ON OPTIMAL FIELDS FOR DIFFERENTIAL GAMES

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

JOHN WALTER WINGATE

1968

This is to certify that the

thesis entitled

ON OPTIMAL FIELDS FOR DIFFERENTIAL GAMES

presented by

John Walter Wingate

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Electrical Engineering

Theodore Guinn
Major professor

Date June 6, 1968

O-169

ABSTRACT

ON OPTIMAL FIELDS FOR DIFFERENTIAL GAMES

by John Walter Wingate

The study of differential games is the study of game theory as applied to processes of the type considered in optimal control theory. Almost all differential games studied have been two-person zero-sum games. This is due partly to limitations in general game theory and partly to the type of differential games most often studied--pursuit-evasion games.

The process in a differential game is modeled by vector differential equations

$$\frac{dx}{dt} = f\ (t,\ x,\ u,\ v) \tag{1}$$

where the independent variable t is called the time, x the state, and u and v are called control variables. These variables, u and v, are chosen by two opposing players, one of whom wishes to maximize and the other to minimize a functional

$$J = K\ (t_1,\ x(t_1)) + \int_{t_0}^{t_1} L\ (t,\ x(t),\ u(t),\ v(t))dt \tag{2}$$

which depends on a solution to (1) on a time interval $t_0 \leq t \leq t_1$. The initial point $(t_0,\ x(t_0))$ is in a region $\underline{F}$ in tx-space, while the terminal point $(t_1,\ x(t_1))$ belongs to

a set T which may be taken to be part of the boundary of F. Functions $U(t,x)$ and $V(t,x)$ which give choices of the control variables u and v to use at each point of the region F are called strategies. Given a pair of strategies and an initial point in F, the payoff (2) is determined.

In an optimal field one assumes that there exist strategies U and V optimal in some sense (this sense being specified for a particular type of optimal field) and a value function W, also defined on F. The value function is closely related to the payoff functional (2). The optimal strategies in an optimal field are taken to be piecewise continuous and have piecewise continuous first partial derivatives. The value function is assumed to be continuous and have piecewise continuous first partial derivatives.

Two types of optimal fields are considered: one of which requires the value function to satisfy a saddle-point condition, and the other of which requires the value function to satisfy a maximin (or, alternatively, a minimax) condition. The saddle-point condition is the more stringent requirement. It corresponds to a solution to the differential game in pure strategies (that is, those chosen directly by the player, without the assistance of a random device). The maximin or minimax optimal fields are applicable to differential games which do not have solutions of this type.

The results obtained are extensions of optimal field and Hamilton-Jacobi theory for optimal control problems. The extension is to fields defined by saddle-point conditions and to fields defined by maximin conditions. Several useful discontinuity conditions, distinguished by the behavior of the optimal trajectories in the neighborhood of the discontinuity, are also obtained.

ON OPTIMAL FIELDS FOR DIFFERENTIAL GAMES

By

John Walter Wingate

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering

1968

ACKNOWLEDGMENTS

The author wishes to thank the members of the guidance committee, Dr. R. C. Dubee, Dr. T. Guinn, Dr. H. G. Hedges, Dr. H. E. Koenig, and Dr. P. K. Wong for their advice, aid and understanding during the time of his doctoral candidacy. He wishes to express particular thanks to Dr. Theodore Guinn for the able direction he gave in carrying out the research for this thesis.

The author also wishes to thank the National Science Foundation for fellowship support and the Department of Electrical Engineering for assistantship support while he was a graduate student.

TABLE OF CONTENTS

LIST OF FIGURES

I. INTRODUCTION

1.1 Game Theory, Optimal Control Theory, and Differential Games

The theory of differential games brings together two originally separate branches of applied mathematics--the theory of games and optimal control theory. It draws on ideas and concepts from both of these fields.

Game theory had its origins in the work of von Neumann who wrote a pioneering article in 1928 [30]. During the Second World War he wrote, in collaboration with Morgenstern, the classic book on the subject, Theory of Games and Economic Behavior [31]. Almost all later workers take as a basis the theory developed by von Neumann and Morgenstern.

A game is a situation in which several persons make decisions. The essential feature of a game is that these decisions must be made on the basis of conflicting interests. (One could consider a game in which the decision makers have completely parallel interests as a degenerate case.) A game must have well-defined outcomes which depend on the decisions made and perhaps on chance factors. Each of the players (the decision-makers) evaluates these outcomes according to some criterion. In general, the players will not agree in their evaluations; this is the source of conflict. Normally the evaluation assigns a real number to each outcome of the game. If a player prefers outcomes with higher numerical values, the evaluation is called his payoff; if he prefers lower numerical values, the term used is cost. Games are usually presented in terms of payoffs rather than in terms of costs. The decisions made by the players, and perhaps chance occurrences, determine the outcome of the game, and hence the payoffs. No one player controls the game completely. In general his payoff is as much determined by the actions of the other players as by his own. Game theory addresses itself to the problem of prescribing, in some fashion, rational behavior under those circumstances.

Most games are defined by a set of rules. The rules prescribe the structure of the game, the manner in which it is played, which player must make a decision at any particular stage of the game, the information available to this player--in fact everything but the actual choices made by the players (and by chance). Games which are described in this manner are said to be in extensive form. After a long careful development, von Neumann and Morgenstern give a precise, axiomatic definition of a game in extensive form [31, section 10]. In a game in extensive form, a plan detailing what choice to make, on the basis of the available information, under every situation which could arise, is called a strategy. If each player chooses a strategy, the course and outcome of the game are determined, except for chance effects. These chance effects can be eliminated by considering the expected values of the payoffs instead of the payoffs themselves. An equivalent game can be generated in which each player makes one choice only, from the set of all strategies available to him, in complete ignorance of the particular choices of the other players. He will, however, be aware of the strategies available to the others (through knowing the rules of the game, for instance). This equivalent game is in the normalized form. More precisely, a game for N players in normalized form consists of:

N sets of strategies S_i, $i = 1, \ldots, N$;

N real-valued payoff functions P_i, $i = 1, \ldots, N$,

with the domain of each of these functions being

$$S \triangleq S_1 \times \ldots \times S_N.$$

Each player chooses independently a strategy from his set of strategies. If $s_1, s_2, \ldots, s_N$ are the strategies chosen, $s = (s_1, s_2, \ldots, s_N)$ is the corresponding point in S, and $P_i(s)$ gives the (expected value of the) payoff to the i^{th} player corresponding to the play of the game in which these strategies are used.

Each of the two formulations has its advantages: the normalized form is most useful when considering features common to all games, and the extensive form emphasizes the peculiarities of individual games.

A satisfactory solution theory does not exist for general games. One does exist for certain types of two-person zero-sum games. A zero-sum game is one in which

$$\sum_{i=1}^{N} P_i(s) = 0 \quad \text{for all } s \varepsilon S.$$

An N-person game which is not a zero-sum game can be made into one by adding another player who has the payoff

$$P_{N+1}(s) = -\sum_{i=1}^{N} P_i(s).$$

The set of strategies for this player, S_{N+1} is, of course, empty. N-person games are generally studied by dividing the players into two coalitions and considering the resulting two-person games for various divisions of this sort. It can be seen then that the theory of two-person zero-sum games plays a large part in the theory of games as a whole.

The basic idea behind the solution to a two-person zero-sum game is guaranteed payoff. If the first player chooses a strategy s_1 his payoff could be as little as

$$\min_{s_2 \varepsilon S_2} P_1 (s_1, s_2).$$

However, by choosing his strategy to be $\bar{s}_1$, the maximin strategy, where

$$\min_{s_2 \varepsilon S_2} P_1 (\bar{s}_1, S_2) = \max_{s_1 \varepsilon S_1} \min_{s_2 \varepsilon S_2} P_1 (s_1, s_2) = \underline{w},$$

he can guarantee that his payoff will be at least $\underline{w}$. Likewise the second player can guarantee that his payoff is at least

$$\max_{s_2 \varepsilon S_2} \min_{s_1 \varepsilon S_1} P_2 (s_1, s_2) = - \min_{s_2 \varepsilon S_2} \max_{s_1 \varepsilon S_1} P_1 (s_1, s_2) = - \bar{w}.$$

In Chapter II it is shown that $\underline{w} \leq \bar{w}$. If $\underline{w} = \bar{w}$, the first player can gain, and the second player can lose, neither more nor less than $\bar{w}$, provided both players use their maximin strategies. In this case, the solution consists of the maximin strategies, $\bar{s}_1$, $\bar{s}_2$ and the payoff $P_1 (\bar{s}_1, \bar{s}_2)$, called the value of the game.

If $\underline{w} < \bar{w}$, in certain types of two-person zero-sum games, such as those in which both players have a finite number of strategies, the sets of strategies can be extended in such a way that with respect to the extended strategies the game has a solution. Usually one considers the set of probability distributions over the original set of strategies (called pure strategies) as the extended set of strategies (called mixed strategies). Every two-person zero-sum game in which each player has a finite number of strategies has a solution in mixed strategies. This result was first obtained by von Neumann [30].

Game theory is only one of the two areas which converge in differential games. The other is optimal control theory.

Optimal control theory is concerned with finding a maximum or a minimum (usually a minimum) of a functional such as

$$K\ (t_1,\ x(t_1)) + \int_{t_0}^{t_1} L\ (t,\ x(t),\ u(t))\ dt,$$

where x(t) and u(t) satisfy the differential equation

$$\frac{dx}{dt} = f\ (t,\ x,\ u).$$

The points $(t_0,\ x(t_0))$ and $(t_1,\ x(t_1))$ are constrained by boundary conditions and the functions u must belong to a certain class of functions (such as measurable functions, or piecewise continuous functions) and must satisfy certain constraints. Optimal control problems can be translated into problems in the Calculus of Variations, and optimal control theory developed by interpreting variational theory. This has been done, for example, by Hestenes [15] and Berkovitz [1]. Alternatively, the theory can be developed more or less independently of Calculus of Variations as it is usually formulated. This is the approach taken by Pontryagin and his colleagues in the Soviet Union [36]. In recent years optimal control theory has expanded rapidly.

From the point of view of game theory an optimal control problem is a one player game. If additional players are added to problems to this type, a differential game results.

A differential game is a game in extensive form, or rather a family of such games indexed by points (initial conditions for the differential equations) in a region F to tx-space. Instead of having a single value (if the game has a solution), the game has a value function whose domain is this region; that is, the game associated with the point (t, x) has the value W(t, x). A play of a differential game evolves as a process continuous in time modeled by a system of differential equations in which certain parameters, called the control variables, are chosen by the players. The solution to the system of differential equations is a curve lying in the region F, and the transition between points on this curve provides the link between members of the family of games. The players receive payoffs which are functionals of the solution to the differential equations and of the control variables. These functionals are of the same type as those considered for performance criteria in optimal control theory.

For reasons which have been mentioned previously, almost all differential games studied have been two-person zero-sum games. The differential games covered in the succeeding chapters are all two-person zero-sum games. Accordingly, only one payoff functional is introduced, and the concepts of maximin points and equilibrium points are discussed with respect to this single payoff. This is not an important restriction since most of the solution concepts for general games rely on equilibrium points or maximin points of the type considered [14, 26, 29, 43]. The appropriate modifications for more general games readily suggest themselves.

The study of differential games was initiated by Isaacs in a series of Rand reports [19]. These were later collected with additional material and published as a book [20]. Differential games were treated in a series of articles in the Contributions to the Theory of Games, volume III by Fleming [7], Scarf [42], and Berkovitz and Fleming [5]. Berkovitz, in later articles [2, 3] in 1964 and 1967 entended the class of games covered. His approach is to show that under certain restrictions on the types of discontinuity allowed in the controls that the differential game has an associated optimal field. In this thesis, such optimal fields are considered *per se*, and the necessary conditions obtained for these fields can be applied to any differential game which has an associated optimal field, whether or not it satisfies the discontinuity conditions of Berkovitz. These optimal fields are an extension of the type of optimal field considered by Hestenes for optimal control problems [16].

Differential games can be approximated by difference games. Fleming [8, 9] has investigated conditions under which the solutions to a sequence of approximating games converge to the solution of the differential game. Meschler [27] has used a method based on Fleming's work to compute the solution to a specific differential game.

Many of the differential games studied have been pursuit-evasion games, in which one controlled object pursues another controlled object taking evasive action. Most of the work done in the Soviet Union has dealt with pursuit games. Kelendzheridze, in an article [21], and also in a section of the well-known book by Pontryagin _et al_. [36], considered a class of pursuit-evasion games. His work was followed by further developments by Pontryagin [37, 38], Petrosyan [82, 33, 34, 35] and others [23, 24, 25]. A differential pursuit game of prescribed duration was studied by Ho, Bryson and Baron [18] to illustrate conjugate point conditions and the use of the matrix Ricatti equation in differential games. Since in most cases in pursuit games, the capture will occur on the boundaries of the attainable sets for the pursuing and evading objects, it is useful, as in optimal control theory, to study boundary arcs in differential games. This has been done by Guinn [13]. More general (i.e., those not necessarily involving differential equations) pursuit games have been studied by Zięba [47, 48], Mycielski [28], Ryll-Nardzewski [40] and Varaiya [46].

Differential games have also been studied in connection with minimax problems in optimal control [10, 11, 12, 22], and with optimal control problems involving uncertainty [39, 41]. In the latter case it would seem useful to consider one of the players, "Nature," as being indifferent to the outcome. In such a two-person nonzero-sum game the payoff to Nature may be taken to be a constant function--say zero. Optimal control problems of this type are problems of guaranteed performance rather than problems requiring a saddle-point for a solution. This also appears to be the approach taken by Pontryagin [37, 38], and Krasovskii, Repin and Tret'yakov [25] to pursuit problems.

Optimal fields can be defined for such problems also, even when a saddle-point does not exist. This is done in Chapter IV of this thesis. As was mentioned previously, the optimal field theory presented in Chapter III is closely related to the work of Berkovitz [3] and is a direct extension of the optimal field theory used by Hestenes [16]. The optimal fields are studied by themselves without direct reference to a differential game.

However, to apply to a differential game the optimal strategies for the field must be the optimal strategies for a differential game--that is they must be functions with domain F giving the optimal control at each point (t, x) of F. Also the value function for the field must be the value function for the game. This game must have the same integral payoff functional and satisfy the same constraints as those used in the optimal field. Berkovitz [3] obtains conditions equivalent to the existence of an optimal field when the optimal strategies satisfy certain conditions on the form of their discontinuities. Other strategies, however, need not satisfy these conditions. The discontinuity conditions are not needed, if one starts with the optimal field rather than with the differential game.

An excellent overview of differential games is the survey paper by Simakova appearing in Automation and Remote Control [44]. In the article she pays particular attention to the work of Isaacs [20], the convergence theory of Fleming [8, 9], and the pursuit games considered by Pontryagin [37, 38].

The second chapter of this thesis presents a definition of a differential game with discussions of the concepts introduced in the definition, and considers the solution of a differential game in terms of maximin points and equilibrium points (which are saddle-points in two-person zero-sum games). In the third chapter, after a preliminary section introducing optimal fields for control problems (following Hestenes [16]), optimal fields for saddle-points are defined and necessary conditions for such fields are derived. In the fourth chapter, a similar procedure is carried out when one of the optimal strategies is a maximin strategy and the other is a strategy which minimizes the payoff when this maximin strategy is used. In the chapter following this one, transversality conditions and several conditions applicable to discontinuities in the strategies are obtained. Hestenes [16] does not consider optimal fields with discontinuous strategies. Strategies which are piecewise continuous and have piecewise continuous derivatives are considered in Chapters III, IV and V. This represents an extension of Hestenes' work. Berkovitz [3] considers games with saddle-points. Consequently, many of the theorems in Chapter III are similar to theorems obtained by him under somewhat different hypotheses. He does not however consider maximin strategies apart from saddle points. Nor, in connection with optimal fields, does anyone else. The results in Chapter IV, are, to the author's knowledge, new, and similar results have not appeared elsewhere. Some of the conditions in Chapter V were obtained by Berkovitz; others are new.

The next section of this chapter contains several definitions and theorems on differential equations.

1.2 Auxiliary Theorems

The definition of a piecewise continuous function of a scalar variable is well-known. Perhaps not so well-known is a definition applicable to functions defined on a region $\underline{X}$ of points x in E^n. The definition follows that of Berkovitz [3] and is based on a decomposition of the region $\underline{X}$.

Definition: A decomposition of a region $\underline{X}$ is a finite collection of subregions $\{\underline{X}^{(\alpha)}\}$ such that

(i) $\bigcup_{\alpha} \overline{\underline{X}^{(\alpha)}} = \overline{\underline{X}}$

(ii) $\underline{X}^{(\alpha)} \cap \underline{X}^{(\beta)} = \phi$ if $\alpha \neq \beta$

(iii) each $\underline{X}^{(\alpha)}$ is connected and has a piecewise smooth boundary.

By a piecewise smooth boundary it is meant that the boundary consists of the union of the closures of a finite number of (n-1)-dimensional manifolds each of which can be described parametrically by equations of the form (where x is a point of one of these manifolds):

$$x = X(a),\ a \in A,\ \text{a region in } E^{n-1},$$

where the function X is $C^{(1)}$ on A.

<u>Definition</u>: A real-valued function g defined on $\underline{X}$ is <u>piecewise continuous</u> on $\underline{X}$ if there exists a decomposition $\{\underline{X}^{(\alpha)}\}$ of $\underline{X}$ such that for each α there is a continuous function $g^{(\alpha)}$ defined on $\overline{\underline{X}^{(\alpha)}}$ for which

$$g(x) = g^{(\alpha)}(x), \quad x \epsilon \underline{X}^{(\alpha)}.$$

The function g is piecewise $C^{(m)}$ if the functions $g^{(\alpha)}$ are $C^{(m)}$ on the $\overline{\underline{X}^{(\alpha)}}$.

If g is a vector function with real-valued components it is piecewise $C^{(m)}$ if there is a decomposition of $\underline{X}$ for which each component satisfies the above definition. If $g_1, g_2, \ldots g_p$ are several vector or scalar functions with the domain $\underline{X}$, the $C^{(m)}$-decomposition associated with these functions $\{\underline{X}^{(\alpha)}\}$, is the "coarsest" decomposition for which each component of these functions satisfies the above definition. By "coarsest" it is meant that any other decomposition contains a decomposition of at least one of the $\underline{X}^{(\alpha)}$.

The following Lagrange multiplier rule is Theorem 10.1 of Chapter 1 of <u>Calculus of Variations and Optimal Control Theory</u> by M. R. Hestenes [16]. The statement has been slightly modified.

Consider the minimization of a function f on a set S of points u in E^q. The set S is defined by the relations

$$\begin{aligned} &\phi^{\alpha}(u) \leq 0 \quad \alpha = 1, \ldots, m' \\ &\phi^{\alpha}(u) = 0 \quad \alpha = m' + 1, \ldots, m \end{aligned} \tag{1.2.1}$$

Let u_0 afford a local minimum to f on S. It is assumed that f and ϕ^{α}, $\alpha = 1, \ldots, m$, are continuous on a neighborhood of u_0 and are differentiable at u_0. Further if $\phi^{\alpha}(u_0) < 0$ for some particular α, this strict inequality holds on a neighborhood of u_0 and thus does not locally constrain the points u. Therefore at u_0, the set S is locally determined by those α for which $\phi^{\alpha}(u_0) = 0$. It is assumed in the theorem that this holds for $\alpha = 1, \ldots, m$.

The theorem is stated without proof.

Theorem 1.1

Let u_0 afford a local minimum to f on S, and suppose that the matrix

$$\left[\frac{\partial \phi^{\alpha}}{\partial u^k} (u_0) \right] \tag{1.2.2}$$

has rank m. Then there exist unique multipliers $\lambda_1, \ldots, \lambda_m$, with $\lambda_{\alpha} \geq 0$ for $\alpha = 1, \ldots, m'$, such that

$$F_{u^k}(u_0) = 0, \quad k = 1, \ldots, q, \tag{1.2.3}$$

where

$$F = f + \lambda_{\alpha} \phi^{\alpha}.$$

If $\phi^{\bar\alpha}(u_0) < 0$ for some $\bar\alpha$, one could include this constraint in F by defining the corresponding multiplier $\lambda_{\bar\alpha}$ to be zero. In this way one obtains the corollary:

Corollary

Suppose that at u_0, $\phi^{\bar\alpha}(u_0) = 0$ for $r \leq m$ indices $\bar\alpha$. Then if the matrix (1.2.2) has rank r where the index α runs over those $\bar\alpha$ for which $\phi^{\bar\alpha}(u_0) = 0$, the conclusions of the theorem hold, and in addition

$$\lambda_\alpha \phi^\alpha(u_0) = 0, \ \alpha \text{ not summed.} \qquad (1.2.4)$$

Theorem 1.2

Suppose f, $\phi^1, \ldots, \phi^m$ are real-valued functions on a region $\underline{R}$ of points $(x,u) = x^1, \ldots, x^p, u^1, \ldots, u^q)$. The functions f, ϕ^α, f_{u^k} and $\phi^\alpha_{u^k}$ are continuous on $\underline{R}$. Let $\underline{R}_0$ be the set of points satisfying the relations

$$\begin{aligned} &\phi^\alpha(x,u) \leq 0 \qquad \alpha = 1, \ldots, m' \\ &\phi^\alpha(x,u) = 0 \qquad \alpha = m'+1, \ldots, m. \end{aligned} \qquad (1.2.5)$$

Let α_j, $j = 1, \ldots, r$ be the indices α for which $\phi^\alpha(x,u) = 0$. (These indices need not be the same for each point (x,u)). It is assumed that the matrix

$$\left| \frac{\partial \phi^\alpha(x,u)}{\partial u^k} \right| \qquad \begin{aligned} &\alpha = \alpha_j, \ j = 1, \ldots, r; \\ &k = 1, \ldots, q \end{aligned} \qquad (1.2.6)$$

has rank r at each point (x,u) of $\underline{R}_0$. Let $\underline{X}$ be a set contained in the projection of $\underline{R}_0$ into x-space, and let U be a piecewise continuous function with domain $\underline{X}$ for which $(x, U(x))\epsilon\underline{R}_0$ whenever $x\epsilon\underline{X}$, with the property that

$$f(x, U(x)) \leq f(x,u), x\epsilon\underline{X}, (x,u)\epsilon\underline{R}_0. \tag{1.2.7}$$

Then there exists a unique set of multipliers

$$\lambda_\alpha(x), x\epsilon\underline{X}, \alpha = 1, \ldots, m, \tag{1.2.8}$$

such that if one makes the definition

$$F(x,u,\lambda) = f(x,u) + \lambda_\alpha\phi^\alpha(x,u)$$

then

$$F_u(x, U(x), \lambda(x)) = 0, x\epsilon\underline{X} \tag{1.2.9}$$

Further, for $\alpha = 1, \ldots, m'$

$$\left.\begin{array}{l} \lambda_\alpha(x) \geq 0, x\epsilon\underline{X} \\ \text{and} \\ \lambda_\alpha(x) = 0 \text{ whenever } \phi^\alpha(x, U(x)) < 0. \end{array}\right\} \tag{1.2.10}$$

The multipliers λ_α are piecewise continuous on $\underline{X}$ and are continuous at each point of continuity of U.

<u>Proof</u>.--The existence and uniqueness of the multipliers $\lambda_\alpha(x)$ at each point x of $\underline{X}$ follow from the rank properties of the matrix (1.2.6) and the Lagrange multiplier rule, Theorem 1.1, and its corollary. This theorem also gives the sign properties (1.2.10).

The continuity properties of the multipliers remain to be proved. To do this let $\bar{x}$ be a point of $\underline{X}$ and let $\bar{u} = U(\bar{x})$. If $\bar{x}$ is a point of discontinuity of U, one of the limiting values can be chosen. Let $\alpha_1, \ldots, \alpha_r$ be the values of α for which

$$\phi^{\alpha}(\bar{x},\bar{u}) = 0$$

There is a subset of $\underline{X}$ containing $\bar{x}$ on which U and hence $\phi^{\alpha}(\bar{x}, U(\bar{x}))$ are continuous. This subset can be restricted to a set N containing $\bar{x}$ such that

$$\phi^{\beta}(x, U(x)) < 0, \; x \epsilon N, \; \beta \neq \alpha_j, \; j = 1, \ldots, r.$$

This gives

$$\lambda_{\beta}(x) = 0 \quad x \epsilon N, \quad \beta \neq \alpha_j, \; j = 1, \ldots, r.$$

Define

$$A_j^k = \frac{\partial \phi^{\alpha_j}}{\partial u^k}(\bar{x},\bar{u}), \; j = 1, \ldots, r_i \quad k = 1, \ldots, q.$$

Then

$$\left| \phi_k^{\alpha}(x) \; A_j^k \right| \neq 0 \quad \alpha = \alpha_1, \ldots, \alpha_r; \; j = 1, \ldots, r, \quad (1.2.11)$$

at the point $\bar{x}$, where $\phi_k^{\alpha}(x) = \phi_{u^k}^{\alpha}(x, U(x))$.

