THE GENUS OF CARTESIAN PRODUCTS OF GRAPHS

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This is to certify that the

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E. A. Nordhaus
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ABSTRACT

THE GENUS OF CARTESIAN PRODUCTS OF GRAPHS

By

Arthur Thomas White II

A graph G is said to have genus n if G can be imbedded in a compact orientable 2-manifold of genus n, where n is minimal. In 1968 Ringel and Youngs completed the determination of the genus of the complete graph on p vertices, and by doing so solved the long-standing Heawood map-coloring problem. There are very few other families of graphs for which the genus is known. The main purpose of this thesis is to determine the genus for several infinite families of graphs.

The general method is to establish a lower bound for the genus of a given graph, usually by using a form of the Euler formula, and then to construct an imbedding of the graph that attains the lower bound. The construction often employs Edmonds' permutation technique. The structure of the cartesian product of two graphs suggests, in certain cases, a form that the desired construction might take.

The first three chapters of this thesis introduce the subject, define basic terms and notation, and survey known results concerning genus problems in graph theory. In Chapter 4 upper and lower bounds for the genus of the cartesian product G_1xG_2 in terms of the genera of G_1 and G_2 are developed, and asymptotic results are established for the cases where G_1 and G_2 are both regular complete k-partite graphs.

In Chapter 5 some general results are presented in connection with the genus of cartesian products of bipartite graphs. The techniques developed here are applied in Chapters 6 and 7, which contain some of the main results of this thesis.

In Chapter 6 the genus of the cartesian product of the complete bipartite graph $K_{2m,2m}$ with itself is computed to be $\gamma(K_{2m,2m} \times K_{2m,2m}) = 1 + 8m^2(m-1)$. As an extension of this result, let $Q_1^{(s)} = K_{s,s}$ and recursively define $Q_n^{(s)} = Q_{n-1}^{(s)} \times K_{s,s}$ for $n \geq 2$. Then it is shown that $\gamma(Q_n^{(s)}) = 1 + 2^{n-3}s^n(ns-4)$, for s even and n any natural number, or for $s = 1$ or 3 and $n \geq 2$.

Repeated cartesian products of certain cycles and paths are taken in Chapter 7, and the corresponding genus formulae are developed. For example, with $G_1 = C_{2m_1}$ and $G_n = G_{n-1} \times C_{2m_n}$ for $n \geq 2$, where $m_i \geq 2$ for $i = 1,\ldots,n$, it is shown that: $\gamma(G_n) = 1 + 2^{n-2}(n-2) \prod_{i=1}^{n} m_i$.

Complete tripartite graphs are investigated in Chapter 8, and it is shown that $\gamma(K_{p,q,r}) \geq \left\{\frac{(p-2)(q+r-2)}{4}\right\}$, where $p \geq q \geq r$, and that equality holds if $q+r \leq 6$. It is also shown that $\gamma(K_{mn,n,n}) = \frac{(mn-2)(n-1)}{2}$, for all natural numbers m and n.

THE GENUS OF CARTESIAN PRODUCTS OF GRAPHS

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Arthur Thomas White II

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For Liz,

Mom and Dad,

and Mom and Dad Siber

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CHAPTER 1

INTRODUCTION

A famous unsolved problem in the literature of mathematics is the four color conjecture, which states that four colors will suffice to color the countries of any map on a sphere. It is not difficult to show that five colors are sufficient, but whether five colors are necessary is not known. One of the oddities of mathematics is that the corresponding coloring question has been completely answered for spheres with γ handles, for all positive integral values of γ.

In 1890 Heawood [11] proved that the chromatic number $\chi(S_\gamma)$ of a sphere S_γ with γ handles satisfies the inequality

$$\chi(S_\gamma) \leq \left[\frac{7 + \sqrt{1 + 48\gamma}}{2}\right] ,$$

where $\gamma > 0$ and $[x]$ denotes the greatest integer less than or equal to x. The _Heawood map-coloring conjecture_ was that equality always holds. In 1968 Ringel and Youngs [21] settled this long-standing conjecture in the affirmative by establishing an equivalent formulation in terms of the genus of the complete graph K_p with p vertices. To see the connection between these two problems, note that the dual of an imbedding of K_p on the surface of a sphere with γ handles is a map in which each of the p countries shares a border with each of the other countries, so that $\chi(S_\gamma) \geq p$.

Despite the intuitive appeal of the concept of the genus of a graph and its application to coloring problems such as those mentioned above, there are very few non-trivial families of graphs for which the genus is known. The main purpose of this thesis is to extend the number of these families for which the genus is precisely determined.

Definitions of terms which are basic to the study of genus problems in graph theory are given in Chapter 2, together with much of the notation that will be employed. In Chapter 3 a brief survey of known results on the genus of graphs is presented.

Several elementary results are presented in Chapter 4, particularly those pertaining to the genus of cartesian products of graphs. Upper and lower bounds are established for the genus of the cartesian product $G_1 x G_2$ in terms of the genera of G_1 and G_2. Asymptotic results are developed for the cases where G_1 and G_2 are both regular complete k-partite graphs.

In Chapter 5 some general results concerning the genus of cartesian products of bipartite graphs are presented. In Chapters 6 and 7 some of the main results of this thesis are developed. In Chapter 6 the genus of cartesian products of complete bipartite graphs is studied. The genera of cartesian products of cycles and paths are treated in Chapter 7. For several infinite families of graphs in each of these chapters, the genus is completely determined.

In Chapter 8 the genus of the complete tripartite graph $K_{p,q,r}$ is investigated. A lower bound is established, and it is shown that the lower bound is attained for some special cases involving infinite families of complete tripartite graphs.

CHAPTER 2

DEFINITIONS AND NOTATION

In this chapter we define some of the terms which are fundamental to the study of genus problems in graph theory. We also present some of the notation that will be employed throughout this thesis.

A _graph_ G is a set of _vertices_ V(G) and a set E(G) of unordered pairs of vertices called _edges_. If the elements of E(G) are ordered pairs, G is called a _directed graph_. For vertices a and b in V(G), (a,b) represents the corresponding edge (if present) in E(G). If G is a directed graph, [a,b] denotes an edge directed from vertex a to vertex b. (This notation conforms to that employed by Youngs [27].) An edge of the type (a,a) is called a _loop_. If any edge (a,b) appears more than once in E(G), G is said to be a _multigraph_. The graph G is called _finite_ if V(G) and E(G) are both finite. In this thesis, unless otherwise stated, all graphs are assumed to be finite, connected, undirected, and without loops or multiple edges.

The _degree_ of a vertex is the number of edges to which the vertex belongs. The graph H is called a _subgraph_ of the graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The _complement_ $\overline{G}$ of a graph G is a graph having $V(\overline{G}) = V(G)$ and exactly those edges which are missing in G. The _first Betti number_ (or _Betti number_) for a graph G is defined to be $\beta(G) = E - V + 1$, where E and V denote the number of edges and vertices of G respectively. This number is also frequently called the cyclomatic number of G.

Additional terms from graph theory may be found in Harary [9]. For topological terms, one may consult Dugundji [5], Massey [13], and Spanier [22].

A graph may be thought of as a collection of vertices, some pairs of which are related to one another. It is when we attempt to give a geometric realization to a graph that we encounter imbedding problems. Any finite graph can be realized in Euclidean 3-space, but the situation becomes more involved if we insist that the imbedding occur in a 2-manifold.

The graph G is said to be imbedded in the 2-manifold M if the geometric realization of G as a one-dimensional simplicial complex is homeomorphic to a subspace of M. Equivalently, if G is a graph where $V(G) = \{v_1,\ldots,v_n\}$ and $E(G) = \{e_1,\ldots,e_m\}$, an imbedding of G in M is a subspace G(M) of M such that

$$G(M) = \cup\, v_i(M) \cup \cup\, e_j(M),$$

where

(i) $v_1(M),\ldots,v_n(M)$ are n distinct points of M.

(ii) $e_1(M),\ldots,e_m(M)$ are m mutually disjoint open arcs in M.

(iii) $e_j(M) \cap v_i(M) = \emptyset$, $i = 1,\ldots,n$; $j = 1,\ldots,m$.

(iv) if $e_j = (v_{j_1}, v_{j_2})$ then the open arc $e_j(M)$ has $v_{j_1}(M)$ and $v_{j_2}(M)$ as end points; $k = 1,\ldots,m$.

In the above definition, an arc in M is a homeomorphic image of the closed unit interval; an open arc is an arc less its two end points, the images of 0 and 1. In the remainder of this thesis, we consider only compact, orientable 2-manifolds. We designate such a manifold by the term "surface". If a graph G is imbedded in a surface M of genus n but cannot be imbedded in any surface of lower genus, the imbedding is called minimal, and the genus of the graph is defined to be n; we write $\gamma(G) = n$. If $\gamma(G) = 0$, we say that G is planar.

Given an imbedding of a graph G in a surface M, each component of the complement of G in M is called a face of the imbedding. If a face is homeomorphic to the open unit disk, it is said to be a 2-cell. If every face in an imbedding is a 2-cell, we say that we have a 2-cell imbedding. The total number of faces for an imbedding is designated by F. For a 2-cell imbedding, F_i denotes the number of i-sided faces, and V_i denotes the number of vertices of degree i. The imbedding is maximal if no other imbedding of the same graph has more faces. The vertices and edges of G which belong to the boundary of a given face are said to belong to the face itself.

Given two graphs G_1 and G_2, the cartesian product $G_1 x G_2$ has for its vertex set

$$V(G_1 x G_2) = \{[u_1, u_2]: u_1 \in V(G_1),\ u_2 \in V(G_2)\}$$

and for its edge set

$$E(G_1 x G_2) = \{([u_1, u_2], [v_1, v_2]): u_1 = v_1 \text{ and } (u_2, v_2) \in E(G_2),$$
$$\text{or } u_2 = v_2 \text{ and } (u_1, v_1) \in E(G_1)\}.$$

It is often convenient to regard $G_1 x G_2$ as being constructed by replacing each vertex of G_2 with an entire copy of G_1, and then joining each corresponding pair of vertices in two copies of G_1 by an edge in exactly those cases for which the corresponding edge was present in G_2.

The graph with p vertices and all $\binom{p}{2}$ possible edges is called the _complete graph_ of order p and is denoted by K_p. The _complete bipartite graph_ $K_{p,q}$ is the complement of the disjoint union of K_p and K_q. Similarly, the _complete tripartite graph_ $K_{p,q,r}$ is the complement of the disjoint union of K_p, K_q, and K_r. A _cycle_ of length n, denoted by C_n, is a connected regular graph having n vertices of degree two. A _path_ of length n, denoted by P_n, is the graph C_n with one edge removed.

The least integer greater than or equal to x is written as $\{x\}$.

CHAPTER 3

A SURVEY OF KNOWN RESULTS

In this chapter the known results concerning imbedding problems in graph theory are surveyed, and the established formulae giving genera of graphs are listed.

The 2-manifolds in which a given graph may be imbedded are understood to be compact orientable 2-manifolds. Such manifolds have been completely classified [13].

Classification Theorem. Compact orientable 2-manifolds are homeomorphic to spheres or to spheres with handles.

For brevity, we refer to compact orientable 2-manifolds as surfaces. The genus γ of a surface may be regarded as the number of handles present. If the genus of a graph G is zero, the graph may be imbedded in the surface of a sphere, and is said to be planar. To see that such a graph may also be imbedded in the plane, take any point in the interior of any face of an imbedding of G in the sphere as the north pole and perform a stereographic projection of the sphere onto the plane. The image of G is a copy of G, now imbedded in the plane, with the image of the face containing the north pole forming the exterior region. In this manner graphs representing the five regular polyhedra, for example, may be pictured in the plane. The familiar Euler polyhedral formula has the following important generalization:

<u>Theorem of Euler</u>. Let F be the number of faces into which a surface of genus γ is separated by a 2-cell imbedding of a graph G, where V and E are the number of vertices and edges of G respectively. Then $F + V = E + 2(1 - \gamma)$.

It is important to remember that we are assuming that G is finite, connected, undirected, and has no loops or multiple edges. The Euler formula is particularly useful in obtaining lower bounds for the genera of graphs. Another useful result, due to Youngs [27], is stated next.

<u>Characterization Theorem</u>. An imbedding of a graph G is minimal if and only if it is a maximal 2-cell imbedding.

A major significance of this result is that it establishes the applicability of the Euler formula to any minimal imbedding. A trivial consequence of Youngs' theorem is that any imbedding of a graph in the plane must be a 2-cell imbedding. That not all imbeddings in surfaces of higher genus are 2-cell imbeddings is evident from Figure 3.1, which shows a planar graph G imbedded in the torus, or sphere with one handle.

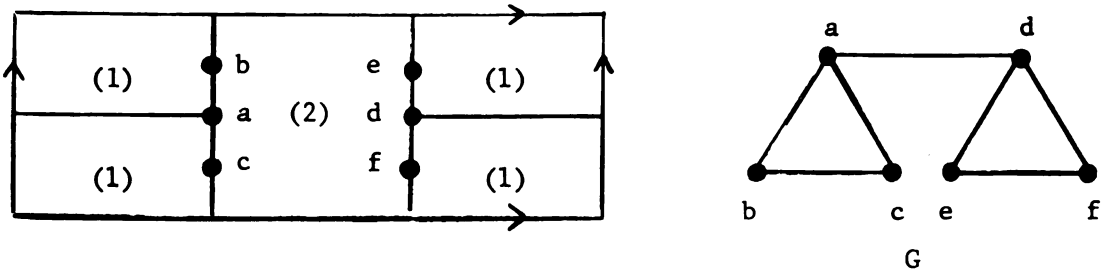

Figure 3.1 A planar graph imbedded in the torus.

This imbedding has two faces, one of which (face number (2)) is a cylinder, and hence is not a 2-cell. The imbedding is not minimal. Not only is the Euler theorem not applicable here, but indeed the formula does not hold.

As another consequence of Youngs' theorem, we have the following: if a graph G is imbedded in a surface M of genus h in such a manner that not all the faces are 2-cells, then $\gamma(G) \leq h - 1$. Using his characterization theorem, Youngs also shows that if G has a triangular imbedding (one in which every face has three sides; i.e. $F = F_3$), then this imbedding must be minimal.

The following theorem of Battle, Harary, Kodama, and Youngs [2] is also helpful in establishing lower bounds for the genera of certain graphs:

__Theorem__. If G is a connected graph having k blocks $B_1,\ldots,B_k$, then $\gamma(G) = \sum_{i=1}^{k} \gamma(B_i)$. Furthermore, in any minimal imbedding of G, $F_G = 1 - k + \sum_{i=1}^{k} F_{B_i}$, where F_G and F_{B_i} denote the number of faces for G and for B_i respectively.

An important corollary to the above theorem is that the genus of a disconnected graph is the sum of the genera of its components.

The following celebrated theorem of Kuratowski [12] completely characterizes planar graphs:

__Theorem__. A graph G is planar if and only if G does not contain a subgraph isomorphic, to within vertices of degree two, to either K_5 or $K_{3,3}$.

No such characterization of toroidal graphs (those of genus one) is as yet known. Only recently has it been shown (by Vollmerhaus [24]) that the class of exceptional graphs (corresponding to K_5 and $K_{3,3}$ in Kuratowski's theorem) is finite for any sphere with a prescribed number of handles.

It is convenient to represent imbeddings of a graph G in the following manner. Suppose a connected graph G has n vertices; we write $V(G) = \{1,\ldots,n\}$. Let $V(i) = \{k : (i,k) \in E(G)\}$. Let $p_i : V(i) \to V(i)$ be a cyclic permutation of V(i) of length $n_i = |V(i)|$, where $i = 1,\ldots,n$. The following theorem of Edmonds [7] (see also Youngs [27]) indicates the correspondence between 2-cell imbeddings and choices of the p_i.

<u>Theorem</u>. Each choice $(p_1,\ldots,p_n)$ determines a 2-cell imbedding G(M) of G in a compact orientable 2-manifold M, such that there is an orientation on M which induces a cyclic ordering of the edges (i,k) at i in which the immediate successor to (i,k) is $(i,p_i(k))$, $i = 1,\ldots,n$. In fact, given $(p_1,\ldots,p_n)$, there is an algorithm which produces the determined imbedding. Conversely, given a 2-cell imbedding G(M) in a compact orientable 2-manifold M with a given orientation, there is a corresponding $(p_1,\ldots,p_n)$ determining that imbedding.

