambda_i I - A)t_i = B \tag{2.6}$$

where t_i is the i^{th} column of T. If λ_i is a repeated eigenvalue, then (2.6) becomes

$$(\lambda_i I - A)t_i = B - t_{i-1} \tag{2.7}$$

for all but the first column of T associated with λ_i.

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For multi-control system, the problem of finding G for a given set of eigenvalues is generally nonlinear and has many possible solutions. In order to obtain a linear solution, the control u in (2.2) is restricted to the form

$$u(t) = \alpha G\, x(t) \tag{2.8}$$

so that the closed-loop system (2.3) becomes

$$\dot{x}(t) = (A + B\alpha G)x(t) \quad , \tag{2.9}$$

where α is an m-vector with all elements equal to unity. If we define

$$B\alpha = b \quad , \tag{2.10}$$

then (2.9) becomes

$$\dot{x}(t) = (A + bG)x(t) \quad , \tag{2.11}$$

which is the closed-loop system equation of a single-control system.

Since the eigenvalues of the system matrix govern the pattern of time paths (in particular, the convergence of the time paths, the speed of the convergence, and the damping ratio, etc.) of the state variables, the solution of the pole assignment problem enables the designers of control systems to effect the "desired" characterization of the system by proper choice of feedback controls.

2.3 Inverse Optimal Control Problem

Recent trends in control theory deal with the fundamental understanding of large scale systems and decentralized decision making. Social and economic processes, which are characterized by complex and often unknown relationships among their constitutent components, have such problems amenable to control theory application.

The research tools of optimal control theory can be used to study the dynamic responses of the social and economic systems and to evaluate their performance. Most of the social and economic decisions usually practiced at their basic levels consist of examining various actions together with their associated consequences and choosing the particular action which would generate the most desirable outcome. Such elementary decision making could be improved upon by using dynamic models for the purpose of examining various decision rules and their associated results in terms of the time paths of the variables generated.

The decision making process could be further improved by specifying some performance indices to generate certain optimal decision rules because the ad hoc decision rules may not be optimal for certain reasonable performance criteria and some better rules might be discovered by the optimal control approach which would otherwise remain unnoticed by the elementary method.

In this case, it is hardly expected that certain performance indices can be generally agreed upon. In fact, it is frequently argued that the choice of the performance index to be optimized is arbitrary and subjective, perhaps only a matter of taste. The argument is even greater in the social and economic regulatory systems compared to the physical and technological systems where the relative merits among various components of trajectories are better understood and clearer, and sometimes they are measured specifically in terms of energy and cost expenses. This suggests that the design of socio-economic systems as well as physical and technological systems involves the so-called "inverse optimal control problem" -- instead of asking for a control policy corresponding to a given performance index, one might seek to determine all performance criteria, if any, for which a given control policy is optimal.

By solving this problem, one might be able to discover general properties shared by all optimal control policies. Furthermore, if it can be discovered that there exist many performance indices for a single optimal control policy, then the preceding criticism about the choice of performance index will be irrelevant since the important aspects of optimality will hold independently of the specific choice of a performance index.

Also, as the inverse optimal control problem is the opposite of the optimal control problem, the solution must distinguish between control policies which are optimal and those which are not, and perhaps disclose practical advantages of using specific control policies in combination with specific performance indices. For instance, if the given control policy has some undesirable effects on the closed-loop system, the policy maker may want to find another control policy. But once the system equation and the specific performance index are given, the control policy is uniquely determined and consequently the properties of the closed-loop system cannot be changed. This means that given a system equation, different performance indices should be specified in order to get different control policies. Here the solution of the inverse optimal control problem may help to find out the appropriate performance index together with the control policy by which the closed-loop system can achieve the desired characteristics of "goodness" (e.g., moderate overshoot, high loop gain, and flat frequency response).

One important class of this problem is how to design optimal systems with prescribed closed-loop eigenvalues. We know that if the system is controllable, it is always possible to find a feedback gain matrix which will assign an arbitrary set of eigenvalues to the closed-loop system matrix (cf. Section 2.2). That is, if some prescribed set of eigenvalues is assigned to the closed-loop system matrix such that the closed-loop system can reveal the "desirable" characteristics, the corresponding feedback gain matrix can always be found. Once this "desirable" feedback gain matrix is obtained, the corresponding performance index can be found by the method of inverse optimal control.

As an example for the inverse optimal control problem, consider a simple dynamic macroeconomic model.

$$\begin{aligned} \dot{y} &= \alpha(c + i - y) \\ \dot{c} &= \beta(ay - c) \end{aligned} \tag{2.12}$$

where

$y = Y - Y_0$

$c = C - C_0$

$i = I - I_0$

α, β ... dynamic adjustment coefficients

Y = GNP

C = consumption

I = investment

a = marginal propensity to consume.

Subscript "0" denotes the equilibrium values.

The dynamic system (2.12) can be written as

$$\dot{x} = \begin{bmatrix} -\alpha & \alpha \\ \beta a & -\beta \end{bmatrix} x + \begin{bmatrix} \alpha \\ 0 \end{bmatrix} u , \tag{2.13}$$

where $x = (y,c)'$, a vector of state variables, and $u = i$, a control variable. Here the prime denotes the transpose.

Let the performance index for the system (2.13) be

$$J = \int_0^T (x_t' Q x_t + k u_t^2)\, dt \tag{2.14}$$

where Q is symmetric and positive semidefinite and k is a positive constant. Then the optimal control policy in the feedback form is given by[1)]

$$u^* = g'x \tag{2.15}$$

where g is the time-varying feedback gain vector which can be uniquely determined by the method of optimal control theory.

The inverse optimal control problem for the example is: given an optimal control policy (2.15), find all performance functional of the form (2.14), if any.[2)] Specifically, determine the weighting matrices, Q and k in (2.14).

In the above example, if the given control policy for the investment has some undesirable effects on the closed-loop system, e.g., too drastic change in consumption level over the

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1) A * indicates the control is optimal with respect to (2.14).

2) We would like to find only one, but this is not always the case.

planning period, the policy maker may want to find another control policy. But once the system equation (2.13), and weighting matrices Q and k in (2.14) are given, the closed-loop system cannot be changed.

Here, the solution of the inverse optimal control problem may help to find out the appropriate weighting matrices Q and k, and the control policy by which the closed-loop system can achieve the desired characteristics. The solution to the problem described above will be the subject of Chapter Four below.

CHAPTER THREE

THE INVERSE OPTIMAL CONTROL PROBLEMS -- LITERATURE SURVEY

Since the inverse optimal problem is to find the performance indices (if any), given an optimal control policy, solution of the problem, in general, starts with the assumption that there exists a solution for the Hamilton-Jacobi equation (which provides a sufficient condition for optimality) or for the matrix Riccati equation which can be derived from the Hamilton-Jacobi equation.

The main purpose of this survey is of a rather technical nature, i.e., to examine the solution process for the inverse problems, which could be utilized for a broader class of the inverse problem.

3.1 Inverse Control Problem for the Continuous-Time Systems

Kalman (1964) considers, for the first time, the inverse optimal control problem for the following case:

(a) The system is governed by a linear differential equation with constant coefficients (linear, time-invariant, continuous-time system).

$$\dot{x}(t) = A\, x(t) + b\, u(t) \ , \tag{3.1}$$

where x is a real n-vector, the state of the system, $u(t)$ is a continuous, real-valued function of time, the control function, A is a real constant $n \times n$ matrix, and b is a real constant n-vector.

(b) The control policy is linear and constant,

$$u(t) = -g'x(t) \ , \tag{3.2}$$

where g is a real, constant n-vector of feedback coefficients.

(c) The performance index is a quadratic form with constant coefficients in the state and control variables.

$$J = \lim_{t_1 \to \infty} \int_0^{t_1} L(x(t),u(t))dt = \lim_{t_1 \to \infty} \tfrac{1}{2} \int_0^{t_1} (x'H'Hx + u^2)dt \ ^{1)} \tag{3.3}$$

where H is a $1 \times n$ matrix with rank = 1.

(d) There is only one control variable (or single-input system).

The basic assumptions employed are

(i) The system (3.1) is completely controllable; $\text{rank}(b, Ab, \ldots, A^{n-1}b) = n$, i.e., all the state variables can be affected by some suitable choice of the control function $u(t)$.

(ii) The pair (A,H) is completely observable; $\text{rank}(H', A'H', \ldots, (A')^{n-1}H') = n$. This assures $y = Hx$ must not vanish identically along any free motion of the system unless the initial state $x_0 = 0$.

Then the inverse optimal control problem in this case is as follows:

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1) This particular form of the quadratic functional $L(x,u)$ can be said to represent a more general form. Consider $L(x,u) = \tfrac{1}{2}\{x'Qx + 2(r'x)u + \sigma u^2\}$ (Q, r, σ are constants, $Q = Q'$). Without loss of generality, set $\sigma = 1$. Then $2\bar{L}(x,u) = x'Qx + 2(r'x)u + u^2 = x'(Q - rr')x + (u + r'x)^2$. Let $Q - rr' = H'H$ and $\bar{u} = u + r'x$. Then $2\bar{L}(x,u) = x'H'Hx + \bar{u}^2$ and the system $\dot{x} = Ax + bu$ must be changed to $\dot{x} = \bar{A}x + b\bar{u}$ where $\bar{A} = A - br'$, and g in $u(t) = -g'x(t)$ is to be replaced by $\bar{g} = g - r$.

Given a completely controllable constant linear system (3.1) and constant linear control law (3.2), determine all loss functions L in (3.3) such that the control law minimizes the performance index (3.3).

A necessary and sufficient condition for $u(t) = -g'x(t)$ to be a stable optimal control law is that there exists a matrix P which satisfies the following algebraic relations (Kalman, 1964)

$$
\begin{aligned}
&\text{(a)} \quad P = P' \text{ is positive definite} \\
&\text{(b)} \quad Pb = g \\
&\text{(c)} \quad -PA_g - A'_g P = H'H + gg' \quad \text{(Riccati equation)},
\end{aligned}
\tag{3.4}
$$

where $A_g = A - bg'$.

Since P in the above relations is unknown, it is desirable to eliminate P and to get a simple relation connecting the control law g and the "representative" performance weighting matrix H. For this, Kalman shows that a necessary and sufficient condition for g to be an optimal control law is that g be a stable control law and that the condition

$$|1 + g'\Phi(iw)b|^2 = 1 + |H\Phi(iw)b|^2 \quad \text{holds for all real } w,^{1)} \tag{3.5}$$

where $\Phi(s) = (sI - A)^{-1}$, $i^2 = -1$, and s is a complex variable.

If the condition (3.5) holds, H may be obtained by factorizing the non-negative polynomial

$$|1 + g'\Phi(iw)b|^2 - 1 = |H\Phi(iw)b|^2 \quad . \tag{3.6}$$

Since g is assumed to be stable, the rational function $H\Phi(iw)b$

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1) It is assumed that H is a matrix consisting of one row only. Otherwise, the condition (A,H) is completely observable is not sufficient to guarantee that the optimal control law is completely observable.

must not possess any common cancelable factor which has a zero with non-negative real part. Using the canonical forms on matrices A and b (Wonham and Johnson, 1964), it is possible to identify the components of an n-vector $H = (h_1, h_2, \ldots, h_n)$ with the numerator coefficients of the rational function

$$H\Phi(iw)b = \frac{h_n(iw)^{n-1} + \ldots + h_1}{(iw)^n + \alpha_n(iw) + \ldots + \alpha_1}$$

where α_i's, $(i = 1, \ldots, n)$ are constant scalars.

Then the matrix H so constructed is the solution for the inverse optimal control problem.

Thau (1967) extends the Kalman's inverse problem to the multiple input system and a class of non-linear control system:

(a) The dynamic system equation considered is

$$\dot{x} = f(x) + Bu \,, \tag{3.7}$$

where x is an n-vector state variables and u(t) is an m-vector control variable continuous in t. B is an $n \times m$ constant matrix.

(b) The integrand of the performance criterion is a sum of a functional of the state variables and a functional of the control. That is, the performance index is given by

$$V(x(0), u) = \int_0^\infty [q(x) + h(u)]dt \,, \tag{3.8}$$

where q(x) and h(u) are smooth functions of their arguments.

(c) It is assumed that

$$\eta(u) \equiv \frac{dh}{du} \quad \text{is a one-to-one mapping,}$$

$h(0) = 0$ and $\dfrac{d^2h}{du^2} > 0$.

(d) The origin $x = 0$ is considered the target set and the feedback control law given by

$$u^*(t) = \phi[x(t)] \qquad (3.9)$$

is assumed to be such that the resulting closed-loop system, $\dot{x} = f(x) + Bu^*(t) \equiv F(x) + B\phi(x)$, is asymptotically stable.

Then assuming an optimal control law $u^*(t)$ exists, the inverse problem is as follows:

Given a control law (3.9) with the above-mentioned properties, find the most general performance functional (if any) of the form (3.8) which is minimized by the control law (3.9).

For the control problem (a) - (d), the necessary and sufficient condition for the optimality is that the value of the optimum performance index (3.8) $V^0(x)$ satisfies the Hamilton-Jacobi equation

$$\max_u \{-(q(x) + h(u)) - \frac{\partial V^0}{\partial x} (f(x) + Bu)\} = 0, \; V^0(0) = 0. \qquad (3.10)$$

From (3.10), the optimal control $\phi(x)$ satisfies

$$\eta(\phi(x)) = -B' \frac{\partial V^0}{\partial x} \qquad (3.11)$$

and

$$q(x) = -h(\phi(x)) - \frac{\partial V^0}{\partial x} [f(x) + B\phi(x)] \; . \qquad (3.12)$$

To obtain more explicit results, Thau considers a completely controllable multiple input linear time-invariant system

$$\dot{x} = Ax + Bu \; , \qquad (3.13)$$

where A is an $(n \times n)$ matrix and B an $(n \times m)$ matrix, and the performance index in the form of

$$V = \tfrac{1}{2} \int_0^\infty (x'H'Hx + u'u)dt \; . \qquad (3.14)$$

The given control law which drives the system toward the origin is

$$u = -Gx , \tag{3.15}$$

where G is a known constant matrix. It follows from (3.11) and (3.12) that

$$G = B'P , \tag{3.16}$$

where P satisfies the algebraic Riccati equation,

$$H'H + G'G = -PA - A'P + PBG + G'BP ,$$

or

$$-PA_g - A_g'P = H'H + G'G , \quad A_g = A - BG . \tag{3.17}$$

Following Kalman (1964), Thau derives the frequency-domain characterization of optimality for the multiple-input system. Using (3.16) and (3.17),

$$T'(iw)T(iw) = I + \Gamma'(iw)\Gamma(iw) \quad \text{for all real } w, \tag{3.18}$$

where $T(iw) = I + G\Phi(iw)B$ and $H\Phi(iw)B = \Gamma(iw)$.

There is, however, no general way at the present to find the explicit expression for H from the equation (3.18).

Thau further considers a class of nonlinear systems[1) given as

$$\dot{x} = Ax + b\theta(u) \tag{3.19}$$

$$u = g'x , \tag{3.20}$$

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1) The development here incorporates Panda's (1971) corrected version of Thau's solution.

where x,b, and g are n-vectors and A is an $n \times n$ matrix. Here $\theta(u)$ is considered to be a known scalar function, defined and continuous for all u, $\theta(0) = 0$, $u\theta(u) > 0$ for all $u \neq 0$ and $\int_0^{+\infty} \theta(u)du$ diverges.

With the additional assumptions that (i) the value of the optimum performance index is given by $V^0(x) = \frac{1}{2}x'Px$, where P is positive definite, (ii) $\theta(u)$ in (3.19) can be expressed as a power series in odd powers of u with all positive coefficients, i.e., $\theta(u) = \sum_{i=1}^{\infty} a_i u^i$, $i \in (I_0^+)$,[1] all $a_i > 0$, and (iii) the inverse function of η in (3.11) is also expressed as a power series $\eta^{-1}(u) = \sum_{i=1}^{\infty} c_i u^i$, $i \in (I_0^+)$; explicit expressions for all performance criteria of the form (3.8) for the system (3.19) and the control law (3.20) can be obtained as follows (Panda, 1971).

$$\theta(g'x) = \eta^{-1}(-b'Px)$$

$$q(x) = -\tfrac{1}{2}\, x'(PA + A'P)x + \frac{a_1}{c_1} \sum_{i=1}^{\infty} \frac{a_i}{i+1} (g'x)^{i+1}, \quad i \in (I_0^+). \qquad (3.21)$$

In order to have non-negative $q(x)$ in (3.21), the condition, $|1 - g'\Phi(iw)b|^2 \geq 1$, should hold for all real w, where $\Phi(iw) = (iwI - A)^{-1}$.

It is seen above that Thau gets the explicit algebraic conditions for the solution of the inverse optimal control problem by introducing the assumption that the optimum performance is of the form $V^0(x) = \frac{1}{2}\, x'Px$, where P is a (positive definite) symmetric matrix.

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1) I_0^+ represents the set of all integers that are positive and odd.

Yokoyama and Kinnen (1972) show the necessary and sufficient conditions for optimized performance indices of a general class of controllable and uncontrollable systems with weakened assumption about the form of the optimum performance.

Yokoyama and Kinnen have the following problem:

(a) The system,

$$\dot{x} = G(x) + Bu \equiv Ax + F(x) + Bu \ . \qquad (3.22)$$

(b) A feedback control law, u(x). (3.23)

(c) The functional form of the performance indices is restricted to the general structure,

$$\int_0^\infty \{L(x) + u'Ru\}dt \ . \qquad (3.24)$$

The assumptions for the problem are

(i) x is an n-vector of state variables. G(x) is an n-dimensional vector valued function of class C^2 satisfying G(0) = 0, and A is an $n \times n$ matrix such that Ax is the first-degree homogeneous term of G(x).

