ON MINIMIZATION OF SOME NON-SMOOTH CONVEX FUNCTIONALS ARISING IN MICROMAGNETICS

By

Hongli Gao

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ABSTRACT

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This thesis is motivated by studying the properties of ferromagnetic materials using the Landau-Lifshitz theory of micromagnetics. In this theory the state of a ferromagnetic material is described by the magnetization vector $\mathbf{m}$ in terms of a total micromagnetic energy that consists of several competing sub-energies: exchange energy, anisotropy energy, external interaction energy and magnetostatic energy. For large ferromagnetic materials and under some limiting regimes of the model, the exchange energy can be negligible and the total energy becomes a reduced model. Our investigations focus on the study of such a reduced model of Landau-Lifshitz theory.

The primary focus of the thesis includes two parts: the minimization (static) study and the evolution (dynamic) study. We investigate a new method for the existence of minimizers of the reduced micromagnetic energy based on a duality method. In this method, the reduced micromagnetic energy is closely related to a convex functional (the dual functional) on the curl-free vector functions. Our minimization and dynamics studies are based on the study of the minimization and gradient flow of this dual functional. Much of the thesis is focused on the minimization problem of two special cases: soft case and uniaxial case on the annulus domain; in particular, in the soft case, for some range of the parameter, the energy minimizers of the original micromagnetic energy are constructed through the Euler-Lagrange equation of the dual functional using the characteristics method for a reduced Eikonal type equation. The second direction of our study of this thesis is an attempt to obtain certain reasonable

dynamic process for the evolution of $\mathbf{m}$, where the asymptotic behavior of the gradient flow of the reduced energy functional is investigated.

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Introduction

1.1 Landau-Lifshitz theory of micromagnetics

Our research is based on the well-known Landau-Lifshitz theory of micromagnetics; see Brailsford [7], Brown [11] and Landau et al [36]. Under this theory, observable magnetic properties of a ferromagnetic material are described by a magnetization vector $\mathbf{m}$ through a formulation of a total micromagnetic energy including several competing energies:

$$\mathcal{E}(\mathbf{m}) = \frac{\alpha}{2}\int_\Omega |\nabla\mathbf{m}(x)|^2\,dx + \int_\Omega \varphi(\mathbf{m}(x))\,dx - \int_\Omega H(x)\cdot\mathbf{m}(x)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |F_{\mathbf{m}}(z)|^2\,dz, \quad (1.1)$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n = 2, 3$ in practice) occupied by the ferromagnetic material, $F_{\mathbf{m}} \in L^2(\mathbb{R}^n;\mathbb{R}^n)$ is a magnetic field induced by $\mathbf{m}$ on the whole $\mathbb{R}^n$ that is determined by the simplified Maxwell's equations:

$$\operatorname{curl} F_{\mathbf{m}} = 0, \quad \operatorname{div}(-F_{\mathbf{m}} + \mathbf{m}\chi_\Omega) = 0 \quad \text{in } \mathbb{R}^n, \quad (1.2)$$

$\varphi(\mathbf{m})$ is a given function representing the anisotropy energy density that is minimized along certain preferred crystallographic directions, and $H(x)$ is a given vector function representing the external applied field. Here $\alpha > 0$ is a material constant. Under this theory, when below certain critical temperature, the magnetization $\mathbf{m}$ should have constant magnitude:

$$|\mathbf{m}(x)| = M_s, \tag{1.3}$$

where $M_s > 0$ is a saturation constant.

The first term in the energy $\mathcal{E}(\mathbf{m})$ is called the *exchange energy*, the second term the *anisotropy energy*, the third term the *external interaction energy*, and the last term is a non-local energy and is usually called the *magnetostatic energy*. The non-locality and non-convexity of the total energy $\mathcal{E}(\mathbf{m})$ not only present a major and challenging mathematical problem but also provide a concrete example for some other physical problems of a similar nature.

The Landau-Lifshitz model has been at the center of much of current active research; see the survey by Kruzìk and Prohl [35]. On one hand, the static Landau-Lifshitz theory is postulated by minimization of energy $\mathcal{E}(\mathbf{m})$ under the saturation condition (1.3). On the other hand, the dynamic theory for time evolution of magnetization $\mathbf{m}$ is governed by the Landau-Lifshitz equation:

$$\partial_t \mathbf{m} = \gamma \mathbf{m} \times \mathbf{F}_{\text{eff}} + \beta \frac{\gamma}{|\mathbf{m}|} \mathbf{m} \times (\mathbf{m} \times \mathbf{F}_{\text{eff}}) \tag{1.4}$$

on $\Omega \times [0, \infty)$, where $\gamma < 0$ is the electron gyromagnetic ratio, $\beta > 0$ is the Landau-Lifshitz phenomenological damping parameter, and F_{eff} is the total *effective magnetic field* defined by the functional derivative of $\mathcal{E}(\mathbf{m})$ as

$$\mathbf{F}_{\text{eff}} = -\frac{\partial \mathcal{E}}{\partial \mathbf{m}} = \alpha \Delta \mathbf{m} - \varphi'(\mathbf{m}) + H(x) + F_{\mathbf{m}}.$$

This equation is also equivalent to the so-called Landau-Lifshitz-Gilbert equation. Many results, such as existence ([2, 3, 16, 17, 19, 30, 32, 50]), stability ([14]) and asymptotic behavior have been well established for the Landau-Lifshitz equations that include the so-called exchange energy (when $\alpha > 0$). Such exchange energy provides the magnetization $\mathbf{m}$ with $\mathbf{m} \in L^\infty((0,\infty); H^1(\Omega))$ which allows us to have some compactness and stability that are needed for using the standard methods.

1.2 Reduced model of Landau-Lifshitz theory

For large ferromagnetic materials, it has been justified by DeSimone [22] (see also James and Kinderlehrer [33]) that the total micromagnetic energy can be approximated by the following reduced form (ignoring the exchange energy):

$$I(\mathbf{m}) = \int_\Omega \varphi(\mathbf{m}(x))\,dx - \int_\Omega H(x)\cdot \mathbf{m}(x)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |F_{\mathbf{m}}(z)|^2\,dz. \tag{1.5}$$

Throughout this thesis, we assume the magnetization $\mathbf{m}$ has unit length:

$$|\mathbf{m}(x)| = 1.$$

Due to the saturation constraint $|\mathbf{m}| = 1$ and the anisotropy energy, the existence of minimizers of this energy $I(\mathbf{m})$ is not guaranteed; so more careful analysis should be carried out.

A new method for minimization of this functional $I(\mathbf{m})$ has been introduced by Pedregal and Yan [43, 44] based on the idea of duality; see also [33]. The main idea of this method is motivated by rewriting the magnetostatic energy as

$$\frac{1}{2}\int_{\mathbb{R}^n} |F_{\mathbf{m}}|^2 = \min_{\operatorname{div} G=0} \frac{1}{2}\int_{\mathbb{R}^n} |\mathbf{m}\chi_\Omega - G|^2$$

by (1.2), where the minimum is taken over all divergence-free fields G in $L^2(\mathbb{R}^n;\mathbb{R}^n)$. In [44] it has been proved that

$$\inf_{\substack{\mathbf{m}\in L^2(\Omega;\mathbb{R}^n)\\ |\mathbf{m}(x)|=1}} I(\mathbf{m}) = -\min_{\substack{F\in L^2(\mathbb{R}^n;\mathbb{R}^n)\\ \operatorname{curl} F=0}} J^*(F),$$

where $J^*(F)$ is a convex functional defined by

$$J^*(F) = \int_\Omega \Phi\Big(F(x)+H(x)\Big)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |F(x)|^2\,dx \tag{1.6}$$

with a convex function Φ defined by

$$\Phi(\xi) = \max_{h\in\mathbb{R}^n,|h|=1} (\xi\cdot h - \varphi(h)) \quad (\xi\in\mathbb{R}^n).$$

It is easily seen that J^* is strictly convex on $\mathbb{V} = L^2(\mathbb{R}^n;\mathbb{R}^n)\cap\{\operatorname{curl} F = 0\}$. The existence of the minimizer $\bar{F}\in\mathbb{V}$ is guaranteed by the general theory. Using this unique minimizer $\bar{F}$ of $J^*(F)$, a necessary and sufficient condition for the existence of minimizers of $I(\mathbf{m})$ has been given in [44, 52]. In [52], a notion of *generalized minimizers* of $I(\mathbf{m})$ has been also defined. A similar dual formulation to the functional $J^*(F)$ has also been used by Melcher [38] to approach some regularity problems for thin films.

We follow this line of investigations to study some concrete problems regarding the minimization of functional $I(\mathbf{m})$. Our primary results will be the construction of minimizers of some special energy $I(\mathbf{m})$ on an annulus domain Ω, which may have some physical applications in studying magnetic nanorings [15].