Now, let $D = \{[a,b] : (a,b) \in E(G)\}$, and define $P: D \to D$ by: $P([a,b]) = [b,p_b(a)]$. Then P is a permutation on the set D of directed edges of G (where each edge of G is associated with two oppositely-directed directed edges), and the orbits under P determine the faces of the corresponding imbedding. This result is extremely useful, as will be seen throughout this thesis.

It then follows that the orientable genus of any connected graph G may be effectively computed, by selecting, from the $\prod_{i=1}^{n} (n_i-1)!$ possible permutations P, one which gives the maximal number of orbits, and hence determines the genus of the graph. Since a minimal imbedding must be a 2-cell imbedding, it corresponds to some P; then by Youngs' characterization theorem, F will be maximal for this minimal imbedding. The obvious difficulty arising is that of selecting a suitable permutation P from the vast number of possible permutations.

As an illustration of these concepts, we consider an imbedding of the complete graph K_5 in the surface S_2, as shown in Figure 3.2. Here,

$$V(K_5) = \{1,2,3,4,5\}$$

$$V(i) = \begin{cases} \{2,3,4,5\}, & i = 1 \\ \{1,3,4,5\}, & i = 2 \\ \{1,2,4,5\}, & i = 3 \\ \{1,2,3,5\}, & i = 4 \\ \{1,2,3,4\}, & i = 5 \end{cases}$$

$$n(i) = 4, \quad i = 1,2,3,4,5$$

The vertex permutations are seen to be:

p_1: (2,3,4,5)

p_2: (1,3,4,5)

p_3: (1,2,4,5)

p_4: (1,2,3,5)

p_5: (1,2,3,4)

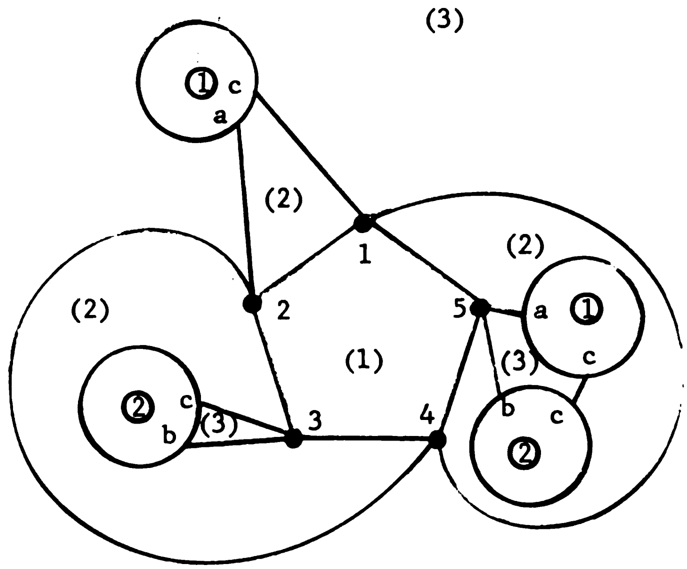

Figure 3.2 A 2-cell imbedding of K_5 in S_2.

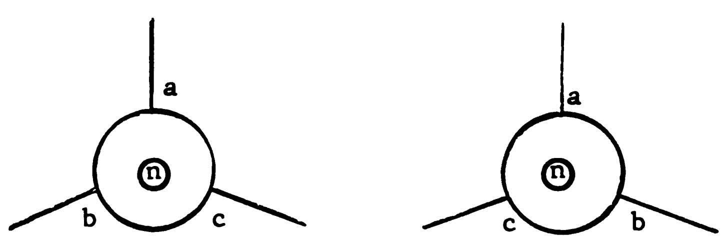

Figure 3.3 Non-intersecting edges on a handle.

This imbedding is a 2-cell imbedding, as guaranteed by Edmond's theorem, but it is not a minimal imbedding. In the imbedding, as shown in Figure 3.2, the two handles are only indicated at their intersections with the surface of the sphere, which may be thought of as curving around to meet itself in the exterior face, (3). Handle (1), for instance, meets the sphere in two places, as indicated. The "missing" portion of the handle extends outward from the plane of the page. In general, care must be taken so that a handle carrying three or more edges reverses the order of entrance of these edges onto the handle from one end to the other, as shown in Figure 3.3. This insures that the edges do not intersect on the handle.

Returning to the imbedding of K_5 given above, we find that the orbits under P determine the faces of the imbedding, as required. There are three faces, two being 5-sided and one 10-sided. Note that, in face (2), each vertex of K_5 is repeated and the boundary is not a simple closed curve.

(1) [1,2], [2,3], [3,4], [4,5], [5,1]

(2) [1,3], [3,2], [2,4], [4,3], [3,5], [5,4], [4,1], [1,5], [5,2], [2,1]

(3) [1,4], [4,2], [2,5], [5,3], [3,1]

In general, for an orbit of length k beginning with directed edge [a,b], we must have $P^k([a,b]) = [a,b]$, where $P^n = P(P^{n-1})$. For example, for the first orbit above, since $P^5([1,2]) = P([5,1]) = [1,2]$, we have an orbit of length 5, corresponding to a 5-sided face. Each edge of G appears as two directed edges in D, so that the sum of the orbit lengths is 2E. This corresponds to the trivial formula $2E = \sum_{i\geq 3} iF_i$. Since the above imbedding of K_5 is 2-cell, we may verify the Euler formula $F + V = E + 2(1 - \gamma)$, where $\gamma = 2$ is the genus of the surface, and not the genus of K_5. Note that if the surface had been given the opposite orientation, with the same permutations p_i, the imbedding would have been the mirror image of that pictured in Figure 3.2.

An orbit will henceforth be represented in the abridged form 1-2-3-4-5, instead of by the more cumbersome notation [1,2], [2,3], [3,4], [4,5], [5,1]. Note that when we write 1-2-3-4-5 for an orbit of length five, it is implied that $p_5(4) = 1$ and $p_1(5) = 2$.

For completeness, we next list the known genus formulae. The <u>n-cube</u> Q_n is defined as follows: let $Q_1 = K_2$, and for $n \geq 2$, recursively define $Q_n = Q_{n-1} x K_2$. In 1955, Ringel [16] showed that $\gamma(Q_n) = 1 + 2^{n-3}(n-4)$, for $n \geq 2$. Every face in a minimal imbedding for Q_n is a quadrilateral. This formula was also established independently by Beineke and Harary [4] in 1965.

In 1963, Auslander, Brown, and Youngs [1] produced a family of graphs G_n for which $\gamma(G_n) = n$. A graph G is said to be <u>n-irreducible</u> if $\gamma(G) = n$, but for any $x \in E(G)$, with G_x the graph obtained by removing edge x from G, $\gamma(G_x) < n$. In his doctoral thesis, Duke [6] showed that the graph G_n of Auslander, Brown, and Youngs is n-irreducible, for $n \geq 2$. Duke also conjectured that, for any minimal imbedding of a graph G, $F \geq 2\gamma(G) + 1$. This conjecture is valid for all genus formulae developed or listed in this thesis.

In 1965 Ringel [18] showed that $\gamma(K_{p,q}) = \left\{\frac{(p-2)(q-2)}{4}\right\}$, where all faces in the minimal imbeddings produced are quadrilateral with at most one exception. Ringel and Youngs [21] settled the Heawood map-coloring conjecture in the affirmative in 1968, by showing that $\gamma(K_p) = \left\{\frac{(p-3)(p-4)}{12}\right\}$. Here, all faces in a minimal imbedding are triangular, with at most five exceptions. The proof is quite complicated, employing various techniques such as the theory of current graphs (see Gustin [8]) and also vortex theory (see Youngs [25]), depending upon the residue of p modulo 12. (See also Mayer [14].)

Ringel and Youngs [20] have recently shown that $\gamma(K_{p,p,p}) = \frac{(p-1)(p-2)}{2}$, producing minimal imbeddings in which every face is a triangle, by the means of current-graph theory.

All of the above results are obtained for orientable surfaces. However, Youngs' characterization theorem also applies for compact non-orientable 2-manifolds, and Youngs [28 and 29] and Ringel [17] have shown that $\tilde{\gamma}(K_p) = \left\{\frac{(p-3)(p-4)}{6}\right\}$ for $p \geq 5$ and $\neq 7$, with $\tilde{\gamma}(K_7) = 3$, where $\tilde{\gamma}$ denotes the non-orientable genus of a graph. Ringel [19] has also shown that $\tilde{\gamma}(K_{p,q}) = \left\{\frac{(p-2)(q-2)}{2}\right\}$.

In what follows, only the orientable genus of a graph is considered. Since many of the graphs to be studied in this thesis are cartesian products of bipartite graphs, the following well-known theorems are frequently used.

Theorem. A graph is bipartite if and only if it contains no odd cycles.

Theorem. Let C be the set of all finite undirected graphs without loops or multiple edges, and for G_1 and G_2 belonging to C define the binary operation "x" on G_1 and G_2 to give the cartesian product $G_1 x G_2$. Then $\{C, x\}$ is a commutative semigroup with identity K_1.

It is also clear that if G_i has V_i vertices and E_i edges, $i = 1,2$, then $G = G_1 x G_2$ has $V = V_1 V_2$ and $E = V_1 E_2 + V_2 E_1$. Furthermore, if vertex v_i has degree d_i in G_i, $i = 1,2$, then vertex $[v_1, v_2]$ has degree $d_1 + d_2$ in $G_1 x G_2$.

In Theorem 5.1 of Chapter 5, we also prove that the cartesian product of two bipartite graphs is bipartite.

CHAPTER 4

ELEMENTARY RESULTS ON THE GENUS OF CARTESIAN PRODUCTS OF GRAPHS

Given two graphs G_1 and G_2, with genera $\gamma(G_1)$ and $\gamma(G_2)$ respectively, how is the genus of the cartesian product $G_1 x G_2$ related to $\gamma(G_1)$ and $\gamma(G_2)$? In general, this appears to be a difficult question to answer, one reason being that there are very few graphs whose genus is known precisely. In this chapter we compute the exact value of $\gamma(G_1 x G_2)$ for certain classes of cartesian products of graphs and lay the foundation for the more intricate computations of chapters 6 and 7. We also derive upper and lower bounds for the genus of $G_1 x G_2$, where G_1 and G_2 are arbitrary graphs. These bounds are sharpened for the special case of the genus of $K_m x K_n$. Finally, we obtain asymptotic results for the genera of cartesian productsof regular complete k-partite graphs.

One elementary result which is frequently used in establishing a lower bound for the genus of a graph is that the genus of a subgraph H of a graph G cannot exceed the genus of G. To see this, let G be minimally imbedded in a surface M of genus $\gamma(G)$. Remove from this imbedding those vertices and edges of G which are not in H. The graph H is now imbedded in the same surface M, so that $\gamma(H) \leq \gamma(G)$. The imbedding obtained for H need not be minimal, nor even a 2-cell imbedding. It also follows from this observation that if G is any graph of order p, then $\gamma(G) \leq \gamma(K_p)$.

Another result useful in determining a lower bound for the genus of a graph with minimum degree $\delta \geq 3$ is a variation on the

Euler formula $V + F = E + 2(1 - \gamma)$. If this equation is multiplied by 4, and the relations $F = \sum_{i\geq 3} F_i$, $V = \sum_{i\geq 3} V_i$, and $2E = \sum_{i\geq 3} iF_i = \sum_{i\geq 3} iV_i$ are used, we readily obtain

$$(1) \qquad \gamma(G) = 1 + \frac{1}{8} \sum_{i\geq 3} (i-4)(F_i + V_i).$$

Multiplying the Euler formula by 3 and using the same relations, we obtain

$$(2) \qquad \gamma(G) = 1 - \frac{E}{6} + \frac{1}{6} \sum_{i\geq 4} (i-3)(F_i + V_i).$$

This second variation is also useful in determining a lower bound for $\gamma(G)$, particularly if G contains a large number of 3-cycles.

Two familiar classes of graphs are the cycles C_m, $m \geq 3$, and the paths P_m, $m \geq 2$. It is relatively easy to compute the genus of the cartesian product of two cycles or of two paths, as shown in Theorems 4.1 and 4.2.

Theorem 4.1. The genus of $C_m x C_n$ is given by $\gamma(C_m x C_n) = 1$, for $m \geq 3$, $n \geq 3$.

Proof: Since $C_m x C_n$ can be imbedded in the torus, as Figure 4.1 illustrates for the case $C_4 x C_6$, then $\gamma(C_m x C_n) \leq 1$. (The numbers (1) and (2) appearing in Figure 4.1 are referred to in the proof of Theorem 7.1.) We now claim that $C_3 x C_3$ is not planar. It would suffice to find a subgraph of $C_3 x C_3$ isomorphic to within vertices of degree two to the Kuratowski graph $K_{3,3}$, but the following alternative approach is offered. In the graph $C_3 x C_3$, the number of 3-cycles is given by $T_3 = \frac{(9)(2)}{3} = 6$, since each vertex is in exactly two 3-cycles, and each 3-cycle is counted three times. It follows that $F_3 \leq 6$ in any imbedding of $C_3 x C_3$. If $\gamma(C_3 x C_3) = 0$, then from the Euler-type formula (1) developed above, since $V = V_4$, we have $0 = 1 + \frac{1}{8} \sum_{i \geq 3} (i-4)F_i$, so that $F_3 \geq 8$, a contradiction. Hence $\gamma(C_3 x C_3) = 1$. It follows that $C_3 x C_3$ contains a Kuratowski subgraph, and therefore so does $C_m x C_n$, so that $\gamma(C_m x C_n) \geq 1$. Equality then follows.

<u>Theorem 4.2</u>. The genus of $P_m x P_n$ is given by $\gamma(P_m x P_n) = 0$, for $m \geq 2$, $n \geq 2$.

<u>Proof</u>: Figure 4.2 illustrates the case m = 8, n = 6, and it is evident that in general $P_m x P_n$ is planar. (The numbers (1) and (2) appearing in Figure 4.2 are referred to in the proof of Theorem 7.4.)

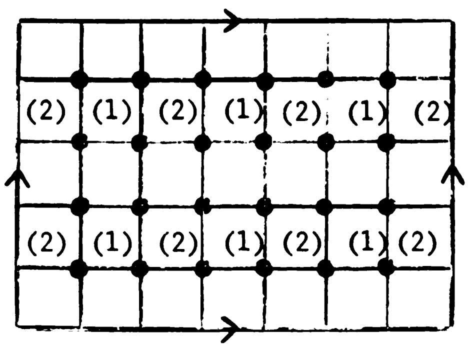

Figure 4.1 An imbedding of $C_4 x C_6$ in the torus.

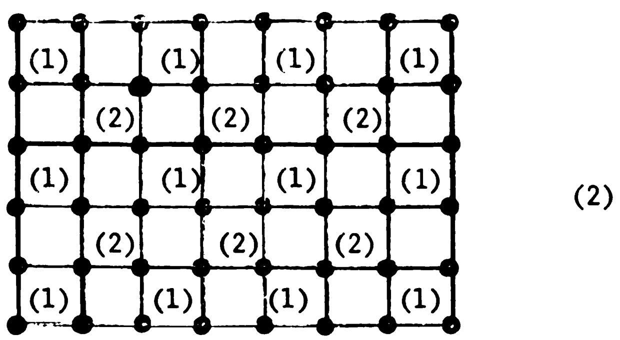

Figure 4.2 An imbedding of $P_6 x P_8$ in the plane.

We will now establish a more general result. The technique embodied in the following theorem is employed to great advantage throughout this thesis.

Theorem 4.3. Let a graph G be given, together with a particular minimal imbedding of G having a distinguished face containing no repeated vertices. Let G^* be the graph consisting of two disjoint copies of G, with corresponding vertices in the two copies of the distinguished face also adjacent. Then $\gamma(G^*) = 2\gamma(G)$.