(ii) u is an m-vector of control variables and B is an $n \times m$ matrix of rank r such that $0 < r \leq m \leq n$.

(iii) u(x) is an m-dimensional vector valued function of class C^2 such that u(0) = 0 and the origin of the synthesized control system (3.22) is asymptotically stable in the large.

(iv) R is restricted to an $m \times m$ matrix, symmetric and positive definite and $L(x)$ to class C^2 such that $L(0) = 0$ and

$$\int_0^\infty \{L(\phi(t,x)) + u[\phi(t,x)]'Ru[\phi(t,x)]\}dt \qquad (3.25)$$

is well defined, where $\phi(t,x)$ is a solution of (3.22) from $x \in R^n$.

Given a system equation (3.22) and a feedback control law (3.23) a priori, the problem is to seek $L(x)$ in the performance indices (3.24) optimized in the synthesized feedback control system.

It is assumed that the inverse problem is considered for the control equivalent canonical form (Luenberger, 1967). Thus

$$A = \underbrace{\begin{bmatrix} A_1 \\ \cdots \\ A_2 \end{bmatrix}}_{n} \begin{matrix} n-r \\ \\ r \end{matrix} = \begin{bmatrix} A_{(e)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_{(1,2)} & \cdots & 0 \\ \vdots & & & & \vdots \\ & & & & 0 \\ 0 & \cdots & & 0 & A_{(r-1,r)} \\ A_{(r,e)} & A_{(r,1)} & \cdots & & A_{(r,r)} \end{bmatrix} \begin{matrix} \ell_e \\ \ell_1 \\ \\ \\ \ell_{r-1} \\ \ell_r \end{matrix}$$

$$\qquad\qquad\qquad \ell_e \quad \ell_1$$

and

$$B = \begin{bmatrix} 0 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & I \end{bmatrix} \begin{matrix} n-\ell_r \\ \\ \ell_r \end{matrix}$$

$$\qquad n-\ell_r \quad \ell_r$$

where the following is true:

(i) $\ell_e, \ell_1, \ldots, \ell_{r-1}$, and ℓ_r are integers determined by A and B such that $\ell_e + \sum_{i=1}^{r} \ell_i = n$, $0 < \ell_1 \le \ell_2 \le \ldots \le \ell_r = r$,

$$\ell_e \begin{cases} = 0; \text{ if } A,B \text{ is a controllable pair} \\ \neq 0; \text{ if } A,B \text{ is not a completely controllable pair,} \end{cases}$$

(ii) $A_{(i,i+1)}$ is an $\ell_i \times \ell_{i+1}$ matrix such that

$$A_{(i,i+1)} = [0 \vdots I_{\ell_i}], \ i = 1,2,\ldots,r-1,$$

$A_{(r,i)}$'s are unspecified.

For convenience, define the following:

(i) $$x = \begin{bmatrix} x_1 \\ \cdots \\ x_2 \end{bmatrix} \begin{matrix} n-r \\ \\ r \end{matrix} \equiv \begin{bmatrix} x_{(e)} \\ x_{(1)} \\ \vdots \\ x_{(r)} \end{bmatrix}$$

the components of $x_2 = x_{(r)}$ are directly dependent on u from the structure of B. Those of x_1 are indirectly controlled state variables.

(ii) $$G(x) \equiv \begin{bmatrix} G_1(x) \\ \\ G_2(x) \end{bmatrix} \begin{matrix} n-r \\ \\ r \end{matrix} \quad , \quad F(x) = \begin{bmatrix} F_1(x) \\ \\ F_2(x) \end{bmatrix} \begin{matrix} n-r \\ \\ r \end{matrix}$$

(iii) $$u(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} \begin{matrix} m-r \\ r \end{matrix} \tag{3.26}$$

(iv) $$R = \begin{bmatrix} R_{11} & R_{12} \\ \\ R_{12}' & R_{22} \end{bmatrix} \begin{matrix} m-r \\ \\ r \end{matrix} \tag{3.27}$$

and $R_0 = R_{22} - R_{12}' R_{11}^{-1} R_{12}$.

For the optimal performance index $V(x)$, the Hamilton-Jacobi equation

$$\min_u \{L(x) + u'Ru + (\frac{\partial V(x)}{\partial x})'[G(x) + Bu]\} = 0 \tag{3.28}$$

must be uniquely realized at each $x \in R^n$ by u such that

$$u = -\tfrac{1}{2} R^{-1} B' (\frac{\partial V(x)}{\partial x}) . \tag{3.29}$$

Identifying (3.29) with the specified $u(x)$, it follows from (3.26), (3.27) that

$$u_1(x) = -R_{11}^{-1} R_{12} u_2(x) \tag{3.30}$$

and

$$\frac{\partial V(x)}{\partial x_2} = -2 R_0 u_2(x) . \tag{3.31}$$

With the symmetry property of $\frac{\partial^2 V(x)}{\partial x \partial x}$, the following is derived.

$$\frac{\partial V(x)}{\partial x} \equiv \begin{bmatrix} \frac{\partial V(x)}{\partial x_1} \\ \frac{\partial V(x)}{\partial x_2} \end{bmatrix} \equiv \begin{bmatrix} -2 \int_0^{x_2} \left[\frac{\partial u_2(x)}{\partial x_1} \right]' R_0 dx_2 + W(x_1) \\ -2 R_0 u_2(x) \end{bmatrix} , \tag{3.32}$$

where $W(x_1)$ is an (n-r)-dimensional vector-valued function of class C^1 (i.e., with continuous first derivatives) to provide the symmetry for $\frac{\partial^2 V(x)}{\partial x_1 \partial x_1}$, and the integral $\int_0^{x_2} \frac{\partial u_2(x)}{\partial x_1} dx$ is defined (for an r-dimensional row vector function $\frac{\partial u_2(x)}{\partial x_1} \equiv E(x)$) as

$$\int_0^{x_2} E(x)dx = \int_0^{x_{n-r+1}} e_1(x_1,x_2,\ldots,x_{n-r},r_1,0,\ldots,0)dr_1 + \int_0^{x_{n-r+2}} e_2(x_1,x_2,\ldots,x_{n-r+1},r_2,0,\ldots,0)dr_2 + \ldots + \int_0^{x_n} e_r(x_1,x_2,\ldots,x_{n-1},r_r)dr_r .$$

The optimal performance index has an expression

$$V(x) = \int_0^{x} \left(\frac{\partial V(x)}{\partial x}\right)' dx . \tag{3.33}$$

Yokoyama and Kinnen (1972) show that a performance index (3.24) can be optimized by the specified $u(x)$ if and only if the following conditions are satisfied:

(a) $$u_1(x) = -R_{11}^{-1} R_{12} u_2(x),$$

(b) $$R_0\left(\frac{\partial u_2(x)}{\partial x_2}\right) \text{ is symmetric,}$$

(c) there exists an (n-r)-dimensional vector-valued function $W(x_1)$ of class C^1 insuring the symmetry of

$$-2 \frac{\partial}{\partial x_1}\left\{\int_0^{x_2} \left(\frac{\partial u_2(x)}{\partial x}\right)' R_0 dx_2\right\} + \frac{\partial W(x_1)}{\partial x_1}$$

(d) $L(x)$ and R are related by

$$L(x) = u_2'(x)R_0u_2(x) + 2u_2'(x)R_0G_2(x) + 2\left\{\int_0^{x_2}\left[\frac{\partial u_2(x)}{\partial x_1}\right]' R_0 dx_2\right\}' G_1(x) - W'(x_1)G_1(x) \quad (3.34)$$

The above procedure does not guarantee that the solution is unique and the resulting performance index $V(x)$ is positive-semi definite. However, Yokoyama and Kinnen develop a method which insures the positive definiteness of $V(x)$ as well as $-\dot{V}(x)$ by adjusting $V(x)$ and $L(x)$ that are obtained as (3.33) and (3.34).

All the inverse optimal problems mentioned so far consider the performance index of a special form, i.e., the integrand of the index does not explicitly depend on time.

Kurz (1969) and Bellman (1970) consider the inverse problem which involves the performance index with time-dependent integrand.

Kurz's system equation is

(a) $$\dot{x} = f(x) - u, \; x(0) = x_0 , \quad (3.35)$$

with performance criterion of the form

$$J = \int_0^\infty e^{-\delta t} h(u)\,dt \ . \tag{3.36}$$

(c) The control law is given by

$$u = \phi(x) \ . \tag{3.37}$$

It is assumed that $f(x)$ in (3.35) is strictly concave with $f'(x) > 0$, $f''(x) < 0$ for all $x(t)$; $h(u)$ in (3.36) is strictly concave belonging to the class C^2, and $\delta > 0$ is constant. The control law u is assumed to be monotonic, continuously differentiable function with $\phi'(x) > 0$.

Then, the inverse problem is to find the function $h(u)$ and δ in the performance index such that the given $u = \phi(x)$ is the optimal control law for the system (3.35).

For the Hamiltonian defined by

$$H(x,u,t,p)e^{\delta t} = h(u) + p[f(x) - u] \ ,$$

where $p(t)\cdot e^{-\delta t}$ is a "costate" variable, the optimality condition is given by

$$\dot{p}(t) = p(t)(\delta - f'(x)) \tag{3.38}$$

$$h'(u) = p(t) \ . \tag{3.39}$$

Assuming that $\dot{x} = f(x) - \phi(x)$ has at most one stationary solution x^*, $f'(x^*) > 0$, and $f'(x^*) - \phi'(x^*) < 0$, then the inverse problem of $u = \phi(x)$ has a solution. In fact, since the system is a simple scalar equation, an explicit solution can be obtained analytically as follows.

From (3.38) and (3.39),

$$\frac{\dot{u}}{u} = \frac{f'(x) - \delta}{-\frac{h''(u)}{h'(u)} \cdot u} \quad . \tag{3.40}$$

Moreover, since $u = \phi(x)$,

$$\frac{\dot{u}}{u} = \frac{\phi'(x)}{\phi(x)} \dot{x} \quad . \tag{3.41}$$

Thus from (3.40) and (3.41),

$$-\frac{h''(u)}{h'(u)} = \frac{f'(x) - \delta}{\phi'(x)[f(x) - \phi(x)]} \quad . \tag{3.42}$$

With $x(u) = \phi^{-1}(u)$, the equation (3.42) can be considered as an equation in u only. The general solution of (3.42) may be formally written as

$$h'(u) = M \exp\left\{-\int \frac{f'x(u) - \delta}{\phi'[x(u)][f(x(u)) - \phi(x(u))]} du\right\}, \quad (M > 0). \tag{3.43}$$

At the stationary point x^*, it must be that $\dot{x} = 0$, which implies that $u^* = \phi(x^*) = f(x^*)$ is constant and $\dot{p}(t) = 0$ in (3.39), and therefore (3.38) gives

$$f'(x^*) = \delta \quad . \tag{3.44}$$

The common features of the inverse problems discussed so far are:

(1) the system matrices are constant (time-invariant systems),

(2) the optimization period is infinite.

The above-type of the inverse problem is of particular interest, mainly because the problems involve a constant feedback gain matrix for the control law.

More interesting optimization problems, however, are formulated in terms of the time-varying system matrices and involve a finite time period of optimization.

Jameson and Kreindler (1973) consider the following dynamic system

(a) $$\dot{x} = Ax + Bu \ , \ x(t_0) = x_0 \ , \tag{3.45}$$

(b) a given control law

$$u = -Gx \ , \tag{3.46}$$

and a performance index

(c) $$J = x'(N)Fx(N) + \int_{t_0}^{N} (x'Qx + u'Ru)dt \ , \tag{3.47}$$

where x is an n-vector of state variables, u an m-vector of controls, N is a fixed terminal time. The matrices, A,B,G,Q, and R are time-varying and assumed to be uniformly bounded and continuous on $[t_0,N]$. In addition, $Q = Q'$ and $F = F'$ are positive semi-definite and $R = R'$ is positive definite.

The explicit expression for G in (3.46) in this problem is given by

$$G = R^{-1}B'P \tag{3.48}$$

where P is the positive semi-definite solution of the Riccati equation

$$-\dot{P} = PA + A'P - PBR^{-1}B'P + Q$$
$$P(N) = F \ . \tag{3.49}$$

The minimum value J^* of the performance index (3.47) is

$$J^{*} = x(t_0)'P(t_0)x(t_0) \; . \tag{3.50}$$

The inverse problem here is to find F, Q, and R in the performance index (3.47), given the system equation (3.45) and the control law (3.46).

Note that the existence of symmetric P, R, Q, and F satisfying (3.48) and (3.49) is a necessary condition for a closed-loop system $\dot{x} = (A - BG)x$ to be optimal with respect to the performance index (3.47).

The solution of inverse problem is obtained by considering (3.48), i.e.,

$$RG = B'P \tag{3.51}$$

or equivalently

$$RGB = B'PB \; , \tag{3.52}$$

$$G'RG = G'B'P \; . \tag{3.53}$$

Writing $R = L'L$, (3.52) implies

$$(L^{-1})'B'PBL^{-1} = (L^{-1})'RGBL^{-1} = LGBL^{-1} \; . \tag{3.54}$$

If $R = R'$ is positive definite and $P = P'$ is positive semi-definite, then (3.52), (3.53) and (3.54) imply

$$RGB = B'G'R \; , \tag{3.55}$$

$$\text{rank}\,(BG) = \text{rank}\,(G) \; , \tag{3.56}$$

$$GB \text{ has non-negative real eigenvalues}, \tag{3.57}$$

respectively. Therefore, if $R = R'$ (positive definite) and $P = P'$ (positive semidefinite) could be constructed such that (3.55) - (3.57) hold, then the chosen R and P satisfy (3.51).

Jameson and Kreindler (1973) developed a procedure for constructing such R and P as follows.

By suitably choosing Γ, which is a positive definite, real, symmetric matrix such that $\Gamma\Lambda = \Lambda\Gamma$, R can be chosen as

$$R = V\Gamma V' , \tag{3.58}$$

where V is a matrix of eigenvectors of $B'G'$ and Λ is the diagonal matrix of corresponding eigenvalues.

The matrix P can be constructed, by choosing a symmetric, positive semidefinite matrix Y, as

$$P = G'R(RGB)^{+}RG + Y , \tag{3.59}$$

where $(RGB)^{+}$ denotes the Penrose generalized inverse of RGB, i.e., $(RGB)(RGB)^{+}(RGB) = (RGB)$.

Once R and P are constructed as (3.58) and (3.59), F and Q can be found from the Riccati equation (3.49).

It should be noted that Q so determined may not be non-negative definite. However, the performance index of the form (3.47) with the chosen F, Q, R, and P attains its absolute minimum J^* over all square-integrable controls for all $x(t_0)$ and all $t_0 < N \leq \infty$.

3.2 Inverse Control Problem for Discrete-Time Systems

All the above inverse optimal control problems are concerned with the continuous-time system equations. But there exist the problems, especially in such fields as biology and economics, etc., for which discrete-time models are the natural ones to assume.

For instance, many of the functions of time representing responses with time-lags that are met with in economics do not correspond with continuous-time equations of known and simple form, as analogous relationships usually do in, say, engineering fields. The economic responses are not given "experimentally" and must, mathematically, be considered to be of arbitrary form, and so are conveniently specified by the discrete-time models.

In other problems, where computation must be done on a digital computer, the discrete-time model often results as suitable approximation to a continuous-time system. A final important class of discrete-time systems comes from sampled-data system. In sampled-data systems, a continuous system is driven by an input specified at discrete-time points and has state and output variables available only at discrete-time points.

Wu and Schroeder (1968) solve the inverse problem for time-invariant, single-input, discrete-time system. They consider the dynamic system given by

$$x_{t+1} = Ax_t + Bu_t \ , \ x_{t_0} = x_0 \ , \tag{3.60}$$

with performance index,

$$J(x_j) = \sum_{t=j}^{\infty} (x_t' H' H x_t + u_t^2) \ , \tag{3.61}$$

and the control law given by

$$u_t = -g' x_t \ . \tag{3.62}$$

It is assumed that the given control law has the stability property, i.e., $x_{t+1} = (A - bg')x_t$ is asymptotically stable.

The system equation (3.61) and the performance index (3.63) are assumed to be in the canonical form (Wonham and Johnson, 1964; Tuel, 1967). That is,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & 0 & 1 \\ -a_1 & -a_2 & & \cdots & -a_n \end{bmatrix} , \qquad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$H'H = \begin{bmatrix} & & & q_{1n} \\ & \mathbf{0} & & q_{2n} \\ & & & \vdots \\ q_{1n} & q_{2n} & \cdots & q_{nn} \end{bmatrix} . \tag{3.63}$$

Then, the inverse problem is to determine n elements of $H'H$, q_{in}, $i = 1,2,\ldots,n$, given the system equation (3.60) and the control law (3.62).

The Riccati difference equation for this problem is given by

$$P = H'H + A'P(A - bg') , \tag{3.64}$$

where $g' = (b'Pb + 1)^{-1}b'PA.$ (3.65)

Considering the canonical forms in (3.63), (3.65) can be written as

$$g' = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} = \frac{1}{P_{nn} + 1} \begin{bmatrix} -a_1 P_{nn} \\ P_{1n} - a_2 P_{nn} \\ \vdots \\ P_{n-1,n} - a_n P_{nn} \end{bmatrix} , \tag{3.66}$$

where $P = [p_{ij}]$, $i,j = 1,2,\ldots,n$. This gives an explicit solution for the last column of the matrix P,

$$p_{i-1,n} = (p_{nn} + 1)g_i + a_i p_{nn} \;, \quad i = 2,3,\ldots,n, \tag{3.67}$$

where

$$p_{nn} = - \frac{g_1}{a_1 + g_1} \;. \tag{3.68}$$

The elements in the matrix $H'H$ can be easily obtained by substituting the canonical forms (3.63) into the Riccati equation (3.64), and by using (3.67), (3.68).

