2-MANIFOLDS IN EUCLIDEAN 4-SPACE

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ABSTRACT

2-MANIFOLDS IN EUCLIDEAN 4-SPACE

by James Lee Murphy

This thesis is a study of polyhedral and almost polyhedral 2-manifolds in Euclidean 4-space.

In chapter I a special 2-disk, $\mathcal{D}$, is constructed in E^4 such that $\mathcal{D}$ is locally polyhedral at every point except one on the boundary and is universal in the sense of the:

Main Theorem: Given any 2-disk, D, embedded in E^4 in such a way that D is locally polyhedral at every point except at one boundary point P, then there exists a space homeomorphism $h: E^4 \to E^4$ such that $h(D) \subseteq \mathcal{D}$ and $h(P)$ is the point at which $\mathcal{D}$ fails to be locally polyhedral.

We show that $\pi_1(E^4-\mathcal{D})$ is trivial. Then we extend the results on the disk to 2-manifolds which fail to be locally polyhedral at just one point.

In chapter II we discuss a polyhedral 2-disk in E^4 which has a flat triangle for boundary but which can not be moved by a space homeomorphism into a three-dimensional hyperspace. Finally, we construct a polyhedral 2-sphere in E^4 which fails to be locally flat at exactly one point.

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INTRODUCTION

In [6] Doyle and Hocking construct a wild 2-disk in E^4 by taking cones over an infinite number of polygonal trefoil knots in such a way that the cone points and the trefoil knots approach a point P. They join the cones with polygonal strips and add the point P. Their disk is, by construction, locally polyhedral at all points except P which lies on the boundary.

We show that all 2-disks in E^4 which are locally polyhedral except at one boundary point arise in this way, i.e., they are equivalently embedded with a disk of the construction of Doyle and Hocking using perhaps polygonal knot types other than that of the trefoil knot.

In particular, we construct in section 1 of Chapter I a type of universal 2-disk, $\mathcal{D}$, in E^4 which is locally polyhedral at all points except one boundary point P. We then show that for any 2-disk, D, embedded in E^4 which is locally polyhedral at all points except a boundary point P' there is a space homeomorphism $h: E^4 \to E^4$ such that $h(D) \subseteq \mathcal{D}$ and $h(P') = P$.

In [10] Gugenheim, restricting himself to polyhedral objects and PL maps, shows that the embedding classes for q-disks in 2q-space are in 1-1 correspondence with finite sequences of q-1-spheres in 2q-1-space independent of order. We note that for q = 2, having removed the polyhedral requirement at one boundary point necessitates changing the finite sequence of 1-spheres in 3-space to an infinite sequence of 1-spheres in 3-space, still independent of order.

Results appear to be forthcoming which would trivialize this result of Gugenheim for $q \geq 3$, i.e., for $q \geq 3$ all q-disks in 2q-space may turn out to be equivalently embedded. In Chapter I section 3 we relate our results to almost polyhedral 2-manifolds in E^4.

Finally, we turn our attention to the polyhedral category in Chapter II to point out that results of Gugenheim in [9] can be used to answer two questions posed at Madison in [12] p. 55. We then construct a polyhedral 2-sphere which fails to be locally flat at exactly one point.

CHAPTER 0

NOTATION AND TERMINOLOGY

Our entire discussion will take place in E^4. By E^n we mean all n-tuples of real numbers with the topology induced by the euclidean metric $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$, while $1/2\ E^n = \{x \in E^n: x_n \geq 0\}$.

For a and b in E^n we denote by ab the segment from a to b, i.e., $ab = \{ta+(1-t)b \in E^n: 0 \leq t \leq 1\}$.

A topological <u>n-manifold</u>, M^n, is a second countable Hausdorff space with an open covering $\{U_\sigma\}$ and a set of homeomorphisms $\{h_\sigma\}$ such that $h_\sigma: U_\sigma \to 1/2\ E^n$ and $h_\sigma(U_\sigma)$ is open in $1/2\ E^n$.

The set of all points with E^n neighborhoods is the <u>interior of M^n</u>, written Int M^n, and the <u>boundary of M^n</u> is Bd $M^n = \dot{M}^n = M^n - \text{Int } M^n$.

We will use the piecewise linear terminology of Hudson in [11], Gugenheim in [9], or Zeeman in [15]. Polyhedra will be the spaces in E^n underlying locally finite rectilinear complexes, i.e., finite unions of convex linear cells.

If X and Y are homeomorphic spaces embedded in E^n we say that they are <u>equivalently embedded</u> if there exists a homeomorphism $h: E^n \to E^n$ such that $h(X) = Y$. This is clearly an equivalence relation on a particular class of homeomorphic spaces and allows us to speak of the equivalence classes of embeddings.

If X is a space embedded in E^n and $P \in X$ we say X is <u>locally polyhedral at P</u> if there exists a neighborhood N of P such that $Cl(N \cap X)$ is a polyhedron. X is <u>locally tame at P</u> if there exists a neighborhood N and homeomorphism $h_p: \bar{N} \to E^n$ such that $h_p(\bar{N})$ is a polyhedron and $h_p(\bar{N} \cap X)$ is also a polyhedron.

If M^k is a k-manifold embedded in E^n and $P \in M^k$ we say M^k is <u>locally flat at P</u> if there exists a neighborhood U of P and a homeomorphism of the pair $(U, U \cap M^k)$ onto (E^n, E^k) or $(E^n, 1/2\ E^k)$ depending on whether $P \in \text{Int } M^k$ or $P \in \text{Bd } M^k$.

Given a 2-disk D with simple disjoint spanning arcs A and B, the disk bounded by A and B, and pieces of the boundary of D is denoted $[A;B]$.

$B(P,\epsilon) = \{x \in E^n: d(x,P) < (\leq)\ \epsilon\}$ is the open (closed) ball in E^n centered at P with radius ϵ.

When using cube neighborhoods, $N(P,\epsilon) = \{x \in E^n: |x_i - P_i| < \epsilon$ for $i = 1, 2, \ldots n\}$, ϵ is still referred to as the radius of the cube neighborhood N.