The determinant in (1.2.11) is continuous in x on N. Consequently, N can be diminished so that (1.2.11) holds on N

Setting

$$f_k(x) = f_{u^k}(x, U(x)),$$

one obtains

$$F_{u^k}(x, U(x), \lambda(x)) = f_k(x) + \lambda_\alpha(x)\ \phi_k^\alpha\ (x) = 0,\ x\varepsilon N.$$

Since

$$\lambda_\alpha(x) = 0 \text{ when } \phi^\alpha(x, U(x)) < 0,$$

$$F_{u^k}(x, U(x), \lambda(x)) = f_k(x) + \lambda_{\alpha_i}(x)\phi_k^{\alpha_i}(x) = 0 \quad \text{on } N.$$

Multiplying by A_j^k gives

$$f_k(x)\ A_j^k + \lambda_{\alpha_i}(x)\phi_k^{\alpha_i}(x)\ A_j^k = 0,\ x\varepsilon N.$$

Since the matrix $[\phi_k^{\alpha_i}(x)A_j^k]$ is nonsingular on N, these equations can be solved uniquely for the multipliers $\lambda_{\alpha_i}(x)$. Furthermore, if f_{u^k}, $\phi_{u^k}^{\alpha_i}$ and U are of class $C^{(n)}$ on N, the λ_{α_i} are also of class $C^{(n)}$. In particular, it follows that the multipliers λ_α have the stated continuity properties.

This theorem is an extension of the theorem given by Hestenes in [16; Theorem 4.1 of chapter 5], which covered the case of x a scalar.

Theo

modi

are

Ass

is

set

f a

pos

Whi

In

bou

or

Whe

fro

tin

Theorem 1.3

Suppose that the hypotheses of Theorem 1.2 are modified so that the functions f and f_{u^k}, $k = 1, \ldots, q$, are continuous in u and piecewise continuous in x on $\underline{R}$. Associated with these piecewise continuous functions there is a decomposition of $\underline{R}$ and also a decomposition of the set $\underline{X}$ introduced in the previous theorem, $\{\underline{X}_i\}$, such that f and f_{u^k} are continuous in x on each set of these decompositions. To denote the functions continuous in x on $\overline{\underline{X}}_i$ which agree with f and f_{u^k} on $\underline{X}_i$, one may use $f^{(i)}$ and $f_{u^k}^{(i)}$.

Then multipliers λ_α exist with the properties stated in theorem 1.2 provided that a point $\bar{x}$ on the common boundary of $\underline{X}_i$ and $\underline{X}_j$ one defines $F(\bar{x},u,\lambda)$ to be either

$$f^{(i)}(\bar{x},u) + \lambda_\alpha^{(i)}\phi^\alpha(\bar{x},u)$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (1.2.12)

$$f^{(j)}(\bar{x},u) + \lambda_\alpha^{(j)}\phi^\alpha(\bar{x},u)$$

where $\lambda_\alpha^{(i)}$ and $\lambda_\alpha^{(j)}$ are the limiting values of λ_α at $\bar{x}$ from $\underline{X}_i$ and $\underline{X}_j$ respectively. The multipliers λ_α are continuous at each point of continuity in x of U, f, and f_{u^k}.

Proof

One may apply Theorem 1.2 directly to the case where $\underline{X}$ is replaced by $\underline{\bar{X}}_i$, f by $f^{(i)}$ and f_{u^k} by $f_{u^k}^{(i)}$ for each $\underline{X}_i$ in the decomposition of $\underline{X}$. Since the resulting multipliers $\lambda_\alpha^{(i)}$ are piecewise continuous on $\underline{\bar{X}}_i$ they can be combined to form piecewise continuous functions on the set $\underline{X}$. This method of combination requires the interpretation (1.2.12) at points of discontinuity of f and f_{u^k}.

Some of the theorems in Chapters III and IV make use of existence, embedding and differentiability theorems for differential equations. The following theorems are taken without proof from the appendix to Hestenes' Calculus of Variations and Optimal Control Theory [16]. The hypotheses are weaker than are required for the applications in the later chapters. Similar theorems with stronger hypotheses can be found in Bliss [6] and any differential equations text.

The differential equations, in vector form are

$$\dot{x} = f(t,x,\lambda), \tag{1.2.13}$$

where x is a vector in n-dimensional euclidean space $\underline{E}^n$, $\dot{x} = dx/dt$, and λ is an element of a normed linear space. For example, λ may be a control function

$$u: \quad u(t), \quad a \leq t \leq b$$

with the norm $||u|| = \sup |u(t)|$ on $a \leq t \leq b$. However, the parameter λ is not restricted to control functions.

Although the results are independent of the norm used, it is convenient to consider the norm

$$|x| = \max |x^i|, \quad i = 1, \ldots, n.$$

With this norm, a δ- neighborhood of a point (α,β) in tx-space is

$$\{(t,x) \mid |t-\alpha|<\delta \text{ and } |x-\beta|<\delta\}$$

Hypotheses

It is assumed that the real-valued functions f^i in the differential equations (1.2.13) are defined for all (t,x) in a region $\underline{F}$ of tx-space and λ in a subset of a Λ of a normed linear space.

Moreover, to each $(\alpha,\beta) \in \underline{F}$, assume there is a constant δ and two integrable functions $M(t)$, $K(t)$ such that

1. the δ-neighborhood of (α,β) is in $\underline{F}$;
2. For each x in the δ-neighborhood β_δ of β and for each $\lambda \in \Lambda$, the functions $f^i(t,x,\lambda)$ are measurable in t on the δ-neighborhood α_δ of α and satisfy

$$|f(t,x,\lambda)| \leq M(t) \tag{1.2.14}$$

on α_δ. Thus $f(t,x,\lambda)$ is integrable on $\alpha-\delta < t < \alpha + \delta$ for each x in β_δ and λ in Λ;

3. For each x and y in β_δ and each λ in Λ, the inequality

$$|f(t,x,\lambda)-f(t,y,\lambda)| \leq K(t)\,|\,x-y\,| \qquad (1.2.15)$$

holds on α_δ;

4. For each x in β_δ and λ_0 in Λ

$$\lim_{\lambda\to\lambda_0} \int_{\alpha-\delta}^{\alpha+\delta} |f(t,x,\lambda)-f(t,x,\lambda_0)|\,dt = 0. \qquad (1.2.16)$$

<u>Lemma</u>

Let S be a compact subset of $\underline{F}$ which is convex in x. Then there is a δ-neighborhood S_δ of S in $\underline{F}$ and integrable functions M(t), K(t) such that

$$|f(t,x,\lambda)| \leq M(t) \qquad (1.2.17a)$$

$$|f(t,x,\lambda)-f(t,y,\lambda)| \leq K(t)\,|\,x-y\,| \qquad (1.2.17b)$$

hold for all points (t,x), (t,y) in S_δ and all $\lambda\varepsilon\Lambda$.
In the following theorem S is a compact subset of $\underline{F}$ convex in x. M, K and δ are related to S as described in the previous lemma.

<u>Theorem 1.4</u>

There exists a constant $\rho>0$ with $\rho<\delta$ such that to each point (α,β) in S and λ in Λ there exists a unique solution

$$x(t,\alpha,\beta,\lambda), \quad \alpha - \rho \leq t \leq \alpha + \rho$$

of the initial value problem

$$\frac{dx}{dt} = f(t,x,\lambda), \quad x(\alpha) = \beta.$$

The function $x(t,\alpha,\beta,\lambda)$ is a continuous function of its arguments on the set

$$|t-\alpha| \leq \rho, \ (\alpha,\beta) \in S, \ \lambda\in\Lambda$$

Let λ_0 be a fixed element of Λ and let

$$\underline{x}: \ x(t), \quad a \leq t \leq b$$

be a solution of the differential equations

$$\frac{dx}{dt} = f(t,x,\lambda_0). \tag{1.2.18}$$

This solution must lie in $\underline{F}$. The closure S of an ε-neighborhood of the points $(t, x(t))$ of the arc $\underline{x}$ is in $\underline{F}$. The Lemma and Theorem 1.4 can be applied to this set S. Using the existence theorem, the function $x(t)$ can be extended uniquely so as to satisfy (1.2.18) on $a-\rho\leq t\leq b+\rho$, where $\rho\leq\varepsilon$. Define

$$G(x,\lambda,\lambda_0) = \int_{a-\rho}^{b+\rho} |f(t,x(t),\lambda) - f(t,x(t),\lambda_0)| \, dt.$$

One can establish the following embedding theorem.

Theorem 1.5

There is a positive number σ such that through each point (α,β) satisfying with λ

$$a-\rho\leq\alpha\leq b+\rho, \quad |\beta-x(\alpha)| < \sigma, \quad |\lambda-\lambda_0| < \sigma \tag{1.2.19}$$

there passes a unique solution

$$y(t,\alpha,\beta,\lambda), \quad a-\rho\leq t\leq b+\rho,$$

of the equations

$$\frac{dx}{dt} = f(t,x,\lambda)$$

containing the arc $\underline{x}$ for $a\leq t\leq b$, $\lambda=\lambda_0$, $\beta=x(\alpha)$.

The function y is continuous in its arguments.

There is a constant C such that

$$|y(t,\alpha,\beta,\lambda) - x(t)| \leq C\,|\beta-x(\alpha)| + CG(x,\lambda,\lambda_0)$$

on $a-\rho\leq t\leq b+\rho$. Moreover if $(\alpha',\beta',\lambda')$ is on the set (1.2.19), this inequality holds if $x(t)$ is replaced by $y(t,\alpha',\beta',\lambda')$ and λ_0 by λ'. In addition

$$|\beta-y(\alpha,\alpha',\beta',\lambda')| \leq |\beta-\beta'| + \left|\int_{\alpha'}^{\alpha} M(s)ds\right|$$

Linear differential equations

$$\frac{dx^i}{dt} = A^i_j(t)\, x^j + v^i(t)$$

or

$$\dot{x} = Ax + v$$

where A^i_j and v^i are integrable on an interval $a \leq t \leq b$ have unique solutions, on this interval through each point (α,β) with $a \leq \alpha \leq b$ and $\beta \varepsilon \underline{E}^n$. If A^i_j and v^i are extended so that they are integrable on the real line $-\infty < t < \infty$, by defining them to be zero outside $[a,b]$ for example, these solutions exist and are unique on $-\infty < t < \infty$.

Suppose now that Hypothesis 3 is replaced by:

3a. At each point (t,x) in the δ-neighborhood of (α,β) and for each $\lambda \varepsilon \Lambda$, the partial derivatives $f^i{}_{x^j}(t,x,\lambda)$ exist and satisfy

$$|f^i{}_{x^j}(t,x,\lambda)| \leq K(t). \tag{1.2.20}$$

It is easily seen that this hypothesis implies the original hypothesis 3 with $K(t)$ replaced by $nK(t)$.

With this hypothesis the following differentiability theorem can be obtained.

Theorem 1.6

Under the additional hypothesis 3a, the solution

$y(t,\alpha,\beta,\lambda) \quad a-\rho \leq t \leq b+\rho$

of the equations $\dot{x} = f(t,x,\lambda)$ is differentiable with respect to β^i at each point (t,α,β,λ) on the set

$$a-\rho \leq t \leq b+\rho, \quad a-\rho \leq \alpha \leq b+\rho, \quad |\beta - x(\alpha)| < \sigma \quad |\lambda-\lambda_0| < \sigma \qquad (1.2.21)$$

and is differentiable with respect to α at each point (t,α,β,λ) on this set at which $\dot{y}(t,\alpha,\beta,\lambda)$ exists. Moreover, the determinant

$$\left| \frac{\partial y^i(t,\alpha,\beta,\lambda)}{\partial \beta^j} \right| \neq 0$$

on the interval $a-\rho \leq t \leq b+\rho$.

One may also note that on the set (1.2.21)

$y(\alpha,\alpha,\beta,\lambda) = \beta$ and the matrix

$\frac{\partial y(\alpha,\alpha,\beta,\lambda)}{\partial \beta} = I$, the identity.

The theorems quoted are Lemma 2.1, Theorem 3.1, Theorem 4.1, and Theorem 7.1 of [16, appendix].

II. DEFINITIONS AND SOLUTION CONCEPTS FOR DIFFERENTIAL GAMES

2.1 Definition of a Differential Game

Playing Space

Unless otherwise stated, x is a vector in n-dimensional euclidean space, $\underline{E}^n$, u is a vector in $\underline{E}^p$, and v is a vector in $\underline{E}^q$. Two regions are also considered. The first, $\underline{R}$, is a region in the (1+n+p+q)-dimensional space of points (t,x,u,v), and the second, $\underline{F}$ is a region of points (t,x) in $\underline{E}^{n+1}$. In later sections other regions will be defined as they are needed. It is assumed that the region $\underline{F}$ is contained in the projection of $\underline{R}$ into tx-space. $\underline{F}$ is known as the playing space, for it is in this region that the solution curves of the differential equations introduced below lie, and this region is the domain of the value function and the strategy functions.

The scalar t is called time. The vector x is known as the state, a quantity which characterizes the process which is under the competing control of the two players. The time derivative of the state is given explicitly by the differential equations modeling the process. One of the players, henceforth called player One has the variable u at his disposal. Likewise, player Two controls v. The vectors u and v are consequently called the <u>control variables</u>.

<u>Differential Equations</u>

The game proceeds in accordance with a system of differential equations

$$\frac{dx^i}{dt} = f^i(t,x^1,x^2,\ldots,x^n,u^1,u^2,\ldots,u^p,v^1,v^2\ldots,v^q)$$

$$i=1,2,\ldots,n \tag{2.1.1a}$$

where the f^i are real-valued continuous functions with domain $\underline{R}$. These differential equations may be expressed in the vector form

$$\dot{x} = f(t,x,u,v) \tag{2.1.1b}$$

The history of choices made by the first player over a time-interval is a vector function u of time t, $t_0 \leq t \leq t_1$. Player Two likewise chooses a function v. A solution of (2.1.1) together with the functions u and v is called an <u>arc</u>. Specifically, a <u>differentiably admissible</u> arc is the entity

$$\underline{x}: \quad x(t),\ u(t),\ v(t), \quad t_0 \leq t \leq t_1$$

where x is a solution of (2.1.1) with the control functions u and v, which are required to be piecewise continuous functions of t.

Since the hypotheses to some theorems are more restrictive than for others, differentiability conditions for the functions f^i are not stated here but are introduced as needed.

Initial Condition

The initial condition is a point (t_0,x_0) in the playing space $\underline{F}$. Separate games start from each point in $\underline{F}$; the differential game is a family of such games indexed by the initial conditions (t_0,x_0). The differential equations link members of the family and allow it to be studied as a whole.

A play of the game continues for $t \geq t_0$ according to the equations (2.1.1) with the initial condition $x(t_0)=x_0$ until termination occurs, that is until the path x intersects the terminal surface for the first time.

Terminal Surface

A play of the game ends when the path first intersects an n-dimensional manifold $\underline{T}$ which is part of the boundary of $\underline{F}$. The surface $\underline{T}$ is parametrized by a vector σ in a region $\underline{K}$ of $\underline{E}^n$ by

$$(t,x)\epsilon\underline{T} \Longleftrightarrow t=T(\sigma),\ x^i=X^i(\sigma),\ i=1,\dots,n,\ \sigma\epsilon\underline{K} \quad (2.1.2)$$

The functions T and x^i are assumed to be continuous, and T_{σ^j} and $X^i_{\sigma^j}$ to be piecewise continuous.

The need for a surface to terminate the game is related to the concepts of strategy and solution of the game and is considered in the section on solutions.

Payoff

A payoff is defined for each differentiably admissible arc

$$\underline{x}: \quad x(t),\ u(t),\ v(t), \quad t_0 \le t \le t_1$$

having

$$(t,\ x(t))\ \varepsilon \underline{F}, \quad t_0 \le t \le t_1$$

and

$$(t_1,\ x(t_1))\ \varepsilon \underline{T};$$

i.e. $t_1 = T(\sigma)$, $x(t_1) = X(\sigma)$ for some $\sigma \varepsilon \underline{K}$ (2.1.3)

This payoff, the payoff to player One, is the real-valued functional

$$J(\underline{x}) \triangleq K(\sigma) + \int_{t_0}^{t_1} L(t, x(t),\ u(t),\ v(t))\ dt \qquad (2.1.4)$$

The payoff to player Two is $-J(\underline{x})$. Player One, consequently wishes to maximize J, player Two to minimize it.

In the next subsection piecewise continuous strategies U and V which are functions with the domain $\underline{F}$ are introduced. With such strategies, the arc $\underline{x}$ is given by $u(t) = U(t, x(t))$, $v(t) = V(t, x(t))$; that is the arc is given by

$$\underline{x}: \quad x(t),\ U(t,x(t)),\ V(t,x(t)),\quad t_0 \le t \le t_1 \qquad (2.1.5)$$

Such an arc may not be unique. There may be several arcs (2.1.5), depending on which limiting values of U or V are assigned to U or V at their manifolds of discontinuity. If there are ν such arcs, they can be designated $\underline{x}_1$, $\underline{x}_2$, ..., $\underline{x}_\nu$, with corresponding payoffs $J(\underline{x}_1)$, $J(\underline{x}_2)$, ..., $J(\underline{x}_\nu)$. To indicate the dependence of the payoff on the initial condition and the strategies, the following notation is useful:

$$J^{(\alpha)}(t_0,x_0;\ U,V) \triangleq J(\underline{x}_\alpha)$$

$$= K(\sigma) + \int_{t_0}^{t_1} L(t,x(t),\ U(t,x(t)),\ V(t,x(t)))\,dt \qquad (2.1.6)$$

where

$$\dot{x}(t) = f(t,x(t),\ U(t,x(t)),\ V(t,x(t)),\quad x(t_0) = x_0.$$

Strategy and Information

A strategy, in the sense introduced by von Neumann and Morgenstern [31], is a plan detailing, on the basis of the information available to a particular player, what action he is to take under any conceivable situation in the game. If the information given to the player is the same for several different situations, he will not be able to distinguish one from another, and a strategy, based on the information, will prescribe the same course of action in each of the indistinguishable circumstances.

A differential game in which each player is continuously informed of t and x--the time and state variables--is said to be a game of perfect information. This is usually understood to mean that any further information is superfluous and does not aid in choosing the control variables. Isaacs [20, p. 26] states this idea as follows (Isaacs considers autonomous differential equations; consequently the state x provides sufficient information): "The x^i are descriptive in the following sense. If a play of a differential game is halted before completion, the values of $x^1,\ldots,x^n$ at the time of interruption supply all the data needed to resume the partie. We mean that if a new partie is commenced starting with these x^i, it will be tantamount to the part of the original that would have occurred after the interruption." It is also tantamount to assuming that Bellman's principle of optimality, or, as Isaacs calls it, the tenet of transition, holds. In a game of perfect information, for payoffs of the type (2.1.4) and dynamic systems modeled by (2.1.1) this is indeed so. In such games it is also assumed that both players know the equations (2.1.1), the payoff functional (2.1.4) and any constraints which are operating. They are not aware of their opponent's choice of control.

Under these circumstances, a strategy could be defined to be a real-valued vector function with the domain $\underline{F}$. The choice of the control variable u for player One at the point (t,x) in $\underline{F}$ would then be given by $u = U(t,x)$, where U is a strategy. Player Two, likewise, would choose $v = V(t,x)$ for a strategy V defined on $\underline{F}$. This thesis treats games of perfect information with strategies of this type.

If less information is available, strategies could be defined to correspond to this reduced state of knowledge. For example, if the players are given only the initial condition and not the subsequent state history, a strategy could reasonably be only a function of time, say u(t), $t \geq t_0$. Some of the consequences of games of this type are considered in the examples in section 2.3.

The players are not normally allowed a completely free choice in the functions chosen for strategies. They are usually restricted to belong to some particular class of functions--the class of piecewise continuous functions, for example--and to be constrained so that the points (t,x,u,v) lie in a specified subset of $\underline{R}$. The set of strategies satisfying these conditions is the set of admissible strategies.

Constraints

The constraint subset of $\underline{R}$ may be defined in several ways. One is to define it as the set $\underline{R}_0$ of points in $\underline{R}$ satisfying constraints of the form

$$\phi^\alpha (t,x,u) \leq 0 \quad \alpha = 1, 2, \ldots, r' \qquad (2.1.7a)$$

$$\phi^\alpha (t,x,u) = 0 \quad \alpha = r'+1, \ldots, r \qquad (2.1.7b)$$

$$\psi^\beta (t,x,v) \leq 0 \quad \beta = 1, 2, \ldots, s' \qquad (2.1.7c)$$

$$\psi^\beta (t,x,v) = 0 \quad \beta = s'+1, \ldots, s \qquad (2.1.7d)$$

The ϕ^α and ψ^β, which are $C^{(1)}$ functions of this arguments on $\underline{R}$, must satisfy, at each point (t,x,u,v) in $\underline{R}_0$, the conditions that the matrices

$$\left[\frac{\partial\phi^\alpha}{\partial u^j}\right], \quad \alpha = \alpha_1, \alpha_2, \ldots, \alpha_\pi; \; j = 1, 2, \ldots p \qquad (2.1.8a)$$

and

$$\left[\frac{\partial\psi^\beta}{\partial v^k}\right] \quad \beta = \beta_1, \beta_2, \ldots, \beta_\rho; \; k = 1, 2, \ldots, q \qquad (2.1.8b)$$

have ranks π and ρ respectively, where $\alpha_1, \ldots, \alpha_\pi$ are the indices α for which $\phi^\alpha (t,x,u) = 0$, and $\beta_1, \ldots, \beta_\rho$ are the indices β for which $\psi^\beta(t,x,v) = 0$. It is clear that this requires $\pi \leq p$ and $\rho \leq q$.

In a true game, the constraints for each player operate independently, for otherwise the players would have to cooperate in selecting strategies, an evident absurdity in two-person zero-sum games, where the players are completely opposed. Constraints of the form

$$\phi(t,x,u,v) \leq 0 \tag{2.1.8}$$

which are not independent of either u or v (but not both) are not allowed. While such constraints are not part of a game as described here, constraints of this type may be of interest in other situations, for example, in discriminatory games, in which one player makes his choice of control variable while cognizant of the other players choice.

If the players are to make independent choices, the set $\underline{R}_0$, whether or not it is determined by constraints of the form (2.2.7), must be of the form

$$\underline{R}_0 = (\underline{R}_1 \times \underline{E}^q) \cap (\underline{E}^p \times \underline{R}_2) \tag{2.1.9}$$

where $\underline{R}_1$ and $\underline{R}_2$ are prescribed sets in txu- and txv-space respectively. Let $\underline{F}_1$ ($\underline{F}_2$) be the projection of $\underline{R}_1$ ($\underline{R}_2$) into tx-space. Then it is assumed that the playing space, $\underline{F}$ is a subset of $\underline{F}_1 \cap \underline{F}_2$. Point-to-set functions Φ and Ψ can be defined on $\underline{F}_1$ and $\underline{F}_2$ by

$$\Phi(t,x) \triangleq \{ u \mid (t,x,u) \,\varepsilon \underline{R}_1 \} \tag{2.1.10a}$$

$$\Psi(t,x) \triangleq \{ v \mid (t,x,v) \,\varepsilon \underline{R}_2 \} \tag{2.1.10b}$$

The domain of Φ is $\underline{F}_1$ and that of Ψ is $\underline{F}_2$. Constraints of the form (2.1.7) can obviously be restated in the present form, since $\underline{R}_1$ is the set of (t,x,u) satisfying (2.1.7a) and (2.1.7b), and $\underline{R}_2$ is the set of (t,x,v) satisfying (2.1.7c) and (2.1.7d). The statement that (t,x,u,v) is an element of the prescribed set $\underline{R}_0$ is equivalent to the statements

$$u \,\varepsilon\, \Phi\,(t,x) \tag{2.1.11a}$$

$$v \,\varepsilon\, \Psi\,(t,x) \tag{2.1.11b}$$

Elements (t,x,u,v) in $\underline{R}_0$ are called <u>admissible elements</u>, and a differentiably admissible arc whose elements (t,x(t), u(t), v(t)) are all admissible and for which $(t,x(t)\varepsilon\underline{F}$ is called simply an <u>admissible arc</u>.