Proof: First note that G^* is a graph, since the distinguished face has no repeated vertices, and no multiple edges have been introduced. By the theorem of Battle, Harary, Kodama, and Youngs, the genus of the two disjoint copies of G is given by $\gamma(2G) = 2\gamma(G)$. Since 2G is a subgraph of G^*, then $2\gamma(G) \leq \gamma(G^*)$. Next consider one copy of G with its imbedding specified by the permutations p_i of Edmonds' theorem and a given orientation. Imbed the second copy of G with the same imbedding, as specified by the same permutations p_i, in a second copy of the same surface, but with the reverse orientation. (The second imbedding may be thought of intuitively as a mirror image of the first.) Cut an open disk from the interiors of both copies of the distinguished face, and attach the two surfaces by means of a hollow tube, one end of which is sewn onto the boundary of each disk. The edges needed to complete the graph G^* may now be added along this tube, as indicated in Figure 4.3. The addition of this tube results in a surface of genus $2\gamma(G)$. Although this is intuitively clear, it may also be seen using the Euler formula. Suppose that for G we have $F + V = E + 2(1 - \gamma)$. Then for G^*, which is now 2-cell imbedded, $V^* = 2V$, $E^* = 2E + m$, where the distinguished face has m sides, and $F^* = 2F + m - 2$. It follows that $2F + m - 2 + 2V = 2E + m + 2 - 2\gamma^*$, where γ^* is the genus of the new surface, upon which G^* is imbedded. That is,

$$2F + 2V = 2E + 4 - 2\gamma^* = 2E + 4 - 4\gamma,$$

so that $\gamma^* = 2\gamma$. Hence $\gamma(G^*) \leq 2\gamma(G)$, and so $\gamma(G^*) = 2\gamma(G)$.

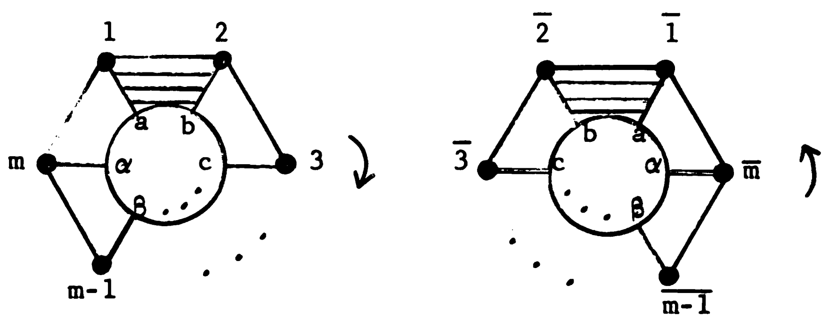

Figure 4.3 A tube carrying m non-intersecting edges.

Reversing the orientation for the second imbedding allows the attached tube to carry the required m edges without their intersecting each other. We observe also that each new face is a quadrilateral. One method for establishing the genus of a graph is to find a lower bound for that genus, and then to construct an imbedding of the graph for which the lower bound is attained. This method is discussed in detail in Chapter 5. Many of the constructions to follow in this thesis employ generalizations of the construction of Theorem 4.3.

Before stating an immediate corollary to Theorem 4.3, we introduce the following definition.

Definition 4.1. A graph G is said to be outer-imbeddable if it has a minimal imbedding with a distinguished face in which every vertex of the graph appears exactly once.

Corollary 4.4. If G is outer-imbeddable, then $\gamma(K_2 x G) = 2\gamma(G)$.

For examples of outer-imbeddable graphs, we have all cycles C_n in the plane, K_5, K_6, and $K_{3,3}$ in the torus, and $K_{5,5}$ in the sphere with three handles.

The Euler formula may also be employed to establish the following two theorems, which we will find most useful.

Theorem 4.5. Let the graphs G_i be minimally imbedded in surfaces M_i respectively, i = 1,2. Let the new surface M be formed by adding n tubes t_i $(i = 1,\dots,n)$ between n distinct faces of M_1 and n distinct faces of M_2 respectively (where $n \leq \min(F_1,F_2)$), with tube t_i carrying $e_i > 0$ edges. Then $\gamma(M) = \gamma(M_1) + \gamma(M_2) + (n-1)$.

Proof: Since the graphs G_i are minimally imbedded, the Euler formula applies, and we have $2\gamma(M_i) = 2 - V_i + E_i - F_i$, i = 1,2, where, in this context, V_i and F_i give the number of vertices and faces respectively for the minimal imbedding of graph G_i. But the new imbedding in M is also a 2-cell imbedding; hence

$$\begin{aligned} 2\gamma(M) &= 2 - V_M + E_M - F_M \\ &= 2 - (V_1+V_2) + (E_1+E_2 + \sum_{i=1}^{n} e_i) - (F_1+F_2-2n + \sum_{i=1}^{n} e_i) \\ &= 2\gamma(M_1) + 2\gamma(M_2) - 2 + 2n. \end{aligned}$$

Therefore, $\gamma(M) = \gamma(M_1) + \gamma(M_2) + (n-1)$.

Corollary 4.6. Let G be the graph representing the 1-skeleton of the surface M described above. Then $\gamma(G) \leq \gamma(G_1) + \gamma(G_2) + (n-1)$.

Proof: $\gamma(G) \leq \gamma(M) = \gamma(M_1) + \gamma(M_2) + (n-1)$

$$= \gamma(G_1) + \gamma(G_2) + (n-1).$$

Theorem 4.7. Let the graphs G_i be minimally imbedded in surfaces M_i respectively, $i = 1,\dots,n$. Let H be a graph of order n, such that vertex i is associated with surface M_i, and having m edges. Let m tubes t_i, $i = 1,\dots,m$ be attached between the surfaces M_i in correspondence with the edges of H, with no two tubes attached within the same face, where we are assuming that F_i is at least as large as the degree of vertex i. Let tube t_i carry $e_i > 0$ edges, so as to form an imbedding of a new graph G in the new surface M. Then $\gamma(G) \leq \gamma(M) = \sum_{i=1}^{n} \gamma(G_i) + \beta(H)$, where $\beta(H)$ is the Betti number of the graph H.

Proof: We have $V_i + F_i = E_i + 2(1 - \gamma(G_i))$, $i = 1,\dots,n$; and $V + F = E + 2(1 - \gamma(M))$. But $V = \sum_{i=1}^{n} V_i$, $E = \sum_{i=1}^{n} E_i + \sum_{i=1}^{m} e_i$, and $F = \sum_{i=1}^{n} F_i - 2m + \sum_{i=1}^{m} e_i$. Therefore,

$$\sum_{i=1}^{n} V_i + \sum_{i=1}^{n} F_i - 2m + \sum_{i=1}^{m} e_i = \sum_{i=1}^{n} E_i + \sum_{i=1}^{m} e_i + 2(1 - \gamma(M)).$$

It follows that

$$\gamma(M) = \sum_{i=1}^{n} \gamma(G_i) + (m - n + 1) = \sum_{i=1}^{n} \gamma(G_i) + \beta(H).$$

It is clear that $\gamma(G) \leq \gamma(M)$.

In Theorem 4.7, corresponding to each edge in H two surfaces were joined by a single tube. If each "join" had been made instead by several tubes running between the appropriate two surfaces, the total effect upon the genus of the resulting surface could be computed by combining Theorems 4.5 and 4.7 in the obvious manner. In this case, we say that each "join" is made by a "bundle" of tubes.

We next consider the problem mentioned earlier of estimating $\gamma(G_1 x G_2)$ in terms of $\gamma(G_1)$ and $\gamma(G_2)$. A graph is said to be 1-factorable if it has a spanning subgraph which is regular of degree one. Given the graphs G_i, i = 1,2, with V_i vertices, E_i edges, genera $\gamma(G_i)$, and 1-factorable subgraphs H_i maximal with respect to order, of order $2h_i$ respectively, let

$$m_1 = V_1\gamma(G_2) + \gamma(G_1)$$
$$m_2 = V_2\gamma(G_1) + \gamma(G_2)$$
$$M_1 = V_1(\gamma(G_2) - 1) + E_1(V_2 - h_2) + 1$$
and
$$M_2 = V_2(\gamma(G_1) - 1) + E_2(V_1 - h_1) + 1.$$

We can now state the following theorem:

Theorem 4.8. The genus of the cartesian product $G_1 x G_2$ is bounded by: Max $(m_1, m_2) \le \gamma(G_1 x G_2) \le \min (M_1, M_2)$.

Proof: (i) Consider V_1 disjoint copies of G_2, each with vertex set $\{1, \ldots, V_2\}$. Add edges connecting the V_1 copies of vertex 1 so as to form a copy of G_1. The resulting graph H is a subgraph of $G_1 x G_2$, and the block decomposition of H may be partitioned into blocks of G_1 and V_1 copies of each block of G_2. The theorem of Battle, Harary, Kodama, and Youngs then applies, to give $\gamma(H) = V_1\gamma(G_2) + \gamma(G_1) = m_1$. Clearly $\gamma(H) \le \gamma(G_1 x G_2)$, as H is a subgraph of $G_1 x G_2$. Interchanging the roles of G_1 and G_2, recalling that

$G_1 x G_2 = G_2 x G_1$, we see also that $m_2 \leq \gamma(G_1 x G_2)$, and the left-hand inequality of the theorem has been verified.

(ii) An imbedding (probably not minimal) of $G_1 x G_2$ may always be constructed as follows: replace each vertex of G_1 with a copy of G_2, minimally imbedded, and then make the required joins between copies. Consider each join as a bundle of tubes, and count the contribution of these bundles to the genus. By Theorem 4.7, this is just $\beta(G_1) = E_1 - V_1 + 1$. Now we must count the contribution to the genus of each bundle individually. Assuming all V_1 copies of G_2 are minimally imbedded in surfaces of like orientation, we can run no more than two edges over a given tube attached to corresponding faces in two copies, so that these edges do not intersect. If H_2 is a 1-factorable subgraph of G_2, maximal with respect to order, of order $2h_2$, h_2 tubes will suffice for these $2h_2$ vertices. We can then join the remaining $(V_2 - 2h_2)$ vertices of G_2 to their counterparts in a second copy over $(V_2 - 2h_2)$ additional tubes, each carrying one edge. By Theorem 4.5, the contribution to the genus of this bundle is $h_2 + V_2 - 2h_2 - 1 = V_2 - h_2 - 1$. But there are E_1 such bundles in all. Hence,

$$\begin{aligned}\gamma(G_1 x G_2) &\leq V_1 \gamma(G_2) + E_1 - V_1 + 1 + E_1(V_2 - h_2 - 1) \\ &= V_1(\gamma(G_2) - 1) + E_1(V_2 - h_2) + 1 \\ &= M_1.\end{aligned}$$

Interchanging the roles of G_1 and G_2 again, we see that $\gamma(G_1 x G_2) \leq M_2$, and the right-hand inequality of the theorem holds also.

The bounds in Theorem 4.8 are in general not sharp. The above proof assumes very little about the structure of the graph G_1. For a particular G_1, it may be possible to sharpen the upper bound considerably, by orienting the surfaces containing copies of G_2 more expeditiously, so that some or all of the tubes added in the construction can be used to carry more than two edges.

We now turn our attention to the graphs $K_m x K_n$, and attempt to sharpen the upper bound for $\gamma(K_m x K_n)$ as stated in Theorem 4.8. In the approach that follows, we employ the concepts of line graph and clique graph. The <u>line graph</u> L(G) of a given graph G has as its vertex set the edges of G, and two vertices in L(G) are adjacent if and only if the corresponding edges in G are adjacent. The <u>clique graph</u> C(G) has as its vertex set the cliques of G, where a <u>clique</u> is a complete subgraph of G contained in no larger complete subgraph; and two vertices of C(G) are adjacent if and only if the corresponding cliques in G intersect. It is well-known that $L(K_{m,n}) = K_m x K_n$ [15], and this indicates a connection with the question at hand. The following theorem gives a relationship between clique graphs and line graphs, in certain cases.

<u>Theorem 4.9</u>. Let G be a graph with no triangles or vertices of degree less than two. Then the graph C(L(G)) is isomorphic to G.

<u>Proof</u>: Let $u \in V(G)$, with degree $d(u) = n \geq 2$. Then in L(G) there is a complete subgraph K_n^u associated with u. We claim that K_n^u is a clique in L(G). For suppose to the contrary that there is a vertex x in L(G) adjacent to all n vertices of K_n^u, giving (n+1) distinct and mutually adjacent vertices in L(G). Then in G the edge x has an end vertex in common with each of the n edges

associated with the vertices of K_n^u. Since G has no triangles, and $n \geq 2$, the edge x must have u as one end vertex. Now, the other end vertex of x is not one of the first n vertices, since these are (n+1) distinct vertices in L(G). But then $d(u) \geq n + 1$, a contradiction. Hence, K_n^u is a clique in L(G).

Now, in C(L(G)), K_n^u is replaced by a vertex, say v_u. We are ready to set up the isomorphism required by the theorem; define $\theta\colon V(G) \to V(C(L(G)))$ by $\theta(u) = v_u$. The remarks above show that θ is well-defined.

That θ is onto follows from the observation that $\{K_{d(u)}^u : u \in V(G)\}$ includes every clique of L(G). To see this, note that: (i) every vertex in L(G) is contained in some (in fact, exactly two) $K_{d(u)}^u$. (ii) Every edge in L(G) is in exactly one $K_{d(u)}^u$. (iii) No triangle in L(G) can arise from a triangle in G, since G has no triangles; hence every triangle in L(G) arises from a $K_{1,3}$ configuration in G and hence is in exactly one $K_{d(u)}^u$. (iv) Any n-clique K_n in L(G), $n \geq 4$, contains a triangle, and hence, by (iii), this triangle is in some $K_{d(u)}^u$; but a fourth vertex in K_n is adjacent to all three vertices in the triangle, and hence the edge in G this vertex represents has u as an end vertex; that is, this fourth vertex is in $K_{d(u)}^u$. It follows that K_n is a subgraph of $K_{d(u)}^u$ and hence must equal $K_{d(u)}^u$, since both are cliques. So, θ is an onto mapping.

It now follows that θ is one-to-one, since the sets V(G), $\{K_{d(u)}^u : u \in V(G)\}$, the set of all cliques of L(G), and V(C(L(G))) all have the same cardinality. We have only to show that adjacency is preserved. Suppose $(u,w) \in E(G)$; then $(u,w) \in V(L(G))$ and hence $(u,w) \in K_{d(u)}^u \cap K_{d(w)}^w$. It follows that $(v_u, v_w) \in E(C(L(G)))$.

Conversely, let $(v_u, v_w) \in E(C(L(G)))$, so that in L(G) there is a vertex $x \in K^u_{d(u)} \cap K^w_{d(w)}$. Then x is an edge in G with one end vertex u and the other end vertex w; that is, $(u,w) \in E(G)$. This completes the proof.

<u>Corollary 4.10</u>. Let G be a graph with no triangles or vertices of degree less than two. Then C(G) = L(G) and C[C(G)] is isomorphic to G.

<u>Proof</u>: If G has no triangles or isolated vertices, then the cliques of G are precisely the edges of G. Hence C(G) = L(G). Now, since C(L(G)) is isomorphic to G, we have that C[C(G)] is isomorphic to G.

<u>Corollary 4.11</u>. If m and n are ≥ 2, then $C(K_m x K_n)$ is isomorphic to $K_{m,n}$.

<u>Proof</u>: $C(K_m x K_n) = C(L(K_{m,n}))$, which is isomorphic to $K_{m,n}$, since $K_{m,n}$ has no odd cycles, and in particular no triangles.

We can now improve the upper bound of Theorem 4.8, for the graphs $K_m x K_n$. Recall that the Betti number of the graph $K_{m,n}$ is given by $\beta(K_{m,n}) = mn - (m + n) + 1$.

<u>Theorem 4.12</u>. The genus of $K_m x K_n$ is bounded above by:

$$\gamma(K_m x K_n) \leq n\gamma(K_m) + m\gamma(K_n) + \beta(K_{m,n}).$$

<u>Proof</u>: Every vertex in $K_m x K_n$ belongs to exactly two cliques. The graph $K_m x K_n$ has m n-cliques and n m-cliques, giving a total of 2mn vertices, each of which is counted twice. Since $C(K_m x K_n)$ is isomorphic to $K_{m,n}$, the (m + n) cliques of $K_m x K_n$ correspond to the

vertices of $K_{m,n}$. This viewpoint motivates much of what follows. Note also that each edge of $K_m x K_n$ is in exactly one clique.