For any space X we denote by $id.: X \to X$ the identity map defined by $id.(x) = x$ for all $x \in X$.

When writing $i \in N$, N denotes the natural numbers.

CHAPTER I

ALMOST POLYHEDRAL 2-MANIFOLDS IN E^4

Section 1. Construction of a Special 2-Disk $\mathcal{D}$ in E^4

There are countably many different polygonal knot types in E^3 with representatives, say $\{K_i'\}$. Diagonalize the infinite matrix $\{a_{ij}\}$, with elements $a_{ij} = j$, to give a sequence $\{a_k\}$ in the order 1, 1, 2, 1, 2, 3,... Define a new sequence of knots $\{K_i\}$ where $K_i = K'_{a_i}$, note that the K_i' occur frequently in the sequence $\{K_i\}$.

Let T_i denote the trapezoidal cube in E^4 determined by the inequalities $1/i+1 \leq x_1 \leq 1/i$ and $-x_1/2 \leq x_j \leq x_1/2$ for $j = 2, 3, 4$ and for all $i \in N$. $\dot{T}_i$ is a PL 3-Sphere. Let $a_i = (1/i, 1/2i, 0, 0)$ and $b_i = (1/i, -1/2i, 0, 0)$ for all $i \in N$. Let K_i be in $\dot{T}_i$ in such a way as to include the segments $a_i b_i$ and $b_{i+1} a_{i+1}$ in the order $a_i a_{i+1} b_{i+1} b_i$ and such that $K_i \cap$ (hyperplane $x_1 = 1/i$) $= a_i b_i$ exactly. Let $m_i = (2i+1/2i(i+1), 0, 0, 0)$ and form polyhedral disks D_i by joining m_i to K_i for each i separately. Finally, form $\mathcal{D}$ by taking the union $\left(\bigcup_{i=1}^{\infty} D_i \right) \cup (0,0,0,0)$. By construction this disk is locally polyhedral at every point except possibly the origin. That it actually fails to be locally polyhedral at the origin follows from the result of Doyle and Hocking in [6].

Section 2. $\mathcal{D}$ is a Universal Disk of its Kind

We show that the disk $\mathcal{D}$ constructed in section 1 is "Universal" in the sense of the following:

MAIN THEOREM 2.1: Given any 2-disk, D, embedded in E^4 in such a way that D is locally polyhedral at all points except at one boundary point P, then there exists a space homeomorphism $h: E^4 \to E^4$ such that $h(D) \subset \mathcal{D}$ and $h(P) =$ origin.

Proof: The theorem is proved in the course of the following six lemmas.

Lemma 2.2: Given a 2-disk, D, embedded in E^4 in such a way that D is locally polyhedral at every point except at one boundary point P, then there is a space homeomorphism $f: E^4 \to E^4$, fixed outside a compact set, such that $f(D)$ is locally polyhedral at every point except $f(P)$ and locally flat at all but at most a sequence of points lying on a polygonal arc spanning $f(D)$.

Proof: Let $D' = \{(x,y): x,y \in R,\ x + y \leq 1,\ x \geq 0, y \geq 0\}$ and let $\varphi: D' \to E^4$ be an embedding such that $\varphi(D') = D$ and $\varphi(0,0) = P$. In D' denote segments $\{(x,y): x + y = 1/n\} \cap D' = 1_n$. $\{\varphi(1_n)\}$ form a sequence of arcs spanning D. For $k \geq 1$, consider the disks on D between $\varphi(1_{2k})$ and $\varphi(1_{2k+2})$. On these disks consider the arcs $\varphi(1_{2k+1})$. For each point x on $\varphi(1_{2k+1})$ there exists a neighborhood U_x such that $\bar{U}_x \cap D$ is a polyhedral disk and is equal to $\bar{U}_x \cap [\varphi(1_{2k});\varphi(1_{2k+2})]$.

There exist a finite number of these open U_x which cover $\varphi(l_{2k+1})$. Order these as $\varphi(l_{2k+1})$ traverses them say from $\varphi(0,1/2k+1)$ to $\varphi(1/2k+1,0)$, consecutive disks having a point in common. In the first disk there is a polygonal arc from $\varphi(0,1/2k+1)$ to this common point, in the second disk a polyhedral arc from this point to the point in common to the second and third disks etc. to $\varphi(1/2k+1,0)$. Drop any loops that occur leaving a simple polygonal arc A_k from $\varphi(0,1/2k+1)$ to $\varphi(1/2k+1,0)$ and contained in D between $\varphi(l_{2k})$ and $\varphi(l_{2k+2})$ for $k \geq 1$. Let A_0 denote the polygonal arc $\varphi(\{(x,y): x + y = 1\})$ from $\varphi(0,1)$ to $\varphi(1,0)$.

Each disk $[A_i;A_{i+1}]$ is locally polyhedral at every point and thus by the proof of Bing's lemma 1 in [1], is polyhedral. Compact polyhedra have only a finite number of vertices so these disks $[A_i;A_{i+1}]$ can fail to be locally flat at at most a finite number of points which, by Theorem 4.2 of Tindell in [13], must be interior points.

Starting at $\varphi(1/2,1/2)$ we pass a simple polygonal arc through all points of $[A_0;A_1]$ which fail to be locally flat ending at a point interior to a segment of A_1. Starting at this point continue similarly to A_2 and so to A_i so traversing each $[A_i;A_{i+1}]$. Let the arc traversing $[A_0;A_1]$ be the image of the interval from 2 to 1, and the arc traversing $[A_i;A_{i+1}]$ be the image of the interval from $1/i$ to $1/i+1$ for $i \geq 1$. Let $\varphi(0,0)$ be the image of 0 giving an arc A which is the image of the interval $[0,2]$ and is locally polyhedral at every point except possibly P. Taking this arc together with $\varphi(0,y)$ for $0 \leq y \leq 1$ and $\varphi(x,y)$ for $x + y = 1$ and $0 \leq x \leq 1/2$ yields a simple closed curve C which is locally polyhedral except possibly at P. By lemma 2 of Cantrell and Edwards in [3], given $\epsilon > 0$ there exists a homeomorphism $f: E^4 \to E^4$ such that:

a. f is the identity on $E^4 - B(P, \epsilon)$

b. f is piecewise linear except at P

c. f(C) is polyhedral,

f(D) is locally polyhedral except perhaps at f(P) and is locally flat except perhaps at a sequence of points which lie on the polygonal arc, f(A), spanning f(D). f is the homeomorphism required to satisfy the lemma.