The evolution model based on the reduced energy $I(\mathbf{m})$ leads to a corresponding reduced Landau-Lifshitz equation (1.4) with $\alpha = 0$ and has been recently studied in [26, 27, 53, 54]. In this case, one only has $\mathbf{m} \in L^\infty((0,\infty); L^\infty(\Omega))$ which leads to the lack of compactness and stability. Yan [54] discussed stability and asymptotic behaviors of solutions for a degenerate Landau-Lifshitz equation in micromagnetics involving only the nonlocal magnetostatic energy. He showed that the Cauchy problems for such an equation are not stable under the weak$*$ convergence of initial data. For the asymptotic behaviors of weak solutions, he established an estimate on the weak $*\,\omega$-limit sets that is valid for all initial data satisfying the saturation condition. Deng and Yan [27] have presented a new method for the existence of global weak solution to the reduced Landau-Lifshitz equation. In addition, they also established higher time regularity when the initial value $\mathbf{m}_0$ is constant. They studied the weak ω-limit sets for the soft case and the asymptotic behaviors in the case when Ω is ellipsoid and initial value $\mathbf{m}_0$ is constant.

In attempt to obtain other reasonable dynamic processes for the evolution of $\mathbf{m}$, we study the gradient flow of the convex functional $\mathcal{L} : L^2(\mathbb{R}^n;\mathbb{R}^n) \to \bar{\mathbb{R}} := (-\infty, \infty]$,

$$\mathcal{L}(F) = \begin{cases} \displaystyle\int_\Omega \Phi\Big(F(x) + H(x)\Big)dx + \frac{1}{2}\int_{\mathbb{R}^n} |F(x)|^2\,dx, & F \in \mathbb{V}, \\ +\infty, & F \in L^2(\mathbb{R}^n;\mathbb{R}^n)\setminus\mathbb{V}. \end{cases} \tag{1.7}$$

1.3 Main results

Our results consist mainly of two parts: minimization of the functional $\mathcal{L}$ and the asymptotic behavior of the gradient flow.

Since every $F(x) \in \mathbb{V}$ can be written as $F(x) = \nabla u(x)$, where $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$, we introduce the following variational functional:

$$L(u) = L_\Omega(u) \equiv \int_\Omega \Phi(\nabla u(x) + H(x))\, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(x)|^2\, dx \tag{1.8}$$

for all $u \in H^1_{loc}(\mathbb{R}^n)$ with $\nabla u \in L^2(\mathbb{R}^n; \mathbb{R}^n)$. For simplicity and when there is no confusion, we simply use $L(u)$ to denote $L_\Omega(u)$.

Note that $L(u + c) = L(u)$ for all constants $c \in \mathbb{R}$. To fix the idea, we define the linear space $\mathcal{X}$ by

$$\mathcal{X} = \left\{ u \in H^1_{loc}(\mathbb{R}^n) \;\middle|\; \nabla u \in L^2(\mathbb{R}^n; \mathbb{R}^n),\ \int_{\partial\Omega} \Gamma u\, dS = 0 \right\}, \tag{1.9}$$

where $\Gamma u = u|_{\partial\Omega}$ is the well-defined trace in $H^{1/2}(\partial\Omega)$ (see [1]). It is easily seen that L is strictly convex on $\mathcal{X}$. Hence L has a *unique minimizer* on $\mathcal{X}$; we denote this unique minimizer by $\bar{u} = \bar{v}\chi_\Omega + \bar{w}\chi_{\Omega^c}$. Certainly, this function $\bar{u}$ depends on the domain Ω, the anisotropy function φ (in terms of function Φ), and the applied field $H(x)$. Pedregal and Yan [44] have shown that $\bar{u}$ is uniquely determined by its boundary data $\bar{g} = \bar{u}|_{\partial\Omega}$ and, in particular, that $\bar{w}$ is harmonic on Ω^c. They have also established a necessary and sufficient condition for the existence of minimizers of energy $I(\mathbf{m})$ in terms of the unique minimizer $\bar{u} = \bar{v}\chi_\Omega + \bar{w}\chi_{\Omega^c}$ of functional $L(u)$. For example, they established the following theorem.

Theorem 1.3.1. *Let $\bar{u} = \bar{v}\chi_\Omega + \bar{w}\chi_{\Omega^c} \in \mathcal{X}$ be the unique minimizer of functional L defined above. Then, the energy $I(\mathbf{m})$ has a minimizer if and only if there exists a function $G \in$*

$L^2(\Omega;\mathbb{R}^n)$ *that satisfies*

$$\begin{cases} \operatorname{div}(\nabla\bar{u}+G\chi_\Omega)=0 & \textit{in } \mathbb{R}^n, \\ G(x)\in\Sigma(\nabla\bar{v}(x)+H(x)) & \textit{a.e. } x\in\Omega. \end{cases} \tag{1.10}$$

Here, $\Sigma(\eta)=\{h\in\mathbb{S}^{n-1} \mid \Phi(\eta)=\eta\cdot h-\varphi(h)\}$. *In addition any* $\bar{\mathbf{m}}\in L^\infty(\Omega;\mathbf{S}^{n-1})$ *satisfying* $\bar{\mathbf{m}}(x)=G(x)$ *a.e. on* Ω *is a minimizer of energy* I.

We focus on how to find the minimizer $\bar{u}$ of $L(u)$. The following result has been proved in [44]. We provide in Chapter 3 a different proof for it.

Theorem 1.3.2. (Chapter 3, section 3.1.2) *A function* $\bar{u}=\bar{v}\chi_\Omega+\bar{w}\chi_{\Omega^c}\in\mathcal{X}$ *is a minimizer of* $L(u)$ *if and only if there exists a vector function* $G\in L^2(\Omega;\mathbb{R}^n)$ *such that*

$$\begin{cases} \operatorname{div}(\nabla\bar{u}+G\chi_\Omega)=0 & \textit{in } \mathbb{R}^n, \\ G\in\partial\Phi(\nabla\bar{v}+H(x)) & \textit{a.e. } \Omega, \end{cases} \tag{1.11}$$

where $\partial\Phi(\xi)$ *denotes the sub-differential of* Φ *at* ξ. *Any such function* G *is called a* generalized minimizer *of the functional* $I(\mathbf{m})$.

Depending on the different anisotropy density functions φ, the functional $L(u)$ takes a different form in terms of the convex function Φ defined above.

We say that the material is in the *soft case* if $\varphi\equiv 0$; in this case $\Phi(\xi)=|\xi|$ on $\xi\in\mathbb{R}^n$. We say the material is in the *uniaxial case* if $\varphi(h)=\beta(1-|h\cdot\mathbf{e}|)$, where $\beta>0$ is a constant and $\mathbf{e}\in\mathbb{R}^n$ is a given unit vector; in this case, the function Φ can be explicitly computed and the sub-differential set $\partial\Phi(\xi)$ has a special structure (see below), which affects the existence of the solution to the problem (1.11).

We are also interested in the dependence of $\bar{u}$ on the domain Ω. When a small region E is removed from the domain Ω, we want to study how the minimizer of $L_{\Omega\setminus E}(u)$ (or ultimately the minimizers of $I(\mathbf{m})$) should change. In particular, can the minimizers of $L_\Omega(u)$ and $L_{\Omega\setminus E}(u)$ be the same on $\Omega \setminus E$?

We have the following result.

Theorem 1.3.3. (Chapter 3, Section 3.1.3) *Let $\bar{u} = \bar{v}\chi_\Omega + \bar{w}\chi_{\Omega^c}$ be the minimizer of the functional $L_\Omega(u)$ and $E \subset\subset \Omega$. Define $\tilde{w}$ by*

$$\begin{cases} \Delta \tilde{w} = 0 \quad \textit{in } E, \\ \tilde{w} = \bar{u} \quad \textit{on } \partial E. \end{cases} \tag{1.12}$$

Suppose that there exists $\tilde{G} \in L^2(\Omega \setminus E; \mathbb{R}^n)$ satisfying

$$\begin{cases} \operatorname{div}(\nabla \bar{v} + \tilde{G}) = 0 & \textit{in } \Omega \setminus E, \\ (\tilde{G} + \nabla \bar{v}) \cdot \nu = \frac{\partial \bar{w}}{\partial \nu} & \textit{on } \partial\Omega, \\ (\tilde{G} + \nabla \bar{v}) \cdot \nu = \frac{\partial \tilde{w}}{\partial \nu} & \textit{on } \partial E, \\ \tilde{G} \in \partial\Phi(\nabla \bar{v} + H(x)) & \textit{a.e. } \Omega \setminus E. \end{cases} \tag{1.13}$$

Then $\tilde{u} = \bar{u}\chi_{\mathbb{R}^n\setminus E} + \tilde{w}\chi_E$ is the minimizer of $L_{\Omega\setminus E}(u)$.

