# THE SET OF GENERATING DOMAINS FOR CERTAIN MANIFOLDS

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# ABSTRACT

## THE SET OF GENERATING DOMAINS FOR CERTAIN MANIFOLDS

by Richard John Tondra

Let X be a topological space. A collection G* of non-empty, connected topological spaces is called a set of generating domains for X if each proper domain (open, connected subset) of X is an open, monotone union of some element g(D) of G*; that is, $D = \bigcup_{k=1}^{\infty} D_k$ where each $D_k$ is an open set homeomorphic to g(D) and $D_k \subset D_{k+1}$ for all $k \in Z^+$. The domain rank of X, denoted by DR(X), is the cardinal number of a set of generating domains for X that has a minimal number of elements.

Let $M^n$ denote a connected n-manifold, $n \geq 2$. The principal theorems characterize those manifolds which have the smallest possible domain rank. Let us say that $M^n$ has Euclidean compact subsets if for each proper, compact subset C of $M^n$ there is a homeomorphism h of the pair $(C, C \cap \dot{M}^n)$ into the pair $(\frac{1}{2}R^n, \frac{1}{2}\dot{R}^n)$, where $\frac{1}{2}R^n = \{x \in R^n | x_n \geq 0\}$. In chapter II it is shown that if $\dot{M}^n = \emptyset$, then $DR(M^n) = 1$ if and only if $M^n$ has Euclidean compact subsets. If $\dot{M}^n \neq \emptyset$, then it is shown in chapter III that $DR(M^n) = 2$ if and only if $M^n$ has Euclidean compact subsets. Chapter IV gives a characterization of those $M^n$ with $\dot{M}^n$ an n-1 sphere that have domain rank 3. In the final chapter are found results concerning the domain rank of spaces which are the open monotone

union or the finite product of those manifolds considered in chapters II through IV.

# THE SET OF GENERATING DOMAINS FOR CERTAIN MANIFOLDS

By

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To

Mom and Dad

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# CHAPTER I

# INTRODUCTION

In [3], Morton Brown proved that any open monotone union of an open n-cell is an open n-cell. This thesis is concerned with a problem quite opposite to that considered by Brown - proving the existence of certain n-manifolds from which many non-homeomorphic spaces can be obtained as open monotone unions. In particular it is shown that there exists a connected n-manifold $M^n$ such that each open connected subset of Euclidean n-space can be obtained as an open monotone union of $M^n$.

## 1. Notational conventions

Let A, B, and X be sets. If A is a subset of X, then this will be denoted by $A \subset X$; if $A \subset X$ but $A \neq X$, then this will be denoted by $A \subsetneq X$. If $A \neq \emptyset$ and $A \neq X$, then A is called a proper subset of X. If A and B are subsets of X, then $A - B$ will denote the set theoretic difference of A and B in X.

Let X be a topological space, $A \subset X$. $\mathrm{Int}_X A$, $\mathrm{Cl}_X A$, and $\mathrm{Fr}_X A$ will denote the *interior*, *closure*, and *frontier* of A in X respectively. The set $A - \mathrm{Int}_X A$ will be denoted by $\mathrm{Ed}_X A$ and is called the *edge* of A in X. Note that if $A = \mathrm{Cl}_X A$, then $\mathrm{Fr}_X A = \mathrm{Ed}_X A$. When there is no possibility of ambiguity, the subscript "X" will be omitted.

If X is a topological space, we will denote by H(X) the set of all homeomorphisms of X onto itself. CH(X) will denote the subset of H(X) consisting of all those homeomorphisms which are the identity on the complement of some proper compact subset of X.

If X and Y are topological spaces, a homeomorphism of X into Y will be called an <u>embedding</u>. If there is a homeomorphism f of X onto Y, we will write $X \overset{T}{=} Y$.

Let X and Y be disjoint topological spaces, and let X + Y denote the disjoint union of X and Y with the weak topology. Suppose that $A \subset X$ is closed, $A \neq \emptyset$, and that f is a continuous function from A into Y. $X \cup_f Y$ will denote the space obtained by attaching X to Y by f where $p : X + Y \to X \cup_f Y$ is the identification.

The following notation will be used for certain sets and topological spaces:

$Z = \{n | n \text{ is an integer}\}$;

$Z^+ = \{n \in Z | n > 0\}$; and

$R^n = \{x | x = (x_1, \ldots, x_n)$, an n-tuple of real numbers, $n \in Z^+\}$.

$R^n$ is assumed to have the topology determined by the Euclidean metric $d_n$ on $R^n$. The subsets

$\frac{1}{2}R^n = \{x \in R^n | x_n \geq 0\}$;

$E^n = \{x \in R^n | d_n(x,0) \leq 1\}$;

$E^n(r) = \{x \in R^n | d_n(x,0) \leq r$, r a real number $> 0\}$; and

$S^{n-1} = \{x \in R^n | d_n(x,0) = 1\}$ are assumed to have the subspace topology induced by the topology of $R^n$.

## 2. Sets of generating domains

Throughout this section, X will denote a fixed non-empty topological space.

Definition 1.2.1 A non-empty set $D \subset X$ is called a domain of X if D is open and connected.

Definition 1.2.2 A non-empty set D* of subsets of X is called a dominion of X if each element of D* is a proper domain of X.

Definition 1.2.3 Let $O \subset X$ be a non-empty open set. A topological space g(O) is called a generator of O if there exists a countable collection of sets $\{O_k\}_{k \in Z^+}$ such that $O = \bigcup_{k=1}^{\infty} O_k$ and such that for all $k \in Z^+$,

i) $O_k \subset X$ is open and homeomorphic to g(O) and

ii) $O_k \subset O_{k+1}$.

If g(O) generates O, then O is called an open, homogeneous monotone union of g(O).

Definition 1.2.4 Let $K(X) = \{D | D$ is a proper domain of $X\}$ and suppose that $K(X) \neq \emptyset$. A non-empty set B* of non-empty, connected topological spaces is called a set of generating domains for X, if for each $D \in K(X)$ there is some $B \in B^*$ such that B is a generator of D.

If B* is a set of generating domains for X, then each $B \in B^*$ is homeomorphic to a proper domain of X. Therefore a set of generating domains with a minimal number of elements can be found among the dominions of X. This consideration leads to the following definition.

<u>Definition 1.2.5</u> Suppose that $K(X) \neq \emptyset$ and let $G^* = \{B^* | B^*$ is a dominion of X which is also a set of generating domains for X$\}$. The <u>domain rank</u> of X, denoted by DR(X), is defined by $DR(X) = \text{g.l.b.}\{|B^*| \,|\, B^* \in G^*\}$ where $|B^*|$ denotes the cardinal number of the set B*. If DR(X) = 1, then X is said to have a <u>generating domain</u>. If $K(X) = \emptyset$, we define DR(X) = 0.

The following theorem is an immediate consequence of the foregoing definitions.

<u>Theorem 1.2.6</u> If $Y \subset X$ is a domain of X, then DR(Y) is less than or equal DR(X).

## 3. Collared manifolds

<u>Definition 1.3.1</u> A topological space X is called an <u>n-dimensional manifold</u>, $n \in Z^+$, and is denoted by $M^n$, if X is a separable metric space and for each $x \in X$ there is an open neighborhood $U_x$ of x such that $U_x \overset{T}{=} R^n$ or $U_x \overset{T}{=} \frac{1}{2}R^n$. A <u>0-dimensional manifold</u> $M^0$ is defined to be an at most countable, descrete topological space.

Definition 1.3.2 Let $M^n$ be an n-dimensional manifold, $n \in Z^+$. The set $\overset{o}{M}{}^n = \{x \in M^n | x$ has an open neighborhood homeomorphic to $R^n\}$ is called the interior of $M^n$. The set $\dot{M}^n = M^n - \overset{o}{M}{}^n$ is called the boundary of $M^n$. If $\dot{M}^n = \emptyset$ $[\dot{M}^n \neq \emptyset]$, then $M^n$ is called a manifold without [with] boundary. An n-manifold without boundary is called closed if $M^n$ is compact; otherwise, $M^n$ is called open.

Definition 1.3.3 A space X which is homeomorphic to $E^n$, $\overset{o}{E}{}^n$, or $S^n$ is called an n-cell, open n-cell, or n-sphere respectively.

Definition 1.3.4 Let $M^n$ be an n-dimensional manifold, $n \in Z^+$. An (n-1)-manifold $L^{n-1} \subset M^n$ is said to be collared [bi-collared] in $M^n$ if there is an embedding h of $L^{n-1} \times [0,1)$ $[L^{n-1} \times (-1,1)]$ into $M^n$ such that

i) $h(x,0) = x$ for all $x \in L^{n-1}$ and

ii) $h(L^{n-1} \times [0,1))$ $[h(L^{n-1} \times (-1,1))]$ is open in $M^n$.

Definition 1.3.5 Let $M^n$ be an n-dimensional manifold with boundary, $n \in Z^+$. The n-manifold $M^n_c = \dot{M}^n \times [0,1) \cup_c M^n$ where $c : \dot{M}^n \times \{0\} \to M^n$ is defined by $c(x,0) = x$ for all $x \in \dot{M}^n$ is called an abstract collaring of $M^n$.

$M^n$ and $\dot{M}^n \times [0,1)$ will always be considered as embedded in $M^n_c$ in the usual way under the identification map $p : \dot{M}^n \times [0,1) + M^n \to M^n_c$.

Definition 1.3.6 Suppose that $M^n$ and $L^n$ are n-manifolds,

$n \in Z^+$, such that $\dot{L}^n \neq \emptyset$ and $L^n \subset M^n$. $L^n$ is called a <u>relative $M^n$ n-manifold</u> if

i) $L^n \cap \dot{M}^n \neq \emptyset$,

ii) $L^n \cap \dot{M}^n$ is an (n-1)-manifold, and

iii) $r\dot{L}^n = L^n - \mathrm{Int}L^n$ is empty or an (n-1)-manifold.

Note that if $L^n$ is a relative $M^n$ n-manifold, then the boundary of $(L^n \cap \dot{M}^n)$ = boundary of $r\dot{L}^n$. Also, since $L^n$ is an n-manifold contained in $M^n$, $r\dot{L}^n = \mathrm{Ed}L^n = L^n - \mathrm{Int}L^n$ and boundary $\mathrm{Ed}L^n = \mathrm{Ed}L^n \cap \dot{M}^n$.

<u>Example</u>. Let $M^2 = \frac{1}{2}R^2$ and $L^2 = [-1,1] \times [0,1)$. Then $L^2$ is a relative $M^2$ 2-manifold with $r\dot{L}^2 = \{-1\} \times [0,1) \cup \{1\} \times [0,1)$.

<u>Definition 1.3.7</u> Suppose that $M^n$, $L^n$ are n-manifolds, $n \in Z^+$, such that $\dot{L}^n \neq \emptyset$ and $L^n \subset M^n$. $L^n$ is said to be collared in $M^n$ if $L^n \subset \overset{\circ}{M}^n$ or $L^n$ is a relative $M^n$ n-manifold, and if there is an embedding h of the pair $(\mathrm{Ed}L^n \times [0,1), (\mathrm{Ed}L^n \cap \dot{M}^n) \times [0,1))$ into the pair $(M^n, \dot{M}^n)$ such that

i) $h(x,0) = x$ for all $x \in \mathrm{Ed}L^n$ and

ii) $h(\mathrm{Ed}L^n \times [0,1)) \cap L^n = h(\mathrm{Ed}L^n \times \{0\})$.

The set $h(\mathrm{Ed}L^n \times [0,1))$ is called a <u>collar</u> of $L^n$ in $M^n$ and is denoted by $cL^n$. The set $CL^n = L^n \cup cL^n$ is called a <u>collaring</u> of $L^n$ in $M^n$

<u>Definition 1.3.8</u> Let $M_i^n$, $L_i^n$ be n-manifolds, $n \in Z^+$, such that $L_i^n \subset M_i^n$, $i = 1,2$. A homeomorphism h of $L_1^n$ onto $L_2^n$ is

called a <u>relative</u> <u>homeomorphism</u> if h induces a homeomorphism of the pair $(L_1^n, L_1^n \cap \dot{M}_1^n)$ onto the pair $(L_2^n, L_2^n \cap \dot{M}_2^n)$.

<u>Lemma 1.3.9</u> Let $M_i^n$, $L_i^n$ be n-manifolds, $n \in Z^+$, such that $L_i^n$ is collared in $M_i^n$ and let $CL_i^n$ be a collaring of $L_i^n$ in $M_i^n$, $i = 1,2$. If h is a relative homeomorphism of $L_1^n$ onto $L_2^n$, then h extends to a relative homeomorphism of $CL_1^n$ onto $CL_2^n$.

<u>Proof</u>. Assume that $EdL_1^n \neq \emptyset$; otherwise, the required extension of h is h itself. Note that both $L_i^n$ and $cL_i^n$ are closed in $CL_i^n$ and that $L_i^n \cap cL_i^n = EdL_i^n$, $i = 1,2$. Let $f_i : EdL_i^n \times [0,1) \to M_i^n$ give a collar $cL_i^n$ of $L_i^n$ in $M_i^n$, $i = 1,2$. If $y \in cL_i^n$, then $y = f_i(x,t)$ for a unique pair $(x,t) \in EdL_i^n \times [0,1)$, $i = 1,2$. Define $g_1 : cL_1^n \to cL_2^n$ by $g_1(y) = g_1(f_1(x,t)) = f_2(h(x),t)$; that is, $g_1(y) = f_2(h|EdL_1^n, id)f_1^{-1}(y)$. Since $h|EdL_1^n$ induces a homeomorphism of $(EdL_1^n, EdL_1^n \cap \dot{M}_1^n)$ onto $(EdL_2^n, EdL_2^n \cap \dot{M}_2^n)$, $g_1$ is a relative homeomorphism of $cL_1^n$ onto $cL_2^n$. If $y \in EdL_1^n$, then $g_1(y) = f_2(h(y),0) = h(y)$. Define g by

$$g(y) = \begin{cases} h(y), & y \in L_1^n \\ g_1(y), & y \in cL_1^n \end{cases}.$$

Then $g : CL_1^n \to CL_2^n$ is the required extension of h.

<u>Definition 1.3.10</u> Let X be a metric space, A and B subsets of X with $A \subset B$. Suppose that f and g are bounded continuous functions from B into $R^1$ such that $f(a) \leq g(a)$ for all $a \in A$. The <u>prism</u> <u>on</u> <u>A</u> <u>determined</u> <u>by</u> <u>f</u> <u>and</u> <u>g</u> is denoted by

$P(f,g;A)$ and defined by $P(f,g;A) = \{(x,t) \in X \times R^1 | x \in A$ and $f(x) \leq t \leq g(x)\}$. If $f(a) < g(a)$ for all $a \in A$, then the <u>topless prism on A determined by f and g</u>, denoted by $TP(f,g;A)$, is defined by $TP(f,g;A) = \{(x,t) \in X \times R^1 | x \in A$ and $f(x) \leq t < g(x)\}$. If $f(a) = c$, $c$ a constant, for all $a \in A$, then $c$ will denote the function $f$. The graph of $g$ restricted to $A$ will be denoted by $G(g;A)$. Note that $G(g;A) = P(g,g;A)$.

The following lemma is a summation of the remarks found on page 556 of [13].

<u>Lemma 1.3.11</u> Let $X$ be a metric space, $A \subset B \subset X$, and let $f_1$, $f_2$ be bounded continuous functions from $B$ into $R^1$ such that $f_1(a) \leq f_2(a)$ for all $a \in A$. Suppose that $g_1$ and $g_2$ are continuous functions from $A$ into $R^1$ such that

i) $f_1(a) \leq g_i(a) \leq f_2(a)$ for all $a \in A$, $i = 1,2$,

ii) $f_1(a) = g_1(a)$ if and only if $f_1(a) = g_2(a)$, and

iii) $f_2(a) = g_1(a)$ if and only if $f_2(a) = g_2(a)$.

Then there is a homeomorphism $h[f_1,f_2;g_1,g_2]$ of $P(f_1,f_2;A)$ onto itself such that

iv) $h[f_1,f_2;g_1,g_2](a,t) = (a,t)$ for all $(a,t) \in (G(f_1;A) \cup G(f_2;A))$ and

v) for each $a \in A$, $h[f_1;f_2;g_1,g_2]$ carries the segment $P(f_1,g_1;a)$ linearly onto the segment $P(f_1,g_2;a)$ and the segment $P(g_1,f_2;a)$ linearly onto the segment $P(g_2,f_2;a)$.