We modify Doyle's proof of theorem 3.2 in [7] to give a PL homeomorphism $f_1: E^4 \to E^4$ such that $f_1(A)$ is a segment where A is a polygonal arc. Thus we have:

<u>Lemma 2.3</u>: Given the disk f(D) of the conclusion of lemma 2.2 we can find a PL homeomorphism $f_1: E^4 \to E^4$ such that $f_1(f(A))$ is a segment.

<u>Proof</u>: Using an inductive argument, we only need to define maps which will reduce the number of segments of the polygonal arc by one. Let v be an end-point of the polygonal arc and let v_1 be the other end-point of the segment containing v. Find a cube neighborhood of v_1, U_1, which intersects the arc f(A) only in the two segments adjacent to v_1. Let W_v be a conical neighborhood of v such that $\bar{W}_v$ intersects f(A) in precisely the segment vv_1. Let v' be a point of the segment vv_1 which is contained in U_1, $v' \neq v_1$. Define PL homeomorphisms h_1, h_2, and h_3 as determined by the simplicial maps indicated in figure 1. These maps are fixed on and outside the boundary of each figure shown.

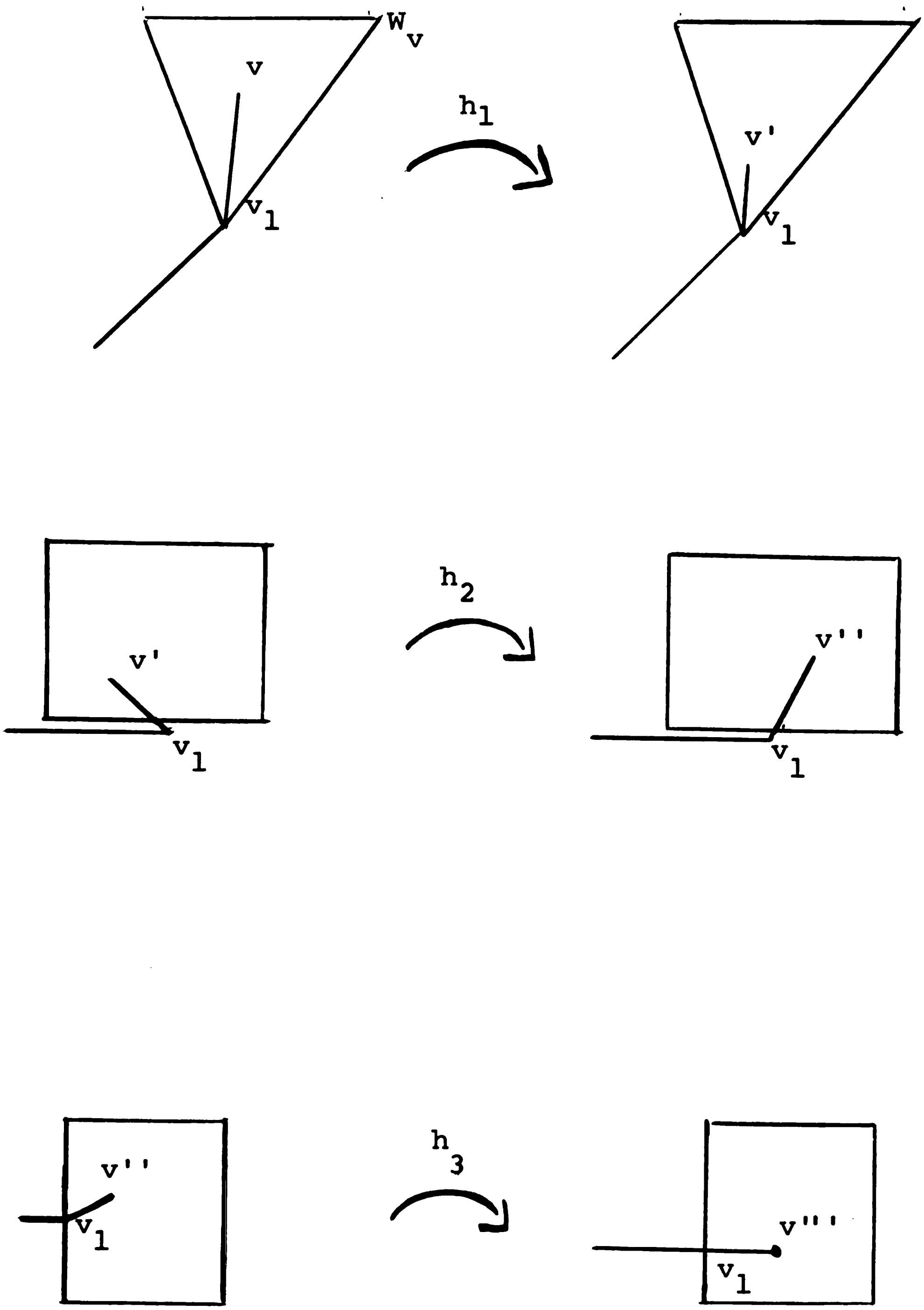

Figure 1

Lemma 2.3 could have been proved easily by using the general result of Gugenheim in Theorem 5 of [9]. He shows that for any q-dimensional polyhedra P and Q in E^n with $2q + 2 \leq n$ there exists a PL homeomorphism $h: E^n \to E^n$ such that $h(P) = Q$, i.e., polygonal arcs are equivalent to segments in E^n for $n \geq 4$. But it seems instructive to use the above proof for our particular case.

