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M-GROUPS AND $\widetilde{M}$ -GROUPS

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ABSTRACT

M-Groups and $\tilde{M}$ -Groups

By

Paul Francis Murphy

The purpose of this paper is to investigate group theoretic properties of M-groups. Interest in this area has been renewed by the work of Dade, Dornhoff and Seitz.

The idea in this paper is to consider solvable groups which have no section of a certain type. The study of these types of sections is motivated by well known examples of groups which are not M-groups.

A group G is a D-group if and only if:

- i) there are primes $p, q, p \neq q$, such that G is a (p,q) group.
- ii) G contains a normal non-abelian p -Sylow subgroup P such that $Z(P)$ is cyclic and $P/Z(P)$ is elementary abelian.
- iii) a Sylow q -subgroup of G has order q and acts trivially on $Z(P)$ and non-trivially on $P/Z(P)$.

A group G is an E-group if and only if:

- i) for some prime p , G has a non-abelian, normal p -subgroup P such that $P/Z(P)$ is elementary abelian and $Z(P)$ is cyclic.
- ii) G acts irreducibly on $P/Z(P)$ and $Z(P) \leq Z(G)$.

A group G is an R-group if and only if G is an E-group, and G has a non-linear, faithful, primitive, irreducible representation.

The main results of the paper are:

- i) Theorem 2.6: If G is a finite solvable group such that no section of G is a D-group, then G is an M-group.

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- ii) Theorem 3.2: Let G be a finite solvable group. G is an $\overline{M}$ -group (i.e. every section of G is an M -group) if and only if G has no section which is an R -group.

The sufficient condition of Dornhoff and Huppert does not imply Theorem 2.6 and their condition is proved as a corollary to Theorem 3.2. Since Theorem 3.2 is clearly not a purely group-theoretic characterization, the final section of the paper considers a possible improvement.

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INTRODUCTION

In this paper, all arbitrary groups are finite and all representations are over the field of complex numbers $\mathbb{C}$. M-groups are characterized in the following representation theoretic manner: G is an M-group if and only if each irreducible representation of G is induced from a one-dimensional representation of a subgroup of G . The problem of proving a purely group-theoretic characterization of M-groups is unsolved. Vaguely, M-groups are (by set inclusion) between the class of solvable groups and the class of supersolvable groups.

This thesis is concerned with the problem of learning more about the group theoretic properties of M-groups. In studying this problem, we are led to the investigation of a subclass of the class of M-groups. G is an $\widetilde{M}$ -group if and only if every section of G is an M-group. The main idea of the thesis is that by requiring that a solvable group have no section of a certain type, we can then conclude that the group is an M-group. These certain types of sections are called D-groups and E-groups. They are motivated by examples in Chapter 1 of solvable groups which are not M-groups.

The organization of this thesis is as follows. The first chapter and part of the second chapter are expository in nature and provide the reader with most of the background needed to understand the results of the paper, and the development of the problem. The remainder of Chapter 2 is the statement and proof of a group theoretic sufficient

condition for a group to be an M-group; and an example to show that this condition is not implied by the sufficient condition of Huppert [10], and Dornhoff [4]. Chapter 3 contains a characterization of $\widetilde{M}$ -groups, from which we obtain the sufficient condition of Dornhoff as a corollary. This indicates that this characterization may be useful in proving further group theoretic sufficient conditions. Finally, because the characterization of M-groups is not purely group theoretic, Chapter 4 investigates the possibility of improving the results of Chapter 3.

Chapter I

Background from Representation Theory and Character Theory

Definition: A representation T , of a group G , is a homomorphism of G into the group, $GL(V)$, of nonsingular linear transformations of some finite dimensional vector space V over $\mathbb{C}$. The dimension of V is the degree of the representation T , and V is called the representation space of T .

Remark: For the purposes of this paper, we shall always assume that the ground field is the field $\mathbb{C}$ of complex numbers.

Definition: A representation T of the group G , with representation space V , is irreducible if there are no proper subspaces V_1 of V , such that $V_1 T(g) = V_1$ for all $g \in G$.

Definition: a) The character χ , afforded by the representation T of G , is the function of G given by: $\chi(g) = \text{trace } (T(g))$ for $g \in G$.
b) The kernel of χ , $\ker \chi = \{g \in G \mid T(g) = I_n\} = \{g \in G \mid \chi(g) = n\}$ where n is the degree of T . χ is a faithful character if and only if $\ker \chi = 1$.
c) χ is irreducible if and only if T is irreducible.
d) Degree of χ = Degree of T ; in notation $\deg \chi = \deg T$.
e) λ is a linear character if and only if λ is afforded by a one dimensional representation.
f) 1_G is the character afforded by the representation T that maps G onto 1 and 1_G is called the principal character.
g) If χ is a character of G , and H is a subgroup of G , $\chi|_H$ is the restriction of the function χ to H .
h) If χ and ϕ are characters of a group G

and $\chi = \phi + \psi$, where ψ is also a character of G , then ϕ is called a constituent of χ . If ϕ is irreducible, then ϕ is an irreducible constituent of χ .

Definition: Let H be a subgroup of G and let $G = \bigcup_{i=1}^m Hg_i$. If S is a representation of H , we define the mapping S^G of G , where $S^G(g)_{ij}$ is the submatrix in the (i,j) position of $S^G(g)$:

$$S^G(g)_{ij} = \begin{cases} S(g_i g g_j^{-1}) & \text{if } (g_i g g_j^{-1}) \in H \\ 0 & \text{if } (g_i g g_j^{-1}) \notin H \end{cases}$$

It can be shown that S^G is a representation of G . S^G is called an induced representation.

Definition: Let H be a subgroup of G and let $G = \bigcup_{i=1}^m Hg_i$. If ϕ is the character afforded by the representation S of H , then we define the function ψ_0 on G : $\psi_0(g) = \begin{cases} \psi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$. Then the induced character ψ^G , afforded by S^G , is defined: $\psi^G(g) = \sum_{i=1}^m \psi_0(g_i g g_i^{-1})$ for $g \in G$.

Definition: A monomial matrix is a square matrix with exactly one non-zero entry in each row and column. A permutation matrix is a monomial matrix with the non-zero entries equal to 1.

Definition: A representation T of a group G is called a monomial (permutation) representation if T maps G into a group of monomial (permutation) matrices.

Definition: Let T be a permutation representation of a group G and let V be the representation space of T . $T(G)$ permutes the basis elements of V over $\mathbb{C}$. If $T(G)$ acts transitively on the basis of V , we say T is a transitive permutation representation.