The following rather complicated lemma is used to establish the existence of certain nice collarings of a collared manifold.

<u>Lemma 1.3.12</u> Let X be a metric space such that there is an embedding of X into $R^n$ for some $n \in Z^+$. Assume that X is embedded in $R^n$ and that A is a proper subset of X such that

i) A is locally compact,

ii) $Cl_X A$ is compact and $C = Cl_X A - A$ is either empty or closed in X and hence compact,

iii) there is an embedding h of $A \times [0,1)$ into X such that $h(a,0) = a$ for all $a \in A$.

Then there is a continuous function $f : Cl_X A \to [0,\frac{1}{2})$ such that $f(x) = 0$ if and only if $x \in C$ and the following hold:

iv) there is an embedding $h_1$ of $P(0,f;Cl_X A)$ into X such that $h_1(y,0) = y$ for all $y \in Cl_X A$ and $h_1(x) = h(x)$ for all $x \in P(0,f;A)$ and

v) if G is open in X, $A \subset G$, then there is a homeomorphism g of $h_1(P(0,f;Cl_X A))$ onto itself such that $g(h_1(TP(0,\frac{1}{2}f;A))) \subset G$ and g restricted to $h_1(G(0;Cl_X A) \cup G(f;Cl_X A))$ is the identity.

<u>Proof</u>. Let d be the Euclidean metric on $R^n$; then $d|X$ is a metric equivalent to the metric of X. For $r \in (0,\infty)$, $D \neq \emptyset$, and $D \subset R^n$, let $B(D,r) = \{x \in R^n | d(x,D) < r\}$. Then $Cl_{R^n} B = \{x \in R^n | d(x,D) \leq r\}$, where $B = B(D,r)$.

Suppose that $C \neq \emptyset$. Define $A_1 = \{x \in Cl_X A | d(x,C) > \frac{1}{2}\}$ and for $k \in Z^+$, $k \geq 2$, define

$A_k = \{x \in Cl_X A | 1/(k+1) < d(x,C) \leq 1/k\}$. Then $Cl_{Cl_X A} A_k \subset A$ and therefore $F_k = Cl_A A_k = A \cap Cl_{Cl_X A} A_k = Cl_{Cl_X A} A_k$ for $k \geq 2$. Thus $F_k$ is compact and we have that $A = \bigcup_{k=1}^{\infty} A_k = A_1 \cup (\bigcup_{k=2}^{\infty} F_k)$. Let $d^*$ be the metric on $Cl_X A \times R^1$ defined by $d^*((x,t),(y,u)) = d(x,y) + |t - u|$. Since $F_k$ is compact for $(k-1) \in Z^+$, there is a real number $a_k$, $0 < a_k < \frac{1}{2}$, such that if $x \in F_k$, then $d(h(x,0),h(x,t)) < 1/k$ for $0 \leq t \leq a_k$. Therefore it is possible to construct a sequence $\{b_k\}_{k \in Z^+}$ of real numbers such that $0 < b_{k+1} < b_k < \frac{1}{2}$ and $b_{k+1} < a_{k+1}$ for all $k \in Z^+$, and $\lim_{k \to \infty} b_k = 0$. Define $g_1 : A \to [0,\frac{1}{2})$ by $g_1(x) = b_k$ if $x \in A_k$. Then if $r \in R^1$, the set $L(r) = \{x \in A | g_1(x) \leq r\}$ equals $A$ or $\bigcup_{k=p}^{\infty} A_k$ for some $(p-1) \in Z^+$. Therefore $L(r)$ is closed in $A$ and thus $g_1$ is lower semi-continuous and positive on $A$. It follows from a theorem due to Dowker (see page 170 of [9]) that there is a continuous function $g_2 : A \to [0,\frac{1}{2})$ such that $0 < g_2(a) < g_1(a)$ for all $a \in A$.

Define $f : Cl_X A \to [0,1)$ by

a) $f(x) = \frac{1}{4}$ for all $x \in Cl_X A$ if $C = \emptyset$ and

b) $f(x) = \left\{\begin{matrix} 0, & x \in C \\ \min(d(x,C),g_2(x)), & x \in A \end{matrix}\right\}$ if $C \neq \emptyset$.

Since $C$ is compact, $f(x) = 0$ if and only if $x \in C$. If $C = \emptyset$, then $f$ is clearly continuous. Suppose that $C \neq \emptyset$. If $x \in A$, then $f$ is continuous at $x$ since $A$ is open in $Cl_X A$ and $f|A$ is clearly continuous. Suppose that $x \in C$ and that $\epsilon > 0$. Let $y \in Cl_X A$ such that $d(y,x) < \epsilon$. Then

$|f(y) - f(x)| = |f(y)| = f(y) \leq d(y,C) \leq d(y,x) < \epsilon$.
Therefore f is continuous at x and hence on $Cl_X A$.

Define $h_1 : P(0,f;Cl_X A) \to X$ by

$$h_1(x,t) = \begin{Bmatrix} x, & x \in C \\ h(x,t), & (x,t) \in P(0,f;A) \end{Bmatrix} .$$

If $C = \emptyset$, $h_1$ is certainly continuous. Suppose that $C \neq \emptyset$. Since C is compact, $P(0,f;Cl_X A) - C \times \{0\} = P(0,f;A)$ is open in $P(0,f;Cl_X A)$ and therefore $h_1$ is continuous at points $(x,t) \in P(0,f;A)$. Suppose that $c \in C$ and that $0 < \epsilon < b_2$. Choose $k \in Z^+$ such that $1/k < \frac{1}{2}\epsilon$. Then $U(c,\epsilon) = ((B(c,1/k) \cap Cl_X A) \times [0,b_k)) \cap P(0,f;Cl_X A)$ is an open neighborhood of $(c,0)$ in $P(0,f;Cl_X A)$. Suppose that $(y,t) \in U(c,\epsilon)$. Since $d(y,C) \leq d(y,c)$, $y \in A_j$ for some $j \geq k \geq 2$. Therefore $d(h_1(c,0),h_1(y,t)) \leq d(h_1(c,0),h_1(y,0)) + d(h_1(y,0),h_1(y,t)) \leq d(c,y) + 1/j < 1/k + 1/j \leq 2/k < \epsilon$. Therefore $h_1$ is continuous at $(c,0)$. Since $h : A \times [0,1) \to X$ is an embedding and $h(a,0) = a$ for all $a \in A$, $C \cap h(A \times [0,1)) = \emptyset$. Therefore $h_1$ is injective. Since X is Hausdorff and $P(0,f;Cl_X A)$ is compact, $h_1$ is an embedding.

Now suppose that G is open in X and that $A \subset G$. Since C is compact or $C = \emptyset$, $G - C = G_1$ is open in X and $A \subset G_1$. Let $P = P(0,f;Cl_X A)$, and define $G_2 = G_1 \cap h_1(P)$ and $F_2 = (X - G_1) \cap h_1(P)$. Then $h_1^{-1}(G_2) = G_3$ is open in P, $h_1^{-1}(F_2) = P - G_3 = F_3$, $A \times \{0\} \subset G_3$, and $C \times \{0\} \subset F_3$. Let $f_3 : Cl_X A \to R^1$ be defined by $f_3(x) = d^*((x,0),F_3)$.

Then $f_3$ is continuous and $f_3(x) = 0$ if and only if $x \in C$. Let $g_1 = \min(f_3, \frac{1}{4}f)$; then $g_1$ is continuous. Furthermore, if $0 \leq t < g_1(y)$ where $y \in A$, then $(y,t) \in G_3$. Since $g_1(y) = 0$ if and only if $x \in C$, it follows from 1.3.11 that there is a homeomorphism $g_2 = h[0,f;\frac{1}{2}f,g_1]$ of P onto itself such that $g_2|(G(0;Cl_XA) \cup G(f;Cl_XA)) = id$ and $g_2$ carries $P(0,\frac{1}{2}f;x)$ linearly onto $P(0,g_1;x)$ for all $x \in Cl_XA$. Therefore $g_2(TP(0,\frac{1}{2}f;A)) \subset G_3$. Define $g : h_1(P) \to h_1(P)$ by $g = h_1 g_2 h_1^{-1}$. Then g is a homeomorphism of $h_1(P(0,f;Cl_XA))$ onto itself such that $g(h_1(TP(0,\frac{1}{2}f;A)))$ is contained in G, and such that g restricted to $h_1(G(0;Cl_XA) \cup G(f;Cl_XA))$ is the identity.

Now suppose that $M^n$ is an n-manifold with boundary, $n \in Z^+$. If $\dot{M}^n$ is not compact, then let X be the one point compactification of $\dot{M}^n$; otherwise, let $X = \dot{M}^n$. In either case X can be embedded in $R^p$ for some $p \in Z^+$. Consider X as embedded in $R^p$ and note that i) $\dot{M}^n$ is locally compact; ii) $Cl_X\dot{M}^n$ is compact and $C = Cl_X\dot{M}^n - \dot{M}^n$ is compact or empty; and iii) as a consequence of theorem 2 of [4], there is an embedding h of $\dot{M}^n \times [0,1)$ into X such that $h(x,0) = x$ for all $x \in \dot{M}^n$. Using 3.1.11 and 3.1.12, we can easily establish the following well known results.

<u>Corollary 1.3.13</u> Suppose that $M^n$ is an n-manifold with boundary, $n \in Z^+$. Then there is an embedding h of $\dot{M}^n \times [0,1]$ into $M^n$ such that $h(x,0) = x$ for all $x \in \dot{M}^n$ and $h(\dot{M}^n \times [0,1])$ is closed in $M^n$.

<u>Corollary 1.3.14</u> Suppose that $M^n$ is an n-manifold with boundary, $n \in Z^+$. Then an abstract collaring $M_c^n$ of $M^n$ is homeomorphic to $\overset{o}{M}{}^n$.

<u>Definition 1.3.15</u> Suppose that $L^n$ is collared in $M^n$. A collaring $CL^n$ of $L^n$ in $M^n$ is called a <u>tapered collaring with support F</u>, if given an open set G, $L^n \subset G \subset M^n$, then there is a homeomorphism h of $M^n$ onto itself such that

i) $L^n \subset h(CL^n) \subset G$ and

ii) $h(x) = x$ for all $x \in (L^n \cup (M^n - F))$.

<u>Theorem 1.3.16</u> If $L^n$ is collared in $M^n$, then $L^n$ has a tapered collaring $CL^n$ in $M^n$. Furthermore, if the closure of $EdL^n$ in $M^n$ is compact, then the support of $CL^n$ may be chosen to be compact.

<u>Proof</u>. Let $L^n$ be collared in $M^n$ and let $A = Ed_M nL^n$. We may assume that $A \neq \emptyset$, since otherwise $CL^n = L^n$. If $CL_M nA$ is compact, then set $X = M^n$; otherwise, let X be the one point compactification of $M^n$. In any case X can be embedded in $R^p$ for some $p \in Z^+$. Consider X and all subspaces of X as embedded in $R^p$. We note that i) A is locally compact; ii) $Cl_X A$ is compact and $C = Cl_X A - A$ is either empty or compact (see p. 245 of [9]); and iii) there is an embedding h of $A \times [0,1)$ into X such that $h(a,0) = a$ for all $a \in A$. Thus there is a continuous function $f : Cl_X A \to [0,\frac{1}{2})$ such that $f(x) = 0$ if and only if $x \in C$ and iv) and v) of 1.3.12 hold. Suppose that G is open in

$M^n$ and that $L^n \subset G$. Since $M^n$ is locally compact, $M^n$ is embedded as an open subset in X and thus G is open in X and $A \subset G$. Let $h_1$ be an embedding of $P(0,f;Cl_X A)$ into X such that $h_1(y,0) = y$ for all $y \in Cl_X A$ and such that $h_1(x) = h(x)$ for all $x \in P(0,f;A)$. Then there is a homeomorphism g of $h_1(P(0,f;Cl_X A))$ onto itself such that $g(h_1(TP(0,\frac{1}{2}f;A))) \subset G$ and g restricted to $h_1(G(0;Cl_X A) \cup G(f;Cl_X A))$ is the identity. Let $F = h_1(P(0,f;Cl_X A)) \cap M^n$. Then F is closed in $M^n$ and $Fr_{M^n}F = h_1(G(0;Cl_X A) \cup G(f;Cl_X A)) \cap M^n$. Therefore g extends to a homeomorphism $g_1$ of $M^n$ onto itself such that $g_1(x) = x$ for all $x \in (L^n \cup (M^n - F))$. The required tapered collaring $CL^n$ of $L^n$ is obtained by setting $CL^n = L^n \cup h_1(TP(0,\frac{1}{2}f;A)) = L^n \cup h(TP(0,\frac{1}{2}f;Ed_{M^n}L^n))$. If $Cl_{M^n}A$ is compact, then clearly F is compact and the theorem is established.

Corollary 1.3.17 If $L^n$ is collared in $M^n$ and G is an open set such that $L^n \subset G \subset M^n$, then $L^n$ is collared in G. Also, if $L^n$ is collared in $M^n$ and the pair $(M^n,\dot{M}^n)$ is contained in the pair $(Q^n,\dot{Q}^n)$, then $L^n$ is collared in $Q^n$.

The following lemmas lead to a theorem which gives sufficient conditions for $L^n$ to be collared in $M^n$.

Lemma 1.3.18 Let X be a topological space. There is a homeomorphism h of $X \times ([-1,0] \times [0,1))$ onto $X \times ([-1,1) \times [0,1))$ such that h restricted to

$(X \times \{-1\} \times [0,1) \cup X \times [-1,0] \times \{0\})$ is the identity and $h$ carries $X \times \{0\} \times [0,1)$ homeomorphically onto $X \times [0,1) \times \{0\}$.

Proof. There exists a homeomorphism $g$ of $[-1,0] \times [0,1)$ onto $[-1,1) \times [0,1)$ such that $g$ restricted to $(\{-1\} \times [0,1) \cup [-1,0] \times \{0\})$ is the identity and $g$ carries $\{0\} \times [0,1)$ homeomorphically onto $[0,1) \times \{0\}$. The map $h : X \times ([-1,0] \times [0,1)) \to X \times ([-1,1) \times [0,1))$ given by $h(x,y) = (x,g(y))$, $x \in X$, $y \in ([-1,0] \times [0,1))$ is the required homeomorphism.

Lemma 1.3.19 Let $L^n$ be a relative $M^n$ manifold, $n \geq 2$, such that $r\dot{L}^n = EdL^n \neq \emptyset$. Then there is an embedding $h$ of the pair $(EdL^n \times [0,1), (EdL^n \cap \dot{M}^n) \times [0,1))$ into $(M^n, \dot{M}^n)$ such that

i) $h(EdL^n \times [0,1)) \subset L^n$ and

ii) $h(x,0) = x$ for all $x \in EdL^n$.