The set of all p-dimensional vector functions U which are piecewise $C^{(1)}$ on $\underline{F}$ and which satisfy $U(t,x)\,\varepsilon\,\Phi(t,x)$ on $\underline{F}$ is defined to be the set $\underline{U}$ of <u>admissible strategies</u> for player One. The set $\underline{V}$ of admissible strategies for player Two is similarly defined as the set of all q-dimensional piecewise $C^{(1)}$ functions V with domain $\underline{F}$ satisfying $V(t,x)\varepsilon\Psi(t,x)$ on $\underline{F}$.

An assumed property of $\underline{R}_0$ is that to each element $(\bar{t},\bar{x},\bar{u})\varepsilon\underline{R}$, with $(\bar{t},\bar{x})\varepsilon\underline{F}$ there exists a function $u(t)$ with $u(\bar{t}) = \bar{u}$ continuous on the set

$$A = \{ t \mid \bar{t} - \delta \le t \le \bar{t}, \quad \delta < 0\}$$

or on the set

$$B = \{ t \mid \bar{t} \le t \le \bar{t} + \delta, \quad \delta < 0\}$$

such that the arcs

$$\underline{x}: \quad x(t),\ u(t),\ V(t,x(t)) \begin{cases} t\varepsilon A \ \forall\ V\varepsilon\underline{V} \\ \text{or} \\ t\varepsilon B \ \forall\ V\varepsilon\underline{V} \end{cases} \tag{2.1.12}$$

with $x(\bar{t}) = \bar{x}$, exist and are admissible.
There is a similar condition on $\underline{R}_2$. If there is more than one arc corresponding to a given $V\varepsilon\underline{V}$, each one must be admissible.

Attainable Set

Definition: Let (t_0,x_0) be a point in tx-space. The attainable set $A^+(t_0,x_0)$ is the set of points (t_1,x_1) such that there exists a differentiably admissible arc

$$\underline{x}: \quad x(t),\ u(t),\ v(t) \quad t_0 \le t \le t_1 \tag{2.1.13}$$

with the properties

$$(t,\ x(t),\ u(t),\ v(t)) \ \varepsilon\ \underline{R}_0,\quad t_0 \leq t \leq t_1,$$

$$x(t_0) = x_0,\quad x(t_1) = x_1.$$

Likewise, the set $A^-(t_0,x_0)$ is the set of points (t_1,x_1) such that there exists a differentiably admissible arc

$$\underline{x}:\ x(t),\ u(t),\ v(t)\qquad t_1 \leq t \leq t_0 \tag{2.1.14}$$

with the properties

$$(t,\ x(t),\ u(t),\ v(t)) \ \varepsilon\ \underline{R}_0 \text{ for } t_1 \leq t \leq t_0,$$

$$x(t_0) = x_0,\quad x(t_1) = x_1.$$

The set $A(t_0,x_0) \ \underline{\Delta}\ A^-(t_0,x_0) \cup A^+(t_0,x_0)$ is called the <u>extended attainable set</u>. The <u>attainable set at time τ</u>, $A\ (\tau;\ t_0,x_0)$ is the intersection of $A(t_0,x_0)$ with the plane $t = \tau$.

It can be seen that $A^-(t_0,x_0)$ can be defined as

$$A^-(t_0,x_0) = \{(\bar{t},\bar{x})\ |\ (t_0,x_0)\ \varepsilon\ A^+(\bar{t},\bar{x})\}.$$

Let S_0 be a given set in tx-space. Then the extended attainable set for S_0 is defined by

$$A\ (S_0)\ \underline{\Delta} \bigcup_{(t_0,x_0)\varepsilon S_0} A(t_0,x_0).$$

Similarly

$$A^{\pm}(S_0) \triangleq \bigcup_{S_0} A^{\pm}(t_0,x_0).$$

If the arcs $\underline{x}$ defining the attainable set all have $v(t) = V(t,x(t))$ for some particular $V\varepsilon\underline{V}$, this restriction is indicated by the notations $A(t_0,x_0,V)$, $A^+(t_0,x_0,V)$, etc. $A(t_0,x_0,U)$ is similarly defined.

If the arcs $\underline{x}$ are also required to satisfy

$$u(t) = U(t,x(t)) \text{ for some } U\varepsilon\underline{U}$$

$$v(t) = V(t,x(t)) \text{ for some } V\varepsilon\underline{V}$$

on $[t_0,t_1]$ (or $[t_1,t_0]$), the resulting attainable sets can be distinguished by designating them $A^{*+}(t_0,x_0)$, $A^*(t_0,x_0)$, etc. It is clear that each of the attainable sets so obtained are subsets of the corresponding attainable obtained without this restriction. For example

$$A^* (S_0) \subseteq A (S_0)$$

$$A^{*+}(S_0) \subseteq A^+(S_0)$$

$$A^*(\tau,S_0) \subseteq A(\tau,S_0).$$

Playable Pairs

A pair of strategies (U, V), $U \varepsilon \underline{U}$, $V \varepsilon \underline{V}$, is said to be a playable pair if for every $(t_0, x_0) \varepsilon \underline{F}$, each solution curve of the differential equation

$$\dot{x} = f(t,x,U(t,x), V(t,x)), \quad x(t_0) = x_0 \tag{2.1.15}$$

intersects the terminal surface $\underline{T}$ at some finite time t_1 and is interior to $\underline{F}$ on the interval $[t_0, t_1]$. Not every pair of admissible strategies is necessarily playable.

In order to define a solution to a game, a well-defined payoff for each pair of admissible strategies is needed. If the pair of strategies is playable, the payoff functional (2.1.6) can be used to evaluate the payoff. But the question of how to treat admissible pairs which are not playable arises. This can be approached in a number of ways.

One is the obvious method of defining it directly for nonplayable pairs. An example will illustrate this procedure. Suppose that player One, the evader, controls the motion of a point x_E in x-space, and that player Two, the pursuer, controls the motion of a point x_P in the same space, according to the differential equations:

$$\dot{x}_E = f_E(t, x_E, U(t, x_E)), \quad U \varepsilon \underline{U}$$

$$\dot{x}_P = f_P(t, x_P, V(t, x_P)), \quad V \varepsilon \underline{V}.$$

Termination--capture--is said to occur when the distance between x_E and x_P drops to a set value r. That is the terminal surface in $x_P x_E$-space is the set of points x_P, x_E satisfying $| x_P - x_E | = r$, assuming that initially the distance between x_P and x_E is greater than r. The payoff is the time at which capture occurs. If for a pair of strategies (U,V) and a particular starting point (t_0, x_{E_0}, x_{P_0}), capture does not occur for any finite time, the payoff for this pair and this initial condition can be given the value $+ \infty$. This does not necessarily mean that $| x_P(t) - x_E(t) | \to r$ as $t \to \infty$.

A second approach, one used by Berkovitz [2,3], is to further restrict the admissible strategies to subsets $\underline{U}_1$ of $\underline{U}$ and $\underline{V}_1$ of $\underline{V}$ which have the property that for every $U \in \underline{U}_1$, $V \in \underline{V}_1$, the pair (U,V) is playable. Its major weakness, as Berkovitz states, is that the restricted sets $\underline{U}_1$ and $\underline{V}_1$ are not necessarily unique, and that there may be no clear way of obtaining them from $\underline{U}$ and $\underline{V}$. On the other hand, there may be obvious candidates. In the pursuit-evasion example above, $\underline{U}_1$ could be set equal to $\underline{U}$ and $\underline{V}_1$ defined to be the set of all V playable against every U in $\underline{U}$. That is, player Two restricts his attention to only those V which guarantee termination in a finite time against every evader strategy. The set $\underline{V}_1$ may be vacuous. In other cases it may not be as clear how to choose $\underline{U}_1$ and $\underline{V}_1$; perhaps one could include them as part of the information given to both players.

A further way around this difficulty is to require the game to be formulated, by suitable modification if necessary, in such a way that (U,V) is a playable pair for every $U\epsilon\underline{U}$ and every $V\epsilon\underline{V}$. An equivalent statement is that $\underline{T}$ must divide the attainable set $A^{*+}(\underline{F})$ into two disjoint sets B and C such that between any point $b\epsilon B$ and any point $c\epsilon C$ every continuous path joining them lying entirely in $A^{*+}(\underline{F})$ will intersect $\underline{T}$. If one uses the attainable set resulting from pairs in the restricted sets $\underline{U}_1$ and $\underline{V}_1$ of the previous paragraph, $\underline{T}$ has this property with respect to this attainable set. Since $\underline{T}$ divides the attainable set into two disjoint sets, it will in general be a surface and thus require n parameters $\sigma^1, \ldots, \sigma^n$ in $\underline{E}^{n+1}$ to specify it. (Since $\underline{T}$ is part of the boundary of $\underline{F}$ one may take these sets to be $\underline{F}$ and $A^{*+}(\underline{F})\sim\underline{F}$.) In the same pursuit-evasion example a stop rule could be introduced by adjoining the surface $t = T$ to the original terminal surface and requiring that play take place for $t \leq T$, where T is some suitably large number. This assumption--that every admissible pair of strategies is playable--is used hereafter unless explicitly stated otherwise.

The need for playable pairs arises from the independence of the players' decisions. There is nothing corresponding to it in optimal-control problems--one player games--where one can ignore arcs which do not satisfy the terminal conditions and confine the optimization to those arcs which do. In games which are inherently discriminatory, the requirement is that the player benefitting from the discrimination has, for each strategy of the opponent, a strategy playable against it. In this case the terminal manifold may have dimension $<n$. It must however have a nonempty intersection with each $A^{*+}(t,x,\underline{U})$, $(t,x)\varepsilon\underline{F}$, $U\varepsilon\underline{U}$, where (for definiteness) One is the player discriminated against.

2.2 Solution Concepts

Maximin and Minimax Points

In this subsection, a discriminatory game starting from some point (t,x) in $\underline{F}$ is considered. Player One picks a strategy U which is communicated to player Two, who, since he is interested in minimizing the payoff, will choose a strategy $\overline{V}$ depending on U such that

$$J\ (t,x;\ U,\ \overline{V}(U)) \leq J\ (t,x;\ U,\ V) \quad \forall\ V\ \varepsilon\ \underline{V}$$

or

$$J\ (t,x;\ U,\ \overline{V}(U)) = \min_{V\varepsilon\underline{V}} J\ (t,x;\ U,\ V) \tag{2.2.1}$$

In order to minimize the loss due to the discriminatory situation, player One can choose a strategy U* such that

$$\underline{W}(t,x) \triangleq J(t,x;\ U^*,\ \overline{V}(U^*)) = \max_{U\varepsilon\underline{U}} J(t,x;\ U,\ \overline{V}(U))$$

$$= \max_{U\varepsilon\underline{U}} \min_{V\varepsilon\underline{V}} J(t,x,\ U,\ V) \tag{2.2.2}$$

$\underline{W}(t,x)$ is called the lower value. (If the maximizing U, U^* does not exist, or if $\overline{V}(U)$ does not exist for each $U \epsilon \underline{U}$, $\underline{W}(t,x)$ could be defined to be $\sup_{U \epsilon \underline{U}} \inf_{V \epsilon \underline{V}} J(t,x;\ U,\ V)$.

However if one wishes to have a determinate optimal strategy for player One, one must have at least $\underline{W}(t,x) = \max_{U \epsilon \underline{U}} \inf_{V \epsilon \underline{V}} J(t,x;\ U,\ V)$.) The strategy U^* is called player One's maximin strategy. By playing it, player One can guarantee a payoff of at least $\underline{W}(t,x)$, regardless of what player Two does.

Player Two likewise has a minimax strategy V^* which guarantees that his loss will be not more than

$$\overline{W}(t,x) = \min_{V \epsilon \underline{V}} \max_{U \epsilon \underline{U}} J(t,x;\ U,\ V) \tag{2.2.3}$$

(i.e., his payoff will be at least $\max_{V \epsilon \underline{V}} \min_{U \epsilon \underline{U}} (-J(t,x;\ U,\ V))$.)

$\overline{W}(t,x)$ is called the upper value. Now $\underline{W}(t,x) \leq \overline{W}(t,x)$, provided (U^*, V^*) is playable--which it is by assumption. This is so because

$$\max_{U \epsilon \underline{U}} \min_{V \epsilon \underline{V}} J(t,x;\ U,\ V) = \min_{V \epsilon \underline{V}} J(t,x;\ U^*,\ V) \leq J(t,x;\ U^*,\ V^*) \tag{2.2.4}$$

and

$$\min_{V\epsilon \underline{V}} \max_{U\epsilon \underline{U}} J(t,x;\ U,\ V) = \max_{U\epsilon \underline{U}} J(t,x:\ U,\ V^*) \leq J(t,x;\ U^*,\ V^*) \tag{2.2.5}$$

The inequalities (2.2.4) and (2.2.5) combine to give

$$\max_{U\epsilon \underline{U}} \min_{V\epsilon \underline{V}} J(t,x;\ U,\ V) \leq \min_{V\epsilon \underline{V}} \max_{U} J(t,x;\ U,\ V) \tag{2.2.6}$$

The notation $\overline{V}(U)$ $(\overline{U}(V))$ will be used to denote, as in (2.2.1), the strategy which minimizes (maximizes) the payoff when the opponent's strategy $U(V)$ is known. If inequality holds in (2.2.6), either

$$\underline{W}(t,x) < J(t,x;\ U^*,\ V^*)$$

or

$$J(t,x;\ U^*,\ V^*) < \overline{W}(t,x)$$

or both. In the first case V^* is not optimal against U^*, for player Two can use $\overline{V}(U^*)$ and lose only $\underline{W}(t,x)$. But he runs the risk of losing more than $\overline{W}(t,x)$ for player One, in anticipation of the use of $\overline{V}(U^*)$, can decide to use $\overline{U}(\overline{V}\ (V^*))$, and

$$J(t,x;\ \overline{U}(\overline{V}(U^*)),\ \overline{V}(U^*)) \geq J(t,x;\ \overline{U}(V^*),\ V^*) = \overline{W}(t,x).$$

This argument is symmetrical in the players, and the process of outguessing the opponent can continue for any number of stages. Any definition of solution would be somewhat arbitrary in this situation. The best that can be done, although not wholly satisfactory, is to consider the solution to consist of the set of maximin strategies (U^*, V^*) with the associated payoff. The pair (U^*, V^*) is then called a maximin point.

Equilibrium Point

If player One plays his maximin strategy U^* he may obtain more than the lower value $\underline{W}$ if his opponent does not play $\overline{V}(U^*)$. That is he may gain if the opponent deviates from the strategy optimal with respect to U^*.

One may also consider the situation in which a pair of strategies has the property that deviations by one player cannot produce any gain provided the opponent does not change his strategy. That is the pair of strategies $(\hat{U}, \hat{V})$ has the property, where P_1 is the payoff to player One, P_2 the payoff to player Two:

$$\begin{aligned} P_1\,(U, \hat{V}) &\leq P_1\,(\hat{U}, \hat{V}) \quad \forall\ U \in \underline{U} \\ P_2\,(\hat{U}, V) &\leq P_2\,(\hat{U}, \hat{V}) \quad \forall\ V \in \underline{V} \end{aligned} \qquad (2.2.7)$$

The pair $(\hat{U}, \hat{V})$ is known as an equilibrium point. In a two-person zero-sum differential game, (2.2.7) becomes

$$J(t,x;\ U, \hat{V}) \leq J(t,x;\ \hat{U}, \hat{V}) \leq J(t,x;\ \hat{U}, V) \qquad (2.2.8)$$

Examination of (2.2.8) shows that $(\hat{U}, \hat{V})$ is also a maximin point and that

$$\max_{U \in \underline{U}} \min_{V \in \underline{V}} J(t,x;\, U,V) = \min_{V \in \underline{V}} \max_{U \in \underline{U}} J(t,x;\, U,V) = J(t,x;\, \hat{U},\hat{V}) \tag{2.2.9}$$

On the other hand, suppose that (U^*, V^*) is a maximin point with $\underline{W}(t,x) = \overline{W}(t,x)$. Is (U^*, V^*) also an equilibrium point? The answer to this question is yes, for

$$\underline{W}(t,x) \le J(t,x;\, U^*,V)$$

$$J(t,x;\, U,\, V^*) \le \overline{W}(t,x)$$

$$\underline{W}(t,x) = \overline{W}(t,x) \Rightarrow J(t,x;\, U^*,\, V^*) = \underline{W}(t,x) = \overline{W}(t,x).$$

Combining these relationships gives

$$J(t,x;\, U,V^*) \le J(t,x;\, U^*,\, V^*) \le J(t,x;\, U^*,\, V) \tag{2.2.10}$$

showing that (U^*, V^*) is an equilibrium point. Thus in two-person zero-sum games the existence of an equilibrium point and the condition that max min J = min max J are equivalent. The term for a pair (U^*, V^*) satisfying (2.2.10) is <u>saddle point</u>. A two-person zero-sum game is said to have a <u>solution</u> if it has a saddle point. At a saddle point a player cannot gain and may lose by deviating from his maximin strategy. This is the reason for identifying the solution of the game with its saddle point. It can also be seen that if a saddle point exists in the discriminatory game described at the beginning of this section the extra information gives no advantage to player Two.

Saddle Points in Differential Games

No mention has yet been made in this section of the dynamic character of a differential game. Maximin and equilibrium strategies have been mentioned with respect to only one point of the playing space $\underline{F}$. If there exists a pair of strategies, $U^* \varepsilon \underline{U}$ and $V^* \varepsilon \underline{V}$ such that (2.2.10) holds for all (t,x) in $\underline{F}$, all U in $\underline{U}$ and all V in $\underline{V}$, then (U^*, V^*) is a saddle point for the differential game. The corresponding payoff, $J(t,x; U^*, V^*)$, the common value of $\underline{W}(t,x)$ and $\overline{W}(t,x)$, is the value function $W(t,x)$. W is a function with domain $\underline{F}$.

It was previously mentioned that there may be several arcs satisfying

$$\dot{x} = f(t,x, U(t,x), V(t,x)), \; x(t_0) = x_0 \tag{2.2.11}$$

for a given pair (U, V). The saddle point condition must be restated to encompass such cases. Suppose that for a given point $(t,x)\varepsilon\underline{F}$ there are λ arcs corresponding to the pair (U, V^*), μ to the pair (U^*, V^*) and ν to the pair (U^*, V). Then the saddle point condition is that

$$J^{(\alpha)}(t,x; U, V^*) \leq J^{(\beta)}(t,x; U^*, V^*) \leq J^{(\gamma)}(t,x; U^*, V)$$

$$\alpha = 1, \ldots, \lambda; \; \beta = 1, \ldots, \mu; \; \gamma = 1, \ldots, \nu. \tag{2.2.12}$$

Setting $U = U^*$ and $V = V^*$ in (2.2.12) shows that

$$J^{(\alpha)}(t,x;\ U^*,\ V^*) = J^{(\beta)}(t,x;\ U^*,\ V^*) \qquad (2.2.13)$$

$$\alpha\ ,\ \beta = 1,\ 2,\ \ldots,\ \mu,$$

that is that $J(t,x;\ U^*,\ V^*)$ has the same value on each of the μ arcs. This value is $W(t,x)$. It is clear that W is continuous along arcs satisfying (2.2.11) with $U = U^*$ and $V = V^*$. Under suitable restrictions (see Berkovitz [3]), W can be shown to be continuous and piecewise $C^{(1)}$ on $\underline{F}$. This however, is not the approach taken in the next chapter, where it is assumed that a function W exists with the required continuity properties.

2.3 Examples

The following examples can all be solved through simple geometric analysis. Their purpose is to exhibit maximin points, saddle points, and the effect of imperfect information. They are all pursuit-evasion games with the same differential equations, viz.

for the evader:

$$\dot{x}_E = u_1 \cos u_2,$$

$$\dot{y} = u_1 \sin u_2, \quad 0 \leq u_1 \leq 1 \qquad (2.3.1)$$

and for the pursuer:

$$\dot{x}_P = v_1 \cos v_2 \qquad (2.3.2)$$

$$\dot{y}_P = v_1 \sin v_2 \quad 0 \leq v_1 \leq a, \ a > 1.$$

The equations are similar, with the pursuer having an advantage in speed. When $u_1 = 1$ and $v_1 = a =$ const., this type of motion is called <u>simple</u> <u>motion</u> by Isaacs, who originated the first of the four examples [20]. The symbol z will mean (x_E, y_E, x_P, y_P).

<u>Example 1</u>

The initial time, t_0, is fixed. The payoff is time to capture--capture meaning $x_E(t_1) = x_P(t_1)$, $y_E(t_1) = y_P(t_1)$, when this occurs for the least time t_1. There is no loss in generality in taking $t_0 = 0$. At each time t, each player knows t and z as well as the equations (2.3.1) and (2.3.2). By considering the rate of change of the distance between the evader and the pursuer, this problem can easily be solved. Figure (2.3.1) and the Theorem of Pythagoras show that

$$r^2(t) = (x_E(t) - x_P(t))^2 + (y_E(t) - y_P(t))^2. \tag{2.3.3}$$

Differentiation yields

$$2r(t)\,\dot{r}(t) = 2(x_E(t)-x_P(t))(\dot{x}_E(t)-\dot{x}_P(t)) + 2(y_E(t)-y_P(t))(\dot{y}_E(t)-\dot{y}_P(t))$$

Upon rearrangement, this becomes

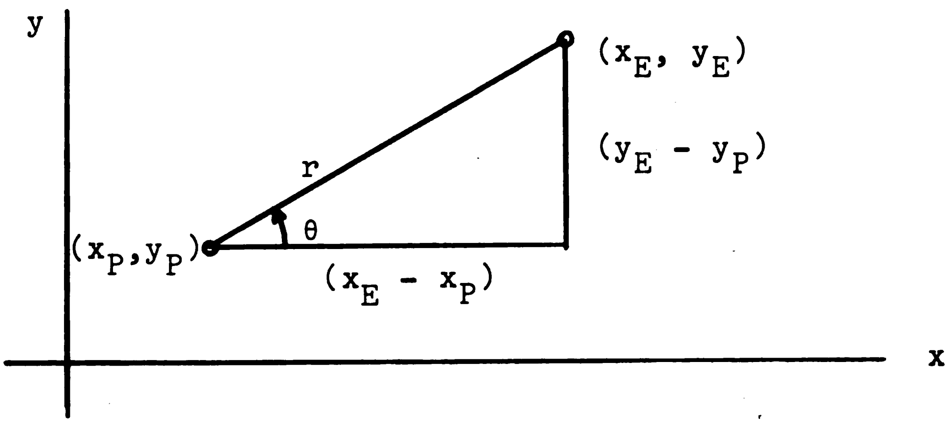

Fig. 2.3.1

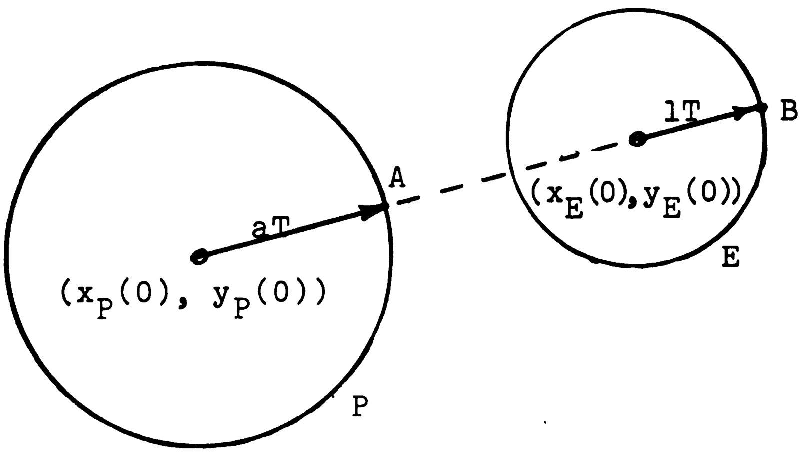

Fig. 2.3.2

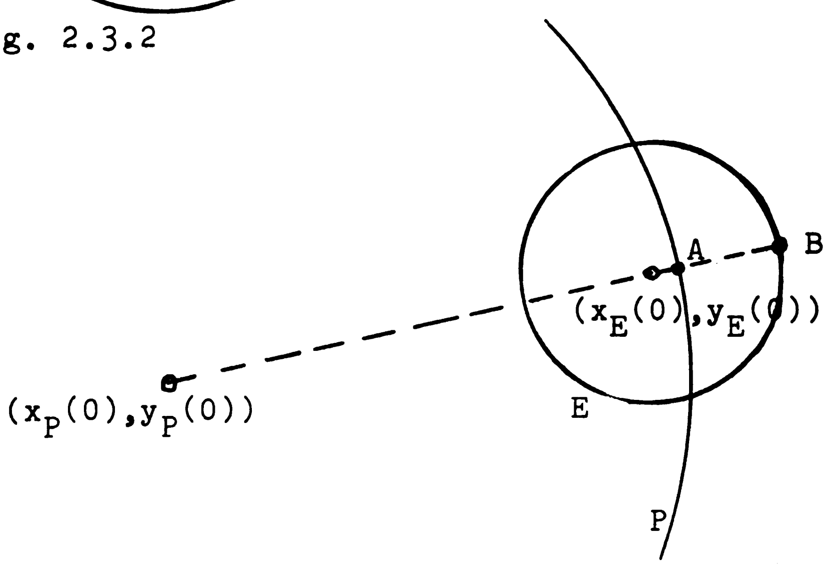

Fig. 2.3.3

$$\dot{r}(t) = \cos\theta(z(t))(\dot{x}_E(t) - \dot{x}_P(t))$$

$$+ \sin\theta(z(t))(\dot{y}_E(t) - \dot{y}_P(t)), \qquad (2.3.4)$$

where

$$\theta(z(t)) = \tan^{-1}[(y_E(t) - y_P(t))/(x_E(t) - x_P(t))].$$

Player One wishes to delay the time t_1 at which $r(t_1)$ is zero as long as possible. Consequently he should strive to maximize $\dot{r}(t)$. Player Two's interest is in minimizing $\dot{r}(t)$.