It is seen from the above literature survey that the general solution of the inverse problems has not yet been found. In particular, no explicit solution of the inverse problem is known for the discrete-time multi-control systems.

Considering the fact that the market framework of the economic system is working through the myriad individual decisions and the economic problems are generally formulated in terms of discrete-time difference equations, it is desirable for the economists that the solution to the inverse problem be found for the discrete-time multi-control systems. The solution to this subclass of the inverse problem is the subject of the following chapter.

CHAPTER FOUR

THE INVERSE CONTROL PROBLEM OF DISCRETE-TIME MULTIVARIATE CONTROL SYSTEMS

4.1 The Solution of a General Inverse Problem

In this section a general inverse control problem will be formulated for an autonomous discrete-time linear system with an *a priori* linear feedback control law, and then a sufficient condition will be developed for the solution of the inverse problem. More specifically, the inverse problem is considered for the system

$$x_{t+1} = Ax_t + Bu_t \ , \ x_0 = x^0 \tag{4.1}$$

a feedback control law

$$u_t = -Gx_t \ , \tag{4.2}$$

and with the performance indices restricted to the general quadratic structure

$$J = \sum_{t=0}^{\infty} [x_t'Qx_t + u_t'Ru_t] \ . \tag{4.3}$$

Here, x is an n-vector of state variables; u is an m-vector of control variables; A and B are constant $n \times n$ and $n \times m$ matrices; and Q and R are constant symmetric positive semidefinite and definite matrices.[1] The following assumptions will be made:

(A.1) The matrix A is nonsingular.

(A.2) The matrix B is of full rank, m.

(A.3) The closed-loop system (4.1) with (4.2) is asymptotically stable in large.[2]

(A.4) The pair (A,B) is stabilizable and the pair (D,A) is observable, where $Q = D'D$.

The assumptions (A.1) and (A.2) are made mainly for simplicity, and (A.3) and (A.4) are for the existence of positive semi-definite solution of matrix Riccati equation (4.9) below. (Aoki, 1973).

By introducing a new control variable

$$\bar{u}_t = u_{t+1} - u_t , \tag{4.4}$$

the problem (4.1) - (4.3) can be transformed into an equivalent problem:

$$\bar{x}_{t+1} = \bar{A}\bar{x}_t + \bar{B}\bar{u}_t , \tag{4.5}$$

$$\bar{u}_t = -\bar{G}\bar{x}_t , \tag{4.6}$$

and

$$J = \sum_{t=0}^{\infty} x_t'\bar{Q}x_t . \tag{4.7}$$

---

1) Optimal control of this problem involves maintaining the state variables close to the zero state (origin) after some initial disturbance away from the zero state. This problem is commonly referred to as a regulator problem (Dorato and Levis, 1971).

2) "Asymptotically stable in large" means that any initial points away from origin approaches origin as time goes to infinity (La Salle, 1962-63).

Where $\bar{x} = \begin{bmatrix} x_t \\ u_t \end{bmatrix}$, $\bar{A} = \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix}$,

$\bar{B} = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$, $\bar{Q} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$, and

$\bar{G} = [G, I_m]\cdot\bar{A}$. Note that the assumptions (A.1), (A.2), (A.3), and (A.4) are preserved for the equivalent problem (4.5) - (4.7). It will be seen that the inverse problem can be solved easily for the equivalent system compared to the original system (4.1) - (4.3).[1)]

Since the control policy (4.6) is optimal, the feedback gain matrix is characterized by (Kleinman and Athans, 1966)

$$\bar{G} = (\bar{B}'P\bar{B})^{-1}\bar{B}'P\bar{A} \quad , \tag{4.8}$$

where P satisfies the following matrix Riccati equation[2)]

$$P = \bar{Q} + \bar{A}'P(\bar{A} - \bar{B}\bar{G}) \; . \tag{4.9}$$

In order to express $\bar{G}$ explicitly in terms of P, let P be partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P'_{12} & P_{22} \end{bmatrix} \quad , \tag{4.10}$$

where P_{22} is an $m \times m$ nonsingular matrix. Then $\bar{G}$ in (4.8) can be written as follows.

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1) Note that the performance index (4.7) for the transformed system does not contain a quadratic term involving the newly defined control vector (4.6).

2) Refer to Kleinman and Athans (1966) or Kuo (1970) for derivation of discrete Riccati equation.

$$\begin{aligned}\bar{G} &= P_{22}^{-1}[P_{12}', P_{22}]\bar{A} \\ &= [P_{22}^{-1} P_{12}', I_m]\cdot\bar{A} \\ &= [G, I_m]\bar{A},\end{aligned} \tag{4.11}$$

which shows that $G = P_{22}^{-1} P_{12}'$, the feedback gain matrix for the system (4.1) - (4.3).

The closed-loop optimal system has the following property.

Lemma 1: The $(n+m)^{th}$-order closed-loop system matrix $\bar{A} - \overline{BG}$ for (4.5) - (4.7) is singular, and if $\bar{A}$ is non-singular, then rank $(\bar{A} - \overline{BG}) = n$.

Proof: From (4.8)

$$\bar{A} - \overline{BG} = \bar{A} - \bar{B}(\bar{B}'P\bar{B})^{-1}\bar{B}'P\bar{A} = E\bar{A}, \tag{4.12}$$

where $E \equiv I_{n+m} - \bar{B}(\bar{B}'P\bar{B})^{-1}\bar{B}'P$. Since $EE = E$, the idempotent matrix,

$$\begin{aligned}\text{rank }(E) &= \text{trace }(E) \\ &= \text{trace }(I_{n+m}) - \text{trace }\{(\bar{B}'P\bar{B})^{-1}\bar{B}'P\bar{B}\} \\ &= n + m - m = n .\end{aligned}$$

Therefore, rank $(E\bar{A}) \leq n$ and thus $E\bar{A}$ is singular. Furthermore if $\bar{A}$ is nonsingular, then rank $(E\bar{A}) = n$, completing the proof.

The solution of the inverse problem for (4.5) - (4.7) can be obtained as follows. Equations (4.8), (4.11), and (4.12) imply[1)]

$$\bar{B}'P(\bar{A} - \overline{BG}) = 0 , \tag{4.13a}$$

---

1) Due to the transformation of the problem carried out in (4.4) - (4.7), we have the following homogeneous equation (4.13) to solve for P.

or

$$\bar{B}'1 \begin{bmatrix} I_n & 0 \\ -G & 0 \end{bmatrix} \bar{A} = 0 \ . \qquad (4.13b)$$

Since A is non-singular, $\bar{A}$ is also non-singular. Thus (4.13) is equivalent to

$$\bar{B}'P \begin{bmatrix} I_n & 0 \\ -G & 0 \end{bmatrix} = 0 \ . \qquad (4.14)$$

By inspection, if P is chosen to be

$$P = W[G, I_m] \ , \qquad (4.15)$$

where W is an $(n+m) \times m$ arbitrary non-zero matrix still to be decided,[1] then

$$P \begin{bmatrix} I_n & 0 \\ -G & 0 \end{bmatrix} = W[G, I_m] \begin{bmatrix} I_n & 0 \\ -G & 0 \end{bmatrix} = 0 \ . \qquad (4.16)$$

On the other hand, since $\bar{B} = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$, P may be chosen to be

$$P = \begin{bmatrix} M \\ 0 \end{bmatrix} \ , \qquad (4.17)$$

where M is an $n \times (n+m)$ non-zero matrix yet to be determined.

Then

$$\bar{B}'P = [0, I_m] \begin{bmatrix} M \\ 0 \end{bmatrix} = 0 \ . \qquad (4.18)$$

---

1) In fact, the matrix $[G, I_m]$ is an $m \times (n+m)$ complete left annihilator (with rank m) of the matrix $(\bar{A} - \bar{B}G)$ in (4.13a), i.e. $[G, I_m](\bar{A} - \bar{B}G) = 0$ (Bodewig, 1959).

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Therefore, the solution P of (4.13) is, from (4.15) and (4.17),

$$P = \begin{bmatrix} M \\ 0 \end{bmatrix} + W[G, I_m] \ . \tag{4.19}$$

The matrix P obtained by (4.19) still needs to satisfy (4.9) for optimality, and thus the following theorem leads to the solution of the inverse problem.

Theorem 1: A sufficient condition for the solution of the inverse problem for (4.1) - (4.3) is that there exist $M \neq 0$, $W \neq 0$ in (4.19) such that

$$P = P' \ ,$$

and

$$\bar{Q} = P - \bar{A}'P(\bar{A} - \overline{BG})$$

is symmetric and positive semidefinite.

As it is intractable at the moment to find the appropriate M and W in (4.19), a solution is sought for a special case in the next section more explicitly.

4.2 The Solution of a Class of Inverse Problem

For the economic system, in contrast to physical and technological systems, the relative weight on the state variables is more significant and heavier than the weight on the control efforts to be implemented. In fact, the existing government structure, for example, requires a fixed (basic) operational budget and does not require additional expenses in implementing a designed optimal control policy, because it is quite possible that the only costs associated with government policy are associated with levels and not with any changes in the policy.

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Thus this section is devoted to the case where the weighting matrix R for the control variables in the performance index is zero. This leads to an explicit formula for the solution of the inverse problem.

The inverse problem is considered for the system

$$x_{t+1} = Ax_t + Bu_t \ , \ x_0 = x^0 \tag{4.20}$$

a feedback control law

$$u_t = -G_t x_t \tag{4.21}$$

and with a modified performance indices of the form[1)]

$$J = \sum_{t=0}^{N} x_t' Q x_t \tag{4.22}$$

where rank (Q) = rank (B).

Since the system (4.20) is controllable, the above problem (4.20) - (4.22) can be transformed into the control equivalent canonical form by applying a nonsingular transformation T to the state variable x. We assume that (4.20) - (4.22) are in the canonical form (Tuel, 1967)[2)]; that is

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \\ A_{21} & A_{22} \end{bmatrix} \ , \quad B = \begin{bmatrix} 0 \\ \\ B_2 \end{bmatrix} \ , \tag{4.23}$$

with the following properties:

(i) B_2 is a nonsingular $(m \times m)$ triangular matrix,

---

1) Now problem involves a finite-time optimization in contrast to the previous section 4.1.

2) This transformed form is convenient for simplifying subsequent work and obtaining compact results. See (4.31) - (4.33) below.

(ii) the matrices A_{21} and A_{22} are $m \times (n-m)$ and $m \times m$ respectively but are otherwise arbitrary,

(iii) there exists a set of m positive integers $\{\ell_i\}$ depending on the structure of the system such that $\sum_{i=1}^{m} \ell_i = n$,

(iv)

$$A_{11} = \begin{bmatrix} W_1 & 0 & 0 \dots & 0 \\ 0 & W_2 & 0 \dots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ 0 & \dots\dots\dots & 0 & W_m \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} e_1 & 0 & \dots\dots & 0 \\ 0 & e_2 & 0 \dots & 0 \\ \vdots & & & \vdots \\ & & & 0 \\ 0 & \dots\dots\dots & 0 & e_m \end{bmatrix}$$

where A_{11} is $(n-m) \times (n-m)$, A_{12} is $(n-m) \times m$; W_i is a $(\ell_i - 1) \times (\ell_i - 1)$ matrix of the form

$$W = \begin{bmatrix} 0 & 1 & 0 & \dots & & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & 1 & 0 \\ 0 & \dots\dots\dots & & & 0 & 1 \\ 0 & \dots\dots\dots\dots & & & & 0 \end{bmatrix}$$

and e_i is a $(\ell_i - 1)$ column vector of the form

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} .$$

Since the control policy (4.21) is optimal, the feedback gain matrix is characterized by (Kleinman and Athans, 1966)

$$G_t = (G'P_{t+1} B)^{-1} B'P_{t+1} A , \tag{4.24}$$

where P_t satisfies the following matrix Riccati equation

$$\begin{aligned} P_t &= Q + A'P_{t+1} (A - BG_t) , \quad t = 0,1,\dots,N-1, \\ P_N &= Q , \end{aligned} \tag{4.25}$$

It is always possible to factorize a positive semi-definite matrix Q into the product DD' where D is a matrix of full rank corresponding to the rank of Q. Thus (4.24) and (4.25) lead to

$$\begin{aligned} P_{N-1} &= Q + A'Q[A - B(B'QB)^{-1}B'QA] \\ &= Q + A'DD'[A - B(B'DD'B)^{-1}B'DD'A] \\ &= Q + A'DD'A - ADD'B(D'B)^{-1}(B'D)^{-1}B'DD'A \\ &= Q , \end{aligned} \tag{4.26}$$

provided that the indicated inverse exists. Note that (4.26) implies, for all $t = 0,1,\dots,N$,

$$P_t = Q , \tag{4.27}$$

$$G_t = G = (D'B)^{-1}D'A , \tag{4.28}$$

which implies that the admissible control law (4.21) must be a constant feedback control. Moreover, (4.28) implies

$$D'(A - BG) = 0 , \tag{4.29}$$

which is analogous to (4.13) for the general inverse problem.

Since $A - BG$ is singular by Lemma 1, (4.29) has a non-trivial solution for D'. The matrix solution D can be chosen as

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$$D' = ZS \tag{4.30}$$

where S is a complete left annihilator of $(A - BG)$, i.e., $S(A - BG) = 0$, and Z is an arbitrary $m \times m$ nonsingular matrix.

Note that, for P defined in (4.10)

$$\begin{aligned} A - BG &= A - B(B'PB)^{-1}B'PA \\ &= [I_n - B(B'PB)^{-1}B'P]A \\ &= \begin{bmatrix} I_{n-m} & 0 \\ -P_{22}^{-1}P_{12}' & 0 \end{bmatrix} A \; . \end{aligned} \tag{4.31}$$

It is easy to see from (4.31) that

$$S = [P_{22}^{-1} P_{12}' \; , \; I_m] \; . \tag{4.32}$$

Moreover,

$$D'B = ZSB = Z(P_{22}^{-1} P_{12}' \; , \; I_m) \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = ZB_2 \; . \tag{4.33}$$

Since both Z and B_2 are nonsingular in (4.33), $D'B$ is also nonsingular. Thus, $(D'B)^{-1}$ exists as is necessary in (4.26).

The solution of the inverse problem for the canonical from (4.21) - (4.23) is therefore

$$Q = DD' = S'Z'ZS \; . \tag{4.34}$$

The solution for the original system is then simply

$$\hat{Q} = T'QT = T'S'Z'ZST \; , \tag{4.35}$$

where T is a non-singular matrix for the canonical transformation leading to (4.21) - (4.23).

The rank of Q and $\hat{Q}$ can be determined as follows by using the Sylvester's inequality

$$\text{rank }(S'Z') + \text{rank }(ZS) - m \le \text{rank }(S'Z'ZS) \le \{\text{rank }(S'Z'), \text{rank }(ZS)\}. \tag{4.36}$$

Since rank $(S'Z')$ = rank (ZS) = m, (4.36) implies that rank $(\hat{Q})$ = rank (Q) = m = rank (B).

It has been observed that the solution of the inverse problem can be obtained by determining $P_{22}^{-1} P_{12}'$ from the known matrices A, B and G in (4.31). The computational procedure will be illustrated through examples in Chapter Five. One must also note that the solution of the inverse problem is not unique because it depends upon the choice of a nonsingular matrix Z as is shown in (4.34).

4.3 The Inverse Optimal Control Problem in Macroeconomic Policy Models

The basic approach developed in the previous sections will be extended to a macroeconomic model. Although the system is in a more complicated form than what was discussed in the previous sections, it will be seen that the solution of the inverse problem developed in Section 4.2 also holds true for this economic system.

For many economic policy problems, linear macroeconomic models can be reduced to the following equation (Pindyck, 1972)

$$y_{t+1} = Ay_t + Bu_t + Cz_t + v_t \tag{4.37}$$

where y_t ... n-vector of endogenous variables,

u_t ... m-vector of control variables,

z_t ... r-vector of other exogenous variables outside the policy consideration,

v_t ... n-vector of random disturbances with zero mean vector and a constant variance-covariance matrix,

A,B,C, ... constant matrices.

The deterministic policy question is formulated by examining the conditional behavior of y_{t+1} in (4.37) under the influence of u_t given that the policy-maker sets the disturbance v_t equal to zero and utilizes a projection of values of z_t, $t = 0,1,\ldots,N$. Then, for the deterministic policy problem, (4.37) can be written as

$$y_{t+1} = Ay_t + Bu_t + Cz_t \, . \tag{4.38}$$

The performance index is given by

$$J = \tfrac{1}{2}(y_N - \hat{y}_N)'Q(y_N - \hat{y}_N) + \tfrac{1}{2}\sum_{t=0}^{N-1}(y_t - \hat{y}_t)'Q(y_t - \hat{y}_t) + u_t'Ru_t, \tag{4.39}$$

where $\hat{y}_t$ is an n-vector of target values of y_t, which is specified for the entire planning period.[1] The assumptions (A.1) - (A.3) in Section 4.1 will also be made here for simplicity of the analysis.