Lemma 2.4: The disk of the conclusion of lemma 2.3, call it D, is equivalent under a space homeomorphism to a disk which is locally polyhedral except possibly at a boundary point, P, the only non-locally flat points are on a spanning segment with end-point P, and all interior vertices of the disk lie on this segment.

Proof: Assume the segment of lemma 2.3 lies on the x_1 axis with $f_1(f(P))$ at the origin and the other end-point at (1,0,0,0). Introduce by starring if necessary a vertex between any two consecutive non-locally flat vertices on our segment and let $\{v_i\}$ denote the vertices on the segment in the order traversed from $v_0 = (1,0,0,0)$ to the origin. If a last v_K, denote by v_{K+k} the point $x_1 = |v_K| / k+1$ $x_2 = x_3 = x_4 = 0$.

There are at most countably many vertices in the disk (countable union of finite numbers of vertices in the Pl image of the $[A_i;A_{i+1}]$ of lemma 2.2). Construct cube neighborhoods N_i' centered at the origin with radius $r_i' = |v_i + v_{i+1}| /2$. Replace the N_i' with cube neighborhoods N_i of radius r_i such that $|r_i - r_i'| < |v_i - v_{i+1}| /2$ and the

3-hyperplanes $x_j = \underline{+}r_i$ ($j = 1, 2, 3, 4$; for all i) miss all the vertices of the disk. These hyperplanes are, of course, the hyperplanes containing the boundary of the N_i. This choice of the N_i assures that the intersection of the disk with the boundary of the N_i are one dimensional.

Denote the intersection of the segment $v_0(0,0,0,0)$ with N_i by R_i. Then the intersection of $\dot{N}_i$ with the disk has a component containing R_i which we denote by Q_i.

Note that Q_i is a polygonal arc spanning the disk. Q_i is one dimensional as noted above and is polygonal since given r_i we can find an A_j in lemma 2.2 such that the image of $(D - [A_j;A_0]) \subseteq B(0,r_i)$. Thus Q_i is in fact in the intersection of a polyhedral disk with $\dot{N}_i$. Q_i cannot be a loop since if it were it would necessarily intersect the spanning segment at least twice but this is impossible as $Q_i \subseteq \dot{N}_i$ which intersects the segment in precisely R_i.

The segment $v_0(0,0,0,0)$ divides the boundary of the disk into two arcs. Denote one as "upper" the other as "lower" fixed for the remainder of the discussion. We will work with the disk bounded by the segment and the upper arc a similar argument serving the other half of the disk.

Given Q_i there exists an $\epsilon_i > 0$ with $\epsilon_i < \min\{|v_i - R_i|/2, |v_{i+1} - R_i|/2\}$ and such that the distance from any vertex of the disk to Q_i is greater than ϵ_i. We can always do this since Q_i is compact and contains no vertices of the disk and only finitely many vertices are within $|v_{i+1} - R_i|$ of Q_i. Within ϵ_i of Q_i we can span the disk with a pair of non-intersecting polygonal arcs on each side of Q_i and thus find a polyhedral 4-ball neighborhood M_i of the disk bounded by the inner arcs which does not intersect the disk outside the outer arcs. Then with a space homeomorphism $h_i: E^4 \to E^4$ such that $h_i|_{E^4 - M_i} = \text{id.}$, $h_i(D) \subseteq D$, and the component of $h_i(D) \cap \dot{M}_i$ containing R_i is at most a pair of segments of length less than ε_i. This is possible since upper disk $\cap \bar{M}_i$ is polyhedral and flat and so equivalent by $g_i: E^4 \to E^4$ to D_i' in Figure 2. Take D_i' to D_i'' in Figure 2 by a PL $F_i: E^4 \to E^4$ fixed outside a small cube neighborhood of g_i(upper disk $\cap \bar{M}_i$) contained in $g_i(\bar{M}_i)$ and disjoint from the remainder of $g_i(D)$.

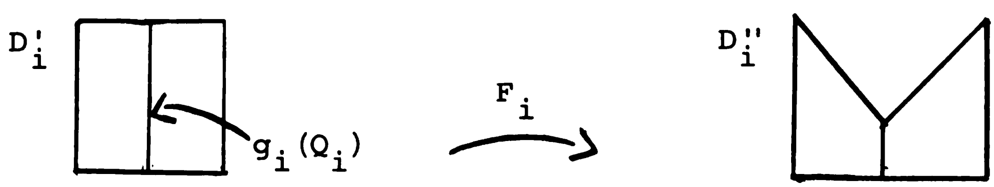

Figure 2

Let $h_i = g_i^{-1} \circ f_i \circ g_i$. Similarly for the lower disk.

Clearly $\bar{M}_i \cap \bar{M}_j = \emptyset$ for $i \neq j$. Thus define a space homeomorphism $h: E^4 \to E^4$ by setting $h = h_i$ on M_i and the identity outside the union of the M_i. $h(D) \cap \dot{N}_i$ has as the component containing R_i a pair of sements meeting at R_i, denote the end points u_i and l_i for end points in the upper and lower disks respectively.

Now in h(upper disk) u_iR_i and $u_{i+1}R_{i+1}$ together with R_iR_{i+1} and Bd D from u_i to u_{i+1} bound a polyhedral disk D_i. Join u_i to u_{i+1} by a simple polygonal arc P_i such that $P_i \cap \text{Bd } D_i = \{u_i, u_{i+1}\}$ and P_i is contained in $M_i' \cup M_{i+1}' \cup C_i$, C_i is a cylindrical neighborhood of the segment R_iR_{i+1} of radius $\epsilon_i' < \min\{u_iR_i, u_{i+1}R_{i+1}, \delta_i\}$ where δ_i is less than the distance from the segment R_iR_{i+1} to the vertices of the disk not on R_iR_{i+1} and less than r_i. M_i' denotes M_i intersect the cylinder about the segment $R_{i+1}R_{i-1}$ of radius u_iR_i.

