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A CHARACTERIZATION OF CERTAIN CLOSED 3-MANIFOLDS

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ABSTRACT

A CHARACTERIZATION OF CERTAIN CLOSED 3-MANIFOLDS

by Gerhard Walter Knutson

Let M be a closed connected combinatorial 3-manifold. A compact subcomplex A of M is a residual set of M if M is the disjoint union $A \cup U$ of an open 3-cell U dense in M and a non-separating continuum A of dimension less than 3. The singular set of A, $S(A)$, is the set of points of A that do not have an open 1- or 2-dimensional euclidean neighborhood in A.

In this thesis we examine the relationship between A and M. In particular we show that if A does not contain a wild arc then we may pick A so that $S(A)$ is a point. Then we prove the following theorem: M has a residual set that contains no wild arc if and only if M is the connected sum of closed 3-manifolds each of which is topologically the 3-sphere, real projective 3-space, $S^1 \times S^2$, or the twisted S^2 bundle over S^1.

We also show that A may be picked so that $A - S(A)$ is the disjoint union of open arcs and the interiors of compact 2-manifolds with connected boundaries. Under this assumption, if $S(A)$ is a simple closed curve, M is the connected sum of closed 3-manifolds each of which is topologically $S^1 \times S^2$, the 3-sphere, real projective 3-space, the twisted S^2 bundle over S^1, or a lens space.

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To My

Mother and Father

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CHAPTER I

INTRODUCTION

In 1962 Doyle and Hocking established a decomposition of a closed n-manifold into an open n-cell and a non-separating continuum of dimension less than n. In this thesis we start with the continuum and under certain conditions reconstruct the manifold. Since our concentration is on connected 3-manifolds, we assume that all our manifolds are combinatorial and connected. Furthermore, all subsets are simplicial and all maps are piecewise linear.

In this chapter we establish some elementary relations between the manifold and its decomposition.

1. Homology and Homotopy of Residual Sets

Definition 1.1.1: Let M be a compact n-manifold. A compact subcomplex A of M is a residual set of M if M is the disjoint union $M = A \cup U$ of an open n-cell U dense in M and a non-separating continuum A of dimension less than n. $A \cup U$ is called a decomposition of M.

We remark that if M has non-empty boundary, the boundary of M is contained in A. Therefore we must not confuse the residual set with a spine [17]. However, if A is a residual set of M then A is a spine of M less an open n-ball of the interior of M.

Note that we will assume that a residual set does not collapse onto any proper subset of itself.

In [5] Doyle and Hocking prove that every compact n-manifold has a decomposition. The Brown-Casler Theorem [2] asserts the existence of a continuous function f from the closed n-ball B^n onto M such that $f|\text{Int } B^n$ is a homeomorphism, $f^{-1}f(\text{Bd } B^n) = \text{Bd } B^n$, and $\dim f(\text{Bd } B^n) < n$. Thus if $M = A \cup U$ is a decomposition of M, we will always assume we are given the map $f\colon (B^n, \text{Bd } B^n) \longrightarrow (M, A)$.

It will be useful to establish the relationship between the homology and homotopy groups of A and M. Since M is the adjunction space obtained by attaching B^n to A by means of f, the pair (M,A) is a relative n-cell. Hence, $H_q(M,A) = \hat{H}_{q-1}(S^{n-1})$ for all q and $\pi_q(M,A) = 0$ for $0 < q < n$ [10]. Note that we will use equality to mean group isomorphism or space homeomorphism whenever no confusion is likely.

If $h_q\colon H_q(M,A) \longrightarrow \hat{H}_{q-1}(S^{n-1})$ is the above isomorphism and $H_q(f) : H_q(S^{n-1}) \longrightarrow H_q(A)$ is the homomorphism induced by f, we obtain the commutative diagram:

$$\begin{tikzcd}
H_q(M) \arrow[r] & H_q(M,A) \arrow[r] \arrow[d, "h_q"'] & H_{q-1}(A) \arrow[r] & H_q(M) \\
& \hat{H}_{q-1}(S^{n-1}) \arrow[ur, "H_{q-1}(f)"'] & &
\end{tikzcd}$$

where the unnamed maps are the maps of the exact homology sequence of the pair (M,A).

Using these definitions we state two well known theorems and an immediate corollary.

Theorem 1.1.2[8]: The following hold;

i) $H_q(M) = H_q(A)$ for $q \neq n, n-1$

ii) $H_{n-1}(M) = H_{n-1}(A)/\text{Im } H_{n-1}(f)$

and

iii) $0 \longrightarrow H_n(A) \longrightarrow H_n(M) \longrightarrow \ker H_{n-1}(f) \longrightarrow 0$

is exact.

Corollary 1.1.3: If $\dim A < n-1$, M is orientable.

Proof: If $\dim A < n-1$, $H_n(A) = 0 = H_{n-1}(A)$ and so $\text{Im } H_{n-1}(f) = 0$. Thus $\ker H_{n-1}(f) = Z$ and so $H_n(M) = Z$. Hence M is orientable.

Theorem 1.1.4: $\pi_q(M) = \pi_q(A)$ for $0 \leq q < n-1$.

Theorem 1.1.5: Let M be a closed n-manifold. Let A be a residual set of M with $\dim A < n/2$. If n is odd suppose that $H_{(n-1)/2}(A)$ is torsion free. Then M is a homology n-sphere.

Proof: If $n = 1$ or 2, A is a point and so M is a sphere. If $n = 3$, A is a 1-complex and hence A is the homotopy type of an r-leafed rose. By Corollary 1.1.3 M is orientable. Thus $\ker H_2(f) = Z$ and $\text{Im } H_2(f) = 0$. By Theorem 1.1.2, $H_2(M) = H_2(A)$. By Poincaré duality $H_1(M) = H_2(A) = 0$. Thus $H_1(A) = 0 = H_1(M)$ and so A is contractable. Hence M is a 3-sphere.

Suppose that $n > 3$. Then M is orientable and $H_{n-1}(A) = H_{n-1}(M)$. By Theorem 1.1.2, we obtain:

$$H_q(M) = \begin{cases} Z & q = 0 \text{ or } q = n \\ 0 & n/2 \leq q \leq n-1 \\ H_q(A) & 1 \leq q < n/2. \end{cases}$$

By Poincaré duality $H^q(M) = H_{n-q}(M)$. Hence

$$H^q(M) = \begin{cases} Z & q = 0 \text{ and } q = n \\ 0 & 1 \leq q < n/2 \\ H_{n-q}(A) & n/2 \leq q \leq n-1. \end{cases}$$

By Theorem 5.5.3 of [14], $H_q(M) = \mathrm{Hom}(H^q(M), Z) \oplus \mathrm{Tor}(H^{q+1}(M))$. Hence $H_q(M) = 0$ if $q \neq 0$ or n and $H_0(M) = 0 = H_n(M)$.

Corollary to the proof: If M is a closed 3-manifold with a residual set of dimension 1, M is a 3-sphere.

2. Local Connectivity of A Relative to M

Let X be a separable metric space and A a subset of X. Let x be an element of X. We say A is locally p-connected in the sense of homotopy at x (p-LC at x) if for every $\zeta > 0$ there is a $\delta > 0$ such that each map $f : S^p \longrightarrow S_x(\delta) \cap A$ is null homotopic in $S_x(\zeta) \cap A$ [7], where $S_x(\zeta)$ is an ζ-ball centered at x. A is locally p-connected in the sense of homotopy in relation to X (A is p-LC rel X) if A is p-LC at x for each x in X.

Lemma 1.2.1: Let M be a closed n-manifold with a decomposition $M = A \cup U$. If U is p-LC rel M for $0 \leq p \leq k$, then $\dim A \leq n - (k+2)$.

Proof: Let B be a simplex of A with maximal dimension m. Let x be an interior point of B. Then x has a neighborhood N in M with $(N, N \cap U) = (R^n, R^n - R^m)$. Thus $N \cap U$ contains an $(n - (m + 1))$-sphere that does not bound in $N \cap U$. Hence U is not $(n - (m + 1))$-LC rel M. Since $k < n - (m + 1)$ by definition of p-LC, and $m = \dim A$, it follows that $\dim A \leq n - (k + 2)$.

Corollary 1.2.2: If U is 0-LC rel M, then M is orientable.

Proof: From Lemma 1.2.1, $\dim A \leq n-2$ and so, by Corollary 1.1.3, M is orientable.

Corollary 1.2.3: If M is a closed 2- or 3-manifold and U 0-LC rel M, then M is a sphere.

Corollary 1.2.4: Let M be a closed 4-manifold with U 0-LC rel M. If M is not a 4-sphere $\dim A = 2$.

To see that Corollary 1.2.4 cannot be strengthened, consider $S^2 \times S^2$. This manifold has a residual set that is topologically the one-point union of two 2-spheres.

Corollary 1.2.5: Let M be a closed n-manifold and suppose that U is p-LC rel M for $0 \leq p \leq n-3$. Then M is an n-sphere.

A concept similar to p-LC is obtained using singular chains and cycles. Using the corresponding definitions we obtain similar results.

CHAPTER II

TOROIDAL MANIFOLDS

In this chapter we investigate the relationship between the residual set of a connected sum and the residual sets of the summands and between the residual set of a disk sum and the residual sets of the summands. Finally we will investigate the residual set of a toroidal manifold.

1. Residual Sets of Connected Sums

Definition 2.1.1: Let M and M' be two closed combinatorial n-manifolds. The connected sum $M \# M'$ is obtained by removing the interior of a closed n-ball from each manifold and matching the resulting boundaries by means of a piecewise linear homeomorphism. If the manifolds are orientable this sum is not always well defined unless the homeomorphism is orientation reversing. When we write $M \# M'$ we will imply that the sum is well defined.

In latter chapters we will use the connected sum of 3-manifolds. In the construction it will follow that the homeomorphism will be orientation reversing whenever necessary. We note that if M has an orientation reversing self homeomorphism, M is homogeneous in the sense of Brown and Gluck, and so $M \# M'$ is well defined. We remark that S^3, $S^1 \times S^2$, and RP^3 (real projective 3-space) have orientation reversing self homeomorphisms.

<u>Theorem 2.1.2</u>: If A and A' are residual sets of the closed n-manifolds M and M', $M \# M'$ has a residual set homeomorphic to the one-point union of A and A' (written $A \vee A'$).

<u>Proof</u>: If we pick the n-balls of the connected sum to be n-simplexes of some triangulation of M and M' that meet the respective residual sets at a point, the theorem follows.

2. <u>Residual Sets of Disk Sums</u>

<u>Definition 2.2.1</u>: Let M and M' be connected compact n-manifolds with connected non-empty boundaries. The <u>disk sum</u> $M \Delta M'$ is obtained by pasting an $(n-1)$-ball of Bd M onto an $(n-1)$-ball of Bd M'.

<u>Theorem 2.2.2</u>: If A and A' are residual sets of the compact n-manifolds M and M', where Bd M and Bd M' are connected and non-empty, $M \Delta M'$ has a residual set homeomorphic to the space obtained by removing the interior of an $(n-1)$-ball from both A and A' and sewing the resulting sets together along the boundaries of the removed balls.

<u>Proof</u>: If we pick the $(n-1)$-balls of the disk sum to be $(n-1)$-simplexes of Bd M and Bd M', under some triangulation of M and M', the theorem follows.