We now form the graph G* from $K_m x K_n$ by splitting each vertex of $K_m x K_n$ into two new vertices. One of these new vertices has exactly those adjacencies in G* that it had within one of the two cliques it belonged to in $K_m x K_n$; the second corresponding new vertex has those adjacencies that it had within the second clique it belonged to in $K_m x K_n$. The graph G* has the same number of edges as $K_m x K_n$, but twice as many vertices. We write $V(G^*) = \{v_{ij}: i = 1,\ldots,mn; j = 1,2\}$. Also, G* consists exactly of m n-cliques and n m-cliques, now all mutually disjoint. Hence, by the theorem of Battle, Harary, Kodama, and Youngs, $\gamma(G^*) = n\gamma(K_m) + m\gamma(K_n)$. Consider G* to be imbedded, not on one surface of genus $\gamma(G^*)$, but on (n+m) surfaces, in the obvious manner.

We now regain $K_m x K_n$, imbedded in one surface constructed from these (n+m) surfaces. Instead of attaching these surfaces by tubes as in the proof of Theorem 4.8, we extend the method of Battle, Harary, Kodama, and Youngs. For each i, i = 1,...,mn, we identify v_{i1} with v_{i2} as follows. Corresponding to vertex v_i in $K_m x K_n$, we now have vertices v_{ij} contained in cliques G_{ij} minimally imbedded in surfaces M_{ij}, j = 1,2 respectively. Take an open 2-cell C_{ij} in M_{ij} with simple closed boundary curve J_{ij} such that $(C_{ij} \cup J_{ij}) \cap G_{ij} = v_{ij}$. The situation is pictured in Figure 4.4.

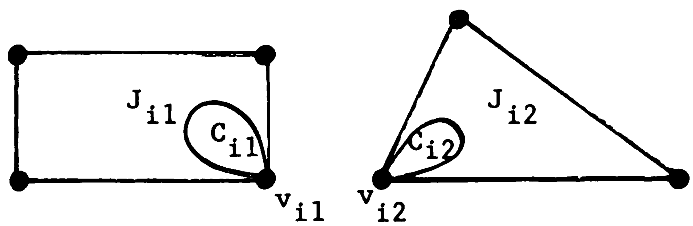

Figure 4.4 Identifying two vertices.

Identify J_{i1} of $(M_{i1} - C_{i1})$ with J_{i2} of $(M_{i2} - C_{i2})$ so that v_{i1} identifies with v_{i2}. This gives a surface $M_i = (M_{i1}-C_{i1}) \cup (M_{i2}-C_{i2})$, with a 2-cell imbedding of cliques G_{i1} and G_{i2}, sharing vertex v_i. We make the required mn such identifications, to regain the graph $K_m x K_n$ from G*. Since each identification corresponds to adding a required edge in $C(K_m x K_n)$, which is isomorphic to $K_{m,n}$ (where we are regarding the (m+n) surfaces we started with as the vertices of $K_{m,n}$), this complete process has the effect of increasing the genus of G* by exactly $\beta(K_{m,n})$. Hence we have $K_m x K_n$ imbedded in a surface of genus $n\gamma(K_m) + m\gamma(K_n) + \beta(K_{m,n})$, and the inequality of the theorem is established.

The upper bound still may not be sharp, but it can be used to give the following asymptotic result:

<u>Theorem 4.13</u>. For $n \geq m^{1+\epsilon}$, where $\epsilon > 0$, and n tending to infinity, $\gamma(K_m x K_n) \sim m\gamma(K_n)$.

<u>Proof</u>: Combining Theorems 4.8 and 4.12, we have:

$$\max\,(m\gamma(K_n) + \gamma(K_m),\ n\gamma(K_m) + \gamma(K_n)) \leq \gamma(K_m x K_n) \leq n\gamma(K_m) + m\gamma(K_n) + \beta(K_{m,n}).$$

Divide through by $m\gamma(K_n)$, and recall that $\gamma(K_n) = \left\{\frac{(n-3)(n-4)}{12}\right\}$.

Taking the limit as $n \to \infty$, we have $\max(1, \frac{1}{m}) \leq \lim_{n\to\infty} \frac{\gamma(K_m x K_n)}{m\gamma(K_n)} \leq 1$,

so that $\lim_{n\to\infty} \frac{\gamma(K_m x K_n)}{m\gamma(K_n)} = 1$.

In the proof of Theorem 4.12, it was not required that the construction employed give rise to a 2-cell imbedding of $K_m x K_n$. If the imbedding is 2-cell, the genus of the surface constructed can also be computed by using the Euler formula, as in the proofs of Theorems 4.5 and 4.7. The construction fails to give a 2-cell imbedding, however, if any cycle of identifications (in $K_{m,n}$) is completed entirely within the same face wherein that cycle began. In this event, at the stage at which the cycle is closed, the new face formed is a cylinder. The upper bound of Theorem 4.12 can be further reduced by making the mn identifications so as to insure the maximum number of cylinders. This is equivalent to determining the maximum number of edge-disjoint 4-cycles in $K_{m,n}$, which is $F(m,n) = \min([\frac{m}{2}[\frac{n}{2}]], [\frac{n}{2}[\frac{m}{2}]])$. Instead of closing a given 4-cycle by the usual identification procedure, since the first and last vertex of the cycle are in the same face, they can be identified within that face without affecting the genus, as indicated in Figure 4.5.

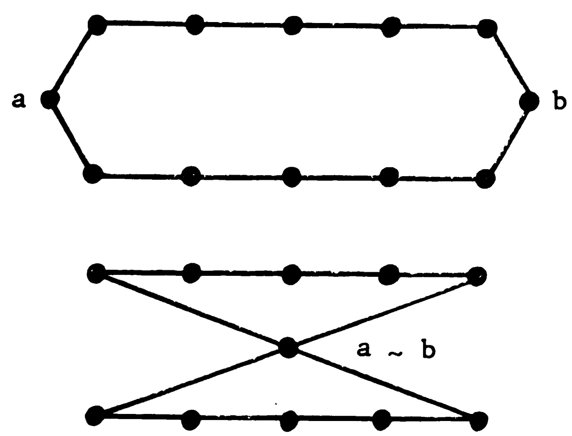

Figure 4.5 Two vertices identified within a face.

In this manner, the upper bound of Theorem 4.12 can be reduced by F(m,n):

Theorem 4.14. The genus of $K_m x K_n$ is bounded above by:

$$\gamma(K_m x K_n) \le m\gamma(K_n) + n\gamma(K_m) + \beta(K_{m,n}) - F(m,n).$$

The technique of Theorem 4.12 can be applied to the general cartesian product $G_1 x G_2$. We obtain the following result.

Theorem 4.15. Let the graph G_i have V_i vertices, i = 1,2. Then $\gamma(G_1 x G_2) \le V_1\gamma(G_2) + V_2\gamma(G_1) + \beta(K_{V_1,V_2})$.

This upper bound may be sharper than that of Theorem 4.8, as in the case of $K_5 x K_5$; or it may not be as sharp, as in the case of $K_{3,3} x K_{3,3}$. We can use the upper bound of Theorem 4.15 to obtain an asymptotic expression for the genus of the cartesian product of two regular complete k-partite graphs. Denote the regular complete k-partite graph with ks vertices by $K_{k(s)}$, with $K_{s(1)} = K_s$ also denoted by $K_{1(s)}$.

Theorem 4.16. For $n \ge m^{1+\epsilon}$, where $\epsilon > 0$, and n tending to infinity, $\gamma(K_{k(m)} x K_{k(n)}) \sim km\gamma(K_{k(n)})$, for all natural numbers k.

Proof: The case k = 1 has been established by Theorem 4.13. For k ≥ 2, we combine Theorems 4.8 and 4.15 to obtain

$$\max(km\gamma(K_{k(n)}) + \gamma(K_{k(m)}),\ kn\gamma(K_{k(m)}) + \gamma(K_{k(n)}))$$
$$\le \gamma(K_{k(m)} x K_{k(n)}) \le km\gamma(K_{k(n)}) + kn\gamma(K_{k(m)})$$
$$+ (k^2mn - k(m+n) + 1).$$

We now divide through by $km\gamma(K_{k(n)})$ and take the limit as $n \to \infty$. We claim that $\lim_{n\to\infty} \frac{n\gamma(K_{k(m)})}{m\gamma(K_{k(n)})} = 0$. To see this, note first that, by the Euler-type formula (2) developed at the beginning of the chapter,

$$\gamma(K_{k(n)}) \geq \frac{k(k-1)n^2 - 6kn + 12}{12}$$

On the other hand,

$$\begin{aligned}\gamma(K_{k(m)}) &\leq \gamma(K_{km}) \\ &= \left\{\frac{(km-3)(km-4)}{12}\right\} \\ &< \frac{k^2m^2 - 7km + 24}{12} .\end{aligned}$$

It now follows that, for $n \geq m^{1+\varepsilon}$, with $\varepsilon > 0$,

$$\begin{aligned}0 &\leq \lim_{n\to\infty} \frac{n\gamma(K_{k(m)})}{m\gamma(K_{k(n)})} \\ &\leq \lim_{n\to\infty} \frac{n(k^2m^2 - 7km + 24)}{m(k(k-1)n^2 - 6kn + 12)} = 0.\end{aligned}$$

The other limits involved are easily evaluated, and we have

$$\max\ (1, \frac{1}{km}) \leq \lim_{n\to\infty} \frac{\gamma(K_{k(m)} \times K_{k(n)})}{km\ \gamma(K_{k(n)})} \leq 1,$$

so that $\lim_{n\to\infty} \dfrac{\gamma(K_{k(m)} \times K_{k(n)})}{km\ \gamma(K_{k(n)})} = 1.$

CHAPTER 5

THE GENUS OF CARTESIAN PRODUCTS OF BIPARTITE GRAPHS

One method for obtaining the genus of a graph is to calculate a lower bound for the genus and then to construct an imbedding for which the lower bound is actually attained. The structure of the cartesian product $G_1 x G_2$ suggests a construction in which the graph G_1 is minimally imbedded in V_2 disjoint surfaces, and then these surfaces are joined together as prescribed by the graph G_2. For each edge in G_2, the two corresponding surfaces are joined by a bundle of tubes which carry edges between all V_1 corresponding vertices in these two copies of G_1. The challenge is to make these joins using the fewest possible tubes. As noted following Theorem 4.3, an efficient use of a tube results in each new face intersecting the tube being a quadrilateral. This suggests that we should seek quadrilateral imbeddings of a graph; that is, imbeddings in which every face is a quadrilateral.

In the proof of Theorem 4.3, we gave the two surfaces we were joining by a tube opposite orientations. It would be convenient if every time we wished to join two surfaces with a bundle of tubes, the two surfaces had opposite orientations. Then every tube attached to two copies of the same face of the same minimal imbedding of G_1 in the two surfaces can carry an edge for every corresponding pair of vertices in the two faces. This construction would be an improvement on that employed in Theorem 4.8, where no tube carried more than two edges. This construction is possible provided that

G_2 has no odd cycles; that is, provided that G_2 is bipartite. Then the V_2 surfaces with their minimal imbeddings of G_1 can be oriented in accordance with the vertex set partition of $V(G_2)$. It is then clear that any join which must be made, corresponding to an edge of G_2, will be between surfaces of opposite orientation, as desired. If we further require G_1 to be bipartite, then $G_1 x G_2$ is bipartite, as proved in Theorem 5.1 below, and a quadrilateral imbedding will be minimal. To establish this, we prove the following two theorems.

Theorem 5.1. The cartesian product of two bipartite graphs is bipartite.

Proof: Equivalently, we show that if neither G_1 nor G_2 contains an odd cycle, then $G_1 x G_2$ cannot contain an odd cycle. So, consider a cycle of length m in $G_1 x G_2$; then h edges of the cycle are taken from one or more copies of G_1, $0 \leq h \leq m$; and (m-h) edges join corresponding vertices of two copies of G_1, corresponding to edges in G_2. Superimpose the V_2 copies of G_1 onto one copy of the graph G_1. The (m-h) edges above disappear, and the h edges now form a cycle in G_1. Since G_1 has no odd cycles, h is even. Similarly, by superimposing the V_1 copies of G_2 onto one copy of the graph G_2, we see that (m-h) is even. Hence m is even.

Let the graphs G_i, i = 1,2,... be bipartite, and define the graph H_n as follows: $H_1 = G_1$, and recursively $H_n = H_{n-1} x G_n$. Then $H_n = G_1 x G_2 x \ldots x G_n$, and the following corollary of Theorem 5.1 is established by a routine application of mathematical induction:

Corollary 5.2. The graph H_n is bipartite.

Theorem 5.3. A quadrilateral imbedding of a bipartite graph is a minimal imbedding.

Proof: A bipartite graph has no odd cycles, and in particular no 3-cycles. Hence in any imbedding, $F_3 = 0$. Recall that $2E = \sum_{i\geq 3} iF_i$, with $F = \sum_{i\geq 3} F_i$, and that a 2-cell imbedding of a graph is minimal when F is maximal, by the characterization theorem of Youngs. Then if $F = F_4$ for a bipartite graph, the imbedding must be minimal.

The next theorem has frequent applications in Chapters 6 and 7.

Theorem 5.4. If a bipartite graph G with V vertices and E edges has a quadrilateral imbedding, then $\gamma(G) = 1 + \frac{E}{4} - \frac{V}{2}$.

Proof: By Theorem 5.3, the imbedding is minimal, and hence is a 2-cell imbedding. The Euler-type formula

$$\gamma(G) = 1 + \frac{1}{8}\sum_{i\geq 3}(i-4)(F_i+V_i) \qquad (1)$$

discussed in Chapter 4 applies. Noting that $F_i = 0$ for $i \neq 4$, with $2E = \sum_{i\geq 3} iV_i$ and $V = \sum_{i\geq 3} V_i$, the result follows immediately.

To construct quadrilateral imbeddings for $G_1 \times G_2$, where both G_1 and G_2 are bipartite, we will follow the procedure outlined at the beginning of the chapter. The contributions to the genus of the surface which arises from this procedure are of three types: (i) V_2 surfaces of genus $\gamma(G_1)$ each, with which we start our construction; (ii) the contribution of the tubes within each bundle; and (iii) the contribution of the bundles taken collectively. The first contribution is known once the genus of G_1 is known. The second and third contributions can be computed using Theorems 4.5 and 4.7 respectively.

It is intuitively clear that all the tubes required by this construction may be added so that they do not intersect each other. This fact is also evident from the following theorem, which may be established by standard topological arguments; the proof is omitted. By a generalized 2-manifold is meant the union of a finite collection of compact orientable 2-manifolds in Euclidean 3-space, each of which is exterior to any other.

Theorem 5.5. If M is a generalized 2-manifold, with C_1 and C_2 two disjoint simple closed curves on M such that C_1 is homotopic to zero on $M-C_2$ and C_2 is homotopic to zero on $M-C_1$, then there exists a topological cylinder K with bases C_1 and C_2 such that $K \cap M = C_1 \cup C_2$.

Adding tubes one at a time and applying mathematical induction, we see that all the tubes required by the construction of this chapter may be added without intersecting one-another.

We are now prepared to construct quadrilateral imbeddings for certain cartesian products of bipartite graphs, and will thus be able to determine their genus exactly.

CHAPTER 6

THE GENUS OF CARTESIAN PRODUCTS OF COMPLETE BIPARTITE GRAPHS

In this chapter we present one of the main results of this thesis: the computation of the genus of the graph $K_{s,s} x K_{s,s}$ for the cases s = 1, s = 3, and for all even s. We then generalize this result by taking the cartesian product of arbitrarily many copies of $K_{s,s}$ and computing the genus of the resulting graph. The approach developed in Chapter 5 is useful in accomplishing this.

Recall that, for s = 2m, $\gamma(K_{2m,2m}) = (m-1)^2$, with $F = F_4 = 2m^2$. The particular minimal imbedding given by Ringel [18] for this case is:

$$V(i) = \begin{cases} \{j:\ 2m+1 \le j \le 4m\}, & 1 \le i \le 2m \\ \{j:\ 1 \le j \le 2m\}, & 2m+1 \le i \le 4m \end{cases}$$

$p_1, p_3, \ldots, p_{2m-1}$: $(2m+1,\ 2m+2, \ldots, 4m)$

$p_2, p_4, \ldots, p_{2m}$: $(4m,\ 4m-1, \ldots, 2m+1)$

$p_{2m+1}, p_{2m+3}, \ldots, p_{4m-1}$: $(1, 2, \ldots, 2m)$

$p_{2m+2}, p_{2m+4}, \ldots, p_{4m}$: $(2m,\ 2m-1, \ldots, 1)$

The following lemma is used to compute the genus of $K_{2m,2m} x K_{2m,2m}$:

<u>Lemma 6.1</u>. For the imbedding of $K_{2m,2m}$ given above, the set of

$2m^2$ quadrilateral faces may be partitioned into 2m subsets of m faces each so that each subset of m faces contains all 4m vertices of the graph.