Proof. Let $L^n$ be a relative $M^n$ n-manifold, $n \geq 2$, such that $EdL^n \neq \emptyset$. Let $E^{n-1} = EdL^n$. If $\dot{E}^{n-1} = \emptyset$, then since $E^{n-1} \subset \dot{L}^n$, the result follows easily from 1.3.13. Now suppose that $\dot{E}^{n-1} \neq \emptyset$ and let $Q^{n-1} = L^n \cap \dot{M}^n$. Then $\dot{E}^{n-1} = \dot{Q}^{n-1}$. It follows from 1.3.13 that there is an embedding $g_1 : \dot{E}^{n-1} \times [-1,0] \to E^{n-1}$ such that $g_1(\dot{E}^{n-1} \times [-1,0])$ is closed in $E^{n-1}$ and $g_1(x,0) = x$ for all $x \in \dot{E}^{n-1}$. Also there is an embedding $g_2 : \dot{E}^{n-1} \times [0,1) \to Q^{n-1}$ such that $g_2(x,0) = x$ for all $x \in \dot{E}^{n-1} = \dot{Q}^{n-1}$. Let $P^{n-1} = E^{n-1} \cup g_2(\dot{E}^{n-1} \times [0,1))$. Note that since

$g_1(\dot{E}^{n-1} \times [-1,0]) = F$ is closed in $E^{n-1}$, $F$ is closed in $P^{n-1}$. It follows from 1.3.13 that there is an embedding $g_3$ of $P^{n-1} \times [0,1)$ into $L^n$ such that $g_3(x,0) = x$ for all $x \in P^{n-1}$ and $g_3(P^{n-1} \times [0,1)) \cap \dot{L}^n = g_3(P^{n-1} \times \{0\})$. Let $f$ be a homeomorphism of $\dot{E}^{n-1} \times ([-1,0] \times [0,1))$ onto $\dot{E}^{n-1} \times ([-1,1) \times [0,1))$ with the properties given in 1.3.18. Define $g : \dot{E}^{n-1} \times [-1,1) \to \dot{L}^n$ by

$$g(x,t) = \begin{Bmatrix} g_1(x,t), & t \in [-1,0] \\ g_2(x,t), & t \in [0,1) \end{Bmatrix}.$$

Define a homeomorphism $h_1$ of $(g_1(\dot{E}^{n-1} \times [-1,0])) \times [0,1)$ onto $(g(\dot{E}^{n-1} \times [-1,1))) \times [0,1)$ by $h_1 = (g,id)f(g_1^{-1},id)$, where id is the identity map on $[0,1)$. Note that $h_1(y,t) = (y,t)$ if i) $y \in g_1(\dot{E}^{n-1} \times \{-1\})$ or ii) if $t = 0$. Let $F_1 = E^{n-1} - g_1(\dot{E}^{n-1} \times (-1,0])$ and $F_2 = g_1(\dot{E}^{n-1} \times [-1,0])$. Then $F_1$ and $F_2$ are closed in $E^{n-1}$ and $F_1 \cap F_2 = F_3 = g_1(\dot{E}^{n-1} \times \{-1\})$ which is also closed in $E^{n-1}$. Define $h_2 : E^{n-1} \times [0,1) \to P^{n-1} \times [0,1)$ by

$$h_2(x,t) = \begin{Bmatrix} (x,t), & x \in F_1 \\ h_1(x,t), & x \in F_2 \end{Bmatrix}.$$

Then $h_2$ is a homeomorphism onto $P^{n-1} \times [0,1)$, and $h = g_3h_2$ is the required embedding of $(EdL^n \times [0,1), (EdL^n \cap \dot{M}^n) \times [0,1))$ into $(M^n, \dot{M}^n)$.

<u>Remark</u>. Note that it follows from 1.3.13 or 1.3.19 that a collaring $CL^n$ of $L^n$ in $M^n$ is an open subset of $M^n$.

<u>Theorem 1.3.20</u> Let $L^n$ and $M^n$ be n-manifolds, $n \geq 2$, such

that $L^n$ is closed in $M^n$. Then $L^n$ is collared in $M^n$ if

i) $L^n \subset \overset{o}{M}{}^n$ and $P^n = M^n$ - Int $L^n$ is an n-manifold or

ii) $L^n$ is a relative $M^n$ n-manifold, $\mathrm{Ed}L^n \neq \emptyset$, and $P^n = M^n$ - Int $L^n$ is a relative n-manifold.

Proof. If i) holds, then $\mathrm{Ed}L^n \subset \dot{P}^n$ and the result follows from 1.3.13. If ii) holds, then $\mathrm{Ed}L^n = \mathrm{Fr}L^n = \mathrm{Fr}P^n = \mathrm{Ed}P^n$. Since $\mathrm{Ed}L^n = \mathrm{Ed}P^n$, the result follows by applying 1.3.19 to $P^n$.

Suppose that $x = (x_1,\ldots,x_{n-1}) \in E^{n-1}$, $n \geq 2$. Define $f : E^{n-1} \to R^1$ by setting $f(x) = (1 - \sum_{k=1}^{n-1} x_k^2)^{\frac{1}{2}}$. Then f is continuous and $f(x) = 0$ if and only if $x \in S^{n-2}$. For each $t \in [0,1]$ define $f_t : E^{n-1} \to R^1$ by $f_t(x) = tf(x)$.

Definition 1.3.21 Let $\frac{1}{2}R_1^n = \{x \in R^n | x_1 \geq 0\}$, $n \geq 2$. For each $t \in [0,1]$ define $B^n(t) = P(0,f_t;E^{n-1})$, $\frac{1}{2}B^n(t) = B^n(t) \cap \frac{1}{2}R_1^n$, and $B_1^{n-1}(t) = \frac{1}{2}B^n(t) \cap \frac{1}{2}\dot{R}_1^n$.

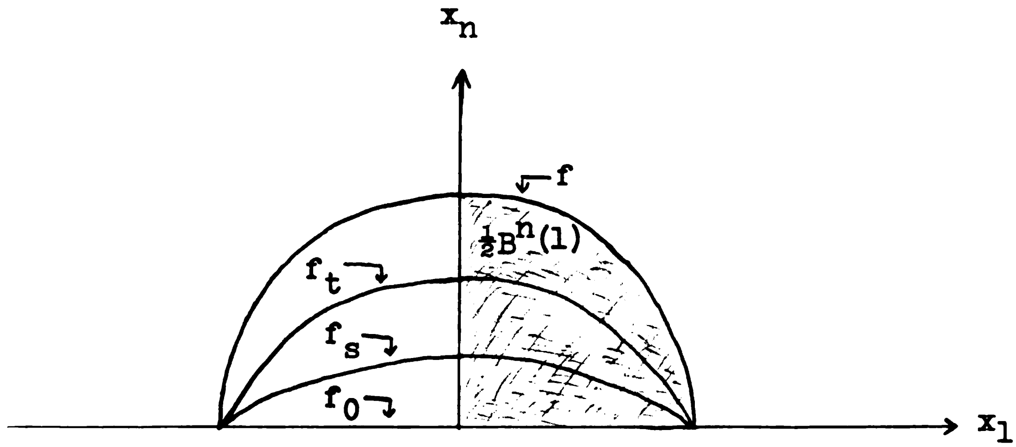


Figure 1.1

Theorem 1.3.22 Let s, t $\in$ [0,1] such that $0 < s < t < 1$. Then there is a homeomorphism h(s,t) of $B^n(1)$ onto itself such that

i) $h(s,t)|\dot{B}^n(1) = id$,

ii) h(s,t) carries $B^n(t)$ onto $B^n(s)$,

iii) $g(s,t) = h(s,t)|\frac{1}{2}B^n(1)$ is a homeomorphism of $\frac{1}{2}B^n(1)$ onto itself and $g(s,t)|\ B_1^{n-1}(1)$ is a homeomorphism of $B_1^{n-1}(1)$ onto itself,

iv) g(s,t) carries $\frac{1}{2}B^n(t)$ onto $\frac{1}{2}B^n(s)$.

Proof. This is an immediate consequence of 1.3.11.

Remark. Note that for any s, t $\in$ (0,1), $P(f_s,f;E^{n-1})$ is homeomorphic to $P(0,f_t;E^{n-1})$ and that $B^n(1) = E^n \cap \frac{1}{2}R^n$.

## 4. Piecewise linear manifolds

The terminology that will be used for simplicial complexes is essentially that used by Zeeman in [15], but is modified to agree with the terminology used by Hudson and Zeeman in [11].

By an n-simplex $t_n$, $0 \leq n$, is meant the convex hull of n+1 linearly independent points (vertices) $\{v_j\}_{j=0}^n$ in $R^p$, $n \leq p$. By an r-face $t_r$ of $t_n$, denoted by $t_r < t_n$, is meant the convex hull of r+1 distinct points of $\{v_j\}_{j=0}^n$.

A simplicial complex K of $R^p$, $p \geq 1$, is an at most countable collection of simplexes of $R^p$ such that i) if t $\in$ K, then all faces of t are in K, ii) if s, t $\in$ K, then s $\cap$ t is a common face of s and t, and iii) each vertex of

K is the face of at most a finite number of elements of K. L is called a subcomplex of K if L is a simplicial complex and $L \subset K$. If $t_n$ is an n-simplex, $n \geq 0$, let $\bar{t}_n = \{s | s < t_n\}$ and $\dot{t}_{n+1} = \{s | s < t_{n+1}, s \neq t_{n+1}\}$. If s, t are simplexes in $R^p$, then s and t are said to be joinable if the union of their vertices forms a linearly independent set of points in $R^p$. If s and t are joinable, then the join of s and t, denoted by st, is defined to be the simplex spanned by the union of their vertices. For $t \in K$, the set $st(t,K) = \{s \in K | t < s\}$ is called the star of t in K; the subcomplex $lk(t,K) = \{s \in K | s$ is joinable to t and $st \in K\}$ is called the link of t in K.

Let K be a simplicial complex in $R^p$. The polyhedron determined by K, denoted by $|K|$, is the set $|K| = \bigcup_{t \in K} t$ with the weak topology determined by the simplexes of K. A complex K' is called a subdivision of K if $|K'| = |K|$ and each simplex of K' is contained in some simplex of K.

Let K and L be simplicial complexes in $R^n$ and $R^p$ respectively. A continuous function $f : |K| \to |L|$ is called simplicial if $f(s) \in L$ for all $s \in K$; f is called piecewise linear, denoted by PL, if there are subdivisions K' of K and L' of L such that $f : |K'| \to |L'|$ is simplicial. If g is a homeomorphism of $|L|$ onto X and h is a homeomorphism of $|K|$ onto Y, then $f : X \to Y$ is called piecewise linear if $h^{-1}fg$ is a PL map of $|L|$ into $|K|$.

Henceforth, PL will be used for the term piecewise linear.

Definition 1.4.1 Let $t_n$ be an n-simplex, $n \geq 0$, and K a complex in $R^p$. $|K|$ is called a combinatorial n-cell, if there is a PL homeomorphism f of $|\bar{t}_n|$ onto $|K|$. $|K|$ is called a combinatorial n-sphere, if there is a PL homeomorphism f of $|\dot{t}_{n+1}|$ onto $|K|$.

Definition 1.4.2 An n-manifold $M^n$, $n \geq 0$, is called a PL n-manifold if there is a homeomorphism f of $|K|$ onto $M^n$ where K is a complex in $R^p$, and such that if $n \geq 1$, then $|lk(v,K)|$ is a combinatorial (n-1)-cell or a combinatorial (n-1)-sphere for all vertices v of K. The pair (K,f) is called a PL triangulation of $M^n$.

It is a well known fact that if (K,f) is a PL triangulation of $M^n$, then any subdivision K' of K yields a PL triangulation (K',f) of $M^n$. Also if $M^n$ is an open subset of $R^n$ or $\frac{1}{2}R^n$, then there is a complex K of $R^n$ such that (K,id) gives a PL triangulation of $M^n$ where id is the identity map.

Definition 1.4.3 Let $M^n$ be a PL n-manifold with PL triangulation (K,f). An m-manifold $L^m$, $0 \leq m \leq n$, which is embedded as a subset of $M^n$ is called PL in $M^n$ if there is a subdivision K' of K and a subcomplex L' of K' such that $(L', f||L'|)$ is a PL triangulation of $f(|L'|) = L^m$. Note that if $L^m$ is PL in $M^n$, then $L^m$ is necessarily closed in $M^n$.

A manifold $L^m$ embedded in $M^n$ may be a PL m-manifold, but may not be PL in $M^n$. For example, let (K,id) be a PL triangulation of $R^2$, where K is a simplicial complex in $R^2$ and id is the identity map. Then $S^1$ is a PL 1-manifold, but $S^1$ is not PL in $R^2$.

<u>Theorem 1.4.4</u> Let $M^n$, $n \geq 2$, be a PL n-manifold and let $L^n$ be a PL n-manifold in $M^n$. If i) $L^n \subset \overset{o}{M}{}^n$ or ii) $L^n \cap \dot{M}^n = Q^{n-1}$ is a PL (n-1)-manifold in $M^n$, then $L^n$ is collared in $M^n$.

<u>Proof</u>. This result is easily established and the method of proof is only outlined. The first step is to use the method employed in the proof of lemma 17 of chapter 3 of [15] to show that $M^n - \mathrm{Int}_M L^n$ is a PL n-manifold in $M^n$. If ii) holds, then i) applied to $Q^{n-1} \subset \dot{M}^n$ and $Q^{n-1} \subset \dot{L}^n$ shows that both $L^n$ and $M^n - \mathrm{Int}_M L^n$ are relative $M^n$ n-manifolds. Since $L^n$ is closed in $M^n$, the fact that $L^n$ is collared in $M^n$ follows from 1.3.20.

<u>Lemma 1.4.5</u> Let $M^n$ be a connected n-manifold, $n \geq 2$, and $C \subset M^n$ a proper compact subset. Then there exists a domain $D \subset M^n$ such that $C \subset D$ and ClD is a proper compact subset of $M^n$. Furthermore, if $C \subset \overset{o}{M}{}^n$, then D may be chosen so that $\mathrm{Cl}D \subset \overset{o}{M}{}^n$.

<u>Proof</u>. This is easily established. Let $x \in M^n - C$. Then $M^n - x$ is a connected n-manifold and the result follows from the fact that $M^n - x$ is connected, locally connected, and locally compact. Note that the result is not true for $n = 1$.

If K is a simplicial complex, let $Sd_j K$, $j \in Z^+$ denote the jth barycentric subdivision of K.

<u>Theorem 1.4.6</u> Let $M^n$, $n \geq 2$, be a connected, PL n-manifold with triangulation (K,f), D a domain of $M^n$, and C a proper compact subset of D. Then there is a compact, connected, PL n-manifold $L^n$ in $M^n$ such that

i) $C \subset \mathrm{Int}_{M^n} L^n \subset L^n \subset D$, $L^n \neq D$,

ii) $L^n = f(|L|)$ where L is a subcomplex of $Sd_j(K)$, $j \in Z^+$,

iii) $L^n \subset \overset{o}{M}{}^n$ if $C \subset \overset{o}{M}{}^n$, and

iv) $L^n \cap \dot{M}^n = L^n \cap \dot{D}$ is a PL (n-1)-manifold in $M^n$ if $C \cap \dot{M}^n \neq \emptyset$.

<u>Proof</u>. Since K is locally finite, it may be assumed that $M^n$ is embedded in some $R^p$ as a closed subset and that $M^n = |K|$. It follows from 1.4.5 that there is a compact, connected set $C_1$ such that $C \subset \mathrm{Int}_{M^n} C_1 \subset C_1 \subset D$, $C_1 \neq D$. Since $n \geq 2$, it may be assumed that $D \neq M^n$. Let $0 < \epsilon \leq d(C_1, M^n - D)$ and $Q = \{x \in M^n | d(x, C_1) < \epsilon\}$, where d is the Euclidean metric on $R^p$ restricted to $M^n$. Note that $Q \subset D$ and that if $C \cap \dot{M}^n = \emptyset$, then $\epsilon$ may be chosen so that $Q \subset \overset{o}{M}{}^n$. The existence of $L^n$ is now established using the terminology and results of [11].

If J is a simplicial complex, and $X \subset |J|$, let $N(X,J) = \{s \in J | s < t,\ t \cap X \neq \emptyset\}$. Let $L_1 = N(C_1, K)$. Since $C_1$ is compact, $L_1$ is a finite simplicial complex with $C_1 \subset \mathrm{Int}_{M^n}|L_1|$. Therefore there is a $q \in Z^+$ such that if

$L = N(C_1, Sd_q L_1)$, then for all $j \geq q$

i) mesh $Sd_j L_1 < \epsilon/4$ and

ii) $N(|L|, Sd_j L_1) = N(|L|, Sd_j K)$ and $|N(|L|, Sd_j L_1)| \subset Q$.