Remark: Let M be the group of all $n \times n$ monomial matrices over $\mathbb{C}$, and let D be the subgroup of M consisting of all the diagonal matrices. Then, $D \triangleleft M$, $M = DS_n$ and $D \cap S_n = I$, where S_n is the group of all $n \times n$ permutation matrices. See Ore [13]. If M_1 is a subgroup of M and if η is the natural homomorphism of M onto S_n , then we say $\eta(M_1)$ is the group of permutation matrices associated with M_1 . Thus, if T is a monomial representation of G of degree n , we call $\eta \cdot T$ the associated permutation representation of T , since $\eta T(G)$ is a group of permutation matrices.

Definition: A monomial representation T of the group G is transitive if the associated permutation representation of T is transitive.

Definition: Two representations T and T' , are equivalent if there exists a non-singular matrix S with entries in $\mathbb{C}$ such that $S^{-1}T(g)S = T'(g)$ for all $g \in G$.

The following is an important characterization of equivalent representations. (See Feit [5], page 12)

Theorem 1.1: Two representations T, T' of the group G which afford characters ϕ, ϕ' respectively are equivalent if and only if $\phi = \phi'$.

Definition: A group G is an M -group if and only if every irreducible representation of G is equivalent to a monomial representation.

The next theorem and corollary yield a representation theoretic characterization of M -groups. (For proofs, see Scott [15] page 364.)

Theorem 1.2: A representation T of a group G is equivalent to a transitive monomial representation if and only if T is induced

from a one-dimensional representation S of some subgroup H of G .

Corollary 1.2.1: A group G is an M-group if and only if every irreducible representation T of G is induced from a one-dimensional representation S of some subgroup H of G .

Corollary 1.2.1 is often used as the definition of M-group, as in Huppert [9].

Definition: If ϕ and ψ are characters of a group G , then we define the inner product $(\phi, \psi)_G : (\phi, \psi)_G = \frac{1}{|G|} \sum_{g \in G} \phi(g)\psi(g)$.

Calculations using this inner product are very useful when working with induced characters. Two results that indicate this are:

1) If χ is an irreducible character of G , then $(\chi, \psi)_G$ equals the number of times that χ appears as an irreducible constituent of any character ψ of G .

2) Frobenius Reciprocity Law: If H is a subgroup of the group G and ζ is a character of H ; χ , a character of G , then $(\chi|_H, \zeta)_H = (\chi, \zeta^G)_G$.

The problem of finding a group theoretic characterization of M-groups is unsolved. In the next chapter, some of the difficulties that arise in working with the class of M-groups will be mentioned. Several mathematicians have proved sufficient conditions for a group G to be an M-group. The following condition is the only necessary condition for M-groups and gives us the only information that we have about an arbitrary M-group.

Theorem 1.3: (Taketa [18]) Every M-group is solvable.

Because of the importance of this theorem, we include its proof. First we extract two elementary but useful facts found in the proof

and write them as lemmas.

Lemma 1.3.1: If H is a non-trivial subgroup of G , then H is not contained in the kernel of some irreducible character of G .

Proof: See Feit [5] for definition and properties of the right regular representation of G .

Lemma 1.3.2: If λ is a linear character of a subgroup H of the group G , then

$$\ker \lambda^G = \text{Core}(\ker \lambda) = \bigcap_{y \in G} y^{-1}(\ker \lambda)y$$

Proof: Let $G = \bigcup_{i=1}^m Hx_i$ and let $g \in \ker \lambda^G$.

If S is a representation of H which affords λ , then $(S^G(g))_{ij} = S(x_i g x_j^{-1}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. It follows that $x_i g x_i^{-1} \in \ker \lambda$, for all i , $1 \leq i \leq m$. Thus $g \in \bigcap_{i=1}^m x_i^{-1}(\ker \lambda)x_i$.

Now let y be an arbitrary element in G . Since $y = hx_k$ for some k , $1 \leq k \leq m$, and since $\ker \lambda \triangleleft H$,

$$y^{-1}(\ker \lambda)y = x_k^{-1}h^{-1}(\ker \lambda)h x_k = x_k^{-1}(\ker \lambda)x_k.$$

Therefore $\bigcap_{i=1}^m x_i^{-1}(\ker \lambda)x_i = \bigcap_{y \in G} y^{-1}(\ker \lambda)y$ and $g \in \text{Core}(\ker \lambda)$.

On the other hand, if $g \in \text{Core}(\ker \lambda)$,

$$\lambda^G(g) = \sum_{i=1}^m \lambda_0(x_i^{-1} g x_i) = m = \lambda^G(1) \text{ since } (x_i^{-1} g x_i) \in \ker \lambda.$$

Thus $g \in \ker \lambda^G$.

Proof of Theorem 1.3: Let G be an M -group which is not solvable. Then $1 \neq G^{(i)} = G^{(i+1)}$ for some i . By Lemma 1.3.1, let T be an irreducible representation of minimal degree such that $G^{(i)} \not\subseteq \ker T$, and let χ be the character afforded by T .

Since G is an M-group, $\chi = \lambda^G$ where λ is a linear character of some subgroup H of G . Thus $\deg \chi = |G:H|$. Let $\mu = 1_H = 1_G|_H$. By Frobenius Reciprocity, $(1_G, \mu^G)_G = (1_H, \mu)_H = 1$. Therefore 1_G is an irreducible constituent of μ^G . Since $\deg \mu^G = |G:H|$, $\deg \zeta < \deg \mu^G = \deg \chi$, for any irreducible constituent ζ of μ^G . Hence $G^{(i)} \leq \ker \zeta$, for all irreducible constituents ζ of μ^G , so $G^{(i)} \leq \ker \mu^G$.

By Lemma 1.3.2, $\ker \mu^G = \text{Core}(\ker \mu)$. Since $\mu = 1_H$, $\text{Core}(\ker \mu) = \text{Core}(H)$ and $G^{(i)} \leq H$. Since $G^{(i)} = G^{(i+1)}$, $G^{(i)} \leq H'$. Since H' is contained in the kernel of any linear character of H , $G^{(i)} \leq \ker \lambda$. Finally, since $G^{(i)} \trianglelefteq G$ and $G^{(i)} \leq \ker \lambda$, then $G^{(i)} \leq \text{Core}(\ker \lambda) = \ker \lambda^G = \ker \chi$. However this contradicts our choice of χ .