Proof: We write out the orbits (each corresponding to a quadrilateral face) determined by the permutation P as defined by the permutations p_i, $1 \le i \le 4m$ (see Chapter 3), given above:

$(2g-1)-(2h-1)-2g-(2h-2)$, $1 \le g \le m$; $m+1 < h \le 2m$

$(2g-1)-(2h-1)-2g-4m$, $1 \le g \le m$; $h = m+1$

$2j-(2k-1)-(2j+1)-2k$, $m+1 \le k \le 2m$, $1 \le j < m$

$2j-(2k-1)-1-2k$, $m+1 \le k \le 2m$, $j = m$.

We now assign these $2m^2$ faces to parts of the partition. For fixed i, the m faces of part (2i-1) are determined by selecting $h = m + g + i$, with $1 \le g \le m$, where we reduce $(g + i)$ modulo m and write m instead of 0. The m faces of part 2i are determined by taking $k = m + j + i$, with $1 \le j \le m$, where we reduce $(j + i)$ modulo m and again write m instead of 0. Letting i run between 1 and m, we obtain 2m sets of m faces each, the sets being mutually disjoint by the manner in which they were selected. Furthermore, each set of m faces contains all 4m vertices of the graph $K_{2m,2m}$.

We are now in a position to prove the following theorem:

Theorem 6.2. The genus of $K_{s,s} x K_{s,s}$ is given by $\gamma(K_{s,s} x K_{s,s}) = 1 + s^2(s-2)$, if s is even or if s = 1 or 3.

Proof: We consider three cases:

Case (i). For s = 1, $K_{1,1} x K_{1,1} = K_2 x K_2 = C_4$, and $\gamma(C_4) = 0$. For the other cases, it suffices to produce a quadrilateral imbedding, as a result of Theorems 5.1 and 5.3. Then by Theorem 5.4, since $V = 4s^2$ and $E = 4s^3$, will follow that $\gamma(K_{s,s} \times K_{s,s}) = 1 + s^2(s-2)$.

Case (ii). For s = 3, we use an imbedding of $K_{3,3}$ for which $F = F_6 = 3$:

$$V(i) = \begin{cases} \{4,5,6\}, & i = 1,2,3 \\ \{1,2,3\}, & i = 4,5,6 \end{cases}$$

$$p_1, p_2, p_3: \quad (4,6,5)$$

$$p_4, p_5, p_6: \quad (1,3,2)$$

For this imbedding, each face contains each vertex of the graph exactly once. Designate the three faces by (1), (2), and (3). Now, take three copies of $K_{3,3}$ imbedded in this fashion, and three additional copies imbedded with reverse orientation. Designate the corresponding faces in the latter three copies by $(\underline{1})$, $(\underline{2})$, and $(\underline{3})$. At this stage, we have six tori, each with a copy of $K_{3,3}$ imbedded as described by the vertex permutations given above; label these tori by a, b, c; $\underline{d}$, $\underline{e}$, and $\underline{f}$. To imbed $K_{3,3} \times K_{3,3}$, we must join each vertex in each of a, b, and c to its counterparts in $\underline{d}$, in $\underline{e}$, and in $\underline{f}$. To accomplish these joins, we add the following nine tubes, each of which will carry the necessary six edges and itself corresponds to an edge in $K_{3,3}$: a(1) to $\underline{d}(\underline{1})$, a(2) to $\underline{e}(\underline{2})$, a(3) to $\underline{f}(\underline{3})$; b(1) to $\underline{e}(\underline{1})$, b(2) to $\underline{f}(\underline{2})$, b(3) to $\underline{d}(\underline{3})$; and c(1) to $\underline{f}(\underline{1})$, c(2) to $\underline{d}(\underline{2})$, c(3) to $\underline{e}(\underline{3})$. As in the proof of Theorem 4.3, since every tube is attached at two corresponding faces of opposite orientation, all six edges that the tube must carry may be added across it without intersecting one another. (Refer to Figure 4.3, with m = 6.) As every new face thus formed is a quadrilateral, and each hexagonal face in the original imbedding of any copy of $K_{3,3}$ has been destroyed, we have constructed a quadrilateral imbedding of $K_{3,3}xK_{3,3}$.

Case (iii). For s = 2m, imbed 4m copies of $K_{2m,2m}$ in 4m surfaces, each of genus $(m-1)^2$, using the imbedding of Lemma 6.1. We choose one of the two possible orientations for 2m of these surfaces, and the reverse orientation for the remaining 2m surfaces. This partition corresponds to the vertex set partition for $K_{2m,2m}$. Between each pair of oppositely oriented surfaces, we must add 4m edges in order to imbed $K_{2m,2m}xK_{2m,2m}$. We add these 4m edges over m tubes, each carrying four edges. There are 2m such joins that must be made from each surface. Lemma 6.1 establishes that Ringel's imbedding for the copy of $K_{2m,2m}$ at each surface is ideally suited for this purpose. We need only check that we can match corresponding parts of the face partitions appropriately. At copy j, $1 \le j \le 2m$, of $K_{2m,2m}$ minimally imbedded with common orientation, match part i of the face partition with part $\underline{i}$ in copy $\underline{j+i}$, $\underline{1} \le \underline{j+i} \le \underline{2m}$ (mod 2m) of $K_{2m,2m}$ minimally imbedded with the opposite orientation. If $\underline{j+i} = \underline{j'+i'}$ with $\underline{i} = \underline{i}'$, then $\underline{j} = \underline{j}'$, so that each part of each partition has exactly one tube attached at each face in that part. As in case (ii), each new face is a quadrilateral. We have constructed a quadrilateral imbedding of $K_{2m,2m}xK_{2m,2m}$, completing the proof.

For the above construction, using Theorems 4.5 and 4.7, we can compute the genus of the resulting surface directly, without recourse to the Euler-type formula of Theorem 5.4. The contributions to the genus are of three types: (i) $4m\gamma(K_{2m,2m}) = 4m(m-1)^2$, representing the collective genera of the surfaces with which we began our construction; (ii) $4m^2(m-1)$, representing an increase of $(m-1)$ in the genus for each of the $4m^2$ joins, due to the addition of m tubes; and (iii) $\beta(K_{2m,2m}) = (2m-1)^2$, representing the contributions of the bundles taken collectively. Adding, we see that

$$\begin{aligned}\gamma(K_{2m,2m} x K_{2m,2m}) &= 1 + (2m)^2(2m-2)\\ &= 1 + s^2(s-2), \text{ where } s = 2m.\end{aligned}$$

This computation provides an unusually intuitive connection between the formula for the genus and the realization of the imbedding in Euclidean 3-space.

We can use Theorem 6.2 to prove the following corollary, which is actually a generalization of this theorem:

<u>Corollary 6.3</u>. The genus of $K_{2m,2m} x K_{r,s}$ is given by $\gamma(K_{2m,2m} x K_{r,s}) = 1 + m[(m-2)(r+s) + rs]$, if $r \leq 2m$ and $s \leq 2m$.

<u>Proof</u>: Imbed $K_{2m,2m} x K_{2m,2m}$ as in the proof of Theorem 6.2, with $F = F_4$. Remove $(4m - (r+s))$ surfaces containing copies of $K_{2m,2m}$, together with all tubes and edges issuing from these surfaces, so as to leave an imbedding of $K_{2m,2m} x K_{r,s}$. This imbedding is also quadrilateral, since each copy of $K_{2m,2m}$ was initially imbedded quadrilaterally, and the removal of any tube re-introduces only quadrilateral faces. Hence this imbedding of $K_{2m,2m} x K_{r,s}$ is a minimal imbedding; and by Theorem 5.4, with $V = 4m(r+s)$ and $E = 4m^2(r+s) + 4mrs$, we obtain

$$\gamma(K_{2m,2m} \times K_{r,s}) = 1 + (m^2(r+s) + mrs) - 2m(r+s)$$
$$= 1 + m[(m-2)(r+s) + rs].$$

From Corollary 6.3, we can readily deduce the following:

Corollary 6.4. The genus of $K_{2m,2m} \times K_{2n,2n}$ is given by $\gamma(K_{2m,2m} \times K_{2n,2n}) = 1 + 4mn(m + n - 2)$, for all positive integers m and n.

Now we define a class of graphs which generalize the n-cube Q_n, as follows: let $Q_1^{(s)} = K_{s,s}$, and recursively define $Q_n^{(s)} = Q_{n-1}^{(s)} \times K_{s,s}$, for $n \geq 2$. The constructions of Theorem 6.2 can be extended, as developed in the next three theorems.

Theorem 6.5. $\gamma(Q_n^{(2m)}) = 1 + 2^{2n-2} m^n (mn-2)$

Proof: By Corollary 5.2, $Q_n^{(2m)}$ is a bipartite graph. We produce a quadrilateral imbedding for $Q_n^{(2m)}$, and compute $\gamma(Q_n^{(2m)})$ using Theorem 5.4. It is clear that $V = 4^n m^n$ for $Q_n^{(2m)}$. We establish the values of E and F, showing that $F = F_4$, by mathematical induction. Let the statement S(n) be as follows: There is an imbedding of $Q_n^{(2m)}$ with $E = n2^{2n} m^{n+1}$ and $F = F_4 = n2^{2n-1} m^{n+1}$, including 2m mutually disjoint sets of $2^{2n-2} m^n$ mutually vertex-disjoint quadrilateral faces each, each set containing all $4^n m^n$ vertices of $Q_n^{(2m)}$. We claim that S(n) is true for all natural numbers n. That S(1) is true follows immediately from Ringel's imbedding of $K_{2m,2m}$ and Lemma 6.1.

Now, assuming S(n) to be true, we establish S(n+1), for $n \geq 1$. So, consider a large copy of $K_{2m,2m}$, each vertex of which is replaced with a small copy of $Q_n^{(2m)}$ imbedded as described by S(n) and with respective orientations determined by the vertex set partition for $K_{2m,2m}$. Label the 2m copies of one orientation by j, $1 \le j \le 2m$, and the 2m copies of opposite orientation by $\underline{j}$, $\underline{1} \le \underline{j} \le \underline{2m}$. Now, by the induction hypothesis, each copy of $Q_n^{(2m)}$ has 2m sets of faces available, one set for each of the 2m joins that must be made from that copy. Furthermore, each set contains each vertex of the graph $Q_n^{(2m)}$ exactly once. As in the proof of Theorem 6.2, at copy j, $1 \le j \le 2m$, match set i with set $\underline{i}$ in copy $\underline{j+i}$, $\underline{1} \le \underline{j+i} \le \underline{2m}$ (mod 2m). For each matching a tube carrying four edges is attached between each pair of corresponding quadrilateral faces. In this manner the required $4m^2$ joins are completed, so that we have a quadrilateral imbedding of $Q_{n+1}^{(2m)}$.

Now, for fixed j, pair off copy i of $Q_n^{(2m)}$ with copy $\underline{i+j}$, where $\underline{1} \le \underline{i+j} \le \underline{2m}$ (mod 2m). For each such pairing, with copy i joined to copy $\underline{i+j}$ by $2^{2n-2}m^n$ tubes, we have (for fixed j and i = 1,...,2m) a total of $4(2^{2n-2})m^n(2m) = 2^{2n+1}m^{n+1}$ quadrilateral faces on $2^{2n-1}m^{n+1}$ tubes. For each tube, select one pair of opposite faces. The $2^{2n}m^{n+1}$ faces thus selected are mutually vertex-disjoint and contain all $4^{n+1}m^{n+1}$ vertices of $Q_{n+1}^{(2m)}$. Now letting j vary between 1 and 2m, we obtain 2m mutually disjoint such sets of quadrilateral faces, as claimed by S(n+1).

The imbedding of $Q_{n+1}^{(2m)}$ we have obtained has $F^{(n+1)} = F_4^{(n+1)}$, since $F^{(n)} = F_4^{(n)}$ and the attaching of each new tube with the four edges it carries eliminates two quadrilaterals and introduces four new quadrilaterals. Now, $F^{(n+1)} = 4mF^{(n)} + \Delta F$, where ΔF is twice the number of tubes added at this stage. But the number of tubes added is $(4m^2)\frac{(4^n m^n)}{4} = 4^n m^{n+2}$, where $4m^2$ is the number of edges in $K_{2m,2m}$ (corresponding to the number of joins we made), $4^n m^n$ is the number of edges per join, and there are four edges per tube. Hence

$$F^{(n+1)} = 4m(n2^{2n-1}m^{n+1}) + 2(4^n m^{n+2})$$
$$= (n+1)2^{2n+1}m^{n+2}.$$

Also,

$$E^{(n+1)} = 4mE^{(n)} + 4m^2 V^{(n)}$$
$$= 4m(n2^{2n}m^{n+1}) + 4m^2(4^n m^n)$$
$$= (n+1)2^{2n+2}m^{n+2}.$$

We have established that S(n+1) follows from S(n), for all $n \geq 1$. Thus S(n) holds, for all natural numbers n. Now, by Theorem 5.4,

$$\gamma(Q_n^{(2m)}) = 1 + \frac{n2^{2n}m^{n+1}}{4} - \frac{4^n m^n}{2}$$
$$= 1 + 2^{2n-2}m^n(mn-2).$$

As with the construction of Theorem 6.2, we can also compute the genus of $Q_{n+1}^{(2m)}$ directly, given the genus of $Q_n^{(2m)}$. We have:

$$\gamma Q_{n+1}^{(2m)} = 4m\gamma(Q_n^{(2m)}) + 4m^2(4^{n-1}m^n-1) + \beta(K_{2m,2m})$$

$$= 4m(1 + 2^{2n-2}m^n(mn-2)) + 4m^2(4^{n-1}m^n-1) + (4m^2-4m+1)$$

$$= 1 + 2^{2n}m^{n+1}[m(n+1)-2].$$

We have also the following generalization of Theorem 6.5, the proof of which is analogous to that of Corollary 6.3:

Corollary 6.6. The genus of $Q_n^{(2m)} \times K_{r,t}$ is given by $\gamma(Q_n^{(2m)} \times K_{r,t}) = 1 + 2^{2n-2}m^n[(r+t)(mn-2) + rt]$, for $r \leq 2m$ and $t \leq 2m$.

In Theorem 6.5, it was convenient to consider $K_{s,s}$ for s even, since $K_{2m,2m}$ has $F = F_4$ in its minimal imbedding. The arguments of Theorem 6.5, with minor modifications, apply also to the cases $s = 1$ and $s = 3$. We can therefore state:

Theorem 6.7. The genus of $Q_n^{(s)}$ is given by $\gamma(Q_n^{(s)}) = 1 + 2^{n-3}s^n(ns-4)$, for s even and any positive integer n, or for $s = 1$ or 3 and $n \geq 2$.

The genus formula given in Theorem 6.7 includes as two of its special cases $\gamma(K_{2m,2m}) = (m-1)^2$ and $\gamma(Q_n) = 1 + 2^{n-3}(n-4)$, two of the familiar results in the literature.

The constructions of minimal imbeddings for $Q_{n+1}^{(s)}$ employed in the proof of Theorem 6.7 used only half of the available faces on each tube in the given minimal imbedding of $Q_n^{(s)}$, for $n \geq 2$. Using every face on every tube, a similar argument can be employed to construct a quadrilateral imbedding for the following graph: let $G_1 = K_{s,s}$, and recursively define $G_n = G_{n-1} \times K_{t,t}$, where $t = 2^{n-2}s$, and $n \geq 2$. Noting that, for G_n,

$$V^{(n)} = 2^{\frac{n(n-1)}{2} + 1} s^n,$$

while

$$E^{(n)} = 2^{\frac{(n+2)(n-1)}{2}} s^{n+1}$$

(using the fact that G_n is regular of degree $d^{(n)} = 2^{n-1}s$), we use Theorem 5.4 to compute the following result:

Theorem 6.8. The genus of G_n is given by $\gamma(G_n) = 1 + 2^{\frac{n(n-1)}{2}} s^n(2^{n-3}s-1)$, for s even and any positive integer n, or for $s = 1$ or 3 and $n \geq 2$.

CHAPTER 7

THE GENUS OF CARTESIAN PRODUCTS OF CYCLES AND PATHS

Every path is a bipartite graph, as are all cycles of even length. We have computed the genus of the cartesian product of two cycles, or of two paths, in Chapter 4. In this chapter we take repeated cartesian products of certain of these graphs and compute their genera. The approach is similar to that employed in Chapter 6.