Suppose that $C \cap \dot{M}^n \neq \emptyset$. Then there is an n-simplex $t_n \in L$ such that $t_n$ has an (n-1)-face $t_{n-1} \subset \dot{M}^n$. Let b be the barycenter of $t_{n-1}$. There exists an n-simplex $t_n' \in Sd_2 L$ such that $b \in t_n'$, $t_n' \cap \dot{M}^n = t_{n-1}'$ is an (n-1)-face of $t_n'$, and $|\overline{t}_n'| \subset Int_{M^n}|\overline{t}_n|$. If $C \cap \dot{M}^n \neq \emptyset$, let $R = Sd_2 L - \{t_n', t_{n-1}'\}$ and $S = \dot{t}_{n-1}'$; if $C \cap \dot{M}^n = \emptyset$, let $R = Sd_2 L$ and $S = \emptyset$. In either case, $|R|$ is link collapsible on $|S|$. Furthermore, if $C \cap \dot{M}^n \neq \emptyset$, then $|R| \cap \dot{M}^n$ is link collapsible on $|S| \cap \dot{M}^n = |\dot{t}_{n-1}'|$. Let $J = N(|R| - |S|, Sd_{q+4}K)$. Then J is a subcomplex of $N(|L|, Sd_{q+4}L_1)$ and thus $|J| \subset Q$. From theorem 1 of [11] it follows that $|J|$ is a compact, connected PL n-manifold in $M^n$ such that $|J| \cap \dot{M}^n = \emptyset$ if $C \cap \dot{M}^n = \emptyset$, and $|J| \cap \dot{M}^n$ is a PL (n-1)-manifold in $M^n$ if $C \cap \dot{M}^n \neq \emptyset$. If $C \cap \dot{M}^n = \emptyset$, let $L^n = |J|$; otherwise, let $P = J \cup Sd_2\overline{t}_n'$ and set $L^n = |P|$. If it is the case that $C \cap \dot{M}^n \neq \emptyset$, then $C \subset Int_{M^n}L^n \subset L^n \subset Q$ and $L^n$ is a compact, connected PL n-manifold in $M^n$ such that $L^n \cap \dot{M}^n$ is a PL (n-1)-manifold in $M^n$. Since P and J are subcomplexes of $Sd_{q+4}K$, $L^n$ satisfies ii) through iv). Since $n \geq 2$, in the proof we have assumed that $D \neq M^n$. Thus since $M^n$ is connected and $L^n$ is compact, $L^n \neq D$ and i) is also satisfied.

# CHAPTER II

## CONNECTED MANIFOLDS WHICH HAVE A GENERATING DOMAIN

In this chapter we will give a characterization of those connected manifolds which have domain rank 1. It is clear that such a manifold must be without boundary.

### 1. Characterization

<u>Definition 2.1.1</u> An n-manifold $M^n$, $n \geq 1$, is said to have <u>Euclidean compact subsets</u> if for each proper compact set $C \subset M^n$, there is a homeomorphism h of the pair $(C, C \cap \dot{M}^n)$ into the pair $(\frac{1}{2}R^n, \frac{1}{2}\dot{R}^n)$.

For $n \geq 1$, let (T,id) be a fixed PL triangulation of $R^n$, where T is a simplicial complex in $R^n$ such that $|T| = R^n$ and id denotes the identity map. Throughout the remainder of this work, it will be assumed that $R^n$ has this fixed PL triangulation.

<u>Definition 2.1.2</u> Let (T,id) be the given fixed PL triangulation of $R^n$, $n \geq 1$. The set $M(T) = \{L | L$ is a subcomplex of some $Sd_k T$ and $|L|$ is a compact, connected, PL n-manifold in $R^n\}$ is called the <u>set of regular submanifolds</u> of T.

If $L \in M(T)$, then L is a finite simplicial complex in

$R^n$. Therefore it is easily seen that M(T) is a countably infinite set.

Theorem 2.1.3 Let $M^n$, $n \geq 2$, be a connected, non-compact n-manifold without boundary such that $M^n$ has Euclidean compact subsets. There exists a sequence $\{M_k^n\}_{k=1}^{\infty}$ of compact, connected n-manifolds such that

i) $M_k^n$ is collared in $M^n$, $k \in Z^+$,

ii) there is an $L_k \in M(T)$ and a relative homeomorphism $h_k$ of $|L_k|$ onto $M_k^n$, $k \in Z^+$,

iii) $M_k^n \subset \mathrm{Int} M_{k+1}^n$, $k \in Z^+$, and

iv) $M^n = \bigcup_{k=1}^{\infty} M_k^n$.

Proof. Since $M^n$ is a non-compact, connected n-manifold, there exists a sequence $\{F_k\}_{k=1}^{\infty}$ of non-empty compact subsets of $M^n$ such that $F_k$ is a proper subset of $F_{k+1}$ for all $k \in Z^+$ and $M^n = \bigcup_{k=1}^{\infty} F_k$. We now obtain the sequence $\{M_k^n\}_{k=1}^{\infty}$ by recursive construction.

a) Suppose that $k = 2$. Since $M^n$ has Euclidean compact subsets and $\dot{M}^n = \emptyset$, it follows from 1.4.5 that there is a proper domain $H_2$ of $M^n$ with $F_2 \subset H_2$ and a relative homeomorphism $f_2$ of $H_2$ onto a domain $f_2(H_2)$ of $R^n$. As a consequence of 1.4.6, there is an $L_2 \in M(T)$ such that $f_2(F_2) \subset \mathrm{Int}_{R^n}|L_2| \subset |L_2| \subset f_2(H_2)$. Define $h_2 = f_2^{-1} | |L_2|$ and set $M_2^n = h_2(|L_2|)$. Since $|L_2|$ is collared in $R^n$ and $f_2$ is a relative homeomorphism of $H_2$ into $R^n$, it follows that $h_2$ is a relative homeomorphism of $|L_2|$ onto $M_2^n$, and that $M_2^n$ is collared in $M^n$. Now there exists an n-cell $C^n \subset \mathrm{Int}_{R^n}|L_2|$

such that $C^n = |L_1|$ where $L_1 \in M(T)$. Let $h_1 = h_2 | |L_1|$ and set $M_1^n = h_1(|L_1|)$. Then $h_1$ is a relative homeomorphism of $|L_1|$ onto $M_1^n$, $M_1^n$ is collared in $M^n$, and $M_1^n \subset \mathrm{Int}_{M^n} M_2^n$.

b) Suppose that $k > 2$ and that a finite sequence $\{M_j^n\}_{j=1}^{k-1}$ of compact, connected n-manifolds has been constructed such that

v) i) and ii) are satisfied, $1 \leq j \leq k-1$,

vi) $F_j \subset \mathrm{Int}_{M^n} M_j^n$, $1 < j \leq k-1$, and

vii) iii) is satisfied, $1 \leq j \leq k-2$.

Let $C_k = F_k \cup M_{k-1}^n$; then $C_k$ is compact and $C_k \neq M^n$. Again, it follows from 1.4.5 that there is a proper domain $H_k$ of $M^n$ with $C_k \subset H_k$ and a relative homeomorphism $f_k$ of $H_k$ onto a domain $f_k(H_k)$ of $R^n$. By 1.4.6 there is a complex $L_k \in M(T)$ such that $f_k(C_k) \subset \mathrm{Int}_{R^n}|L_k| \subset |L_k| \subset f_k(H_k)$. Define $h_k = f_k^{-1} \,| |L_k|$ and set $M_k^n = h_k(|L_k|)$. Then $h_k$ is a relative homeomorphism of $|L_k|$ onto $M_k^n$, $M_k^n$ is collared in $M^n$, and $M_{k-1}^n \subset \mathrm{Int}_{M^n} M_k^n$. It is clear that the finite sequence of compact, connected n-manifolds $\{M_j^n\}_{j=1}^{k}$ satisfies the recursive hypothesis for k+1. Therefore it is possible to construct a sequence $\{M_k^n\}_{k=1}^{\infty}$ of compact, connected n-manifolds with properties i) through iv).

<u>Lemma 2.1.4</u> Let (T,id) be the given fixed PL triangulation of $R^n$, $n \geq 2$, and let $\{L_j\}_{j=1}^{\infty}$ be an enumeration of M(T). There exists a domain $D_1^n$ of $R^n$ such that if G is an open set and $|L_j| \subset G$ for some $j \in Z^+$, then there is a homeomorphism h of $R^n$ onto itself which is the identity outside a

proper compact set and $|L_j| \subset h(D_1^n) \subset G$.

<u>Proof</u>. Let $(K,id)$ be a fixed PL triangulation of $Q^n = \{x \in R^n | x_1 > 0\}$, where $|K| = Q^n$ and id is the identity map. For each $j \in Z^+$, let $C(j) = \{x \in R^n | x_1 = 1/j\}$ and for $i,j \in Z^+$, $i < j$, let $Sl(j,i) = \{x \in R^n | 1/j \leq x_1 \leq 1/i\}$. Set $q = (1,0,\ldots,0)$ and choose an $(n-1)$-simplex $t_{n-1} \subset C(1)$ such that $q \in |\overset{o}{t}_{n-1}|$. Let $t_n$ be the n-simplex which is the convex hull of $p \cup |\overline{t}_{n-1}|$, where $p = (0,\ldots,0)$. For each $j \in Z^+$, let $E_j = t_n \cap Sl(j+1,j)$. Then for all $j \in Z^+$, $E_j$ is a PL n-cell in $Q^n$ and in $R^n$.

We now begin the construction of $D_1^n$. For each $j \in Z^+$, there exists a PL homeomorphism $l_j$ of $R^n$ onto itself which is the identity outside a proper compact set such that $l_j(|L_j|) = M_j^n \subset \mathrm{Int}_{R^n} E_j$. Note that $M_j^n$ is a PL n-manifold in both $R^n$ and $Q^n$. Let $I^n$ be the unit n-cube, and set $A = \{x \in I^n | x_n = 0\}$ and $B = \{x \in I^n | x_n = 1\}$. By recursive construction a sequence $\{f_j\}_{j=1}^{\infty}$ of embeddings of $I^n$ into $R^n$ can easily be constructed such that

a) $\{f_j(I^n)\}_{j=1}^{\infty}$ is a disjoint collection of PL n-cells in $R^n$ and $Q^n$;

b) for all $j \in Z^+$, $f_j(I^n) \subset \mathrm{Int}_{R^n}(E_j \cup E_{j+1})$;

c) for all $j \in Z^+$, $f_j(I^n) \cap E_j = E'_j$ is a PL n-cell in $R^n$ and $Q^n$, and $f_j(I^n) \cap E_{j+1} = F'_j$ is a PL n-cell in $R^n$ and $Q^n$;

d) for all $j \in Z^+$, $f_j(I^n) \cap C(j+1)$ is a PL (n-1)-cell in $R^n$ and $Q^n$;

e) for all $j \in Z^+$, $f_j(I^n) \cap M_j^n = f_j(A)$ is a PL $(n-1)$-cell in $R^n$ and $Q^n$, $f_j(A) \subset \dot{M}_j^n$; and

f) for all $j \in Z^+$, $f_j(I^n) \cap M_{j+1}^n = f_j(B)$ is a PL $(n-1)$-cell in $R^n$ and $Q^n$, $f_j(B) \subset \dot{M}_{j+1}^n$.

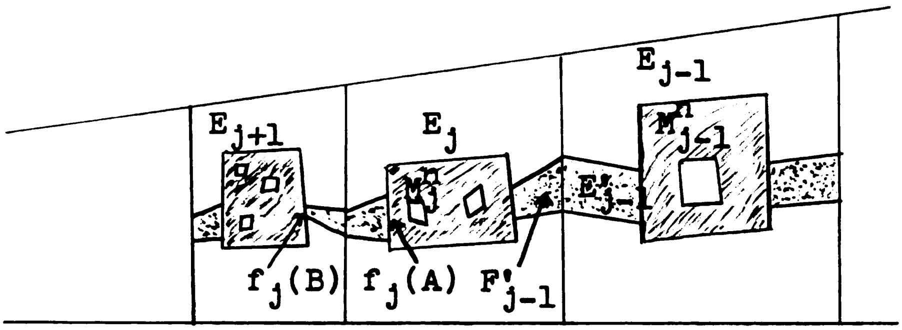


Figure 2.1

Let $M^n = (\bigcup_{j=1}^{\infty} M_j^n) \cup (\bigcup_{j=1}^{\infty} f_j(I^n))$. It is easily seen that $M^n$ is a PL n-manifold in $Q^n$. It follows from 1.4.4 that $M^n$ is collared in $Q^n$ and hence from 1.3.17 that $M^n$ is collared in $R^n$. Since $Cl_{R^n}(Ed_{R^n}M^n) = Ed_{R^n}M^n \cup p$ is compact, we can choose a tapered collaring $D_1^n$ of $M^n$ in $R^n$ such that the support of $D_1^n$ is compact. Now suppose that $j \in Z^+$ and that G is open in $R^n$, $|L_j| \subset G \subset R^n$. Let $G_1 = l_j(G)$; then $G_1$ is open in $R^n$ and $M_j^n \subset G_1$. Define

$$P_j^1 = \begin{cases} \emptyset, & j = 1 \\ f_{j-1}(I^n) \cup (\bigcup_{q=1}^{j-1} E_q) = F'_{j-1} \cup (\bigcup_{q=1}^{j-1} E_q), & j > 1 \end{cases},$$

$$P_j^2 = f_j(I^n) \cup (\bigcup_{q=j+1}^{\infty} E_q) \cup p = E'_j \cup (\bigcup_{q=j+1}^{\infty} E_q) \cup p, \text{ and}$$

$N_j^n = R^n - Int_{R^n}M_j^n$. Since $|L_1| = |L_k|$ for some $k > 1$, we may assume that $j > 1$. Then $N_j^n$ is a PL n-manifold in $R^n$ and $P_j^1$ and $P_j^2$ are PL n-cells in $N_j^n$ and $R^n$. Furthermore $Q_j^k = P_j^k$ - (boundary of $Ed_{N_j^n}P_j^k$) is a collared relative $N_j^n$ n-manifold and $Cl_{N_j^n}(Ed_{N_j^n}Q_j^k) - Ed_{N_j^n}Q_j^k$ = (boundary of $Ed_{N_j^n}P_j^k$) $= Ed_{N_j^n}P_j^k \cap \dot{N}_j^n = S_k^{n-1}$, $k = 1,2$, where $S_1^{n-1}$ = boundary of $f_{j-1}(B)$ and $S_2^{n-1}$ = boundary of $f_j(A)$.

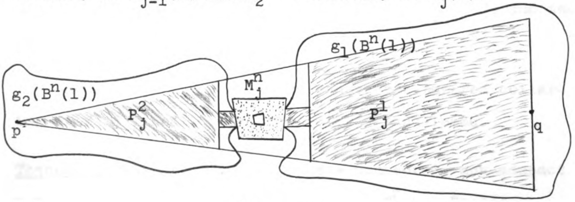


Figure 2.2

Therefore it follows from 1.3.12 and the remark following 1.3.22 that there exist embeddings $g_1$ and $g_2$ of $B^n(1)$ into $N_j^n$ such that

a) $g_1(B^n(1)) \cap g_2(B^n(1)) = \emptyset$;

b) $g_k(B^n(0)) = P_j^k \cap \dot{M}_j^n = P_j^k \cap \dot{N}_j^n$, $k = 1,2$;

c) $g_k(B^n(1)) \cap M_j^n = g_k(B^n(0))$, $k = 1,2$; and

d) $g_k(B^n(\frac{1}{2})) = P_j^k$, $k = 1,2$.

Note that $M^n \subset M_j^n \cup P_j^1 \cup P_j^2$. Now since $G_1$ is open and $g_k(B^n(0)) \subset G_1$, $k = 1,2$, there is a $t \in (0,\frac{1}{2})$ such that $g_k(B^n(t)) \subset G_1$, $k = 1,2$. Define $g : R^n \to R^n$ by

$$g(x) = \begin{Bmatrix} x, & x \notin g_1(B^n(1)) \cup g_2(B^n(1)) \\ g_k h(t,\frac{1}{2}) g_k^{-1}(x), & x \in g_k(B^n(1)),\ k = 1,2 \end{Bmatrix}.$$

Then g is a homeomorphism of $R^n$ onto itself which is the identity outside a proper compact set; that is $g \in CH(R^n)$. Furthermore $g(M^n) \subset G_1$ and $g|M_j^n = id$. Let $G_2 = g^{-1}(G_1)$; then $G_2$ is open and $M^n \subset G_2$. Since $D_1^n$ is a tapered collaring of $M^n$ with compact support, there is a homeomorphism $h_1 \in CH(R^n)$ such that $h_1|M^n = id$ and $h_1(D_1^n) \subset G_2$. Let $h = l_j^{-1}gh_1$. Then $h \in CH(R^n)$ and $|L_j| \subset h(D_1^n) \subset G$; and the proof is complete.

Henceforth $D_1^n$ will always denote the tapered collaring of $M^n$ defined in the proof of the last lemma.

<u>Theorem 2.1.5</u> Let $M^n$, $n \geq 2$, be a connected, non-compact n-manifold without boundary such that $M^n$ has Euclidean compact subsets. Then $D_1^n$ is a generator of $M^n$.