Solvability is certainly not a sufficient condition for a group G to be an M-group, as is illustrated by the following examples. The two examples are given in detail, since they will be used to motivate ideas in later chapters.

Example 1.1: Let $Q = \langle x, y | x^4 = y^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1} \rangle$, the quaternion group of order 8, and let $\sigma \in \text{Aut } Q = S_4$, with $|\sigma| = 3$. σ is defined on Q : $\sigma(x) = y$, $\sigma(y) = xy$, $\sigma(xy) = x$.

Now consider $G = Q \langle \sigma \rangle$, the relative holomorph of G and $\langle \sigma \rangle$. The only elements of Q fixed by $\langle \sigma \rangle$ are 1 and x^2 . Thus $x^2 = Z(Q) = Z(G)$. G/Q is abelian, so $G' \leq Q$.

If $|G'| = 2$, $G' = Z(G)$ and $G/Z(G)$ is abelian, so G is nilpotent, which implies $G = Q \times \langle \sigma \rangle$, a contradiction.

If $|G'| = 4$, $G' = \langle x \rangle$ or $\langle y \rangle$ or $\langle xy \rangle$. However, none of these groups is normal in G , since σ permutes x , y , and xy , a contradiction.

Therefore $|G'| = 8$ and $G' = Q$. Hence G has 3 linear characters.

$G/Z(G)$ contains no element of order 6, since σ permutes x, y , and xy , so $G/Z(G) \cong A_4$.

Therefore G has irreducible characters of degree 1, 1, 1, 3. Since the sum of the squares of the irreducible characters of G is 24, the degrees of the remaining characters must be 2. However $G' = Q$ implies that G has no subgroup of index 2. Thus G is not an M-group.

Before proceeding to Example 1.2, we introduce the following concepts.

Definition: If $H \trianglelefteq G$ and S is a representation of H which affords the character ϕ , then we define the following:

a) For a fixed $g \in G$, S^g , the conjugate representation is defined: $S^g(h) = S(g^{-1}hg)$ for any $h \in H$. The conjugate character ϕ^g is the character afforded by S^g .

b) The inertial group of ϕ , $I(\phi)$ is defined: $I(\phi) = \{g \in G \mid \phi^g = \phi\}$. Since ϕ is constant on conjugacy classes of H , $H \leq I(\phi) \leq G$.

The following results are due to Clifford [2], and are essential in working with induced representations.

Theorem 1.4: Let $H \trianglelefteq G$, and χ be an irreducible character of G . Then $\chi|_H = k(\phi + \phi^{g_2} + \cdots + \phi^{g_t})$ where ϕ is an irreducible character of H , and $\{\phi^{g_i}, 1 \leq i \leq t\}$ is a complete set of conjugate characters of ϕ and k is a positive integer.

Theorem 1.5: Let $H \trianglelefteq G$ and let χ be an irreducible character of G . Let ϕ be an irreducible constituent of $\chi|_H$ and let ζ be an irreducible constituent of $\chi|_{I(\phi)}$ such that $\phi \leq \zeta|_H$. Then $\chi = \zeta^G$.

Theorem 1.6: Let $H \trianglelefteq G$ and $|G:H| = p$, a prime. If ϕ is an

irreducible character of H and $I(\phi) = G$, then there is an irreducible character χ of G such that $\chi|_H = \phi$.

Finally, we use the following result about extraspecial p -groups. (Feit [5], page 128).

Theorem 1.7: If P is an extraspecial p -group of order p^3 , then every non-linear character of P has degree p .

Example 1.2: Let P be an extraspecial p -group of order p^3 and exponent p , with p , an odd prime, such that $p + 1$ is divisible by an odd prime, q . $P = \langle x, y, z \mid x^p = y^p = z^p = 1, [x, z] = [y, z] = 1, [x, y] = z \rangle$. $SL(2, p)$ is the group of outer automorphisms of P which fix $Z(P)$, (Winter [19]). Since $|SL(2, p)| = p(p-1)(p+1)$, P has an automorphism, σ , of order q which fixes $Z(P)$. σ acts non-trivially on $P/Z(P)$, an elementary abelian group of order p^2 , (Gorenstein [7], page 178).

Every non-zero element of $P/Z(P)$ generates a one dimensional subspace of $P/Z(P)$, but $p-1$ elements generate the same one dimensional subspace. Thus there are $\frac{p^2-1}{p-1} = p+1$ one dimensional subspaces of $P/Z(P)$. $SL(2, p)$ acts transitively on the one-dimensional subspaces of $P/Z(P)$.

Let K be the subgroup of $SL(2, p)$ which fixes a one dimensional subspace of $P/Z(P)$. Since $SL(2, p)$ acts transitively on the one-dimensional subspaces, $|SL(2, p):K| = p+1$, the number of such subspaces. Since $|\sigma| \nmid p+1$, $\sigma \notin K$, i.e., σ fixes no one dimensional subspace of $P/Z(P)$.

Now consider the group, $G = P\langle\sigma\rangle$, $|G| = p^3q$. Since $|P'| = p$, P has p^2 linear characters. By Theorem 1.7, the remaining irreducible characters of G have degree p , and there are $(p-1)$ such characters

since $|P| = p^3 = p^2(1)^2 + (p-1)(p)^2$. Let ψ be a non-linear irreducible character of P . Since $P \triangleleft G$, and $|G:P| = q$, a prime, using Theorems 1.5 and 1.6, either

- a) ψ^G is irreducible, or
- b) $\chi|_P = \psi$ for some irreducible character χ , of G .

Suppose G is an M-group.

If a) holds, G has a irreducible character of degree pq , namely ψ^G . Since G is an M-group, there is a subgroup, H of G and a linear character λ of H such that $\lambda^G = \psi^G$. Since $|H| = p^2$, $Z(P) = Z(G) \leq H$ and $|\psi^G(z)| = |\lambda^G(z)| = pq$ for each $z \in Z(G)$. Therefore $\sum_{g \in G} |\psi^G(g)| \geq p(pq)^2 > |G|$. This contradicts the assumption that ψ^G is irreducible.

If b) holds, then since G is an M-group, G contains a subgroup H of order p^2q , since $\deg \chi = p$. If S is a Sylow p -subgroup of H , $S \triangleleft H$ since $p > q$. Since $\langle \sigma \rangle < H$ and $S \triangleleft P$, $S \triangleleft G$. Therefore $S/Z(P) \triangleleft G/Z(P)$. Since $S/Z(P) < P/Z(P)$, σ fixes a one dimensional subspace of $P/Z(P)$, a contradiction. Therefore $G = P \langle \sigma \rangle$ is not an M-group.