We can find a closed neighborhood B_i of D_i such that $(\bigcup_{k<i} B_k \cup D) \cap \dot{B}_i = u_iR_i \cup R_iR_{i+1} \cup u_{i+1}R_{i+1}$ and $B_i \cap D = D_i$. By a PL homeomorphism $\varphi_i: E^4 \to E^4$ such that $\varphi_i|_{E^4-B_i}$ is the identity and $\varphi_i(D) \subseteq D$ take the polygonal boundary from u_i to u_{i+1} in the interior of B_i to P_i. Let $\varphi = \varphi_i$ on B_i. Similarly for the lower disk find h' and φ'. Then $\varphi' \circ h' \circ \varphi \circ h(D)$ satisfies the lemma.

<u>Lemma 2.5</u>: The disk resulting in lemma 2.4, call it D, is equivalent to one with interior vertices on the segment and the intersection of this disk with $\dot{N}_i$ is a segment lying in the plane $\mathcal{P}$, defined by $x_3 = 0$, $x_4 = 0$, and all upper end points are on the same side of the segment $v_0(0,0,0,0)$.

<u>Proof</u>: Define two PL homeomorphisms $f,g: E^4 \to E^4$, one to correct the intersection with $\dot{N}_i$ for odd i the other for even i. Let T_i be the trapezoidal neighborhood from R_{2i} to R_{2i+2} with base at R_{2i} of radius δ_{2i}. Picking one side of the segment in this plane $\mathcal{P}$, call it upper $\mathcal{P}$, in $\dot{N}_{2i+1} \cap \mathcal{P}$ let u_{2i+1}' be a point in upper $\mathcal{P}$ at a distance $\rho_{2i+1/2}$ from R_{2i+1}, $\rho_i = \min\{\delta_i, |R_i|\}$ similarly let l_{2i+1}' be in lower $\mathcal{P}$ at a distance $\rho_{2i+1/2}$ from R_{2i+1}. Define a vertex map which fixes all vertices of D except u_{2i+1} and l_{2i+1} which map to

u'_{2i+1} and l'_{2i+1} respectively and extend this linearly to obtain f_i. Define f to be f_i on T_i as described for all i. Define g_i and g similarly resulting in a disk locally polyhedral at all points except the origin and $\dot{N}_i$ intersects this disk in the segment $u'_i l'_i$. Note all interior vertices of the disk are on the segment from the origin to (1,0,0,0).

Using the segments $u'_i l'_i$ from the proof of lemma 2.5, define trapezoidal cube neighborhoods by $T'_i = \{(x_1,x_2,x_3,x_4)$: $|R_{i+1}| \leq x_1 \leq |R_i|, -(\rho_i-\rho_{i+1})/2|R_i-R_{i+1}|(x_1-|R_i|) - \rho_i/2 \leq x_j \leq \rho_i-\rho_{i+1}/2|R_i-R_{i+1}|\ (x_1-|R_i|) + \rho_i/2$ for $j = 2, 3, 4\}$.

<u>Lemma 2.6</u>: The disk of the conclusion of lemma 2.5 is equivalently embedded with a disk D with the same intersections with $\dot{N}_i$ and within the region $|R_{i+1}| \leq x_1 \leq |R_i|$ $\dot{D} \subseteq \dot{T}'_i$.

<u>Proof</u>: At each v_i a pseudo radial projection fixed on and outside the boundary of the rectangular cube defined by $|R_{i+1}| \leq x_1 \leq |R_i|$ and $-|R_{i+1}| \leq x_j \leq |R_{i+1}|$ for $j = 2,$ $3,4$, takes the boundary of D within the region $|R_{i+1}| \leq x_1 \leq |R_i|$ to $\dot{T}'_i$. This can be extended to a space homeomorphism fixed outside the union of these cubes and locally PL except at the origin.

The intersection of D and $\dot{T}'_i$ is a polygonal S^1 in a PL 3-sphere and thus is equivalent to some knot K'_i which is trivial if and only if the disk D is locally flat at v_i so we have:

<u>Lemma 2.7</u>: Given the disk D of lemma 2.6 there exists a homeomorphism $g: E^4 \to E^4$ such that $g(D) \subseteq \mathcal{D}$.

<u>Proof</u>: Let g_1 be fixed in the complement of N_1. Let v_{k_1} be the first vertex of the sequence $\{v_i\}$ at which D is not locally flat. There is a knot $K_{k_1'}$ in the sequence $\{K_i\}$ such that $K_{k_1'}$ is equivalent to $D \cap \dot{T}'_{k_1}$. Then let g_1 send $N_1 - N_{k_1+1}$ linearly to $C(0,1) - C(0,1/k_1')$ and send $N_{k_1} - N_{k_1+1}$ to $C(0,1/k_1') - C(0,1/k_1'+1)$. Let v_{k_2} be the next vertex at which D fails to be locally flat then there exists $K_{k_2'}$ in $\{K_i\}$ such that $k_2' > k_1'$ and $K_{k_2'}$ is equivalent to $D \cap \dot{T}'_{k_2}$. Let g_1 send $N_{k_1+1} - N_{k_2}$ to $C(0,1/k_1'+1) - C(0,1/k_2')$ and $N_{k_2} - N_{k_2+1}$ to $C(0,1/k_2') - C(0,1/k_2'+1)$ and continue in this way to define $g_1: E^4 \to E^4$ a locally PL homeomorphism except at the origin. Let $g_2, g_3: E^4 \to E^4$ be PL homeomorphisms defined in such a way as to make the segments $g_1(u_{k_i} l_{k_i})$ coincide with the segments $a_{k_i'} b_{k_i'}$ in $\mathcal{D}$, by changing the size of the trapezoidal cubes, g_2 for the k_{2i}' and g_3 for the k_{2i+1}'. Now for the v_{k_i} which are not locally flat

$g_3 g_2 g_1(T'_{k_i}) = T_{k_i'}$. By a PL homeomorphism g_4 fixed on and outside $\dot{T}_{k_i'}$ take $g_3 g_2 g_1(v_{k_i})$ to $m_{k_i'}$. There exists a PL homeomorphism of $\dot{T}_{k_i'}$ onto itself which keeps the segments $a_{k_i'} b_{k_i'}$ fixed and takes $\dot{T}_{k_i'} \cap g_3 g_2 g_1(D)$ to $\dot{T}_{k_i'} \cap \mathcal{D}$. Define