Substituting the differential equations (2.3.1) and (2.3.2) into (2.3.4) shows that One should maximize

$$u_1 (\cos\theta(z(t)) \cos u_2 + \sin\theta(z(t)) \sin u_2) \qquad (2.3.5)$$

and that Two should minimize

$$-v_1 (\cos\theta(z(t)) \cos v_2 + \sin\theta(z(t)) \sin v_2) \qquad (2.3.6)$$

The optimum conditions are

$$u_2(t) = v_2(t) = \theta(z(t)) \qquad (2.3.7)$$

$$u_1(t) = 1, \; v_1(t) = a, \qquad (2.3.8)$$

that is, a direct chase at maximum speed.

The payoff is $r(0)/(a-1)$. If the capture condition is $r(t_1) = \ell$ (where $r(0) > \ell$), the optimal strategies remain the same, but the payoff becomes

$(r(0) - \ell)/(a-1)$

The optimal strategies:

$$U(t,z) = \begin{bmatrix} u_1(t,z) \\ u_2(t,z) \end{bmatrix} = \begin{bmatrix} 1 \\ \theta(z) \end{bmatrix} \tag{2.3.9}$$

and

$$V(t,z) = \begin{bmatrix} v_1(t,z) \\ v_2(t,z) \end{bmatrix} = \begin{bmatrix} a \\ \theta(z) \end{bmatrix} \tag{2.3.10}$$

are independent of t. This is to be expected, since the differential equations (2.3.1) and 2.3.2) are autonomous.

<u>Example 2</u>

This example differs from Example 1 in that the terminal surface in tz-space is given by the equation $t = T$ $(T>0)$. and in that the payoff is $r(T)$.

For $T < r(0)/(a-1)$, the strategies (2.3.9) and (2.3.10) which provide a saddle point for $\dot{r}(t)$ on $[0,T]$ are optimal. For $T > r(0)/(a-1)$, $r(t) = 0$ for $r(0)/(a-1) \leq t \leq T$, and consequently $\theta(z)$ is indeterminate. However, even if no strategies are defined when $r(t) = 0$, any deviation from this condition results in its immediate restoration.

The payoff for $T < r(0)/(a-1)$ is $r(0) + (1-a)T$. For $T \geq r(0)/(a-1)$ it is zero.

Example 3

This example differs from Example 1 in that the information includes only t and z(0). That is, the players do not know the state after the play has commenced. In this case the maximin payoff is less than the minimax payoff, and the game has no saddle point.

Suppose the pursuer announces his strategy. Then, regardless of what it is, the evader can choose his strategy so that the paths

$$x_P(t),\ y_P(t), \quad t \geq 0$$

and

$$x_E(t),\ y_E(t), \quad t \geq 0$$

never intersect. The minimax payoff is $+\infty$.

On the other hand, if the evader announces his strategy, the pursuer can choose any strategy which brings him to the point where the evader's path first intersects his attainable set at the same time as the evader. This can be done since the pursuer is faster.

To minimize his loss, the evader must choose his announced strategy to delay this encounter as long as possible. Since for $t < \bar{T} = r(0)/(a-1)$, the evader can reach points which the pursuer cannot and since for $t \geq \bar{T}$, the pursuer can reach any point the evader can, the maximin payoff is $\bar{T}$ and the evader's maximin strategy is $u_1 = 1$, $u_2 = \theta(z(0))$. The play in this case is the same as in Example 1.

Example 4

This example differs from Example 2 in that the same restriction of information occurs as in Example 3.

The cross-sections of the attainable sets for fixed times are circles centered on $x_i(0)$, $y_i(0)$, $i = E, P$, with radii of $1 \cdot t$ for the evader, and $a \cdot t$ for the pursuer. If $T < r(0)/a$ the situation in Figure (2.3.2) occurs.

The cross-section P(E) of the attainable set at time T is the circle of radius aT(1T) centered on the pursuer's (evader's) initial point. The point A is closer to the point B than any other point in P; the point B is further from A than any other point in E. Thus the controls which bring the pursuer to A, the evader to B, constitute a saddle point. The value is $r(0) + (1-a)T$.

For $T \geq r(0)/a$, the saddle point condition breaks down. In this case, Figure (2.3.3), the point $(x_E(0), y_E(0))$ is within the circle P of radius aT. Suppose the pursuer's strategy is known to the evader. The evader can choose his strategy to take him to the point on the circle E of radius T furthest from $(x_P(T), y_P(T))$. The resulting payoff, $r(T)$

is clearly $\geq T$, the radius of E. The pursuer's minimax strategy is any one which makes $(x_P(T), y_P(T)) = x_E(0), y_E(0))$, for in this case $r(T) \leq T$, regardless of what the evader does.

If the evader's strategy is known to the pursuer, the latter can use the strategy which brings him to the point within P closest to $(x_E(T), y_E(T))$. In this case,

(a) $r(T) \leq r(0) + (1-a)T$ if $r(0) + (1-a) T > 0$, or

(b) $r(T) = 0$ if $r(0) + (1-a) T \leq 0$.

The distance $r(0) + (1-a)T$ is the distance between the points A and B in the figure, which is drawn for the case (a). The evader's maximin strategy is the one which brings him to the point B, for with this strategy $r(T) \geq r(0) + (1-a)T$. Case (b) corresponds to the situation where T is sufficiently large for the circle P to enclose the circle E. For this situation, any of the evader's strategies may be considered a maximin strategy, since the payoff is zero, independent of the strategy used by the evader.

The principle of optimality does not hold for the two examples with incomplete information, although it does hold partially in Example 4 for $T < r(0)/a$. The optimal fields introduced in the next chapter have the principle of optimality stated as part of their definition. One may view the principle of optimality as a consequence of the existence of an optimal field. Since the principle is not, in general, valid for games of imperfect information, as these examples illustrate, all games to which these fields are applied are assumed to be games of perfect information.

III. OPTIMAL FIELDS FOR SADDLE POINTS

3.1 Optimal Fields in Control Problems

The results in this chapter are extensions of those stated by Hestenes [16, ch. 6, sec. 10] on optimal fields for optimal control problems. In this section the definition of an optimal field for the case of a one-player game is presented, and the basic theorem--a version of the Weierstrass necessary condition--is proved. The terminology developed in the previous chapter, with the maximizing player, One, suppressed, is used. Consequently u is not one of the arguments of L and the f^i, and $\underline{R}$ and $\underline{R}_0$ are sets in txv-space.

The fundamental assumption about an optimal field is that at each point (t,x) in a region $\underline{F}$, there exists a choice V(t,x) of the control variable v optimal in some sense. That is solutions of

$$\dot{x} = f(t,x, V(t,x)) \tag{3.1.1}$$

are optimal paths in tx-space. In what follows L and the f^i are assumed to be $C^{(1)}$ on $\underline{R}$. Consequently if V is $C^{(1)}$ on a neighborhood of a point $(\bar{t},\bar{x}) \in \underline{F}_2$, there is a unique arc $\underline{x}$ satisfying (3.1.1) on an interval $[\bar{t}-\delta, \bar{t}+\delta]$, $\delta > 0$, with $x(\bar{t}) = \bar{x}$. This is a consequence of a theorem on the existence and uniqueness of solutions to differential equations, Theorem 1.4.

Definition

An optimal field $\underline{F}$ is a region $\underline{F}$ in tx-space, a vector control function V, $C^{(1)}$ on $\underline{F}$, and a function W also $C^{(1)}$ on $\underline{F}$, such that

(i) $(t,x, V(t,x)) \varepsilon \underline{R}_0 \quad \forall (t,x) \varepsilon \underline{F}$

(ii) The inequality

$$W(\alpha,x(\alpha)) \leq \int_{\alpha}^{\beta} L(t,x(t), v(t))\,dt + W(\beta,x(\beta)) \tag{3.1.2}$$

holds for every admissible arc

$$\underline{x}: \quad x(t), v(t), \quad \alpha \leq t \leq \beta, \tag{3.1.3}$$

equality holding in case

$$v(t) = V(t,x(t)), \quad \alpha \leq t \leq \beta. \tag{3.1.4}$$

If (3.1.4) holds, $\underline{x}$ is an extremal or a characteristic arc of the optimal field $\underline{F}$. Surfaces in tx-space defined by equations of the form $W(t,x) = \text{constant}$ are called transversals of the field $\underline{F}$. If $\underline{x}$ is an extremal of the field $\underline{F}$, then

$$I(\underline{x}) = \int_{\alpha}^{\beta} L(t,x(t), v(t))\, dt = W(\alpha,x(\alpha)) - W(\beta,x(\beta)). \tag{3.1.5}$$

Two extremals $\underline{x}$ and $\hat{\underline{x}}$ whose initial points are on the transversal $W = C_1$ and whose endpoints are on the transversal $W = C_2$ have the property that

$$I(\underline{x}) = I(\hat{\underline{x}}) = C_1 - C_2 \tag{3.1.6}$$

The following theorem is taken from [16] and the proof follows that given by Hestenes.

<u>Theorem 3.1</u>

At each point $(t,x) \varepsilon \underline{F}$ of an optimal field $\underline{F}$ one has

$$L(t,x,v) + W_t(t,x) + W_x(t,x)\, f(t,x,v) \geq 0 \tag{3.1.7a}$$

for all v such that $(t,x,v) \varepsilon \underline{R}_0$. Moreover, for $v = V(t,x)$ equality holds:

$$L(t,x, V(t,x)) + W_t(t,x) + W_x(t,x)\, f(t,x, V(t,x)) = 0. \tag{3.1.8a}$$

For convenience, $E(t,x,v)$ can be defined as

$$E(t,x,v) \underline{\Delta} L(t,x,v) + W_t(t,x) + W_x(t,x)\, f(t,x,v)$$

so that (3.1.7) can be re-expressed as

$$E(t,x,v) \geq 0, \tag{3.1.7b}$$

and (3.1.8) as

$$E(t,x,\ V(t,x)) = 0 \tag{3.1.8b}$$

<u>Proof</u>

Let $(\bar{t},\bar{x})$ be a point of $\underline{F}$ and choose $\bar{v}\varepsilon\Psi(\bar{t},\bar{x})$. Let

$\underline{x}$: $x(t), v(t)$

be a continuous admissible arc in $\underline{F}$ on either

$$\left.\begin{array}{ll} & \text{(i) } \bar{t} \leq t \leq \bar{t} + \delta \\ \text{or} & \text{(ii) } \bar{t} - \delta \leq t \leq \bar{t} \end{array}\right\} \delta > 0$$

with $x(\bar{t}) = \bar{x}$, $v(\bar{t}) = \bar{v}$.

In case (i) consider the function

$$g(\varepsilon) = \int_{\bar{t}}^{\bar{t}+\varepsilon} L(t,\ x(t),\ v(t))\ dt + W(\bar{t}+\varepsilon,\ x(\bar{t}+\varepsilon))$$

and in case (ii)

$$h(\varepsilon) = -\int_{\bar{t}-\varepsilon}^{\bar{t}} L(t,\ x(t),\ v(t))\ dt + W(t-\varepsilon,\ x(t-\varepsilon)),$$

where $0 \leq \varepsilon \leq \delta$ in both cases. By the inequality (3.1.2), $g(\varepsilon)$ has a minimum and $h(\varepsilon)$ has a maximum at $\varepsilon = 0$. It follows that

case (i) $g'(0) \geq 0$

case (ii) $h'(0) \leq 0$.

For case (i), one has

$$g'(0) = L(\bar{t},\bar{x},\bar{v}) + W_t(\bar{t},\bar{x}) + W_x(\bar{t},\bar{x})\ \dot{x}\ (\bar{t}) \geq 0.$$

Substituting $\dot{x}(\bar{t}) = f(\bar{t},\bar{x},\bar{v})$ yeilds

$$L(\bar{t},\bar{x},\bar{v}) + W_t(\bar{t},\bar{x}) + W_x(\bar{t},\bar{x})\ f(\bar{t},\bar{x},\bar{v}) \geq 0 \qquad (3.1.11a)$$

For case (ii),

$$h'(0) = -L(\bar{t},\bar{x},\bar{v}) - W_t(\bar{t},\bar{x}) - W_x(\bar{t},\bar{x})\ \dot{x}\ (\bar{t}) \leq 0.$$

Again substituting $\dot{x}(\bar{t}) = f(\bar{t},\bar{x},\bar{v})$ yields

$$L(\bar{t},\bar{x},\bar{v}) + W_t(\bar{t},\bar{x}) + W_x(\bar{t},\bar{x})\ f(\bar{t},\bar{x},\bar{v}) \geq 0 \qquad (3.1.11b)$$

Thus in both cases

$$E(t,x,v) \geq 0 \qquad (3.1.11c).$$

If $v(t) = V(t,x(t))$ on the intervals (i) or (ii),

(i) $g(\varepsilon) = W(t,x)$

or (ii) $h(\varepsilon) = W(t,x)$.

That is, $g(\varepsilon)$ and $h(\varepsilon)$ are constant on $0 \leq \varepsilon \leq \delta$. Consequently, $g'(0) = h'(0) = 0$ in this situation and equality holds in (3.1.11). This proves Theorem 3.1.

The proof is based on the inequality (3.1.2). If this inequality is reversed, then so are all inequalities (except those defining intervals) in the proof. This consideration leads to the corollary:

<u>Corollary 1</u>

If the inequality (3.1.2) is replaced by

$$W(\alpha, x(\alpha)) \geq \int_{\alpha}^{\beta} L(t,x(t), v(t))\, dt + W(\beta,x(\beta)), \qquad (3.1.12)$$

Theorem 3.1 is unchanged except for (3.1.7), which becomes

$$L(t,x,v) + W_t(t,x) + W_x(t,x)\, f(t,x,v) \leq 0. \qquad (3.1.13)$$

The theorem can be extended to cover cases where L and f are $C^{(1)}$ in v, piecewise $C^{(1)}$ in (t,x), V is piecewise $C^{(1)}$ on $\underline{F}$ and W is continuous and piecewise $C^{(1)}$ on $\underline{F}$. In this case if $(\overline{t},\overline{x})$ is a point of discontinuity of one of these functions the Lipschitz condition in the existence theorem (i.e., Hypothesis 3 in the section on differential equations in Chapter I) does not hold, and one cannot prove that there is a unique solution to the initial value problem with the initial condition $x(\bar{t}) = \bar{x}$. However, with the proper interpretation at manifolds of discontinuity, the theorem still holds.

Corollary 2

Let $\{\underline{F}^{(\alpha)}\}$ be a decomposition of $\underline{F}$ such that L, f, V and W are all $C^{(1)}$ on each $\underline{F}^{(\alpha)}$. Then these functions and their first partial derivatives agree on each $\underline{F}^{(\alpha)}$ with functions $L^{(\alpha)}$, $f^{(\alpha)}$, $V^{(\alpha)}$, $W^{(\alpha)}$, etc. which are continuous on $\bar{\underline{F}}^{(\alpha)}$ (from the definition of piecewise continuity). If $(\bar{t},\bar{x})$ is a point on the boundary of ν regions $\underline{F}^{(\alpha_1)}, \ldots, \underline{F}^{(\alpha_\nu)}$, Theorem (3.1) holds at $(\bar{t},\bar{x})$ if (3.1.7) is replaced by

$$L^{(\alpha)}(\bar{t},\bar{x},v) + W_t^{(\alpha)}(\bar{t},\bar{x}) + W_x^{(\alpha)}(\bar{t},\bar{x})\ f^{(\alpha)}(\bar{t},\bar{x},v) \geq 0 \tag{3.1.14}$$

$$\alpha = \alpha_j,\ j = 1, \ldots, \nu,$$

and if (3.1.8) is replaced by

$$L^{(\alpha)}(t,x,\ V^{(\alpha)}(t,x)) + W^{(\alpha)}(t,x)$$

$$+ W^{(\alpha)}(t,x) + W^{(\alpha)}(T,x)f^{(\alpha)}(t,x,\ V^{(\alpha)}(t,x)) = 0$$

$$\alpha = \alpha_j,\ j = 1, \ldots, \nu. \tag{3.1.15}$$

Proof

The function E is continuous on the set G of points (t,x,v) such that (t,x) is in one of the regions $\underline{F}^{(\alpha)}$ and $(t,x,v)\varepsilon\underline{R}_0$. By Theorem 3.1,

$$E(t,x,v) \geq 0 \text{ on } G$$

Furthermore the function $E^{(\alpha)} = L^{(\alpha)} + W_t^{(\alpha)} + W_x^{(\alpha)} f^{(\alpha)}$ is continuous on the set (t,x,v) such that $(t,x)\varepsilon\overline{F}^{(\alpha)}$ and $(t,x,v)\varepsilon\underline{R}_0$ and agrees with E on G. Consider the point $(\bar{t},\bar{x})$ on the boundary of $F^{(\alpha)}$ and let $\bar{v}$ be such that $(\bar{t},\bar{x},\bar{v})\varepsilon\underline{R}_0$. Let $\{(t_i, x_i, v_i)\}$ be a sequence of points in G converging to $(\bar{t},\bar{x},\bar{v})$. Since for each i,

$$E^{(\alpha)}(t_i, x_i, v_i) \geq 0,$$

$$\lim_{i\to\infty} E^{(\alpha)}(t_i, x_i, v_i) \geq 0$$

But the continuity of $E^{(\alpha)}$ shows that

$$\lim_{i\to\infty} E^{(\alpha)}(t_i, x_i, v_i) = E^{(\alpha)}(\bar{t},\bar{x},\bar{v}).$$

Thus

$$E^{(\alpha)}(t,x,v) \geq 0.$$

This expression expanded gives (3.1.14). Likewise

$$E^{(\alpha)}(t_i,\ x_i,\ V^{(\alpha)}(t_i,\ x_i)) = 0 \text{ for all } i$$

In the limit this gives

$$E^{(\alpha)}(t,x,\ V^{(\alpha)}(t,x)) = 0,$$

an expression equivalent to (3.1.15).

The function E is one form of the Weierstrass E-function. This can be demonstrated by setting

$$E(t,x,V,v) = E(t,x,v) - E(t,x,\ V(t,x)) \geq 0,$$

which can be rewritten as

$$L(t,x,v) - L(t,x,V(t,x)) + W_x(t,x)[f(t,x,v) - f(t,x,V(t,x)] \geq 0$$

which corresponds more closely to the usual form of the Weierstrass condition.

3.2 Optimal Field for a Saddle Point

In this section it is assumed that a differential game of the type defined in Chapter II has a saddle point, that there exist optimal strategies $U \epsilon \underline{U}$ and $V \epsilon \underline{V}$ which are piecewise $C^{(1)}$ on $\underline{F}$ and that there exists a value function W which is continuous and piecewise $C^{(1)}$ on $\underline{F}$. Solutions of

$$\dot{x} = f(t,x, U(t,x), V(t,x)) \tag{3.2.1}$$

are then optimal paths in tx-space.

It is assumed that L and f are $C^{(1)}$ on $\underline{R}$.

The previous definition of an optimal field can be extended to cover saddle points.

Definition

An optimal field F is a region $\underline{F}$ in tx-space, vector strategies U and V which are piecewise $C^{(1)}$ on $\underline{F}$, and a function W continuous and piecewise $C^{(1)}$ on $\underline{F}$ such that

$$\left.\begin{array}{ll} \text{(i)} & (t,x, U(t,x)) \in \underline{R}_1 \\ \text{(ii)} & (t,x, V(t,x)) \in \underline{R}_2 \end{array}\right\} \forall\, (t,x) \in \underline{F}$$

(iii) The inequality

$$W(\alpha,x(\alpha)) \leq \int_\alpha^\beta L(t,x(t), U(t,x(t)), v(t))dt + W(\beta,x(\beta)) \tag{3.2.2a}$$

holds for every admissible arc

$$\underline{x}:\ x(t),\ U(t,x(t)),\ v(t),\quad \alpha \leq t \leq \beta,$$

equality holding in case

$$v(t) = V(t,x(t)),\ \alpha \leq t \leq \beta; \tag{3.2.2b}$$

and the inequality

$$\int_{\alpha}^{\beta} L(t,x(t),\ u(t),\ V(t,x(t)))dt + W(\beta,x(\beta)) \leq W(\alpha,x(\alpha)) \tag{3.2.3a}$$

holds for every admissible arc

$$x:\ x(t),\ u(t),\ V(t,x(t)),\quad \alpha \leq t \leq \beta,$$

equality holding in case

$$u(t) = U(t,x(t)),\ \alpha \leq t \leq \beta. \tag{3.2.3b}$$

If there is more than one admissible arc

$$\underline{x}:\quad x(t),\ U(t,x(t)),\ v(t)\quad \alpha \leq t \leq \beta$$

for each v(t), (3.2.2a) holds for each of these arcs. Likewise (3.2.3a) holds when the arcs

$$\underline{x}:\quad x(t),\ u(t),\ V(t,x(t))\quad \alpha \leq t \leq \beta$$

are not unique for a given u(t), $\alpha \leq t \leq \beta$.

On a manifold of discontinuity of U or V one may or may not have a definition of these functions differing from their limiting values from neighboring regions of continuity. If the optimal paths lie on such a manifold, the optimal strategies need to be defined there, but if the optimal paths merely cross the manifold from one region to another this is not necessary (and indeed superfluous). Optimal arcs lying on manifolds are considered in the corollary to Theorem 3.4.

Surfaces $W(t,x)$ = constant arc <u>transversals</u> of the field $\underline{F}$. If (3.2.3) holds, the arc $\underline{x}$ is a <u>characteristic arc</u> or an <u>extremal of the optimal field</u> $\underline{F}$. If the arc

$$\underline{x}: \quad x(t),\ u(t),\ v(t), \qquad \alpha \leq t \leq \beta$$

is an extremal of the field, then

$$I(\underline{x}) = \int_{\alpha}^{\beta} L(t,x(t),u(t),v(t))dt = W(\alpha,x(\alpha))-W(\beta,x(\beta)) \tag{3.2.5}$$

Two extremals $\underline{x}$ and $\hat{\underline{x}}$ whose initial points are on the transversal surface $W(t,x)=C_1$ and whose final points are on $W = C_2$ have the property that

$$I(\underline{x}) = I(\hat{\underline{x}}) = C_1 - C_2.$$

In the problem of Mayer, $L(t,x,u,v) \equiv 0$, and the value function W is constant along optimal paths. Such paths, then lie in transversal surfaces. Along arcs

$$\underline{x}: \quad x(t),\ U(t,x(t)),\ v(t), \qquad \alpha \leq t \leq \beta$$

$$W(\alpha,x(\alpha)) \leq W(\beta,x(\beta))$$

and along arcs

$$\underline{x}: \quad x(t),\ u(t),\ V(t,x(t)), \qquad \alpha \leq t \leq \beta$$

$$W(\alpha,x(\alpha)) \geq W(\beta,x(\beta)).$$

Transversal surfaces in this case have the property of <u>semipermeability</u> used by Isaacs [20]. That is, if player One uses the strategy U at a point (t,x) no arc can penetrate the transversal surface containing (t,x) in the direction of decreasing W, and if player Two uses V, no arc can penetrate in the direction of increasing W.