<u>Proof</u>. As a consequence of 2.1.3 there is a sequence $\{M_k^n\}_{k=1}^{\infty}$ of compact, connected n-manifolds such that i) $M_k^n$ is collared in $M^n$, $k \in Z^+$; ii) there is an $L_k \in M(T)$ and a relative homeomorphism $h_k$ of $|L_k|$ onto $M_k^n$, $k \in Z^+$; iii) $M_k^n \subset \mathrm{Int}M_{k+1}^n$, $k \in Z^+$; and iv) $M^n = \bigcup_{k=1}^{\infty} M_k^n$. It follows from 1.3.17 that there is a collaring $CM_k^n$ of $M_k^n$ in $\mathrm{Int}M_{k+1}^n$ and a collaring $C|L_k|$ of $|L_k|$ in $R^n$. From 2.1.4 it follows that there is a homeomorphism $f_k \in CH(R^n)$ such that $|L_k| \subset f_k(D_1^n) \subset C|L_k|$; and from 1.3.9 it follows that there is a relative homeomorphism $h_k'$ of $C|L_k|$ onto $CM_k^n$ which is an extension of $h_k$. Then for all $k \in Z^+$, $M_k^n \subset h_k'(f_k(D_1^n)) \subset \mathrm{Int}M_{k+1}^n$. Therefore $D_1^n$ generates $M^n$.

<u>Theorem 2.1.6</u> Let $M^n$, $n \geq 2$, be a connected n-manifold without boundary. Then $M^n$ has a generating domain if and only if $M^n$ has Euclidean compact subsets.

<u>Proof</u>. Suppose that $M^n$ has a generating domain D, and let $K \subset M^n$ be a proper compact subset and $x \in M^n - K$. Since $n \geq 2$, $G_1 = M^n - x$ is a domain of $M^n$. Since D is a generating domain for $M^n$, there is a domain $D_1$, with $K \subset D_1 \subset G_1$, and a homeomorphism $f_1$ of D onto $D_1$. Since $M^n$ is an n-manifold, there is a proper domain $G_2$ of $M^n$ and a homeomorphism g of $\frac{1}{2}H^n - \frac{1}{2}\dot{H}^n$ onto $G_2$. Since D also generates $G_2$, there is a domain $D_2 \subset G_2$ and a homeomorphism $f_2$ of D onto $D_2$. Let $f = g^{-1}f_2f_1^{-1}$. Then $f|K$ induces a homeomorphism of $(K,\emptyset)$ into $(\frac{1}{2}H^n, \frac{1}{2}\dot{H}^n)$, and thus $M^n$ has Euclidean compact subsets.

Now suppose that $M^n$ has Euclidean compact subsets. If D is a proper domain of $M^n$, then D is a connected, non-compact n-manifold without boundary which has Euclidean compact subsets. It follows from 2.1.5 that $D_1^n$ generates D. Since D was arbitrary, $D_1^n$ is a generating domain for $M^n$.

<u>Corollary 2.1.7</u> Suppose that $M^n$, $n \geq 2$, is a closed connected n-manifold. Then $M^n$ has a generating domain if and only if $M^n$ is an n-sphere.

<u>Proof</u>. If $M^n$ is an n-sphere, then clearly $M^n$ has a generating domain. Now suppose that $M^n$ has a generating domain. Let $B^n$ be an n-cell in $M^n$ such that $\dot{B}^n$ is bi-col-

lared in $M^n$. Since $M^n$ has Euclidean compact subsets, it follows from 1.4.5 that there is an embedding h of $L^n = M^n - \text{Int}B^n$ into $S^n$ such that $h(\dot{L}^n) = h(\dot{B}^n)$ is a bi-collared (n-1)-sphere. Since $\text{Fr}_{S^n}h(L^n) = h(\dot{B}^n)$, it follows from [2] that $h(L^n)$ and $L^n$ are n-cells. Therefore $M^n$ is an n-sphere.

Although each open connected subset of $S^n$ has $D_1^n$ for a generating domain, it is not true that every connected n-manifold with a generating domain is homeomorphic to a domain of $S^n$. Examples of such manifolds are considered in [12].

## 2. Compact, connected n-manifolds with $\dot{M}^n \neq \emptyset$ and $DR(\overset{o}{M}{}^n) = 1$

<u>Theorem 2.2.1</u> Let $M^n$, $n \geq 2$, be a compact, connected n-manifold such that $\dot{M}^n \neq \emptyset$ and $DR(\overset{o}{M}{}^n) = 1$. Then $M^n$ can be embedded in $R^n$ such that $\dot{M}^n$ is bi-collared in $R^n$.

<u>Proof</u>. Let $M_c^n = \dot{M}^n \times [0,1) \cup_c M^n$ be an abstract collaring of $M^n$. It follows from 1.3.14 that $M_c^n$ has a generating domain. Since $M^n \subset M_c^n$ is a proper compact subset and $\dot{M}^n$ is bi-collared in $M_c^n$, it follows from 2.1.6 and 1.4.5 that $M^n$ can be embedded in $R^n$ in the required manner.

<u>Definition 2.2.2</u> Let $\{B_k^n\}_{k=1}^q$, $q \in Z^+$, be a disjoint collection of n-cells contained in $S^n$, $n \geq 2$, such that $\dot{B}_k^n$, $1 \leq k \leq q$, is bi-collared in $S^n$. A space X which is homeo-

morphic to $S^n - (\bigcup_{k=1}^{q} B_k^n)$ is called a punctured n-sphere (with q holes). A space X which is homeomorphic to $S^n - (\bigcup_{k=1}^{q} \mathrm{Int} B_k^n)$ is called a compact punctured n-sphere (with q holes).

The following result follows immediately from 2.2.1, 2.1.7, and the fact that a 1-sphere is the only closed 1-manifold with domain rank 1.

<u>Corollary 2.2.3</u> Let $M^n$, $n \geq 2$, be a compact, connected n-manifold such that $\dot{M}^n \neq \emptyset$ and $DR(\overset{o}{M}{}^n) = 1$. If $DR(C) = 1$ for all components C of $\dot{M}^n$, then $M^n$ is a compact, punctured n-sphere.

# CHAPTER III

## CONNECTED MANIFOLDS WITH BOUNDARY WHICH HAVE DOMAIN RANK 2

Because of invariance of domain in manifolds, it is clear that an n-manifold $M^n$ with boundary has $DR(M^n) \geq 2$. In this chapter we give a characterization of those connected n-manifolds with boundary that have domain rank 2.

### 1. Characterization

For $n \geq 1$, let $(T_1, id)$ be a fixed PL triangulation of $\frac{1}{2}R^n$, where $T_1$ is a simplicial complex of $R^n$ such that $|T_1| = \frac{1}{2}R^n$ and id denotes the identity map. Throughout the remainder of this work, it will be assumed that $\frac{1}{2}R^n$ has this fixed PL triangulation.

<u>Definition 3.1.1</u> Let $(T_1, id)$ be the given fixed PL triangulation of $\frac{1}{2}R^n$, $n \geq 1$. The set $R(T_1) = \{L | L$ is a subcomplex of some $Sd_k T_1$, $|L|$ is a compact, connected PL n-manifold in $\frac{1}{2}R^n$ which is a relative $\frac{1}{2}R^n$ n-manifold, and $|L| \cap \frac{1}{2}\dot{R}^n$ is a PL (n-1)-manifold in $\frac{1}{2}R^n\}$ is called the set of <u>regular</u>, <u>relative</u> <u>submanifolds</u> of $T_1$.

Note that $R(T_1)$ is a countably infinite set.

<u>Theorem 3.1.2</u> Let $M^n$, $n \geq 2$, be a connected, non-compact

n-manifold with boundary such that $M^n$ has Euclidean compact subsets. There exists a sequence $\{M_k^n\}_{k=1}^{\infty}$ of compact, connected, relative $M^n$ n-manifolds such that

i) $M_k^n$ is collared in $M^n$, $k \in Z^+$;

ii) there is an $L_k \in R(T_1)$ and a relative homeomorphism $h_k$ of $|L_K|$ onto $M_k^n$, $k \in Z^+$;

iii) $M_k^n \subset \mathrm{Int} M_{k+1}^n$, $k \in Z^+$; and

iv) $M^n = \bigcup_{k=1}^{\infty} M_k^n$.

Proof. Since $M^n$ is a non-compact, connected n-manifold with boundary, there exists a sequence $\{F_k\}_{k=1}^{\infty}$ of non-empty compact subsets of $M^n$ such that $F_k \subset F_{k+1}$ and $F_k \cap \dot{M}^n \neq \emptyset$ for all $k \in Z^+$; and $M^n = \bigcup_{k=1}^{\infty} F_k$. We now obtain the sequence $\{M_k^n\}_{k=1}^{\infty}$ by recursive construction.

a) Suppose that $k = 2$. Since $M^n$ has Euclidean compact subsets and $\dot{M}^n \neq \emptyset$, it follows from 1.4.5 that there is a proper domain $H_2$ of $M^n$ with $F_2 \subset H_2$ and a relative homeomorphism $f_2$ of $H_2$ onto a domain $f_2(H_2)$ of $\frac{1}{2}R^n$; that is, $f_2$ induces a homeomorphism of the pair $(H_2, H_2 \cap \dot{M}^n)$ into the pair $(\frac{1}{2}R^n, \frac{1}{2}\dot{R}^n)$. As a consequence of 1.4.6, there is an $L_2 \in R(T_1)$ such that $f_2(F_2) \subset \mathrm{Int}_{\frac{1}{2}R^n}|L_2| \subset |L_2| \subset f_2(H_2)$. Define $h_2 = f_2^{-1}||L_2|$ and set $M_2^n = h_2(|L_2|)$. Since $|L_2|$ is collared in $\frac{1}{2}R^n$ and $f_2$ is a relative homeomorphism of $H_2$ into $\frac{1}{2}R^n$, it follows that $h_2$ is a relative homeomorphism of $|L_2|$ onto $M_2^n$, and that $M_2^n$ is a relative $M^n$ n-manifold which is collared in $M^n$. Now there exists an n-cell $C^n \subset \mathrm{Int}_{\frac{1}{2}R^n}|L_2|$ such that $C^n = |L_1|$ and $|L_1| \cap \frac{1}{2}\dot{R}^n$ is a PL (n-1)-cell in $\frac{1}{2}R^n$, where $L_1 \in R(T_1)$. Let $h_1 = h_2 | |L_1|$ and set $M_1^n = h_1(|L_1|)$. Then $h_1$ is a relative homeomorphism of $|L_1|$ onto $M_1^n$, $M_1^n$ is a relative $M^n$ n-manifold which is collared in $M^n$, and $M_1^n \subset \mathrm{Int}_{M^n} M_2^n$.

b) Suppose that $k > 2$ and that a finite sequence $\{M_j^n\}_{j=1}^{k-1}$ of compact, connected, relative $M^n$ n-manifolds has been constructed such that

v) i) and ii) are satisfied, $1 \leq j \leq k-1$;

vi) $F_j \subset \mathrm{Int}_{M^n} M_j^n$, $1 < j \leq k-1$; and

vii) iii) is satisfied, $1 \leq j \leq k-2$.

Let $C_k = F_k \cup M_{k-1}^n$; then $C_k$ is compact, $C_k \cap \dot{M}^n \neq \emptyset$, and $C_k \neq M^n$. Since $M^n$ has Euclidean compact subsets, it follows from 1.4.5 that there is a proper domain $H_k$ of $M^n$ with $C_k \subset H_k$ and a relative homeomorphism $f_k$ of $H_k$ onto a domain $f_k(H_k)$ of $\frac{1}{2}R^n$. By 1.4.6 there is a complex $L_k \in R(T_1)$ such that $f_k(C_k) \subset \mathrm{Int}_{\frac{1}{2}R^n}|L_k| \subset |L_k| \subset f_k(H_k)$. Define $h_k = f_k^{-1} | |L_k|$ and set $M_k^n = h_k(|L_k|)$. Then $h_k$ is a relative homeomorphism of $|L_k|$ onto $M_k^n$, $M_k^n$ is a relative $M^n$ n-manifold which is collared in $M^n$, and $M_{k-1}^n \subset \mathrm{Int}_{M^n} M_k^n$. It is clear that the finite sequence $\{M_j^n\}_{j=1}^{k}$ of compact, connected, relative $M^n$ n-manifolds satisfies the recursive hypothesis for k+1. Therefore it is possible to construct a sequence $\{M_k^n\}_{k=1}^{\infty}$ of compact, connected, relative $M^n$ n-manifolds with properties i) through iv).

<u>Lemma 3.1.3</u> Let $T_1$ be the given fixed PL triangulation of $\frac{1}{2}R^n$, $n \geq 2$, and let $\{L_j\}_{j=1}^{\infty}$ be an enumeration of $R(T_1)$.

There exists a domain $D_2^n$ of $\frac{1}{2}R^n$, $D_2^n \cap \frac{1}{2}\dot{R}^n \neq \emptyset$, such that if G is an open set and $|L_j| \subset G$ for some $j \in Z^+$, then there is a homeomorphism h of $\frac{1}{2}R^n$ onto itself which is the identity outside a proper compact set and $|L_j| \subset h(D_2^n) \subset G$.

Proof. Let (K,id) be a fixed PL triangulation of $Q^n = \{x \in \frac{1}{2}R^n | x_1 > 0\}$ where $|K| = Q^n$ and id is the identity map. For each $j \in Z^+$, let $c(j) = \{x \in \frac{1}{2}R^n | x_1 = 1/j\}$ and for $i, j \in Z^+$, $i < j$, let $sl(j,i) = \{x \in \frac{1}{2}R^n | 1/j \leq x_1 \leq 1/i\}$. Set $q = (1,0,\ldots,0)$ and choose an (n-1)-simplex $t_{n-1} \subset c(1)$ such that $t_{n-1} \cap \frac{1}{2}\dot{R}^n = t_{n-2}$ is an (n-2)-face of $t_{n-1}$ and $q \in |\mathring{\bar{t}}_{n-2}|$. Let $t_n$ be the n-simplex which is the convex hull of $\{p\} \cup |\bar{t}_{n-1}|$, where $p = (0,\ldots,0)$. For each $j \in Z^+$, let $E_j = t_n \cap sl(j+1,j)$. Then for all $j \in Z^+$, $E_j$ is a PL n-cell in $\frac{1}{2}R^n$ and $Q^n$; and $E_j \cap \frac{1}{2}\dot{R}^n$ is a PL (n-1)-cell in $\frac{1}{2}R^n$ and $Q^n$.

We now begin the construction of $D_2^n$. For each $j \in Z^+$, there exists a PL homeomorphism $l_j \in CH(\frac{1}{2}R^n)$ such that $l_j(|L_j|) = M_j^n \subset Int_{\frac{1}{2}R^n}E_j$. Note that $M_j^n$ is a PL, relative $\frac{1}{2}R^n$ n-manifold and also a PL, relative $Q^n$ n-manifold. Let $I^n$ be the unit n-cube, $A = \{x \in I^n | x_n = 0\}$, $B = \{x \in I^n | x_n = 1\}$, and $C = \{x \in I^n | x_{n-1} = 0\}$. By recursive construction a sequence $\{f_j\}_{j=1}^{\infty}$ of embeddings of $I^n$ into $\frac{1}{2}R^n$ can easily be constructed such that

a) $\{f_j(I^n)\}_{j=1}^{\infty}$ is a disjoint collection of PL n-cells in $\frac{1}{2}R^n$ and $Q^n$;

b) for all $j \in Z^+$, $f_j(I^n) \cap \frac{1}{2}\dot{R}^n = f_j(C)$ is a PL (n-1)-

cell in $\frac{1}{2}R^n$ and $Q^n$;

c) for all $j \in Z^+$, $f_j(I^n) \subset \mathrm{Int}_{\frac{1}{2}R^n}(E_j \cup E_{j+1})$;

d) for all $j \in Z^+$, $f_j(I^n) \cap E_j = E_j'$ is a PL n-cell in $\frac{1}{2}R^n$ and $Q^n$, and $f_j(I^n) \cap E_{j+1} = F_j'$ is a PL n-cell in $\frac{1}{2}R^n$ and $Q^n$;

e) for all $j \in Z^+$, $E_j' \cap \frac{1}{2}\dot{R}^n = E_j''$ is a PL (n-1)-cell in $\frac{1}{2}R^n$ and $Q^n$, and $F_j' \cap \frac{1}{2}\dot{R}^n = F_j''$ is a PL (n-1)-cell in $\frac{1}{2}R^n$ and $Q^n$;

f) for all $j \in Z^+$, $f_j(I^n) \cap c(j+1) = E_j^{n-1}$ is a PL (n-1)-cell in $\frac{1}{2}R^n$ and $Q^n$, and $E_j^{n-1} \cap \frac{1}{2}\dot{R}^n = E_j^{n-2}$ is a PL (n-2)-cell in $\frac{1}{2}R^n$ and $Q^n$.

g) for all $j \in Z^+$, $f_j(I^n) \cap M_j^n = f_j(A)$ is a PL (n-1)-cell in $\frac{1}{2}R^n$ and $Q^n$, $f_j(A) \subset \mathrm{Ed}_{\frac{1}{2}R^n}M_j^n$, and $f_j(A) \cap \frac{1}{2}\dot{R}^n$ is a PL (n-2)-cell in $\frac{1}{2}R^n$ and $Q^n$;

h) for all $j \in Z^+$, $f_j(I^n) \cap M_{j+1}^n = f_j(B)$ is a PL (n-1)-cell in $\frac{1}{2}R^n$ and $Q^n$, $f_j(B) \subset \mathrm{Ed}_{\frac{1}{2}R^n}M_{j+1}^n$, and $f_j(B) \cap \frac{1}{2}\dot{R}^n$ is a PL (n-2)-cell in $\frac{1}{2}R^n$ and $Q^n$.