Define the graph G_n as follows: let $G_1 = C_{2m_1}$, the cycle on $2m_1$ vertices, and recursively define $G_n = G_{n-1} \times C_{2m_n}$, for $n \geq 2$, where we require each m_i to be ≥ 2. Let $M^{(n)}$ denote $\prod_{i=1}^{n} m_i$. The following genus formula contains (n+1) parameters:

<u>Theorem 7.1</u>. The genus of G_n is given by $\gamma(G_n) = 1 + 2^{n-2}(n-2)M^{(n)}$, for $n \geq 2$.

<u>Proof</u>: By Corollary 5.2, G_n is a bipartite graph. We produce a quadrilateral imbedding for G_n, and compute $\gamma(G_n)$ using Theorem 5.4. For G_n, $V^{(n)} = 2^n M^{(n)}$; and since G_n is regular of degree 2n, it is a simple matter to compute $E^{(n)} = 2^n n M^{(n)}$. Now, let the statement S(n) be: There is an imbedding of G_n for which $F = F_4 = n2^{n-1}M^{(n)}$, including two disjoint sets of $2^{n-2}M^{(n)}$ mutually vertex-disjoint quadrilateral faces each, both sets containing all $2^n M^{(n)}$ vertices of G_n. We claim that S(n) is true for all $n \geq 2$. We verify this by mathematical induction.

That $S(2)$ is true is apparent from Figure 4.1, with the faces designated by (1) making up one set and those designated by (2) making up the other. Now we assume $S(n)$ to be true and establish $S(n+1)$, for $n \geq 2$. For the graph G_{n+1}, we start with $2m_{n+1}$ copies of G_n, minimally imbedded as described by $S(n)$. We partition the corresponding surfaces into m_{n+1} copies of one orientation and m_{n+1} copies of the reverse orientation, corresponding to the vertex set partition of $C_{2m_{n+1}}$. From each copy, two joins must be made, both to copies of opposite orientation. From the statement $S(n)$, it is clear that these two joins can be made, each one over $2^{n-2}M^{(n)}$ tubes carrying four edges each. Each new face formed is a quadrilateral. In this fashion the required $2m_{n+1}$ joins can be made to imbed G_{n+1}, with $F = F_4$. Now form one set of faces by selecting opposite quadrilaterals from each tube added in alternate joins in this construction. Form the second set by selecting the remaining quadrilaterals on the same tubes. It is clear that the two sets of faces thus selected are disjoint, and that each contains $(2)(m_{n+1})(2^{n-2}M^{(n)}) = 2^{n-1}M^{(n+1)}$ mutually vertex-disjoint quadrilaterals; both sets contain all $2^{n+1}M^{(n+1)}$ vertices of G_{n+1}. Furthermore, $F^{(n+1)} = 2m_{n+1}F^{(n)} + \Delta F$, where $\Delta F = (2m_{n+1})(2^{n-2}M^{(n)})(2)$, where $2m_{n+1}$ joins have been made, with $2^{n-2}M^{(n)}$ tubes per join, and a net increase in F of 2 per tube. Hence,

$$\begin{aligned} F^{(n+1)} &= 2m_{n+1}(n2^{n-1}M^{(n)}) + 2^nM^{(n+1)} \\ &= (n+1)2^nM^{(n+1)}, \end{aligned}$$

and we have established that $S(n+1)$ follows from $S(n)$. Therefore, $S(n)$ holds, for all $n \geq 2$. We can now compute:

$$\gamma(G_n) = 1 + \frac{2^n n M^{(n)}}{4} - \frac{2^n M^{(n)}}{2}$$
$$= 1 + 2^{n-2}(n-2)M^{(n)}.$$

For the special case where $m_i = m$, $i = 1,\ldots,n$, we have $M^{(n)} = m^n$, and:

Corollary 7.2. The genus of $G_n^{(m)}$ is given by $\gamma(G_n^{(m)}) = 1 + 2^{n-2}(n-2)m^n$.

Furthermore, if $m = 2$ in the above formula, since $C_4 = K_{2,2} = K_2 x K_2$, $G_n^{(2)}$ is the 2n-cube, and we obtain the familiar result:

Corollary 7.3. $\gamma(Q_{2n}) = 1 + 2^{2n-2}(n-2)$.

We now turn our attention from cycles to paths. Let $H_1 = P_{m_1}$, a path on m_1 vertices, and recursively define $H_n = H_{n-1} x P_{m_n}$, for $n \geq 2$. All paths are bipartite graphs, but we nevertheless restrict m_1, m_2, and m_3 to be even in the theorem to follow. The complexity of this genus formula in comparison with that of Theorem 7.1 is largely due to the fact that the graphs G_n are regular, whereas the graphs H_n are not (unless $m_i = 2$, for all $i = 1,\ldots,n$). Since Theorem 4.2 covers the case $n = 2$, we consider here only $n \geq 3$. We again let $M^{(n)} = \prod_{i=1}^{n} m_i$.

Theorem 7.4. The genus of H_n is given by $\gamma(H_n) = 1 + \frac{M^{(n)}}{4}(n-2-\sum_{i=1}^{n}\frac{1}{m_i})$, for $n \geq 3$ and m_1, m_2, and m_3 all even.

Proof: By Corollary 5.2, H_n is a bipartite graph. We construct a quadrilateral imbedding for H_n, and compute $\gamma(H_n)$ using Theorem 5.4. For H_n, $V^{(n)} = M^{(n)}$. Now, let the statement S(n) be: There is an imbedding of H_n for which $F = F_4 = \frac{n}{2}M^{(n)} - \frac{1}{2}M^{(n)}\sum_{i=1}^{n}\frac{1}{m_i}$, including two disjoint sets of $\frac{1}{4}M^{(n)}$ mutually vertex-disjoint quadrilateral faces each, both sets containing all $M^{(n)}$ vertices of H_n; furthermore, for H_n, $E^{(n)} = nM^{(n)} - M^{(n)} \sum_{i=1}^{n} \frac{1}{m_i}$. We claim that S(n) is true for all $n \geq 3$. We verify this by mathematical induction.

To see that S(3) is true, refer to Figure 4.2, which shows that $\gamma(P_6 x P_8) = 0$. Every face but the exterior face is a quadrilateral for this imbedding. We see that two joins may be made at each copy of $P_{m_1} x P_{m_2}$ in general, provided m_1 and m_2 are both even. One join employs the faces designated by (1), and the other join uses the faces designated by (2), as in Figure 4.2. Provided m_3 is even also, we can arrange the two end copies of $P_{m_1} x P_{m_2}$ so that the faces (2), including the exterior face, are employed in the single join that must be made from each end copy. Partition the m_3 copies of $P_{m_1} x P_{m_2}$ into $\frac{m_3}{2}$ copies of one orientation and $\frac{m_3}{2}$ copies of the other orientation, with the two end copies in different parts of this partition (corresponding to the vertex set partition of P_{m_3}.) The graph $H_3 = P_{m_1} x P_{m_2} x P_{m_3}$ thus has a quadrilateral imbedding, since a tube attached between two oppositely oriented copies of the exterior face (2) replaces those two faces of $2(m_1+m_2-2)$ sides each with $2(m_1+m_2-2)$ quadrilaterals, once the required edges are added over the tube. Now,

$$\begin{aligned} E^{(2)} &= (m_1-1)m_2 + (m_2-1)m_1 \\ &= 2m_1m_2 - m_1 - m_2, \end{aligned}$$

so that

$$\begin{aligned} E^{(3)} &= m_3E^{(2)} + (m_3-1)V^{(2)} \\ &= 3m_1m_2m_3 - m_1m_3 - m_2m_3 - m_1m_2. \end{aligned}$$

Also, $F^{(3)} = m_3F^{(2)} + \Delta F$, where ΔF is the increase in faces accounted for by the tubes we have added. This increase is of two types, corresponding to tubes attached within faces designated by (1) and to tubes attached within faces designated by (2). We have:

$$F^{(3)} = m_3[(m_1-1)(m_2-1) + 1] + 2(\frac{m_3}{2} - 1)(\frac{m_1}{2}\frac{m_2}{2}) + \frac{m_3}{2}[2(\frac{m_1}{2} - 1)(\frac{m_2}{2} - 1) + 2(m_1 + m_2 - 3)]$$
$$= \frac{3}{2}m_1m_2m_3 - \frac{m_1m_2}{2} - \frac{m_1m_3}{2} - \frac{m_2m_3}{2}\ .$$

Furthermore, consider the set of faces obtained by taking, from each tube joining faces designated by (2), every second face. These faces are mutually vertex-disjoint, and contain all $m_1m_2m_3$ vertices of H_3. Now, form a second set of faces consisting of the remaining faces on the tubes joining faces designated by (2). These faces are also mutually vertex-disjoint, and contain all $m_1m_2m_3$ vertices of H_3. Moreover, the two sets of faces we have selected are clearly disjoint. Therefore, S(3) is true.

Now we assume S(n) to be true, and establish S(n+1), for $n \geq 3$. Given the graph H_{n+1}, we give the m_{n+1} copies of H_n minimal imbeddings as described by S(n), with orientation as determined by the vertex set partition of $P_{m_{n+1}}$. It is clear that we can make the required $(m_{n+1} - 1)$ joins so as to obtain a quadrilateral imbedding for H_{n+1}. We have

$$E^{(n+1)} = m_{n+1}E^{(n)} + (m_{n+1} - 1)V^{(n)}$$
$$= m_{n+1}(nM^{(n)} - M^{(n)}\sum_{i=1}^{n}\frac{1}{m_i}) + (m_{n+1} - 1)M^{(n)}$$
$$= (n+1)M^{(n+1)} - M^{(n+1)}\sum_{i=1}^{n+1}\frac{1}{m_i}\ .$$

Also, $F^{(n+1)} = m_{n+1}F^{(n)} + \Delta F$, where $\Delta F = (m_{n+1} - 1)(\frac{1}{4}M^{(n)})(2)$, where $m_{n+1} - 1$ is the number of joins, $\frac{1}{4}M^{(n)}$ is the number of tubes per join, and there is a net increase in F of two for each tube.

We have

$$F^{(n+1)} = m_{n+1}(\frac{n}{2}M^{(n)} - \frac{1}{2}M^{(n)}\sum_{i=1}^{n}\frac{1}{m_i}) + \frac{1}{2}M^{(n+1)} - \frac{1}{2}M^{(n)}$$
$$= \frac{(n+1)}{2}M^{(n+1)} - \frac{1}{2}M^{(n+1)}\sum_{i=1}^{n+1}\frac{1}{m_i}.$$

To complete the verification of S(n+1), we must find two disjoint sets of $\frac{M^{(n+1)}}{4}$ mutually vertex-disjoint quadrilateral faces each, both sets containing all $M^{(n+1)}$ vertices of H_{n+1}. We have two cases to consider:

Case (i). If m_{n+1} is even we choose opposite faces on each tube of alternate joins to form one set, and the remaining faces on the same tubes to form the second set, as indicated in Figure 7.1.

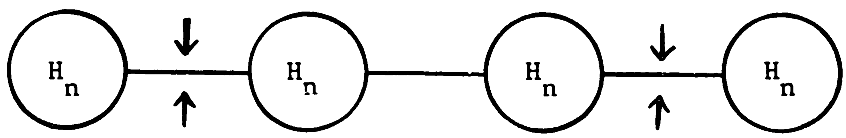

Figure 7.1 Selecting faces for m_{n+1} even.

Case (ii). If m_{n+1} is odd, we make our selection as indicated in Figure 7.2, using at each end copy of H_n the remaining set of $\frac{1}{4}M^{(n)}$ mutually vertex-disjoint quadrilaterals. As in Figure 7.1, an arrow at a join indicates that opposite faces on each tube of the join have been selected.

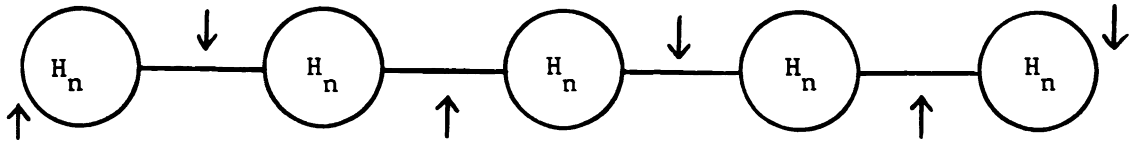

Figure 7.2 Selecting faces for m_{n+1} odd.

We have shown that S(n+1) follows from S(n), and hence that S(n) holds for all $n \geq 3$. It only remains to compute the genus of H_n. But by Theorem 5.4,

$$\gamma(H_n) = 1 + \frac{1}{4}(nM^{(n)} - M^{(n)} \sum_{i=1}^{n} \frac{1}{m_i}) - \frac{1}{2}M^{(n)}$$

$$= 1 + \frac{M^{(n)}}{4}(n - 2 - \sum_{i=1}^{n} \frac{1}{m_i}).$$

Since the operation of taking the cartesian product is commutative, Theorem 7.4 can be applied if any three or more of the m_i are even. Elementary probability considerations show that this fails to happen in only $\frac{n^2 + n+2}{2^{n+1}}$ of the possible cases, for fixed n. For $n > 5$, this probability will be less than one half.

For the special case where $m_i = m$, $i = 1,\ldots,n$, for m even, we have:

Corollary 7.5. The genus of the graph $H_n^{(m)}$ is given by $\gamma(H_n^{(m)}) = 1 + \frac{m^{n-1}}{4}(mn - 2m - n)$, for m any even positive integer.

Futhermore, if m = 2 in the above formula, $H_n^{(2)}$ is the n-cube, since $P_2 = K_2$, and we have the familiar result:

Corollary 7.6. $\gamma(Q_n) = 1 + 2^{n-3}(n-4)$.

We have now generalized the genus of the n-cube in three different directions: in Theorem 6.7, regarding $K_{1,1}$ as K_2; in Theorem 7.1, regarding C_4 as $K_2 x K_2$ to obtain the genus of the 2n-cube, and in Theorem 7.4, regarding P_2 as K_2.

We have thus far studied the genus of cartesian products of bipartite graphs and in particular of complete bipartite graphs, even cycles, and paths. The techniques we have developed can also be applied to cartesian products of certain combinations of these graphs. The following theorems are a sample of results in this direction. The proofs, being similar to those already given, are omitted.

<u>Theorem 7.7</u>. The genus of the graph $K_2 x P_{2m} x C_{2n}$ is given by: $\gamma(K_2 x P_{2m} x C_{2n}) = 1 + n(m-1)$, for $n \geq 2$.

<u>Theorem 7.8</u>. The genus of the graph $K_2 x P_{2m} x C_{2n} x K_{4,4}$ is given by: $\gamma(K_2 x P_{2m} x C_{2n} x K_{4,4}) = 1 + 8n(5m-1)$, for $n \geq 2$.

<u>Theorem 7.9</u>. The genus of the graph $C_{2m} x C_{2n} x K_{4,4}$ is given by: $\gamma(C_{2m} x C_{2n} x K_{4,4}) = 1 + 16mn$, for $m \geq 2$ and $n \geq 2$.

CHAPTER 8

THE GENUS OF COMPLETE TRIPARTITE GRAPHS

Since the genus has been determined for the complete graphs K_p and for the complete bipartite graphs $K_{p,q}$, it seems appropriate to investigate next the genus of the complete tripartite graphs $K_{p,q,r}$. This problem appears to be very difficult, and in this chapter we will be content to establish a lower bound for $\gamma(K_{p,q,r})$ and to show that the lower bound is attained in certain special cases, each of which includes infinitely many graphs of this family.

The graph $K_{p,q,r}$ has $(p + q + r)$ vertices, which are partitioned into three sets P, Q, and R, containing p, q, and r vertices respectively. We assume throughout this chapter that $p \geq q \geq r \geq 1$. The edges of $K_{p,q,r}$ are precisely those edges which join a vertex in one of the three sets to a vertex in one of the other two sets. In order to distinguish the three types of edges which occur, we make the following definition:

Definition 8.1. An edge of the graph $K_{p,q,r}$ which joins a vertex in set R with a vertex in set Q is called an edge of type I. Similarly, an edge joining sets R and P is called an edge of type II, and one joining sets Q and P an edge of type III.