$f_{v_{k_i'}} : E^4 \to E^4$ by extending over $T_{k_i'} = m_{k_i'}\dot{T}_{k_i'}$ and over the region $G_{k_i'}$ defined by $1/k_i'+1 \leq x_1 \leq 1/k_i'$ such that $f_{v_{k_i'}}$ is the identity on the boundary of this region. Define $g_5: E^4 \to E^4$ to be $f_{v_{k_i'}}$ on $G_{k_i'}$ and the identity outside these regions. For vertices between v_{k_i} and $v_{k_{i+1}}$ at which D is locally flat define a map g_6 which takes the flat disk bounded by $a_{k_i'+1}b_{k_i'+1}$ and $a_{k_{i+1}'-1}b_{k_{i+1}'-1}$ to a ribbon along the upper boundary of $\mathcal{D}$ between $a_{k_i'+1}$ and $a_{k_{i+1}'-1}$. The composition of these maps $g = g_6g_5g_4g_3g_2g_1$ satisfies the lemma.

Lemmas 2.2 to 2.7 prove our main theorem 2.1.

Section 3. $\pi_1(E^4-\mathcal{D}) \cong 0$.

In [8], Fox and Artin give several examples of arcs or 1-disks which are wildly embedded in S^3, some whose complements fail to be simply connected, some whose complements are simply connected but fail to be E^3, and some whose complements are E^3.

We show that any 2-disk D in S^4, locally polyhedral at every point except at one boundary point P, has a complement S^4-D which is E^4. Let S^4 be the boundary of the 5-ball $\{(x_1x_2,\ldots x_5) \in E^5: 0 \leq x_5 \leq 2, -2 \leq x_j \leq 2$ for $j = 1, 2, 3, 4\}$.

<u>Lemma 3.1</u>: If D is a polyhedral 2-disk in S^4 and C a compact set in S^4-D then there exists a PL 4-ball neighborhood N such that $D \subseteq N$ and $\bar{N} \cap C = \emptyset$.

<u>Proof</u>: By induction on the number of triangles in D. Trivial for n = 1 where D is a triangle and is at a distance ϵ from C. Fatten D to a 4-ball with thickness $\epsilon/2$ in two orthogonal directions.

Assume true for disks of n-1 triangles and let D have n triangles. Let T be a triangle of D with at least one edge on the boundary of D and such that D - T is a polyhedral disk D' of n-1 triangles. Then by the inductive hypothesis D' has a 4-ball neighborhood N and T has a 4-ball neighborhood N_T of the type above and $\bar{N} \cap C = \emptyset = \bar{N}_T \cap C$.

$T \cap D'$ is a segment or pair of segments and in either case the distance from $T \cap D'$ to $\dot{N}$ is greater than some $\epsilon > 0$. Then there is a PL homeomorphism h fixed outside $\bar{N}_T$ and fixed on D' such that $h(D) \subseteq N$. Then $h^{-1}(N)$ is a PL 4-ball such that $D \subseteq h^{-1}(N)$ and $h^{-1}(N) \cap C = \emptyset$.

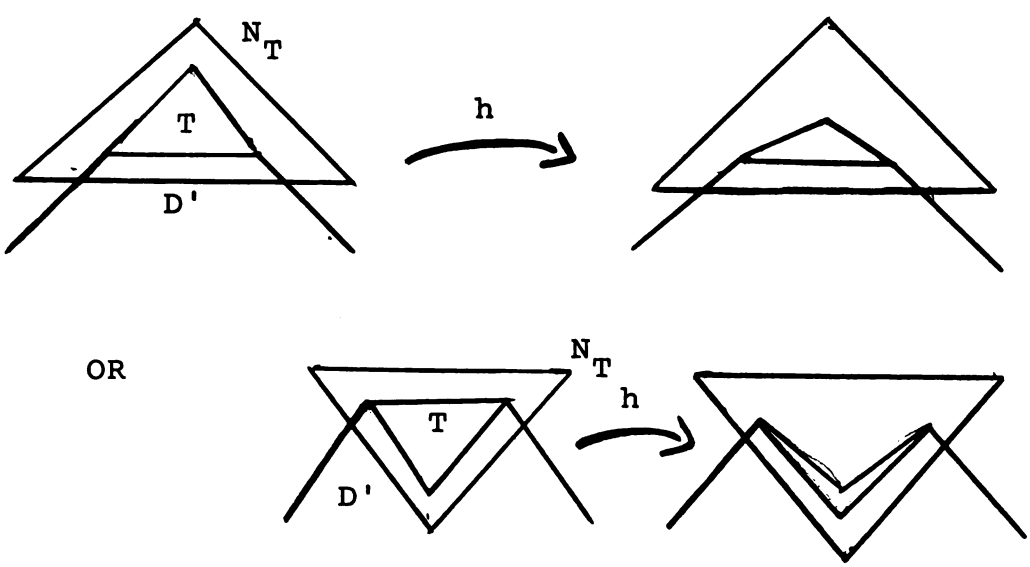

Figure 3

<u>Corollary 3.2</u>: Given a polyhedral 2-disk D in S^4 and compact $C \subseteq S^4-D$ there exists a topological 4-cell neighborhood N of C such that $N \subseteq S^4-D$. That is, polyhedral 2-disks are cellular in S^4 and have complements E^4.

<u>Theorem 3.3</u>: Given a 2-disk D in S^4 which is locally polyhedral at every point except at one boundary point P and compact set C in S^4-D, then there is a PL 4-ball N containing D such that $N \cap C = \emptyset$.