3. Toroidal 3-Manifolds

It is well known that each closed connected orientable 3-manifold M may be obtained by sewing two solid tori of the same genus together by a boundary homeomorphism. We investigate M when we know how M is obtained from two tori. In particular we investigate toroidal manifolds.

Theorem 2.3.1: Let M be a closed orientable 3-manifold. Suppose that T_1 and T_2 are two solid tori of genus n, and h is a homeomorphism of $\text{Bd}\ T_1$ onto $\text{Bd}\ T_2$ with $M = T_1 \cup_h T_2$. Let $T_i = A_i \cup U_i$ be a nice decomposition of T_i. Then there is a 2-cell C in A_1 with $\text{Int}\ C$ open in A_1 and $h(\text{Int}\ C)$ open in A_2 such that M has a decomposition $M = A \cup U$, where $U = (U_1 \cup \text{Int}\ C) \cup_h (U_2 \cup h(\text{Int}\ C))$ and $A = (A_1 - \text{Int}\ C) \cup_h (A_2 - h(\text{Int}\ C))$.

Proof: A nice decomposition of a solid torus T of genus n, $T = A \cup U$, is obtained by taking A to be the boundary of T plus n disjoint 2-cells $C_1, \cdots, C_n$ where $C_i \cap \text{Bd}\ T = \text{Bd}\ C_i$ and $\text{Bd}\ T - \bigcup_{i=1}^{n} \text{Bd}\ C_i$ is a sphere with 2n holes. For example in the torus of genus one, $T = S^1 \times B^2$, A would be $(S^1 \times \text{Bd}\ B^2) \cup (p \times B^2)$ where p is a point of S^1.

We will consider T_1 and T_2 as submanifolds of M with $M = T_1 \cup T_2$ and $T_1 \cap T_2 = \text{Bd}\ T_1 = \text{Bd}\ T_2$. Then $M = A_1 \cup A_2 \cup U_1 \cup U_2$. Since $A_1 \cup A_2$ is 2-dimensional, there is a 2-cell C in $T_1 \cap T_2$, such that C is the carrier of a 2-simplex of some triangulation of M. Then M has a decomposition of the desired form, namely $M = ((A_1 - \mathrm{Int}\ C) \cup (A_2 - \mathrm{Int}\ C)) \cup (U_1 \cup \mathrm{Int}\ C \cup U_2)$.

Corollary 2.3.2: Any closed orientable 3-manifold has a residual set which is an orientable surface of genus n, less an open 2-cell, to which 2n 2-disks are attached by means of homeomorphisms of their 1-sphere boundaries.

If M is a 3-manifold obtained by attaching two solid tori of genus n by a boundary homeomorphism, we will call M an n-tuple toroidal manifold or an n-TM. Bing [1] has shown that any 1-TM is either the 3-sphere, $S^1 \times S^2$, or a lens space. In Chapter IV we will need to know the residual set of a 1-TM, so we turn our attention to that goal.

Each 1-TM is obtained by attaching two solid tori T_1 and T_2 by an appropriate boundary homeomorphism. We now describe such a homeomorphism.

Let M_i and L_i be meridianal and longitudinal simple closed curves on $\mathrm{Bd}\ T_i$, for $i = 1$ or 2. Suppose that n and m are relatively prime positive integers. Let $a_1, \cdots, a_n$ be n points on M_2, cyclicly ordered by their subscripts. Let $J(n,m)$ be a simple closed curve on $\mathrm{Bd}\ T_2$ that meets M_2 at the n points a_i, with the a_i cyclicly ordered on $J(n,m)$ as $a_1, a_{m+1}, \ldots, a_{(n-1)m+1}$. Let h be a homeomorphism of $\mathrm{Bd}\ T_1$ onto $\mathrm{Bd}\ T_2$ such that $h(M_1) = J(n,m)$. Define $T(n,m)$ to be the adjunction space $T_1 \cup_h T_2$. Set $J(1,0) = M_2$ and $J(0,1) = L_2$. Then $S^1 \times S^2 = T(1,0)$ and $S^3 = T(0,1)$. Since isotopic maps yield homeomorphic 1-TM's and since each isotopy class of homeomorphisms of Bd T_1 onto Bd T_2 has a representative that maps M_1 onto $J(n,m)$, each 1-TM is a $T(n,m)$ manifold.

To obtain a decomposition for $T(n,m)$, we consider T_1 and T_2 as submanifolds of $T(n,m)$ with $T_1 \cup T_2 = T(n,m)$ and $T_1 \cap T_2 = \text{Bd } T_1 = \text{Bd } T_2$. In T_1, $J(n,m)$ is a meridianal simple closed curve and so bounds a disk D in T_1 with $\text{Int } D \subset \text{Int } T_1$. Then T_1 has a residual set $\text{Bd } T_1 \cup D$. Considering T_2 as $B^2 \times S^1$, where B^2 is the closed unit 2-ball, let C be the simple closed curve in T_2 corresponding to $(0) \times S^1$. Let B be the singular annulus obtained by pushing $J(n,m)$ onto C by a radial projection; that is, B corresponds to the image of the function $F : J(n,m) \times I \longrightarrow B^2 \times S^1$ defined by $F(((x,y),s),t) = (((1-t)x,(1-t)y),s)$, where (x,y) is a point of $J(n,m)$ and s lies on S^1. Since $T(n,m)-(B \cup D)$ is an open 3-cell $B \cup D$ is a residual set of $T(n,m)$. Notice that $B \cup D$ is topologically the quotient space of an n-gon obtained by identifying each edge with a simple closed curve in an orientation preserving manner.

In [3] Casler defines a standard spine of a 3-manifold with non-empty boundary. Following this definition we will define a standard residual set of a closed 3-manifold.

Let K be a 2-complex. A vertex v of K is of type I if v has a 2-cell neighborhood, of type II if v is not of type I and has a 3-book neighborhood, and of type III if v is not of type I or II and has a neighborhood homeomorphic to the cone over a set consisting of a circle together with three of its radii. K is a <u>standard 2-complex</u> if the following hold:

i) each vertex of K is of type I, II or III.

ii) K less its singular 1-skeleton, K_1, is a countable number of disjoint open disks, and

iii) K_1 less the singular 0-skeleton of K_1 is the sum of a countable number of pairwise disjoint open arcs.

If A is a standard 2-complex and if A is a residual set of a closed 3-manifold M, then A is defined to be a <u>standard residual set</u>. Likewise A is a <u>standard spine</u> if A is a standard 2-complex and A is also a spine.

The main result of [3] is:

<u>Theorem 2.3.3</u>: If K is a standard spine of a compact 3-manifold M with non-empty boundary and K' is a standard spine of a compact 3-manifold M' with non-empty boundary, and if K and K' are homeomorphic, then M is homeomorphic to M'.

Recall that we are in the piecewise linear category so that the above homeomorphisms are piecewise linear.

<u>Corollary 2.3.4</u>: If two closed 3-manifolds have homeomorphic standard residual sets, then the manifolds are homeomorphic.

<u>Proof</u>: We need only note that a standard residual set of a closed 3-manifold is a standard spine of the manifold less an open 3-simplex and then apply Theorem 2.3.3.

We would now like to find a standard residual set for the manifold $T(n,m)$. If $n = 2$, $B \cup D$ is a copy of the real projective plane. By a result of Hocking and Kwun [9], $T(2,m)$ is real projective 3-space. Since $T(3,1)$ and $T(3,2)$ are homeomorphic we need only consider $T(3,1)$. For $T(3,1)$, $B \cup D$ is a 3-book with its ends identified after a twist of 120 degrees. Let v be a vertex of $B \cup D$ of type II. Let N be a 3-book neighborhood of v in $B \cup D$, with pages P_1, P_2 and P_3. Swell up v to a 3-cell C that meets $B \cup D$ in a disk E contained in $P_1 \cup P_2$ with E contained in the boundary of C. By collapsing C onto a copy of Bing's house with two rooms leaving E fixed, we obtain a standard residual set for $T(3,1)$. If $n \geq 4$, we do not consider the residual set $B \cup D$, but rather start all over. As before $T(n,m) = T_1 \cup_h T_2$, and D is the same nice disk in T_2. We now decompose T_1 into two open sets U_1 and U_2, each topologically the upper half 3-space, and a continuum H. Then $U_1 \cup U_2 \cup \mathrm{Int}(T_2 - D)$ is an open 3-cell and if H is sufficiently nice $H \cup D$ is a standard residual set for $T(n,m)$. To construct H, consider T_1 as being obtained as the identification space of $B^2 \times I$, under the action of a homeomorphism f of $B^2 \times (0)$ onto $B^2 \times (1)$, defined by $f(te^{i\theta},0) = (te^{i\theta+(2\pi/m)},1)$. Then T_1 is homeomorphic to $(B^2 \times I)/R$, where R is the equivalence relation $x \sim f(x)$. Let p be the composite map $p : B^2 \times I \longrightarrow (B^2 \times I)/R \longrightarrow T_1$, where the first map is the quotient map and the second is the above homeomorphism.

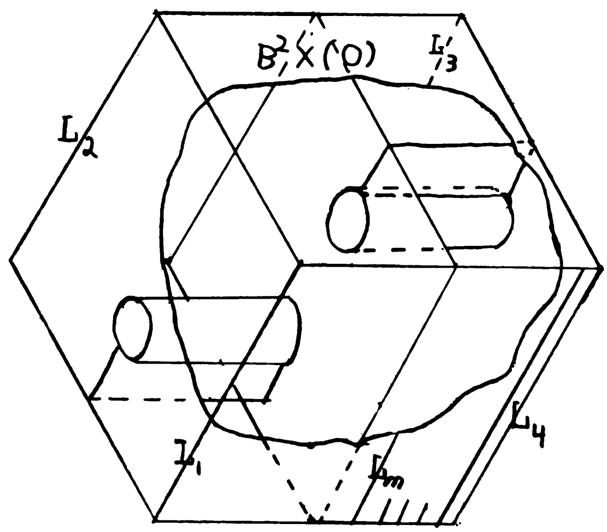

Figure 2.1

Let $L_1, \cdots, L_n$ be n arcs in $Bd\,(B^2 \times I)$ such that $p(\bigcup_1^n L_i) = J(n,m)$. Now consider $B^2 \times I$ as a cube with $B^2 \times (0)$ as top and $B^2 \times (1)$ as bottom. Furthermore consider L_1, L_2, L_3 and L_4 as the four edges on the sides of the cube. The remaining arcs, $L_5, \cdots, L_n$, are on the side that has L_1 and L_4 as edges. Now collapse $B^2 \times I$ onto a copy of Bing's house with two rooms as in Figure 2.1. Setting H equal to the image of the house under the map p, it follows that $H \cup D$ is a residual set of $T(n,m)$.

However, $H \cup D$ was so constructed that, if a little care is taken as to how we collapse onto the house, $H \cup D$ will be a standard residual set for $T(n,m)$.

CHAPTER III

A CHARACTERIZATION OF CLOSED 3-MANIFOLDS WITH RESIDUAL SETS CONTAINING NO WILD ARCS

In this chapter we establish that a closed 3-manifold has a residual set containing no wild arc if and only if it is the connected sum of closed 3-manifolds each of which is homeomorphic to a 3-sphere, real projective 3-space, $S^1 \times S^2$, or the twisted S^2 bundle over S^1. Finally we establish a similar characterization for compact 3-manifolds with boundary.