In Chapter 4, we apply this result to the minimization of $I(\mathbf{m})$ and $L(u)$ in the soft case (when $\Phi(\xi) = |\xi|$) with constant applied field $H = \lambda \mathbf{e}_1$ for an annulus $\Omega = \{x \in \mathbb{R}^n \mid a < |x| < 1\}$, where $0 < a < 1$. Such condition leads to the study of Lipschitz solutions to the boundary value problem of the Eikonal equation below:

$$\begin{cases} |\nabla\psi(s,t)| = t^{n-2} & \text{in } a^2 < s^2+t^2 < 1,\ t>0, \\ \psi(s,t) = 0 & \text{on } s^2+t^2 = a^2,\ t \geq 0, \\ \psi(s,t) = \frac{n\lambda}{n-1}t^{n-1} & \text{on } s^2+t^2=1,\ t\geq 0, \\ \psi(s,t) = 0 & \text{on } t=0,\ a \leq |s| \leq 1. \end{cases} \tag{1.14}$$

We prove that the boundary value problem (1.14) has a Lipschitz solution if and only if

$$0 \leq \lambda \leq \frac{1}{n}(1-a^{n-1}), \tag{1.15}$$

and that, in this case, we can construct infinitely many Lipschitz solutions. This construction is the primary goal of Chapter 4 of the thesis. We summarize the result as follows:

Theorem 1.3.4. (Chapter 4, Section 4.1) *If $0 \leq \lambda \leq \frac{1}{n}(1-a^{n-1})$, then the problem* (1.14) *has infinitely many Lipschitz solutions $\psi(s,t)$, constructed in Theorems 4.4.6 and 4.4.9 in Chapter 4. In this case, the minimizers $\bar{\mathbf{m}}$ of $I(\mathbf{m})$ obtained from the constructed solution ψ will be the constant $\pm\mathbf{e}_1$ in certain subdomains $\Omega_0 = \{(x_1,x') \in \Omega \mid (|x_1|,|x'|) \in \mathcal{Z}_0\}$ away from the boundary $\partial\Omega$.*

The second direction of our study is an attempt to obtain certain reasonable dynamic process for the evolution of $\mathbf{m}$. We study the gradient flow of the convex functional $\mathcal{L}$ on $L^2(\mathbb{R}^n;\mathbb{R}^n)$:

$$\dot{F}(t) \in -\partial\mathcal{L}(F(t)) \ \ (t>0), \quad F(0) = F_0, \tag{1.16}$$

where the subdifferential $\partial\mathcal{L}(F)$ is given by

$$\partial\mathcal{L}(F) = F + P_{\mathbb{V}}\big(\partial\Phi(F(x) + H(x))\chi_\Omega\big) + \mathbb{V}^\perp \quad \forall\, F \in \mathbb{V},$$

with $P_{\mathbb{V}}$ being the orthogonal projection on $\mathbb{V}$.

The existence of the gradient flow is standard; see [10, 28]. For the asymptotic behavior as $t \to \infty$, Bruck [8] has shown that in general the gradient flow of a strictly convex functional converges weakly to a minimizer, while Baillon [9] has given a counterexample showing that in general the gradient flow does not strongly converge to a minimizer.

Due to the fact that the minimizer of $\mathcal{L}$ is harmonic outside Ω, we obtain the strong convergence of the gradient flow for the function L outside of Ω.

Theorem 1.3.5. (Chapter 3, section 3.2) *For each $F_0 \in \mathbb{V}$ there exists a unique solution to the gradient flow* (1.16)*. Furthermore, $F(t) \rightharpoonup \bar{F}$ as $t \to \infty$ and $F(t) \to \bar{F}$ in $L^2(\tilde{\Omega}^c; \mathbb{R}^n)$ for all compact sets $\tilde{\Omega}$ containing $\bar{\Omega}$.*

Note that the gradient flow (1.16) determines a (nonunique) time-dependent vector function $\mathbf{m}(t) = \mathbf{m}(x,t)$ with the property

$$\begin{cases} \operatorname{div}(\dot{F}(t) + F(t) + \mathbf{m}(t)\chi_\Omega) = 0 & (t > 0), \\ \mathbf{m}(x,t) \in \partial\Phi(F(x,t) + H(x)) & a.e.\, x \in \Omega, \;\; t > 0. \end{cases}$$

This, along a subsequence $t_k \to \infty$, determines a vector function $\bar{\mathbf{m}}(x)$ satisfying $\operatorname{div}(\bar{F} + \bar{\mathbf{m}}\chi_\Omega) = 0$. If, in addition, one has that $F(t) \to \bar{F}$ strongly in $L^2(\mathbb{R}^n; \mathbb{R}^n)$ then it would follow that $\bar{\mathbf{m}}(x) \in \partial\Phi(\bar{F}(x) + H(x))$ for a.e. $x \in \Omega$ and thus $\bar{\mathbf{m}}(x)$ would be a generalized minimizer for energy $I(\mathbf{m})$.

At the end of Chapter 3, we will discuss a particular example for the soft case with a constant applied field H on the unit ball in $\mathbb{R}^3$. In this case, the gradient flow can be expressed as an ordinary differential equation, where we have the strong convergence for gradient flow both inside and outside of Ω.

Chapter 2

Preliminaries

In this chapter, we review some preliminary definitions and results in order to present our results in Chapter 3 and Chapter 4.

2.1 Notations and definitions

Throughout this thesis, we use $\mathbb{H}$ to denote the real Hilbert space $L^2(\mathbb{R}^n;\mathbb{R}^n)$ with usual L^2-inner product $F \cdot G$ and norm $||\cdot||$. When using the convergence notation in this thesis, " $\to$ " denotes the strong convergence in $L^2(U;\mathbb{R}^n)$; " $\rightharpoonup$ " denotes the weak convergence in $L^2(U;\mathbb{R}^n)$ provided

$$\int_U u_k v\,dx \to \int_U uv\,dx \quad \text{as} \quad k \to \infty.$$

for each $v \in L^2(U;\mathbb{R}^n)$. Here $U \subseteq \mathbb{R}^n$ is any set in $\mathbb{R}^n$.

Let $\mathbb{V}$ be the subspace of $\mathbb{H}$ defined by

$$\mathbb{V} = \{F \in \mathbb{H} \mid \operatorname{curl} F = 0 \ \text{ in the sense of distributions on } \mathbb{R}^n\}.$$

Then each element $F \in \mathbb{V}$ can be represented as $F = \nabla u$ for some function $u \in H^1_{loc}(\mathbb{R}^n)$. Moreover the orthogonal complement of $\mathbb{V}$ is exactly given by

$$\mathbb{V}^\perp = \{G \in \mathbb{H} \mid \operatorname{div} G = 0 \ \text{ in the sense of distributions on } \mathbb{R}^n\};$$

that is, $G \in \mathbb{V}$ if and only if

$$\int_{\mathbb{R}^n} G(x) \cdot \nabla \zeta(x)\, dx = 0 \quad \forall\, \zeta \in C_c^\infty(\mathbb{R}^n).$$

We refer to the books [47, 49] for the proof of these results.

We next review some notations and definitions in convex analysis (see, e.g., [6, 45]). Let $p \colon \mathbb{H} \to \bar{\mathbb{R}}$ be a given functional on $\mathbb{H}$.

Definition 2.1.1. *The* **(convex) conjugate** *or the* **Legendre transform** p^* *of* p *and the* convexification $p^\#$ *of* p *are, respectively, defined by*

$$p^*(G) = \sup_{F \in \mathbb{H}} \{F \cdot G - p(F)\}, \quad p^\#(G) = \sup_{F \in \mathbb{H}} \{F \cdot G - p^*(F)\};$$

that is, $p^\# = (p^*)^*$. *Both are convex functionals on* $\mathbb{H}$ *and it also follows that* $p^* = (p^\#)^*$.

Definition 2.1.2. *The* **sub-differential** *of* p *at* $G \in \mathbb{H}$ *is defined to be the set*

$$\partial p(G) = \{F \in \mathbb{H} \mid p(A) \geq p(G) + F \cdot (A - G) \ \ \forall\ A \in \mathbb{H}\}. \tag{2.1}$$

Note that $\partial p(G) \neq \emptyset$ only if $p(G) < \infty$, and that if $\partial p(G) \neq \emptyset$ then it is a convex subset of $\mathbb{H}$. Moreover, $0 \in \partial p(G)$ if and only if $p(G)$ is the absolute minimum of p on $\mathbb{H}$.

We also have the following property:

$$F \in \partial p(G) \ \text{ if and only if } \ p^*(F) = F \cdot G - p(G). \tag{2.2}$$

Moreover, if q is a convex functional on $\mathbb{H}$, then

$$\|F\| \leq \sup_{\|A\|\leq 1} \{q(G+A) - q(G)\} \quad \forall\, F \in \partial q(G). \tag{2.3}$$

2.2 The duality method for micromagnetics

Assume that $\varphi\colon \mathbb{S}^{n-1} \to \mathbb{R}$ is a given function representing the anisotropy energy density, Ω is a given bounded domain with piece-wise smooth boundary occupied by the ferromagnetic material, and $H \in L^2(\Omega;\mathbb{R}^n)$ is a given applied magnetic field. Consider the (reduced) micromagnetic energy introduced above

$$I(\mathbf{m}) = \int_\Omega [\varphi(\mathbf{m}(x)) - H(x)\cdot \mathbf{m}(x)]\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |F_{\mathbf{m}}|^2\,dx,$$

where $F_{\mathbf{m}} \in \mathbb{V}$ is defined by Maxwell's equation (1.2) above.