Using the discrete-time maximum principle and the matrix identity, $I - (Y + X)^{-1}X = (Y + X)^{-1}Y$, the optimal control policy for the system (4.38) - (4.39) can be expressed as (See Appendix):

$$\begin{aligned} u_t^* = &-(R + B'K_{t+1}B)^{-1}B'K_{t+1}Ay_t - (R + B'K_{t+1}B)^{-1}B'g_{t+1} \\ &-(R + B'K_{t+1}B)^{-1}B'K_{t+1}Cz_t \, , \end{aligned} \tag{4.40}$$

where K_t and g_t satisfy the following equations

---

[1] Optimal control problem for (4.38), (4.39) is often referred to as a linear tracking problem (Athans and Falb, 1966).

$$K_t = Q + A'K_{t+1}[A - B(R + B'K_{t+1} B)^{-1}B'K_{t+1} A] , \qquad (4.41)$$

$$g_t = -A'[K_{t+1} B(R + B'K_{t+1} B)^{-1}B' - I]g_{t+1}$$
$$+ A'(K_{t+1} - K_{t+1} B(R + B'K_{t+1} B)^{-1}B'K_{t+1})Cz_t - Q\hat{y}_t . \qquad (4.42)$$

Now, consider the case where $R = 0$ in the above model. Let the positive semi-definite matrix Q be factorized into the product DD' where rank (D) = rank (Q) and $(D'B)$ is nonsingular. Then, the matrix Riccati equation (4.41) becomes

$$K_t = Q + A'K_{t+1}[A - B(B'K_{t+1} B)^{-1}B'K_{t+1} A] , \; K_N = Q, \qquad (4.43)$$

and as in Section 4.2,

$$K_{N-1} = Q + A'DD'[A - B(B'DD'B)^{-1}B'DD'A] \qquad (4.44)$$
$$= Q + A'DD'A - A'DD'B(D'B)^{-1}(B'D)^{-1}B'DD'A = Q,$$

which implies $K_t = Q$ for all $t = 0,1,2,\dots,N$. Similarly, (4.42) is reduced to

$$g_t = -A'[K_{t+1} B(B'K_{t+1} B)^{-1}B' - I]g_{t+1} \qquad (4.45)$$
$$+ A'[K_{t+1} - K_{t+1} B(B'K_{t+1} B)^{-1}B'K_{t+1}]Cz_t - Q\hat{y}_t,$$

and

$$g_{N-1} = -A'[DD'B(B'DD'B)^{-1}B' - I](-Q\hat{y}_N)$$
$$+ A'[DD' - DD'B(B'DD'B)^{-1}B'DD']Cz_N - Q\hat{y}_N$$
$$= A'[D(B'D)^{-1}B' - I]DD'\hat{y}_N$$
$$+ A'[DD' - DD'B(D'B)^{-1}(B'D)^{-1}B'DD']Cz_N - Q\hat{y}_{N-1}$$
$$= -Q\hat{y}_{N-1}, \qquad (4.46)$$

which implies that

$$g_t = -Q\hat{y}_t , \; t = 0,1,\dots,N . \qquad (4.47)$$

Again, consider the optimal control policy u_t^* in (4.40) for the period N-1, i.e.,

$$\begin{aligned} u_{N-1}^* &= -(B'DD'B)^{-1}B'DD'Ay_{N-1} - (B'DD'B)^{-1}B'g_N - (B'DD'B)^{-1}B'DD'Cz_{N-1} \\ &= -(D'B)^{-1}D'Ay_{N-1} + (D'B)^{-1}(B'D)^{-1}B'DD'\hat{y}_N - (D'B)^{-1}D'Cz_{N-1} \\ &= -(D'B)^{-1}D'Ay_{N-1} + (D'B)^{-1}D'\hat{y}_N - (D'B)^{-1}D'Cz_{N-1}\,, \qquad (4.48) \end{aligned}$$

which implies that

$$\begin{aligned} u_t^* &= -(D'B)^{-1}D'Ay_t + (D'B)^{-1}D'\hat{y}_{t+1} - (D'B)^{-1}D'Cz_t \qquad (4.49) \\ &\equiv -Gy_t + E\hat{y}_{t+1} - Fz_t\,, \quad t = 0,1,\ldots,N-1. \end{aligned}$$

Thus the optimal feedback gains are

$$\begin{aligned} (D'B)^{-1}D'A &= G, \\ (D'B)^{-1}D' &= E, \qquad (4.50) \\ (D'B)^{-1}D'C &= F\,. \end{aligned}$$

The equations in (4.50) are equivalent to

$$\begin{aligned} D'(A - BG) &= 0, \\ D'(I - BE) &= 0, \qquad (4.51) \\ D'(C - BF) &= 0, \end{aligned}$$

which implies that

$$\begin{aligned} D'(A - BG) &= D'[A - B(D'B)^{-1}D'A] = D'[I - B(D'B)^{-1}D']A = 0 \\ D'(I - BE) &= D'[I - B(D'B)^{-1}D'] = 0 \qquad (4.52) \\ D'(C - BF) &= D'[C - B(D'B)^{-1}D'C] = D'[I - B(D'B)^{-1}D']C = 0\,. \end{aligned}$$

Equations in (4.52) show that a solution for D' for any equation will also satisfy remaining two equations. This implies that the solution of the inverse problem for a linear regulator model in Section 4.2 still holds true for this linear tracking problem. Using the similar arguments that were made in Section 4.2, the solution for D' in (4.52) is

$$D' = ZS \ , \qquad (4.53)$$

where $S[I - B(D'B)^{-1}D'] = 0$, and Z is an arbitrary $m \times m$ non-singular matrix. Thus the required solution for the weighting matrix Q is given by

$$Q = S'Z'ZS \ . \qquad (4.54)$$

The steps obtaining the solution of the inverse problem were studied in Section 4.2.

Recall that the inverse problem (4.38) - (4.40) in this section was set up for the macroeconomic policy problems. Thus, the solution of the problem provides the explicit procedure to quantify the performance index (or preference function) for the policy making processes, enabling the decision maker to get the numerical measure of the relative "welfare" weights to be given to the competing policy goals.

Note, however, the solution (4.54) is not unique in this problem where the performance index does not contain the costs of adjusting the control policies. This implies that the optimal control policies are fairly robust against different performance indices under the conditions stated in this inverse problem.

The developed techniques so far will be illustrated through examples in the following chapter.

CHAPTER FIVE

APPLICATIONS OF THE INVERSE OPTIMAL CONTROL TO MACROECONOMIC MODELS

The dynamic economy under study is characterized by the following short-run macroeconomic model.[1)]

$$\begin{aligned} p_{t+1} &= a_1 p_t + a_2 w_t - a_2 y_t + a_3 m_t \\ r_{t+1} &= b_1 p_t + b_2 r_t + b_3 y_t - b_4 m_t \\ u_{t+1} &= c_1 p_t + c_2 u_t + c_3 w_t - c_4 y_t \end{aligned} \tag{5.1}$$

where

p; price

r; interest rates

u; unemployment rate

m; money supply

y; real production

w; money wage rate

a_i, b_i, c_i; constant coefficients (positive),

(all the variables are in terms of quarterly change rates.)

The price equation in (5.1) is basically an inflationary model which involves the cost-push effects of wages and the aggregate demand-pull effect. That is, price increases are

1) This is a submodel of Pindyck's short-run macroeconomic policy model of the U.S. (1972), slightly modified by Shupp's inflationary model of the U.S. (1972).

positively related to increases in average unit labor cost and to increases in money supply.

The second equation in (5.1) expresses the interest adjustments to the disequilibrium in the markets. Since demand for bonds changes together with changes in real value of bonds and money stock (hence, inversely to the price changes) and the supply of bonds (investments) are positively related to the changes in income, interest rate variations are positively related to price and income changes, and inversely to the money supply.

The last equation of the model explains the changes in unemployment rates in the labor market.

Now, define a performance index for the system (5.1) as follows:

$$J = \sum_{t=0}^{N} (x_t - x_t^*)' Q (x_t - x_t^*) \ , \tag{5.2}$$

where $x_t = (p_t, r_t, u_t)'$ and x_t^* is the target trajectory of x_t, and $Q = Q'$ is positive semi-definite.

For the purpose of illustration, we assign numerical values to the coefficients of the system (5.1) based on the models of Pindyck (1972) and Shupp (1972).

$$\begin{bmatrix} p_{t+1} \\ r_{t+1} \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} 0.77 & 0 & 0 \\ 0.48 & 0.37 & 0 \\ -0.01 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} p_t \\ r_t \\ u_t \end{bmatrix} + \begin{bmatrix} 0.02 & 0.23 \\ -0.16 & 0 \\ 0 & 0.002 \end{bmatrix} \begin{bmatrix} m_t \\ w_t \end{bmatrix} + \begin{bmatrix} -0.23 \\ 0.03 \\ -0.0004 \end{bmatrix} y_t \ . \tag{5.3}$$

Eigenvalues of the system (5.3) are $\lambda_1 = 0.77$, $\lambda_2 = 0.37$, and $\lambda_3 = 0.8$. Since all the eigenvalues are real and less than unity in absolute value, the basic homogeneous solution for the time paths of state variables, p_t, r_t, u_t, apart from the influence of exogenous variables and of initial conditions, are damped, non-oscillatory movements, i.e., all responses of the open-loop economic system converge to their equilibrium values without oscillations (Kuo, 1970).

5.1 A Single-Control Model

We consider the first two equation submodel of the system (5.3), assuming that the labor market is always in equilibrium and also noting that unemployment rate u_t influences neither the price changes p_t nor the interest rate variations r_t.

$$\begin{bmatrix} p_{t+1} \\ r_{t+1} \end{bmatrix} = \begin{bmatrix} 0.77 & 0 \\ 0.48 & 0.37 \end{bmatrix} \begin{bmatrix} p_t \\ r_t \end{bmatrix} + \begin{bmatrix} 0.02 \\ -0.16 \end{bmatrix} m_t + \begin{bmatrix} 0.23 & -0.23 \\ 0 & 0.03 \end{bmatrix} \begin{bmatrix} w_t \\ y_t \end{bmatrix} . \quad (5.4)$$

In the submodel (5.4), w_t and y_t are treated as variables outside the policy considerations. For the purpose of simplicity, it is assumed that the goal of policy is to maintain a stable price and interest rate, i.e., $p^* = r^* = 0$, and at the beginning of the planning period, both price and interest rates have been stabilized; $p_0 = r_0 = 0$. In addition, as the model is of the short-run nature, constant values are given to w_t and y_t; $w_t = 0.9\%$, $y_t = 0.38\%$.

For the case with zero control, $m_t = 0$, i.e., keeping money supply constant, the behavior of p_t and r_t over the period is shown in Figure 1.

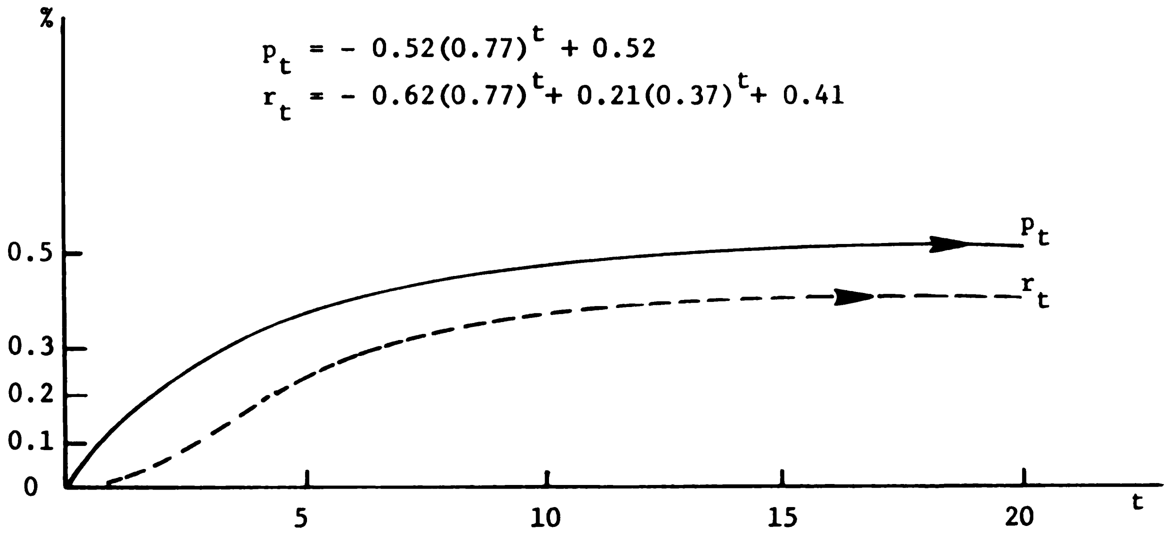

Figure 1. Responses of System (5.4) with Zero Control

It is seen from Figure 1 that when the money supply is kept constant, the policy goal, $p_t = r_t = 0 \quad (t > 0)$, is not attained.

Now, assume that policy makers want to achieve price stability using a feedback control of the form (4.49) in Chapter Four:

$$m_t^* = g_1 p_t + g_2 r_t + k , \tag{5.5}$$

where g_1, g_2, and k are constant scalars.

Specifically, it is desired that zero steady-state value of p_t be attained by using a control of the form (5.5).

A digression is in order at this point. Let the system (5.4) be written as

$$x_{t+1} = Ax_t + Bm_t + Cz_t , \tag{5.6}$$

where $x_t = (p_t , r_t)'$, $z_t = (w_t , y_t)' = (0.9, 0.38)'$,

$$A = \begin{bmatrix} 0.77 & 0 \\ 0.48 & 0.37 \end{bmatrix}, \quad B = \begin{bmatrix} 0.02 \\ -0.16 \end{bmatrix}, \quad C = \begin{bmatrix} 0.23 & -0.23 \\ 0 & 0.03 \end{bmatrix}.$$

Note in this example, Cz_t is a constant vector.

From (4.49) in Chapter Four, the feedback control policy (5.5) is also written as

$$m_t = gx_t + gA^{-1}Cz_t \quad , \tag{5.7}$$

where $g = (g_1, g_2)$.

Then the system under the feedback control (or closed-loop system) is governed by the relation

$$x_{t+1} = (A + Bg)x_t + (BgA^{-1} + I)Cz_t \quad . \tag{5.8}$$

The steady-state values of x_t in the closed-loop system (5.8) is given by[1)]

$$x^e = (A + Bg)x^e + (BgA^{-1} + I)Cz_t$$

or

$$x^e = (I - A - Bg)^{-1}(BgA^{-1} + I)Cz_t \ . \tag{5.9}$$

From (5.9), we derive

$$Bg(A^{-1}Cz_t + x^e) = (I - A)x^e - Cz_t \quad . \tag{5.10}$$

By solving (5.10) for g, it is possible to find the required feedback control rule for a prescribed steady-state value vector x^e of the state variables.

Note, however, that $g(A^{-1}Cz_t + x^e)$ in (5.10) is a scalar. Define

$$g(A^{-1}Cz_t + x^e) = k, \tag{5.11}$$

where k is a scalar.

---

1) Since the closed-loop system (5.8) is assumed to be stable, the indicated inverse in (5.9) exists.

Then it is seen that in order to solve (5.10) for g, x^e should be chosen such that

$$x^e = (I - A)^{-1}(Bk + Cz_t) \qquad (5.12)$$

holds.[1)]

If x^e in (5.12) is considered as an arbitrarily chosen known vector, (5.12) becomes a two-equation system with only one unknown k, and in general, cannot be solved. One element of x^e should be set "free" to be determined together with k by the system. This implies that only one element of x^e can be specified arbitrary if k is a scalar.[2)] That is, if there is only one control variable in the model, only one state variable of the closed-loop system can be steered to have the prescribed steady-state value. This can be generalized that the number of the policy goal variables which can be controlled to attain the steady-state values is equal to the number of the control variables in the model (cf. Tinbergen, 1967).

As the state variable $x_t = (p_t , r_t)'$ in (5.8) are required to have the steady-state values, the closed-loop system must be stable, and eigenvalues of the closed-loop system should be restricted to be less than unity in absolute magnitude. This implies that the feedback gain vector of the required control for the price stabilization should be determined so as to make the closed-loop system be stable ("arbitrary pole assignment problem" in Chapter Two).

---

1) Since all the eigenvalues of A are less than unity in absolute value, $(I - A)^{-1}$ exists.

2) k in (5.11) is a scalar since the system (5.6) has one control, m_t.

That is, from (2.5) in Chapter Two, the feedback gain vector g needs to satisfy

$$g = h'T^{-1} \quad \text{or} \quad gT = h' , \tag{5.13}$$

where $h' = (1, 1)$ and a nonsingular matrix $T = (t_1 , t_2)$. The columns of matrix T, t_1 and t_2 are determined for the prescribed closed-loop eigenvalues λ_i as follows.

$$(\lambda_i I - A)t_i = B , \; i = 1,2 \quad . \tag{5.14}$$

Equations (5.13) and (5.14) imply

$$gt_i = 1 . \tag{5.15}$$

Since the solution for the inverse problem requires that the closed-loop system matrix, $A + Bg$, is singular (Lemma 1, Chapter Four), we assign $\lambda_1 = 0$ for (5.14) and find the corresponding t_1.

Then, from (5.11) and (5.15), the following relations should be satisfied simultaneously for the required feedback gain vector g:

$$\begin{aligned} -0.026535\, g_1 - 0.215176\, g_2 &= 0.9428 \\ -0.026\, g_1 + 0.4661\, g_2 &= 1 . \end{aligned} \tag{5.16}$$

Solving (5.16) for g_1 and g_2, we find the following "desired" feedback control policy:

$$m_t^* = -36.4\, p_t + 0.11\, r_t - 5.7 \quad . \tag{5.17}$$

The behavior of the resulting closed-loop system is shown in Figure 2.