1. A Is a One-Point Union

Definition 3.1.1: An arc B in a complex X is wild if there does not exist a homeomorphism of X onto itself carrying B onto a polyhedral arc of X.

Since a trefoil knot may be embedded in a 3-book, a 3-book contains a wild arc [14]. Thus if A is a residual set of a closed 3-manifold that does not contain a wild arc, then A does not contain a 3-book. Recall that A does not collapse onto any subcomplex of itself. If v is a vertex of A and $N(v,A)$ the second derived neighborhood of v in A [16], we may classify the vertices of A into three disjoint types:

i) $N(v,A)$ is an arc,

ii) $N(v,A)$ is a disk, and

iii) $N(v,A)$ is the one-point union of arcs and disks.

<u>Lemma 3.1.2</u>: Let A be a residual set for the closed 3-manifold M which contains no wild arcs. Then there is another residual set A' for M that is the one-point union of closed 2-manifolds and 1-spheres.

<u>Proof</u>: If the dimension of A is less than two, M is a 3-sphere and so A is a point. Suppose that A has dimension two and that a is a vertex of A. Let $St(x,X)$ and $Lk(x,X)$ be the star and link of x in the second derived subdivision of a complex triangulating X [16]. Then $St(a,M)$ is a 3-ball with $St(a,A)$ contained in $St(a,M)$ as the join of a with $Lk(a,A)$.

Since $Lk(a,A) = Lk(a,M) \cap A$ is the disjoint union of $p \geq 0$ 1-spheres and $q \geq 0$ points, we may associate with a the pair (p,q) and a will be called a (p,q)-point. We will define a series of moves that change A into A', where A' contains only $(1,0)$-points, $(0,2)$-points and one (m,n)-point.

<u>MOVE A</u>: Let a be a (p,q)-point of A with $pq > 2$. Let x be an isolated point of $Lk(a,A)$. In $Lk(a,M)$ there is an arc C with $Bd\ C = A \cap C = x \cup y$, where y is a point of a 1-sphere of $Lk(a,A)$. There is a 2-cell B in $St(a,M)$ with $A \cap B = A \cap Bd\ B = aox \cup aoy$ and $Bd\ B = aox \cup aoy \cup C$. Here "o" denotes the join operator. An A-move expands A to $A \cup B$ and collapses from aox across B onto $Cl(A-aox) \cup C$.

<u>MOVE B</u>: Let a be a $(p,0)$-point of A with $p > 1$. There is a 1-sphere S of $Lk(a,A)$ that bounds a 2-disk D in $(Lk(a,M) - Lk(a,A)) \cup S$. Thus there is a 3-cell C in $St(a,M)$ with $A \cap C = A \cap Bd\ C = aoS$ and $Bd\ C = aoS \cup D$. A B-move expands A to $A \cup C$ and collapses onto $(A-aoS) \cup aoy \cup D$, where y is a point of S.

<u>MOVE C</u>: Let a be a $(0,q)$-point of A with $q > 2$. Suppose that x is a point of $Lk(a,A)$ and aox may be extended to an arc B in A with $Bd\ B = a \cup y$, where y is a (p,q)-point of A with $pq \neq 0$, such that $Int\ B$ contains only $(0,2)$-points. Let $z \neq x$ be a point of $Lk(a,A)$. There is a 2-cell C in M with $A \cap C = A \cap Bd\ C = B \cup aoz$. Let $D = Cl(Bd\ C - (aoz \cup B))$. A C-move expands A to $A \cup C$ and collapses from aoz across C onto $(A-aoz) \cup a \cup D$.

<u>MOVE D</u>: Let a be a $(1,1)$-point of A with x the isolated point of $Lk(a,A)$. Suppose that y is a (p,q)-point of A in the same 2-chainable component of A as a. There is an arc C in A, with $Bd\ C = a \cup y$ and $Int\ C$ containing only $(1,0)$-points, and a 2-cell B in M with $A \cap B = A \cap Bd\ B = C \cup aox$. Let $D = Cl(Bd\ B - (C \cup aox))$. A D-move expands A to $A \cup B$ and collapses from aox across B onto $(A-aox) \cup a \cup D$.

<u>MOVE E</u>: Let a be a $(1,1)$-point of A with x the point and S the 1-sphere of $Lk(a,A)$. Suppose that aox may be extended to an arc B in A with $Bd\ B = a \cup y$,

where y is a (p,q)-point of A with $pq \neq 0$, and Int B containing only $(0,2)$-points. There is a 3-cell C of M with $A \cap C = B \cup aoS$ and $A \cap \mathrm{Bd}\, C = y \cup aoS$, such that C collapses onto $B \cup aoS$. Let $D = \mathrm{Cl}(\mathrm{Bd}\, C - aoS)$. An E-move expands A to $A \cup C$ and collapses from aoS across C onto $(A - (B \cup aoS)) \cup D$.

We observe that each move transforms A into a residual set of M. By a finite series of A-moves we may assume that each vertex of A is either a $(1,1)$-point, a $(0,q)$-point or a $(p,0)$-point. By A- and B-moves we may assume that each $(p,0)$-point of A has $p = 1$. By A- and C-moves each vertex of A is a $(1,1)$-, $(1,0)$- or a $(0,2)$-point. By D- and E-moves we obtain the desired form.

<u>Lemma 3.1.3</u>: If A is as in the conclusion of Lemma 3.1.2, where A is the one-point union of n 1-spheres and m closed 2-manifolds, $n \leq m$.

<u>Proof</u>: Since Theorem 1.1.2 holds for arbitrary coefficients, it follows that $H_2(M;Z_2) = H_2(A;Z_2)$ and $H_1(M;Z_2) = H_1(A;Z_2)$. By Poincaré duality and the universal coefficient theorem for cohomology, $H_1(A;Z_2) = H_2(A;Z_2)$. Since $H_2(A;Z_2) = \overset{m}{\underset{1}{\oplus}} Z_2$, and $\overset{n}{\underset{1}{\oplus}} Z_2 \subset H_1(A;Z_2)$, the lemma follows.

<u>Remark 3.1.4</u>: Given $m \geq n \geq 0$ there is a closed 3-manifold with a residual set that is the one-point union of n 1-spheres and m closed 2-manifolds. The connected sum of n copies of $S^1 \times S^2$ and $m-n$ copies of RP^3 has the desired residual set.

Remark 3.1.5: By swelling up a principal simplex of a residual set and collapsing onto a copy of Bing's house with two rooms we see that every 3-manifold has a residual set that contains an arc that is wild.

If A is as in the conclusion of Lemma 3.1.2, and M is not a 3-sphere, then $m \neq 0$ and so M has a non-trivial second homology group with Z_2 coefficients. Thus we obtain:

Corollary 3.1.6: A residual set of a counter example to the 3-dimensional Poincaré conjecture must contain a wild arc.

2. The 2-Manifolds of A

If A is a residual set that is the one-point union of n 1-spheres S_i and m 2-manifolds P_i we will call A an (n,m)-residual set. Again $N(X,M)$ will be the second derived neighborhood of X in M. We will set $N(X) = N(X,M)$ if the manifold M is understood.

Remark 3.2.1: If S is a 1-sphere embedded in a closed 3-manifold M, $N(S,M)$ is either a solid Klein bottle or a solid torus. Notice that if M is non-orientable $N(S^1)$ may be either a solid torus or a solid Klein bottle. For example in J, the twisted S^2 bundle over S^1, both types of neighborhoods are easily found.

<u>Lemma 3.2.2</u>: A regular neighborhood of a compact 2-manifold embedded in the interior of an orientable 3-manifold is topologically independent of the 3-manifold.

<u>Proof</u>: Suppose that P and Q are isomorphic, (that is, P and Q are homeomorphic under a simplicial map), compact 2-manifolds simplicially embedded in the interior of two orientable 3-manifolds M and N respectively. Triangulate M and N so that under the induced triangulation P is isomorphic to Q.

Now $N(P,M)$ is a solid torus H of genus n plus some 3-cells attached to H along annuli. Also $N(Q,N)$ is a solid torus K of genus m plus some 3-cells attached to K along annuli. Since P and Q are isomorphic, we may take H and K as the second derived neighborhood of the respective 1-skeletons so that $n = m$. Moreover the isomorphism of P onto Q extends to an isomorphism of $P \cup H$ onto $Q \cup K$. By collapsing $P \cup H$ and $Q \cup K$ carefully, we obtain P' and Q', standard spines of $N(P,M)$ and $N(Q,N)$ respectively. Furthermore, P' and Q' will be isomorphic. By Theorem 2.3.3, $N(P,M)$ is isomorphic to $N(Q,N)$. The above collapse is obtained by collapsing each neighborhood of a vertex to a copy of Bing's house with two rooms, so that the house meets the tubes of H and K in disks whose interiors are open in the house. Then collapse the tubes by pushing the disks into the middle of the tubes.

Since an orientable 2-manifold embeds in R^3, we obtain:

<u>Corollary 3.2.3</u>: A regular neighborhood of a compact orientable 2-manifold embedded in the interior of an orientable 3-manifold is a product neighborhood.

<u>Lemma 3.2.4</u>: Let A be an (n,m)-residual set of a closed orientable 3-manifold M. Then each P_i is either a 2-sphere or a real projective plane.

<u>Proof</u>: Consider $N(A) = (\bigcup_{i=1}^{n} N(S_i)) \cup (\bigcup_{i=1}^{m} N(P_i))$. Since $N(A)$ is topologically M less an open 3-cell, $\mathrm{Bd}\ N(A)$ is a 2-sphere. If a is the join point of A, $N(a,M) = B$ is a 3-ball with $N(A) - B$ the disjoint union of the n sets $N(S_i) - B$ and the m sets $N(P_i) - B$. By definition of the second derived neighborhood, it is clear that $\mathrm{Bd}\ (N(P_i) - B)$ is $\mathrm{Bd}\ N(P_i)$ less two disks. Since $\mathrm{Bd}\ N(P_i)$ is a 2-manifold and $\mathrm{Bd}\ (N(P_i) - B)$ is contained in $\mathrm{Bd}\ N(A)$, a 2-sphere, $\mathrm{Bd}\ N(P_i)$ is either one or two 2-spheres. Suppose that P_1 is orientable. Then $N(P_1)$ is topologically $P_1 \times I$ and so $\mathrm{Bd}\ N(P_1)$ is two disjoint copies of P_1. Hence if P_1 is orientable, P_1 is a 2-sphere. Suppose that P_1 is non-orientable. If P_1 has an orientable handle, $\mathrm{Bd}\ N(P_1)$ must contain a torus with a hole. Since a torus with a hole does not embed in a 2-sphere, P_1 does not have an orientable handle. Since the Klein bottle embeds in $S^1 \times S^2$ with a regular neighborhood having a torus boundary,

Lemma 3.2.2 implies that P_1 is not a Klein bottle. Hence if P_1 is non-orientable, P_1 is a real projective plane.

<u>Lemma 3.2.5</u>: Let A be an (n,m)-residual set for a closed 3-manifold M. Then each P_i is either a 2-sphere or a real projective plane.