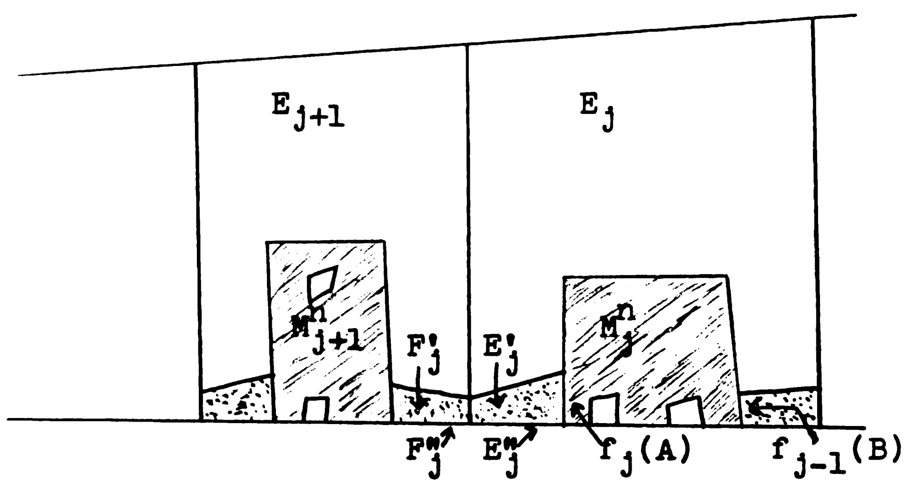


Figure 3.1

Let $M^n = (\overset{\infty}{\underset{j=1}{\cup}} M_j^n) \cup (\overset{\infty}{\underset{j=1}{\cup}} f_j(I^n))$. Then it is easily seen that $M^n$ is a PL n-manifold in $Q^n$ and that $M^n \cap \dot{Q}^n = M^n \cap \tfrac{1}{2}\dot{R}^n$ is a PL (n-1)-manifold in $Q^n$. Therefore it follows from 1.4.4 that $M^n$ is collared in $Q^n$ and hence from 1.3.17 that $M^n$ is collared in $\tfrac{1}{2}R^n$. Since $Cl_{\frac{1}{2}R^n}(Ed_{\frac{1}{2}R^n}M^n) = Ed_{\frac{1}{2}R^n}M^n \cup p$ is compact, we can choose a tapered collaring $D_2^n$ of $M^n$ in $\tfrac{1}{2}R^n$ such that the support of $D_2^n$ is compact. Now suppose that G is open in $\tfrac{1}{2}R^n$, $|L_j| \subset G \subset \tfrac{1}{2}R^n$. Let $G_1 = l_j(G)$; then $G_1$ is open in $\tfrac{1}{2}R^n$ and $M_j^n \subset G_1$. Define

$$P_j^1 = \begin{cases} \emptyset, & j = 1 \\ f_{j-1}(I^n) \cup (\overset{j-1}{\underset{q=1}{\cup}} E_q) = F'_{j-1} \cup (\overset{j-1}{\underset{q=1}{\cup}} E_q), & j > 1 \end{cases},$$

$$P_j^2 = f_j(I^n) \cup (\overset{\infty}{\underset{q=j+1}{\cup}} E_q) \cup p = E'_j \cup (\overset{\infty}{\underset{q=j+1}{\cup}} E_q) \cup p, \text{ and}$$

$$N_j^n = \tfrac{1}{2}R^n - Int_{\frac{1}{2}R^n} M_j^n.$$

Since $|L_1| = |L_k|$ for some $k > 1$, we may assume that $j > 1$. Then $N_j^n$ is a PL n-manifold in $\tfrac{1}{2}R^n$ and $N_j^n$ is a relative $\tfrac{1}{2}R^n$ n-manifold. Furthermore, $Q_j^k = P_j^k - (Ed_{N_j^n}P_j^k \cap Ed_{\frac{1}{2}R^n}N_j^n)$ is a collared, relative $N_j^n$ n-manifold and $Cl_{N_j^n}(Ed_{N_j^n}Q_j^k) - Ed_{N_j^n}Q_j^k = Ed_{N_j^n}P_j^k \cap Ed_{\frac{1}{2}R^n}N_j^n$, $k = 1,2$. Therefore it follows from 1.3.12 and the remark following 1.3.22 that there exist embeddings $g_1$ and $g_2$ of $(\tfrac{1}{2}B^n(1), B_1^{n-1}(1))$ into $(N_j^n, N_j^n \cap \tfrac{1}{2}\dot{R}^n)$ such that

a) $g_1(\tfrac{1}{2}B^n(1)) \cap g_2(\tfrac{1}{2}B^n(1)) = \emptyset$;

b) $g_k(\tfrac{1}{2}B^n(0)) = P_j^k \cap M_j^n = P_j^k \cap Ed_{\frac{1}{2}R^n}N_j^n$, $k = 1,2$;

c) $g_k(\tfrac{1}{2}B^n(1)) \cap M_j^n = g_k(\tfrac{1}{2}B^n(0))$, $k = 1,2$; and

d) $g_k(\tfrac{1}{2}B^n(\tfrac{1}{2})) = P_j^k$, $k = 1,2$.

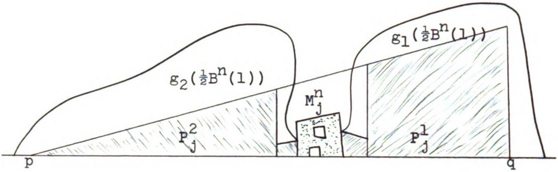


Figure 3.2

Note that $M^n \subset M^n_j \cup P^1_j \cup P^2_j$. Since $G_1$ is open in $\frac{1}{2}R^n$ and $g_k(\frac{1}{2}B^n(0)) \subset G_1$, $k = 1,2$, there is a $t \in (0,\frac{1}{2})$ such that $g_k(\frac{1}{2}B^n(t)) \subset G_1$, $k = 1,2$. Define $f : \frac{1}{2}R^n \to \frac{1}{2}R^n$ by

$$f(x) = \begin{Bmatrix} x, \; x \notin g_1(\frac{1}{2}B^n(1)) \cup g_2(\frac{1}{2}B^n(1)) \\ g_1 g(t,\frac{1}{2}) g_1^{-1}(x), \; x \in g_1(\frac{1}{2}B^n(1)) \\ g_2 g(t,\frac{1}{2}) g_2^{-1}(x), \; x \in g_2(\frac{1}{2}B^n(1)) \end{Bmatrix} .$$

Then $f \in CH(\frac{1}{2}R^n)$ such that $f(M^n) \subset G_1$ and $f|M^n_j = id$. Let $G_2 = f^{-1}(G_1)$; then $G_2$ is open and $M^n \subset G_2$. Since $D^n_2$ is a tapered collaring of $M^n$ with compact support, there is a homeomorphism $h_1 \in CH(\frac{1}{2}R^n)$ such that $h_1|M^n = id$ and $h_1(D^n_2) \subset G_2$. Let $h = l_j^{-1} f h_1$. Then $h$ is a homeomorphism of $\frac{1}{2}R^n$ onto itself which is the identity outside a proper compact set and $|L_j| \subset h(D^n_2) \subset G$.

Henceforth $D^n_2$ will always denote that tapered collaring of $M^n$ defined in the proof of the last lemma.

<u>Theorem 3.1.4</u> Let $M^n$, $n \geq 2$, be a connected, non-compact n-manifold with boundary such that $M^n$ has Euclidean compact subsets. Then $D^n_2$ is a generator of $M^n$.

<u>Proof</u>. As a consequence of 3.1.2, there is a sequence $\{M_k^n\}_{k=1}^{\infty}$ of compact, connected, relative $M^n$ n-manifolds such that i) $M_k^n$ is collared in $M^n$, $k \in Z^+$; ii) there is an $L_k \in R(T_1)$ and a relative homeomorphism $h_k$ of $|L_k|$ onto $M_k^n$, $k \in Z^+$; iii) $M_k^n \subset \mathrm{Int} M_{k+1}^n$, $k \in Z^+$; and iv) $M^n = \bigcup_{k=1}^{\infty} M_k^n$. It follows from 1.3.17 that there is a collaring $CM_k^n$ of $M_k^n$ in $\mathrm{Int} M_{k+1}^n$ and a collaring $C|L_k|$ of $|L_k|$ in $\frac{1}{2}R^n$. From 3.1.3 it follows that there is a homeomorphism $f_k \in CH(\frac{1}{2}R^n)$ such that $|L_k| \subset f_k(D_2^n) \subset C|L_k|$, and from 1.3.9 it follows that there is a relative homeomorphism $h_k'$ of $C|L_k|$ onto $CM_k^n$ which is an extension of $h_k$. Then for all $k \in Z^+$, $M_k^n \subset h_k'(f_k(D_2^n)) \subset \mathrm{Int} M_{k+1}^n$. Therefore $D_2^n$ generates $M^n$ since $M^n = \bigcup_{k=1}^{\infty} h_k'(f_k(D_2^n))$ and $h_k'(f_k(D_2^n))$ is open in $M^n$, $k \in Z^+$.

<u>Theorem 3.1.5</u> Let $M^n$, $n \geq 2$, be a connected n-manifold with boundary. Then $DR(M^n) = 2$ if and only if $M^n$ has Euclidean compact subsets.

<u>Proof</u>. Suppose that $DR(M^n) = 2$ and that $\{D_1, D_2\}$ is a set of generating domains for $M^n$. Since $DR(M^n) = 2$ and $\dot{M}^n \neq \emptyset$, we may assume that $D_1$ is an n-manifold without boundary and that $D_2$ is an n-manifold with boundary. Let $K \subset M^n$ be a proper compact set. If $x \in \dot{M}^n$, then $K \cup x = K_1$ is still a proper compact set. Let $y \in M^n - K_1$ and set $G_1 = M^n - y$. Since $n \geq 2$, $G_1$ is a domain of $M^n$ such that $G_1 \cap \dot{M}^n \neq \emptyset$. Therefore $D_2$ must generate $G_1$ and thus there is a domain $H_1$, $K_1 \subset H_1 \subset G_1$, and a homeomorphism $f_1$ of the pair $(D_2, \dot{D}_2)$ onto $(H_1, H_1 \cap \dot{M}^n) = (H_1, H_1 \cap \dot{G}_1)$. Since

$\dot{M}^n \neq \emptyset$, there is a homeomorphism g of the pair $(\frac{1}{2}R^n, \frac{1}{2}\dot{R}^n)$ onto the pair $(G_2, G_2 \cap \dot{M}^n)$ where $G_2$ is a proper domain of $M^n$. Since $D_2$ must also generate $G_2$, there is a domain $H_2 \subset G_2$ and a homeomorphism $f_2$ of $(D_2, \dot{D}_2)$ onto $(H_2, H_2 \cap \dot{M}^n)$. Let $f = g^{-1} f_2 f_1^{-1}$. Then $f|(K, K \cap \dot{M}^n)$ gives an embedding of $(K, K \cap \dot{M}^n)$ into $(\frac{1}{2}R^n, \frac{1}{2}\dot{R}^n)$. Thus $M^n$ has Euclidean compact subsets.

Now suppose that $M^n$ has Euclidean compact subsets. If D is a proper domain of $M^n$, then D is a connected, non-compact, n-manifold which has Euclidean compact subsets. If $\dot{D} = \emptyset$, then it follows from 2.1.5 that $D_1^n$ generates D. If $\dot{D} \neq \emptyset$, it follows from 3.1.4 that $D_2^n$ generates D. Therefore $DR(M^n) = 2$.

<u>Remark</u>. It follows easily from the last theorem that if an n-manifold $M^n$, $n \geq 2$, with boundary has $DR(M^n) = 2$, then each component of $\dot{M}^n$ is an open (n-1)-manifold.

## 2. Some special manifolds of domain rank 2

<u>Definition 3.2.1</u> Let $M^n$, $n \geq 2$, be an n-manifold such that $\overset{\circ}{M}{}^n \overset{T}{=} R^n$ and $\dot{M}^n \overset{T}{=} R^{n-1}$. Then $M^n$ is called a <u>K-R manifold</u>.

<u>Theorem 3.2.2</u> If $M^n$ is a K-R manifold, then $DR(M^n) = 2$.

<u>Proof</u>. Suppose that $M^n$ is a K-R manifold, $n \neq 3$. Then it follows from [7] that $M^n \overset{T}{=} \frac{1}{2}R^n$ and thus $DR(M^n) = 2$. Now suppose that n = 3 and let (L',f') be a PL triangulation of $M^n$ (the existence of such a triangulation follows from [1]). As remarked in [8], it is possible to extend (L',f')

to a PL triangulation (L,f) of an abstract collaring $M^3_c = \dot{M}^3 \times [0,1) \cup_c M^3$ of $M^3$. Let $M^3$, $\dot{M}^3 \times [0,1)$, and $\dot{M}^3$ be identified with their respective embeddings under the identification $p : \dot{M}^3 \times [0,1) + M^3 \to M^3_c$. Suppose that K is a compact set, $K \subset M^3$. Since $(L,f)|f^{-1}(\dot{M}^3 \times [0,1))$ gives a PL triangulation of $\dot{M}^3 \times [0,1) \overset{T}{=} \frac{1}{2}R^3$, there exists a PL 3-cell $D^3$ in $M^3_c$ such that $D^3 \subset \dot{M}^3 \times [0,1)$, $D^3 \cap \dot{M}^3$ is a PL 2-cell $D^2$ in $M^3_c$, and $K \cap \dot{M}^3 \subset \overset{o}{D}{}^2$. Let $S^3_1$ denote the 1-point compactification of $M^3_c$, and consider $M^3_c$ as embedded in the 3-sphere $S^3_1$. Since $D^3$ is PL in $M^3_c$, $\dot{D}^3$ is a bi-collared 2-sphere in $S^3_1$. Therefore $S^3_1 - \text{Int}_{S^3_1} D^3$ is thus a 3-cell $B^3$ and $\dot{B}^3 = \dot{D}^3$. Furthermore, $K \subset B^3$ and $K \cap \dot{M}^3 = K \cap \dot{B}^3$. Since $K \cap \dot{B}^3 \subset \overset{o}{D}{}^2 \subset \dot{B}^3$, there is a point $x \in \dot{B}^3 - K$ such that $K \subset B^3 - x$. Therefore, there is an embedding h of $(K, K \cap \dot{M}^3) = (K, K \cap \dot{B}^3)$ into $(B^3 - x, \text{boundary}(B^3 - x))$. Since $B^3 - x \overset{T}{=} \frac{1}{2}R^3$, $M^3$ has Euclidean compact subsets and it now follows from 3.1.5 that $DR(M^3) = 2$.

<u>Definition 3.2.3</u> Let $M^n$, $n \geq 2$, be an n-manifold such that $\overset{o}{M}{}^n \overset{T}{=} R^n$ and $\dot{M}^n$ has two components, both of which are homeomorphic to $R^{n-1}$, then $M^n$ is called a <u>pseudo</u> <u>n-slab</u>. If $M^n$ is homeomorphic to $R^{n-1} \times [0,1]$, then $M^n$ is called an <u>n-slab</u>.