<u>Proof</u>: Let D be in the position of the conclusion of lemma 2.6. There exists a cube neighborhood $N'(P,r)$ such that $N'(P,r) \cap C = \emptyset$. There exists a natural number i such that $R_i \in N'(P,r)$ and $N(P, |R_i|) \cap C = \emptyset$, $D \cap \dot{N}(P, |R_i|)$ is the segment a_ib_i while a_ib_i divides D into two disks D' which is polyhedral and D" which contains P. By lemma 3.1 there is a PL 4-ball B containing D' which is disjoint from C and $\dot{B}$ is at a distance from C greater than some $\epsilon > 0$. Then there is a 4-cube neighborhood $U \subseteq N(P, |R_i|)$ such that $D'' \subseteq U$, $\dot{D}'' \cap U = a_ib_i$, $\dot{N}(P, |R_i|) \cap U = a_ib_i$, and a PL homeomorphism h fixed outside $N(P, |R_i|)$ such that $h(U) \subseteq B$. Then $h^{-1}(B)$ is a PL 4-ball containing D and disjoint from C.

Then $S^4-h^{-1}(B)$ is topologically a 4-cell containing C and itself contained as an open set in S^4-D. If every compact set of the countably compact space S^4-D is contained in an open 4-cell, then S^4-D is a monotone union of open 4-cells and by the result of Morton Brown in [2] we have:

Corollary 3.4: If D is a disk as in theorem 3.3, then S^4-D is topologically E^4.

And then since E^4-D is topologically E^4 with a point removed we have:

Corollary 3.5: $\pi_1(E^4-\dot{D}) \cong 0$.

Section 4: 2-Manifolds in E^4

The 2-disk is the building block of the 2-manifold so we make a few remarks relating our results to 2-manifolds in E^4.

Let M^2 be a compact 2-manifold with boundary $\dot{M}^2$ embedded in E^4 in such a way that M^2 is locally polyhedral at every point except at one boundary point P. Using the technique of the proof of lemma 2.2 we can find a sequence of polygonal arcs $\{A_i\}$ with end points on the boundary of M^2 on each side of P converging to P and such that each arc A_i divides M^2 into a polyhedral 2-manifold and a 2-disk locally polyhedral except at P. Let A_0 divide M^2 into M_0^2 and D_0. M_0^2 will fail to be locally flat at at most a finite number of points, $v_1,\ldots,v_k$, all in the Int M_0^2. In a finite number of steps we can add these v_i to D_0 by forming a new disk by adding a finite chain of triangles in Int M_0^2 reaching from A_0 to say v_1 thus forming a new disk D_{-1} with complement in M^2 a polyhedral 2-manifold M_{-1}^2 divided by a polygonal arc A_{-1}. Continue in this way to form the disk D_{-n} and polyhedral 2-manifold M_{-n}^2 divided by polygonal arc A_{-n}. M_{-n}^2 is also locally flat at every point. By lemma 2.3 there is a PL homeomorphism h: $E^4 \to E^4$ such that $h(A_{-n})$ is a segment giving the following:

<u>Theorem 4.1</u>: Given a compact 2-manifold M^2 in E^4 which fails to be locally polyhedral at just one point $P \in \dot{M}^2$, there exists a PL space homeomorphism $h: E^4 \to E^4$ and a segment, S, on $h(M^2)$ such that S divides $h(M^2)$ into a locally flat, polyhedral 2-manifold $M^{2\prime}$ and a 2-disk D' which is locally polyhedral except at the boundary point h(P).

If D" is any other 2-disk which arises in this way from M^2, there is a space homeomorphism $g: E^4 \to E^4$, locally PL except at h(P) and such that g(D') = D", since these embeddings depend only on the knot types at the vertices at which M^2 fails to be locally flat.

Now suppose the point P at which M^2 fails to be locally polyhedral is in Int M^2. Using an argument similar to that used above, we can find a polygonal S^1 in Int M^2 which divides M^2 into a locally flat polyhedral 2-manifold $M^{2\prime}$ and a 2-disk D' which fails to be locally polyhedral at just one interior point P. On this disk there is a sequence $\{C_i\}$ of non intersecting polygonal simple closed curves converging to P, with all points at which M^2 fails to be locally flat in the interior of the annuli $[C_i;C_{i+1}]$. We can start at points a and b on S^1 with a pair of arcs A and B which on each $[C_i;C_{i+1}]$ are polygonal and bound a 2-disk which misses the finite number of points of $[C_i;C_{i+1}]$ at which M^2 fails to be locally flat adding in the point P. A and B together with the appropriate half of S^1 between a and b form a simple closed curve C which is locally polyhedral except at P. Then as in lemma 2.2, there is an almost PL space homeomorphism $f: E^4 \to E^4$ such that f(C) is polygonal. f(C) divides $f(M^2)$ into a 2-disk D which is locally polyhedral except at one boundary point f(P) and a 2-manifold M_1^2 which is locally polyhedral except at one boundary point $f(P)$ and which is locally flat at every point. This last since it is locally flat except possibly at the boundary point $f(P)$; but then by theorem 4.2 of Tindell in [13], M_1^2 is locally flat at $f(P)$ also. Note that M^2 is not necessarily locally flat at $f(P)$. This gives us:

Theorem 4.2: Given a compact 2-manifold M^2 in E^4 which fails to be locally polyhedral at just one point $P \in \text{Int } M^2$, there exists a space homeomorphism $f: E^4 \to E^4$ and a polygonal simple closed curve $f(C)$ in $\text{Int } f(M^2)$ which divides $f(M^2)$ into a locally flat 2-manifold M_1^2 which is locally polyhedral except at the boundary point $f(P)$ and a 2-disk D which is locally polyhedral except at the boundary point $f(P)$.

CHAPTER II

POLYHEDRAL 2-MANIFOLDS IN E^4

Section 1: Madison Problems

Using the results of Gugenheim throughout the investigations for Chapter I stimulated interest in the polyhedral category. It was noticed that two problems which were posed in Madison in [12] could be answered negatively using results obtained by Gugenheim.

The first of these problems appeared in [12] as number 5 on page 56 as:

If C is a polygonal simple closed curve in E^2 and D a polyhedral disk in E^4 of which C is the boundary, does there exist a PL space homeomorphism $h: E^4 \to E^4$ such that $h(C) = C$ and $h(D) \subseteq E^3$? The answer is no.