Note that, by (1.2), the magnetostatic energy can be expressed as a variational problem

$$\frac{1}{2}\int_{\mathbb{R}^n} |F_{\mathbf{m}}|^2 = \min_{G\in\mathbb{V}^\perp} \frac{1}{2}\int_{\mathbb{R}^n} |\mathbf{m}\chi_\Omega - G|^2.$$

Introduce an auxiliary functional $\mathcal{A}(\mathbf{m},G)$ for $\mathbf{m}\in L^\infty(\Omega;\mathbb{S}^{n-1}), G\in\mathbb{H} = L^2(\mathbb{R}^n;\mathbb{R}^n)$ by

$$\mathcal{A}(\mathbf{m},G) = \int_\Omega \varphi(\mathbf{m}) - \int_\Omega H(x)\cdot\mathbf{m} + \frac{1}{2}\int_{\mathbb{R}^n} |\mathbf{m}\chi_\Omega - G|^2, \tag{2.4}$$

which leads to $I(\mathbf{m}) = \min\limits_{G\in\mathbb{V}^\perp} \mathcal{A}(\mathbf{m},G)$. Therefore

$$\inf_{|\mathbf{m}|=1} I(\mathbf{m}) = \inf_{|\mathbf{m}|=1}\left[\inf_{G\in\mathbb{V}^\perp} \mathcal{A}(\mathbf{m},G)\right] = \inf_{G\in\mathbb{V}^\perp}\left[\inf_{|\mathbf{m}|=1} \mathcal{A}(\mathbf{m},G)\right].$$

Now, for fixed $G \in \mathbb{H}$, define

$$J(G) = \inf_{|\mathbf{m}|=1} \mathcal{A}(\mathbf{m}, G),$$

where the infimum (in fact a minimum) is taken over all $\mathbf{m} \in L^\infty(\Omega; \mathbf{S}^{n-1})$. Then one easily has

$$\inf_{|\mathbf{m}|=1} I(\mathbf{m}) = \inf_{G \in \mathbb{V}^\perp} J(G).$$

An elementary computation shows that

$$J(G) = \int_\Omega \psi(x, G(x))\, dx + \frac{1}{2} \int_{\Omega^c} |G|^2\, dx, \tag{2.5}$$

where

$$\psi(x, \xi) = \frac{1}{2}(|\xi|^2 + 1) - \Phi(\xi + H(x))$$

with Φ denoting the convex function defined above by

$$\Phi(\eta) = \max_{h \in \mathbb{S}^{n-1}} [\eta \cdot h - \varphi(h)], \quad (\eta \in \mathbb{R}^n). \tag{2.6}$$

Define

$$\Sigma(\eta) = \{h \in \mathbb{S}^{n-1} \mid h \cdot \eta - \varphi(h) = \Phi(\eta)\}.$$

Then

$$\Sigma(\eta) = \partial\Phi(\eta) \cap \mathbb{S}^{n-1} \quad (\eta \in \mathbb{R}^n).$$

Remark 2.2.1. *(1) If anisotropy energy density φ is given by $\varphi = 0$, which is the soft case, then $\Phi(\eta) \equiv |\eta|$.*

(2) If anisotropy energy density φ is given by

$$\varphi(h) = \beta(1 - |h \cdot \mathbf{e}|), \tag{2.7}$$

where $\beta > 0$ and $\mathbf{e} \in \mathbb{S}^{n-1}$ are given constants. Then $\varphi(h) \geq 0$ and equals 0 if and only if $h \in \{\mathbf{e}, -\mathbf{e}\}$; these are the so-called easy axes. *This is the* uniaxial case. *In this case the function Φ defined above can be easily found as follows:*

$$\begin{aligned}
\Phi(\eta) &= \max_{|h|=1} (\eta \cdot h + |h \cdot \beta\mathbf{e}| - \beta) \\
&= \max_{|h|=1} \max_{t=\pm 1} \{\eta \cdot h + t\beta\mathbf{e} \cdot h\} - \beta \\
&= \max_{t=\pm 1} \max_{|h|=1} (\eta + t\beta\mathbf{e}) \cdot h - \beta \\
&= \max_{t=\pm 1} |\eta + t\beta\mathbf{e}| - \beta \\
&= (|\eta|^2 + 2\beta|\eta \cdot \mathbf{e}| + \beta^2)^{1/2} - \beta.
\end{aligned}$$

Therefore,

$$\partial\Phi(\eta) = \begin{cases} \frac{\eta + \beta\,\mathrm{sgn}(\eta\cdot\mathbf{e})\mathbf{e}}{|\eta + \beta\,\mathrm{sgn}(\eta\cdot\mathbf{e})\mathbf{e}|}, & \textit{if } \eta \cdot \mathbf{e} \neq 0, \\ \left\{ \frac{\eta + t\beta\mathbf{e}}{(|\eta|^2 + \beta^2)^{1/2}} \;:\; -1 \leq t \leq 1 \right\}, & \textit{if } \eta \cdot \mathbf{e} = 0. \end{cases}$$

We study these two special cases for minimization or gradient flow.

In general, the Legendre transform of $\psi(x,\cdot)$ can be computed as follows:

$$
\begin{aligned}
\psi^*(x,\lambda) &= \sup_{\xi\in\mathbb{R}^n}\{\lambda\cdot\xi-\psi(x,\xi)\} \\
&= \sup_{\xi\in\mathbb{R}^n}\left\{\lambda\cdot\xi-\frac{1}{2}(|\xi|^2+1)+\Phi(\xi+H(x))\right\} \\
&= \sup_{\xi\in\mathbb{R}^n}\left\{\lambda\cdot\xi-\frac{1}{2}|\xi|^2-\frac{1}{2}+\max_{h\in\mathbb{S}^{n-1}}\{(\xi+H(x))\cdot h-\varphi(h)\}\right\} \\
&= \sup_{\xi\in\mathbb{R}^n}\left[\max_{h\in\mathbb{S}^{n-1}}\left\{(\lambda+h)\cdot\xi-\frac{1}{2}|\xi|^2-\frac{1}{2}+H(x)\cdot h-\varphi(h)\right\}\right] \\
&= \sup_{h\in\mathbb{S}^{n-1}}\left[\sup_{\xi\in\mathbb{R}^n}\left\{(\lambda+h)\cdot\xi-\frac{1}{2}|\xi|^2-\frac{1}{2}+H(x)\cdot h-\varphi(h)\right\}\right] \\
&= \sup_{h\in\mathbb{S}^{n-1}}\left[\frac{1}{2}|\lambda+h|^2-\frac{1}{2}+H(x)\cdot h-\varphi(h)\right] \\
&= \sup_{h\in\mathbb{S}^{n-1}}\left[\frac{1}{2}|\lambda|^2+(\lambda+H(x))\cdot h-\varphi(h)\right] \\
&= \frac{1}{2}|\lambda|^2+\Phi(\lambda+H(x)).
\end{aligned}
$$

Therefore, the Legendre transform of J can be written as

$$
J^*(F)=\int_\Omega \Phi\Big(F(x)+H(x)\Big)\,dx+\frac{1}{2}\int_{\mathbb{R}^n}|F(x)|^2\,dx \tag{2.8}
$$

for all $F(x)\in\mathbb{H}$. Let $\mathcal{L}(F)$ be defined by (1.7) on $\mathbb{H}$.

Theorem 2.2.2. [44, Theorem 1.2] *Let $\bar{F}$ be a minimizer of $\mathcal{L}(F)$. A vector field $\bar{\mathbf{m}}$ is a minimizer of $I(\mathbf{m})$ if and only if*

$$
\begin{cases}
\operatorname{div}(\bar{F}+\bar{\mathbf{m}}\chi_\Omega)=0 & \text{in } \mathbb{R}^n, \\
\bar{\mathbf{m}}(x)\in\Sigma(\bar{F}(x)+H(x)) & \text{a.e. } \Omega.
\end{cases} \tag{2.9}
$$

Since there exists u, such that $\bar{F} = \nabla u$. Accordingly, $\operatorname{div}(\nabla u + \bar{\mathbf{m}}\chi_\Omega) = 0$, which yields that

$$\Delta u = -\operatorname{div}(\bar{\mathbf{m}}\chi_\Omega).$$

Hence, $u(x)$ can be solved by Newton's potential:

$$u(x) = \int_\Omega \mathbf{m}(y) \cdot \nabla\Gamma(x-y)\, dy = c_n \int_\Omega \frac{\mathbf{m}(y) \cdot (y-x)}{|y-x|^n}\, dy,$$

where $\Gamma(z)$ is the fundamental solution of Laplace's equation and c_n is a constant.

Remark 2.2.3. *If Ω is the unit ball in $\mathbb{R}^n$ and $\mathbf{m} \equiv K \in \mathbb{R}^n$ is a constant, then $u(x)$ can be expressed explicitly by (see [34] or [44, Lemma 4.2]):*

$$u(x) = \begin{cases} \dfrac{K \cdot x}{n} & x \in \Omega, \\ \dfrac{K \cdot x}{n|x|^n} & x \in \Omega^c, \end{cases}$$

which will be later applied to the calculation of gradient flow in the soft case in Chapter 3.

2.3 Constant constraint problem in $\mathbb{V}^\perp$

In this section, we review some existing results that are helpful to understand the main results in this thesis.