Chapter II

M-Groups

This chapter is partly expository and is designed to show how the results of this thesis fit in with the current knowledge of M-groups. The literature on M-groups is rather limited and new results have been infrequent. The recent work of Dade provides a partial explanation. (See Huppert [9], page 583 for proofs).

Theorem 2.1: (Dade) If G is an M-group and C is a cyclic group of prime order, then the regular wreath product $G \wr C$ is an M-group.

Corollary 2.1.1: (Dade) Every solvable group can be embedded in an M-group.

Example 2.1: If $G = Q \langle \sigma \rangle$, as in Example 1.1 and $|C| = |G/Q| = 3$, then $G \leq Q \wr C$. (See Huppert [9], page 98). By Theorem 2.1, $Q \wr C$ is an M-group which contains a subgroup $Q \langle \sigma \rangle$, which is not an M-group.

Example 2.2: (Dornhoff [4]) The group in this example is also an M-group which contains $Q \langle \sigma \rangle$. Let Q, Q_1 be two quaternion groups of order 8, and let QQ_1 be their central product of order 32. Let σ be an automorphism of order 3, such that $Q^\sigma = Q, Q_1^\sigma = Q_1, \sigma$ non-trivial on both Q and Q_1 ; σ is described in Example 1.1. Then the relative holomorph $G = QQ_1 \langle \sigma \rangle$ is an M-group, and $Q \langle \sigma \rangle \leq G$.

The results of Dade give us an idea of the size of the class of M-groups, and further provide us with examples of M-groups, which have

subgroups which are not M-groups. It is not known whether a normal subgroup of an M-group is an M-group, but there is the following theorem.

Theorem 2.2: (Dornhoff [4]). If G is an M-group and H is a normal Hall subgroup of G , then H is an M-group.

We now move to a consideration of sufficient conditions, and introduce the needed terminology.

Definition: A section of a group G is a homomorphic image of a subgroup H of G .

Definition: Let H_2 be the quaternion group of order 8 and let H_p , for p an odd prime, be the extraspecial p -group of order p^3 and exponent p . Then a group G is H_p -free if G has no section isomorphic to H_p .

Definition: An irreducible representation T of a group G , with representation space V , is imprimitive if $V = V_1 \oplus V_2 \oplus \dots \oplus V_r$ where V_i , $1 \leq i \leq r$, are subspaces of V and $r > 1$, and each $V_i T(g) = V_j$ for $1 \leq j \leq r$ and $g \in G$.

An irreducible representation is primitive if it is not imprimitive. An irreducible character χ is primitive if it is afforded by an irreducible primitive representation.

Definition: G is a primitive linear group, if and only if G has a faithful, primitive, non-linear, irreducible representation. Thus, for the purposes of this paper, all primitive linear groups are over $\mathbb{C}$.

Theorem 2.3: (Curtis-Reiner [3], pp. 346-48). Let T be an irreducible representation of a group G . T is primitive if and only if T is not induced from any irreducible representation S of a subgroup H of G .

Definition: A group G is an A-group if G is solvable and every

Sylow p -subgroup of G is abelian for all $p \mid |G|$.

In the following theorem, Huppert generalized the sufficient conditions of Itô [11], and Zassenhaus [21].

Theorem 2.4: (Huppert [10]). If a solvable group G has a normal subgroup N , such that G/N is supersolvable, and N is an A-group, then G is an M-group.

In particular, all supersolvable groups, and all A-groups are M-groups. Theorem 2.4 is implied by the following theorem of Dornhoff. However, Seitz and Wright [17], showed that the two theorems are actually equivalent.

Theorem 2.5: (Dornhoff [4]). If a solvable group G has a normal subgroup N such that G/N is supersolvable, and the Sylow p -subgroups of N are H_p -free for each prime p , then G is an M-group.

We will present a proof of Theorem 2.5 in Chapter 3, as a corollary to another result. Now we present a collection of lemmas which are often used in this paper. Specifically, they will be used in the proof of the next theorem. Proofs of known results are presented when it is thought that the proof is different than existing proofs. For convenience, we will call a faithful, primitive, non-linear irreducible representation (character) a restrained representation (character). Thus, for example, G is a primitive linear group if and only if G has a restrained representation.

Lemma 2.6.1: (Blichfeldt [1]). If G is a primitive linear group, then all normal abelian subgroups of G are contained in $Z(G)$.

Proof: Let H be a normal, abelian subgroup of G , and let ζ be a restrained character of G . By Theorem 1.4, $\zeta|_H = k(\phi + \phi^{g_2} + \dots + \phi^{g_t})$.

If $I(\phi) \ntriangleleft G$, by Theorem 1.5, $\zeta = \psi^G$ where ψ is an irreducible constituent of $\zeta|_{I(\phi)}$, such that $\phi \leq \psi|_H$, a contradiction.

So $I(\phi) = G$. This implies $\zeta|_H = k\phi$. Since H is abelian, ϕ is linear and $\zeta(1) = k$. Therefore $\zeta|_H = \zeta(1)\phi$, and for all $h \in H$, $\zeta(h) = \zeta(1)\omega, \omega \in \mathbb{C}$. If T is a representation of G that affords ζ , $T(h)$ is a scalar matrix for each $h \in H$. Thus $T(H) \leq Z(T(G))$. Since T is faithful, $H \leq Z(G)$.

Lemma 2.6.2: (Feit [5], page 46). If $H \leq K \leq G$, and θ is a character of H , then $(\theta^K)^G = \theta^G$.

Lemma 2.6.3: Let χ be an irreducible character of the group G and let $H < \ker \chi < G$ with $H \ntriangleleft G$. Let χ' be the character of G/H associated with χ (i.e. $\chi'(Hg) = \chi(g)$); if $\chi' = \lambda'^{G/H}$, then $\chi = \lambda^G$, (where λ' is a character of some subgroup K/H of G/H and λ is the associated character of K).

Proof: Let $G = \bigcup_{i=1}^m Kg_i$. Then $G/H = \bigcup_{i=1}^m (K/H)Hg_i$.

$$\lambda^G(g) = \sum_{i=1}^m \lambda_o(g_i^{-1}gg_i) = \sum_{i=1}^m \lambda'_o(Hg_i^{-1}gg_i) = \lambda'^{G/H}(Hg).$$

$\lambda'^{G/H}(Hg) = \chi'(Hg) = \chi(g)$. Therefore $\chi(g) = \lambda^G(g)$.