Since there are qr edges of type I, pr edges of type II, and pq edges of type III, the total number of edges is $E = qr + pr + pq$.

Lower bounds for genus formulae are ordinarily obtained by the use of the Euler formula and certain properties of the graph in question. Theorem 8.1, which follows, can be established in this way, but a simpler proof is presented which used Ringel's result for the genus of complete bipartite graphs.

Theorem 8.1. The genus of the graph $K_{p,q,r}$ is bounded below by: $\gamma(K_{p,q,r}) \geq \left\{\frac{(p-2)(q+r-2)}{4}\right\}$.

Proof: Consider any minimal imbedding of $K_{p,q,r}$ in a surface M. By the removal of all edges of type I from this imbedding, we obtain an imbedding of $K_{p,q+r}$ in the same surface M. Hence $\gamma(K_{p,q,r}) = \gamma(M) \geq \gamma(K_{p,q+r}) = \left\{\frac{(p-2)(q+r-2)}{4}\right\}$, by Ringel's formula for the genus of complete bipartite graphs.

Much of the remainder of this chapter is devoted to showing that equality holds in Theorem 8.1 when $q + r \leq 6$, and we conjecture that it holds for all complete tripartite graphs.

Conjecture. $\gamma(K_{p,q,r}) = \left\{\frac{(p-2)(q+r-2)}{4}\right\}$.

The result of Ringel and Youngs that $\gamma(K_{p,p,p}) = \frac{(p-1)(p-2)}{2}$ is seen to be consistent with this conjecture. We will also show that $\gamma(K_{mn,n,n}) = \frac{(mn-2)(n-1)}{2}$, which likewise agrees with this conjecture. The other cases where $q + r > 6$ remain open.

To show that equality holds when $q + r \leq 6$, it is sufficient to construct an imbedding of $K_{p,q,r}$ in a surface of genus $\left\{\frac{(p-2)(q+r-2)}{4}\right\}$, so that $\gamma(K_{p,q,r}) \leq \left\{\frac{(p-2)(q+r-2)}{4}\right\}$. The following lemmas will assist us in investigating the face distributions of such an imbedding.

<u>Lemma 8.2</u>. In any imbedding of $K_{p,q,r}$, $F_3 \leq 2qr$.

<u>Proof</u>: Any 3-cycle in $K_{p,q,r}$ must be composed of one edge of each of the three types, since otherwise two vertices in the same vertex set would be adjacent, a contradiction. Hence any triangle in an imbedding of this graph contains one edge of each type, and in particular an edge of type I. But there are only qr edges of type I, and each edge appears in at most two faces in any imbedding of the graph. Hence $F_3 \leq 2qr$.

<u>Lemma 8.3</u>. If an imbedding of $K_{p,q,r}$ has $F_3 = 2qr$, then $F_{2i+1} = 0$, for $i = 2,3,\ldots$

<u>Proof</u>: If $F_3 = 2qr$, the qr edges of type I each appear in two triangular faces. Any other face must then include only edges of type II or of type III. Since the vertices of $K_{p,q,r}$ are partitioned into three sets P, Q, and R of p, q, and r vertices respectively, the boundary of any non-triangular face is a subgraph of the bipartite graph $K_{p,q+r}$, which has its vertex set partitioned into sets P and $Q \cup R$. Any such subgraph is itself a bipartite graph and hence cannot contain any odd cycle. We observe that a face could contain a given vertex more than once, but in this case each cycle formed must be even, implying that the face has an even number of sides.

<u>Theorem 8.4</u>. If $F_3 = 2qr$ in à 2-cell imbedding of $K_{p,q,r}$ in a surface M, then $\gamma(K_{p,q,r}) \leq \gamma(M) = \frac{(p-2)(q+r-2)}{4} + \frac{1}{4}\sum_{i\geq 3}(i-2)F_{2i}$.

Proof: We use the Euler-type formula (1) discussed in Chapter 4: $\gamma(G) = 1 + \frac{1}{8} \sum_{i\geq 3} (i-4)(F_i+V_i)$. In particular, for $G = K_{p,q,r}$, since $V_{p+q} = r$, $V_{p+r} = q$, $V_{q+r} = p$, and since we are assuming that $F_3 = 2qr$, we have, using Lemma 8.3,

$$\begin{aligned}\gamma(K_{p,q,r}) &\leq \gamma(M) \\ &= 1 + \frac{1}{8}(-2rq + (p+q-4)r + (p+r-4)q + (q+r-4)p) \\ &\qquad + \frac{1}{8} \sum_{i\geq 5} (i-4)F_i \\ &= \frac{(p-2)(q+r-2)}{4} + \frac{1}{4} \sum_{i\geq 3} (i-2)F_{2i}.\end{aligned}$$

As a result of Theorem 8.4, it is possible to show that equality holds in Theorem 8.1, provided we produce a 2-cell imbedding of $K_{p,q,r}$ for which $F_3 = 2qr$ and $\frac{1}{4}(F_6 + 2F_8 + 3F_{10}) = \left\{\frac{(p-2)(q+r-2)}{4}\right\} - \frac{(p-2)(q+r-2)}{4}$, with all other faces being quadrilateral, for then $\gamma(K_{p,q,r}) \leq \left\{\frac{(p-2)(q+r-2)}{4}\right\}$. In particular, if $\frac{(p-2)(q+r-2)}{4}$ is an integer, we seek a 2-cell imbedding with $F_3 = 2qr$ and $F_4 = F - F_3$. This search utilizes Edmond's permutation technique, which produces only 2-cell imbeddings.

Before proceeding with this plan, let us state the following corollary of Theorem 8.4, which is not employed in the remainder of this chapter, but is of interest in its own right, since it indicates that a minimal imbedding of $K_{p,q,r}$ cannot in general be triangular.

Corollary 8.5. A minimal imbedding of $K_{p,q,r}$ is triangular if and only if $p = q = r$.

Proof: (i) Ringel and Youngs have shown that $\gamma(K_{p,p,p}) = \frac{(p-1)(p-2)}{2}$, with $F = F_3$.

(ii) Suppose $K_{p,q,r}$ has a triangular imbedding. This imbedding is therefore minimal, by a result of Youngs, and hence is a 2-cell imbedding. Then $F = F_3 = 2qr$, since $F_3 \leq 2qr$; and each edge of type I lies in exactly two triangular faces for this imbedding, so that $F_3 \geq 2qr$ also. Then, by Theorems 8.1 and 8.4, $\gamma(K_{p,q,r}) = \frac{(p-2)(q+r-2)}{4}$. Now, from the Euler formula,

$$2qr = F = -V + E + 2(1-\gamma)$$
$$= -(p+q+r) + (pq + pr + qr) + 2 - \frac{1}{2}(p-2)(q+r-2),$$

so that $pq + pr = 2qr$. Since $p \geq q \geq r$, then $pq \geq qr$ and $pr \geq qr$. It follows that $pq = qr = pr$, and $p = q = r$.

We have established the preliminary results needed to show that $\gamma(K_{p,q,r}) = \left\{\frac{(p-2)(q+r-2)}{4}\right\}$, when $2 \leq q + r \leq 6$. We treat the nine cases: $(q,r) = (1,1)$; $(2,1)$; $(3,1)$, $(2,2)$; $(4,1)$, $(3,2)$; $(5,1)$, $(4,2)$, and $(3,3)$ in the theorems that follow. We first note that if $q = r = 1$, the graph is planar and the genus formula clearly holds; see Figure 8.1. In this case, $F_3 = 2$, $F_4 = p - 1$, and $F = F_3 + F_4$. Euler's formula is satisfied, with $\gamma = 0$.

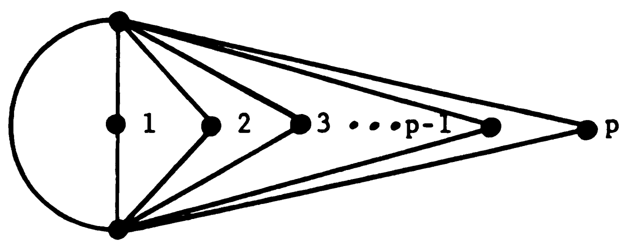

Figure 8.1 An imbedding of $K_{p,1,1}$ in the plane.

The remaining cases are more involved and are handled by the use of Edmonds' permutation technique, discussed in Chapter 3.

<u>Theorem 8.6</u>. The genus of the graph $K_{p,2,1}$ is given by $\gamma(K_{p,2,1}) = \left\{\frac{p-2}{4}\right\}$, for $p \geq 2$.

<u>Proof</u>: By Theorem 8.1, $\gamma(K_{p,2,1}) \geq \left\{\frac{p-2}{4}\right\}$. We show that $\gamma(K_{p,2,1}) \leq \left\{\frac{p-2}{4}\right\}$ by exhibiting, for each $p \equiv 2 \pmod 4$, an appropriate imbedding. The result will then follow, since if $p_o \equiv 2 \pmod 4$ and $m = p_o$, p_o-1, p_o-2, or p_o-3, then $\gamma(K_{m,2,1}) \leq \gamma(K_{p_o,2,1}) \leq \left\{\frac{p_o-2}{4}\right\} = \left\{\frac{m-2}{4}\right\}$. So, we claim that $\gamma(K_{p,2,1}) \leq \left\{\frac{p-2}{4}\right\}$, for $p \equiv 2 \pmod 4$. By Theorem 8.4, it suffices to produce a 2-cell imbedding with $F_3 = 4$ and $F_4 = F - F_3$. We employ Edmonds' permutation technique:

$$V(i) = \begin{cases} \{2,3;4,\ldots,p+3\}, & i = 1 \\ \{1;4,\ldots,p+3\}, & i = 2,3 \\ \{1;2,3\}, & i = 4,\ldots,p+3 \end{cases}$$

$$n(i) = \begin{cases} p+2, & i = 1 \\ p+1, & i = 2,3 \\ 3, & i = 4,\ldots,p+3 \end{cases}$$

p_1: $(p+3,p+2,\ldots,6,5;3,4,2)$

p_2: $(1,4;p+1,p+2;p-3,p-2;\ldots;7,8:5,6;9,10,\ldots,p-1,p;p+3)$

p_3: $(1,5;8,9,6,7;12,13,10,11;\ldots;p+2,p+3,p,p+1;4)$

$p_4,p_6,\ldots,p_{p+2}$: $(1,3,2)$

$p_5,p_7,\ldots,p_{p+3}$: $(1,2,3)$

The permutation P determined by these p_i, $i = 1,\ldots,p+3$, (see Chapter 3) applies for all $p \equiv 2 \pmod 4$ except for $p = 2$; but then $0 \leq \gamma(K_{2,2,1}) \leq \gamma(K_{2,2,2}) = 0$, by the result of Ringel and Youngs, so that this case is trivial. Now, we compute the orbits under P as defined above:

(i) $F_3 = 4$: 1-(p+3)-2; 1-2-4; 1-3-5; 1-4-3.

We note that at this stage all edges of type I have been accounted for.

(ii) $F_4 = F - F_3$: 1-(2m-1)-2-2m, $m = 3,\ldots,\frac{p}{2}+1$

1-2m-3-(2m+1), $m = 3,\ldots,\frac{p}{2}+1$

(Now all edges of tyep II have been accounted for.)

2-5-3-8

2-(p+1)-3-4

2-(2m-1)-3-(2m-4), if $(2m-1) \equiv 1 \pmod 4$

2-(2m-1)-3-(2m+4), if $(2m-1) \equiv 3 \pmod 4$

$(m = 4,\ldots,\frac{p}{2}+2,\ m \neq \frac{p}{2}+1)$

Now all edges of type III have also been taken into account. We have shown that every directed edge not in a triangular face is in a quadrilateral face. This completes the proof.

The above representation allows us to count the number of faces directly: $F = 6 + 2(\frac{p}{2} + 1 - 2) + (\frac{p}{2} + 2 - 4) = \frac{3p}{2} + 2$, which is consistent with the Euler formula, as indeed it must be. We observe also that removing vertices p+i,...,p+3, i = 1,2, or 3, gives a minimal imbedding for $K_{p+i-4,2,1}$.

<u>Theorem 8.7</u>. The genus of the graph $K_{p,3,1}$ is given by $\gamma(K_{p,3,1}) = \left\{\frac{p-2}{2}\right\}$, for $p \geq 3$.

<u>Proof</u>: By reasoning similar to that employed in the proof of Theorem 8.6, it suffices to produce a 2-cell imbedding of $K_{p,3,1}$, for all $p \equiv 0 \pmod 2$, for which $F_3 = 6$ and $F_4 = F - F_3$. Such an imbedding is given by:

p_1: $(p+4, p+3, \ldots, 9, 8; 4, 7, 3, 6, 2, 5)$

p_2: $(1, 6; p+4, p+3, \ldots, 8, 7; 5)$

p_3: $(1; 7, 8, \ldots, p+3, p+4; 5, 6)$

p_4: $(1; 8, 9, \ldots, p+3, p+4; 6, 5, 7)$

p_5: $(1, 2, 4, 3)$

p_6: $(1, 3, 4, 2)$

$p_7, p_9, \ldots, p_{p+3}$: $(1, 4, 2, 3)$

$p_8, p_{10}, \ldots, p_{p+4}$: $(1, 3, 2, 4)$

The orbits determined by P for these p_i, $i = 1, \ldots, p+4$, are:

(i) $F_3 = 6$: 1-2-6; 1-3-7; 1-4-8;

1-5-2; 1-6-3; 1-7-4.

(ii) $F_4 = F - F_3$: 1-(p+4)-3-5

1-(2m-1)-4-2m, $m = 5, \ldots, \frac{p}{2}+2$

1-(2m-2)-3-(2m-1), $m = 5, \ldots, \frac{p}{2}+2$

2-(p+4)-4-6

3-6-4-5

4-7-2-5

2-(2m-1)-3-2m, $m = 4, \ldots, \frac{p}{2}+2$

2-(2m-2)-4-(2m-1), $m = 5, \ldots, \frac{p}{2}+2$

Here we have $F = F_3 + F_4 = 10 + 3(\frac{p}{2} + 2 - 4) + (\frac{p}{2} + 2 - 3) = 2p + 3$, which checks with the Euler formula. Removing vertex (p+4) in the above representation gives a minimal imbedding of $K_{p-1,3,1}$.

<u>Theorem 8.8</u>. The genus of the graph $K_{p,2,2}$ is given by $\gamma(K_{p,2,2}) = \left\{\frac{p-2}{2}\right\}$, for $p \geq 2$.

<u>Proof</u>: We have only to add two suitably chosen edges in the

imbedding of Theorem 8.7, and delete an appropriate edge, to obtain a minimal imbedding for $K_{p,2,2}$, for all $p \neq 2$. Recall that $\gamma(K_{2,2,2}) = 0$, by the result of Ringel and Youngs. For $p \neq 2$, delete edge (1,2); and add edge (2,3) in face 2-7-3-8, and edge (2,4) in face 2-(p+4)-4-6. The permutation representation becomes:

P_1: (p+4,p+3,...,9,8;4,7,3,6,5)

P_2: (6,4;p+4,p+3,...,9,8;3,7,5)

P_3: (1,7,2;8,9,...,p+3,p+4;5,6)

P_4: (1;8,9,...,p+3,p+4;2,6,5,7)

P_5: (1,2,4,3)

P_6: (1,3,4,2)

$P_7, P_9, \ldots, P_{p+3}$: (1,4,2,3)

$P_8, P_{10}, \ldots, P_{p+4}$: (1,3,2,4)

For p odd, delete vertex p+4 from the above imbedding.

The general method of proof for the remaining cases $q+r \leq 6$ follows that employed above. We give the pertinent permutations for each case, but omit the straightforward counting of orbits.

<u>Theorem 8.9</u>. The genus of the graph $K_{p,4,1}$ is given by $\gamma(K_{p,4,1}) = \left\{\frac{3(p-2)}{4}\right\}$, for $p \geq 4$.