<u>Theorem 3.2.4</u> If $M^3$ is a pseudo 3-slab, then $DR(M^3) = 2$.

<u>Proof</u>. Suppose that $M^3$ is a pseudo 3-slab and that $R^2_1$ and $R^2_2$ are the components of $\dot{M}^3$. Let (L',f') be a PL tri-

angulation of $M^3$. It is possible to extend $(L',f')$ to a PL triangulation $(L,f)$ of an abstract collaring $M_c^3 = \dot{M}^3 \times [0,1) \cup_c M^3$ of $M^3$. Consider $M^3, R_1^2, R_2^2, R_1^2 \times [0,1)$, and $R_2^2 \times [0,1)$ as embedded in $M_c^3$ under the identification $p : \dot{M}^3 \times [0,1) + M^3 \to M_c^3$. Note that $(L,f)|f^{-1}(R_k^2 \times [0,1))$ gives a PL triangulation of $R_k^2 \times [0,1) \overset{T}{=} \frac{1}{2}R^3$, $k = 1,2$. Let $K \subset M^3$ be a compact set. There exist PL 3-cells $D_i^3$ in $M_c^3$, such that $D_i^3 \subset R_i^2 \times [0,1)$, $D_i^3 \cap R_i^2 = D_i^2$ is a PL 2-cell in $M_c^3$, and $K \cap R_i^2 \subset \overset{o}{D}{}_i^2$ for $i = 1,2$. Let $S_1^3$ be the 1-point compactification of $M_c^3$ and consider $M_c^3$ as embedded in the 3-sphere $S_1^3 = M_c^3 \cup p$. Since $K_1 = K \cup D_1^3 \cup D_2^3$ is compact in $M_c^3 \overset{T}{=} R^3$, there is a PL 3-cell $B^3$ in $M_c^3$ such that $K_1 \subset \mathrm{Int}_{M_c^3} B^3$. Therefore we may assume that $S_1^3$ has a PL triangulation $(J,g)$ obtained by extending $(L^2, f||L^2|)$ where $|L^2| = f^{-1}(B^3)$ and $L^2$ is a subcomplex of some subdivision of L. Let $F_1 = K \cup D_2^3$, $F_2 = K \cup D_1^3$, $D_3^3 = S_1^3 - \mathrm{Int}B^3$, $S_1^2 = R_1^2 \cup p$, and $S_2^2 = R_2^2 \cup p$. For $i = 1,2$, $A_i^3 = B^3 - \mathrm{Int}D_i^3$ is a PL 3-annulus in $S_1^3$. Since $F_i \cap \dot{D}_i^3 \subset \overset{o}{D}{}_i^2 \subset D_i^2 = D_i^3 \cap S_i^2$ and $S_i^2 \cap \dot{B}^3 \neq \emptyset$, $\dot{A}_i^3 - D_i^2$ lies in some component of $A_i^3 - F_i$, $i = 1,2$. Therefore there exist PL 3-cells $B_i^3$ in $S_1^3$, $B_i^3 \subset A_i^3 - F_i$ such that $B_i^3 \cap \dot{D}_i^3$ and $B_i^3 \cap \dot{B}^3$ are PL 2-cells in $S_1^3$, $i = 1,2$. Furthermore, we can construct $B_1^3$ and $B_2^3$ so that $B_1^3 \cap B_2^3 = \emptyset$. Therefore it follows from the results of chapter 3 of [15], that $E_1^3 = D_1^3 \cup B_1^3 \cup D_3^3 \cup B_2^3 \cup D_2^3$ is a PL 3-cell in $S_1^3$. Let $E_2^3 = S_1^3 - \mathrm{Int}E_1^3$; then $E_2^3$ is a PL 3-cell in $S_1^3$ such that $K \subset E_2^3$, $K \cap \dot{M}^3 = K \cap \dot{E}_2^3$, and

$K \cap \dot{E}_2^3 \neq \dot{E}_2^3$. Let $x \in \dot{E}_2^3 - K$ and set $L^3 = E_2^3 - x$. Then there is an embedding h of $(K, K \cap \dot{M}^3) = (K, K \cap \dot{E}_2^3)$ into $(L^3, \dot{L}^3)$. Since $L^3 \overset{T}{=} \frac{1}{2}R^3$, $M^3$ has Euclidean compact subsets and it follows from 3.1.5 that $DR(M^3) = 2$.

In contrast to 3.2.2 we have the following theorem about pseudo n-slabs.

<u>Theorem 3.2.5</u> Let $M^n$, $n \geq 4$, be a pseudo n-slab. Then $M^n$ is an n-slab if and only if $DR(M^n) = 2$.

<u>Proof</u>. If $M^n$ is an n-slab, then there is an embedding h of $(M^n, \dot{M}^n)$ into $(\frac{1}{2}R^n, \frac{1}{2}\dot{R}^n)$ and thus $DR(M^n) = 2$.

Now suppose that $DR(M^n) = 2$. Let $M_c^n = \dot{M}^n \times [0,1) \cup_c M^n$ be an abstract collaring of $M^n$. Since $M_c^n \overset{T}{=} R^n$, we may assume that $M_c^n = R^n$, $M^n \subset R^n$ and that the components $R_1^2$ and $R_2^2$ of $\dot{M}^n$ are bi-collared, closed subsets of $R^n$. For $n \geq 0$, let $S_0^n = \{x \in R^{n+1} | \sum_{k=1}^{n+1} |x_k| = 1\}$. Then $S_0^n$ is a combinatorial n-sphere in $R^{n+1}$, and the suspension of $S_0^n$ in $R^{n+2}$ may be taken to be $S_0^{n+1}$. We now employ the technique which Greathouse used to prove the theorem of [10]. Let $S_1^n = R^n \cup \{p\}$ be the 1-point compactification of $R^n$ and let $S_i^{n-1} = R_i^{n-1} \cup \{p\}$, $i = 1,2$. It follows from the corollary to theorem 2 of [6] that $S_i^{n-1}$ is bi-collared in $S_1^n$ for $i = 1,2$. Therefore we may assume that $S_1^n = S_0^n$, $p = (0,\ldots,0,1,0)$, $S_1^{n-1} = S_0^{n-1}$, and that $S_2^{n-1}$ lies in the northern hemisphere of $S_0^n$ with $S_1^{n-1} \cap S_2^{n-1} = \{p\}$. Let $B^{n-1}$ be a combinatorial (n-1)-cell such that $B^{n-1} \subset S_0^{n-1}$ and

$p \in \overset{o}{B}^{n-1}$. Let r be the south pole of $S_0^n$, q the midpoint of the line segment joining p to r in $S_0^n$, L the line segment joining p to q in $S_0^n$, and $B_q^n$, $B_r^n$ the cones (n-cells) in $S_0^n$ with base $B^{n-1}$ and cone points q, r respectively. Let $S_3^{n-1} = (S_1^{n-1} \cup \dot{B}_q^n) - \overset{o}{B}^{n-1}$. Then $S_3^{n-1}$ is a bi-collared (n-1)-sphere in $S_0^n$ and $S_2^{n-1} \cap S_3^{n-1} = \emptyset$. Let $u \in R_2^{n-1}$ and $v \in R_1^{n-1} - B^{n-1}$. Then there exists an embedding $f : I \to M^n$ such that $f(0) = u$, $f(1) = v$, and $f(0,1) \subset \overset{o}{M}^n$. Since $DR(M^n) = 2$, it follows from 3.1.5 and 1.4.5 that there is a proper domain $G \subset M^n$, $f(I) \subset G$, and an embedding h of $(G, G \cap \dot{M}^n)$ into $(\frac{1}{2}R^n, \frac{1}{2}\dot{R}^n)$. Therefore there exists and n-cell $F^n \subset M^n$ such that $F^n \cap S_i^{n-1}$ is an (n-1)-cell, i = 2,3. Thus $A^n = M^n \cup B_q^n$ is an n-annulus. From this point the proof proceeds exactly as the proof of the theorem of [10] to show that $M^n$ is an n-slab; and thus the remainder of the proof will be omitted.

## CHAPTER IV

## MANIFOLDS WITH COMPACT BOUNDARY WHICH HAVE DOMAIN RANK 3

It is clear that if $M^n$ is an n-manifold with boundary, $n \geq 2$, such that some component of $\dot{M}^n$ is compact, then $DR(M^n) \geq 3$. In this chapter we will characterize those n-manifolds with compact boundary which have domain rank 3.

### 1. A generator for a certain dominion of a compact, punctured n-sphere

<u>Definition 4.1.1</u> A homeomorphism h of $S^n$ onto itself is called <u>strictly stable</u> if h is the identity on a non-empty open set. A homeomorphism h of $S^n$ onto itself is called <u>stable</u> if h is the product of a finite number of strictly stable homeomorphisms of $S^n$. $SH(S^n)$ will denote the group of stable homeomorphisms of $S^n$.

<u>Lemma 4.1.2</u> Let $\{B_k^n\}_{k=1}^q$ be a finite disjoint collection of n-cells contained in $S^n$, $n \geq 2$, such that $\dot{B}_k^n$ is bi-collared in $S^n$, $1 \leq k \leq q$. Suppose that $M^n$ is a connected n-manifold, $M^n \subset S^n$, and that $h_1$ and $h_2$ are stable homeomorphisms of $S^n$ onto itself such that $\bigcup_{k=1}^q B_k^n \subset \operatorname{Int} h_i(M^n) = h_i(\overset{o}{M}{}^n)$, $i = 1,2$. Then $h_1(M^n) - \bigcup_{k=1}^q \operatorname{Int} B_k^n \overset{T}{=} h_2(M^n) - \bigcup_{k=1}^q \operatorname{Int} B_k^n$.

<u>Proof</u>. Let $h = h_2 h_1^{-1}$ and $L^n = h_1(M^n)$. Then $h \in SH(S^n)$

and $\bigcup_{k=1}^{q} B_k^n \subset \text{Int}L^n \cap \text{Int}h(L^n)$. It suffices to show that $h(L^n) - \bigcup_{k=1}^{q} \text{Int}h(B_k^n)$ is homeomorphic to $h(L^n) - \bigcup_{k=1}^{q} \text{Int}B_k^n$. For each k, $1 \leq k \leq q$, let $c_k \in \text{Int}B_k^n$. Since $n \geq 2$ and h is a homeomorphism, there exists a disjoint collection $\{D_k^n\}_{k=1}^{q}$ of n-cells contained in $\text{Int}h(L^n)$ such that $\dot{D}_k^n$ is bi-collared in $S^n$ and $\{c_k\} \cup \{h(c_k)\} \subset \text{Int } D_k^n$, $1 \leq k \leq q$. Therefore there exists a homeomorphism $g_1$ of $S^n$ onto itself such that $g_1|(S^n - \bigcup_{k=1}^{q} \text{Int}D_k^n) = \text{id}$ and $g_1\ (h(c_k)) = c_k$, $1 \leq k \leq q$. For $1 \leq k \leq q$, let $U_k$ be a neighborhood of $c_k$, $c_k \in U_k \subset \text{Int}B_k^n$ and let $V_k$ be a neighborhood of $h(c_k)$, $h(c_k) \in V_k \subset \text{Int}h(B_k^n)$ such that $g_1(V_k) \subset U_k$. Since $\{h(B_k^n)\}_{k=1}^{q}$ is a disjoint collection of n-cells with bi-collared boundaries, there is a disjoint collection $\{C_k^n\}_{k=1}^{q}$ of n-cells contained in $\text{Int}h(L^n)$ and a collection of embeddings $\{f_k\}_{k=1}^{q}$ of $E^n$ into $S^n$ such that $f_k(E^n) = C_k^n$, $f_k(E^n(\frac{1}{2})) = h(B_k^n)$, and $f_k(0) = h(c_k)$, $1 \leq k \leq q$, where $E^n(\frac{1}{2}) = \{x \in E^n | d(x,0) \leq \frac{1}{2}\}$. Therefore there exists a homeomorphism $g_2$ of $S^n$ onto itself such that $g_2$ restricted to $(S^n - \bigcup_{k=1}^{q} \text{Int}C_k^n) = \text{id}$, and such that for $1 \leq k \leq q$, $g_2(h(c_k)) = h(c_k)$ and $g_2(h(B_k^n)) \subset V_k$. Then $g = g_1 g_2$ is a stable homeomorphism of $S^n$, $g|h(L^n)$ is a homeomorphism of $h(L^n)$ onto itself, and $g(h(B_k^n)) \subset \text{Int}B_k^n$ for $1 \leq k \leq q$. Since $\dot{B}_k^n$ is a bi-collared (n-1)-sphere in $S^n$, it follows from 9.1 of [5] that $B_k^n - \text{Int}g(h(B_k^n))$ is an n-annulus, $1 \leq k \leq q$. Since $\{B_k^n\}_{k=1}^{q}$ is a disjoint collection of n-cells with bi-collared boundaries contained in $\text{Int}h(L^n)$, there is a homeomorphism $f \in SH(S^n)$ which carries $h(L^n) - \bigcup_{k=1}^{q} Int(B_k^n)$ homeomorphically onto $h(L^n) - \bigcup_{k=1}^{q} Intg(h(B_k^n))$. Since $h(L^n) - \bigcup_{k=1}^{q} Intg(h(B_k^n)) \overset{T}{=} h(L^n) - \bigcup_{k=1}^{q} Int(B_k^n)$ and $g|h(L^n)$ is a homeomorphism of $h(L^n)$ onto itself, $h(L^n) - \bigcup_{k=1}^{q} IntB_k^n \overset{T}{=} h(L^n) - \bigcup_{k=1}^{q} Inth(B_k^n)$ and the desired result is established.

<u>Theorem 4.1.3</u> Let $M^n$ be a compact, punctured n-sphere with q holes, $n \geq 2$. There is a proper domain $D \subset M^n$, $\dot{M}^n \subset D$, such that if G is a proper domain of $M^n$, $\dot{M}^n \subset G$, then D generates G.

<u>Proof</u>. Let p be the north pole of $S^n$. Without loss of generality we may assume that $M^n = S^n - \bigcup_{k=1}^{q} \overset{o}{B}{}_k^n$, where $\{B_k^n\}_{k=1}^{q}$ is a finite disjoint collection of n-cells with bi-collared boundaries, and that $p \in \overset{o}{M}{}^n = Int_{S^n} M^n$. We will consider $R^n = |T|$ as embedded in $S^n$ as the subspace $S^n - \{p\}$ under an embedding e.

Let G be a proper domain of $M^n$, $\dot{M}^n \subset G$. Since G is a proper domain, there exists a $g \in SH(S^n)$ and an $x \in M^n - G$ such that $g(x) = p$ and $g|(\bigcup_{k=1}^{q} B_k^n) = id$. Let $G_1 = g(G)$ and set $G_2 = G_1 \cup (\bigcup_{k=1}^{q} B_k^n)$. Then $G_2$ is a domain of $R^n$. It follows from 1.4.6 and 2.1.4 that there is an $h \in SH(S^n)$ which is the identity in a neighborhood of p, such that $\bigcup_{k=1}^{q} B_k^n \subset h(D_1^n)$. Let $D = h(D_1^n) - \bigcup_{k=1}^{q} IntB_k^n$; then D is a proper domain of $M^n$, $\dot{M}^n \subset D$. Since $G_2$ is a domain of $R^n$, it follows from 1.4.6, 2.1.4, and the fact that $R^n$ has a PL triangulation (T,e), that there is a sequence $\{f_j\}_{j=1}^{\infty}$ of elements of $SH(S^n)$ such that

i) for all $j \in Z^+$, $f_j$ is the identity in a neighborhood $U_j$ of $p$;

ii) for all $j \in Z^+$, $\bigcup_{k=1}^{q} B_k^n \subset f_j(D_1^n) \subset f_{j+1}(D_1^n) \subset G_2$;

iii) $G_2 = \bigcup_{j=1}^{\infty} f_j(D_1^n)$.

For all $j \in Z^+$, define $g_j = g^{-1}f_jh^{-1}$. Then $\{g_j\}_{j=1}^{\infty}$ is a sequence of stable homeomorphisms such that for all $j \in Z^+$, $\bigcup_{k=1}^{q} B_k^n \subset g_j(h(D_1^n)) \subset g_{j+1}(h(D_1^n))$, and such that $G = \bigcup_{j=1}^{\infty}(g_j(h(D_1^n)) - \bigcup_{k=1}^{q} \mathrm{Int}B_k^n)$. It follows from 4.1.2 that for all $j \in Z^+$, $g_j(h(D_1^n)) - \bigcup_{k=1}^{q} \mathrm{Int}B_k^n \overset{T}{=} h(D_1^n) - \bigcup_{k=1}^{q} \mathrm{Int}B_k^n = D$ and thus $D$ generates $G$.