Lemma 2.6.4: If G is a primitive linear group, then $Z(G)$ is cyclic.

Proof: Let T be a restrained representation of G . $T(Z(G))$ is a subgroup of the group of scalar matrices, whose entries are from the cyclic group of n^{th} roots of unity where n is the exponent of G . Thus $T(Z(G))$ is a subgroup of a cyclic group, and thus is cyclic. Since T is faithful, $Z(G)$ is cyclic.

Lemma 2.6.5: Let G be a solvable, primitive linear group. Then G has a normal, non-abelian p -subgroup P such that $P/Z(P)$ is elementary abelian and $|P'| = p$.

Proof: Let $M/Z(G)$ be a minimal normal subgroup of $G/Z(G)$. By Lemma 2.6.1, M is non-abelian and therefore $M/Z(G)$ is elementary abelian of order p^r , $r > 1$. If $(|M/Z(G)|, |Z(G)|) = 1$, $M = AZ = A \times Z$, where A is an elementary abelian subgroup of M of order p^r ; this implies M is abelian, a contradiction. Therefore $p \mid |Z(G)|$.

Let P be a Sylow p -subgroup of M , and let $Z(G) = (P \cap Z(G)) \times Z_1$. Clearly $(P \cap Z(G)) \leq Z(P)$. $|M| = p^r |Z(G)|$, $|P| = p^r |P \cap Z(G)|$ and $|Z| = |P \cap Z(G)| |Z_1|$. We have $Z_1 \triangleleft M$ and $PZ_1 \leq M$. By the above, $|PZ_1| = |M|$. Therefore $M = PZ_1 = P \times Z_1$ and $P \text{ char } M \triangleleft G$, so $P \triangleleft G$. Hence $Z(P) \triangleleft G$, and since $Z(P)$ is abelian $Z(P) \leq Z(G)$, by Lemma 2.6.1. Hence $Z(P) = Z(G) \cap P$. Since $M = PZ(G)$, $P/Z(P) = P/Z(G) \cap P \cong PZ(G)/Z(G) = M/Z(G)$. Therefore $P/Z(P)$ is elementary abelian. Hence $P' \leq Z(P)$ and P' is cyclic since $P' \leq Z(P) \leq Z(G)$, which is cyclic by Lemma 2.6.4. Hence for $x, y \in P$, $x^p \in Z(P)$ and $1 = [x^p, y] = [x, y]^p$. (See Huppert [9], page 253). Since P' is cyclic, $|P'| = p$.

After considering Examples 1.1 and 1.2, we are led to the following class of groups.

Definition: A group G is a D-group if it satisfies the following properties:

- 1) there are primes p, q , $p \neq q$ such that G is a (p, q) group.
- 2) G contains a normal non-abelian p -Sylow subgroup P such that $Z(P)$ is cyclic and $P/Z(P)$ is elementary abelian.
- 3) A Sylow q -subgroup of G has order q and acts trivially on $Z(P)$ and non-trivially on $P/Z(P)$.

Definition: A group G is a D_0 -group if no section of G is a D -group.

Lemma 2.6.6: Subgroups and factor groups of D_0 -groups are D_0 -groups.

Proof: Obvious.

Theorem 2.6: A solvable D_0 -group is an M -group.

Proof: Let G be a minimal counterexample and let ζ be an irreducible non-monomial character of G . By Lemma 2.6.6, every subgroup and factor group of G satisfy the hypothesis of the theorem. Therefore every proper subgroup and factor group of G is an M -group. Using this fact and Lemmas 2.6.2 and 2.6.3, ζ is a restrained character, and G is a primitive linear group.

By Lemma 2.6.5, G has a normal, non-abelian subgroup P such that $P/Z(P)$ is elementary abelian and $Z(P) \leq Z(G)$.

Now let H be the centralizer of $P/Z(P)$ in G . Suppose $H \neq G$. Then, there exists $y \in (G-H)$ such that $\langle Hy \rangle$ has prime order q . Suppose $q \neq p$. Consider $\langle y \rangle P / \langle y^q \rangle$. Notice that we can choose y so that $p \nmid |y|$. If $p \mid |y|$, i.e. $|y| = p^a n$, then since $(Hy)^q = H$, $(Hy)^{p^a} \neq H$. Hence $y^{p^a} \in G-H$ and $(H y^{p^a})^q = H$. So we can use y^{p^a} if needed. Since $Z(P) \leq Z(G)$ and $y^q \in H$, y^q stabilizes the normal series $1 \triangleleft Z(P) \triangleleft P$. Since $(|y^q|, p) = 1$, y^q centralizes P (see Gorenstein [7], page 178). Thus $\langle y^q \rangle \triangleleft \langle y \rangle P$ and $\langle y \rangle P / \langle y^q \rangle$ is a D -group. This contradicts the hypothesis of the theorem, so G/H is a p -group.

Thus, H contains a Sylow q -subgroup of G for each $q \neq p$. Thus for any such q , there is a Sylow q -subgroup, S_q of G such that S_q centralizes $P/Z(P)$. Therefore S_q stabilizes the normal series $1 \triangleleft Z(P) \triangleleft P$ and as above, $S_q \leq C_G(P)$. Since this is true for any $q \neq p$, $G/C_G(P)$ is a p -group.

We now assert that $\zeta|_{C_G(P)}$ is reducible. Let T be a representation of G which affords ζ . If T is irreducible on $C_G(P)$, then $T(P)$ centralizes the irreducible set $T(C_G(P))$. By Schur's Lemma (Feit [5], page 5), $T(P)$ consists of scalar matrices. Since T is faithful, $P \leq Z(G)$, a contradiction.

Since $G/C_G(P)$ is a p -group, we can form the normal series:

$G = H_0 \geq H_1 \geq \dots \geq H_i \geq H_{i+1} \geq \dots \geq H_n = C_G(P)$ such that $|H_k:H_{k+1}| = p$ for $0 \leq k \leq n$. For some i , $0 \leq i \leq n$, $\zeta|_{H_i}$ is irreducible and $\zeta|_{H_{i+1}}$ is reducible.

By Theorem 1.4, $\zeta|_{H_{i+1}} = k(\phi + \phi^{g_2} + \dots + \phi^{g_t})$. If $I(\phi) \neq G$, by Theorem 1.5, $\zeta = \psi^G$ where ψ is an irreducible character of $I(\phi)$ such that $\phi \leq \psi|_{H_{i+1}}$, a contradiction.