<u>Proof</u>: If $N(P_1)$ is orientable, Lemma 3.2.4 produces the desired result. Suppose that $N(P_1) = N$ is non-orientable. As in Lemma 3.2.4, Bd N is one or two 2-spheres. If Bd N is one 2-sphere, let E be a 3-ball attached to N by a boundary homeomorphism. Since N collapses onto P_1 and $N \cup E$ is non-orientable, we obtain the Mayer Vietoris sequence:

$$\longrightarrow H_q(N \cap E) \longrightarrow H_q(N) \oplus H_q(E) \longrightarrow H_q(N \cup E) \longrightarrow$$

Hence $0 \longrightarrow Z \longrightarrow H_2(P_1) \longrightarrow H_2(N \cup E) \longrightarrow 0$ is exact. Thus $H_2(P_1) \neq 0$ and so P_1 is orientable. If P_1 has genus g, $H_1(P_1)$ is the direct sum of $2g$ copies of Z. Since $H_1(N \cap E) = 0$, $H_1(N \cup E) = \bigoplus_1^{2g} Z$. Thus, if $\chi(X)$ is the Euler characteristic of X, $\chi(N \cup E) = 1 - 2g + 0 - 0 = 0$, since $N \cup E$ is a closed 3-manifold. Since g is an integer, we have a contradiction. Thus Bd N is two 2-spheres. Let E and F be two 3-cells attached to N by boundary homeomorphisms. We obtain the M-V sequence

$$\longrightarrow H_3(N \cup E \cup F) \longrightarrow H_2(N \cap (F \cup E)) \longrightarrow H_2(N) \oplus H_2(E \cup F) \longrightarrow.$$

Since $N \cup E \cup F$ is non-orientable and $N \cap (E \cup F) = \text{Bd } N$,

$$0 \longrightarrow Z \oplus Z \longrightarrow H_2(P_1) \longrightarrow H_2(N \cup E \cup F) \longrightarrow \cdots$$

is exact. However $H_2(P_1) = 0$ or Z. This contradiction establishes the lemma.

Corollary to the proof: If A is an (n,m)-residual set for the closed 3-manifold M, $N(P_i,M)$ is orientable for all i.

3. Reconstruction of M

Suppose that A is an $(n,k+m)$-residual set of the closed 3-manifold M, with k of the 2-manifolds of A 2-spheres and m of the 2-manifolds real projective planes. By the argument of Lemma 3.1.3, $H_2(A;Z_2) = H_1(A;Z_2)$. But $H_1(A;Z_2) = \bigoplus_1^{n+m} Z_2$ and $H_2(A;Z_2) = \bigoplus_1^{k+m} Z_2$. Thus $k = n$. Hence A is the one-point union of n 1-spheres $S_1, \cdots, S_n$, n 2-spheres $T_1, \cdots, T_n$ and m real projective planes $P_1, \ldots, P_n$. If A is in this form, we will call A an (n,n,m)-residual set.

Lemma 3.3.1: Let A be an (n,n,m)-residual set for the closed 3-manifold M. Then

$$N(A) = N((\bigvee_{i=1}^{n} S_i) \vee (\bigvee_{i=1}^{n} T_i)) \triangle N(P_1) \triangle \ldots \triangle N(P_m).$$

Proof: Suppose that a is the join point of A. Then the simple closed curve $L = P_1 \cap Lk(a,N)$ bounds two disks D and D' in $Lk(a,M)$. If $Int(D) \cap Lk(a,A)$ is empty, by a B-move we may change A into $A' = (A - aoL) \cup xoL \cup B$, where x is an interior point of aoD and B is the straight line segment from a to x. Let

$P' = (P_1 - aoL) \cup xoL$. Then $A' = (A - P_1) \cup P' \cup B$ and since $N(B)$ is a 3-cell $N(A')$ is homeomorphic to $N((A - P_1) \cup a) \Delta N(P')$. Since $N(P')$ is homeomorphic to $N(P_1)$ the lemma will follow by induction if we are able to justify our initial assumption, that $\text{Int}(D) \cap \text{Lk}(a,A)$ is empty.

Let $L_1, \ldots, L_p$ be the simple closed curves in $A \cap \text{Int}\, D$ and suppose that $x_1, \ldots, x_q$ are the points of $A \cap \text{Int}\, D$. Likewise let $L_{p+1}, \ldots, L_{n+m-1}$ be the simple closed curves of $A \cap \text{Int}\, D'$ and $x_{q+1}, \ldots, x_{2n}$ the points of $A \cap \text{Int}\, D'$. By the elementary moves of Lemma 3.1.2, we may change A into a residual set A'' with $A'' =$

$$(A-\text{St}(a,M)) \cup xo[(\bigcup_{i=1}^{p} L_i) \cup (\bigcup_{i=1}^{q} x_i)] \cup B \cup yo[(\bigcup_{p+1}^{n+m-1} L_i) \cup (\bigcup_{q+1}^{2n} x_i)] \cup B'$$

where x is an interior point of aoD, y is an interior point of aoD', $B = aox$ and $B' = aoy$.

Since $N(P_1,M)$ is orientable, $N(P_1,M)$ is homeomorphic to $N(RP^2,RP^3)$. Since RP^3 is obtained by the antipodal identification of the boundary of the unit 3-ball, we may consider $N(RP^2,RP^3)$ as the quotient space $(S^2 \times I)/R$, where R is the equivalence relation

$$R: \{((x,0),(-x,0)) \mid x \in S^2\} \cup \{((x,t),(x,t)) \mid (x,t) \in S^2 \times I\}$$

(For the definition of quotient space see [6].) Suppose that $q: S^2 \times I \longrightarrow N(P_1,M)$ is the composite of the quotient map and the obvious homeomorphism. Without loss of generality we may assume that $q((1,0,0),0) = a$,

$q((1,0,0) \times I) = B \cap N(P_1)$ and $q((-1,0,0) \times I) = B' \cap N(P_1)$. Let E be the 2-cell in $S^2 \times I$ given by $E = \{((x,y,0),t) \mid x^2 + y^2 = 1,\ y \geq 0,$ and $0 \leq t \leq 1/2\}$. Then $q(E)$ is a singular 2-cell in M. $q(E)$ meets A'' on $(B \cap q(E)) \cup (B' \cap q(E)) \cup (P_1 \cap q(E)$. Let C be the arc of $q(E)$ given by $C = q(\{((x,y,0),1/2) \mid x^2 + y^2 = 1$ and $y \geq 0\})$.

To make the desired change of A'', expand A'' to $A'' \cup q(E)$ and collapse $A'' \cup p(E)$ onto $(A'' - (B \cap q(E))) \cup C \cup a$. Let A''' denote the resulting residual set. Let F be the arc $F = \mathrm{Cl}(B - q(E)) \cup \mathrm{Cl}(B' - q(E)) \cup C$. Consider the relation $R: (F \times F) \cup \{(z,z) \mid z \in M\}$. Let $M' = M/R$. Since F is point-like, M' is homeomorphic to M. Note that A'''/R is a residual set for M' that has the desired form. Thus M itself has a residual set of the desired form and the lemma is established.

<u>Lemma 3.3.2</u>: Suppose that M is a closed 3-manifold with an $(n,n,0)$-residual set. Then M has an $(n,n,0)$-residual set such that for all i,

i) S_i pierces T_i and no other 2-sphere,

ii) T_i is pierced by S_i and no other 1-sphere and

iii) $N(S_i \vee T_i)$ is topologically either $S^1 \times S^2$ less an open 3-cell or the twisted S^2 bundle over S^1 less an open 3-cell.

Moreover, $N(A) = N(S_1 \vee T_1) \triangle \ldots \triangle N(S_n \vee T_n)$.

Proof: Suppose that $n = 1$. If S_1 does not pierce T_1, then $\mathrm{Bd}\, N(S_1 \vee T_1) = \mathrm{Bd}\, N(S_1) \# \mathrm{Bd}\, N(T_1)$. By Corollary 3.2.3, $N(T_1)$ is homeomorphic to $S^2 \times I$ and by the remark after Lemma 3.2.1, $N(S_1)$ is either a solid torus or a solid Klein bottle. Thus if S_1 does not pierce T_1 we will contradict the connectivity of $\mathrm{Bd}\, N(A)$. Therefore S_1 pierces T_1.

Since $N(A) = N(S_1) \cup N(T_1)$, $N(A) = (B^2 \times I) \cup_h (S^2 \times I)$, where h is a homeomorphism of $B^2 \times \mathrm{Bd}\, I$ onto two disks of $\mathrm{Bd}\,(S^2 \times I)$. Since $\mathrm{Bd}\, N(A)$ is connected, h must take each of the disks of $B^2 \times \mathrm{Bd}\, I$ into distinct 2-spheres in $\mathrm{Bd}\,(S^2 \times I)$. If $N(S_1)$ is a solid torus, h is either orientation preserving or orientation reversing on both ends of $B^2 \times I$. If $N(S_1)$ is a solid Klein bottle, then h is orientation preserving on one end and orientation reversing on the other. In the first case $N(A)$ is orientable and by attaching a 3-cell to $N(A)$ by a boundary homeomorphism we obtain $S^1 \times S^2$. In the second case $N(A)$ is non-orientable and by attaching a 3-cell to $N(A)$ we obtain the twisted S^2 bundle over S^1. Thus the Lemma holds if $n = 1$.

Assume the lemma is true for $n = 1, 2, \ldots, k-1$. Define X by $X = \mathrm{Cl}(\mathrm{Bd}[N(a,M) - (\bigcup_{i=1}^{k} N(T_i) \cup \bigcup_{i=1}^{k} N(S_i))])$, where a is the join point of A. X is a 2-sphere with k disjoint open annuli and $2k$ disjoint open disks removed. We may obtain $\mathrm{Bd}\, N(A)$ from X by attaching $2k$ disks to the boundaries of the annuli and k annuli to the boundaries of the $2k$ disks. Note that the annuli may be attached with different orientations on each end. Since X has $k+1$ components and $\mathrm{Bd}\, N(A)$ is a 2-sphere, the annuli must bridge the components of X. Hence each component has a disk removed.

<u>MOVE F</u>: Let a be the join point of A. Suppose x_1 and x_2 are two isolated points of $\mathrm{Lk}(a,A)$ that lie in the same component of $\mathrm{Lk}(a,M)$ less the 2-manifolds of A. Since $\mathrm{Bd}\, N(A,M)$ is a 2-sphere, x_1 and x_2 belong to two distinct 1-spheres, say S_1 and S_2, of A. There is a 2-ball B in M with $A \cap B = A \cap \mathrm{Bd}\, B = aox_1 \cup aox_2$. Let $C = \mathrm{Cl}(\mathrm{Bd}\, B - aox_1 \cup aox_2)$. Then there is a 2-ball D in M with $A \cap D = A \cap \mathrm{Bd}\, D = \mathrm{Cl}(S_2 - aox_2) \cup x_1$ and $B \cap D = B \cap \mathrm{Bd}\, D = C$. Let $E = \mathrm{Bd}\, D - (C \cup \mathrm{Cl}(S_2 - aox_2))$. An F-move expands A to $A \cup B \cup D$ and collapses from aox_1 across $B \cup D$ onto $(A - (aox_1)) \cup a \cup E$.

The effect of an F-move is to slide a disk from one component of X along an annulus to another component. Thus we may assume that k of the disks lie in one component of X and that each of the other components contain exactly one of the disks.