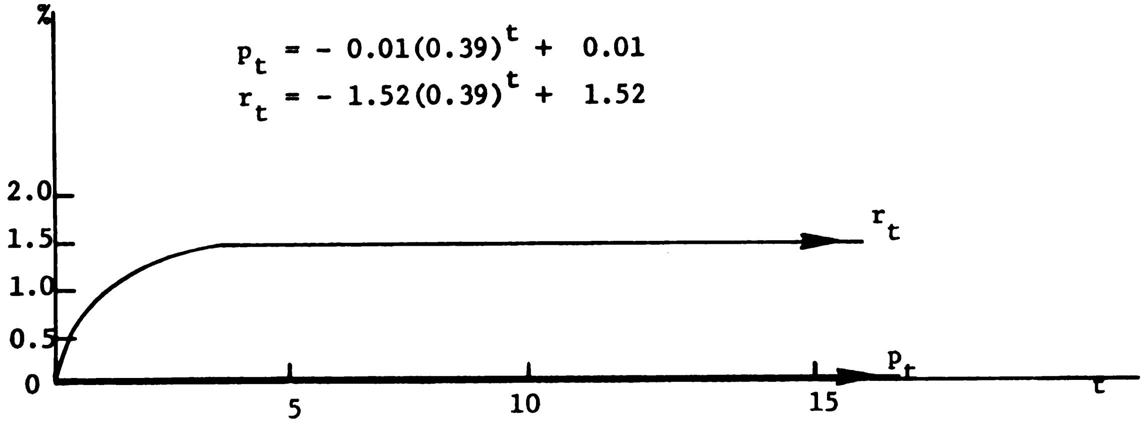

Figure 2. Responses of System (5.4) under Control (5.17)

Given the chosen feedback control policy (5.17), the weighting matrix Q in the performance index can be determined by the method of inverse optimal control problem discussed in the previous chapter.

Recall (4.52) - (4.54) in Chapter Four:

$$D'(A + Bg) = 0,$$

$$Q = DD' .$$

For the given system (5.4) and the feedback gain vector g (5.17),

$$A + Bg = \begin{bmatrix} 0.04 & 0.00224 \\ 6.31 & 0.352 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 157 & 0 \end{bmatrix} E , \tag{5.18}$$

where E is the inverse of the product of the elementary (column) operation matrices. Thus,

$$D' = z(-157 \quad 1) , \tag{5.19}$$

where z is a non-zero, otherwise arbitrary, scalar, Choosing z = 1,

$$Q = \begin{bmatrix} 24649 & -157 \\ -157 & 1 \end{bmatrix} . \qquad (5.20)$$

From Figure 2, it is seen that one of the targets, price stability, is now attained by the prescribed feedback control policy. Also, since the dominant eigenvalue of the closed-loop system, 0.39, is smaller than that 0.77 of the zero-control system, trajectories of both price and interest rate converge to their steady-state values in shorter time in this case.

Note, however, that the steady-state value of the interest rate is higher in the closed-loop system with the chosen control rule (5.17). This can be explained by the fact that the optimum control policy involves the continuously decreasing money supply (Figure 3), effecting the price stability, but on the other hand, causing the interest rate to increase continuously.

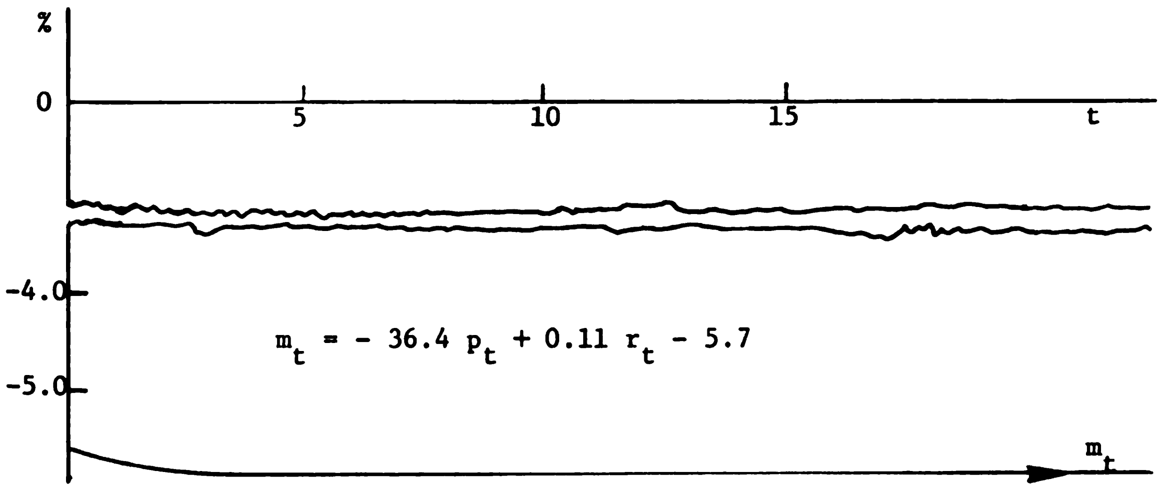

Figure 3. Optimal Feedback Control for Price Stabilization

It is to be noted that the weighting matrix Q in (5.20) gives the very heavy weight to the price variable compared to the interest rate. Negative sign of the off-diagonal elements of Q indicates that there exists "trade-off" between the monetary policy for price regulation and that for interest rate control.

Anologously, another feedback control policy is found for interest rate stabilization:

$$m_t^* = 3p_t + 2.31\ r_t + 0.06\ . \tag{5.21}$$

Time paths of price and interest changes and of the money supply changes under this policy (5.21), are shown in Figure 4 and Figure 5, respectively.

In this case, the closed-loop system matrix is

$$A + bg = \begin{bmatrix} 0.83 & 0.05 \\ -0.001 & -0.00006 \end{bmatrix}$$

and the weighting matrix Q is found to be

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 594441 \end{bmatrix}\ . \tag{5.22}$$

As expected, in (5.22) the interest rate variable is given larger weight compared to the price variable.

One remark concerning the chosen control policies (5.17) and (5.21): the continuously decreasing or continuously increasing money supply policy would not be feasible in reality, for, say, political reasons, if the change rates are big all over the period. It is important, however, to observe that in the examples, the control is not constrained and no cost for conducting control policy

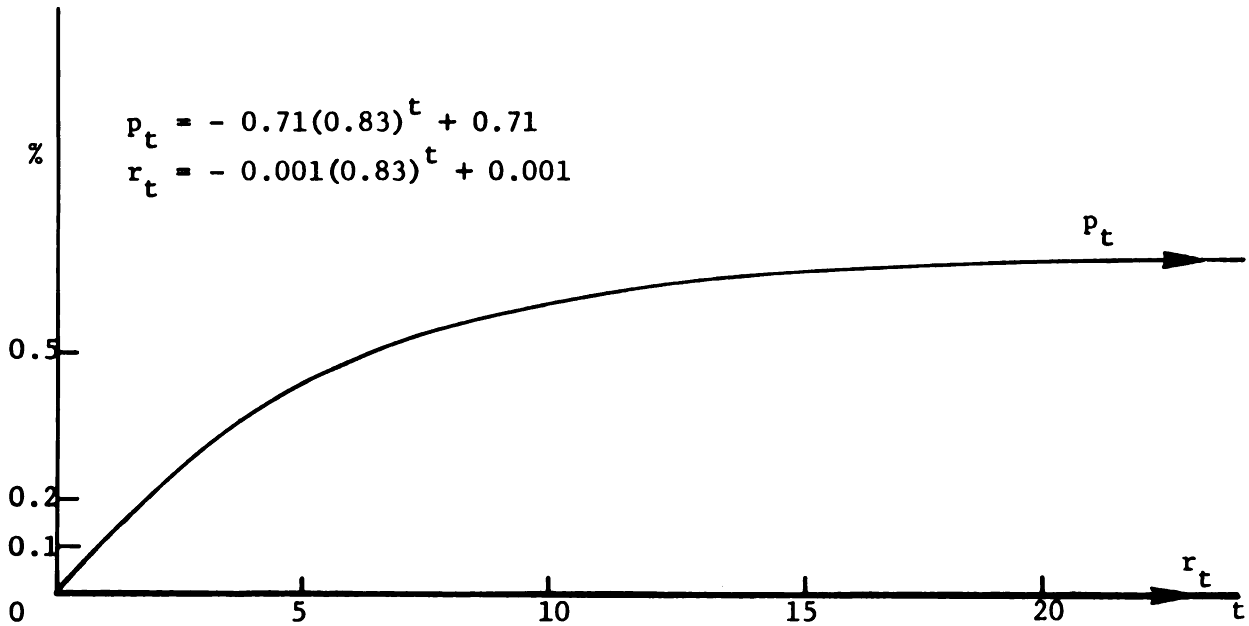

Figure 4. Responses of the System (5.4) under Control (5.21).

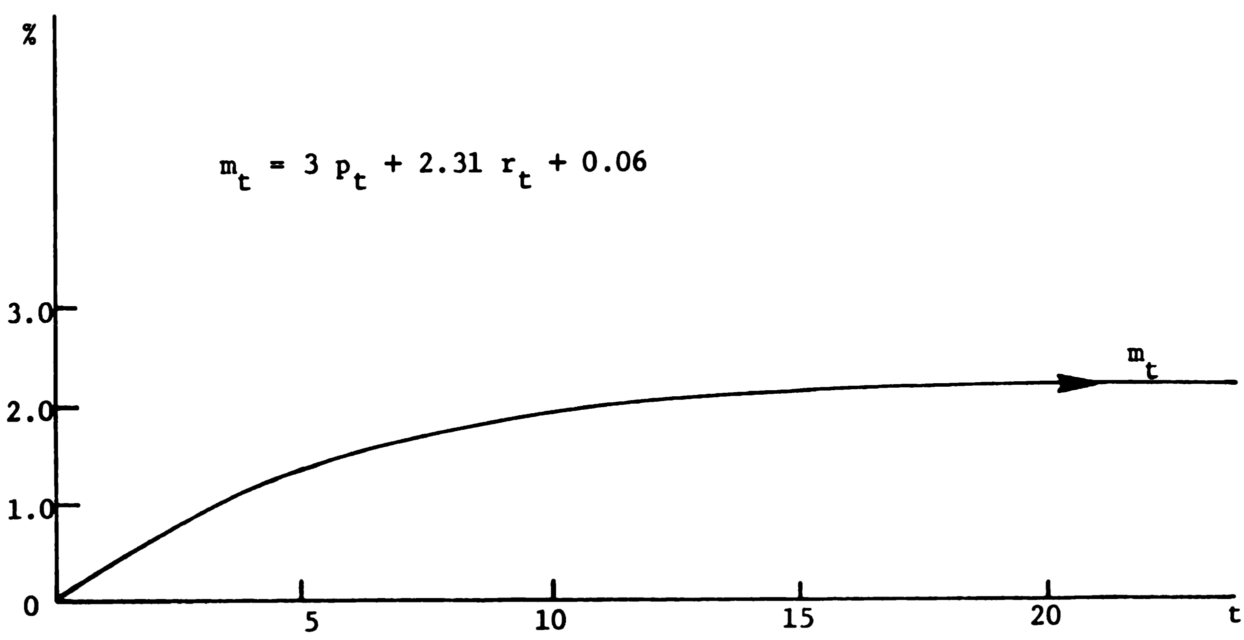

Figure 5. Optimal Feedback Control for Interest Stabilization

is included in the performance criterion, and accordingly the control is free to vary in the control space (i.e., the real line in this case). Obviously, further research needs to be done when the controls have some constraints and/or the costs of conducting the policies are incorporated in the performance index.

5.2 Two-Control Model

Dropping the perfectly competitive, full employment assumption for the labor market, we will consider the overall system given by (5.3).

In this model, we choose w_t as a control variable, since in an assumed imprefectly competitive market, the discretionary pricing power of unions can be used to demand monetary wage increases in excess of productivity gains. To the extent that price expectations are based upon recent price changes, these demands for larger monetary wage increases will persist in the absence of any continuing excess demand. Furthermore, these large increases in money wage rates induce a higher rate of price inflation which confirms the original expectations. In such circumstances, temporary wage-price controls by curbing these expectations may prove effective. Since it is generally agreed that wage controls are the more easily administered, w_t is chosen to be a control variable.

The time paths of state variables for the system with zero controls, $m_t = 0 = w_t$, i.e., when both money supply and money wage rate are kept constant, are shown in Figure 6.

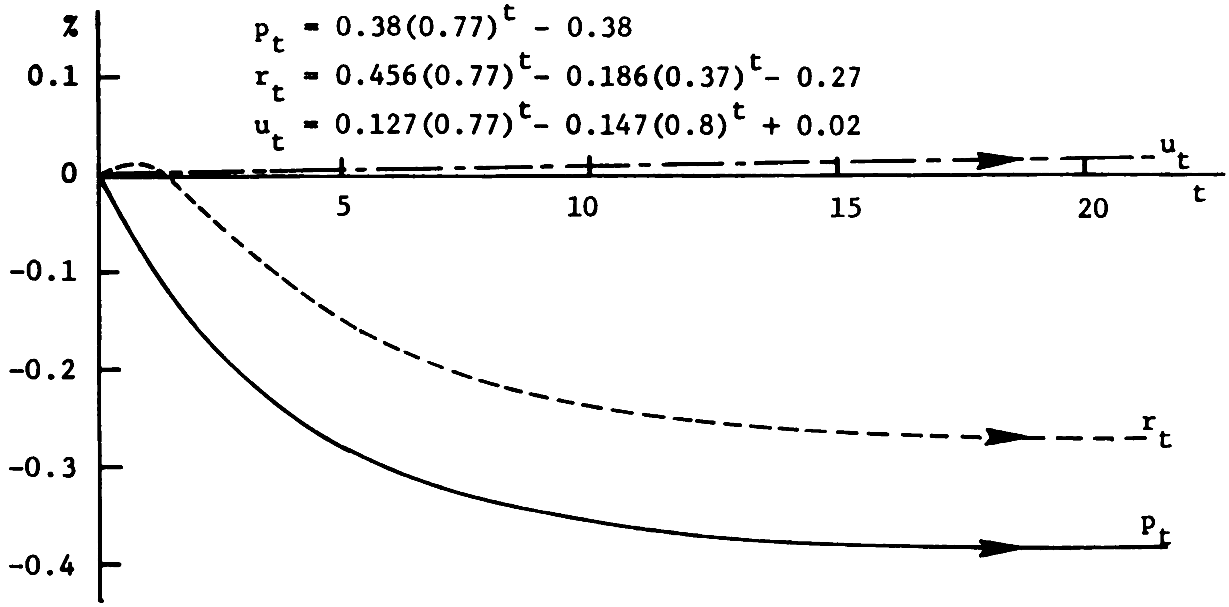

Figure 6. Responses of System (5.3) with Zero Control

In order to have price stability with a linear feedback control, the chosen feedback gain vector $g = (g_1, g_2, g_3)'$ should satisfy the following relations simultaneously [(5.11), (5.15) above]:

$$-0.028125\ g_1 + 0.027015\ g_2 + 0.000348\ g_3 = 0.08763$$

$$-0.3247\ g_1 + 0.8536\ g_2 - 0.00656\ g_3 = 1$$

$$-0.5\ g_1 + 4\ g_2 - 0.01321\ g_3 = 1\ . \qquad (5.23)$$

From (5.23), the following feedback control policy is obtained:

$$\begin{bmatrix} m_t^* \\ w_t^* \end{bmatrix} = \alpha(-3.44\ p_t - 0.21\ r_t - 9.76\ u_t + 0.37) \qquad (5.24)$$

where $\alpha = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.[1)]

The resulting time paths of state variables and of the control variable are shown in Figure 7 and Figure 8, respectively.

With the selected control policy (5.24), the closed-loop system matrix is

$$A + B\alpha g = \begin{bmatrix} -0.09 & -0.053 & -2.44 \\ 1.03 & 0.404 & 1.56 \\ -0.02 & -0.0004 & 0.78 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.35 & -0.05 & 0 \end{bmatrix} E ,$$

where E is a nonsingular matrix.

Applying the inverse optimal control method as discussed in (4.52) - (4.54),

$$D'(A + B\alpha g) = 0,$$

$$DD' = Q ,$$

where $D' = z(0.35,\ 0.05,\ 1)$, (5.25)

the weighting matrix Q in the performance index is found by setting $z = 1$ in (5.25) to be

$$Q = \begin{bmatrix} 0.12 & 0.02 & 0.35 \\ 0.02 & 0.002 & 0.05 \\ 0.35 & 0.05 & 1 \end{bmatrix} . \qquad (5.26)$$

1) For the computational convenience, the system is converted to a single-control model [(2.8) - (2.11), Chapter Two].