If $I(\phi) = G$, then we consider the irreducible character $\zeta|_{H_i}$ of H_i , and we see $\zeta|_{H_{i+1}} = (\zeta|_{H_i})|_{H_{i+1}}$. So we can consider the irreducible character $\zeta|_{H_i}$ being restricted to the normal subgroup H_{i+1} of index p in H_i . Since $I_G(\phi) = G$, $I_{H_i}(\phi) = H_i$. By Theorem 1.6, $(\zeta|_{H_i})|_{H_{i+1}} = \phi$, a contradiction since $(\zeta)|_{H_{i+1}}$ is reducible.

The only remaining possibility is that $G = H = C_G(P/Z(P))$. Therefore a Hall p' -subgroup K of G centralizes $P/Z(P)$ and $Z(P)$, so as before, $K \leq C_G(P)$ and $G/C_G(P)$ is a p -group. We have just shown that this case leads to a contradiction, and the theorem is proved.

It might be asked whether it is possible to extend Theorem 2.6 to a result of the form of Theorem 2.5 i.e.: If G is solvable and $N \triangleleft G$ such that G/N is supersolvable and N is a D_0 -group, then G is an M -group. The answer is negative since $0 < \sigma$ of Example 2.1 satisfies the hypothesis, but is not an M -group.

We now require some concepts from the theory of formations in order to describe the results of Seitz [16]. We shall apply these results as well as our Theorem 2.6 to an example.

Definition: A class $\mathcal{F}$ of finite groups is a formation if:

- 1) $G \in \mathcal{F}$ implies $G^\sigma \in \mathcal{F}$ for all epimorphic images G^σ of G .
- 2) $G/N_1, G/N_2 \in \mathcal{F}$ implies $G/N_1 \cap N_2 \in \mathcal{F}$.

Definition: A formation $\mathcal{F}$ is saturated if $G/\Phi(G) \in \mathcal{F}$ implies $G \in \mathcal{F}$.

Definition: For every prime p , let $\mathcal{F}(p)$ be a formation. A class $\mathcal{F}$ of groups is called a locally defined formation, defined by $\mathcal{F}(p)$, if $\mathcal{F}$ consists of those groups G having the following properties:

- 1) $p \nmid |G|$ if $\mathcal{F}(p) = \emptyset$
- 2) if H/K is a chief factor of G such that $p \mid |H/K|$, then $G/C_G(H/K) \in \mathcal{F}(p)$, where $C_G(H/K) = \{g \in G \mid h^g K = hK \text{ for all } h \in H\} = \{g \in G \mid [H, g] \leq K\}$.

Remark: If $\mathcal{F}$ is a saturated formation of solvable groups and contains every nilpotent group, then $G \in \mathcal{F}$ if and only if $G/Z(G) \in \mathcal{F}$. This fact follows from the theorem of Lubeseder [12], that a saturated formation of solvable groups is locally defined. Classes of groups, X , where $G \in X$ if and only if $G/Z(G) \in X$ are used in the work of Seitz, and saturated formations which contain every nilpotent group provide examples of such classes.

The class $\mathcal{S}$ of supersolvable groups is a saturated formation which is locally defined as follows: for all p , $\mathcal{S}(p)$ consists of all abelian groups with exponent dividing $p-1$. (See Huppert [9], page 712, Hilfsatz 8.3).



With this background, we now proceed to the sufficient condition of Seitz [16], which implies Theorem 2.5.

Definition: Let $\mathcal{C}$ be the union of all classes of solvable groups X such that X is subgroup closed, X is closed under homomorphisms, X consists of M-groups, $G/Z(G)$ is in X if and only if G is in X .

Theorem 2.7: (Seitz [16]). Let G be a solvable group such that $N \trianglelefteq G$ and $G/N \in \mathcal{C}$. Suppose the p -Sylow subgroups of N are H_p -free for each prime p . Then G is an M-group.

Corollary 2.7.1: (Seitz [16]). Let $\mathcal{F}$ be a saturated formation containing all nilpotent groups, and let $\mathcal{F}$ be subgroup closed and consist only of M-groups. If G/N is in $\mathcal{F}$, and the p -Sylow subgroups of N are H_p -free for all primes p , then G is an M-group.

Let $\mathcal{F} = \mathcal{S}$, the formation of supersolvable groups, and we have Theorem 2.5. In the process of obtaining this generalization, Seitz has introduced conditions which are not purely group theoretic; specifically, the condition on $\mathcal{C}$ that $G \in \mathcal{C}$ implies that $G/Z(G)$ is an M-group if and only if G is an M-group.

Proposition 2.8: (Seitz [16]). There exists a saturated formation $\mathcal{F}$ such that $\mathcal{S} \subset \mathcal{F}$, $\mathcal{F}$ is subgroup closed, $\mathcal{F}$ contains every nilpotent group and $\mathcal{F}$ contains only M-groups.

The formation exhibited in this proposition is locally defined as follows:

- a) For p odd, $\mathcal{F}(p)$ is the class of abelian groups with exponent dividing $p-1$.
- b) $\mathcal{F}(2)$ is the class of elementary abelian 7-groups.

Theorem 2.9: (Dornhoff [4]). If P is a non-abelian 2-group with an automorphism α , of odd order n , acting without fixed points on $P/\Phi(P)$, P has a quaternion section.

Theorem 2.9 will be used in the following example which gives groups which satisfy the hypothesis of Theorem 2.6 but not of Theorem 2.5.

Definition: A Suzuki 2-group is a non-abelian 2-group, with more than one involution, having a cyclic group of automorphism which permutes its involutions transitively.

Example 2.3: Let P be a Suzuki 2-group of order 2^{2n} where $(2^n - 1) > 3$ is a Mersenne prime. As described in Higman [8], P has the following properties:

- a) P has $(2^n - 1)$ involutions and P has an automorphism α which permutes the involution of P cyclically; $|\alpha| = 2^n - 1$
- b) $Z(P) = P' = \Omega_1(P) = \Phi(P)$; $|Z(P)| = 2^n$
- c) P has exponent 4.