Since at least one of the 1-spheres of $\mathrm{Lk}(a,A)$ is nullhomotopic in $\mathrm{Lk}(a,M)$ less the other 1-spheres of $\mathrm{Lk}(a,A)$, suppose that $L = T_1 \cap \mathrm{Lk}(a,M)$ is the 1-sphere. Then L bounds a disk D in $\mathrm{Lk}(a,M)$ with $T_i \cap D$ empty for $i \geq 2$. By F-moves we may assume that only S_1 intersects D. Then $D \cap S_1$ is a point x_1. By another series of F-moves, we may assume that each 1-sphere of A meets the component of $Lk(a,M)$ less all the 2-manifolds that has L as one of its boundary components. Thus S_1 pierces T_1. Since any other S_i lies on one side of T_1, S_i cannot pierce T_1. Likewise S_1 lies on one side of the other T_i and so cannot pierce them.

Let $A' = (A - (S_1 \cup T_1)) \cup a$ and suppose that x is an interior point of aox_1 and y is an interior point of $(S_1 - aox_1) \cap St(a,M)$. Let $T' = (T_1 - aoL) \cup xoL$. Then by elementary moves change A to $A' \cup aox \cup T' \cup S_1$. By another series of elementary moves, move aoy along aox to an arc C from y to x. Set $S' = (S_1 - ao(x \cup y)) \cup C$. Thus we may change A into a new residual set $A' \cup S' \cup T' \cup aox = A''$. Now $N(A'')$ is homeomorphic to both $N(A)$ and $N(S' \vee T') \triangle N(A')$. Clearly A' is an $(n-1,n-1,0)$-residual set for a closed 3-manifold, for $N(A')$ is a 3-manifold with 2-sphere boundary. Since $N(S' \vee T')$ is homeomorphic to $N(S_1 \vee T_1)$, the lemma follows by induction.

From Lemma 3.3.1 and Lemma 3.3.2, we obtain:

<u>Lemma 3.3.3</u>: Let A be an (n,n,m)-residual set of a closed 3-manifold M. Then

$$N(A) = N(S_1 \vee T_1) \triangle \dots \triangle N(S_n \vee T_n) \triangle N(P_1) \triangle \dots \triangle N(P_m).$$

<u>Theorem 3.3.4</u>: Let A be an (n,n,m)-residual set and let B be a (p,p,q)-residual set for the same closed 3-manifold M. Then A and B are homeomorphic.

<u>Proof</u>: The second homology groups with Z_2 coefficients of A, B and M are isomorphic and so $n + m = p + q$. Since the rank of $H_1(A)$, $H_1(M)$, and $H_1(B)$ is the same, $n = p$. Thus A and B are homeomorphic (n,n,m)-residual sets for M.

<u>Theorem 3.3.5</u>: Suppose that M and M' are two closed 3-manifolds with the same orientability. If M and M' have homeomorphic (n,n,m)-residual sets A and A', then M is homeomorphic to M'.

<u>Proof</u>: Suppose that the 1-spheres of A are denoted by S_i, the 2-spheres by T_i, and the real projective planes by P_i. Let S_i', T_i', and P_i' denote the corresponding parts of A'. Let $N(X,M) = N(X)$ and $N(X,M') = N'(X)$. By Corollary 3.3.3, $N(A) = N(S_1 \vee T_1) \triangle \ldots \triangle N(S_n \vee T_n) \triangle N(P_1) \triangle \ldots \triangle N(P_m)$ and $N'(A') = N'(S_1' \vee T_1') \triangle \ldots \triangle N'(S_n' \vee T_n') \triangle N'(P_1') \triangle \ldots \triangle N'(P_m')$.

Suppose that X in a subcomplex of M with Bd $N(X)$ a 2-sphere. Then we will denote the closed 3-manifold obtained by attaching a 3-cell to $N(X)$ by $M(X)$. In the same manner define $M'(X)$. Then $M(A)$ is homeomorphic to M and $M'(A')$ is homeomorphic to M'.

Notice that if $N(X \cup Y) = N(X) \Delta N(Y)$, it follows that $M(X \cup Y) = M(X) \# M(Y)$, whenever $N(X \cup Y)$ has a 2-sphere boundary. Thus we obtain:

$M = M(S_1 \vee T_1) \# \cdots \# M(S_n \vee T_n) \# M(P_1) \# \cdots \# M(P_m)$ and

$M' = M'(S_1' \vee T_1') \# \cdots \# M'(S_n' \vee T_n') \# M'(P_1') \# \cdots \# M'(P_m')$.

Since P_i is a 2-manifold that is a residual set of the closed 3-manifold $M(P_i)$ it follows from [9] that $M(P_i)$ is a copy of real projective 3-space. Also $M'(P_i') = RP^3$.

Since $J \# J = J \# S^1 \times S^2$ [13], we may assume that $M(S_i \vee T_i)$ and $M'(S_i' \vee T_i')$ are topologically $S^1 \times S^2$ for $i \neq 1$. Since M and M' are either both orientable or both non-orientable, the same is true for $M(A)$ and $M'(A')$. Hence each term in the connected sum is pairwise homeomorphic. Thus M and M' are homeomorphic.

Combining these results we obtain:

Theorem 3.3.6: A Closed 3-manifold has a residual set that contains no wild arc if and only if it is the connected sum of closed 3-manifolds each of which is homeomoprhic to a 3-sphere, RP^3, $S^1 \times S^2$ or J.

4. Compact 3-Manifolds with Boundary

Lemma 3.4.1: Let M be a compact 3-manifold with or without boundary. Suppose that $M = A \cup U$ is a decomposition of M such that A contains no wild arc. Let C be a 3-cell in Int M. Then M - Int C has a decomposition M - Int C = $A' \cup U'$ such that A' contains no wild arc.

Proof: Since C is point-like, there is a homeomorphism h of M onto itself such that $h(C) \subset U$. After a possible subdivision there is a simplicial arc B in $U \cup x$, where x is a point of A, with $B \cap h(C) = y$ and $\mathrm{Bd}\, B = x \cup y$. Thus we may expand $h(C) \cup B$ to a 3-cell D in $U \cup x$. Then there is a homeomorphism k of M onto itself such that $k(D) = h(C)$. Thus $C = h^{-1}k(D)$ and $M - \mathrm{Int}\, D$ is a compact 3-manifold with a decomposition $M - \mathrm{Int}\, D = (A \cup \mathrm{Bd}\, D) \cup (U - D)$. Since $A \cap \mathrm{Bd}\, D = x$, $A \cup \mathrm{Bd}\, D$ contains no wild arc. Thus $M - \mathrm{Int}\, C$ has a decomposition $M - \mathrm{Int}\, C = h^{-1}k(A \cup \mathrm{Bd}\, D) \cup h^{-1}k(U - D)$, and $h^{-1}k(A \cup \mathrm{Bd}\, D)$ contains no wild arc.

Lemma 3.4.2: Let M_1 and M_2 be compact 3-manifolds with boundary. Suppose that B_1 and B_2 are homeomorphic boundary components of M_1 and M_2 respectively. Let M be obtained by sewing M_1 and M_2 together along B_1 and B_2. Suppose that $M_i = A_i \cup U_i$ is a decomposition of M_i such that A_i contains no wild arc. Then M has a residual set that contains no wild arc.

Proof: Consider M_1 and M_2 as submanifolds of M so that $M = M_1 \cup M_2$ and $M_1 \cap M_2 = B_1 = B_2$. Set $B = B_1$. Since A_i does not contain a wild arc $B \cap \mathrm{Cl}(A_i - B)$ is a finite point set. By elementary moves we may assume that $B \cap \mathrm{Cl}(A_i - B) = x$ for $i = 1$ or 2, where x is a $(1,2)$-point of $A_1 \cup A_2$. Let C be a 2-simplex of B. Then M has a decomposition

$M = ((A_1 - B) \cup (B - \text{Int } C) \cup (A_2 - B)) \cup (U_1 \cup \text{Int}(C) \cup U_2)$. By collapsing $B - \text{Int } C$ to a spine K, we obtain residual set A for M, where $A = (A_1 - B) \cup K \cup (A_2 - B)$. Clearly A contains no wild arc.

With the notation of the above lemma, $N(K,A)$ is K with two whiskers. If M is not a closed 3-manifold, take the double of M. It follows that A less some open disks of A embeds in a residual set of a closed 3-manifold, Thus $\text{Bd } N(K,M)$ less two disks embeds in a 2-sphere. Since K has the homotopy type of an r-leaved rose, $\text{Bd } N(K,M)$ is a 2-sphere with r handles, orientable or not. Thus $r = 0$ and $N(K,M)$ is a 3-ball. Therefore K is a point and B is a 2-sphere. Thus we have established:

Corollary 3.4.3: If M is a compact 3-manifold with boundary having a residual set containing no wild arc, each boundary component is a 2-sphere.

Theorem 3.4.4: Let M be a compact 3-manifold. M has a residual set that contains no wild arc if and only if M is obtained from the connected sum of closed 3-manifolds each of which is S^3, $S^1 \times S^2$, RP^3 or J, by deleting $n \geq 0$ disjoint 3-balls.

Proof: One way follows from Theorem 3.3.6 and Lemma 3.4.1. The converse follows by sewing n 3-balls onto M and applying Lemma 3.4.3 and Theorem 3.3.6.

CHAPTER IV

A CHARACTERIZATION OF CERTAIN CLOSED 3-MANIFOLDS WHOSE SINGULAR SET IS A SIMPLE CLOSED CURVE

In this chapter we define the singular set of a residual set A of a closed 3-manifold, denoted by $S(A)$. We show that A may be chosen so that $A - S(A)$ is the disjoint union of open arcs and the interiors of compact 2-manifolds with connected boundary. We then classify all closed 3-manifolds with $S(A)$ a simple closed curve.

1. The Singular Set

Definition 4.1.1: Let A be a residual set for a closed 3-manifold M. The special singular set $S'(A)$ of A is the set of all points of A that do not have an open 2-dimensional euclidean neighborhood in A. The singular set $S(A)$ of A is the set of all points of A that do not have an open 1- or 2-dimensional euclidean neighborhood in A. If A is already locally euclidean, we will set $S'(A) = S(A) = a$ where a is an arbitrary point of A.

Suppose that M is not a 3-sphere. Then A is a 2-complex. Since $S'(A)$ is a subcomplex of A, $S'(A)$ is contained in the 1-skeleton of A, A^1. Let T be a maximal tree of A^1. Mod out T; that is consider the quotient space M/T obtained by identifying T to a point [6]. Since T is contractable, T is point-like and so M/T

is homeomorphic to M. Notice that A'/T is a t-leafed rose and since $S(A)/T \subset S'(A)/T \subset A^1/T$, $S(A)$ is an r-leafed rose and $S'(A)$ is an s-leafed rose. Since T is contained in A^1, $(M-A)/T$ is an open 3-cell and A/T is a non-separating continuum of dimension two. Thus M/T has a decomposition $M/T = (A/T) \cup ((M-A)/T)$. The singular set of A/T is clearly $S(A)/T$ and $S'(A/T) = S'(A)/T$. Since M and M/T are homeomorphic, we may assume that A is already in the above form, that is $S(A)$ is an r-leafed rose and $S'(A)$ is an s-leafed rose.