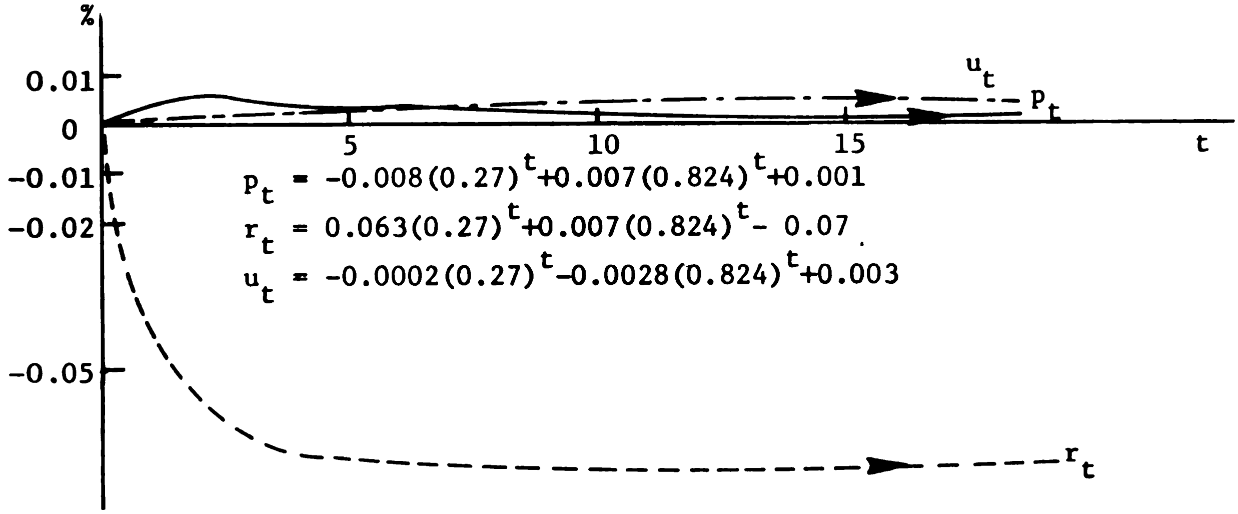

Figure 7. Responses of System (5.3) under Control (5.24).

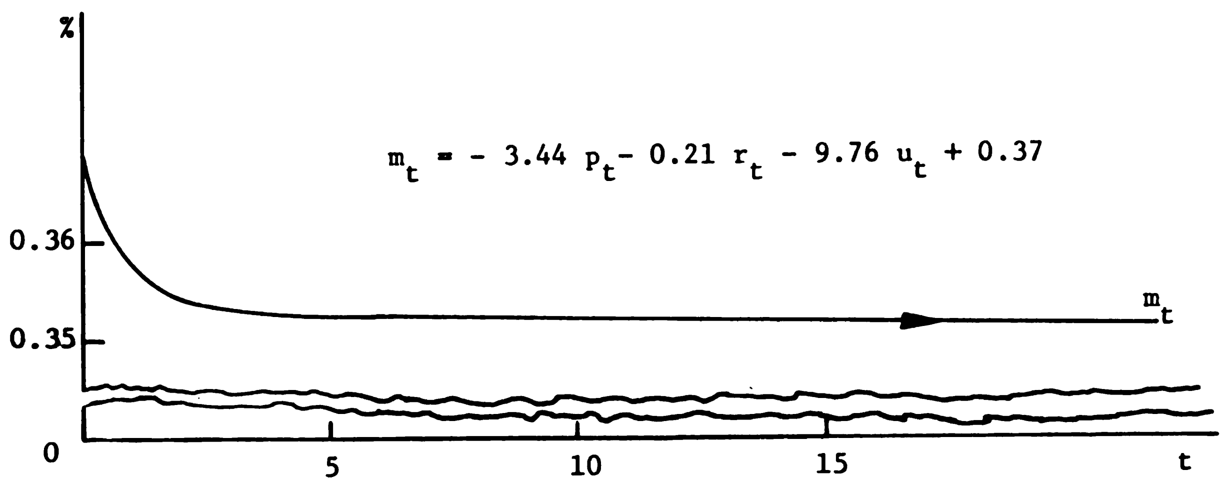

Figure 8. Optimal Feedback Control for Price Stabilization.

Similarly, the control policy which would achieve full employment is:

$$\begin{bmatrix} m_t^* \\ w_t^* \end{bmatrix} = \alpha(-7.7\ p_t + 0.07\ r_t + 237.1\ u_t + 0.5)$$

where $\alpha = (1\ ,\ 1)'$. (5.27)

The corresponding time paths of the state and control variables are shown in Figure 9 and Figure 10.

The weighting matrix for the performance index, in this case is

$$Q = \begin{bmatrix} (0.021)^2 & 0 & -0.021 \\ 0 & (0.0007)^2 & 0.0007 \\ -0.021 & 0.0007 & 1 \end{bmatrix} . \quad (5.28)$$

Comparing the two cases considered so far for the system (5.3) (price stabilization policy and full employment policy), we observed the following.

(i) The model reveals the familiar Phillips Curve relation for the price changes and unemployment variations. That is, when the steady-state value of u_t is reduced from 0.003% to 0.001%, that of p_t increases from 0.001% to 0.045% (Figure 7 and Figure 9).[1)]

(ii) Relative weight for the price variable is higher for the price stabilization policy compared to the full employment policy (vice versa), even though the major weight is found to be given to the unemployment variable in both cases.

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1) Magnitudes of figures may be too small to claim the Phillips curve relation. However, the figures show the essential characteristics of the curve.

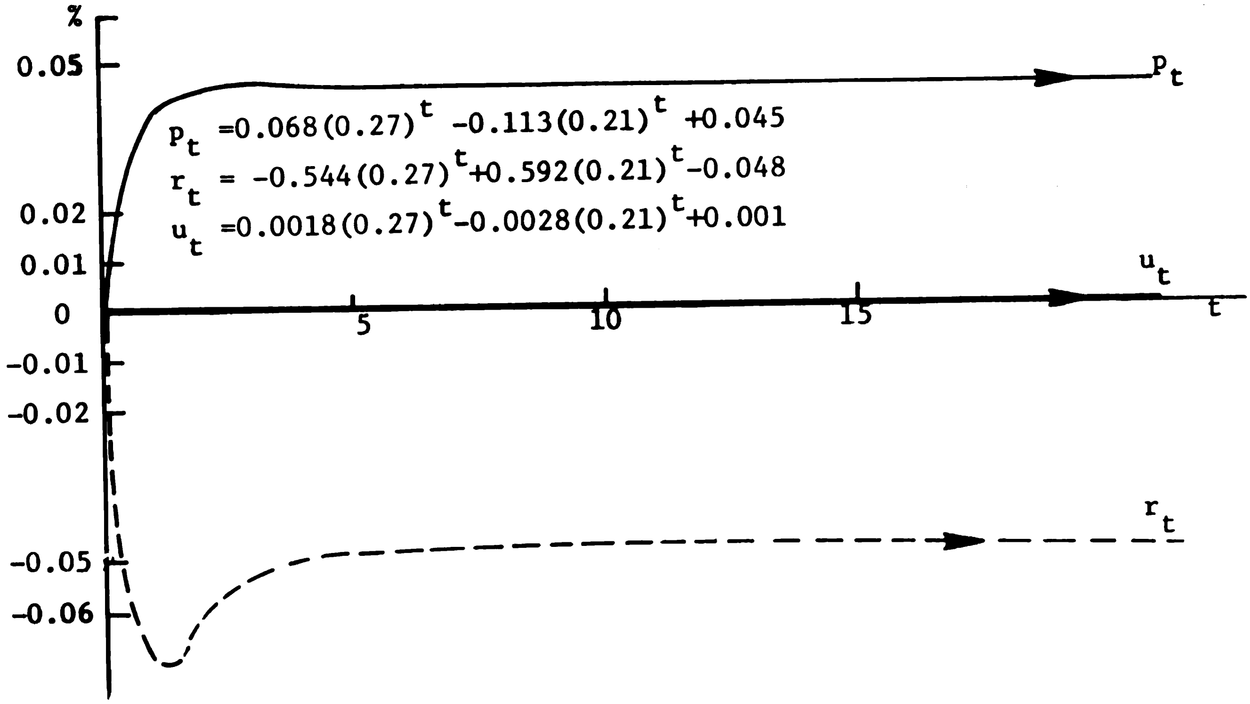

Figure 9. Responses of System (5.3) under Control (5.27).

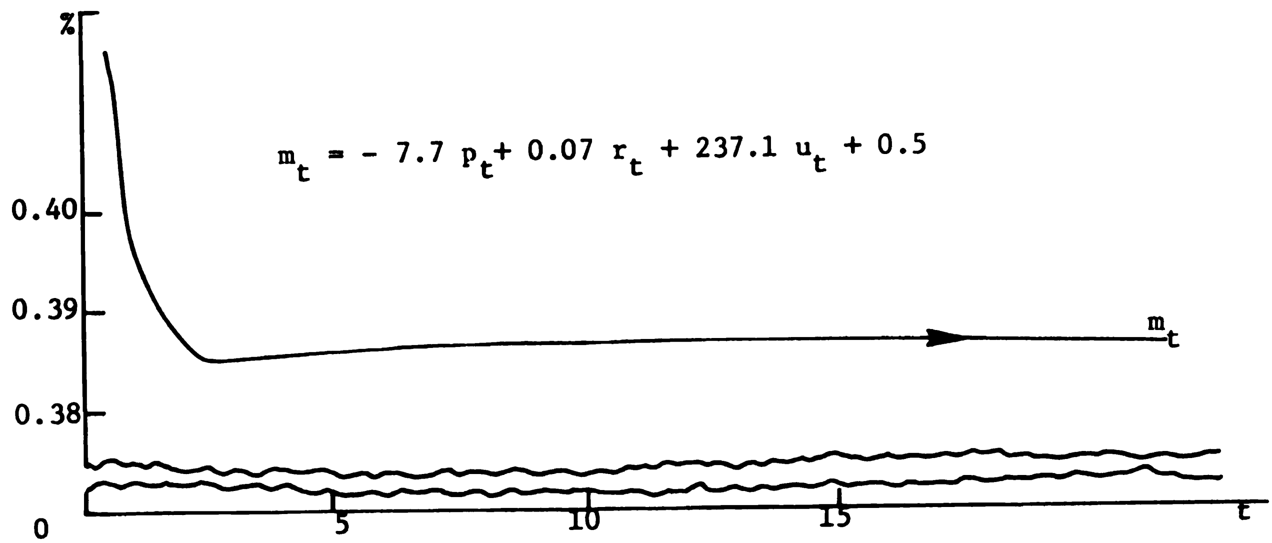

Figure 10. Optimal Feedback Control for Full Employment.

CHAPTER SIX

CONCLUSIONS AND RESEARCH RECOMMENDATIONS

6.1 Summary of Conclusions

By solving the inverse optimal control problem, this study has demonstrated the feasibility of quantifying performance criteria for the economic decision making problems. Economic processes, which are characterized by complex and often unknown relationships among their constituent components, require one to formulate the decision making problems in general, in terms of discrete-time difference equation systems. This demands the solution of the inverse problem for the discrete-time multi-control systems for the choice of the performance indices.

A sufficient condition was developed for the solution of the inverse problem for a linear discrete-time multi-control regulatory process with a quadratic performance index. An explicit solution was obtained for a special case where the performance functional does not contain the control variables, and the result was further extended to a linear tracking problem, which could be easily adapted to the macroeconomic regulation problems.

It has been shown that the solution of the inverse problem is not unique in general, which implies that the optimal control policies are fairly robust against different performance indices under the conditions stated above. Also, it was found that the solution of the inverse problem for a linear regulator holds true without any modification for the linear tracking problem.

In Chapter Five, the illustrations about the applications of the developed techniques for the selection of the performance criteria emphasize three points; the determination of the steady-state values of the trajectories of goal variables which are under control, the computation of the feedback control policy, and the construction of a performance index. It was shown that the number of the policy goal variables that can be made to attain the prescribed steady-state values is equal to the number of the control variables in the model. It was also demonstrated that the relative weights given to the competing policy goals in the performance index can be quantitatively determined by the method of the inverse optimal control problem for a dynamic policy model.

6.2 Recommendations for Further Studies

Further research is needed to find a general solution of the inverse problem for the performance indices which involve the state variables as well as the control variables. This will enable one not only to quantify a more general class of performance functional but also to compare quantitatively the relative merits of different control policies.

Another line of research can be conducted for defining the performance functional of a more general type, which could even reflect the sociological, group dynamic, and game theoretical aspects of the administrative interactions involved in the determination of policy objectives.

Considering the fact that the economic decisions are typically made under uncertainty and decision makers increase their knowledge by the cumulative past experiences, the study of the adaptive control problems would greatly improve our knowledge of the economic decision processes.

Since the adaptive systems include a performance index as an essential function which permits correction of system dynamic response during actual operation, the performance index takes on much greater significance in the multistage adaptive decision processes (Murphy, 1965).

BIBLIOGRAPHY

BIBLIOGRAPHY

Adorno, D.S. (1962), "Optimal Control of Certain Linear Systems with Quadratic Loss," Information and Control, Vol. 5, pp. 1-12.

Advani, R. and R. Mukundan (1970), "Optimal Control Methods Applied to Economic Systems," International J. of Systems Science, Vol. 1, No. 2, pp. 153-72.

Allen, R.G.D. (1966), Mathematical Economics, St. Martin's Press, New York.

Anderson, B.D.O. and J.B. Moore (1971), Linear Optimal Control, Prentice-Hall, Englewood Cliffs.

Aoki, M. (1967), Optimization of Stochastic System, Academic Press, New York.

________ (1973), "Sufficient Conditions for Optimal Stabilization Policies," Review of Economic Studies, Vol. XL(1), No. 121, Jan., pp. 131-8.

Arrow, K.J . S. Karlin, and H. Scarf (eds.) (1958), Studies in the Mathematical Theory of Inventory and Production, Stanford Univ. Press, Stanford.

Arrow, K.J., M. Kurz (1970), Public Investment, the Rate of Return, and Optimal Fiscal Policy, Baltimore.

Athans, M. and P. Falb (1966), Optimal Control -- An Introduction to the Theory and Its Applications, McGraw Hill, New York.

Barnard, G.A. (1959), "Control Charts and Stochastic Processes," J. of Royal Statistical Society, Series B., Vol. 21, pp. 239-71.

Bellman, R.E. (1957), Dynamic Programming, Princeton Univ. Press, Princeton.

Bellman, R. and R. Kalaba (1961), "Dynamic Programming and Feedback Control," 1960 Proc. IFAC, Butterworths, pp. 460-76.

________ (1963), "An Inverse Problem in Dynamic Programming and Automatic Control," Journal of Math. Anal. and Appl., Vol. 7, No. 2, Oct., pp. 322-35.

Bellman, R. (1970), "Dynamic Programming and Inverse Problems in Mathematical Economics," Journal of Math. Anal. and Appl., Vol. 29, No. 2, Feb., pp. 424-28.

Benavie, A. (1970), "Economics of the Maximum Principle," Western Economic Journal, Vol. VIII, No. 4, Dec., pp. 426-30.

Ben-Porath, Y. (1967), "The Production of Human Capital and the Life Cycle of Earnings," J. of Political Economy, Vol. 75, No. 4, Aug., Part I, pp. 353-65.

Bodewig, E. (1959), Matrix Calculus, North-Holland, Amsterdam.

Boltyanski, V.G., R.V. Gamkrelidze, E.F. Mishchenko, and L.S. Pontryagin (1961), "The Maximum Principle in the Theory of Optimal Processes of Control," 1960 Proc. IFAC, London, Butterworths, pp. 454-59.

Brockett, R.W. (1970), Finite Dimensional Linear Systems, John Wiley, New York, London, Sydney, Toronto.

Brown, J.F. and B.A. Kalymon (1971), "A Simulation Programming Approach to Bank Asset Management in an Uncertain Environment," Working Paper No. 171, Western Management Science Institute, UCLA, Los Angeles, March.

Buchanan, L.F. (1969), "Problems in Optimal Control of Macroeconomic Systems," Computing Methods in Optimization Problems, (eds.) M. Beckmann and H.P. Künzi, Springer-Verlag, Berlin, Heidelberg, New York, pp. 30-42.

Bullock, T.E. and J.M. Elder (1971), "Quadratic Performance Index Generation for Optimal Regulator Design," Paper No. W8-3, Proc. 1971 IEEE Conference on Decision and Control, Univ. of Florida, pp. 123-4.

Burmeister, E. and A.R. Dobell (1972), "Guidance and Optimal Control of Free-Market Economics: A New Interpretation," IEEE Trans. Systems, Man, and Cybernetics, Vol. SMC-2, No. 1, Jan., pp. 9-15.

Butkovskii, A.G. (1964), "The Necessary and Sufficient Conditions for Optimality of Discrete Control Systems," Automation and Remote Control, Jan., pp. 963-70.

Canon, M.D., C.D. Cullum, Jr., and E. Polak (1970), Theory of Optimal Control and Mathematical Programming, McGraw-Hill, New York.

Chen, R.T.N. and D.W.C. Shen (1968), "Sensitivity Analysis and Design of Multivariable Regulators Using a Quadratic Performance Criterion," Proc. 1968 Joint-Automatic Control Conference of the AACC. The Univ. of Michigan, pp. 229-37.

Chow, G. (1970), "Optimal Stochastic Control of Linear Economic Systems," J. of Money, Credit, and Banking, II., Aug., pp. 291-302.

________ (1972), "Optimal Control of Linear Econometric Systems with Finite Time Horizon," International Economic Review, Feb., pp. 16-25.

Dobell, A.R. and Y.C. Ho (1967), "Optimal Investment Policy: An Example of Control Problem in Economic Theory," IEEE Trans. on Automat. Contr., Vol. AC-12, No. 1, Feb., pp. 4-14.

Dorato, P. and A.H. Levis (1971), "Optimal Linear Regulators: The Discrete-Time Case," IEEE Trans. Automat. Contr., Vol. AC-16, No. 6, Dec., pp. 613-20.

Dorfman, R. (1969), "An Economic Interpretation of Optimal Control Theory," American Economic Review, Vol. LIX, Dec., pp. 817-31.

Dreyfus, S.E. (1968), "Introduction to Stochastic Optimization and Control," H. Karreman (ed.) Stochastic Optimization and Control, Wiley, New York, pp. 3-24.