Consider the relative holomorph, $G = P \langle \alpha \rangle$. Since α acts fixed-point-free on the involutions of P and $\exp P = 4$, α acts fixed-point-free on P . (i.e., if $p \in P$, $|p| = 4$ and $\alpha(p) = p$, then $\alpha(p^2) = p^2$, a contradiction since p^2 is an involution.) The property of being fixed-point-free is preserved under homomorphic images, so α acts fixed-point-free on $P/\Phi(P)$. (See Gorenstein [7], page 335). By Theorem 2.9 P is not H_p -free. Also $1 \triangleleft \Phi(P) \triangleleft P \triangleleft G$ is a chief series for G , since $|\Phi(P)| = |P/\Phi(P)| = 2^n$ and $|\alpha| = 2^n - 1$, a prime. Any proper subgroup P_1 , normal in G , contains an involution and thus contains $Z(P)$. So P is minimal in G with the property that G/P is supersolvable. Since P is not H_p -free, Theorem 2.5 does not apply.

Suppose G has a section H/K , which is a D -group. Then we can assume $\langle \alpha \rangle \leq H$; so K is fixed by α . If $K \neq 1$, K contains an involution and $Z(P) \leq K$. Since α acts cyclically on $P/Z(P)$, $K = Z(P)$ or $K = P$. But since H/K is a (p,q) group, $K = Z(P)$ and $H = G$. However $P \langle \alpha \rangle / Z(P)$ has an abelian p -Sylow group and is not a D -group. Thus $K = 1$. Thus H has a normal non-abelian p -Sylow subgroup P^* such that $Z(P^*)$ is cyclic and H acts trivially on $Z(P^*)$. This implies α fixes the involution in $Z(P^*)$, a contradiction. So G is a D_0 -group. Thus Theorem 2.6 implies that G is an M -group.

In the case $n = 3$, Seitz [16] shows that G belongs to the formation described in Proposition 2.8, and thus by Corollary 2.7.1, G is an M -group. Although it appears possible to exhibit a formation like that of Proposition 2.8 for the cases where $(2^n - 1)$ is any Mersenne prime, Theorem 2.6 is easier to apply.

Chapter III

$\tilde{M}$ -Groups

Definition: A group G is an $\tilde{M}$ -group if and only if every section of G is an M -group.

In the previous chapter, we considered the following type of theorem:

If G satisfies hypothesis (H) , then G is an M -group. For each such theorem, the following hold:

- 1) every subgroup S of G satisfies (H) .
- 2) every homomorphic image, G/K , of G satisfies (H) .

1) and 2) imply that every homomorphic image of every subgroup of G satisfies H , so every section of G satisfies H . Therefore, since we proved G is an M -group if G satisfies (H) , then every section of G is an M -group. Thus each sufficient condition in the previous chapter, is actually a sufficient condition for G to be an $\tilde{M}$ -group.

In this chapter, we will prove the same type of theorem:

If G satisfies (H) , (where (H) is a hypothesis satisfied by all sections of G), then every section of G has property W .

The following reasoning will be used. Let G be a minimal counter-example. Since every proper section of G satisfies (H) , every proper section has property W . Therefore only G itself does not have property W . Then, we will show that this leads to a contradiction.

Theorem 3.1: G is an $\tilde{M}$ -group if and only if no section of G has a restrained character.

Proof: Certainly, if G is an $\tilde{M}$ -group, every non-linear irreducible character of every section of G is induced from a linear character.

On the other hand, let G be a minimal counterexample, i.e. no section of G has a restrained character and G is not an $\tilde{M}$ -group. Using the reasoning described above, G itself is not an M -group. There is a non-monomial irreducible character, ζ , of G . Since every proper section of G is an M -group, we can use Lemmas 2.6.2 and 2.6.3 to show that ζ is a restrained character of G , a contradiction.

Defintiiion: A finite solvable group G is called an E -group if the following hold:

- 1) For some prime p , G has a non-abelian, normal, p -subgroup P such that $P/Z(P)$ is elementary abelian.
- 2) G acts irreducibly on $P/Z(P)$ and $Z(P) \leq Z(G)$; $Z(P)$ is cyclic.

Definition: An E -group which has a restrained character is called an R -group.

Definition: A solvable group G is called an E_0 -group (respectively R_0 -group) if and only if G has no section isomorphic to an E -group (respectively R -group).

Theorem 3.2: Let G be a finite solvable group. G is an $\tilde{M}$ -group if and only if G is an R_0 -group.

Proof: If G is an $\tilde{M}$ -group, clearly G is an R_0 -group by Theorem 3.1.

On the other hand, let G be a minimal counterexample such that G is an R_0 -group, and G is not an $\tilde{M}$ -group. Thus G itself is not an M -group. Let ζ be a non-monomial irreducible character of G . By the method of Theorem 3.1, ζ is a restrained character of G and G is a primitive linear group, and G has a normal non-abelian p -subgroup P , such that $P/Z(P)$ is elementary abelian, and $Z(P) \leq Z(G)$. Also $Z(G)$

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contains all normal abelian subgroups of G .

If G acts irreducibly on $P/Z(P)$, then G is an R -group, a contradiction. Thus G acts reducibly on $P/Z(P)$ and irreducibly on some subgroup $P_1/Z(P) \neq 1$ of $P/Z(P)$. Thus $P_1/Z(P) \triangleleft G/Z(P)$, so $P_1 \triangleleft G$. If P_1 is abelian, $P_1 \leq Z(G)$, and $P_1 \leq Z(P)$, a contradiction. So $P_1/Z(P)$ is elementary abelian of order p^s , $s > 1$. $Z(P_1) \triangleleft G$ implies $Z(P_1) \leq Z(G)$. Since $Z(P) \leq Z(P_1) \leq P_1 \leq P$, $Z(P) = Z(P_1)$.

Thus G acts irreducibly on $P_1/Z(P_1)$ where P_1 is non-abelian, and $Z(P_1) \leq Z(G)$. Therefore G is an R -group, a contradiction.

Theorem 3.3: (Rigby [14]). If G is a solvable primitive linear group, and P is a normal non-abelian p -subgroup in G , then P is not H_p -free.

Using this result of Rigby, we now prove Theorem 2.5 as a corollary to Theorem 3.2.

Corollary 3.2.1: If a solvable group G has a normal subgroup N such that G/N is supersolvable, and the Sylow p -subgroups of N are H_p -free for each prime p , then G is an R_0 -group (i.e. an $\tilde{M}$ -group).

Proof: Let G be a minimal counterexample. By our previous discussion, and the fact that all proper sections of G satisfy the hypothesis, G is itself an R -group. Therefore, G has a normal, non-abelian subgroup P , such that $Z(P) \leq Z(G)$ and G acts irreducibly on $P/Z(P)$. Also G is a primitive linear group.