<u>Theorem 3.2</u>

At each point (t,x) of an optimal field $\underline{F}$ one has

$$L(t,x,u,V(t,x))+W_t(t,x)+W_x(t,x)f(t,x,u,V(t,x)) \leq 0,$$

$$0 \leq L(t,x,U(t,x)y)+W_t(t,x)+W_x(t,x)f(t,x,U(t,x),v) \quad (3.2.6)$$

for all $u\varepsilon\Phi(t,x)$, $v\varepsilon\Psi(t,x)$. Moreover for $u = U(t,x)$ and $v = V(t,x)$, equality holds:

$$L(t,x,U(t,x),V(t,x))+W_t(t,x)$$
$$+ W_x(t,x)f(t,x,U(t,x), V(t,x)) = 0 \quad (3.2.7)$$

If (t,x) is a point of discontinuity of U, V, W_t or W_x, in the expression $L + W_t + W_x f$, U, V, W_t and W_x must all be evaluated as limits approaching (t,x) from the same region of continuity of these functions (U, V, W_t, W_x). (In succeeding theorems this may be referred to as "the usual interpretation at points of discontinuity.")

Proof.--The right-hand inequality in (3.2.6) follows directly from Theorem 3.1 and the second corollary with $L(t,x,v) = L(t,x, U(t,x), v)$ and $f(t,x,v) = f(t,x, U(t,x),v)$. The left-hand inequality is obtained in a similar manner from Theorem 3.1 and both corollaries. Equation (3.2.7) again comes from Theorem 3.1. It is also an immediate consequence of (3.2.6). The remaining part of the theorem is a restatement of Corollary 2 to Theorem 3.1.

Define

$$P_i(t,x) = W_{x^i}(t,x)$$

$$H(t,x,u,v,p) = L(t,x,u,v) + pf(t,x,u,v). \tag{3.2.8}$$

Here p is an n-dimensional row vector.

Theorem 3.2 is equivalent to:

Theorem 3.3

At each point (t,x) in an optimal field $\underline{F}$ the inequalities

$$H(t,x,u,V(t,x),P(t,x)) \le H(t,x,U(t,x),V(t,x),P(t,x)) \le H(t,x,U(t,x),v,P(t,x) \tag{3.2.9}$$

hold for all $u \varepsilon \Phi(t,x)$, and for all $v \varepsilon \Psi(t,x)$. Moreover U, V, and W satisfy

$$W_t(t,x) + H(t,x, U(t,x), V(t,x), W_x(t,x)) = 0. \tag{3.2.10}$$

The same interpretation at points of discontinuity as in Theorem 3.2 is used in (3.2.9) and (3.2.10) for W_t, W_x, P, H, U and V.

Finally the integral

$$I^* = \int \{P(t,x)dx - H(t,x, U(t,x), V(t,x), P(t,x))dt\} \tag{3.2.11}$$

is independent of the path in $\underline{F}$, provided that any segment of the path lying on a manifold of discontinuity is divided into segments on which P and H take the limiting values of only one adjacent region.

<u>Proof</u>.--Expressions (3.2.9) and (3.2.10) are restatements of Theorem 3.2 using the terminology of (3.2.8). The integral I* is seen to be $\int dW$ when (3.2.10) and (3.2.8) are used to make substitutions. $\int dW$ is clearly independent of the path. The restrictions on the evaluation of the integrand insure its existence. This Theorem corresponds to Theorem 10.2 in Hestenes [16]. The following theorem relates limiting values of functions appearing in Theorem 3.3.

<u>Theorem 3.4</u>

Let $\{\underline{F}^{(\alpha)}\}$ be a $C^{(1)}$-decomposition of $\underline{F}$ based on U, V and W, and let Γ be a set of indices α such that the manifold of discontinuity $\underline{M} \underline{\Delta} \bigcap_{\alpha\epsilon\Gamma} \partial\underline{F}^{(\alpha)}$ is not empty.

The superscript α (e.g. $W_t^{(\alpha)}$) denotes the functions $C^{(1)}$ on $\overline{F^{(\alpha)}}$ which agree on $F^{(\alpha)}$ with the corresponding (unsuperscripted) functions (e.g. W_t). Then if $(\bar{t},\bar{x})\varepsilon\underline{M}$ and (dt, dx) is tangent to $\underline{M}$, for each $\alpha\varepsilon\Gamma$ and $\beta\varepsilon\Gamma$

$$W_t^{(\alpha)}(\bar{t},\bar{x})dt + W_x^{(\alpha)}(\bar{t},\bar{x})dx = W_t^{(\beta)}(\bar{t},\bar{x})dt + W_x^{(\beta)}(\bar{t},\bar{x})dx. \tag{3.2.12}$$

Also

$$\begin{aligned}[P^{(\alpha)}(\bar{t},\bar{x}) - P^{(\beta)}(\bar{t},\bar{x})]dx \\ = [H^{(\alpha)}(\bar{t},\bar{x},U^{(\alpha)}(\bar{t},\bar{x}),\ V^{(\alpha)}(\bar{t},\bar{x}),\ P^{(\alpha)}(\bar{t},\bar{x})) \\ - H^{(\beta)}(\bar{t},\bar{x},U^{(\beta)}(\bar{t},\bar{x}),\ V^{(\beta)}(\bar{t},\bar{x},)\ P^{(\beta)}(\bar{t},\bar{x},))]dt\end{aligned} \tag{3.2.13}$$

<u>Proof</u>.--(3.2.13) follows from (3.2.8), (3.2.10), and (3.2.12). To show (3.2.12), a $C^{(1)}$ curve lying in $\underline{M}$ given parametrically by $t = t(s)$, $x = x(s)$, with $t(0) = \bar{t}$, $x(0) = \bar{x}$, can be defined. Let $\underline{N}$ be a neighborhood of $(\bar{t},\bar{x})$. Since W is piecewise $C^{(\ )}$ on $\underline{F}$ and $C^{(1)}$on $\underline{F}^{(\alpha)}$, it is $C^{(1)}$ on $\underline{N}\cap\overline{F^{(\alpha)}}$. A $C^{(1)}$ function $\tilde{W}$ can be defined on $\underline{N}$ which has $\tilde{W}(t,x) = W^{(\alpha)}(t,x)$ on $\underline{\overline{F^{(\alpha)}}}$. On the curve a further $C^{(1)}$ function can be defined

$$w(s) \underline{\Delta} \tilde{W}(t(s),\ x(s)) = W^{(\alpha)}(t(s),\ x(s)),$$

for which one obtains

$$w'(0) = \tilde{W}_t(\bar{t},\bar{x})t'(0) + \tilde{W}_x(\bar{t},\bar{x})x'(0) = W_t^{(\alpha)}(\bar{t},\bar{x})t'(0)$$

$$+ W^{(\alpha)}(\bar{t},\bar{x})x'(0).$$

Another equation is obtained in the same manner:

$$w'(0) = W_t^{(\beta)}(\bar{t},\bar{x})t'(0) + W_x^{(\beta)}(\bar{t},\bar{x})x'(0).$$

The curve is arbitrary. It follows that $(t'(0), x'(0))ds$ may be any vector tangent to $\underline{M}$. This, together with the two expressions for $w'(0)$, established (3.2.12).

The corollary which follows considers the case where an optimal path starting or terminating at $(\bar{t},\bar{x})$ lies in the manifold $\underline{M}$. In this situation optimal strategies defined on the manifold are not necessarily the same as one of the limiting strategies $U^{(\alpha)}$ or $V^{(\alpha)}$.

<u>Corollary</u>

Let optimal strategies U and V defined on the manifold $\underline{M}$ prescribe optimal paths lying in $\underline{M}$ and let $(\bar{t},\bar{x})$ be a point of $\underline{M}$. Then

$$L(\bar{t},\bar{x}, U(\bar{t},\bar{x}),v) + W_t^{(\alpha)}(\bar{t},\bar{x}) + W^{(\alpha)}(\bar{t},\bar{x})$$

$$+ W^{(\alpha)}(\bar{t},\bar{x}) + W^{(\alpha)}(\bar{t},\bar{x})\ f(\bar{t},\bar{x},U(\bar{t},\bar{x}),v) \geq 0 \qquad (3.2.14a)$$

whenever $(\bar{t},\bar{x},\bar{v})\varepsilon\underline{R}_2$ and $(1, f(t,x, U(t,x),v))$ is tangent to $\underline{M}$;

$$L(\bar{t},\bar{x},u,V(\bar{t},\bar{x}))+W_t^{(\alpha)}(\bar{t},\bar{x})+W_x^{(\alpha)}(\bar{t},\bar{x})f(\bar{t},\bar{x},u,V(\bar{t},\bar{x})) \leq 0$$

$$(3.2.14b)$$

whenever $(\bar{t},\bar{x},u)\varepsilon R_1$ and $(1,f(\bar{t},\bar{x},u,V(\bar{t},\bar{x})))$ is tangent to $\underline{M}$, and

$$L(\bar{t},\bar{x},U(\bar{t},\bar{x}),V(\bar{t},\bar{x}))+W_t^{(\alpha)}(\bar{t},\bar{x})$$

$$+ W_t^{(\alpha)}(\bar{t},\bar{x})f(\bar{t},\bar{x},U(\bar{t},\bar{x}), V(\bar{t},\bar{x})) = 0. \qquad (3.2.14c)$$

Proof.--Let

$$\underline{x}: x(t), U(t,x(t)), v(t), \qquad t_0 \leq t \leq t_1$$

be an admissible arc with $(t,x(t))$ lying in $\underline{M}$ where either $(t_0,x(t_0)) = (\bar{t},\bar{x})$ or $(t_1,x(t_1)) = (\bar{t},\bar{x})$. This arc can be described parametrically with parameter $s = t - \bar{t}$, becoming

$$\underline{x}: \ x(s), U(t(s), x(s)), v(s) \qquad s_0 \leq s \leq s_1$$

Either s_0 or s_1 is zero. The function w defined in the proof of Theorem 3.4 is differentiable along the arc $\underline{x}$. With $g(\varepsilon)$ given by (where $s_0 = 0$):

$$g(\varepsilon) = \int_0^{\varepsilon} L(t(s),x(s), U(t(s),x(s)), v(s))ds + w(\varepsilon),$$

or with $h(\varepsilon)$ given by (where $s_1 = 0$):

$$h(\varepsilon) = -\int_{-\varepsilon}^{0} L(t(s),x(s),\ U(t(s),x(s)),\ v(s))ds + w(-\varepsilon),$$

the arguments used in the proof of Theorem 3.1 establish the first and third of the above expressions. The second is obtained through the use of the first corollary to that theorem. In carrying out these arguments the substitution

$$w'(0) = W_t^{(\alpha)}(\bar{t},\bar{x})t'(0) + W_x^{(\alpha)}(\bar{t},\bar{x})x'(0)$$

obtained in the proof of Theorem 3.4 is used, where in the present case

$$t'(0) = 1,\ x'(0) = f(\bar{t},\bar{x},\ U(\bar{t},\bar{x}),\ v(\bar{t})).$$

The tangency restrictions arise from the corresponding conditions in Theorem 3.4.

The following theorem is true (as can be seen from the proof) for any pair of admissible strategies U and V and function W which satisfies

$$W(t_0,x(t_0)) = \int_{t_0}^{t_1} L(t,x(t)),u(t),v(t))dt + W(t_1,x(t_1))$$

for any admissible arc

$$\underline{x}:\quad x(t),\ u(t),\ v(t) \qquad t_0 \le t \le t_1$$

with $u(t) = U(t,x(t))$, $v(t) = V(t,x(t))$.

the type of field defined in this section or the type defined in Chapter IV.

<u>Theorem 3.5</u>

On each region $\underline{G}$ in $\underline{F}$ on which U and V are $C^{(1)}$, an extremal $\underline{x}$: $x(t)$, $u(t)$, $v(t)$, $\alpha \leq t \leq \beta$ of the field, together with the functions

$$p_i(t) = P_i(t,x(t)) \tag{3.2.15}$$

satisfies the equations

$$\begin{aligned} \dot{x} &= H_p(t,x,\ u(t),\ v(t),\ p) \\ \dot{p} &= -(H_x(t,x,u(t),v(t),p)+H_u(t,x,u(t),v(t),p)\ U_x(t,x) \\ &\qquad +H_v(t,x,u(t),v(t),p)V_x(t,x)). \end{aligned} \tag{3.2.16}$$

Moreover

$$\begin{aligned} \dot{H}(t,x(t),u(t),v(t),p(t)) &= H_t(t,x(t),u(t),v(t),p(t)) \\ &\quad +H_u(t,x(t),u(t),v(t),p(t)U_t(t,x(t) \\ &\quad +H_v(t,x(t),u(t),v(t),p(t))V_t(t,x(t)) \end{aligned} \tag{3.2.17},$$

along the extremal $\underline{x}$.

<u>Proof</u>.-- $x(t)$ satisfies $\dot{x} = H_p$ along $\underline{x}$ since $\underline{x}$ is differentiably admissible.

Consider the differential equation

$$\dot{p} = -(H_x + H_u U_x + H_v V_x)$$

where the right hand side is evaluated along the extremal $\underline{x}$. This equation can be rewritten as

$$\dot{p} = -(L_x + L_u U_x + L_v V_x) - p(f_x + f_u U_x + f_v V_x), \qquad (3.2.18)$$

which is a system of linear differential equations. Furthermore $(L_x + L_u U_x + L_v V_x)$ and $(f_x + f_u U_x + f_v V_x)$ are continuous on $\alpha \leq t \leq \beta$. It follows that for any (τ, π) with $\alpha \leq \tau \leq \beta$, there is a unique $p(t)$ satisfying (3.2.18) on $\alpha \leq t \leq \beta$. This $p(t)$ may be denoted $p(t,\tau,\pi)$ to indicate its dependence on the initial condition.

Consider now the arc $\underline{x}$. It can be imbedded by Theorem 1.5 in set of arcs

$$x(t,\tau,\xi) \text{ for which } x(t,\tau, x(\tau)) = x(t).$$

Let $\overline{x} = x(\overline{t})$ for some $\alpha \leq \overline{t} \leq \beta$. Then

$$W(\overline{t},\overline{x}) = \int_{\overline{t}}^{\beta} L(t, x(t,\overline{t},\overline{x}), U(t, x(t,\overline{t},\overline{x})), V(t, x(t,\overline{t},\overline{x})))dt$$

$$+ W(\beta, x(\beta,\overline{t},\overline{x})) \qquad (3.2.19)$$

and

$$+ W_{\bar{x}}(\beta,x(\beta,\bar{t},\bar{x}))\ x_{\bar{x}}(\beta,\bar{t},\bar{x})$$

$$= \int_{\bar{t}}^{\beta} \{(H_x+H_uU_x+H_vV_x)-p(f_x+f_uU_x+f_vV_x)\}\quad x_{\bar{x}}(t,\bar{t},\bar{x})dt$$

$$+ W_x(\beta,x(\beta,\bar{t},\bar{x}))X_{\bar{x}}(\beta,\bar{t},\bar{x}). \tag{3.2.20}$$

Consider now the particular p satisfying (2.3.18) which has the initial condition $p(\beta) = P(\beta,\ x(\beta))$. Also to be noted is that $Z(t,\bar{t}) \triangleq x_{\bar{x}}(t,\bar{t},x(\bar{t}))$ satisfies

$$\dot{Z} = f_x(t,x(t),u(t),v(t))Z \quad \text{with} \quad Z(\bar{t},\bar{t}) = I.$$

Thus (3.2.20) becomes

$$P(\bar{t},\bar{x}) = -\int_{\bar{t}}^{\beta} \frac{d}{dt}\ p(t,\beta,P(\beta,x(\beta)))x_{\bar{x}}(t,\bar{t},\bar{x})\ dt$$

$$+ P(\beta,x(\beta))x_{\bar{x}}(\beta,\bar{t},\bar{x}),$$

or

$$P(\bar{t},\bar{x}) = p(\bar{t},\beta,P(\beta,x(\beta)))x_{\bar{x}}(\bar{t},\bar{t},\bar{x})$$

$$-p(\beta,\beta,P(\beta,x(\beta)))\ x_{\bar{x}}(\beta,\bar{t},\bar{x})+P(\beta,x(\beta))x_{\bar{x}}(\beta,\bar{t},\bar{x}) \tag{3.2.21}$$

But $x_{\bar{x}}(\bar{t},\bar{t},\bar{x}) = I$ and $p(\beta,\beta,P(\beta,x(\beta))) = P(\beta,x(\beta))$, so that (3.2.21) reduces to

$$P(\bar{t},\bar{x}) = p(\bar{t},\beta,P(\beta,x(\beta))) \tag{3.2.22}$$

$\bar{t}$ is an arbitrary point on the interval $\alpha \leq t \leq \beta$ with $\bar{x} = x(\bar{t})$. Not only does

$$P(\bar{t},x(\bar{t})) = p(\bar{t},\beta,P(\beta,x(\beta)))$$

hold along the extremal $\underline{x}$, but also

$$P(\bar{t},x(\bar{t})) = p(\bar{t},\bar{t},P(\bar{t},x(\bar{t}))),$$

which is an identity. Consequently since $p(t,\bar{t},P(\bar{t},x(\bar{t})))$ satisfies $\dot{p} = -(H_x+H_uU_x+H_vV_x)$, the function $P(t,x(t))$ does also. This establishes (3.2.15) and (3.2.16). Having demonstrated this, it is easy to show that

$$\frac{d}{dt}H = H_t + H_uU_t + H_vV_t.$$

$$\frac{d}{dt}H = \frac{d}{dt}(L + pf)$$

$$= L_t + L_uU_t + L_vV_t + (L_x + L_uU_x + L_vV_x)\dot{x}$$

$$+ \dot{p}f + p(f_t + f_uU_t + f_vV_t + (f_x + f_uU_x + f_vV_x)\dot{x})$$

$$= H_t + H_uU_t + H_vV_t + (L_x + L_uU_x + L_vV_x)f$$

$$- (L_x + L_uU_x + L_vV_x + p(f_x + f_uU_x + f_vV_x))f$$

$$+ p(f_x + f_uU_x + f_vV_x)f$$

$$= H_t + H_uU_t + H_vV_t.$$

All of the above functions are evaluated along the extremal $\underline{x}$, that is that H, L, f and their derivatives have as arguments t, x(t), U(t,x(t)), V(t,x(t)), and (in H) p(t). U, V and their derivatives are functions of (t,x(t)). The proof here is similar to one used by Berkovitz [3].

Under the assumption that W is $C^{(2)}$ on the region $\underline{G}$ this theorem can be proved using necessary conditions from Calculus of Variations. This is the approach used by Hestenes [16].

Theorem 3.3 states that

$$H(t,x,u, V(t,x), P(t,x))$$

considered a function of u has a maximum at u = U(t,x) and that

$$H(t,x, U(t,x), v, P(t,x))$$

considered as a function of v has a minimum at v = V(t,x). If U(t,x) is interior to $\Phi(t,x)$ and v interior to $\Psi(t,x)$, then

$$H_u\,(t,x, U(t,x)\ V(t,x), P(t,x)) = 0$$

and

$$H_v\,(t,x, U(t,x), V(t,x), P(t,x)) = 0$$

This result may be combined with Theorems 3.3 and 3.5 to give

Theorem 3.6

Suppose that on the region $\underline{G}$ of Theorem 3.5 that $\Phi(t,x)$ and $\Psi(t,x)$ are open, or alternatively that $U(t,x)$ and $V(t,x)$ are interior points of $\Phi(t,x)$ and $\Psi(t,x)$ respectively. Then on $\underline{G}$ U, V, and W satisfy the Hamilton-Jacobi equations

$$\left.\begin{aligned} W_t(t,x) + H(t,x,\ U(t,x),\ V(t,x),\ W_x(t,x)) &= 0 \\ H_u(t,x,\ U(t,x),\ V(t,x),\ W_x(t,x)) &= 0 \\ H_v(t,x,\ U(t,x),\ V(t,x),\ W_x(t,x)) &= 0 \end{aligned}\right\} \quad (3.2.23).$$

An extremal of the field

$$\underline{x}:\quad x(t),\ u(t),\ v(t), \qquad \alpha \leq t \leq \beta$$

satisfies with $p(t) = P(t,x(t))$ the canonical Euler equations

$$\dot{x} = H_p,\ \dot{p} = -H_x,\ H_u = 0,\ H_v = 0 \tag{3.2.24}$$

together with

$$\dot{H} = H_t \tag{3.2.25}$$

on the region $\underline{G}$.

Suppose that $\underline{R}_1$ and $\underline{R}_2$ are given by the constraint conditions (2.2.7) stated in the last chapter. For convenience one can make the definition

$$\overline{H}(t,x,u,v,p,\mu\ \nu) = H(t,x,u,v,p) + \mu_\alpha \phi^\alpha(t,x,u) +$$

$$\nu_\beta \psi^\beta(t,x,v). \qquad (3.2.26)$$

Theorem 3.7

Let $\underline{R}_1$ and $\underline{R}_2$ be as stated immediately above. Then there exist multipliers $\mu_\alpha(t,x)$ and $\nu_\beta(t,x)$, piecewise continuous on $\underline{F}$ (piecewise $C^{(1)}$ if ϕ^α and ψ^β are of class $C^{(2)}$) such that on $\underline{F}$

$$\mu_\alpha(t,x) \leq 0 \qquad \alpha = 1, \ldots r'$$

$$\nu_\beta(t,x) \geq 0 \qquad \beta = 1, \ldots s'$$

$$\mu_\alpha(t,x)\phi^\alpha(t,x,\ U(t,x)) = 0 \qquad \alpha = 1, \ldots, r, \text{ and not summed,}$$

$$\nu_\beta(t,x)\phi^\beta(t,x,\ V(t,x)) = 0 \qquad \beta = 1, \ldots, s, \text{ and not summed,}$$

$$\left.\begin{aligned} &\overline{H}_u(t,x,U(t,x),V(t,x),P(t\ x),\ \mu(t,x),\nu(t,x)) = 0 \\ &\overline{H}_v(t,x,U(t,x),V(t,x),P(t,x),\mu(t,x),\nu(t,x)) = 0 \end{aligned}\right\} (3.2.27)$$

The usual interpretation is made at points of discontinuity. Let

$$\underline{x}\!: \ x(t),\ u(t),\ v(t), \quad a \leq t \leq b$$

be an extremal, such that there is a decomposition

$$(t_{i-1}, t_i) \quad i = 1, \ldots, N$$

$$a = t_0 < t_1 < \cdots < t_i < \cdots < t_N = b$$

of the interval [a,b] with (t,x(t)) in a region on which U and V are $C^{(1)}$ on each of the intervals comprising the decomposition. (This excludes arcs with subarcs lying on manifolds of discontinuity of U or V.)

Then $\underline{x}$ satisfies, with

$$p(t) = P(t,x(t))$$

$$\mu(t) = \mu(t,x(t))$$

and $\nu(t) = \nu(t,x(t))$,

the canonical Euler equations

$$\left.\begin{array}{ll}
\dot{x}(t) = \overline{H}_p(t,x(t),u(t),v(t),p(t),\mu(t),\nu(t)) & \\
\dot{p}(t) = -\overline{H}_x(t,x(t),u(t),v(t),p(t),\mu(t),\nu(t)) & \\
\overline{H}_u(t,x(t),u(t),v(t),p(t),\mu(t),\nu(t)) = 0 & \\
\overline{H}_v(t,x(t),u(t),v(t),p(t),\mu(t),\nu(t)) = 0 & \\
\phi^\alpha(t,x(t),u(t)) \leq 0 & \alpha = 1, \ldots\ r' \\
\phi^\alpha(t,x(t),u(t)) = 0 & \alpha = r'+1, \ldots, r \\
\psi^\beta(t,x(t),u(t)) \leq 0 & \beta = 1, \ldots\ s' \\
\psi^\beta(t,x(t),u(t)) = 0 & \beta = s'+1, \ldots, s.
\end{array}\right\} \quad (3.2.28)$$

together with

$$\frac{d}{dt}\,\overline{H}(t,x(t),u(t),v(t),p(t),\mu(t),\nu(t)) = \overline{H}_t(t,x(t),u(t),$$

$$v(t),\ p(t),\mu(t),\nu(t)) \qquad (3.2.29)$$

on each of the intervals $(t_{i-1},\ t_i)$, $i = 1, \ldots, N$.

At the points t_i, $i = 0, \ldots, N$, these expressions hold in the sense of left- and right-hand limits.

Proof.--The existence of the multipliers ν_β with the stated properties follows from Theorem 1.3 with $H(t,x,U(t,x),v,P(t,x))$ the function minimized by $v = V(t,x)$. The second equation in (3.2.27) is established by

$$\overline{H}_v(t,x,U(t,x),v(t,x),P(t,x),\mu(t,x),\nu(t,x))$$

$$= H_v(t,x,U(t,x),V(t,x),P(t,x)) + \nu_\beta\psi^\beta_v(t,x,V(t,x)) = 0.$$

The same theorem establishes the existence of the multipliers μ_α. The nonpositive sign of these multipliers is a result of maximizing, rather than minimizing H with respect to u.