<u>Proof</u>: (i) For $p \equiv 2 \pmod 4$, the following imbedding has $F_3 = 8$ and $F_4 = F - F_3$:

P_1: (p+5,p+4,...,11,10;5,9,4,8,3,7,2,6)

P_2: (1,7;p+5,p+2,p+3,p+4;p+1,p-2,p-1,p;...;15,12,13,14;8,11, 10,9,6)

P_3: (1,8;11,12,...,p+4,p+5;6,9,10,7)

P_4: (1,9,6;p+2,p+5,p+4,p+3;p-2,p+1,p,p-1;...;12,15,14,13;7, 10,11,8)

P_5: (1,10,11,8;14,15;18,19;...;p+4,p+5;7;13,12;17,16;...; p+3,p+2;6,9)

P_6: (1,2,5,4,3)

P_7: (1,3,4,5,2)

P_8: (1,4,2,5,3)

P_9: (1,5,2,3,4)

P_{10}: (1,4,3,2,5)

P_{11}: (1,3,5,2,4)

$P_{12},P_{16},\ldots,P_{p+2}$: (1,2,4,5,3)

$P_{13},P_{17},\ldots,P_{p+3}$: (1,3,5,4,2)

$P_{14},P_{18},\ldots,P_{p+4}$: (1,5,2,4,3)

$P_{15},P_{19},\ldots,P_{p+5}$: (1,3,4,2,5)

(ii) For $p \equiv 1 \pmod 4$, remove vertex p+5 from the imbedding in case (i). The genus is unaffected. Here, $F_6 + 2F_8 + 3F_{10} = 3$.

(iii) For $p \equiv 0 \pmod 4$, remove vertices p+4 and p+5 from the permutation representation of case (i). This lowers the genus by one, which may most readily be seen from the Euler formula, noting that $\Delta V = -2$, $\Delta E = -10$, and $\Delta F = -6$. We have $F_6 + 2F_8 = 2$, unless $p = 4$, in which case $F_3 = 7$, $F_4 = 4$, and $F_5 = F_6 = 1$.

(iv) For $p \equiv 3 \pmod 4$, the following imbedding has $F_3 = 8$, $F_6 = 1$, $F_4 = F - 9$:

p_1: (p+5,p+4,...,11,10;5,9,4,8,3,7,2,6)

p_2: (1,7;p+5,p+2,p+3,p+4;p+1,p-2,p-1,p;...;16,13,14,15;9,8,12,10,11,6)

p_3: (1,8,9,11,10;13,14,...,p+4,p+5;6,12,7)

p_4: (1,9;15,16;19,20;...;p+4,p+5;7,10,11,12,6;14,13;18,17;...;p+3,p+2;8)

p_5: (1,10,7,12,8;p+2,p+5,p+4,p+3;p-2,p+1,p,p-1;...;13,16,15,14;6,11,9)

p_6: (1,2,5,4,3)

p_7: (1,3,5,4,2)

p_8: (1,4,5,2,3)

p_9: (1,5,3,2,4)

p_{10}: (1,2,3,4,5)

p_{11}: (1,4,3,5,2)

p_{12}: (1,2,5,3,4)

$p_{13},p_{17},\dots,p_{p+2}$: (1,2,5,4,3)

$p_{14},p_{18},\dots,p_{p+3}$: (1,3,4,5,2)

$p_{15},p_{19},\dots,p_{p+4}$: (1,4,2,5,3)

$p_{16},p_{20},\dots,p_{p+5}$: (1,3,5,2,4)

<u>Theorem 8.10</u>. The genus of the graph $K_{p,3,2}$ is given by $\gamma(K_{p,3,2}) = \left\{\frac{3(p-2)}{4}\right\}$, for $p \geq 3$.

<u>Proof</u>: For p = 3, we have $\gamma(K_{3,3,2}) \geq 1$ by Kuratowski's theorem, since $K_{3,3}$ is clearly a subgraph. But $\gamma(K_{3,3,2}) \leq \gamma(K_{3,3,3}) = 1$, by the result of Ringel and Youngs. Thus the theorem holds for p = 3. For $p \geq 4$, we add the three edges (2,3), (2,4), and (2,5), and delete the edge (1,2), in the imbedding given in Theorem 8.9. For $p \not\equiv 3 \pmod 4$, we then have:

P_1: $(p+5, p+4, \ldots, 11, 10; 5, 9, 4, 8, 3, 7, 6)$

P_2: $(7; p+5, p+2, p+3, p+4; p+1, p-2, p-1, p; \ldots; 15, 12, 13, 14; 8, 4, 11, 10, 3, 9, 5, 6)$

P_3: $(1, 8; 11, 12, \ldots, p+4, p+5; 6, 9, 2, 10, 7)$

P_4: $(1, 9, 6; p+2, p+5, p+4, p+3; p-2, p+1, p, p-1; \ldots; 12, 15, 14, 13; 7, 10, 11, 2, 8)$

P_5: $(1, 10, 11, 8; 14, 15; 18, 19; \ldots; p+4, p+5; 7; 13, 12; 17, 16; \ldots; p+3, p+2; 6, 2, 9)$

$P_6, \ldots, P_{p+5}$: as in Theorem 8.9 for $p \equiv 2 \pmod 4$.

Recall that for the cases $p \equiv 1 \pmod 4$ and $p \equiv 0 \pmod 4$, we must remove vertices $p+5$ and then $p+4$ respectively. For $p \equiv 3 \pmod 4$, we have:

P_1: $(p+5, p+4, \ldots, 11, 10; 5, 9, 4, 8, 3, 7, 6)$

P_2: $(7; p+5, p+2, p+3, p+4; p+1, p-2, p-1, p; \ldots; 16, 13, 14, 15; 4, 9, 3, 8, 12, 10, 11, 5, 6)$

P_3: $(1, 8, 2, 9, 11, 10; 13, 14, \ldots, p+4, p+5; 6, 12, 7)$

P_4: $(1, 9, 2; 15, 16; 19, 20; \ldots; p+4, p+5; 7, 10, 11, 12, 6; 14, 13; 18, 17; \ldots; p+3, p+2; 8)$

P_5: $(1, 10, 7, 12, 8; p+2, p+5, p+4, p+3; p-2, p+1, p, p-1; \ldots; 13, 16, 15, 14; 6, 2, 11, 9)$

$P_6, \ldots, P_{p+5}$: as in Theorem 8.9 for $p \equiv 3 \pmod 4$.

<u>Theorem 8.11</u>. The genus of the graph $K_{p,5,1}$ is given by $\gamma(K_{p,5,1}) = p-2$, for $p \geq 5$.

<u>Proof</u>: We distinguish two cases, p odd and p even. In either case, the imbedding presented has $F_3 = 10$, $F_4 = F - 10 = 3p - 5$.

Case (i); p odd:

P_1: (p+6,p+5,...,12,11;6,10,5,9,4,8,3,7,2)

P_2: (1,7,10,9,8;11,12,...,p+5,p+6)

P_3: (1,8,9,11,10;13,12;15,14;...;p+6,p+5;7)

P_4: (1,9,10,11;p+6,p+5,...,13,12;7,8)

P_5: (1,10,7;12,13,...,p+5,p+6;11,8,9)

P_6: (1,11,9,8,7;p+5,p+6;p+3,p+4;...;12,13;10)

P_7: (2,1,3,6,4,5)

P_8: (3,1,4,6,5,2)

P_9: (4,1,5,6,3,2)

P_{10}: (5,1,6,3,4,2)

P_{11}: (6,1,2,5,4,3)

$P_{12},P_{14},...,P_{p+5}$: (1,6,3,5,4,2)

$P_{13},P_{15},...,P_{p+6}$: (1,2,4,5,3,6)

Case (ii); p even:

P_1: (p+6,p+5,...,12,11;6,10,5,9,4,8,3,7,2)

P_2: (1,7,9,11,10,8;12,13,...,p+5,p+6)

P_3: (1,8,10,12,11,9;13,14,...,p+5,p+6;7)

P_4: (1,9,7;14,13;16,15;...;p+6,p+5;11,12,10,8)

P_5: (1,10,11;p+5,p+6;p+3,p+4;...;13,14;7,12,8,9)

P_6: (1,11,12,7;p+6,p+5,...,14,13;9,8,10)

P_7: (2,1,3,6,5,4)

P_8: (3,1,4,6,5,2)

P_9: (4,1,5,6,3,2)

P_{10}: (5,1,6,4,3,2)

P_{11}: (6,1,4,5,2,3)

P_{12}: (1,2,5,6,3,4)

$P_{13},P_{15},\ldots,P_{p+5}$: (1,3,6,5,4,2)

$P_{14},P_{16},\ldots,P_{p+6}$: (1,2,4,5,6,3)

Theorem 8.12. The genus of the graph $K_{p,4,2}$ is given by $\gamma(K_{p,4,2}) = p-2$, for $p \geq 4$.

Proof: Here we distinguish three cases. The imbeddings in the first two of these cases are derived from those of Theorem 8.11, deleting edge (1,4) and adding edges (4,2), (4,3), (4,5), and (4,6) within appropriate faces.

Case (i); p odd:

P_1: (p+6,p+5,...,12,11;6,10,5,9,8,3,7,2)

P_2: (1,7,10,4,9,8;11,12,...,p+5,p+6)

P_3: (1,8,9,11,4,10;13,12;15,14;...;p+6,p+5;7)

P_4: (9,2,10,3,11;p+6,p+5,...,13,12;5,7,6,8)

P_5: (1,10,7,4;12,13,...,p+5,p+6;11,8,9)

P_6: (1,11,9,8,4,7;p+5,p+6;p+3,p+4;...;12,13;10)

$P_7,\ldots,P_{p+6}$: as in Theorem 8.11, for p odd.

Case (ii); p even; $p \geq 6$:

P_1: (p+6,p+5,...,12,11;6,10,5,9,8,3,7,2)

P_2: (1,7,4,9,11,10,8;12,13,...,p+5,p+6)

P_3: (1,8,10,4,12,11,9;13,14,...,p+5,p+6;7)

P_4: (9,2,7,5;14,13;16,15;...;p+6,p+5;11,12,3,10,6,8)

P_5: (1,10,11;p+5,p+6;p+3,p+4;...;13,14;4,7,12,8,9)

p_6: (1,11,12,7;p+6,p+5,...,14,13;9,8,4,10)

$p_7,\ldots,p_{p+6}$: as in Theorem 8.11, for p even.

Case (iii); p = 4:

p_1: (10,6,9,5,8,4,7,3)

p_2: (3,9,4,10,5,7,6,8)

p_3: (10,1,7,9,2,8)

p_4: (7,1,8,10,2,9)

p_5: (8,1,9,7,2,10)

p_6: (9,1,10,8,2,7)

p_7: (3,1,4,6,2,5)

p_8: (4,1,5,3,2,6)

p_9: (5,1,6,4,2,3)

p_{10}: (6,1,3,5,2,4)

Theorem 8.13. The genus of the graph $K_{p,3,3}$ is given by $\gamma(K_{p,3,3}) = p-2$, for $p \geq 3$.

Proof: Here we treat four cases. The imbeddings in the first two of these cases are derived from those of Theorem 8.11, deleting edges (1,3) and (1,5), and adding edges (i,2), (i,4), and (i,6), for i = 3,5.

Case (i); p odd, $p \geq 5$:

p_1: (p+6,p+5,...,12,11;6,10,9,4,8,7,2)

p_2: (1,7,5,10,9,3,8;11,12,...,p+5,p+6)

p_3: (8,2,9,6,11,4,10;13,12;15,14;...;p+6,p+5;7)

p_4: (1,9,10,3,11;p+6,p+5,...,13,12;5,7,8)

p_5: (10,2,7,4;12,13,...,p+5,p+6;11,8,6,9)

p_6: (1,11,3,9,5,8,7;p+5,p+6;p+3,p+4;...;12,13;10)

$p_7,\ldots,p_{p+6}$: as in Theorem 8.11, for p odd.

Case (ii); p even, $p \geq 6$:

P_1: (p+6,p+5,...,12,11;6,10,9,4,8,7,2)

P_2: (1,7,9,11,5,10,3,8;12,13,...,p+5,p+6)

P_3: (8,2,10,4,12,6,11,9;13,14,...,p+5,p+6;7)

P_4: (1,9,7,5;14,13;16,15;...;p+6,p+5;11,12,3,10,8)

P_5: (10,2,11;p+5,p+6;p+3,p+4;...;13,14;4,7,6,12,8,9)

P_6: (1,11,3,12,5,7;p+6,p+5,...,14,13;9,8,10)

$P_7,\ldots,P_{p+6}$: as in Theorem 8.11, for p even.

Case (iii); p = 3: $\gamma(K_{3,3,3}) = 1$, by the result of Ringel and Youngs.

Case (iv); p = 4:

P_1: (10,6,9,5,8,4,7)

P_2: (7,6,10,4,9,8,5)

P_3: (5,9,8,6,7,4,10)

P_4: (7,1,8,9,2,10,3)

P_5: (8,1,9,3,10,7,2)

P_6: (9,1,10,2,7,3,8)

P_7: (1,4,3,6,2,5)

P_8: (4,1,5,2,6,3)

P_9: (5,1,6,2,4,3)

P_{10}: (6,1,5,3,4,2)

We now combine Theorems 8.6 through 8.13 into one theorem:

<u>Theorem 8.14</u>. The genus of the graph $K_{p,q,r}$ is given by $\gamma(K_{p,q,r}) = \left\{\frac{(p-2)(q+r-2)}{4}\right\}$, where $p \geq q \geq r$ and $q+r \leq 6$.

We have conjectured that the above result holds for all values of q+r. It is likely that the case q+r = 7 could be handled as above, and then q+r = 8, and so on; but some more general approach would seem to be desirable.

As a result of Theorem 8.14, we can make the following observation:

Theorem 8.15. In any minimal imbedding of $K_{p,q,r}$, for $q+r \le 6$, every handle carries at least one edge of type II or of type III.

Proof: Assume to the contrary that there is a handle of this surface of genus $\left\{\frac{(p-2)(q+r-2)}{4}\right\}$ which carries only edges of type I. Removing all type I edges in the imbedding, we obtain an imbedding of the graph $K_{p,q+r}$ on the same surface. But the handle that formerly carried only edges of type I now contains no part of the graph $K_{p,q+r}$, and hence the face of the new imbedding containing this handle is not a 2-cell. This imbedding of $K_{p,q+r}$, then, is not minimal, so that $\gamma(K_{p,q+r}) < \left\{\frac{(p-2)(q+r-2)}{4}\right\}$, contradicting Ringel's formula for the genus of complete bipartite graphs.

We conclude this chapter with one further result concerning the genus of complete tripartite graphs:

Theorem 8.16. The genus of the graph $K_{mn,n,n}$ is given by $\gamma(K_{mn,n,n}) = \frac{(mn-2)(n-1)}{2}$, for all natural numbers m and n.

Proof: It suffices to produce an imbedding of $K_{mn,n,n}$ for which $F_3 = 2n^2$ and $F_4 = F - F_3$. We start with the following imbedding of $K_{mn,2n}$ (which differs from Ringel's imbedding for the same graph, unless n = 2), having $F = F_4 = mn^2$ and $\gamma = \frac{(mn-2)(n-1)}{2}$:

$$V(i) = \begin{cases} \{2n+1,\dots,2n+mn\}, & i = 1,\dots,2n \\ \{1,\dots,2n\}, & i = 2n+1,\dots,2n+mn \end{cases}$$

$p_1, p_3, \ldots, p_{2n-1}$: $(2n+1, 2n+2, \ldots, 2n+mn)$

$p_2, p_4, \ldots, p_{2n}$: $(2n+mn, 2n+mn-1, \ldots, 2n+1)$

p_{2n+i}: $(1, 2i, 3, 2i-2, 5, 2i-4, 7, \ldots, 2n-1, 2i+2)$, $i = 1, \ldots, mn$;

where arithmetic is modulo 2n, and we write 2n instead of 0.

The orbits (faces) are:

$(2j-1)-(2n+i)-(2i-2j+2)-(2n+i-1)$, for $j = 1, \ldots, n$; $i = 2, \ldots, mn$

$(2j-1)-(2n+i)-(2i-2j+2)-(2n+mn)$, for $j = 1, \ldots, n$; $i = 1$, where the third entry only, in each of the above representations of orbits, is reduced modulo 2n, with 2n being written instead of 0.

We now add edges $(2j-1, 2k)$, $j = 1, \ldots, n$ and $k = 1, \ldots, n$, through the faces determined by $i = k + j - 1$ respectively. These are precisely the n^2 edges of type I needed to convert the graph $K_{mn,2n}$ to the graph $K_{mn,n,n}$. Each such edge destroys one quadrilateral face and creates two triangular faces, so that $K_{mn,n,n}$ is imbedded with $F_3 = 2n^2$ and $F_4 = F - F_3 = mn^2 - n^2$. This is the desired imbedding.

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