In [44] it has been shown that the condition (1.10) is equivalent to the following constrained problem for function $\tilde{G} \in L^2(\Omega; \mathbb{R}^n)$:

$$\begin{cases} \operatorname{div}(\tilde{G}\chi_\Omega) = 0 & \text{on } \mathbb{R}^n; \\ \tilde{G}(x) \in \mathbf{S}(x) & a.e.\ x \in \Omega, \end{cases} \tag{2.10}$$

where $\mathbf{S}(x)$ is some set-valued function. The constrained problem (2.10) for divergence-free fields with constant set $\mathbf{S}(x) = \mathbf{S}$ has been recently studied by many authors; see, e.g., [5, 12, 18, 20, 33, 44]. For example, the following result has been proved in [5, 18].

Theorem 2.3.1. (cf, [5, Theorem 4.15]; [18, Theorem 6.2]) *Let $n = 3$ and let Ω be any bounded open set in $\mathbb{R}^3$, and assume $\mathbf{S}(x) = \mathbf{S}$ is any constant bounded set in $\mathbb{R}^3$. Then problem* (2.10) *has a solution if and only if either $0 \in \mathbf{S}$ or there exists a subset $\mathbf{F} \subseteq \mathbf{S}$ such that $\dim(\operatorname{span} \mathbf{F}) \geq 2$ and $0 \in \operatorname{ri}(\operatorname{con} \mathbf{F})$. Moreover, in this case, a solution $\tilde{G}$ can be obtained by $\tilde{G} = \nabla \times \omega$ with some $\omega \in W_0^{1,\infty}(\Omega; \mathbb{R}^3)$.*

This theorem has its own limitations: (1) $\mathbf{S}(x)$ has to be constant; (2) $\tilde{G} \cdot \nu$ has to be 0 on the boundary but sometimes we do not have such condition. For example, in the case when the domain is annulus, such condition fails and we cannot use this theorem.

Chapter 3

General Results

In this chapter, we present our general results in the minimization and the asymptotic behavior of gradient flow of the functional $\mathcal{L}$. Some special cases will be also discussed.

3.1 Minimization of the dual functional

3.1.1 The minimizer is harmonic outside Ω

Theorem 3.1.1. *Suppose that the functional $\mathcal{L}$ is defined by (1.7). If follows that $\bar{F}$ is harmonic on Ω^c.*

Proof. Let $\zeta \in C_c^\infty(\Omega^c)$ be a test function with compact support in Ω^c. Since $\mathcal{L}(\bar{F}+\varepsilon\nabla\zeta) \geq \mathcal{L}(\bar{F})$ and $\zeta \equiv 0$ on Ω,

$$\int\limits_{\Omega} \Phi\Big(x, \bar{F}\Big)dx + \frac{1}{2}\int\limits_{\Omega^c} |\bar{F}+\varepsilon\nabla\zeta|^2\,dx \geq \int\limits_{\Omega} \Phi\Big(x, \bar{F}\Big)dx + \frac{1}{2}\int\limits_{\Omega^c} |\bar{F}|^2\,dx.$$

Let $\int\limits_{\Omega^c} |\bar{F}+\varepsilon\nabla\zeta|^2\,dx = h(\varepsilon)$, then the above inequality implies that $h(\varepsilon) \geq h(0)$. Therefore $h'(0) = 0$, i.e.

$$\int_{\Omega^c} \bar{F}\cdot\nabla\zeta = 0 \quad \text{for any } \zeta \in C_c^\infty(\Omega^c).$$

Combing the definition of $\bar{F}$, we have the following results

$$\operatorname{div} \bar{F} = 0 \quad \text{on } \Omega^c,$$

$$\operatorname{curl} \bar{F} = 0 \quad \text{on } \Omega^c,$$

in the sense of distribution. We apply the *distributional* identity

$$\operatorname{curl}\operatorname{curl}(N) + \triangle(N) = \nabla(\operatorname{div}(N))$$

to have that

$$\Delta \bar{F} = 0 \quad \text{on } \Omega^c.$$

in distribution and thus in classical sense. □

3.1.2 A necessary and sufficient condition for the minimizer

It has been established in [44] a necessary and sufficient condition for the existence of minimizers of energy $I(\mathbf{m})$ in terms of the unique minimizer $\bar{u} = \bar{v}\chi_\Omega + \bar{w}\chi_{\Omega^c}$ of functional $L(u)$ on Ω, which is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary. $L(u)$ is defined previously,

$$L(u) = \int_\Omega \Phi(\nabla u + H(x))\, dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\, dx$$

Theorem 3.1.2. *A function $\bar{u} = \bar{v}\chi_\Omega + \bar{w}\chi_{\Omega^c} \in \mathcal{X}$ is a minimizer of $L(u)$ if and only if* *there exists a function $G \in L^2(\Omega;\mathbb{R}^n)$ such that*

$$\begin{cases} \operatorname{div}(\nabla \bar{u} + G\chi_\Omega) = 0 & \text{in } \mathbb{R}^n, \\ G \in \partial\Phi(\nabla \bar{v} + H(x)) & \text{a.e. } \Omega. \end{cases} \tag{3.1}$$

Any such function G is called a generalized minimizer *of the functional $I(\mathbf{m})$.*

Remark 3.1.3. *Let ν be the outward unit normal to the boundary $\partial\Omega$ of domain Ω. Then both $G\cdot\nu$ and $\frac{\partial \bar{w}}{\partial \nu}$ are defined as elements in $H^{-1/2}(\partial\Omega)$ (see, e.g., [49, Page 9]). The above necessary and sufficient condition (3.1) can be also reformulated as:*

$$\begin{cases} \operatorname{div}(\nabla \bar{v} + G) = 0 & \text{in } H^{-1}(\Omega), \\ (G + \nabla \bar{v})\cdot \nu = \frac{\partial \bar{w}}{\partial \nu} & \text{on } \partial\Omega, \\ \Delta \bar{w} = 0 & \text{in } \Omega^c, \\ G \in \partial\Phi(\nabla \bar{v}(x) + H(x)) & \text{a.e. } x \in \Omega. \end{cases} \tag{3.2}$$

Here we present a different method to prove Theorem 3.1.2.

Proof. Suppose that there exist G and $\bar{u}$ satisfying (3.1) then for any $v \in \mathcal{X}$, we have

$$\begin{aligned} & L(v) - L(\bar{u}) \\ = \; & \int_\Omega \Phi(\nabla v + H(x)) - \Phi(\nabla \bar{u} + H(x))\,dx + \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla v|^2 - |\nabla \bar{u}|^2\right)\,dx \\ \geq \; & \int_\Omega G\cdot(\nabla v - \nabla \bar{u}) + \int_{\mathbb{R}^n} \nabla \bar{u}\cdot(\nabla v - \nabla \bar{u}) + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v - \nabla \bar{u}|^2\,dx \\ = \; & \int_{\mathbb{R}^n} [G\chi_\Omega + \nabla \bar{u}]\cdot(\nabla v - \nabla \bar{u}) + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v - \nabla \bar{u}|^2\,dx \\ = \; & \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v - \nabla \bar{u}|^2\,dx \geq 0. \end{aligned}$$

Therefore, $\bar{u}$ is the minimizer of (1.8).

Next, assume that $\bar{u}$ is a minimizer of $L(u)$. Introduce a function $\Phi_\varepsilon(\eta)$, for $\varepsilon > 0$, which is defined as

$$\Phi_\varepsilon(\eta) = \min_{\xi \in \mathbb{R}^n} \left\{ \frac{1}{2\varepsilon} |\eta - \xi|^2 + \Phi(\xi) \right\}.$$

$\Phi_\varepsilon(\eta)$ follows the following properties (refer to Brézis[10]):

(1) $\Phi_\varepsilon(\eta)$ is convex;

(2) $\Phi_\varepsilon(\eta)$ is differentiable;

(3) $\Phi_\varepsilon(\eta) \to \Phi(\eta)$ as $\varepsilon \to 0$;

(4) $\left|\Phi'_\varepsilon(\eta)\right| \leq 1$;

(5) $\Phi_\varepsilon(\eta) \leq \Phi(\eta)$.

Consider the functional below

$$\begin{aligned} L_\varepsilon(u) &= \int_\Omega \Phi_\varepsilon(\nabla u + H(x))\, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2\, dx \\ &= \int_\Omega \frac{1}{2\varepsilon} |\nabla u + H(x) - B|^2 + \Phi(B)\, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2\, dx. \end{aligned}$$

Then, functional $L_\varepsilon(u)$ is differentiable, strictly convex on $\mathcal{X}$ and thus has a unique minimizer u_ε in $\mathcal{X}$. Let $L_\varepsilon(u_\varepsilon) = \min L_\varepsilon(u)$. Then we apply the definition of Φ_ε to have

$$L_\varepsilon(u_\varepsilon) = \int_\Omega \Phi_\varepsilon(\nabla u_\varepsilon + H(x))\, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\, dx \tag{3.3}$$

$$= \int_\Omega \frac{1}{2\varepsilon} |\nabla u_\varepsilon + H(x) - B_\varepsilon|^2 + \Phi(B_\varepsilon)\, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\, dx \tag{3.4}$$

Therefore, by Euler-Lagrange equation,

$$\operatorname{div}\left(\Phi'_\varepsilon(\nabla u_\varepsilon + H(x))\chi_\Omega + \nabla u_\varepsilon\right) = 0.$$