The properties of the extremal $\underline{x}$, except for

$$\dot{p} = -\overline{H}_x$$

and

$$\dot{\overline{H}} = \overline{H}_t$$

are a result of (3.2.27) and the fact that $\underline{x}$ is an extremal satisfying the constraints. Let $\bar{t}$ be a point of the interval (t_{i-1}, t_i), and set $\bar{x} = x(\bar{t})$. Then U and V are $C^{(1)}$ on a neighborhood N of $(\bar{t},\bar{x})$ by hypothesis. Let α_j, $j = 1, \ldots, M$ be the indices α for which

$$\phi^{\alpha}(\bar{t},\bar{x}, U(\bar{t},\bar{x})) = 0$$

Then

$$\phi^{\alpha_j}(t,x, U(t,x)) \leq 0 \text{ on } N,$$

(It will be identically zero if $r' < \alpha_j \leq r$), and

$$[\phi_t^{\alpha_j}(\bar{t},\bar{x},U(\bar{t},\bar{x})) + \phi_u^{\alpha_j}(\bar{t},\bar{x},U(\bar{t},\bar{x}))U_t(\bar{t},\bar{x})]dt$$

$$+ [\phi_x^{\alpha_j}(\bar{t},\bar{x},U(\bar{t},\bar{x})) + \phi_u^{\alpha_j}(\bar{t},\bar{x},U(\bar{t},\bar{x}))U_x(\bar{t},\bar{x})]dx \leq 0$$

This holds for each (dt, dx) in $\underline{E}^{n+1}$, since $(\bar{t},\bar{x})$ is interior to the neighborhood N. As a consequence (3.2.20) cannot hold unless

$$\phi_t^{\alpha_j}(\bar{t},\bar{x},U(\bar{t},\bar{x})) + \phi_u^{\alpha_j}(\bar{t},\bar{x},U(\bar{t},\bar{x}))U_t(\bar{t},\bar{x}) = 0$$

$$\phi_x^{\alpha_j}(\bar{t},\bar{x},U(\bar{t},\bar{x})) + \phi_u^{\alpha_j}(\bar{t},\bar{x},U(\bar{t},\bar{x}))U_x(\bar{t},\bar{x}) = 0.$$

From these equations and from

$$\mu_{\alpha}(\bar{t},\bar{x}) = 0 \text{ for those } \alpha \text{ for which}$$

$$\phi^{\alpha}(\bar{t},\bar{x},\ U(\bar{t},\bar{x})) < 0,$$

it follows that

$$\mu_{\alpha}(\bar{t},\bar{x})\ [\phi_t^{\alpha}(\bar{t},\bar{x},U(\bar{t},\bar{x})) + \phi_u^{\alpha}(\bar{t},\bar{x},U(\bar{t},\bar{x}))U_t(\bar{t},\bar{x})] = 0 \tag{3.2.31a}$$

and

$$\mu_{\alpha}(\bar{t},\bar{x})[\phi_x^{\alpha}(\bar{t},\bar{x},U(\bar{t},\bar{x})) + \phi_u^{\alpha}(\bar{t},\bar{x},U(\bar{t},\bar{x}))U_x(\bar{t},\bar{x})] = 0 \tag{3.2.31b}$$

The first equation in (3.2.27) can be rewritten to give, at $(\bar{t},\bar{x})$,

$$H_u(\bar{t},\bar{x},U(\bar{t},\bar{x}),V(\bar{t},\bar{x}),P(\bar{t},\bar{x})) = -\mu_{\alpha}(\bar{t},\bar{x})\phi_u^{\alpha}(\bar{t},\bar{x},U(\bar{t},\bar{x})).$$

Multiplying by $U_t(\bar{t},\bar{x})$ and by $U_x(\bar{t},\bar{x})$ gives

$$H_u\ U_t = -\ \mu_{\alpha}\phi^{\alpha}U_t = \mu_{\alpha}\phi_t^{\alpha}$$

and

$$H_u\ U_x = \ \mu_{\alpha}\phi^{\alpha}U_x = \mu_{\alpha}\phi_x^{\alpha} \tag{3.2.32}$$

where each function is evaluated at $(\bar{t},\bar{x})$. The second equation in each line of (3.2.32) comes from (3.2.31). In the same fashion one obtains

$$H_v V_t = \nu_\beta \ \psi_t^\beta$$

$$H_v V_x = \nu_\beta \ \psi_x^\beta \tag{3.2.33}$$

at $(\bar{t},\bar{x})$.

Equations (3.2.32) and (3.2.33) may be used to substitute in (3.2.15) and (3.2.16) to yield at $(\bar{t},\bar{x})$:

$$\dot{p} = -H_x - \mu_\alpha \ \phi_x^\alpha - \nu_\beta \ \psi_x^\beta = - \bar{H}_x$$

$$\dot{H} = H_t + \mu_\alpha \ \phi^\alpha + \mu_\beta \ \phi^\beta = \bar{H}_t .$$

Since

$$\mu_\alpha(t,x)\phi^\alpha(t,x,U(t,x)) + \nu_\beta\psi^\beta(t,x,V(t,x)) \equiv 0 \text{ on } \underline{F}$$

$\bar{H} = H$ on $\underline{F}$, and along the arc $\underline{x}$

$$\dot{\bar{H}} = \dot{H}.$$

Thus

$$\dot{\bar{H}} = \bar{H}_t .$$

Since the point $(\bar{t},\bar{x})$ is an arbitrary point of any one of the intervals (t_{i-1},t_i) $i=1, \ldots, N$, the expressions (3.2.28) and (3.2.20) hold on each of those intervals. Further if $\{\tau_j\}$ is any sequence of points in (t_{i-1}, t_i) with limit t_i (t_{i-1}), they hold at each of these points, and hence hold in the sense of left- (right-) hand limits at t_i (t_{i-1}).

IV. MAXIMIN FIELDS

The previous chapter handled optimal fields for a pair of strategies (U,V) forming a saddle point. In the present chapter, it is not assumed that the pair (U,V) is a saddle point, but that U is a maximin strategy and that Vis the opponent's minimizing strategy against U. The theorems stated in this chapter correspond closely to the theorems obtained in section 3.2. The type of optimal field used here is called a maximin field.

Definition

A Maximin Field $\underline{F}$ is a region $\underline{F}$ in tx-space, vector strategies U and V piecewise $C^{(1)}$ on $\underline{F}$ and a function W continuous and piecewise $C^{(1)}$ on $\underline{F}$ such that

(i) $(t,x,\ U(t,x))\varepsilon\underline{R}_1 \qquad \forall\ (t,x)\ \varepsilon\ \underline{F}$

(ii) $(t,x,\ V(t,x))\varepsilon\underline{R}_2 \qquad \forall\ (t,x)\ \varepsilon\ \underline{F}$

(iii) The inequality

$$W(\alpha,x(\alpha)) \le \int_{\alpha}^{\beta} L(t,x(t),U(t,x(t)),v(t))\ dt + W(\beta,x(\beta)) \tag{4.1.1}$$

holds for every admissible arc

$$\underline{x}: \quad x(t),\ U(t,x(t)),\ v(t) \qquad \alpha \le t \le \beta \tag{4.1.2}$$

equality holding in case

$$v(t) = V(t,x(t)). \tag{4.1.3}$$

This is equivalent to

$$W(\alpha,x(\alpha)) = \min_{v} \left\{ \int_{\alpha}^{\beta} L(t,x(t),U(t,x(t)),v(t))dt + W(\beta,x(\beta) \right\} \tag{4.1.4}$$

where the minimum is taken over those v for which (4.1.2) is admissible.

If $\overline{U}$ is any admissible strategy, then

$$W(\alpha,x(\alpha)) \ge \min_{v} \left\{ \int L(t,x(t),\overline{U}(t,x(t)),v(t))dt + W(\beta,x(\beta)) \right\} \tag{4.1.5}$$

the minimum (which is assumed to be attained for at least one v) being taken over admissible arcs

$$\underline{x}: \quad x(t),\ \overline{U}(t,x(t)),\ v(t) \quad \alpha \le t \le \beta. \tag{4.1.6}$$

If v(t), $\alpha \le t \le \beta$, minimized (4.1.5), then it can be shown by contraposition that the same v restricted to $\alpha' \le t \le \beta$, where $\alpha \le \alpha' \le \beta$, minimizes (4.1.5) with α' replacing α.

If there is more than one arc (4.1.2) or (4.1.6) satisfying

$$\dot{x} = f(t,x,U(t,x),v) \quad \text{or} \quad \dot{x} = f(t,x,\overline{U}(t,x),v) \tag{4.1.7}$$

the inequalities (4.1.1) and (4.1.5) are satisfied for each such arc. An admissible arc satisfying

$$\dot{x} = f(t,x,U(t,x),V(t,x))$$

is an <u>extremal</u> of the field. This type of optimal field corresponds to a differential game in which U is player One's maximin strategy, for if $(\beta,x(\beta)) \varepsilon \underline{T}$ and if on $\underline{T}$, $w(T(\sigma),X(\sigma))$ is defined to be $K(\sigma)$, (4.1.4) and (4.1.5) simply state that U is the maximin strategy. Furthermore, if U is player One's maximin strategy and V is the corresponding minimizing strategy for player Two, by setting

$$\widetilde{U}(t,x) = \left\{ \begin{matrix} \overline{U}(t,x) & t < \beta \\ U(t,x) & t \geq \beta \end{matrix} \right\} (t,x) \; \varepsilon \; \underline{F} \cup \underline{T}, \tag{4.1.8}$$

one can show that not only (4.1.4) but also (4.1.5) must hold, with $W(\alpha,x(\alpha))$ given by

$$W(\alpha,x(\alpha)) = \int_{\alpha}^{t_1} L(t,x(t),U(t,x(t)),V(t,x(t))dt + W(t_1,x(t_1)) \tag{4.1.9}$$

in which the arc

$$\underline{x}: \; x(t),\; U(t,x(t)),\; V(t,x(t)) \quad \alpha \leq t \leq t_1, \tag{4.1.10}$$

with $(t_1,x(t_1)) = (T(\sigma),X(\sigma)) \varepsilon \underline{T}$ and $W(t_1,x(t_1)) = K(\sigma)$, is admissible. Equation (4.1.9) defines $W(\alpha,x(\alpha))$ as $J(\alpha,x(\alpha);\ U,V)$. Since V is minimizing

$$J(\alpha,x(\alpha);\ U,\ V) \leq J(\alpha,x(\alpha),\ U,\ \overline{V})$$

for any $\overline{V}\varepsilon\underline{V}$. Suppose that $\overline{V}$ does not differ from V on the set of (t,x) in $\underline{F}$ which have $t \geq \beta$ for some $\beta > \alpha$. Let $\underline{x}$ be an arc corresponding to the pair $(U,\overline{V})$ starting at $t = \alpha$, terminating at $(t_1,x(t_1))\varepsilon\underline{T}$ and let

$$v(t) = \overline{V}(t,x(t)) \qquad \alpha \leq t < \beta$$

Then (if $\beta \leq t_1$)

$$\begin{aligned} W(\alpha,x(\alpha)) \leq J(\alpha,x(\alpha);\ U,\overline{V}) = &\int_\alpha^\beta L(t,x(t),U(t,x(t)),(v(t))dt \\ &+ \int_\beta^{t_1} L(t,x(t),U(t,x(t)),V(t,x(t)))dt \\ &+ W(t_1,x(t_1)), \end{aligned}$$

so that

$$W(\alpha,x(\alpha)) \leq \int_\alpha^\beta L(t,x(t),U(t,x(t)),v(t))dt + W(\beta,x(\beta)). \tag{4.1.11}$$

This is (4.1.4). To show (4.1.5) one starts with

$$W(\alpha,x(\alpha)) = J(\alpha,x(\alpha): U,V) \geq \min_{\bar{V}\varepsilon\underline{V}} J(\alpha,x(\alpha); \tilde{U},V) \qquad (4.1.12)$$

where $\tilde{U}$ is given by (4.1.8). This holds since U is the maximin strategy. Then

$$\min_{\bar{V}\varepsilon\underline{V}} J(\alpha,x(\alpha); \tilde{U},V)$$

$$= \min_{\bar{V}\varepsilon\underline{V}} \{\int_{\alpha}^{\beta} L(t,x(t),U(t,x(t)), V(t,x(t)))dt$$

$$+ \int_{\beta}^{t_1} L(t,x(t),U(t,x(t)), V(t,x(t)))dt$$

$$+ W(t_1,x(t_1))\}$$

where the arc

$$\underline{x}: \quad x(t),\ U(t,x(t)),\ V(t,x(t)) \quad \alpha \leq t \leq t_1$$

intersects $\underline{T}$ at $t = t_1$, and $\beta \leq t_1$. Then

$$\min_{\bar{V}\varepsilon\underline{V}} J(\alpha,x(\alpha): \tilde{U},\bar{V})$$

$$= \min_{\bar{V}_1\varepsilon\underline{V}} \{\int_{\alpha}^{\beta} L(t,x(t),\bar{U}(t,x(t)), \bar{V}_L(t,x(t)))dt$$

$$+ \min_{\bar{V}_2\varepsilon\underline{V}} \{\int_{\beta}^{t_1} L(t,x(t),U(t,x(t)), \bar{V}_2(t,x(t)))dt$$

$$+ W(t_1,x(t_1))\}\}$$

$$= \min_{\bar{V}_1\varepsilon\underline{V}} \{\int_{\alpha}^{\beta} L(t,x(t),\bar{U}(t,x(t)), \bar{V}_1(t,x(t)))dt$$

$$+ W(\beta,x(\beta))\}, \qquad (4.1.13)$$

since the second minimum is given by $\overline{V}_2 = V$. Now

$$\min_{\overline{V}_1 \varepsilon \underline{V}} \left\{ \int_\alpha^\beta L(t,x(t),\overline{U}(t,x(t)),\overline{V}_1(t,x(t)))dt + W(\beta,x(\beta)) \right\}$$

$$\geq \min_{v} \left\{ \int_\alpha^\beta L(t,x(t),\overline{U}(t,x(t)),v(t))dt + W(\beta,x(\beta)) \right\}. \tag{4.1.14}$$

the minimum being taken over admissible arcs (4.1.6). Combining (4.1.12), (4.1.13), and (4.1.14) establishes (4.1.5). The inequality in (4.1.14) arises from considering v's which do not satisfy $v(t) = \overline{V}(t,x(t))$ for any $\overline{V} \varepsilon \underline{V}$, since v is only required to be piecewise continuous and $\overline{V}$ is piecewise $C^{(1)}$. One may also note that $\min\limits_{\overline{V} \varepsilon \underline{V}}$, $\min\limits_{\overline{V}_1 \varepsilon \underline{V}}$ and $\min\limits_{\overline{V}_2 \varepsilon \underline{V}}$ could be replaced by $\inf\limits_{\overline{V} \varepsilon \underline{V}}$, $\inf\limits_{\overline{V}_1 \varepsilon \underline{V}}$, $\inf\limits_{\overline{V}_2 \varepsilon \underline{V}}$ without changing the validity of the argument.

In an analogous fashion a minimax field could be defined in which V is player Two's minimax strategy. While the theorems in this section are stated for maximin fields, they hold also for minimax fields if the obvious changes (e.g. in inequalities) are made.

The fact that the inequalities defining an optimal field can be obtained from the assumption that the players in a differential game have optimal strategies (as has just been demonstrated for a maximin field) motivates the study of such fields in connection with differential games. However, the continuity properties of the function W have not been obtained. It would be necessary to show this if one wanted to demonstrate that the field corresponds to the solution of the differential game. Berkovitz [3] assumes that a differential game has a saddle point and shows that if the decomposition of F corresponding to the optimal strategies U and V is of a certain type, called a "regular decomposition," the payoff $J(t,x;\, U,V)$ has the continuity properties required of W.

As before let E be defined by

$$E(t,x,u,v) = L(t,x,u,v) + W_t(t,x) + W_x(t,x)f(t,x,u,v).$$

Theorem 4.1

The functions U, V, and W satisfy on F

$$E(t,x(t),U(t,x),\, v) \geq E(t,x,U(t,x),V(t,x)) = 0 \tag{4.1.15}$$

where $v \varepsilon \Psi(t,x)$, and

$$E(t,x(t),U(t,x(t)),V(t,x(t))) =$$

$$= \max_{u\varepsilon\Phi(t,x)} \left\{ \inf_{\varepsilon\Psi(t,x)} E(t,x,u,v) \right\} \tag{4.1.16}$$

If $\{\underline{F}^{(\alpha)}\}$ is a $C^{(1)}$-decomposition of $\underline{F}$ corresponding to U, V and W, (4.1.15) and (4.1.16) hold on each $\underline{F}^{(\alpha)}$, and (4.1.15) holds on the manifolds separating regions of the decomposition if interpreted with U, V, W_t and W_x as limits from one of the adjacent regions. If $(\bar{t},\bar{x})\varepsilon\partial\,\underline{F}^{(\alpha)}$ is a point on one of these manifolds such that for all $\bar{u}\varepsilon\Phi(\bar{t},\bar{x})$, $(\bar{t},\bar{x},\bar{u})$ is a point of continuity of $\inf_v E^{(\alpha)}(t,x,u,v)$, (4.1.16) also holds at $(\bar{t},\bar{x})$ in the same limiting sense. The expression

$$E^{(\alpha)}(t,x,U^{(\alpha)}(t,x),V^{(\alpha)}(t,x))$$

$$= \max_{\bar{u}\varepsilon\Phi(t,x)} \{\overline{\lim_{(t,x,u)\to(t,x,u)}} \; \inf_{v\varepsilon\Psi(t,x)} E^{(\alpha)}(t,x,u,v)\} \tag{4.1.17}$$

where $(t,x)\,\varepsilon\,\underline{F}^{(\alpha)}$, $u\varepsilon\Phi(t,x)$, holds regardless of the continuity of $\inf E^{(\alpha)}$.

<u>Proof</u>.--The equations

$$E(t,x,U(t,x),V(t,x)) = \min_{v\varepsilon\Psi(t,x)} E(t,x,U(t,x),v) = 0$$

are consequences of Theorem 3.1 and its second corollary, for with U fixed, $\underline{F}$, W and V form an optimal field of the type considered in section 3.1. This establishes (4.1.15).

It remains to show that (4.1.16) holds. Let $(\bar{t},\bar{x})$ be a point of continuity of U, V, W_t and W_x, and let $\bar{U}$ be any admissible strategy. Arcs to be considered are admissible arcs of the form

$$\underline{x}: \quad x(t),\ u(t),\ v(t) \qquad \bar{t} - \delta \le t \le \bar{t},\ \delta>0 \tag{4.1.18}$$

with $x(\bar{t}) = \bar{x}$, $u(t) = \bar{U}(t,x(t))$. From the definition of the field

$$W(t-\varepsilon,x(t-\varepsilon)) \ge \min_{v} \left\{\int_{\bar{t}-\varepsilon}^{\bar{t}} L(t,x(t),u(t),v(t))dt + W(\bar{t},\bar{x})\right\} \tag{4.1.19}$$

for each ε such that $0 \le \varepsilon \le \delta$. If $\bar{v}(t)$, $\bar{t} - \delta \le t - \bar{t}$, is the v which minimizes the expression in (4.1.19), for $\varepsilon = \delta$ the same $\bar{v}$ minimizes (4.1.19) for ε on $[0,\delta]$. Then

$$W(\bar{t}-\varepsilon,x(\bar{t}-\varepsilon)) \ge \int_{\bar{t}-\varepsilon}^{\bar{t}} L(t,x(t),u(t),\bar{v}(t))dt + W(\bar{t},\bar{x}).$$

But

$$\int_{\bar{t}-\varepsilon}^{\bar{t}} L(t,x(t),u(t),\bar{v}(t))dt = \int_{\bar{t}-\varepsilon}^{\bar{t}} [E(t,x(t),u(t),\bar{v}(t)) - W_t(t,x(t)) - W_x(t,x(t))f(t,x(t),u(t),\bar{v}(t))]dt$$

$$= \int_{\bar{t}-\varepsilon}^{\bar{t}} E(t,x(t),u(t),\bar{v}(t))dt + W(\bar{t}-\varepsilon,X(\bar{t}-\varepsilon)) - W(\bar{t},\bar{x})$$

Since

$$\int_{t-}^{\bar{t}} [W_t(t,x(t)) + W_x(t,x(t)) f(t,x(t),u(t),v(t))]dt = \int_{\underline{x}} dW$$

Thus

$$g(\varepsilon) \triangleq \int_{\bar{t}-\varepsilon}^{\bar{t}} E(t,x(t),u(t),\bar{v}(t))dt \leq 0.$$

The function g has a maximum on $[0, \delta]$ at $\varepsilon = 0$, since $g(0) = 0$. Since g is differentiable

$$0 \geq g'(0) = E(\bar{t},\bar{x},\bar{u},\bar{v}), \tag{4.1.20}$$

where

$$\bar{u} = \lim_{t\to\bar{t}-} u(t), \quad \bar{v} = \lim_{t\to\bar{t}-} \bar{v}(t).$$

Since $\bar{U}$ is any strategy in $\underline{U}$, $\bar{u}$ may be any point in $\Phi(\bar{t},\bar{x})$. Then, from (4.1.20)

$$\inf_{v\varepsilon\Psi(\bar{t},\bar{x})} E(\bar{t},\bar{x},\bar{u},v) \leq 0 \quad \text{for any } \bar{u}\varepsilon\Phi(\bar{t},\bar{x}).$$

Since $\inf_{v\varepsilon\Psi(\bar{t},\bar{x})} E(\bar{t},\bar{x},U(\bar{t},\bar{x}),v) = 0$ by (4.1.15),

equation (4.1.16) holds. Suppose that $(\bar{t},\bar{x})$ is a point on on a manifold of discontinuity which is part of the boundary of the subregion $\underline{F}^{(\alpha)}$, and that $\bar{u}\varepsilon\Phi(\bar{t},\bar{x})$. Since

$$\inf_{v} E^{(\alpha)}(t,x,u,v) \leq 0 \quad \text{on } \underline{F}^{(\alpha)}$$

$$\overline{\lim_{(t,x,u)\to(\bar{t},\bar{x},\bar{u})}} \left\{ \inf_{v\varepsilon\Psi(t,x)} E^{(\alpha)}(t,x,u,v) \right\} \leq 0, \tag{4.1.21}$$

Also

$$E^{(\alpha)}(t,x,U^{(\alpha)}(t,x),V^{(\alpha)}(t,x)) = 0 \quad \text{on } F^{(\alpha)}$$

implies

$$E^{(\alpha)}(\bar{t},\bar{x},U^{(\alpha)}(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x})) = 0 \tag{4.1.22}$$

This equation combines with (4.1.21) to give (4.1.17). If $(\bar{t},\bar{x},\bar{u})$ is a point of continuity of $\inf\limits_{v} E^{(\alpha)}(t,x,u,v)$,

$$\lim_{(t,x,u)\to(\bar{t},\bar{x},\bar{u})} \inf_{v\varepsilon\Psi(t,x)} E^{(\alpha)}(t,x,u,v)$$

$$= \inf_{v\varepsilon\Psi(\bar{t},\bar{x})} E^{(\alpha)}(\bar{t},\bar{x},\bar{u},\bar{v}),$$

and this may be used to substitute in (4.1.17). In $(\bar{t},\bar{x},\bar{u})$ is a point of continuity of $\inf\limits_{v} E^{(\alpha)}(t,x,u,v)$ for each $u\varepsilon\Phi(t,x)$,

$$E^{(\alpha)}(\bar{t},\bar{x},U^{(\alpha)}(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x}))$$

$$= \max_{u\varepsilon\Phi(\bar{t},\bar{x})} \{ \inf_{v\varepsilon\Psi(\bar{t},\bar{x})} E^{(\alpha)}(\bar{t},\bar{x},u,v)\}$$

This cannot be concluded in general, since $\inf\limits_{v} E^{(\alpha)}(t,x,u,v)$ may not be continuous. It is however, upper semicontinuous, which yields

$$\inf_{v\varepsilon\Psi(t,x)} E^{(\alpha)}(\bar{t},\bar{x},\bar{u},v) \geq$$

$$\overline{\lim}_{(t,x,u)\to(\bar{t},\bar{x},\bar{u})} \{ \inf_{v\varepsilon\Psi(t,x)} E^{(\alpha)}(t,x,u,v)\}$$

The next theorem is Theorem 3.3 restated for the present type of field. The definitions of H and P are the same as before--although the U, V and W used in them are different, W now being a maximin value function, U, player One's maximin strategy, and V the strategy optimal against U.