Let $M_1', \ldots, M_n'$ be the components of $A - S'(A)$. Then each M_i' is an open 2-manifold; in fact, each M_i' is the interior of a compact 2-manifold M_i with non-empty boundary. Thus A is obtained by attaching the M_i's to $S'(A)$ by wrapping each boundary component of the M_i's around $S(A)$. To be more precise, let X be the disjoint union of the M_i's and let Y be the disjoint union of the boundaries of the M_i's. Then there is a continuous map $\phi\colon Y \longrightarrow S'(A)$ such that A is topologically the space obtained by attaching X to $S'(A)$ by ϕ. For the definition of the attaching of spaces see [6]. In effect we are sewing the M_i onto $S'(A)$ by the map ϕ. Let $p\colon X \longrightarrow A$ denote the composite of the quotient map and the homeomorphism between $X \cup_\phi S'(A)$ and A.

To obtain a better picture of A let us examine the map Φ. Now Φ is a map onto $S(A)$, an r-leafed rose. Let $L_1, \ldots, L_r$ be the leaves of $S(A)$, where the L_i are given a definite orientation. Let f_i be the map from the unit interval onto L_i given by $f_i(t) = h_i(e^{2\pi it})$, where h_i is a homeomorphism of the unit 1-sphere onto L_i with the induced orientation of $h_i(S^1)$ agreeing with the orientation of L_i and $f_i(\mathrm{Bd}\ I)$ is equal to the join point a of $S(A)$. Suppose that S is a component of the boundary of one of the M_i. Let $\Phi|S = \Phi'$. Since we have modded out a maximal tree in the 1-skeleton of a residual set of M to obtain A, Φ' will induce a subdivision of S into k segments $S_1, \ldots, S_k$ with the interiors of the S_i disjoint and Φ' mapping each S_j onto one of the L_i in the same way that f_i or f_i^{-1} maps the unit interval onto L_i. If Φ' already maps S homeomorphically onto one of the leaves of $S(A)$, we denote Φ' by f_i or f_i^{-1} depending on how Φ' operates. If Φ' is the constant map we denote Φ' by 0. If Φ' is more complex we assume that the S_i are cyclicly ordered on S by their subscripts. By disregarding the obvious homeomorphism between the unit interval and each S_j, we regard f_i as a map from S_j onto L_i. We then denote the action of Φ' by setting $\Phi' = h_1 h_2 \ldots h_k$ where h_j is one of the maps f_i or f_i^{-1} for some i.

<u>Definition 4.1.2</u>: Let A be a residual set for a closed 3-manifold M. With A as above, a <u>presentation</u> P of A

$$P : S(A), S'(A), M_1, \ldots, M_n; \Phi$$

is a set consisting of the singular set, the special singular set, the compact 2-manifolds M_i and the map Φ.

We now establish some properties of A.

Let k_i be the rank of $H_1(M_i)$ and h_i the number of boundary components of M_i. Let V be a second derived neighborhood of $S'(A)$ in A and let W be the closure of $A - V$. Since $A = W \cup V$ and $W \cap V$ is the disjoint union of $\sum_{i=1}^{n} h_i$ simple closed curves, $\chi(A) = \chi(V) + \chi(W) - \chi(V \cap W)$. Since V collapses onto $S'(A)$, an s-leafed rose, $\chi(V) = 1 - s$. Since W is homeomorphic to the disjoint union of the M_i, $\chi(W) = \sum_{i=1}^{n} (1 - k_i)$. Since $V \cap W$ is $\sum_{i=1}^{n} h_i$ disjoint simple closed curves, $\chi(W \cap V) = 0$. If C is a 3-simplex of $M - A$, $M - \text{Int } C$ collapses onto A and so $\chi(A) = \chi(M - \text{Int } C) = \chi(M) + 1$. However, M is a closed 3-manifold and so $\chi(M) = 0$. Hence

$$1 = \chi(A) = (1 - s) + \sum_{i=1}^{n} (1 - k_i) = n + 1 - (s + \sum_{i=1}^{n} k_i).$$

<u>Lemma 4.1.3</u>: $n = s + \sum_{i=1}^{n} k_i$.

<u>Lemma 4.1.4</u>: Let M be a closed 3-manifold and A a residual set for M with a presentation

$$P: S(A), S'(A), M_1, \ldots, M_n; \Phi.$$

Suppose that ϕ restricted to any boundary component S is either the constant map or $\phi|S = h_1h_2 \ldots h_p$, with each h_i either h_1 or h_1^{-1}, where h_1 and p depend on S. Then M has a residual set A' with a presentation

$$p'\colon S(A'),\ S'(A'),\ N_1,\ \ldots,\ N_n; \phi',$$

where N_i is homeomorphic to M_i for all i, and ϕ' restricted to any boundary component S' is either the constant map or $\phi'|S' = h_1h_1 \ldots h_1 = h_1^q$, for $0 < q \leq p$.

<u>Proof</u>: Let S be a boundary component of M_1 and $\phi|S = h_1 \ldots h_i h_{i+1} \ldots h_p$, with $p > 1$ and $h_{i+1} = h_i^{-1}$. Then ϕ induces a subdivision of S into p arcs $S_1, \ldots, S_p$ cyclicly ordered by their subscripts. Let B and C be proper subarcs of S_i and S_{i+1} respectively with $B \cup C$ connected and $p(B) = p(C)$. Then $B \cup C$ lies on a 2-cell D in M_1 with $B \cup C = D \cap \mathrm{Bd}\, M_1$. Clearly $p(D)$ is a 2-cell of A. Moreover $p(D)$ lies on a 3-cell E in M as three sides of a 3-simplex. Let F be the remaining side of E. Suppose that $E \cap A = p(D)$. Swell up A to $A \cup E$ and collapse onto $(A - E) \cup F \cup p(B)$. By modding out the closure of $p(S_i) - p(B)$, we obtain a new residual set A' with a presentation

$$P'\colon S(A'),\ S'(A'),\ N_1,\ \ldots,\ N_n;\ \phi',$$

with N_i homeomorphic to M_i and ϕ' restricted to the component S' of N_1 corresponding to S is given by $\phi'|S' = h_1h_2 \ldots h_{i-1}h_{i+2} \ldots h_p$.

In the above we assumed that $E \cap A = p(D)$. In general this is too much to ask. However, if $E \cap A \neq p(D)$, there is a finite sequence of disks $D_1, \ldots, D_t = D$ in the disjoint union of the M_i, with $p(\bigcup_{i=1}^{t} D_i)$ homeomorphic to the cone over X with vertex v, where X is the planar set that is the union of the simple closed curves $C_i = \{(x,y) \mid (x-i)^2 + y^2 = i^2\}$, for $1 \leq i \leq t$, and voC_i is mapped onto D_i for all i. In M there is a finite sequence of 3-cells $E_1, \ldots, E_t$, with $E_t = E$ and $p(D_i)$ lying on E_i as three sides of a 3-simplex. Moreover the E_i may be so chosen that the above homeomoprhism extends to a homeomorphism of the cone over the bounded region bounded by C_t onto E_t in such a way that the cone over the compact plane set bounded by C_i is mapped onto E_i. Then by collapsing first E_1 as above and continuing for the other E_i we finally collapse E. In each step we obtain a residual set with a presentation closer to the desired presentation. Thus the obvious induction establishes the lemma.

<u>Corollary 4.1.5:</u> Each closed 3-manifold has a residual set A with a presentation

$$P: S(A), S'(A), M_1, \ldots, M_n; \Phi,$$

such that if S is a boundary component of one of the M_j, $\Phi|S = h_1 \ldots h_p$ with h_i distinct from h_{i+1}^{-1} for all i.

<u>Corollary 4.1.6:</u> If the dunce hat is a residual set for a closed 3-manifold M, M is a 3-sphere.

Proof: Let D be the dunce hat. Then D has a presentation

$$P: S^1, S^1, B^2; hhh^{-1}.$$

By Lemma 4.1.4, M has a residual set A with a presentation

$$P': S^1, S^1, B^2; h.$$

Hence A is a disk and so M is a 3-sphere.

Lemma 4.1.7: Each M_i in a presentation for A is either a disk with holes or a Moebius band with holes.

Proof: Suppose that each M_i has connected boundary. Since each $p(M_i)$ contains a topological copy P_i of M_i in $P(M_i) - S(A)$, $Bd(N(P_i,M)-N(Bd\ P_i,M))$ embeds in $Bd\ N(A,M)$, a 2-sphere. Let $N = N(P_1,M)$. If N is orientable, it is unique by Lemma 3.2.2. Since any 2-manifold with non-empty boundary embeds in R^3, N is homeomorphic to $N(P_1,R^3)$. If P_1 is orientable with positive genus, or if P_1 is non-orientable with genus greater than two, $N' = N - N(Bd\ P_1,M)$ has a boundary that contains a torus with a disk removed. If P_1 is a Klein bottle less an open disk, $Bd\ N'$ is two open annuli attached in such a way that $Bd\ N'$ does not embed in a 2-sphere. Thus if $N(P_1,M)$ is orientable, P_1 is either a disk or a Moebius band. Suppose that N is non-orientable. Since N collapses onto P_1, $H_2(N) = 0$ and $H_1(N)$ is free. Since N is non-orientable, $H_3(N,Bd\ N) = 0$. Thus the exact homology sequence of the pair $(N,Bd\ N)$ yields an exact sequence

$$0 \longrightarrow H_2(\mathrm{Bd}\ N) \longrightarrow 0.$$

Since $\mathrm{Bd}\ N$ is a closed 2-manifold, each component of $\mathrm{Bd}\ N$ is non-orientable. Let $Q = N(\mathrm{Bd}\ P_1, M)$, and $N' = N - Q$. Then $\mathrm{Bd}\ N'$ embeds in $\mathrm{Bd}\ N(A,M)$ and so each component of $\mathrm{Bd}\ N'$ is a disk with holes. Since $\mathrm{Bd}\ N = \mathrm{Bd}\ N' \cup (\mathrm{Bd}\ N \cap \mathrm{Bd}\ Q)$, $\mathrm{Bd}\ N \cap \mathrm{Bd}\ Q$ is non-orientable. Thus Q is a solid Klein bottle and $\mathrm{Bd}\ Q$ is a Klein bottle. Since $P_1 \cap \mathrm{Bd}\ Q = C$ is a simple closed curve, $N(C, \mathrm{Bd}\ Q)$ is either an annulus or a Moebius band. In the first case $\mathrm{Bd}\ Q - N(C, \mathrm{Bd}\ Q)$ is a pair of Moebius bands and in the second case it is a single Moebius band. Since $\mathrm{Bd}\ N = \mathrm{Bd}\ N' \cup (\mathrm{Bd}\ Q - N(C, \mathrm{Bd}\ Q))$ and $\mathrm{Bd}\ N$ is a closed 2-manifold, the first case implies that $\mathrm{Bd}\ N$ is either two projective planes or a Klein bottle and the second case implies that $\mathrm{Bd}\ N$ is a projective plane. If $\mathrm{Bd}\ N$ is a projective plane, $2\chi(N) = \chi(\mathrm{Bd}\ N) = 1$. Thus $\chi(N) = 1/2$. This is impossible. If $\mathrm{Bd}\ N$ is two projective planes, we obtain from the homology sequence of the pair $(N, \mathrm{Bd}\ N)$ the exact sequence

$$0 \longrightarrow H_2(N, \mathrm{Bd}\ N) \longrightarrow Z_2 \oplus Z_2 \longrightarrow H_1(N) \longrightarrow \cdots$$

This is impossible since $H_1(N)$ has no torsion and $\mathrm{Tor}(H_2(N, \mathrm{Bd}\ N)) = Z_2$. Thus $\mathrm{Bd}\ N$ is a Klein bottle. The exact homology sequence for the pair $(N, \mathrm{Bd}\ N)$ becomes

$$0 \longrightarrow H_2(N, \mathrm{Bd}\ N) \longrightarrow Z \oplus Z_2 \longrightarrow H_1(N) \longrightarrow H_1(N, \mathrm{Bd}\ N) \longrightarrow 0.$$

Thus $H_1(N) = Z = N_1(P_1)$. Since P_1 has connected boundary P_1 is a Moebius band.