Elder, J.M. (1971), "The Inverse Optimal Linear Regulator Problem," Ph.D. Dissertation, Univ. of Florida.

Erickson, D.L., C.T. Leondes, and F.E. Norton (1970), "Optimal Decision and Control Policies in the National Economy," Proc. 1970 IEEE Symposium on Adaptive Processes, Decision, and Control, Univ. of Texas, Austin.

Erickson, D.L. (1968), "Sensitivity Constrained Optimal Control Policies for a Dynamic Model of the U.S. National Economy," Ph.D. Dissertation in Engineering and Applied Sciences, UCLA.

Fischer, S. and J.P. Cooper (1971), "Stabilization Policy and Lags in a Closed Economy," (Mimeo) Department of Economics, The Univ. of Chicago, Chicago, Aug.

Fox, K., J. Sengupta, and E. Thorbecke (1966), The Theory of Quantitative Economic Policy, North-Holland, Amsterdam.

Friedlaender, A.F. (1973), "Macroeconomic Policy Goals and Revealed Preference," Quarterly J. of Economics, Vol. LXXXLII, No. 1., Feb., pp. 25-43.

Friedman, B.M. (1972), "Optimal Economic Stabilization Policy: An Extended Framework," J. of Political Economy, Vol. 80, No. 5, Sept./Oct., pp. 1002-22.

Fromm, G. and P. Taubman (1968), Policy Simulations with An Econometric Model, The Brookings Institution, Washington, D.C.

Gardner, M.F. and J.L. Barnes (1942), Transients in Linear Systems, Vol. I, John Wiley and Sons, New York.

Gel'fand, I.M. and S.V. Fomin (1963), Calculus of Variations, Prentice-Hall, Englewood Cliffs.

Gilbert, E.G. (1962-63), "Controllability in Multivariable Control Systems," SIAM J. of Control, Ser. A, Vol. 1, No. 1, pp. 128-51.

Hadley, G. and M.C. Kemp (1971), Variational Methods in Economics, North-Holland, Amsterdam, London.

Hakansson, N.H. (1969), "Optimal Investment and Consumption Strategies under Risk, an Uncertain Lifetime, and Insurance," International Economic Review, Vol. 10, No. 3, Oct., pp. 443-466.

________ (1970), "An Induced Theory of the Firm under Risk," J. of Finance and Quantitative Analysis, Jun.

Haley, W.J. (1973), "Human Capital: The Choice between Investment and Income," Forthcoming American Economic Review.

Halkin, H. (1964), "Optimal Control for Systems Described by Difference Equations," Advances in Control Systems, Theory, and Applications. (ed.) C.T. Leondes, Vol. 1, Academic Press, New York, London, pp. 173-96.

Hamada, K. (1969), "Optimum Capital Accumulation by an Economy facing an International Capital Market," Journal of Political Economy, Vol. 77, pp. 684-97.

Haurie, A., M. Polis, and P. Yansouni (1972), "On Optimal Pollution and Consumption Control in a Macroeconomic System," Paper No. WP 3-5, Proc. of 1972 IEEE Conference on Decision and Control, New Orleans, pp. 144-8.

Hestenes, M.R. (1965), "On Variational Theory and Optimal Control Theory," SIAM J. on Control, Series A, Vol. 3, No. 1, pp. 23-48.

Hickman, B.G. (ed.) (1965), Quantitative Planning of Economic Policy, The Brookings Institution, Washington, D.C.

Holbrook, R. (1972), "Instrument Instability Problem," American Economic Review, March.

Holt, C.C., F. Modigliani, and J.F. Muth (1956), "Derivation of a Linear Decision Rule for Production and Employment," Management Science, Vol. 2, Jan., pp. 15-77.

Intriligator, M.D. (1971), Mathematical Optimization and Economic Theory, Prentice-Hall, Englewood Cliffs.

Iyoha, M.A. (1971), "Optimal Balance of Payments Strategy by Stochastic Dynamic Programming," Western Economic Journal, Vol. IX, No. 4, Dec., pp. 429-40.

Jameson, J. and E. Kreindler (1973), "Inverse Problem of Linear Optimal Control," SIAM J. Control, Vol. 11, Feb., pp. 1-19.

Jordan, B.W. and E. Polak (1964), "Theory of a Class of Discrete Optimal Control Systems," Journal of Electronics and Control, Vol. 17, pp. 697-713.

Jury, E.I. and F.W. Semelka (1958), "Time Domain Synthesis of Sampled-Data Control Systems," Trans. ASME, Vol. 80, Nov., pp. 1827-38.

Kalman, R.E. (1960), "Contributions to the Theory of Optimal Control," Boletin de la Sociedad Matematics Mexicana, pp. 102-19.

________ (1961), "On the General Theory of Control," 1960 Proc. IFAC, London, Butterworths, Vol. 1, pp. 481-92.

________ (1962-63), "Mathematical Description of Linear Dynamical Systems," SIAM J. on Control, Series A, Vol. 1, pp. 152-223.

________ (1963), "The Theory of Optimal Control and the Calculus of Variations," Mathematical Optimization Techniques, (ed.) R. Bellman, Berkeley, Univ. of California Press, pp. 309-31.

________ (1964), "When is a Linear Control System Optimal?," Trans. of ASME Journal of Basic Engineering, Vol. 86, Series D, No. 1, Mar., pp. 51-60.

Katz, S. (1962), "A Discrete Version of Pontryagin's Maximum Principle," J. of Electronics and Control, Vol. 13, pp. 179-84.

Keeler, E., M. Spencer, and R. Zeckhauser (1972), "The Optimal Control of Pollution," Journal of Economic Theory, Vol. 4, pp. 19-34.

Kendrick, D.A. and L. Taylor (1970), "Numerical Solution of Non-linear Planning Models," Econometrica, Vol. 38, No. 3, May, pp. 453-67.

Kendrick, D.A. (1971), "Control Theory in Economic Models: Survey," Paper No. W7-1, Proc. of 1971 IEEE Conference on Decision and Control, Univ. of Florida, pp. 80-4.

Kleinman, D.L. and M. Athans (1966), "The Discrete Minimum Principle with Application to the Linear Regulator Problem," Report ESL-R-260, Feb., Electronic Systems Laboratory, Dept. of Electrical Engineering, MIT, Cambridge.

Koenig, H.E., Y. Tokad, H.K. Kesavan, and H.G. Hedges (1967), Analysis of Discrete Physical Systems, McGraw Hill, New York.

Koo, A.Y.C. (1963), "An Empirical Test of Revealed Preference Theory," Econometrica, Vol. 31, Oct., pp. 646-64.

Kozhinskaya, L.I. (1972), "Performance of Linear Controllable Systems," Automation and Remote Control, Vol. 33, No. 1, Pt. I, Jan., pp. 17-25.

Kreindler, E. and A. Jameson (1972), "Optimality of Linear Control Systems," IEEE Transactions on Automatic Control, Vol. AC-17, No. 3, Jun., pp. 349-51.

Ku, Y.H. (1962), Analysis and Control of Linear Systems, International Textbook Co., Scranton.

Kuo, B.C. (1970), Discrete Data Control Systems, Prentice-Hall, Englewood Cliffs.

Kurz, M. (1968), "The General Instability of a Class of Competitive Growth Processes," Review of Economic Studies, Apr., pp. 155-74.

________ (1969), "On the Inverse Optimal Problem," Mathematical Systems Theory and Economics I, (eds.) H.W. Kuhn and G.P. Szegö, Springer-Verlag, Berlin, Heidelberg, New York.

Kushner, H.J. (1965), "Near-Optimal Control in the Presence of Small Stochastic Perturbations," Journal of Basis Engineering, Vol. 87, No. 1, March, pp. 103-8.

Lancaster, K. (1965), "The Theory of Qualitative Linear Systems," Econometrica, Vol. 33, April, pp. 395-408.

Lange, O. (1970), Introduction to Economic Cybernetics, Pergamon Press, PWN, Warszawa.

La Salle, J.P. (1962-63), "Stability and Control," SIAM J. on Control, Series A, No. 1, pp. 3-15.

Lasdon, E.S., S.K. Mitter, and A.D. Warren (1967), "The Conjugate Gradient Method for Optimal Control Problems," IEEE Trans. Auto. Contrl., April, pp. 132-38.

Lee, E.B. and L. Markus (1967), Foundations of Optimal Control Theory, Wiley, New York.

Lee, K.Y., S. Chow, and R.O. Barr (1972), "On the Control of Discrete-Time Distributed Parameter Systems," SIAM J. on Control, Vol. 10, No. 2, May, pp. 361-76.

Lefschetz, S. (1965), Stability of Nonlinear Control Systems, Academic Press, New York.

Leland, H.E. (1972), "Why Profit Maximization May Be Better Assumption than You Think," Technical Report No. 80, The Economics Series, Institute for Mathematical Studies in the Social Sciences, Stanford Univ.

Lele, M.M., D.H. Jacobson, and J.L. McCabe (1971), "Qualitative Application of a Result in Control Theory to Problems of Economic Growth," International Economic Review, Jun., pp. 209-26.

Levhari, D. and T.N. Srinivasan (1969), "Optimal Savings under Uncertainty," Review of Economics and Statistics, Vol. 36, No. 106, Apr., pp. 153-64.

Levis, A.H., M. Athans, and R. Schlueter (1971), "On the Behavior of Optimal Linear Sampled-Data Regulators," International Journal of Control, Vol. 13, Feb., pp. 343-61.

Little, I.M.D. and J.A. Mirrless (1969), Manual of Industrial Project Analysis in Developing Countries, Vol. II, Social Cost Benefit Analysis, Paris.

Livesey, D.A. (1970), "The Modeling and Control of U.K. Economy," Presented at the Second World Congress of the Econometric Society, Cambridge, England, Sept.

________ (1971), "Control Theory and Input-Output Analysis," International Journal of Systems Science, Vol. 2, No. 3, Nov., pp. 307-18.

Lucas, R.E., Jr. (1967), "Optimal Investment Policy and the Flexible Accelerator," International Economic Review, Vol. 8, No. 1, Feb., pp. 78-85.

Luenberger, D.G. (1967), "Canonical Forms for Linear Multivariable Systems," IEEE Transactions on Automatic Control, Vol. AC-12, No. 3, Jun., pp. 290-93.

Markus, L. (1969), "Dynamic Keynesian Economic Systems: Control and Identification," Mathematical Systems Theory and Economics I, (eds.) H.W. Kuhn and G.P. Szegö, Springer-Verlag, Berlin, Heidelberg, New York, pp. 203-220.

Martens, A. and R.S. Pindyck (1972), "An Application of Optimal Control to Investment Allocation for Development Planning," Paper No. WA 2-4, Proc. of 1972 IEEE Conference on Decision and Control, New Orleans, pp. 31-6.

Meditch, J.S. (1964), "The Pontryagin Maximum Principle and Some of Its Applications," C.T. Leondes (ed.) Advances in Control Systems, Academic Press, New York, London, pp. 55-74.

Merton, R.C. (1969), "Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case," Review of Economics and Statistics, Vol. LI, No. 3, Aug., pp. 247-57.

Morgan, B.S. (1964), "The Synthesis of Linear Multivariable Systems by State-Variable Feedback," IEEE Trans. Automat. Contr., Vol. AC-9, No. 4, Oct., pp. 405-11.

Moylan, P.J. and B.D.O. Anderson (1971), "Nonlinear Regulator Theory and an Inverse Optimal Control Problem," Proceedings of 1971 Joint Automatic Control Conference, pp. 462-67.

Mulholland, R.J. and R.P. Rhoten (1972), "Optimal Inner-Product Angular Momentum Controllers," IEEE Transactions on Automatic Control, Vol. AC-17, No. 1, Feb., pp. 95-7.

Mundell, R.A. (1965), "Growth, Stability, and Inflationary Finance," J. of Political Economy, Vol. 73, April, pp. 97-109.

Murphy, R.E. (1965), Adaptive Processes in Economic Systems, Academic Press, New York.

Nelson, W.C. (1964), "On the Use of Optimization Theory for Practical Control System Design," IEEE Trans. Automat. Contr., Vol. AC-9, No. 4, Oct., pp. 469-76.

Norman, A.L. (1971), "Optimal Economic Policy and Econometric Models," Paper No. W7-2, Proc. of 1971 IEEE Conference on Decision and Control, Univ. of Florida, pp. 85-95.

Panda, S.P. (1971), "Comments on "On the Inverse Optimum Control Problem for a Class of Nonlinear Autonomous Systems," IEEE Trans. Automat. Contr., Oct., pp. 509-11.

Paryani, K. (1972), "Optimal Control of Linear Discrete Macro-economic Systems," Ph.D. Dissertation in Systems Science, Michigan State Univ., East Lansing.

Pearson, J.B. and B. Sridhar (1966), "A Discrete Optimal Control Problem," IEEE Trans. Automat. Contr., Vol. AC-11, April, pp. 171-4.

Phillips, A.W. (1954), "Stabilization Policy in a Closed Economy," Economic Journal, Vol. 64, Jun., pp. 290-323.

________ (1957), "Stabilization Policy and Time Form of the Lagged Responses," Economic Journal, Vol. 67, Jun., pp. 265-77.

Pindyck, R.S. (1972), "An Application of the Linear Quadratic Tracking Problem to Economic Stabilization Policy," IEEE Trans. Automat. Contr., Vol. AC-17, Jun., pp. 287-300.

Rasche, R.H. (1973), "Criteria for the Choice of Operating Instruments for Open Market Policy," (Mimeo) Department of Economics, Michigan State Univ., East Lansing.

Rekasius, Z.V. and T.C. Hsia (1964), "On an Inverse Problem in Optimal Control," IEEE Trans. Auto. Contrl., Vol. AC-9, No. 4, Oct., pp. 370-75.

Rozonoer, L.I. (1959), "Pontryagin Maximum Principle in the Theory of Optimum Systems, I, II, III," Automation and Remote Control, Vol.20, Part 2, Jul.-Dec., pp. 1288-1352, 1405-1421, 1517-32.

________ (1961), "On the Variational Methods of Analysis of Automatic Control Systems Performance," 1960 Proc. of IFAC, London, Butterworths, pp. 477-80.

Rugh, W.J. (1971), "On an Inverse Optimal Control Problem," IEEE Transactions on Automatic Control, Vol. AC-16, No. 1, Feb., pp. 87-8.

Sakakibara, E. (1970), "Dynamic Optimization and Economic Policy," American Economic Review, Vol. LX, No. 5, Dec., pp. 826-36.

Sage, A.P. (1968), Optimum Systems Control, Prentice-Hall, Englewood Cliffs.

Samuelson, P.A. (1960), "Efficient Paths of Capital Accumulation in Terms of the Calculus of Variations," Mathematical Methods in the Social Sciences, (eds.) K.J. Arrow, et.al., Stanford.

________ (1969), "Lifetime Portfolio Selection by Dynamic Stochastic Programming," Review of Economics and Statistics, Vol. LI, No. 3, Aug., pp. 239-46.

________ (1972), "Maximum Principles in Analytical Economics," American Economic Review, Vol. LXII, No. 3, Jun., pp. 249-62.

Sarachik, P.E. and K. Kreindler (1965), "Controllability and Observability of Linear Discrete-Time Systems," International Journal of Control, Vol. I, No. 5, May, pp. 419-32.

Schultz, W.C. and V.C. Rideout (1961), "Control System Performance Measures: Past, Present, and Future," IRE Trans. Automat. Control, Vol. AC-6, No. 1, Feb., pp. 22-34.

Sengupta, J.K. (1970), "Optimal Stabilization Policy with a Quadratic Criterion Function," The Review of Economic Studies, Vol. XXXVII(1), No. 109, Jan., pp. 127-46.

Sengupta, J.K., G. Tintner, and B. Morrison (1963), "Stochastic Linear Programming with Applications to Economic Models," Economica, Vol. 30, August, pp. 262-76.

Shell, K. (ed.) (1967), Essays on the Theory of Optimal Economic Growth, MIT Press, Cambridge.

Shupp, F.R. (1972), "Optimal Control, Uncertainty and a Temporary Income Policy," Paper No. WA-2-2, Proc. 1972 IEEE Conference on Decision and Control, New Orleans, pp. 21-25.

Simon, H.A. (1956), "Dynamic Programming under Uncertainty with a Quadratic Criterion Function," Econometrica, Vol. 24, Jan., pp. 74-81.

Simon, J.D. and S.K. Mitter (1968), "A Theory of Modal Control," Information and Control, Vol. 13, No. 4, Oct., pp. 316-53.

Stafford, F.P. and P.E. Stephen (1972), "Labor, Leisure, and Training over the Life Cycle," (Mimeo), Presented at 1972 Meetings of the Econometric Society in Toronto, Ontario, Canada.

Stoleru, L.G. (1965), "An Optimal Policy for Economic Growth," Econometrica, Vol. 33, No. 2, April, pp. 321-48.

Taylor, L. (1970), "The Existence of Optimal Distributed Lags," The Review of Economic Studies, Vol. XXXVII(1), No. 109, Jan., pp. 95-106.

Thau, F.E. (1967), "On the Inverse Optimal Control Problem for a Class of Nonlinear Autonomous Systems," IEEE Transactions on Automatic Control, Vol. AC-12, Dec., pp. 674-81.

Theil, H. (1957), "A Note on Certainty Equivalence in Dynamic Planning," Econometrica, Vol. 25, April, pp. 346-49.