Theorem 4.2

At each point (t,x) in a maximin field $\underline{F}$, the statements

$$H(t,x,U(t,x),V(t,x),P(t,x)) \leq H(t,x,U(t,x),v,P(t,x)),$$

$$v\varepsilon\Psi(t,x), \tag{4.1.23}$$

$$H(t,x,U(t,x),V(t,x),P(t,x))$$

$$= \max_{u\varepsilon\Phi(t,x)} \inf_{v\varepsilon\Psi(t,x)} H(t,x,u,v,P(t,x)) \tag{4.1.24}$$

hold. Moreover, U, V and W satisfy

$$W_t(t,x) + H(t,x,U(t,x),V(t,x), W_x(t,x)) = 0. \tag{4.1.25}$$

The usual limiting sense is given to (4.1.23) and (4.1.25) at manifolds of discontinuity. Equation (4.1.24) must be interpreted with the same caution about limiting values of

$\inf_v H(t,x,u,v,P(t,x))$ as with $\inf_v E(t,x,u,v)$ in (4.1.16).

Finally, the integral

$$I^* = \int \{P(t,x)dx - H(t,x,U(t,x),V(t,x),P(t,x))dt\}$$

is independent of the path in $\underline{F}$.

Proof.--The proof is similar to that given for Theorem 3.3. It amounts to a restatement of Theorem 4.1 in different notation.

Theorem 4.3

Theorem 3.4 holds for maximin fields without restatement.

Proof.--The inequalities defining the type of field are not used in the statement of Theorem 3.4 or in its proof. Theorem 3.4 holds for any continuous, piecewise $C^{(1)}$ function W on a region $\underline{F}$, regardless of its origin. In the present context the statement about H and P (3.2.13) is a consequence of (3.2.8) (defining H and P), (4.1.25) and (3.2.12).

To hold for a maximin field, the corollary to Theorem 3.4 must be reformulated.

Theorem 4.4

Let $\{\underline{F}^{(\alpha)}\}$ be a $C^{(1)}$-decomposition of $\underline{F}$ based on U, V and W, and let Γ be a set of indices α such that the manifold $\underline{M} = \bigcap_{\alpha\varepsilon\Gamma} \partial\underline{F}^{(\alpha)}$ is not empty. If there exist a maximin strategy U and a strategy V optimal against U which prescribe optimal paths lying in $\underline{M}$, then at each point $(\bar{t},\bar{x})$ of $\underline{M}$

$$E^{(\alpha)}(\bar{t},\bar{x},U(\bar{t},\bar{x}),v) \geq E^{(\alpha)}(\bar{t},\bar{x},U(\bar{t},\bar{x}),V(\bar{t},\bar{x})) = 0 \quad (4.4.26)$$

whenever $v\varepsilon\Psi(\bar{t},\bar{x})$, $(1,f(\bar{t},\bar{x},U(\bar{t},\bar{x}),v))$ is tangent to $\underline{M}$ and $\alpha\varepsilon\Gamma$. Further,

$$E^{(\alpha)}(\bar{t},\bar{x},U(\bar{t},\bar{x}),V(\bar{t},\bar{x})) = \max_{u} \inf_{v\varepsilon\Phi(\bar{t},\bar{x})} E^{(\alpha)}(\bar{t},\bar{x},u,v). \quad (4.4.27)$$

The maximum is not necessarily taken over all u in $\Phi(\bar{t},\bar{x})$, but is taken over a subset of $\Phi(\bar{t},\bar{x})$. For each u in this subset there must exist an admissible arc

$$\begin{aligned} &\underline{\hat{x}}: \quad \hat{x}(t),\ \bar{U}(t,\hat{x}(t)),\ \bar{v}(t) \qquad \alpha \leq t \leq \bar{t} \\ &\hat{x}(\bar{t})=\bar{x},\ \bar{U}(\bar{t},\bar{x})=u \qquad (t,\hat{x}(t))\,\varepsilon\, \overline{\underline{F}^{(\alpha)}} \end{aligned} \quad (4.4.28)$$

where $\bar{U}\varepsilon\underline{U}$, which minimizes

$$\int_{\alpha}^{\bar{t}} L(t,x(t),\ \bar{U}(t,x(t)),\ v(t))dt + W(\bar{t},\bar{x})$$

among admissible arcs

$$\left.\begin{array}{l} \underline{x}:\quad x(t),\ \overline{U}(t,x(t)),\ v(t),\quad \alpha \le t \le \overline{t} \\ \text{with} \\ \qquad x(\overline{t}) = \overline{x}. \end{array}\right\} \qquad (4.1.29)$$

The arcs (4.1.29) need not lie in $\overline{\underline{F}^{(\alpha)}}$, although (4.1.28) is required to do so.

Proof.--Since V is the strategy minimizing against U, the sections of the corollary to Theorem 3.4 which apply to minimization yield (4.1.26) immediately. The tangency restriction allows one to state (4.1.26) with $U(\overline{t},\overline{x})$ and $V(\overline{t},\overline{x})$ rather than $U^{(\alpha)}(\overline{t},\overline{x})$ and $V^{(\alpha)}(\overline{t},\overline{x})$.

If, for $\alpha\varepsilon\Gamma$, one defined W_t and W_x on $\underline{M}$ to be $W^{(\alpha)}$ and $W_x{}^{(\alpha)}$, one can obtain (4.1.27) with the same argument as used in the proof of Theorem 4.1 provided that the stated restrictions are satisfied. These allow the use of $E^{(\alpha)}$, $W_t{}^{(\alpha)}$ and $W_x{}^{(\alpha)}$ for E, W_t and W_x in the argument.

The comments preceding Theorem 3.5 indicate that the following theorem is true.

Theorem 4.5

Theorem 3.5 holds without alteration for a maximin field.

Proof.--Theorem 3.5 is independent of the type of optimal field; indeed, it depends only on the existence of a pair of strategies $C^{(1)}$ on a region $\underline{G} \subseteq \underline{F}$, and a function W satisfying

$$W(t_0,x(t_0)) = \int_{t_0}^{t_1} L(t,x(t),U(t,x(t)),V(t,x(t)))dt$$

$$+ W(t_1,x(t_1))$$

for admissible arcs $\underline{x}$ defined by the strategies U and V lying in $\underline{G}$.

One would wish to have results analogous to Theorems 3.6 and 3.7. However, in a maximin field, the strategy U is not optimal--does not maximize--against the strategy V. Since U does not maximize, one would not expect conditions such as $H_u = 0$ or a multiplier rule to hold. Furthermore, while a global extremum is a local extremum, a global saddle point is a local saddle, a global maxmin is not necessarily a local maxmin. Nevertheless some partial results can be obtained. If U is considered fixed, the maximin field is an optimal (minimizing) field in which V is the optimal strategy. This property may be used to obtain the following theorems. The proofs are omitted, since they are similar to the proofs of Theorems 3.6 and 3.7.

Theorem 4.6

Suppose that on the region $\underline{G}$ of Theorem 3.5 that $\Psi(t,x)$ is open, or alternatively that $V(t,x)$ is an interior point of $\Psi(t,x)$. Then on $\underline{G}$, U, V and W satisfy

$$\left.\begin{aligned} &W_t(t,x) + H(t,x,\ U(t,x),\ V(t,x),\ W_x(t,x)) = 0 \\ &H_v(t,x,\ U(t,x),\ V(t,x),\ W_x(t,x)) \quad = 0. \end{aligned}\right\} \quad (4.1.30)$$

An extremal of the field

$$\underline{x}: \quad x(t),\ u(t),\ v(t) \qquad \alpha \leq t \leq \beta$$

satisfies, with $p(t) = P(t,x(t))$ the canonical Euler equations

$$\dot{x} = H_p,\ \dot{p} = -(H_x + H_u U_x),\ H_v = 0 \qquad (4.1.31)$$

together with

$$\dot{H} = H_t + H_u U_t \qquad (4.1.32)$$

on the region $\underline{G}$.

<u>Theorem 4.7</u>

Suppose that $\underline{R}_2$ is given by the constraint conditions (2.2.7c and d), i.e., by

$$\psi^\beta(t,x,v) \leq 0 \quad \beta = 1,\ \ldots,\ s'$$

$$\psi^\beta(t,x,v) = 0 \quad \beta = s'+1,\ldots,\ s.$$

Then there exist multipliers $\nu_\beta(t,x)$, piecewise continuous on $\underline{F}$ (piecewise $C^{(1)}$ if the ψ^β are of class $C^{(2)}$) such that on $\underline{F}$

$$\nu_\beta(t,x) \geq 0$$

$$\nu_\beta(t,x)\psi^\beta(t,x,V(t,x)) = 0 \qquad \beta = 1, \ldots, s, \text{ not summed,}$$

$$H_v(t,x,U(t,x),V(t,x),P(t,x)) + \nu_\beta(t,x)\psi^\beta_v(t,x,V(t,x)) = 0. \tag{4.1.33}$$

The usual interpretation is made at points of discontinuity. Let

$$\underline{x}: \; x(t),\ u(t),\ v(t) \qquad a \leq t \leq b$$

be an extremal, such that there is a decomposition

$$(t_{i-1},\ t_i) \qquad i = 1, \ldots, N$$

$$a = t_0 < t_1 < \cdots < t_i < \cdots < t_N = b$$

of the interval [a,b] with $(t,x(t))$ in a region of continuity of U and V on each of the intervals comprising the decomposition (thus excluding arcs which have subarcs lying on manifolds of discontinuity of U or V). Then $\underline{x}$ satisfies, with

$$p_i(t) = P_i(t,x(t))$$

$$\nu_\beta(t) = \nu_\beta(t,x(t))$$

the canonical Euler equations

$$\dot{x}(t) = H_p(t,x(t),u(t),v(t),p(t)) \tag{4.1.34a}$$

$$\begin{aligned}\dot{p}(t) = -(&H_x(t,x(t),u(t),v(t),p(t)) \\ &+ H_u(t,x(t),u(t),v(t),p(t))\, U_x(t,x(t)) \\ &+ \nu(t)\, \psi_x(t,x(t),v(t)))\end{aligned} \tag{4.1.34b}$$

$$H_v(t,x(t),u(t),v(t),p(t)) + \nu(t)\, \psi_v(t,x(t),v(t))) = 0 \tag{4.1.34c}$$

$$\begin{aligned}&\psi^\beta(t,x(t),v(t)) \leq 0 \quad \beta = 1, \ldots, s' \\ &\psi^\beta(t,x(t),v(t)) = 0 \quad \beta = s'+1, \ldots, s\end{aligned} \tag{4.1.34d}$$

together with

$$\begin{aligned}\frac{d}{dt} H(t,x(t),u(t),v(t),p(t)) = &H_t(t,x(t),u(t),v(t),p(t)) \\ &+ H_u(t,x(t),u(t),v(t),p(t))\, U_t(t,x(t)) \\ &+ \nu(t)\, \psi_t(t,x(t),v(t))\end{aligned} \tag{4.1.35}$$

on each of the intervals (t_{i-1}, t_i) $i = 1, \ldots, N$. At the points t_i, $i = 0, \ldots, N$, these expressions hold in the sense of left- and right-hand limits.

In this last theorem $H_v V_x$ and $H_v V_t$ were replaced by $\nu\psi_x$ and $\nu\psi_t$, as was done in Theorem 3.7. A similar replacement for $H_u U_x$ and $H_u U_t$ could not be made since U does not maximize against V.

One could also consider the situation in which V is player Two's minimax strategy, and not the strategy minimizing against the maximin strategy U. Provided that on $\underline{F}$ a function W continuous and piecewise $C^{(1)}$ is defined which satisfies (on $\underline{F}$)

$$W(t_0, x(t_0)) = \int_{t_0}^{t_1} L(t, x(t), U(t, x(t)), V(t, x(t)))\,dt + W(t_1, x(t_1))$$

for admissible arcs

$$\underline{x}: \quad x(t),\ U(t, x(t)),\ V(t, x(t)) \quad t_0 \leq t \leq t_1,$$

Theorems 3.4 and 3.5 apply in this case also. This is so because these theorems do not depend on the optimality of U or V.

V. TRANSVERSALITY AND DISCONTINUITY CONDITIONS

5.1 Transversality Conditions

Let

$$\underset{\sim}{x}: \quad x(t),\ u(t),\ v(t) \qquad t_0 \le t \le t_1$$

$$(t,x(t)) \in \underline{F}, \qquad t_0 \le t \le t_1, \qquad (5.1.1)$$

$$(t_1,x(t_1)) = (T(\sigma),\ X(\sigma)) \in \underline{T},$$

be an extremal arc. The transversality conditions are obtained by requiring that

$$W(t_1,x(t_1)) = K(\sigma) \qquad (5.1.2)$$

Theorem 5.1

Let $\underline{T}'$ be that subset of $\underline{T}$ which contains the terminal points of all extremal arcs (3.4.1) which are not tangent to $\underline{T}$ at $(t_1,x(t_1))$. Let $\{\underline{K}^{(\alpha)}\}$ be a $C^{(1)}$-decomposition based on K, T, and X, and let $\bar{\sigma}$ be a point of one of the $\underline{K}^{(\alpha)}$ for which

$$(t_1,x_1) = (t(\sigma),\ X(\sigma)) \in \underline{T}',$$

such that there exists a neighborhood N of (t_1,x_1) such that U and V are $C^{(1)}$ on $N \cap \underline{F} = N'$.

Then the transversality conditions

$$W_t(t_1,x_1)\ T_\sigma(\bar\sigma) + W_x(t_1,x_1)\ X_\sigma(\bar\sigma) = K_\sigma(\bar\sigma) \qquad (5.1.3)$$

$$K_\sigma(\bar\sigma) + H(t_1,x_1,U(t_1,x_1),V(t_1,x_1)-P(t_1,x_1))\ T_\sigma(\bar\sigma)$$

$$- P(t_1,x_1)\ X_\sigma(\bar\sigma) = 0 \qquad (5.1.4)$$

hold, where W_t, W_x, H, P, U and V are given their limiting values as $(t,x)\varepsilon N'\to(t_1,x_1)$.

Proof.--Equation (5.1.4) follows from (5.1.3), the definitions of H and P, (3.2.8), and the Hamilton-Jacobi Equation

$$W_t(t,x) + H(t,x,U(t,x),V(t,x),W_x(t,x)) = 0$$

The functions U and V have $C^{(1)}$ extensions to the neighborhood N.

Consider the differential equations

$$\dot{x} = f(t,x,\ U(t,x),\ V(t,x)) \qquad (5.1.5)$$

There is a constant $\rho > 0$ such that there is a unique solution x_0, lying in N, of (5.1.5) through the point (t_1,x_1) on the interval $[\alpha,\beta]$, $\alpha = t_1 - \rho$, $\beta = t_1 + \rho$. Furthermore, there exist constants ρ', $\pi > 0$ such that through each point (τ,ξ) satisfying

$$\alpha - \rho' \le \tau \le \beta + \rho', \quad |\xi - x_0(\tau)| < \pi$$

there is a unique solution

$$x(t,\tau,\xi) \qquad \alpha - \rho' < t < \beta + \rho' \tag{5.1.6}$$

of (5.1.5), containing $\underline{x_0}$ for $\alpha \leq t \leq \beta$, $\xi = x_0(\tau)$.

On N' each of these solutions is an extremal arc. By hypothesis, each one intersects $\underline{T}'$ and is not tangent to it. Let $t(\tau,\xi)$, $x(\tau,\xi)$ be the point of intersection of (3.4.6) with $\underline{T}'$.

$W(\tau,\xi)$ must satisfy, by (5.1.2)

$$W(\tau,\xi) = \int_{\tau}^{t(\tau,\xi)} L(t,x(t,\tau,\xi),U(t,x(t,\tau,\xi)),V(t,x(t,\tau,\xi)))dt+K(\sigma)$$

where $t(\tau,\xi) = T(\sigma)$, $x(\tau,\xi) = X(\sigma)$, or equivalently

$$G(\sigma,\tau,\xi) \triangleq -x(T(\sigma),\tau,\xi) + X(\sigma) = 0. \tag{5.1.7}$$

G is $C^{(1)}$ in σ, τ and ξ, because of the properties of the solution (5.1.6), T and X. If $|G^i{}_{\sigma^j}(\bar{\sigma},t_1,x_1)| \neq 0$, (5.1.7) determines σ as a $C^{(1)}$ function of (τ,ξ) in a neighborhood of (t_1,x_1), which may be taken to be N without loss of generality. The nontangency assumption assures that the determinant is nonzero, since then the matrix

$$\begin{bmatrix} 1 & T_\sigma(\bar{\sigma}) \\ f(t_1,x_1,U(t_1,x_1),\ V(t_1,x_1)) & X_\sigma(\bar{\sigma}) \end{bmatrix}$$

has rank n+1. By elementary transformations this becomes

$$\begin{vmatrix} 1 & 0 \\ f(t_1,x_1,U(t_1,x_1),\ V(t_1,x_1)) & M \end{vmatrix}$$

where

$$M = X_\sigma(\bar{\sigma}) - f(t_1,x_1,U(t_1,x_1),\ V(t_1,x_1))\ T_\sigma(\bar{\sigma})$$

has rank m. But since

$$\frac{\partial}{\partial t}\, x(t,\tau,\xi) = f(t,x(t,\tau,\xi),U(t,x(t,\tau,\xi),V(t,x(t,\tau,\xi)))$$

$$\frac{\partial G}{\partial \sigma}\,(\bar{\sigma},t_1,x_1) = X_\sigma(\bar{\sigma}) - f(t_1,x_1,U(t_1,x_1),V(t_1,x_1))T_\sigma(\bar{\sigma}) = M$$

and the determinant $\left|G^i_\sigma j\right| \neq 0$. Then

$$W(\tau,\xi) = \int_\tau^{T(\sigma(\tau,\xi)} L(t,x(t,\tau,\xi),U(t,x(t,\tau,\xi),V(t,x(t,\tau,\xi)))dt$$

$$+ K(\sigma\ (\tau,\xi)).$$

$W(\tau,\xi)$ is $C^{(1)}$ on $N \cap \underline{\bar{F}}$ since T and K are $C^{(1)}$ on $K^{(\alpha)}$, σ and x are $C^{(1)}$ on N, L is $C^{(1)}$ in (t,x,u,v) and U and V are $C^{(1)}$ on N.

Let

$$w(\tau,\xi) = W(T(\sigma(\tau,\xi)), X(\sigma(\tau,\xi))) = K(\sigma(\tau,\xi))$$

and let $(\overline{\tau},\overline{\xi})$ be a point on the extremal $\underline{x_0}$. Then $\overline{\sigma} = \sigma(\overline{\tau},\overline{\xi})$, $T(\overline{\sigma}) = t_1$, $X(\overline{\sigma}) = x_1$ and

$$\begin{aligned} dw(\overline{\tau},\overline{\xi}) &= w_\tau(\overline{\tau},\overline{\xi})d\tau + w_\xi(\overline{\tau},\overline{\xi})d\xi \\ &= (W_t(t_1,x_1)T_\sigma(\overline{\sigma})+W_x(t_1,x_1)X_\sigma(\overline{\sigma}))(\sigma_\tau(\overline{\tau},\overline{\xi})d\tau \\ &+ \sigma_\xi(\overline{\tau},\overline{\xi})d\xi) \\ &= K_\sigma(\overline{\sigma})\ (\sigma_\tau(\overline{\tau},\overline{\xi})d\tau + \sigma_\xi(\overline{\tau},\overline{\xi})d\xi). \end{aligned}$$

Since $d\tau$ and $d\xi$ are arbitrary

$$W_t(t_1,x_1)T_\sigma(\overline{\sigma}) + W_x(t_1,x_1)\ X_\sigma(\overline{\sigma}) = K_\sigma(\overline{\sigma})$$

as was to be proved.

The proof of this theorem is based on the proof used by Berkovitz [3] for a related result applied to transition surfaces (cf. the following section).

5.2 Manifolds of Discontinuity

A careful application of Theorems 3.2 and 3.4 can yield additional conditions at certain types of manifolds of discontinuity. Three main types are considered: transition surfaces, dispersal manifolds, and universal manifolds. This terminology is due to Isaacs [20].

Let $\underline{M}$ be an n-dimensional manifold of discontinuity of U or of V (or of both), (where U and V are optimal strategies in the case that the game has a saddle point, U is a maximin strategy with V optimal against it, or V is a minimax strategy with U optimal against it). If, for each point $(\bar{t},\bar{x})\varepsilon\underline{M}$, there is an extremal

$$\underline{x}:\quad x(t),\ u(t),\ v(t)\qquad \alpha \leq t \leq \beta \tag{5.2.1}$$

with $x(\bar{t}) = \bar{x}$, $\alpha < \bar{t} < \beta$, and $x(t)\notin\underline{M}$ for $t \neq \bar{t}$, $\underline{M}$ is called a <u>transition surface</u>.

<u>Theorem 5.2</u>

Let $\{\underline{F}^{(\alpha)}\}$ be a $C^{(1)}$-decomposition of $\underline{F}$ based on U, V and W for a game with a saddle point, and suppose that $\underline{M} = \overline{\underline{F}^{(\alpha)}} \cap \overline{\underline{F}^{(\beta)}} \cap \underline{F}$ is a transition surface which is a manifold of discontinuity of at most one of the strategies U and V. If at $(\bar{t},\bar{x})\varepsilon\underline{M}$, the arc (5.2.1) intersecting $\underline{M}$ at $(\bar{t},\bar{x})$ is not tangent to $\underline{M}$ as $t \to \bar{t} - 0$ and $t \to \bar{t} + 0$, $W_t^{(\alpha)}(\bar{t},\bar{x}) = W_t^{(\beta)}(\bar{t},\bar{x})$ and $W_x^{(\alpha)}(\bar{t},\bar{x}) = W_x^{(\beta)}(\bar{t},\bar{x})$. That is, W is $C^{(1)}$ at $(\bar{t},\bar{x})$.

<u>Proof</u>.--From Theorem 3.4, for any vector (dt, dx) tangent to $\underline{M}$ at $(\bar{t},\bar{x})$

$$(W_t^{(\alpha)}(\bar{t},\bar{x})-W_t^{(\beta)}(\bar{t},\bar{x}))dt+(W_x^{(\alpha)}(\bar{t},\bar{x})-W_x^{(\beta)}(\bar{t},\bar{x}))dx=0,$$

which indicates either that

$$W_t^{(\alpha)}(\bar{t},\bar{x})=W_t^{(\beta)}(\bar{t},\bar{x}) \text{ and } W_x^{(\alpha)}(\bar{t},\bar{x})=W_x^{(\beta)}(\bar{t},\bar{x})$$

or that

$$\left.(W_t^{(\alpha)} - W_t^{(\beta)},\ W_x^{(\alpha)} - W_x^{(\beta)})\right|_{(t,x) = \bar{t},\bar{x})}$$

is a nonzero vector orthogonal to $\underline{M}$ at $(\bar{t},\bar{x})$. Let (μ,ν) be any vector orthogonal to $\underline{M}$ at (t,x). Then, since $\underline{M}$ is a transition surface

$$\mu + \nu f(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x}))$$

and

$$\mu + \nu f(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\beta)}(\bar{t},\bar{x}))$$

have the same sign, where, for definiteness, V is taken to be the strategy having the discontinuity. Further, by the nontangency assumption, this sign must be either strictly positive or strictly negative.