To complete the proof observe that if M_1 is a surface with holes, M_1 contains a surface of the same genus with only one hole. In this surface find a P_1 as before. The above argument then implies that M_1 is either a disk with holes or a Moebius band with holes.

<u>Theorem 4.1.8</u>: If M is a closed 3-manifold, M has a residual set A with a presentation P,

$$P: S(A), S'(A), M_1, \ldots, M_n; \Phi,$$

where each M_i has connected boundary.

<u>Proof</u>: Suppose that A is a residual set for M with a presentation where at least one of the M_i, say M_1, does not have connected boundary. Let S and T be two components of $\mathrm{Bd}\, M_1$. Let B be an arc in M_1 connecting S and T such that $\mathrm{Int}\, B \subset \mathrm{Int}\, M_1$ and p maps one of the end points of B onto a, the join point of $S'(A)$. Either $p(B)$ is an arc or a simple closed curve in A. If $p(B)$ is an arc, $M/p(B)$ has a residual set $A/p(B)$. Since M is homeomorphic to $M/p(B)$ we need only show that $A' = A/p(B)$ has a presentation that simplifies the presentation of A in the sense that the number of boundary components is reduced. A' has a presentation P',

$$P': S(A'), S'(A'), N_1, \ldots, N_n; \Phi',$$

where N_i is homeomorphic to M_i for $2 \leq i \leq n$ and $\mathrm{Int}\ N_1 = (\mathrm{Int}\, M_1)/p(B) - p(B)/p(B)$. We may think of N_1 as being derived from M_1 by expanding B to $N(B, M_1)$ and removing the open star of B in M_1.

If $p(B)$ is a simple closed curve for all choices of B, $\Phi|T \cup S$ is the constant map. Let S' be a simple closed curve in the interior of M_1 with $S \cup S'$ bounding an annulus D in M_1. Then $p(D)$ is a disk meeting $S(A)$ only at a. Either $S'(A)$ pierces $p(D)$ or it does not. Suppose that $S'(A)$ does not pierce $p(D)$. By swelling $p(D)$ up to a 3-cell and collapsing as in a B-move and then pushing the resulting arc along $p(M_1 - D)$ as in an E-move, so as to form a simple closed curve, we obtain a new residual set A' for M. A' has a presentation

$$P': S(A'), S'(A'), N_1, \ldots, N_n; \Phi'.$$

Again it is clear that N_i is homeomorphic to M_i for $i \neq 1$. N_1 may be obtained from M_1 by cutting M_1 along S' and sewing in a 2-cell. Note that we then must add one more 1-sphere to $S'(A)$ in order to obtain $S'(A')$. Suppose that $S'(A)$ pierces $p(D)$. Without loss of generality, assume Corollary 4.1.5 has been applied. Swell up $p(D)$ to a 3-cell E as before. Let $F = \mathrm{Cl}(\mathrm{Bd}\, E - p(D))$. $A \cap \mathrm{Int}\, E = C$ is an arc in a 1-sphere of $S'(A)$. Expand A to $A \cup E$ and collapse $A \cup E$ onto $(A - E) \cup F \cup C$. There are two arcs of $(A - E) \cup F \cup C$ that meet $\mathrm{Int}\, F$, C and C'. By E-moves push the end points of C and C' along $F \cup p(M_1 - D)$ to the join point a. Let A' be the resulting residual set. A' has a presentation

$$P': S(A'), S'(A'), N_1, \ldots N_n; \Phi,$$

with N_i homeomoprhic to M_i for $i \neq 1$. N_1 is obtained from M_1 by cutting M_1 along S' and sewing in a 2-cell. Note that we have added a 1-sphere to $S'(A)$ in order to obtain $S'(A')$. Thus no matter what $p(B)$ is we are able to find a new residual set with a presentation that has reduced the number of boundary components of the 2-manifolds. Therefore, by an inductive argument, the lemma is established.

Remark 4.1.9: Lemma 4.1.8 enables us to assume that each M_i has connected boundary. However, we may have to sacrifice a little, for by reducing the number of boundary components we may increase the number of leaves of $S(A)$. As an example, let $M = RP^2 \times S^1$. RP^2 has a residual set RP^1, a 1-sphere. Let p be a point of S^1. M has a residual set $A = (RP^2 \times p) \cup (RP^1 \times S^1)$, with a presentation P,

$$P: RP^1 \times p, RP^1 \times p, M_1, M_2; \Phi,$$

where M_1 is a disk, M_2 is an annulus, $\Phi|\text{Bd } M_1 = ff$ and Φ restricted to either component of $\text{Bd } M_2$ is f. By changing A as in Lemma 4.1.8, we obtain a residual set A' with a presentation P',

$$P': S(A'), S'(A'), N_1, N_2; \Phi',$$

where N_1 and N_2 are disks, $S(A') = S'(A')$ is a 2-leafed rose, $\Phi'|\text{Bd } N_1 = f_1f_2f_1f_2$ and $\Phi'|\text{Bd } N_2 = f_1f_2f_1^{-1}f_2^{-1}$.

2. Residual Sets with $S(A)$ a Simple Closed Curve

In this section we assume that each 2-manifold in a presentation has connected boundary.

Theorem 4.2.1: Let A be a residual set for a closed 3-manifold M. Suppose that A has a presentation that has only one 2-manifold. Then M is either the twisted S^2 bundle over S^1 or a toroidal manifold.

Proof: By Lemma 4.1.3, $1 = s + k_1$. If $S = 0$, $S'(A)$ is a point and so ϕ is the constant map. Thus A is homeomorphic to RP^2 and hence $M = RP^3 = T(2,1)$. If $s = 1$ and $r = 0$, $S'(A)$ is a 1-sphere and $S(A)$ is a point. Thus A is the one-point union of a 1-sphere and a 2-sphere. By Theorem 3.3.6, M is either $S^1 \times S^2 = T(0,1)$ or J.

Thus we may assume that $S(A) = S'(A)$ is a 1-sphere. By Lemma 4.1.4, A has a presentation

$$P\colon S^1,\ S^1,\ B^2;\ h^k,\ k > 0.$$

If $k = 1$, A is a disk and so $M = S^3 = T(1,1)$. If $k = 2$, A is RP^2 and so $M = RP^3$. Thus we assume that $k \geq 3$. Suppose that M is orientable. Since $S(A)$ is a 1-sphere, it follows that the singular points of A lie in n-books. Thus $A \cap N$ is an n-book with its ends identified after a twist of $2\pi/m$ degrees for some integer M. Hence $A \cap \mathrm{Bd}\, N$ is a $J(n,m)$ curve on the boundary of a solid torus. We now proceed as in Chapter II to construct a standard residual set for M. Since the argument goes through exactly as in Chapter II, we find that M and $T(n,m)$ have homeomorphic standard residual sets. By Corollary 2.3.4, M and $T(n,m)$ are homeomorphic if M is orientable.

We now show that M is orientable. As in the proof of Lemma 4.1.3, we obtain the exact sequence:

$$0 \longrightarrow H_2(A) \longrightarrow Z \longrightarrow Z \longrightarrow H_1(A) \longrightarrow 0.$$

We will show that $H_1(A) = Z_k$ for some k. Thus $H_2(A) = 0$ and so M is orientable.

Let $B = N(S(A),A)$. Since $S(A)$ is a simple closed curve, $\pi_1(B) = (b:)$. Since $A - S(A)$ is an open disk $\pi_1(A - S(A))$ is trivial. Since $(A - S(A)) \cap B$ is a half open annulus, $\pi_1((A - S(A)) \cap B) = (a:)$. By the van Kampen Theorem and the observation that $a \simeq b^k$ in B and $a \simeq 0$ in $A - S(A)$, we obtain $\pi_1(A) = (b: b^k = 0) = Z_k$. Thus $H_1(A) = Z_k$.

Suppose that $S(A)$ is a 1-sphere and that $S'(A)$ is an s-leafed rose. By Lemma 4.1.6 and our assumption that each M_i has connected boundary, M_i is either a disk or a Moebius band. Let the M_i be arranged so that the first q are disks and the last $n - q$ are Moebius bands. By Lemma 4.1.4, we may assume that $\Phi|\mathrm{Bd}\ M_i = f^{k(i)}$. If $k(i) = 0$, $p(M_i)$ is either a 2-sphere or a copy of RP^2 attached to $S(A)$ at a, the join point of $S'(A)$. Suppose that $p(M_i) = RP^2$. Let $N = N(p(M_i),M)$ and $B = \mathrm{Bd}\ N$. If N is non-orientable, consider the exact homology sequence of the pair (N,B),

$$0 \longrightarrow H_2(B) \longrightarrow H_2(N) \longrightarrow H_2(N,B) \longrightarrow H_1(B)$$

$$\longrightarrow H_1(N) \longrightarrow H_1(N,B) \longrightarrow \hat{H}_0(B) \longrightarrow 0.$$

Since $H_2(N) = H_2(p(M_i)) = 0$, $H_2(B) = 0$. However, B less two disks embeds in a 2-sphere. Thus B is either one or two 2-spheres and so $H_2(B) \neq 0$. Thus N is orientable. By the proof of Lemma 3.3.1, we may change A into a residual set topologically $(A - p(\mathrm{Int}\ M_i)) \cup B \cup RP^2$, where B is an arc from a point of $(A - p(\mathrm{Int}\ M_i))$ to a point of

RP^2 such that Int B does not meet $(A - p(\text{Int } M_i)) \cup RP^2$. We will say that such an RP^2 has been "put on a stick". In the same way we put each $p(M_i)$ on a stick if $q + 1 \leq i \leq n$ and $k(i) = 0$. If $p(M_i)$ is a 2-sphere, the connectivity of Bd $N(A,M)$ implies that $p(M_i)$ is pierced by a 1-sphere of $S'(A)$. By an argument similar to that of Lemma 3.3.2, we may assume that one and only one of the 1-spheres of $S'(A)$ pierces $p(M_i)$ and that $p(M_i)$ and that 1-sphere may be put on a stick. Hence we obtain:

Lemma 4.2.2: Let A be a residual set for the closed 3-manifold M. Suppose that A has a presentation

$$P: S^1, S'(A), M_1, \ldots, M_n; \Phi,$$

with $M_1, \ldots, M_q$ disks and $M_{q+1}, \ldots, M_n$ Moebius bands. Suppose that $\Phi|\text{Bd } M_i = 0$ for $p+1 \leq i \leq q$ and $t+1 \leq i \leq n$. Then $M = M' \# M''$ where M' has a residual set A' with a presentation

$$P': S^1, S'(A'), N_1, \ldots, N_p, N_{q+1}, \ldots, N_t; \Phi',$$

with N_i homeomorphic to M_i and M'' is the connected sum of $q - p$ copies of $S^1 \times S^2$ or J and $n - t$ copies of RP^3.