Let $G_\varepsilon(x) = \Phi'_\varepsilon(\nabla u_\varepsilon + H(x))$. Then there exist $G(x)$, such that $G_\varepsilon(x) \rightharpoonup G(x)$ weakly in L^∞. Consequently,

$$\operatorname{div}(G\chi_\Omega + \nabla\bar{u}) = 0.$$

Since $L_\varepsilon(u_\varepsilon) \le L_\varepsilon(0) = \int_\Omega \Phi_\varepsilon(H(x))\,dx = \int_\Omega \Phi(H(x))\,dx < \infty$. In addition, $\Phi(\eta)$ is Lipschitz so $|\Phi(\eta)| \le c(|\eta|+1)$. Therefore, $|\Phi_\varepsilon(\eta)| \le |\Phi(\eta)| \le c(|\eta|+1)$, which yields that $|\nabla u_\varepsilon|$ is uniformly bounded from (3.4). Hence ∇u_ε is weakly convergent and there exists $\tilde{u} \in \mathcal{X}$, such that $\nabla u_\varepsilon \rightharpoonup \tilde{u}$ weakly as $\varepsilon \to 0$. Note that L is lower semicontinuous,

$$L(\tilde{u}) \le \lim_{\varepsilon\to 0} L(u_\varepsilon). \tag{3.5}$$

Note that $L_\varepsilon(u_\varepsilon)$ and ∇u_ε are bounded and Φ is Lipschitz, there exist $M_1, M_2 > 0$ such that

$$\begin{aligned}
M_1 \ge L_\varepsilon(u_\varepsilon) &= \int_\Omega \Phi_\varepsilon(\nabla u_\varepsilon + H(x))\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\,dx \\
&= \int_\Omega \frac{1}{2\varepsilon}|\nabla u_\varepsilon + H(x) - B_\varepsilon|^2 + \Phi(B_\varepsilon)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\,dx \\
&\ge \int_\Omega \frac{1}{2\varepsilon}|\nabla u_\varepsilon + H(x) - B_\varepsilon|^2 - |B_\varepsilon|\,dx - M_2.
\end{aligned}$$

Note that $|\nabla u_\varepsilon + H|$ is L^2 bounded and we apply the triangle inequality and Cauchy inequality to have that

$$
\begin{aligned}
& \int_\Omega |\nabla u_\varepsilon + H(x) - B_\varepsilon|^2\, dx \\
\leq \; & 2\varepsilon \int_\Omega |B_\varepsilon|\, dx + 2\varepsilon M_1 \\
\leq \; & 2\varepsilon \int_\Omega |\nabla u_\varepsilon + H(x) - B_\varepsilon|\, dx + 2\varepsilon \int_\Omega |\nabla u_\varepsilon + H(x)|\, dx + 2(M_1 - M_2)\varepsilon \\
\leq \; & 2\varepsilon \left[\frac{1}{2} \int_\Omega |\nabla u_\varepsilon + H(x) - B_\varepsilon|^2\, dx + \frac{1}{2}|\Omega| \right] + \varepsilon M_3
\end{aligned}
$$

for some constant M_3, which yields that $\int_\Omega |\nabla u_\varepsilon + H(x) - B_\varepsilon|^2\, dx \leq \frac{\varepsilon}{1-\varepsilon} \cdot M, \quad$ for some $M > 0$. Thus

$$\int_\Omega |\nabla u_\varepsilon + H(x) - B_\varepsilon|^2\, dx \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.6}$$

Note that Φ is Lipschitz,

$$
\begin{aligned}
L_\varepsilon(u_\varepsilon) &= \int_\Omega \frac{1}{2\varepsilon} |\nabla u_\varepsilon + H(x) - B_\varepsilon|^2 + \Phi(B_\varepsilon)\, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\, dx \\
&\geq \int_\Omega \Phi(B_\varepsilon)\, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\, dx \\
&\geq \int_\Omega \left[\Phi(\nabla u_\varepsilon + H) - |\nabla u_\varepsilon + H - B_\varepsilon| \right]\, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\, dx
\end{aligned}
$$

which yields that

$$L_\varepsilon(u_\varepsilon) \geq L(u_\varepsilon) - \int_\Omega |\nabla u_\varepsilon + H - B_\varepsilon|\, dx. \tag{3.7}$$

Combining (3.5), (3.7)and (3.6), we have that

$$
\begin{aligned}
L(\tilde{u}) &\leq \lim_{\varepsilon \to 0} L(u_\varepsilon) \leq \lim_{\varepsilon \to 0} \left[L_\varepsilon(u_\varepsilon) + \int_\Omega |\nabla u_\varepsilon + H - B_\varepsilon|\, dx \right] \\
&\leq \lim_{\varepsilon \to 0} L_\varepsilon(v) + \lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon + H - B_\varepsilon|\, dx \\
&= L(v), \quad \text{for any } v \in \mathcal{X}.
\end{aligned}
$$

Hence $\tilde{u}$ is the minimizer of L and thus $\tilde{u} = \bar{u}$. In addition,

$$\begin{aligned}
& L_\varepsilon(\bar{u}) - L_\varepsilon(u_\varepsilon) \\
= & \int_\Omega \Phi_\varepsilon(\nabla\bar{u} + H(x))\,dx - \int_\Omega \Phi_\varepsilon(\nabla u_\varepsilon + H(x))\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\bar{u}|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\,dx \\
= & \int_\Omega \Phi_\varepsilon(\nabla\bar{u} + H(x)) - \Phi_\varepsilon(\nabla u_\varepsilon + H(x))\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\bar{u} - \nabla u_\varepsilon|^2\,dx \\
+ & \int_{\mathbb{R}^n} \nabla u_\varepsilon(\nabla\bar{u} - \nabla u_\varepsilon)\,dx \\
\geq & \int_\Omega \Phi'_\varepsilon(\nabla u_\varepsilon + H(x)) \cdot (\nabla\bar{u} - \nabla u_\varepsilon)\,dx + \int_{\mathbb{R}^n} \nabla u_\varepsilon(\nabla\bar{u} - \nabla u_\varepsilon)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\bar{u} - \nabla u_\varepsilon|^2\,dx \\
= & \int_{\mathbb{R}^n} \left[\Phi'_\varepsilon(\nabla u_\varepsilon + H(x))\chi_\Omega + \nabla u_\varepsilon\right] \cdot (\nabla\bar{u} - \nabla u_\varepsilon)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\bar{u} - \nabla u_\varepsilon|^2\,dx \\
= & \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\bar{u} - \nabla u_\varepsilon|^2\,dx
\end{aligned}$$

yields

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^n} |\nabla\bar{u} - \nabla u_\varepsilon|^2\,dx & \leq L_\varepsilon(\bar{u}) - L_\varepsilon(u_\varepsilon) \\
& \leq L(\bar{u}) - L_\varepsilon(u_\varepsilon) \leq L(u_\varepsilon) - L_\varepsilon(u_\varepsilon) \\
& \leq \int_\Omega |\nabla u_\varepsilon + H - B_\varepsilon|\,dx \to 0.
\end{aligned}$$

Hence

$$\nabla u_\epsilon \to \nabla\bar{u} \quad \text{a.e. in } L^2(\mathbb{R}^n). \tag{3.8}$$

By the definition of $G_\varepsilon(x)$,

$$\Phi_\varepsilon(\eta) \geq G_\varepsilon(x) \cdot (\eta - \nabla u_\varepsilon - H) + \Phi_\varepsilon(\nabla u_\varepsilon + H)$$

Let ζ be a test function with $\zeta \in C^\infty(\mathbb{R}^n)$, multiply both sides by ζ and integrate both sides,

$$\int_{\mathbb{R}^n} \zeta \Phi_\varepsilon(\eta) \geq \int_{\mathbb{R}^n} \zeta G_\varepsilon(x) \cdot (\eta - \nabla u_\varepsilon - H)\, dx + \int_{\mathbb{R}^n} \zeta \Phi_\varepsilon(\nabla u_\varepsilon + H)\, dx.$$

Let $\varepsilon \to 0$ which yields that

$$\int_{\mathbb{R}^n} \zeta \Phi(\eta) \geq \int_{\mathbb{R}^n} \zeta G(x) \cdot (\eta - \nabla u - H)\, dx + \int_{\mathbb{R}^n} \zeta \Phi(\nabla u + H)\, dx.$$

i.e.

$$G \in \partial \Phi(\nabla \bar{v} + H(x)).$$

The last term is obtained by the fact that

$$\begin{aligned}
& |\Phi_\varepsilon(\nabla u_\varepsilon + H) - \Phi(\nabla u + H)| \\
\leq\ & |\Phi_\varepsilon(\nabla u_\varepsilon + H) - \Phi_\varepsilon(\nabla u + H)| + |\Phi_\varepsilon(\nabla u + H) - \Phi(\nabla u + H)| \\
\leq\ & |\nabla u_\varepsilon - \nabla u| + + |\Phi_\varepsilon(\nabla u + H) - \Phi(\nabla u + H)| \\
=\ & 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

This completes the sufficient part. □

3.1.3 A domain dependence result for the minimizer

Suppose that the domain Ω has a small region E removed from inside.