Since $(P \cap N) Z(P) \triangleleft G$ and G acts irreducibly on $P/Z(P)$, $(P \cap N) Z(P) = Z(P)$ or $(P \cap N) Z(P) = P$.

Suppose $(P \cap N) Z(P) = P$. Since P is non-abelian, $P \cap N$ is non-abelian. Since $P \cap N \triangleleft G$, $P \cap N$ is described in Theorem 3.3,

and $P \cap N$ is not H_p -free. Since $P \cap N \leq S_p$, a Sylow p -subgroup of N , S_p is not H_p -free, a contradiction.

Therefore $(P \cap N) Z(P) = Z(P)$, so $P \cap N \leq Z(P)$, and since $[P, N] \leq Z(P)$, N acts trivially on $P/Z(P)$. Therefore G/N acts irreducibly on $P/Z(P)$. Also G/N acts irreducibly on $K = [PN/N / Z(P)N/N]$.

If this is not the case, there is P_1 , with $Z(P) \subsetneq P_1 \subsetneq P$, such that $Z(P)N/N \triangleleft P_1N/N \triangleleft PN/N \triangleleft G/N$ is a normal series. Thus $Z(P)N \triangleleft P_1N \triangleleft PN \triangleleft G$ is a normal series. Let $p_1 \in P_1$. Then for $g \in G$ $g^{-1}p_1g = p_1^*n \in P_1N$, since $P_1N \triangleleft G$ and $g^{-1}p_1g = p \in P$ since $P \triangleleft G$. Thus $p_1^*n = p$ and $n \in P \cap N \leq Z(P)$. So $g^{-1}p_1g = zp_1^*$, and G fixes $P_1/Z(P) \subsetneq P/Z(P)$, a contradiction since G acts irreducibly on $P/Z(P)$.

Now $K \cong PN/Z(P)N \cong P/Z(P)$, since $P \cap N \leq Z(P)$; and since G/N acts irreducibly on K , K is a chief factor of G/N which is not cyclic, contradicting the hypothesis that G/N is supersolvable. This is the final contradiction, and therefore G is an R_0 -group.

Corollary 3.2.2: If G is a solvable E_0 -group, then G is an $\widetilde{M}$ -group.

Proof: An E_0 -group is an R_0 -group.

In the characterization of $\widetilde{M}$ -groups in Theorem 3.2, it was necessary to use representation-theoretic conditions. In striving for a purely group theoretic characterization, we are interested in the converse of Corollary 3.2.2.

Chapter IV

Possible Improvement of Results

In this very short chapter, we indicate progress made in studying the question of whether the converse of Corollary 3.2.2 is true.

Theorem 4.1: (Rigby [14]). If P is a p -group such that $Z(P)$ is cyclic, and $P/Z(P)$ is elementary abelian, a basis of $P \bmod Z(P)$ can be chosen, say $x_1, y_1, \dots, x_h, y_h$, such that $[x_i, y_i] = c$, $[i = 1, 2, \dots, h]$ where $P' = \langle c \rangle$, and all other pairs of basis elements commute.

Theorem 4.2: (Rigby [14]). Let P be a p -group such that $Z(P)$ is cyclic and $P/Z(P)$ is elementary abelian. Then P has (p^Z) faithful irreducible representations of degree p^h , where $|Z(P)| = Z$ and $|P:Z(P)| = p^{2h}$.

Theorem 4.3: (Gallagher [6]). If N is a normal Hall subgroup of a group G , and ζ is an irreducible character of N such that $I(\zeta) = G$, then there is an irreducible character of G such that $\chi|_N = \zeta$.

Theorem 4.4: (Winter [20])(Seitz [16]). Let H be a subgroup of G , and let λ be a linear character of H such that λ^G is irreducible. If A is an abelian normal subgroup of G , then there exists a subgroup H_1 of G such that $A \leq H_1$ and H_1 has a linear character ζ such that $\zeta^G = \lambda^G$.

Theorem 4.5: Let G be an M -group, and assume G is also an E -group. Then in the notation of the definition of E -group, $p \mid |G:P|$.

Proof: Let P be the normal non-abelian p -subgroup described in the definition of E -group, and suppose $p \nmid |G:P|$.

By Theorem 4.2, P has an irreducible character ζ of degree p^h where $|P:Z(P)| = p^{2h}$. For $y \in Z(P)$, $\zeta(y) \overline{\zeta(y)} = p^{2h}$, so

$$\sum_{y \in Z(P)} \zeta(y) \overline{\zeta(y)} = p^{Z+2h} = |P| = \sum_{x \in P} \zeta(x) \overline{\zeta(x)}.$$

Thus $\zeta(x) = 0$ for $x \in (P-Z(P))$. Since $Z(P) \leq Z(G)$, $I(\zeta) = G$. Since $p \nmid |G:P|$, Theorem 4.3 implies $\zeta = \chi|_P$ for some irreducible character χ of G . Since G is an M-group, $\chi = \lambda^G$ where λ is a linear character of a subgroup H of G with $|G:H| = p^h$. Consider $P_1 = H \cap P$. By Theorem 4.4, $Z(P) \leq H$ since $Z(P) \triangleleft G$. Thus $Z(P) \leq P_1$. Since P is a normal Sylow p -subgroup of G and since $|P:Z(P)| = p^{2h}$ and $|G:H| = p^h$, then $Z(P) \not\leq P_1 \not\leq P$ and $G = PH$. Since $P \triangleleft G$, and $P/Z(P)$ is elementary abelian, $P_1 \triangleleft H$ and $P_1 \triangleleft P$. Thus $P_1 \triangleleft G$, contradicting the hypothesis that G is an E-group which acts irreducibly on $P/Z(P)$. Therefore χ is not monomial and G is not an M-group, a contradiction. Therefore $p \mid |G:P|$.

Corollary 4.5.1: If L is an $\tilde{M}$ -group, and L has a section G , which is an E-group, then in the sense of the definition of E-group, $p \mid |G:P|$.

Example 4.1: The group $G = QQ_1 \langle \sigma \rangle$ in Example 2.2 is an example of an E-group which is an M-group.

$Z(G) = Z(QQ_1) = Z(Q)$ and $Q \triangleleft G$. G acts irreducibly on $Q/Z(Q)$ since α cyclically permutes the involutions in $Q/Z(Q)$. Also $Q/Z(Q)$ is elementary abelian. Thus G is an E-group.

Since G is not an $\tilde{M}$ -group, the converse of Corollary 4.2.2 is still possible.

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