With the above notation, if $k(i) = 1$ for some i between 1 and p, $p(N_i)$ is a disk. By modding it out we obtain a 3-manifold homeomorphic to M' that has a residual set whose singular set is a point. Since the main theorem of Chapter III classifies all closed 3-manifolds with this property we assume that $k(i) \neq 1$ for $1 \leq i \leq p$.

Suppose that $N = N(S(A'),M')$ is a solid Klein bottle. Let $C_i = p(N_i) \cap \mathrm{Bd}\, N$. Clearly C_i is a simple closed curve for all i. By [12], there are exactly four isotopy classes of simple, closed, orientation preserving paths and exactly four isotopy classes of simple, closed, orientation reversing paths on Bd N. Let 0, a, -a and b be representatives of the orientation preserving classes and p_1, p_2, $-p_1$ and $-p_2$ be representatives of the orientation reversing classes. These may be pictured as in Figure 4.1.

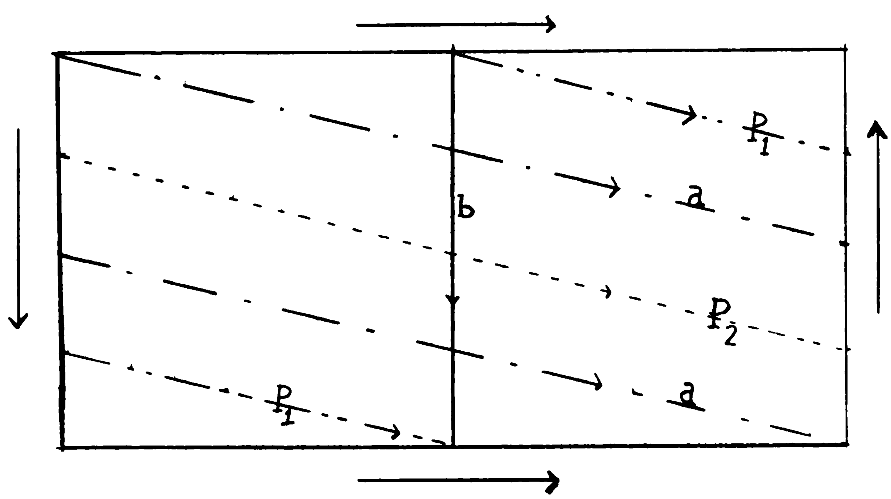

Figure 4.1.

If $C_i \simeq 0$ or b, $k(i) = 0$, a contradiction. If $C_i \simeq \pm p_1$ or $\pm p_2$ for $1 \leq i \leq p$, $k(i) = 1$, a contradiction. Thus $C_i \simeq \pm a$ for $1 \leq i \leq p$. By reversing the orientation of N_i, we may assume that $C_i \simeq a$ for $1 \leq i \leq p$. Likewise $C_i \simeq \pm a$, $\pm p_1$ or $\pm p_2$ for $q+1 \leq i \leq t$. Suppose that $C_t \simeq \pm p_1$. By an isotopy we may assume that $C_t = \pm p_1$. Since Bd N - C_t is a Moebius band, no other

$C_i \simeq \pm p_1$. In the same way, if $C_t \simeq \pm p_1$ and $C_{t-1} \simeq \pm p_2$, no other C_i may be isotopic to $\pm p_1$ or $\pm p_2$. Thus we obtain three cases:

CASE 1: $C_i \simeq \pm a$ for all i,

CASE 2: $C_i \simeq \pm a$ for $i \neq t$ and $C_t \simeq \pm p_1$ and

CASE 3: $C_i \simeq \pm a$ for $i \neq t, t-1$, $C_t \simeq \pm p_1$ and $C_{t-1} \simeq \pm p_2$.

In any case there are two C_i, say C and C', that bound an annulus E on Bd N, with $C_i \subset E$ if $C_i \simeq \pm a$. If there is only one $C_i \simeq \pm a$, set $C_1 = E$. Notice that $E' = N(E, \text{Bd } N)$ is an annulus.

CASE 1: Since Bd $N - E'$ is two open Moebius bands and Bd $N - E'$ less some disks embeds in a 2-sphere, we have a contradiction. It is necessary to remove the disks since an arc of $S'(A') - S(A')$ may intersect Bd $N - E'$.

CASE 2: A similar argument excludes this case.

CASE 3: Let $P = \text{Cl}(p(N_t) - N)$. P is a Moebius band. Clearly $N(P,M')$ is either a solid torus or a solid Klein bottle. Since Bd $N(C_t,N) - N(C_t,\text{Bd } N)$ is a Moebius band embedded in Bd $N(P,M')$, $N(P,M')$ is a solid Klein bottle. However, Bd $N(P,M')$ - Bd $N(C_t,N)$ is a Moebius band that embeds in a 2-sphere, excluding case 3.

<u>Lemma 4.2.3</u>: Let A be a residual set of a closed 3-manifold M with $S(A)$ a simple closed curve. Then $N(S(A),M)$ is orientable.

<u>Proof</u>: As in Lemma 4.2.2, we write $M = M' \# M''$. Notice that $N(S(A),M)$ is homeomorphic to $N(S(A'),M')$. However, the above argument implies that $N(A(S'),M)$ is orientable.

<u>Remark 4.2.4</u>: Our assumption that the M_i have connected boundary is essential in Lemma 4.2.3. Again $M = RP^2 \times S^1$ gives the counter example, for $N(S(A),M)$ is a solid Klein bottle.

<u>Lemma 4.2.5</u>: Suppose that A is a residual set of the closed 3-manifold M and that A has a presentation P

$$P\colon S^1,\ S^1,\ M_1,\ M_2;\ \Phi.$$

Then $M = RP^3 \# T(n,m)$ or $M = RP^3 \# J$.

<u>Proof</u>: Since $2 = 1 + k_1 + k_2$ by Lemma 4.1.3, we may assume that M_1 is a disk and M_2 is a Moebius band. If $\Phi|\text{Bd } M_1$ is the constant map, $p(M_1)$ is a 2-sphere and so $\text{Bd } N(A,M)$ is not connected. If $\Phi|\text{Bd } M_2$ is the constant map, $p(M_2)$ may be put on a stick. Then M has a residual set $RP^2 \vee p(M_1)$ and so $M = RP^3 \# T(n,m)$ or $M = RP^3 \# J$ by Theorem 4.2.1. Thus we assume that Φ is not the constant map on either boundary component. By Lemma 4.2.4, we may assume that $\Phi|\text{Bd } M_1 = f^k$ and $\Phi|\text{Bd } M_2 = f^h$ with $k \neq 0 \neq h$. By Lemma 4.2.3, $N(S(A),M)$ is a solid torus T. Let $C = (\text{Bd } T) \cap p(M_1)$ and $D = (\text{Bd } T) \cap p(M_2)$. Now C is a $J(k,m)$ curve and D is a $J(h,m')$ curve. Since C does not meet D, after changing the orientation of M_1 if

necessary, $k = h$ and $m = m'$. Thus $C \cup D$ bounds an annulus E on Bd T. Swell up $S(A)$ to a singular solid torus T' with boundary $((p(M_1) \cup p(M_2)) \cap T) \cup E$. Thus all the singularities of T' lie on $S(A)$. Collapse T' from $p(M_2) \cap T$ onto $(p(M_1) \cap T) \cup E$. Mod out an arc J on $p(M_1) \cap T$, with one end point x on $p(M_1) \cap \mathrm{Bd}\, T$ and the other end point a on $S(A)$, such that Int $J \subset p(\mathrm{Int}\, M_1) \cap \mathrm{Int}\, T$. Let A' be the resulting residual set. A' has a presentation

$$P': \; S^1 \vee S^1,\ S^1 \vee S^1,\ B_1,\ B_2,\ N;\ \Phi',$$

where the B_i are disks and N is a Moebius band. Moreover, $\Phi'|\mathrm{Bd}\, B_1 = f$, $\Phi'|\mathrm{Bd}\, B_2 = g^k f$ and $\Phi'|\mathrm{Bd}\, N = f$, where f is the map around one of the leaves and g is the map around the other leaf. Thus $p(B_1)$ is a disk. Mod out $p(B_1)$, obtaining a residual set A'' with a presentation

$$P'': \; S^1,\ S^1,\ N_1,\ N_2;\ \Phi'',$$

where N_1 is a disk with $\Phi''|\mathrm{Bd}\, N_1 = g^k$ and N_2 is a Moebius band with $\Phi''|\mathrm{Bd}\, N_2 = 0$. Hence A'' is the one-point union of RP^2 and the residual set for a $T(k,m)$ manifold. Thus the Lemma is established.

<u>Theorem 4.2.6</u>: Let A be a residual set for a closed 3-manifold M. Suppose that A has a presentation

$$P: \; S^1,\ S'(A),\ M_1,\ \ldots,\ M_n;\ \Phi,$$

then either

i) $M = (\#_1^{q-1}\, S^1 \times S^2) \# (\#_1^{n-q}\, RP^3) \# (T(n,m))$ or

ii) $M = (\#_1^{q-1} J) \# (\#_1^{n-q} RP^3) \# T(n,m)$.

<u>Proof</u>: By Theorem 4.1.4, $\Phi|\text{Bd } M_i = f^{k(i)}$. Since $N(S(A),M)$ is a solid torus, the argument of Lemma 4.2.5 implies that either $k(i) = \pm k$ or $\Phi|\text{Bd } M_i$ is the constant map. By Lemma 4.2.2, we may write $M = M' \# M''$, where M' and M'' are as in the lemma. Let A' be the residual set for M'. As in Lemma 4.2.2, A' has a presentation

$$P': S^1, S'(A'), N_1, \ldots, N_p, N_{q+1}, \ldots, N_t; \Phi'.$$

Note that $\Phi'|\text{Bd } N_i = f^{\pm k}$ for all i. Set $C_i = p(N_i) \cap \text{Bd } N(S(A'),M')$. Then there are two of the C_i, say C_1 and C_2, that bound an annulus E on $\text{Bd } N(S(A)',M')$ with $C_i \subset E$ for all i. As in Lemma 4.2.5, there is a singular torus T' in $N(S(A'), M')$ bounded by $((p(N_1) \cup p(N_2)) \cap N(S(A'), M')) \cup E$. Collapse T' from $p(N_1) \cap N(S(A'),M')$ onto $E \cup (p(N_2) \cap N(S(A'),M'))$. Let J be as in Lemma 4.2.5. Mod out J. If there are any arcs in the resulting residual set that do not form a simple closed curve move their end points, by E-moves, to the image of the join point. We thus obtain a residual set A'' with a presentation

$$P'': S^1 \vee S^1, S'(A''), Q, Q_1, \ldots, Q_p, Q_{q+1}, \ldots, Q_t; \Phi'',$$

where Q is a disk with $\Phi''|\text{Bd } Q = g^k f$ and Q_i is homeomorphic to N_i for all i with $\Phi''|\text{Bd } Q_i = f$. Since Q_1 is a disk, $p(Q_1)$ is a disk. By modding out $p(Q_1)$ we obtain a residual set A''' with a presentation

$$p''': S^1, S'(A'''), D, D_2, \ldots, D_p, D_{q+1}, \ldots, D_t; \Phi''',$$

with D homeomorphic to Q and D_i homeomorphic to Q_i, and $\Phi'''|\mathrm{Bd}\, D = g^k$ amd $\Phi'''|\mathrm{Bd}\, D_i = 0$. The theorem follows.

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