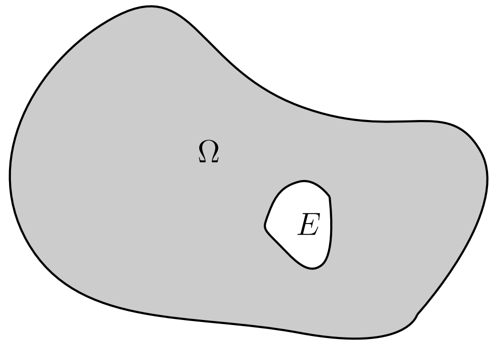

Figure 3.1: The new domain when a small region E is removed

Let us denote

$$L_{\Omega}(u) = \int_{\Omega} \Phi(\nabla u(x) + H(x))\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2\,dx,$$
$$L_{\Omega\setminus E}(u) = \int_{\Omega\setminus E} \Phi(\nabla u(x) + H(x))\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2\,dx.$$

Theorem 3.1.4. *Let $\bar{u} = \bar{v}\chi_{\Omega} + \bar{w}\chi_{\Omega^c}$ be the minimizer of the functional $L_{\Omega}(u)$ and $E \subset\subset \Omega$. Define $\tilde{w}$ by*

$$\begin{cases} \Delta\tilde{w} = 0 \quad \text{in } E, \\ \tilde{w} = \bar{u} \quad \text{on } \partial E. \end{cases} \tag{3.9}$$

Suppose that there exists $\tilde{G} \in L^2(\Omega \setminus E; \mathbb{R}^n)$ satisfying

$$\begin{cases} \operatorname{div}(\nabla\bar{v} + \tilde{G}) = 0 & \text{in } \Omega \setminus E, \\ \left(\tilde{G} + \nabla\bar{v}\right)\cdot\nu = \frac{\partial\bar{w}}{\partial\nu} & \text{on } \partial\Omega, \\ \left(\tilde{G} + \nabla\bar{v}\right)\cdot\nu = \frac{\partial\tilde{w}}{\partial\nu} & \text{on } \partial E, \\ \tilde{G} \in \partial\Phi(\nabla\bar{v} + H(x)) & \text{a.e. } \Omega \setminus E. \end{cases} \tag{3.10}$$

Then $\tilde{u} = \bar{u}\chi_{\mathbb{R}^n\setminus E} + \tilde{w}\chi_E$ *is the minimizer of* $L_{\Omega\setminus E}(u)$.

Proof. This follows directly from Remark 3.1.3. $\square$

3.1.4 The uniaxial anisotropy energy

We consider the easy case where the anisotropy energy density $\varphi(h) = \beta(1 - |h\cdot\mathbf{e}|)$, where $\beta > 0$ and $\mathbf{e} \in \mathbb{S}^{n-1}$ are given constants. In this case, let us recall from Remark 2.2.1,

$$\Phi(\eta) = (|\eta|^2 + 2\beta|\eta\cdot\mathbf{e}| + \beta^2)^{1/2} - \beta, \tag{3.11}$$

and

$$\partial\Phi(\eta) = \begin{cases} \frac{\eta+\beta\,\mathrm{sgn}(\eta\cdot\mathbf{e})\mathbf{e}}{|\eta+\beta\,\mathrm{sgn}(\eta\cdot\mathbf{e})\mathbf{e}|}, & \text{if } \ \eta\cdot\mathbf{e}\neq 0,\\ \left\{\frac{\eta+t\beta\mathbf{e}}{(|\eta|^2+\beta^2)^{1/2}} \ : \ -1\le t\le 1\right\}, & \text{if } \ \eta\cdot\mathbf{e} = 0. \end{cases}$$

Let $\Omega = \{x \in \mathbb{R}^n|\ a < |x| < 1\}$ be an annulus domain in $\mathbb{R}^n$. Consider

$$L(u) = \int_\Omega \Phi(\nabla u + H)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx$$

where $H \in \mathbb{R}^n$ is a constant and $H \neq 0$.

Theorem 3.1.5. *Suppose that* Φ *is defined as (3.11). Then the minimizer of* L *cannot be linear on* Ω.

Proof. Notice that $\eta = 0$ is the only minimizer of Φ, defined in (3.11) on $\mathbb{R}^n$. Suppose that the minimizer $\bar{u}$ of $L(u)$ is linear on $\Omega : \bar{u}(x) = \bar{\lambda}\cdot x$ on Ω. Then recall from Remark 2.2.3,

$\bar{u} = \bar{\lambda} \cdot x \chi_{|x|<1} + \frac{\bar{\lambda} \cdot x}{|x|^n} \chi_{|x|>1}$, and there exists a $G \in L^2(\Omega; \mathbb{R}^n)$ such that

$$\begin{cases} \operatorname{div} G = 0 & \text{in } \ H^{-1}(\Omega), \\ G \cdot \nu = -n\bar{\lambda} \cdot \nu & \text{on } \ |x| = 1, \\ G \cdot \nu = 0 & \text{on } \ |x| = a, \\ G(x) \in \partial\Phi(\bar{\lambda} + H) & \text{a.e. } \ \Omega. \end{cases}$$

We proceed with 2 cases.

Case 1: $\big(\bar{\lambda} + H\big) \cdot \mathbf{e} \neq 0$. Then $G(x) \equiv \Phi'(\bar{\lambda} + H)$. Since $G \cdot \nu = -n\bar{\lambda} \cdot \nu$ on $|x| = 1$, then $\Phi'(\bar{\lambda} + H) = -n\bar{\lambda} = 0$. Hence $\bar{\lambda} = 0$ and $\Phi'(H) = 0$. Accordingly, $H = 0$ and $\bar{\lambda} + H = 0$ which is a contradiction.

Case 2: $\big(\bar{\lambda} + H\big) \cdot \mathbf{e} = 0$. Then $G(x) = \frac{\eta + t(x)\beta\mathbf{e}}{\sqrt{|\eta|^2+\beta^2}}$ for some function $t(x) \in L^\infty(\Omega)$, $-1 \leq t(x) \leq 1$, where $\eta = \bar{\lambda} + H$. Then applying the condition $\operatorname{div} G = 0$ in $H^{-1}(\Omega)$, we have that $\frac{\partial t}{\partial \mathbf{e}}(x) = 0$ in $H^{-1}(\Omega)$, which implies that $t(x) = t(x')$ if $x = x_1\mathbf{e} + x'\mathbf{e}' = x_1\mathbf{e} + x_2\mathbf{e}_2 + ... + x_n e_n$ with $\mathbf{e}_i \perp \mathbf{e}$, $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$, and $|\mathbf{e}_i| = 1$. Since for any $|x| = a$, $G(x) \cdot x = 0$. We have that

$$\eta \cdot x + t(x')\beta\mathbf{e} \cdot x = 0, \quad \forall\, |x| = a.$$

If assume that $\eta \neq 0$ and let $x = \dfrac{\eta}{|\eta|}a$. Then we have $\eta = 0$. Therefore, $\eta = 0$ and $\bar{\lambda} = -H$. Accordingly, $G(x) = t(x)\mathbf{e}$. Let $x_1 \neq 0$, then $0 = G(x) \cdot x = t(x')\beta x_1$ yields that $t(x') = 0$. Thus $G(x) \equiv 0$ and therefore $-n\bar{\lambda} = H = 0$, which yields a contradiction. $\square$

3.2 The gradient flow of the dual functional

In this section, for simplicity, we write $\Phi(x, F) \equiv \Phi(F(x)+H(x))$. Recall that $\mathcal{L}$ is previously defined as:

$$\mathcal{L}(F) = \begin{cases} \displaystyle\int_\Omega \Phi(x, F)dx + \frac{1}{2}\int_{\mathbb{R}^n} |F(x)|^2\, dx, & F \in \mathbb{V}; \\ +\infty, & F \in \mathbb{H}\backslash\mathbb{V}. \end{cases}$$

3.2.1 The subdifferential

To study the asymptotic behavior of gradient flow of $\mathcal{L}(F)$, we first calculate the subdifferential $\partial\mathcal{L}(F)$. Note that this functional is convex on $\mathbb{H}$ with (finite-value) domain $D(\mathcal{L}) = \mathbb{V}$. For each $F \in D(\mathcal{L}) = \mathbb{V}$, the subdifferential of $\mathcal{L}$ is defined as:

$$\partial\mathcal{L}(F) = \{K \in \mathbb{H} \mid \mathcal{L}(X) \geq \mathcal{L}(F) + K \cdot (X - F)\ \forall\ X \in \mathbb{H}\}.$$

Theorem 3.2.1. *For each $F \in D(\mathcal{L}) = \mathbb{V}$, we have*

$$\partial\mathcal{L}(F) = F + P_{\mathbb{V}}\big(\partial\Phi(x, F)\chi_\Omega\big) + \mathbb{V}^\perp,$$

where $P_{\mathbb{V}}\colon \mathbb{H} \to \mathbb{V}$ is the projection operator and $\mathbb{V}^\perp$ is the orthogonal complement of $\mathbb{V}$ in $\mathbb{H}$.