________ (1958), Economic Forecasts and Policy, North-Holland, Amsterdam.

________ (1964), Optimal Decision Rules for Government and Industry, North-Holland, Amsterdam.

________ (1965), "Linear Decision Rules for Macrodynamic Policy Problems," Ch. 2, pp. 18-42, Quantitative Planning of Economic Policy, The Brookings Institution, Washington, D.C. (ed.) B.G. Hickman.

Thompson, R.G., M. George, P. Brown, and M.S. Proctor (1971), "Optimal Production, Investment, and Output Price Controls for a Monopoly Firm of the Evan's Type," Econometrica, Vol. 39, No. 1, Jan., pp. 119-30.

Tinbergen, J. (1967), Economic Policy: Principles and Design, North-Holland, Amsterdam.

Tou, J.T. (1963), Optimal Design of Digital Control Systems, Academic Press, New York, London.

________ (1964), Modern Control Theory, McGraw-Hill, New York, San Francisco, Toronto, London.

Tuel, W.G., Jr. (1967), "An Improved Algorithm for the Solution of Discrete Regulation Problems," IEEE Transactions on Automatic Control, Vol. AC-12, No. 5, Oct., pp. 522-28.

Turnovsky, S.J. (1973), "Optimal Stabilization Policies and Stochastic Linear Economic Systems," Review of Economic Studies, Vol. XL(1), No. 121, Jan., pp. 79-95.

Tustin, A. (1953), The Mechanism of Economic Systems, Harvard Univ. Press, Cambridge.

Tyler, T.S. and F.B. Tuteur (1966), "The Use of a Quadratic Performance Index to Design Multivariable Control Systems," IEEE Trans. Automat. Control, Vol. AC-11, No. 1, Jan., pp. 84-92.

Weiss, L. and R.E. Kalman (1964), Contributions to Linear System Theory, RIAS Technical Report 64-9, A Division of the Martin Co., Baltimore, April.

Whittle, P. (1969), "A View of Stochastic Control Theory," Journal of Royal Statistical Society, Series A, pp. 320-34.

Widodo, R.J. (1972), "Design of Optimal Control Systems with Prescribed Closed-Loop Poles," Electronics Letters, Vol. 8, No. 13, Jun., pp. 339-40.

Willner, L., R.H. Ash, and R.J. Roy (1972), "An Improved General Algorithm for Arbitrary Pole Assignment," IEEE Trans. Automat. Contr., Vol. AC-17, Oct., pp. 734-36.

Wishart, D.M.G. (1969), "A Survey of Control Theory," Journal of Royal Statistical Society, Series A, pp. 293-319.

Wonham, W.M. and C.D. Johnson (1964), "Optimal Bang-Bang Control with Quadratic Performance Index," J. of Basic Engineering, Trans. of the ASME, Ser. D, March, pp. 107-115.

Wonham, W.M.(1967), "On Pole Assignment in Multi-input Controllable Linear Systems," IEEE Transactions on Automatic Control, Vol. AC-12, Dec., pp. 660-65.

________ (1969), "Optimal Stochastic Control," Automatica, Vol. 5, No. 1, Jan., pp. 113-6.

Wolovich, W.A. (1972), "The Use of State Feedback for Exact Model Matching," SIAM J. on Control, Vol. 10, No. 2, Aug., pp. 512-23.

Wu, S.H. and W.J. Schroeder (1968), "Inverse Optimal Control Problem for Discrete Systems," Conference Record of Second Asilomar Conference on Circuits and Systems, Pacific Grove, U.S.A., pp. 561-5.

Wymer, C.R. (1972), "Econometric Estimation of Stochastic Differential Equation Systems," Econometrica, Vol. 40, May, pp. 565-77.

Yaari, M.E. (1964), "On the Consumer's Lifetime Allocation Process," International Economic Review, Vol. 5, No. 3, Sept., pp. 304-17.

Yokoyama, R. and E. Kinnen (1972), "Inverse Problem of Optimal Regulator," IEEE Transactions on Automatic Control, Vol. AC-17, Aug., pp. 497-503.

Zabel, E. (1967), "A Dynamic Model of the Competitive Firm," International Economic Review, VIII, Jun., pp. 199-208.

Zemach, R. (1968), "A State Space Model for Resource Allocation in Higher Education," IEEE Transactions on Systems Science and Cybernetics, Vol. SSC-4, No. 2, Jul., pp. 108-18.

APPENDIX

APPENDIX

THE SOLUTION OF A LINEAR TRACKING PROBLEM

The dynamic system is governed by

$$y_{t+1} = Ay_t + Bu_t + Cz_t \ , \ y_0 = y^0 \ , \tag{A.1}$$

where y_t is an n-vector of state variables, u_t an m-vector of control variables, z_t an r-vector of other exogenous variables outside the control consideration. The matrices, A (non-singular), B (with rank m), and C are constant matrices.

The performance functional is

$$J = \tfrac{1}{2}(y_N - \hat{y}_N)'Q(y_N - \hat{y}_N) + \tfrac{1}{2}\sum_{t=0}^{N-1} \{(y_t - \hat{y}_t)'Q(y_t - \hat{y}_t) + u_t'Ru_t\} \ , \tag{A.2}$$

where $Q = Q'$ is a positive semi-definite matrix with rank m, $R = R'$ a positive-definite matrix, and $\hat{y}_t$ a vector of target values of the state y_t assumed to be specified for the entire optimization period.

The problem is to determine the control sequence $\{u_t^*, \ t = 0,1,2,\ldots,N-1\}$ such that the corresponding state variable sequence $\{y_t^*, \ t = 0,1,\ldots,N\}$ satisfies the given initial condition, $y_0^* = y^0$ such that the performance functional (A.2) is minimized.

In order to get the necessary conditions for the solution, construct the Hamiltonian,

$$H(y_t, P_{t+1}, u_t) = \tfrac{1}{2}(y_t - \hat{y}_t)'Q(y_t - \hat{y}_t)$$

$$+ \tfrac{1}{2} u_t'Ru_t + P_{t+1}'(Fy_t + Bu_t + Cz_t)$$

[Here, the system equation (A.1) is considered to be $y_{t+1} - y_t = Fy_t + Bu_t + Cz_t$. Thus $F + I = A$], where P_t is the vector of "co-states".

The minimization of the Hamiltonian is written

$$\frac{\partial H}{\partial u_t}(y_t^*, P_{t+1}^*, u_t^*) = Ru_t^* + B'P_{t+1}^* = 0 \ . \qquad (A.3)$$

The canonical equations for the problem are

$$y_{t+1}^* - y_t^* = \frac{\partial H}{\partial P_{t+1}}(y_t^*, P_{t+1}^*, u_t^*) = Fy_t^* + Bu_t^* + Cz_t \qquad (A.4a)$$

$$P_{t+1}^* - P_t^* = -\frac{\partial H}{\partial y_t}(y_t^*, P_{t+1}^*, u_t^*) = -Q(y_t^* - \hat{y}_t) - F'P_{t+1}^*, \qquad (A.4b)$$

with the "split" boundary conditions $y_0^* = y^0$ and (A.5)

$$P_N^* = \tfrac{1}{2}\frac{\partial}{\partial y}\{(y_N^* - \hat{y}_N)'Q(y_N^* - \hat{y}_N)\} = Q(y_N^* - \hat{y}_N) \ . \qquad (A.6)$$

We know that (Lee, et.al., 1972)

$$P_t^* = K_t y_t^* + g_t \ . \qquad (A.7)$$

Substituting (A.7) into (A.3),

$$u_t^* = -R^{-1}B'(K_{t+1} y_{t+1}^* + g_{t+1}) \qquad (A.8)$$

and with (A.7), (A.8), (A.4a), and (A.4b),

$$y_{t+1}^* - y_t^* = Fy_t^* - BR^{-1}B'K_{t+1}y_{t+1}^* - BR^{-1}B'g_{t+1} + Cz_t \qquad (A.9)$$

$$P_{t+1}^* - P_t^* = -Q(y_t^* - \hat{y}_t) - F'(K_{t+1} y_{t+1}^* + g_{t+1}) \ . \qquad (A.10)$$

From (A.9),

$$(I + BR^{-1}B'K_{t+1})y^*_{t+1} = (I + F)y^*_t - BR^{-1}B'g_{t+1} + Cz_t$$
$$= Ay^*_t - BR^{-1}B'g_{t+1} + Cz_t \ . \tag{A.11}$$

Substituting (A.7) into (A.10),

$$A'K_{t+1}y^*_{t+1} + Qy^*_t - K_t y^*_t = -A'g_{t+1} + g_t + Q\hat{y}_t \ . \tag{A.12}$$

Define $W = I + BR^{-1}B'K_{t+1}$. Then (A.11) becomes

$$y^*_{t+1} = W^{-1}Ay^*_t - W^{-1}BR^{-1}B'g_{t+1} + W^{-1}Cz_t \ . \tag{A.13}$$

Substituting (A.13) into (A.12),

$$A'K_{t+1}\{W^{-1}Ay^*_t - W^{-1}BR^{-1}B'g_{t+1} + W^{-1}Cz_t\} + Qy^*_t - K_t y^*_t$$
$$=-A'g_{t+1} + g_t + Q\hat{y}_t \ . \tag{A.14}$$

Rearranging (A.14),

$$Qy^*_t + A'K_{t+1}W^{-1}Ay^*_t - A'K_{t+1}W^{-1}BR^{-1}B'g_{t+1} + A'K_{t+1}W^{-1}Cz_t - Q\hat{y}_t + A'g_{t+1}$$
$$= K_t y^*_t + g_t \tag{A.15}$$

for any initial value y^0 and for all y^*_t, which implies that

$$K_t = Q + A'K_{t+1}W^{-1}A \tag{A.16}$$

$$g_t = -A'K_{t+1}W^{-1}BR^{-1}B'g_{t+1} + A'g_{t+1} + A'K_{t+1}W^{-1}Cz_t - Q\hat{y}_t \ . \tag{A.17}$$

By the transversality condition (A.6) and (A.7),

$$P^*_N = Q(y^*_N - \hat{y}_N) = K_N y^*_N + g_N \quad \text{for any} \quad y^*_N \ . \tag{A.18}$$

Then, $K_N = Q$ (A.19)

and $g_N = -Q\hat{y}_N$. (A.20)

Recall $W^{-1} = (I + BR^{-1}B'K_{t+1})^{-1}$

$$= I - B(R + B'K_{t+1}B)^{-1}B'K_{t+1} . \quad (A.21)$$

Substitution (A.21) into (A.16) and (A.17),

$$K_t = Q + A'K_{t+1}[A - B(R + B'K_{t+1}B)^{-1}B'K_{t+1}A] \quad (A.22)$$

and

$$g_t = -A'[K_{t+1} - K_{t+1}B(R + B'K_{t+1}B)^{-1}B'K_{t+1}]BR^{-1}B'g_{t+1} + A'g_{t+1}$$

$$+ A'[K_{t+1} - K_{t+1}B(R + B'K_{t+1}B)^{-1}B'K_{t+1}]Cz_t - Q\hat{y}_t$$

$$\equiv G_t + A'[K_{t+1} - K_{t+1}B(R + B'K_{t+1}B)^{-1}B'K_{t+1}]Cz_t - Q\hat{y}_t . \quad (A.23)$$

$$G_t = -A'[K_{t+1} - K_{t+1}B(R + B'K_{t+1}B)^{-1}B'K_{t+1}]BR^{-1}B'g_{t+1} + A'g_{t+1}$$

$$= -A'[K_{t+1}BR^{-1}B' - K_{t+1}B(R + B'K_{t+1}B)^{-1}B'K_{t+1}BR^{-1}B' - I]g_{t+1}$$

$$= -A'\{K_{t+1}B[I - (R + B'K_{t+1}B)^{-1}B'K_{t+1}B]R^{-1}B' - I\}g_{t+1}$$

$$= -A'\{K_{t+1}B[I - R^{-1}(I + B'K_{t+1}BR^{-1})^{-1}B'K_{t+1}B]R^{-1}B' - I\}g_{t+1}$$

$$= -A'\{K_{t+1}B[I + R^{-1}B'K_{t+1}B]^{-1}R^{-1}B' - I\}g_{t+1}$$

$$= -A'\{K_{t+1}B(R + B'K_{t+1}B)^{-1}RR^{-1}B' - I\}g_{t+1}$$

$$= -A'\{K_{t+1}B(R + B'K_{t+1}B)^{-1}B' - I\}g_{t+1} . \quad (A.24)$$

Then

$$g_t = -A'\{K_{t+1}B(R + B'K_{t+1}B)^{-1}B' - I\}g_{t+1}$$

$$+A'\{K_{t+1} - K_{t+1}B(R + B'K_{t+1}B)^{-1}B'K_{t+1}\}Cz_t - Q\hat{y}_t . \quad (A.25)$$

Substituting (A.13) and (A.21) into (A.8),

$$u_t^* = -R^{-1}[I - B'K_{t+1}B(R + B'K_{t+1}B)^{-1}]B'K_{t+1}Ay_t^*$$
$$+R^{-1}[I - B'K_{t+1}B(R + B'K_{t+1}B)^{-1}]B'K_{t+1}BR^{-1}B'g_{t+1}$$
$$-R^{-1}[I - B'K_{t+1}B(R + B'K_{t+1}B)^{-1}]B'K_{t+1}Cz_t - R^{-1}B'g_{t+1} .$$

By making use of the matrix identity

$$I - X(Y + X)^{-1} = Y(Y + X)^{-1} ,$$

$$u_t^* = -R^{-1}R(R + B'K_{t+1}B)^{-1}B'K_{t+1}Ay_t^* + R^{-1}R(R + B'K_{t+1}B)^{-1}B'K_{t+1}BR^{-1}B'g_{t+1}$$
$$-R^{-1}R(R + B'K_{t+1}B)^{-1}B'K_{t+1}Cz_t - R^{-1}B'g_{t+1}$$
$$= -(R + B'K_{t+1}B)^{-1}B'K_{t+1}Ay_t^*$$
$$-\{I - (R + B'K_{t+1}B)^{-1}B'K_{t+1}B\}\cdot R^{-1}B'g_{t+1} - (R + B'K_{t+1}B)^{-1}B'K_{t+1}Cz_t .$$

Again, using the matrix identity

$$I - (Y + X)^{-1}X = (Y + X)^{-1}Y ,$$

$$u_t^* = -(R + B'K_{t+1}B)^{-1}B'K_{t+1}Ay_t^* - (R + B'K_{t+1}B)^{-1}RR^{-1}B'g_{t+1}$$
$$-(R + B'K_{t+1}B)^{-1}B'K_{t+1}Cz_t$$
$$= -(R + B'K_{t+1}B)^{-1}B'K_{t+1}Ay_t^* - (R + B'K_{t+1}B)^{-1}B'g_{t+1}$$
$$-(R + B'K_{t+1}B)^{-1}B'K_{t+1}Cz_t . \tag{A.26}$$

Now consider the case where $R = 0$ in the above model. Let the positive semidefinite matrix Q be factorized into the product DD' where rank (D) = rank (Q) and $(D'B)$ is non-singular. Then the matrix Riccati equation (A.22) becomes

$$K_t = Q + A'K_{t+1}[A - B(B'K_{t+1}B)^{-1}B'K_{t+1}A]$$

$$K_N = Q \ .$$

$$K_{N-1} = Q + A'DD'[A - B(B'DD'B)^{-1}B'DD'A]$$

$$= Q + A'DD'A - A'DD'B(D'B)^{-1}(B'D)^{-1}B'DD'A$$

$$= Q \ .$$

This implies

$$K_t = Q \quad \text{for all} \quad t = 0,1,2,\ldots,N \ . \tag{A.27}$$

The equation (A.25) becomes

$$g_t = -A'[K_{t+1}B(B'K_{t+1}B)^{-1}B' - I]g_{t+1}$$

$$+A'[K_{t+1} - K_{t+1}B(B'K_{t+1}B)^{-1}B'K_{t+1}]Cz_t - Q\hat{y}_t \ .$$

Then

$$g_{N-1} = -A'[DD'B(B'DD'B)^{-1}B' - I](-Q\hat{y}_N)$$

$$+A'[DD' - DD'B(B'DD'B)^{-1}B'DD']Cz_N - Q\hat{y}_N$$

$$= +A'[D(B'D)^{-1}B' - I]DD'\hat{y}_N$$

$$+A'[DD' - DD'B(D'B)^{-1}\cdot(B'D)^{-1}B'DD']Cz_N - Q\hat{y}_{N-1} = -Q\hat{y}_{N-1}. \tag{A.28}$$

Equation (A.28) implies that

$$g_t = -Qy_t \ , \ t = 0,1,\ldots,N. \tag{A.29}$$

Again, consider the optimal control policy u_t^* (A.26) for the period N-1.

$$u^*_{N-1} = -(B'DD'B)^{-1}B'DD'Ay^*_{N-1} - (B'DD'B)^{-1}B'g_N - (B'DD'B)^{-1}B'DD'Cz_{N-1}$$

$$= -(D'B)^{-1}D'Ay^*_{N-1} + (D'B)^{-1}(B'D)^{-1}B'DD'\hat{y}_N - (D'B)^{-1}D'Cz_{N-1}$$

$$= -(D'B)^{-1}D'Ay^*_{N-1} + (D'B)^{-1}D'\hat{y}_N - (D'B)^{-1}D'Cz_{N-1} \ , \qquad (A.30)$$

which implies that

$$u^*_t = -(D'B)^{-1}D'Ay^*_t + (D'B)^{-1}D'\hat{y}_{t+1} - (D'B)^{-1}D'Cz_t$$

$$= -Gy^*_t + E\hat{y}_{t+1} - Fz_t \ , \ t = 0,1,\ldots,N-1 \ . \qquad (A.31)$$