<u>Proof</u>: Let K be a polygonal trefoil knot in E^3 and let P be a point of $E^4 - E^3$. The join P.K is a polyhedral 2-disk with boundary K, call it D'. By theorem 5 of [8] there is a PL homeomorphism $f: E^4 \to E^4$ such that $f(K)$ is a polygonal simple closed curve in E^2. Let the D of the question be $f(D')$ with polygonal simple closed curve boundary in E^2, $C = f(K)$. If there exists a PL homeomorphism $h: E^4 \to E^4$ such that $h(C) = C$ and $h(D) \subseteq E^3$ (even without the restriction $h(C) = C$) then $h(D)$ would be a 2-disk in E^3. But all polyhedral 2-disks in E^3 are equivalently embedded and hence $h(D)$ would be flat. In particular, $h(D)$ would be locally flat at $h(f(P))$ and D' would be locally flat at P which it is not.

The second problem appeared as number 6 on page 56 of [12] as:

If K is a polyhedral 2-sphere in E^4 does there exist a polyhedral 3-ball in E^4 of which K is the boundary? No.

The suspension of a polygonal trefoil knot in E^3 forms a polyhedral 2-sphere in E^4 which fails to be locally flat at its suspension points. But every PL 3-ball in E^4 is equivalently embedded by 7.33 of [10] and is therefore flat and thus its boundary is locally flat at every point. Thus no PL 3-ball has the suspension of the trefoil knot as its boundary.

Section 2: A Polyhedral 2-Sphere which Fails to be Locally Flat at Just One Point

In [4], Cantrell and Edwards constructed a 2-sphere in E^4 which failed to be locally tame at just one point P and was locally flat at every point except possibly P. In [14], Tindell showed that the example constructed by Cantrell and Edwards did in fact fail to be locally flat at P.

The construction of polyhedral 2-spheres in E^4 which fail to be locally flat at two points by suspending polygonal knots in E^3, together with the facts of the above paragraph concerning a wild sphere in E^4 which fails to be locally flat at just one point, brings us to the following question.

Does there exist a polyhedral 2-sphere in E^4 which fails to be locally flat at just one point? We answer in the affirmative by constructing an example.

Example 2.1: Let K be a polygonal trefoil knot in the 3-dimensional hyperplane E^3 in E^4 defined by $x_4 = 0$. Let K be so situated that of its vertices, $v_0, \ldots, v_n$, $v_0 = (0,0,0,0)$ and the other n vertices have first coordinate greater than 1. Let $a = (0,0,0,1)$ and $b = (0,0,0,-1)$ and form a polyhedral 2-sphere, S, by joining a to K and b to K. This sphere is locally flat except at a and b.

Let H be the 3-dimensional hyperplane defined by $x_1 = 1$. We want to see first that $S \cap H$ is a non-trivial polygonal knot in H. Let $r\colon E^3 \to E^3$ be defined by $r(x_1, x_2, x_3, 0) = (-x_1+2, x_2, x_3, 0)$, i.e., reflection through the plane $x_1 = 1$, $x_4 = 0$. Let $H \cap v_0v_1 = v'$ and $H \cap v_0v_n = v''$ and let A be the polygonal arc on K from v' through v_1 to v''. Denote by K', $r(A) \cup A$, the connected sum of a right and left handed trefoil knot, which is non-trivial since knots under connected sums form a semigroup without inverses (see page 164 of [5]).

Let $p\colon E^3_{x_1 \geq 1} \to H_{0 \leq x_4 < 1}$ denote the restriction of the projection from E^3 into H through a to the points of E^3 with $x_1 \geq 1$. Let $q\colon E^3_{x_1 \geq 1} \to H_{0 \geq x_4 > -1}$ denote the restriction of the projection from E^3 into H through b to the points of E^3 with $x_1 \geq 1$. p and q are homeomorphisms and on $H \cap E^3$ they are both fixed pointwise. Define $h\colon E^3 \to H_{-1<x_4<1}$ to be p on $E^3_{x_1 \geq 1}$ and qr^{-1} on $E^3_{x_1 \leq 1}$, these agree and are fixed on the plane $x_1 = 1$, $x_4 = 0$, which is $H \cap E^3$.

$h(K') = H \cap S$ is a polygonal S^1 which must be knotted in H. Since, there exists a real number R such that $0 < R < 1$ and $h(K') \subseteq S_R = \{(1, x_2, x_3, x_4) \in H\colon -R < x_4 < R\}$ and if $h(K')$ unknots in H, then there is a space homeomorphism

$f\colon H \to H$, fixed outside S_R and such that $f(h(K'))$ is a triangle in $\{(1,x_2,x_3,x_4) \in H\colon 0 < x_4 < R\}$ (may use argument of 3.44 in [9]). Call f restricted to $H_{-1<x_4<1}$, f^*, and this gives a space homeomorphism $h^{-1}f^*h\colon E^3 \to E^3$ such that $h^{-1}f^*h(K')$ is a triangle in E^3. But K' is knotted in E^3 and so $h(K')$ must be knotted in H.

Since $h(K') = H \cap S$ is a polygonal S^1 on S, it divides S into two polyhedral 2-disks say D in $E^4_{x_4 \geq 1}$ and D' in $E^4_{x_4 \leq 1}$. Replace D' with the cone formed by joining $h(K')$ and the origin, call this cone D". Then $D'' \cup D$ forms a polyhedral 2-sphere, S", which fails to be locally flat at the origin and is locally flat elsewhere except possibly at points of $h(K')$.

For points of $h(K')$ other than v' and v", S" is locally PL homeomorphic to S and thus S" is locally flat at points of $h(K') - \{v',v''\}$. v' lies on the edge v_0v_1 joining the two triangles av_0v_1 and bv_0v_1, and v" lies on the edge v_0v_n joining the two triangles av_0v_n and bv_0v_n and hence S" is locally flat at v' and v". S" is a polyhedral 2-sphere in E^4 which fails to be locally flat at just one point.

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