## 2. Characterization

<u>Lemma 4.2.1</u> Let $M^n$, $n \geq 2$, be a connected n-manifold with boundary such that $DR(M^n) = 3$. If $\dot{M}^n$ has a compact component, then $\dot{M}^n$ is an (n-1)-sphere.

<u>Proof</u>. Suppose that $C$ is a compact component of $\dot{M}^n$. Let $\{D_1, D_2, D_3\}$ be a set of generating domains for $M^n$. We may assume that $D_1$ is an n-manifold without boundary, that $D_2$ is an n-manifold with boundary such that all components of $\dot{D}^2$ are open (n-1)-manifolds, and that $D_3$ is an n-manifold with boundary such that $C \subset \dot{D}_3$. Suppose that $\dot{M}^n \neq C$ and let $x \in \dot{M}^n - C$. Since $M^n$ is a connected n-manifold, there exists an embedding $f : I \to M^n$ such that $f(0) \in C$, $f(1) = x$, and $f(0,1) \subset \overset{\circ}{M}{}^n$. Since $n \geq 2$, it follows from 1.4.5 that there is a proper domain $H_1$ such that $f(I) \cup C \subset H_1$. Also it follows from 1.3.13 that there is a

proper domain $H_2$ of $M^n$ such that $H_2 \cap \dot{M}^n = C$. Since $C \subset H_1 \cap H_2$, $D_3$ must generate both $H_1$ and $H_2$, which is impossible. Therefore $C = \dot{M}^n$. Suppose that $x \in C$, then $DR(M^n - x) = 2$ and if $n > 2$ it follows that $DR(C) = 1$. Since $C$ is a closed $(n-1)$-manifold, 2.1.7 shows that $C \overset{T}{=} S^{n-1}$ if $n > 2$. If $n = 2$, then $C \overset{T}{=} S^1$, since $S^1$ is the only closed, connected 1-manifold.

<u>Definition 4.2.2</u> Let $J = [-1,1]$ and set $SA^n = J^n - (0,\ldots,0)$, $n \geq 2$. A space $X$ which is homeomorphic to $SA^n$ is called an <u>n-semi-annulus</u>.

<u>Definition 4.2.3</u> Let $M^n$, $n \geq 2$, be an n-manifold such that $\dot{M}^n$ is an $(n-1)$-sphere. $M^n$ is said to have <u>semi-annular compact subsets</u> if for each proper compact subset $K \subset M^n$, there is an embedding $h$ of $(K, K \cap \dot{M}^n)$ into $(SA^n, \dot{SA}^n)$.

Let $(T_2, id)$ be a fixed PL triangulation of $SA^n$ where $|T_2| = SA^n$ and id is the identity map.

<u>Definition 4.2.4</u> Let $T_2$ be the given fixed PL triangulation of $SA^n$, $n \geq 2$. The set $S(T_2) = \{L \mid L$ is a subcomplex of some $Sd_k T_2$ and $|L|$ is a compact, connected, PL n-manifold in $SA^n$ with $\dot{SA}^n \subset |L|\}$ is called the set of <u>regular, boundary submanifolds</u> of $T_2$.

<u>Theorem 4.2.5</u> Let $M^n$, $n \geq 2$, be a connected, non-compact n-manifold such that $M^n$ has semi-annular compact subsets. There exists a sequence $\{M_k^n\}_{k=1}^{\infty}$ of compact, connected n-

manifolds such that

i) $M_k^n$ is collared in $M^n$, $\dot{M}^n \subset M_k^n \subset M^n$, $k \in Z^+$;

ii) there is an $L_k \in S(T_2)$ and a relative homeomorphism $h_k$ of $|L_k|$ onto $M_k^n$;

iii) $M_k^n \subset \mathrm{Int} M_{k+1}^n$, $k \in Z^+$; and

iv) $M^n = \bigcup_{k=1}^{\infty} M_k^n$.

Proof. The method of proof is similar to that of 2.1.3 and 3.1.2 and will be omitted.

Lemma 4.2.6 Let $(T_2, \mathrm{id})$ be the given fixed PL triangulation of $SA^n$, $n \geq 2$, and let $\{L_j\}_{j=1}^{\infty}$ be an enumeration of $S(T_2)$. There exists a domain $D_3^n$ of $SA^n$, $\dot{SA}^n \subset D_3^n$, such that if G is an open set and $|L_j| \subset G$ for some $j \in Z^+$, then there is a homeomorphism h of $(D_3^n, \dot{D}_3^n)$ into $(SA^n, \dot{SA}^n)$ such that $|L_j| \subset h(D_3^n) \subset G$.

Proof. This is an immediate consequence of 4.1.3.

Henceforth $D_3^n$ will always denote the domain of $SA^n$ referred to in the last lemma.

Theorem 4.2.7 Let $M^n$, $n \geq 2$, be a connected, non-compact n-manifold such that $M^n$ has semi-annular compact subsets. Then $D_3^n$ is a generator of $M^n$.

Proof. The result follows directly from 4.2.5 and 4.2.6. The method of proof is similar to that used in the proof of 3.1.4 and the details of the proof will be omitted.

Theorem 4.2.8 Let $M^n$, $n \geq 2$, be a connected n-manifold with $\dot{M}^n \overset{T}{=} S^{n-1}$. Then $DR(M^n) = 3$ if and only if $M^n$ has semi-annular compact subsets.

Proof. Suppose that $DR(M^n) = 3$ and that $\{D_1, D_2, D_3\}$ is a set of generating domains for $M^n$. Since $DR(M^n) = 3$ and $\dot{M}^n$ is an (n-1)-sphere, we may assume that $D_1$ is an n-manifold without boundary, $D_2$ is an n-manifold with boundary such that each component of $\dot{D}_2$ is an open (n-1)-manifold, and $D_3$ is an n-manifold with $\dot{D}_3 \overset{T}{=} S^{n-1}$. Now suppose that $K \subset M^n$ is a proper compact subset; then $K \cup \dot{M}^n = K_1$ is also a proper compact subset. Let $x \in M^n - K_1$ and set $G_1 = M^n - x$. Since $n \geq 2$, $G_1$ is a proper domain of $M^n$ with $\dot{M}^n \subset G_1$. Since $D_3$ must generate $G_1$ there is a domain $H_1$, $K_1 \subset H_1 \subset G$ and a homeomorphism $f_1$ of $(D_3, \dot{D}_3)$ onto $(H_1, H_1 \cap \dot{M}^n) = (H_1, \dot{M}^n)$. It follows from 1.3.13 that there is a domain $G_2$ of $M^n$ and a homeomorphism $g$ of $(SA^n, \dot{SA}^n) \overset{T}{=} (S^{n-1} \times [0,1), S^{n-1} \times \{0\})$ onto $(G_2, G_2 \cap \dot{M}^n) = (G_2, \dot{M}^n)$. Since $D_3$ must also generate $G_2$ there is a domain $H_2$, $H_2 \subset G_2$ and a homeomorphism $f_2$ of $(D_3, \dot{D}_3)$ onto $(H_2, H_2 \cap \dot{M}^n) = (H_2, \dot{M}^n)$. Let $f = g^{-1} f_2 f_1^{-1}$. Then $f|(K, K \cap \dot{M}^n)$ gives an embedding of $(K, K \cap \dot{M}^n)$ into $(SA^n, \dot{SA}^n)$ and thus $M^n$ has semi-annular compact subsets.

Now suppose that $M^n$ has semi-annular compact subsets, and that $G$ is a proper domain of $M^n$. If $\dot{M}^n \subset G$, then it follows from 4.2.7 that $D_3^n$ generates $G$. If $G \cap \dot{M}^n \neq \emptyset$, $G \cap \dot{M}^n \neq \dot{M}^n$, then since $G$ has Euclidean compact subsets, it follows from 3.1.4 that $D_2^n$ generates $G$. Finally if $G \cap \dot{M}^n = \emptyset$, then since $G$ has Euclidean compact subsets, it follows from 2.1.5 that $D_1^n$ generates G. Therefore $\{D_1^n, D_2^n, D_3^n\}$ is a minimal set of generating domains for $M^n$ and thus $DR(M^n) = 3$.

<u>Corollary 4.2.9</u> Let $M^n$, $n \geq 2$, be a compact, connected n-manifold with boundary. Then $DR(M^n) = 3$ if and only if $M^n$ is an n-cell.

<u>Proof</u>. If $M^n$ is an n-cell, then $M^n$ has semi-annular compact subsets and by 4.2.8 $DR(M^n) = 3$. If $DR(M^n) = 3$, then it follows from 4.2.1 that $\dot{M}^n \overset{T}{=} S^{n-1}$. Since $DR(M^n) = 3$ and $\dot{M}^n \overset{T}{=} S^{n-1}$, $DR(\mathring{M}^n) = 1$. Therefore it follows from 2.2.3 that $M^n$ is a compact punctured n-sphere with 1 hole and consequently theorem 5 of [2] shows that $M^n$ is an n-cell.

<u>Corollary 4.2.10</u> Let $M^n$, $n \geq 2$, be a connected n-manifold such that $\dot{M}^n \overset{T}{=} S^{n-1}$. If $DR(\mathring{M}^n) = 1$, then $DR(M^n) = 3$.

<u>Proof</u>. Let $M_c^n = \dot{M}^n \times [0,1) \cup_c M^n$ be an abstract collaring of $M^n$ and consider $M^n$ and $\dot{M}^n$ as embedded in $M_c^n$. Let K be a proper compact subset of $M^n$. Then $K_1 = K \cup \dot{M}^n$ is also a proper compact subset of $M^n$. Since $DR(M_c^n) = DR(\mathring{M}^n) = 1$, there is an embedding h of $K_1$ into $R^n$ such that $h(\dot{M}^n)$ is a bi-collared (n-1)-sphere in $R^n$. Since $M^n$ is connected, $K_1$ is not an n-cell and thus there is an embedding g of $(h(K_1), h(\dot{M}^n))$ into $(SA^n, S\dot{A}^n)$. Therefore $M^n$ has semi-annular compact subsets and so $DR(M^n) = 3$.

## CHAPTER V

## MONOTONE UNIONS AND PRODUCTS

<u>Definition 5.1.1</u> Let C* be a collection of non-empty topological spaces. A topological space X is said to be an open monotone union of C* if $X = \bigcup_{k=1}^{\infty} X_k$ where

i) for all $k \in Z^+$, $X_k$ is open in X and $X_k$ is homeomorphic to some element $C_k \in C^*$, and

ii) for all $k \in Z^+$, $X_k \subset X_{k+1}$.

X is said to be an open, homogeneous monotone union of C* if X is an open monotone union of C* and for all $k \in Z^+$, $X_k \overset{T}{=} X_1$.

<u>Theorem 5.1.2</u> Let C* be a collection of connected n-manifolds, $n \geq 2$, such that either

i) for all $C \in C^*$, $\dot{C} = \emptyset$ and $DR(C) = 1$;

ii) for all $C \in C^*$, $\dot{C} \neq \emptyset$ and $DR(C) = 2$; or

iii) for all $C \in C^*$, $\dot{C} \overset{T}{=} S^{n-1}$ and $DR(C) = 3$.

Let X be an open monotone union of C*. Then X is a connected n-manifold such that

iv) if i) holds, then $\dot{X} = \emptyset$ and $DR(X) = 1$;

v) if ii) holds, then $\dot{X} \neq \emptyset$ and $DR(X) = 2$; and

vi) if iii) holds, then $\dot{X} \overset{T}{=} S^{n-1}$ and $DR(X) = 3$.

<u>Proof</u>. Let X be an open monotone union of C*. It is clear that X is a connected n-manifold such that if i)

holds, then $\dot{X} = \emptyset$; if ii) holds, then $\dot{X} \neq \emptyset$; and if iii) holds, then $\dot{X} \overset{T}{=} S^{n-1}$. The result now follows easily from the characterizations given in 2.1.6, 3.1.5, and 4.2.8.

Throughout the rest of this chapter the boundary of an n-manifold $M^n$ will be denoted either by $bd(M^n)$ or $\dot{M}^n$.

Theorem 5.1.3 Let $M^n$, $M^k$ be connected n and k manifolds respectively such that $M^n$, $M^k$ have Euclidean compact subsets, and $M^k$ is not compact. If either $k \geq 2$ or $M^n$ is not compact, then $M^n \times M^k$ has Euclidean compact subsets.

Proof. Let C be a proper compact set in $M^n \times M^k$ and let $p_1$ and $p_2$ be the projections onto $M^n$ and $M^k$ respectively. Define $C_i = p_i(C)$, $i = 1,2$. Then $C \subset p_1(C) \times p_2(C)$.

a) Suppose that $k \geq 2$. If $M^n$ is compact, then it follows from the remark after 3.1.5 that $\dot{M}^n = \emptyset$ and thus $M^n \overset{T}{=} S^n$. Therefore either $C_1 \overset{T}{=} S^n$ or there is an embedding $h_1$ of $(C_1, C_1 \cap \dot{M}^n)$ into $(\frac{1}{2}R^n, \frac{1}{2}\dot{R}^n)$. Since $M^k$ is not compact, there is an embedding $h_2$ of $(C_2, C_2 \cap \dot{M}^k)$ into $(\frac{1}{2}R^k, \frac{1}{2}\dot{R}^k)$.

i) Suppose that $C_1 \overset{T}{=} S^n$ and that $h_1$ is a homeomorphism of $C_1$ onto $S^n$. Then $f : C_1 \times C_2 \rightarrow S^n \times \frac{1}{2}R^k$ defined by $f(x_1, x_2) = (h_1(x_1), h_2(x_2))$ induces an embedding h of the pair $(C_1 \times C_2, C_1 \times C_2 \cap bd(M^n \times M^k))$ into the pair $(S^n \times \frac{1}{2}R^k, bd(S^n \times \frac{1}{2}R^k))$. Since $k \geq 2$, $S^n \times R^{k-1}$ can be embedded in $R^{n+k-1}$, and hence there is an embedding g of $(S^n \times \frac{1}{2}R^k, bd(S^n \times \frac{1}{2}R^k))$ into $(\frac{1}{2}R^{n+k}, \frac{1}{2}\dot{R}^{n+k})$. Therefore

$gf|(C, C \cap bd(M^n \times M^k))$ gives an embedding of $(C, C \cap bd(M^n \times M^k))$ into $(\frac{1}{2}R^{n+k}, \frac{1}{2}\dot{R}^{n+k})$ and thus $M^n \times M^k$ has Euclidean compact subsets.

ii) Suppose that $C_1$ is not an n-sphere. Then $C_1 \neq M^n$, $C_2 \neq M^k$ and there are embeddings $h_1$ of $(C_1, C_1 \cap \dot{M}^n)$ into $(\frac{1}{2}R^n, \frac{1}{2}\dot{R}^n)$ and $h_2$ of $(C_2, C_2 \cap \dot{M}^k)$ into $(\frac{1}{2}R^k, \frac{1}{2}\dot{R}^k)$. Define $h : C_1 \times C_2 \to \frac{1}{2}R^n \times \frac{1}{2}R^k$ by $h(x_1, x_2) = (h_1(x_1), h_2(x_2))$. Since $\frac{1}{2}R^n \times \frac{1}{2}R^k \overset{T}{=} \frac{1}{2}R^{n+k}$, h induces an embedding g of $(C, C \cap bd(M^n \times M^k))$ into $(\frac{1}{2}R^{n+k}, \frac{1}{2}\dot{R}^{n+k})$. Therefore $M^n \times M^k$ has Euclidean compact subsets.

b) If $M^n$ is not compact, then an argument similar to that given in ii) above shows that $M^n \times M^k$ has Euclidean compact subsets.

<u>Corollary 5.1.4</u> Let $\{M_k^{h(k)}\}_{k=1}^q$ be a finite collection of connected h(k)-manifolds such that for $1 \leq k \leq q$, $DR(M_k^{h(k)}) = 1$. If for some j, $1 \leq j \leq q$, $M_j^{h(j)}$ is not compact, then $DR(\prod_{k=1}^q M_k^{h(k)}) = 1$.

<u>Proof</u>. This follows immediately from 5.1.3.

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