By Theorem 3.2

$$\begin{aligned} - W_t^{(\alpha)}(\bar{t},\bar{x}) &= L(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x})) \\ &\quad + W_x^{(\alpha)}(\bar{t},\bar{x})f(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x})) \\ &\leq L(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\beta)}(\bar{t},\bar{x})) \\ &\quad + W_x^{(\alpha)}(\bar{t},\bar{x})f(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\beta)}(\bar{t},\bar{x})). \end{aligned}$$

Likewise,

$$-W_t^{(\beta)}(\bar{t},\bar{x}) = L(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\beta)}(\bar{t},\bar{x}))$$

$$+ W_x^{(\beta)}(\bar{t},\bar{x})f(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\beta)}(\bar{t},\bar{x}))$$

$$- L(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x}))$$

$$+ W_x^{(\beta)}(\bar{t},\bar{x},)f(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x})).$$

Consequently, on the one hand

$$W_t^{(\alpha)}(\bar{t},\bar{x}) - W_t^{(\beta)}(\bar{t},\bar{x}) \geq (W_x^{(\beta)}(\bar{t},\bar{x})$$

$$- W_x^{(\alpha)}(\bar{t},\bar{x}))f(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\beta)}(\bar{t},\bar{x}))$$

and on the other hand

$$W_t^{(\alpha)}(\bar{t},\bar{x}) - W_t^{(\beta)}(\bar{t},\bar{x}) \leq (W_x^{(\beta)}(\bar{t},\bar{x})$$

$$- W_x^{(\alpha)}(\bar{t},\bar{x}))f(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x})).$$

By rearranging

$$(W_t^{(\alpha)}(\bar{t},\bar{x}) - W_t^{(\beta)}(\bar{t},\bar{x})) + (W_x^{(\alpha)}(\bar{t},\bar{x})$$

$$- W_x^{(\beta)}(\bar{t},\bar{x}))f(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x})) \leq 0$$

$$(W_t^{(\alpha)}(\bar{t},\bar{x}) - W_t^{(\beta)}(\bar{t},\bar{x})) + (W_x^{(\alpha)}(\bar{t},\bar{x})$$

$$- W_x^{(\beta)}(\bar{t},\bar{x}))f(\bar{t},\bar{x},U(\bar{t},\bar{x}),V^{(\beta)}(\bar{t},\bar{x})) \geq 0$$

(5.2.2)

But, if $(W_t^{(\alpha)} - W_t^{(\beta)}, W_x^{(\alpha)} - W_x^{(\beta)})$ evaluated at $(\bar{t},\bar{x})$ is a nonzero vector orthogonal to $\underline{M}$ at $(\bar{t},\bar{x})$, both of these quantities must be either strictly positive or strictly negative, which is impossible. Therefore

$$W_t^{(\alpha)}(t,x) = W_t^{(\beta)}(t,x)$$

and

$$W_x^{(\alpha)}(t,x) = W_x^{(\beta)}(t,x).$$

This theorem and the method of proof are due to Berkovitz [3].

Corollary

Theorem 5.2 holds for a transition surface $\underline{M}$

(i) in a maximin field, if the maximin strategy is continuous across $\underline{M}$

(ii) in a minimax field, if the minimax strategy is continuous across $\underline{M}$

(iii) in an optimal field for a control problem, if L and f are continuous in (t,x).

Proof.--This is true since the proof of the theorem requires varying only the discontinuous strategy, which is either a minimizing strategy or a maximizing strategy. The proof (given for a minimizing strategy) is actually the proof for an optimal control problem with

$$\bar{L}(t,x,v) = L(t,x,U(t,x),v)$$

and

$$\bar{f}(t,x,v) = f(t,x,U(t,x),v),$$

and holds if $\bar{L}$ and $\bar{f}$ are continuous in (t,x) at $(\bar{t},\bar{x})$.

The next theorem concerns manifolds such that each point $(\bar{t},\bar{x})$ on one of these manifolds is the initial point for several extremal arcs, each of which proceeds, for $t > \bar{t}$ into a different subregion of $\underline{F}$. Manifolds of this type are called dispersal manifolds. The theorem also holds for points on manifolds to which several extremal arcs converge. That is the extremals are distinct for $t < \bar{t}$, but all have the point $(\bar{t},\bar{x})$ in common.

As before $\{\underline{F}^{(\alpha)}\}$ is a $C^{(1)}$-decomposition of $\underline{F}$ based on U, V and W for an optimal field in which U and V provide a saddle point.

Theorem 5.3

Let $\underline{M} = \overline{\underline{F}^{(\alpha)}} \cap \overline{\underline{F}^{(\beta)}} \cap \underline{F}$ be a dispersal manifold. Then at $(\bar{t},\bar{x}) \varepsilon \underline{M}$

$$\begin{aligned}(W_t^{(\alpha)}(\bar{t},\bar{x}) - W_t^{(\alpha)}(\bar{t},\bar{x})) + (W_x^{(\alpha)}(\bar{t},\bar{x}) \\ - W_x^{(\beta)}(\bar{t},\bar{x}))f(\bar{t},\bar{x},U^{(\alpha)}(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x})) \\ \geq 0 \ (\leq 0) \qquad (5.2.3)\end{aligned}$$

and

$$(W_t^{(\alpha)}(\bar{t},\bar{x}) - W_t^{(\beta)}(\bar{t},\bar{x})) + (W_x^{(\alpha)}(\bar{t},\bar{x})$$

$$- W_x^{(\beta)}(\bar{t},\bar{x}))f(\bar{t},\bar{x},U^{(\beta)}(\bar{t},\bar{x}),V^{(\beta)}(\bar{t},\bar{x}))$$

$$\leq 0 \ (\geq 0), \tag{5.2.4}$$

that is, that these expressions have opposite signs.

Furthermore

$$(W_t^{(\alpha)}(\bar{t},\bar{x}) - W_t^{(\beta)}(\bar{t},\bar{x})) + (W_x^{(\alpha)}(\bar{t},\bar{x})$$

$$- W_x^{(\beta)}(\bar{t},\bar{x}))f(\bar{t},\bar{x},U^{(\alpha)}(\bar{t},\bar{x}),V^{(\beta)}(\bar{t},\bar{x})) \geq 0 \tag{5.2.5}$$

and

$$(W_t^{(\alpha)}(\bar{t},\bar{x}) - W_t^{(\beta)}(\bar{t},\bar{x})) + (W_x^{(\alpha)}(\bar{t},\bar{x})$$

$$- W_x^{(\beta)}(\bar{t},\bar{x}))f(\bar{t},\bar{x},U^{(\beta)}(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x})) \leq 0 \tag{5.2.6}$$

If $\underline{M}$ is a surface (n-dimensional manifold), if the players choose between $U^{(\alpha)}$ and $U^{(\beta)}$, $V^{(\alpha)}$ and $V^{(\beta)}$, and if (5.2.3) to (5.2.6) are all strict inequalities, one of the players can choose which extremal arc--the one entering $\underline{F}^{(\alpha)}$ or the one entering $\underline{F}^{(\beta)}$--is taken.

If the dispersal manifold $\underline{M} = \bigcap_{\alpha\epsilon\Gamma} \overline{\underline{F}^{(\alpha)}} \bigcap \underline{F}$ for some set of indices Γ, (5.2.3) to (5.2.6) apply to each pair α, β, $\alpha\epsilon\Gamma$ and $\beta\epsilon\Gamma$.

Proof.--Let (μ,ν) be a vector orthogonal to $\underline{M}$ at $(\overline{t},\overline{x})$. Then the statement that there exist distinct arcs

$$\underline{x}_\gamma: \quad x_\gamma(t), U^{(\gamma)}(t,x_\gamma(t)), V^{(\gamma)}(t,x_\gamma(t)) \quad \overline{t} \leq t \leq t_1,$$

$$\gamma = \alpha, \beta,$$

with

$$(t,x_\gamma(t))\epsilon\underline{F}^{(\gamma)} \text{ for } \overline{t} < t \leq t_1 \tag{5.2.7}$$

is equivalent to

$$\mu + \nu f(\overline{t},\overline{x},U^{(\alpha)}(\overline{t},\overline{x}),V^{(\alpha)}(\overline{t},\overline{x})) \geq 0 \quad (\leq 0) \tag{5.2.8}$$

and

$$\mu + \nu f(\overline{t},\overline{x},U^{(\beta)}(\overline{t},\overline{x}),V^{(\beta)}(\overline{t},\overline{x})) \leq 0 \quad (\geq 0) \tag{5.2.9}$$

that is, the two expressions have opposite signs.

By Theorem 3.4 $(W_t^{(\alpha)}(\overline{t},\overline{x}) - W_t^{(\beta)}(\overline{t},\overline{x}),\ W_x^{(\alpha)}(\overline{t},\overline{x}) - W_x^{(\beta)}(\overline{t},\overline{x}))$ is either zero or a nonzero vector orthogonal to $\underline{M}$ at $(\overline{t},\overline{x})$. If it is zero, the relationships (5.2.3), (5.2.4), (5.2.5) and (5.2.6) are satisfied trivially. If it is not zero, then (5.2.8) and (5.2.9) hold with

$$\mu = (W_t^{(\alpha)}(\overline{t},\overline{x}) - W_t^{(\beta)}(\overline{t},\overline{x})) \text{ and}$$

$$\nu = (W_x^{(\alpha)}(\overline{t},\overline{x}) - W_x^{(\beta)}(\overline{t},\overline{x})),$$

yielding (5.2.3) and (5.2.4). As a matter of notational convenience, set

$$f(U,V) = f(\bar{t},\bar{x},U(\bar{t},\bar{x}),V(\bar{t},\bar{x}))$$

$$L(U,V) = L(\bar{t},\bar{x},U(\bar{t},\bar{x}),V(\bar{t},\bar{x}))$$

$$(W_t,W_x) = (W_t(\bar{t},\bar{x}),W_x(\bar{t},\bar{x})).$$

From Theorem 3.2 one obtains

$$L(U^{(\alpha)},V^{(\beta)}) + W_t{}^{(\alpha)} + W_x{}^{(\alpha)}f(U^{(\alpha)},V^{(\beta)}) \geq 0 \qquad (5.2.10)$$

$$L(U^{(\beta)},V^{(\alpha)}) + W_t{}^{(\alpha)} + W_x{}^{(\alpha)}f(U^{(\beta)},V^{(\alpha)}) \leq 0 \qquad (5.2.11)$$

and

$$L(U^{(\alpha)},V^{(\beta)}) + W_t{}^{(\beta)} + W_x{}^{(\beta)}f(U^{(\alpha)},V^{(\beta)}) \leq 0 \qquad (5.2.12)$$

$$L(U^{(\beta)},V^{(\alpha)}) + W_t{}^{(\beta)} + W_x{}^{(\beta)}f(U^{(\beta)},V^{(\alpha)}) \geq 0 \qquad (5.2.13)$$

Subtracting (5.2.12) from (5.2.10) yields (5.2.5) and subtracting (5.2.13) from (5.2.11) yields (5.2.6).

Suppose that (5.2.3) to (5.2.6) are all strict inequalities, so that none of the arce corresponding to $(U^{(\alpha)},V^{(\alpha)})$, $(U^{(\alpha)},V^{(\beta)})$, $(U^{(\beta)},V^{(\alpha)})$ and $(U^{(\beta)},V^{(\beta)})$ with initial condition $(\bar{t},\bar{x})$ are tangent to M at $(\bar{t},\bar{x})$, and suppose for definiteness that (3.4.10) > 0. Then both of the arcs corresponding to $(U^{(\alpha)},V^{(\alpha)})$ and $(U^{(\alpha)},V^{(\beta)})$ enter $\underline{F}^{(\alpha)}$. In the same manner, both arcs corresponding to

$(U^{(\beta)},V^{(\alpha)})$ and $(U^{(\beta)},V^{(\beta)})$ enter $\underline{F}^{(\beta)}$. Clearly player One can choose to enter either $\underline{F}^{(\alpha)}$ or $\underline{F}^{(\beta)}$. If (3.4.10) < 0, then player Two has the choice.

If $\underline{M} = \bigcap_{\alpha\varepsilon\Gamma} \overline{\underline{F}^{(\alpha)}} \bigcap \underline{F}$, the preceding arguments apply to each pair α, β, $\alpha\varepsilon\Gamma$, $\beta\varepsilon\Gamma$.

Corollary

If two or more arcs converge to $(\bar{t},\bar{x})\varepsilon\underline{M}$ instead of diverging from $(\bar{t},\bar{x})$, then (5.2.3) to (5.2.6) hold in this case also. If $\underline{M} = \bigcap_{\alpha\varepsilon\Gamma} \overline{\underline{F}^{(\alpha)}} \bigcap \underline{F}$, these inequalities hold pairwise for α, $\beta\varepsilon\Gamma$.

Proof.--The only change required is that the arcs (3.4.16) must be defined on some interval $t_0 \leq t \leq \bar{t}$ rather than $t \leq t \leq t_1$. The statement that one player can choose the arc which is taken is inapplicable to this case.

Manifolds of the type in the preceding corollary are called universal surfaces (curves, etc.) by Isaacs [20]. The corollary to Theorem 3.4 applies to such manifolds. If $\underline{M} = \bigcap_{\alpha\varepsilon\Gamma} \overline{\underline{F}^{(\alpha)}} \bigcap \underline{F}$, and $\tilde{U}$, $\tilde{V}$ are strategies which equal U, V on $\underline{M}$, the following theorem holds.

Theorem 5.4

Let $\underline{M}$ be a universal manifold as just described. Then at $(\bar{t},\bar{x})\varepsilon\underline{M}$, for each $\alpha\varepsilon\Gamma$

$$L(\bar{t},\bar{x},\tilde{U}(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x})) + W_t^{(\alpha)}(\bar{t},\bar{x})$$

$$+ W_x^{(\alpha)}(\bar{t},\bar{x})f(\bar{t},\bar{x},\tilde{U}(\bar{t},\bar{x}),V^{(\alpha)}(\bar{t},\bar{x})) \leq 0 \quad (5.2.14)$$

$$L(\bar{t},\bar{x},U^{(\alpha)}(\bar{t},\bar{x}),\tilde{V}(\bar{t},\bar{x})) + W_t^{(\alpha)}(\bar{t},\bar{x})$$

$$+ W_x^{(\alpha)}(\bar{t},\bar{x})f(\bar{t},\bar{x},U^{(\alpha)}(\bar{t},\bar{x}),\tilde{V}(\bar{t},\bar{x})) \geq 0 \quad (5.2.15)$$

The inequality (5.2.14) holds if the field is a minimax field and (5.2.15) holds if it is a maximin field.

Proof.--These inequalities are immediate consequences of Theorems 3.2 and 4.1. For example (5.2.15) follows from

$$L(\bar{t},\bar{x},U^{(\alpha)}(\bar{t},\bar{x}),v) + W_t^{(\alpha)}(\bar{t},\bar{x})$$

$$+ W_x^{(\alpha)}(\bar{t},\bar{x})f(\bar{t},\bar{x},U^{(\alpha)}(\bar{t},\bar{x}),v) \geq 0$$

in either Theorem 3.2 or 4.1.

The situations covered in the above theorems do not exhaust the types of behavior exhibited by optimal trajectories in the neighborhood of discontinuities. However, most of them can be treated with a careful application of these theorems. For example, if the optimal trajectories on one side of a surface were parallel to the surface, and on the other side departed from the surface in a direction not tangent to it, Theorem 5.3 could be applied with (5.2.3) (or (5.2.4)) an equality.

VI. CONCLUSION

6.1 Conclusions

Because of the close relationship between differential games and optimal fields with independent controls, optimal fields can profitably be studied in connection with differential games.

In Chapter III optimal fields with a saddle point were investigated. The necessary conditions obtained included Hamilton-Jacobi equations, Euler equations, and a saddle point in the Hamiltonian function corresponding to the saddle point in the optimal field. Also a multiplier rule was derived for constraints on the controls given by systems of equalities and inequalities. None of these results is particularly surprising; they are an extension of the corresponding results in optimal control theory.

In the fourth chapter a maximin field was introduced. Maximin fields are a type of optimal field not previously treated. In a maximin field one of the players has a strategy which maximizes a functional among a collection of functionals minimized by his opponent. While the second player has a minimizing strategy optimal against the first player's maximin strategy, the maximin strategy is not necessarily optimal against this minimizing strategy. It is optimal when the collection of minimal problems is considered. Because of this, the results obtained for maximin fields are not as strong as those for saddle point optimal fields. In particular, a multiplier rule which applies to the constraints on only one player was obtained.

In Chapter V a transversality condition for extremal arcs terminating on the surface $\underline{T}$ was derived. The behavior of extremal arcs in the vicinity of manifolds of discontinuity was used to derive further conditions at these manifolds.

6.2 Further Research

Further research in optimal fields for differential games could profitably concentrate on strengthening the results obtained for maximin fields. In particular, if one can be obtained, a multiplier rule applicable to both players is a result which would be most useful.

The value function for a differential game is not necessarily continuous on the playing space $\underline{F}$. It may be piecewise continuous, in which case one could consider optimal fields defined on each region of continuity. One would like to obtain conditions relating these optimal fields on the manifolds of discontinuity of the value function.

In differential games in general, rather than in the optimal fields associated with them, a direction of research useful in applications would be into games of imperfect information. To be successful, this would most likely require some sort of mixed strategy. Mixed strategies would lead to the consideration of stochastic differential equations, a difficult subject in itself. Some start in this direction has been made by Ho [17].

The extension to general N-person differential games will have to be deferred until the theory of general games is at a more settled state. Perhaps something could be done in this line for optimal rendezvous and collision avoidance problems, which are two-person non-zero-sum games.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. L. D. Berkovitz, "Variational methods in problems of control and programming," Journal of Mathematical Analysis and Applications, V. 3 (1961), pp. 145-169.

2. L. D. Berkovitz, "A variational approach to differential games," in M. Dresher, L. S. Shapley and A. W. Tucker, eds., Advances in Game Theory, Annals of Mathematics Study 52, Princeton University Press, Princeton, N. J., 1964, pp. 127-174.

3. L. D. Berkovitz, "Necessary conditions for optimal strategies in a class of differential games and control problems," SIAM Journal on Control, V. 5 (1967), pp. 1-24.

4. L. D. Berkovitz, "A differential game without pure strategy solutions on an open set," in M. Dresher, L. S. Shapley and A. W. Tucker, eds., Advances in Game Theory, Annals of Mathematics Study 52, Princeton University Press, Princeton, N. J., 1964, pp. 175-194.

5. L. D. Berkovitz and W. H. Fleming, "On differential games with integral payoff," in M. Dresher, A. W. Tucker and P. Wolfe, eds., Contributions to the Theory of Games, III, Annals of Mathematics Study 39, Princeton University Press, Princeton, N. J., 1957.

6. G. A. Bliss, Lectures on the Calculus of Variations, University of Chicago Press, Chicago, Ill., 1946.

7. W. H. Fleming, "A note on differential games of prescribed duration," in M. Dresher, A. W. Tucker, and P. Wolfe, eds., Contributions to the Theory of Games, III, Annals of Mathematics Study 39, Princeton University Press, Princeton, N. J., 1957.

8. W. H. Fleming, "The convergence problem for differential games," Journal of Mathematical Analysis and Applications, V. 3 (1961), pp. 102-116.

9. W. H. Fleming, "The convergence problem for differential games, II," in M. Dresher, L. S. Shapley and A. W. Tucker, eds., Advances in Game Theory, Annals of Mathematics Study 52, Princeton University Press, Princteon, N. J., 1964.

10. M. Yu. Gadzhiev, "Application of game theory to certain problems in optimal control." Avtomatika i telemekhanika, V. 23 Nos. 8 and 9 (Journal translated as Automation and Remote Control.)

11. V. B. Gindes, Optimal processes for two controlled systems," Proc. 3rd Siberian Conference on Mathematics and Mechanics, Tomsk, USSR, 1964, pp. 260-261.

12. V. B. Gindes, "Optimal control in linear systems under conflict," SIAM Journal on Control, V. 5 (1967), pp. 163-169.

13. T. Guinn, "Boundary arcs for a class of differential games," to appear in Journal of Optimization Theory and Applications.

14. J. C. Harsanyi, "A general solution for noncooperative games based on risk-dominance," in M. Dresher, L. S. Shapley and A. W. Tucker, eds., Advances in Game Theory, Annals of Mathematics Study 52, Princeton University Press, Princeton, N. J., 1964.

15. M. R. Hestenes, "A general problem in the calculus of variations with applications to paths of least time," RAND Corporation Research Memorandum RM-100, February, 1949.

16. M. R. Hestenes, Calculus of Variations and Optimal Control Theory, John Wiley and Sons, Inc., New York, 1966.

17. Y. C. Ho, "Optimal terminal maneuver and evasion strategy," SIAM Journal on Control, V. 4 (1966), pp. 421-428.

18. Y. C. Ho, A. E. Bryson, Jr., and S. Baron, "Differential Games and Optimal Pursuit-Evasion Strategies," IEEE Transactions on Automatic Control, V. AC-10 (1966), pp. 385-389.

19. R. Isaacs, Rand Reports RM-1391 (November, 1954), RM-1399 (November, 1954), RM-1411 (December, 1954), RM-1486 (March, 1955).

20. R. Isaacs, Differential Games, John Wiley and Sons, Inc., New York, 1965.

21. D. L. Kelendzheridze, Theory of an optimal pursuit strategy," Soviet Math. Doklady, V. 2 (1961), pp. 654-656, (translation of Doklady Akad. Nauk SSR, V. 138 (1961)).

22. Yu. A. Kochetkov, "Application of Pontryagin's method to investigation of minimax automatic control problems," Tekhnicheskaya Kibernetika, No. 5 (1965) (journal translated as Engineering Cybernetics).

23. N. N. Krasovskii, "The pursuit problem in the case of identical players," PMM, V. 3 (1966), No. 2.

24. N. N. Krasovskii and V. E. Tret'yakov, "The pursuit problem in the case of constraints on the impulses of the control forces," Differentsial'nye uravneniya, V. 2 (1966), No. 5.

25. N. N. Krasovskii, Yu. M. Rupin and V. E. Tret'yakov, "On certain game situations in the theory of controlled systems," Tekhnicheskaya kibernetika, No. 4 (1964) (translated in Engineering Cybernetics V. 4 (1964), pp. 1-11).

26. R. D. Luce and H. Raiffa, Games and Decisions, John Wiley and Sons, Inc., New York, 1957.

27. P. A. Meschler, "On a goal-keeping differential game," IEEE Transactions on Automatic Control, V. AC-12 (1967), pp. 15-21.

28. J. Mycielski, "Continuous games of perfect information," in M. Dresher, L. S. Shapley and A. W. Tucker, eds., Advances in Game Theory, Annals of Mathematics Study 52, Princeton University Press, Princeton, N. J., 1964.

29. J. F. Nash, "Non-cooperative games," Annals of Mathematics, V. 54 (1951), pp. 286-295.

30. J. von Neumann, "Zur Theorie der Gesellschaftsspiele." Mathematische Annalen V. 100 (1928), pp. 295-320.

31. J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, Princeton, N. J., 1944.

32. L. A. Petrosyan, "On a family of differential survival games in R_n," Dokl. Akad. Nauk SSSR, V. 161 (1965), No. 1.

33. L. A. Petrosyan, "Differential survival games with several participants," Dokl. Akad. Nauk SSSR, V. 161 (1965), No. 2.

34. L. A. Petrosyan, "Reduction of one pursuit game to solution of the Cauchy problem for a first-order partial differential equation," Dokl. Akad, Nauk Arm. SSR, V. 40 (1965), No. 4.

35. L. A. Petrosyan, One Class of Pursuit Games, Candidate's Dissertation, LGU, 1965.

36. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience, New York, 1962.

37. L. S. Pontryagin, "On certain differential games," Dokl. Akad. Nauk SSR, V. 156 (1964), No. 4, pp. 738-741.

38. L. S. Pontryagin, "On some differential games," SIAM Journal on Control, V. 3 (1965), pp. 49-52.

39. R. K. Ragade and I. G. Sarma, "A game theoretical approach to optimal control in the presence of uncertainty," IEEE Transactions on Automatic Control V. AC-12 (1967), pp. 395-401.

40. C. Ryll-Nardzewski, "A theory of pursuit and evasion," in M. Dresher, L. S. Shapley and A. W. Tucker, eds., Advances in Game Theory, Annals of Mathematics Study 52, Princeton University Press, Princeton, N. J., 1964.

41. I. G. Sarma and R. K. Ragade, "Some considerations in formulating optimal control problems as differential games," International Journal of Control V. 4 (1966), pp. 265-279.

42. H. Scarf, "On differential games with survival payoff," Contributions to the Theory of Games, III, Annals of Mathematics Study 39, Princeton University Press, Princeton, N. J., 1957.

43. L. S. Shapley, "A value for N-person games," in H. W. Kuhn and A. W. Tucker, eds., Contributions to the Theory of Games, II, Annals of Mathematics Study 28, Princeton University Press, Princeton, N. J., 1953.

44. E. N. Simakova, "Differential Games," Avtomatika i Telemekhanika, V. 27 (1966), No. 11, pp. 161-178, (translated in Automation and Remote Control, V. 27, pp. 1980-1998).

45. D. D. Sworder, Optimal Adaptive Control Systems, Academic Press, Inc., New York, 1966.

46. P. P. Variaya, "On the existence of solutions to a differential game," SIAM Journal on Control, V. 5 (1967), pp. 153-162.

47. A. Zięba, "An example in pursuit theory," Studia Math. V. 22 (1962), pp. 1-6.

48. A. Zięba, "Fundamental equations of the theory of pursuit," Trans. of the Second Prague Conference in Math. Statistics, 1959.