Proof. Clearly, $F + P_{\mathbb{V}}\big(\partial\Phi(x, F)\chi_\Omega\big) + \mathbb{V}^\perp \subseteq \partial\mathcal{L}(F)$. Now let $K \in \partial\mathcal{L}(F)$. With an abuse of notation, we define

$$L(F) = \int_\Omega \Phi(x, F)dx + \frac{1}{2}\int_{\mathbb{R}^n} |F(x)|^2\, dx, \forall F \in \mathbb{H},$$

namely, $L(F) = J^*(F)$ on $\mathbb{H}$, defined in (2.8) before. We may assume $K \in \mathbb{V}$. Then $L(X) \geq L(F) + K \cdot (X - F)$ for all $X \in \mathbb{V}$. This means that $F \in \mathbb{V}$ is the minimizer of the functional

$$\tilde{L}(X) = L(X) - L(F) - K \cdot (X - F)$$

over $\mathbb{V}$. For each $\epsilon > 0$ consider the functional

$$\tilde{L}_\epsilon(X) = \tilde{L}(X) + \frac{1}{2}\|X - F\|^2 + \frac{1}{2\epsilon}\|P_{\mathbb{V}^\perp}(X)\|^2$$

on $X \in \mathbb{H}$, where $P_{\mathbb{V}^\perp} : \mathbb{H} \to \mathbb{V}^\perp$ is the projection operator. The functional $\tilde{L}_\epsilon$ is convex and thus the standard direct method of the calculus of variations shows that it has a unique minimizer F_ϵ over whole $\mathbb{H}$. For this minimizer F_ϵ, since $\tilde{L}_\epsilon(F_\epsilon) \leq \tilde{L}_\epsilon(F) = \tilde{L}(F) = 0$, we have

$$\tilde{L}(F_\epsilon) + \frac{1}{2}\|F_\epsilon - F\|^2 + \frac{1}{2\epsilon}\|P_{\mathbb{V}^\perp}(F_\epsilon)\|^2 \leq 0. \tag{3.12}$$

From this and the linear growth of $L(X)$, we have that $\{F_\epsilon\}_{\epsilon>0}$ is bounded; therefore, by a subsequence, we assume $F_\epsilon \rightharpoonup \tilde{F}$ as $\epsilon \to 0^+$, weakly in $\mathbb{H}$, for some $\tilde{F} \in \mathbb{H}$. From (3.12) and the lower semicontinuity of $\tilde{L}$, it follows that

$$\tilde{L}(\tilde{F}) + \frac{1}{2}\|\tilde{F} - F\|^2 \leq 0, \quad \|P_{\mathbb{V}^\perp}(\tilde{F})\| \leq \lim_{\epsilon \to 0^+} \|P_{\mathbb{V}^\perp}(F_\epsilon)\| = 0.$$

Therefore, $\tilde{F} \in \mathbb{V}$ and hence $\tilde{L}(\tilde{F}) \geq \tilde{L}(F) = 0$; this implies that $\tilde{F} = F$ and $F_\epsilon \to F$ as $\epsilon \to 0^+$. Finally, from $0 \in \partial\tilde{L}_\epsilon(F_\epsilon)$ the elementary computations yield that

$$K \in 2F_\epsilon - F + \partial\Phi(x, F_\epsilon)\chi_\Omega + \frac{1}{\epsilon}P_{\mathbb{V}^\perp}(F_\epsilon).$$

Let, via subsubsequences if necessary, $\frac{1}{\epsilon}P_{\mathbb{V}^\perp}(F_\epsilon) \rightharpoonup G$ in $\mathbb{H}$ as $\epsilon \to 0^+$. Then $G \in \mathbb{V}^\perp$ and

$$K \in F + \partial\Phi(x, F)\chi_\Omega + G,$$

which proves $K \in F + P_{\mathbb{V}}\big(\partial\Phi(x, F)\chi_\Omega\big) + \mathbb{V}^\perp$. □

3.2.2 The gradient flow

Since $\mathcal{L}$ is convex, proper and lower semicontinuous, we have that for any initial datum $F_0 \in \mathbb{V}$, there exists a unique function $F : [0, \infty) \to \mathbb{V}$ such that

$$\begin{cases} 0 \in \dot{F} + \partial\mathcal{L}(F), \quad t > 0, \\ F(0) = F_0. \end{cases} \tag{3.13}$$

Bruck [8] demonstrated that, for general gradient flow of a strictly convex functional

$$F(t) \rightharpoonup \bar{F} \quad \text{weakly as } t \to \infty,$$

where $\bar{F}$ is the unique minimizer of the functional. Next, we will show that for the functional $\mathcal{L}$ defined above, we also have strong convergence outside of Ω. Let us summarize our discussion above in the theorem below:

Theorem 3.2.2. *For each $F_0 \in \mathbb{V}$ there exists a unique solution to the gradient flow given by (3.13). Furthermore, $F(t) \rightharpoonup \bar{F}$ as $t \to \infty$ and $F(t) \to \bar{F}$ in $L^2(\tilde{\Omega}^c; \mathbb{R}^n)$ for all compact sets $\tilde{\Omega}$ containing $\bar{\Omega}$.*

Proof. Since $F(t) \in \mathbb{V}$, the gradient flow should be reduced to

$$\begin{cases} 0 \in \dot{F} + F + P_{\mathbb{V}}\big(\partial\Phi(x,F)\chi_\Omega\big), \quad t > 0, \\ F(0) = F_0. \end{cases} \tag{3.14}$$

Let $N \in P_{\mathbb{V}}\big(\partial\Phi(x,F)\chi_\Omega\big)$. then we have $\mathrm{curl}(N) = 0$ by definition and $\mathrm{div}(N) = 0$ in Ω^c in the sense of distribution. By the identity $\mathrm{curl}\,\mathrm{curl}(N) + \triangle(N) = \nabla(\mathrm{div}(N))$, we have that $\triangle N = 0$ in Ω^c in the sense of distribution. Hence

$$\triangle(\dot{F} + F) = 0 \quad \text{in } \Omega^c.$$

Solving this evolution equation, we obtain that $F(x,t) = F_0(x)e^{-t} + U(x,t)$ for $x \in \Omega^c$ and $t > 0$, where $\triangle U(x,t) = 0$ in $\Omega^c \times (0,\infty)$. Let $V(x,t) = U(x,t) - \bar{F}(x)$. Then $V(x,t)$ is harmonic in Ω^c and $V(x,t) \rightharpoonup 0$ as $t \to \infty$. For any fixed $x_0 \in \Omega^c$,

$$V(x_0,t) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} V(x,t)dx \to 0 \quad \text{as } t \to \infty.$$

For any ball $\mathbb{R}^n \supset B_R(0) \equiv B_R \supset\supset \Omega$, $\Omega^c = B_R^c + B_R\backslash\Omega$. Let $C \subset B_R\backslash\tilde{\Omega}$ be any compact set with $d(C, \partial\tilde{\Omega}) = d_0 > 0$.

$$V(x,t) = \frac{1}{B_{\frac{d_0}{2}}(x)}\int_{B_{\frac{d_0}{2}}(x)} V(y,t)\,dy, \quad \forall\, x \in C.$$

$\Longrightarrow$

$$|V(x,t)| \leq \frac{1}{B_{\frac{d_0}{2}}(x)} \cdot ||V(y,t)||_{L^2} \cdot \Big(\big|B_{\frac{d_0}{2}}(x)\big|\Big)^{\frac{1}{2}} \leq M.$$

Therefore,

$$U(x,t) \to \bar{F}(x) \quad \text{strongly in } C, \quad \text{when } t \to \infty,$$

by the Bounded Convergence Theorem.

Let's fix t first. For any $x \in B_R^c$,

$$|V(x,t)| = \left| \frac{1}{|B_{|x|-R}(x)|} \int_{B_{|x|-R}(x)} V(y,t)\, dy \right| \le C(|x|-R)^{-n/2}, \tag{3.15}$$

Now let $x \to \infty$, we have that $V(x,t) \to 0$, for all t. Applying the **Kelvin Transform**, (assume $R = \frac{1}{2}$ below) we have that

$$\tilde{V}(y,t) = \frac{1}{|y|^{n-2}} V\Big(\frac{y}{|y|^2}, t\Big), \quad 0 \le |y| \le 1. \tag{3.16}$$

We proceed with three cases when $x \in B_R^c$:

(i) If $n = 3$, let $x = \dfrac{y}{|y|^2}$, so $|x| \ge 1$ and there exists $c > 0$, not depending on t, such that

$$|\tilde{V}(y,t)||y| = |V(x,t)| \le c(|x|-R)^{-3/2} \le c|y|^{3/2}, \tag{3.17}$$

i.e. $|V(y,t)| \le c|y|^{\frac{1}{2}}$. So $y = 0$ is a removable singular point. Therefore, we define

$$W(y,t) = \begin{cases} \tilde{V}(y,t), & y \ne 0, \\ 0, & y = 0. \end{cases}$$

Thus $|W(y,t)| \le c_1|y|$. and combined with (3.17), it follows that

$$|V(x,t)|^2 \le C_1|y|^4 = C_1|x|^{-4} \in L^1(B_R^c).$$

By the Dominant Convergence Theorem, $V(x,t) \to 0$ strongly in $L^2(B_R^c)$ as $t \to \infty$.

(ii) If $n = 4$, the Kelvin Transform is $\tilde{V}(y,t) = \dfrac{1}{|y|^2} V\Big(\dfrac{y}{|y|^2}, t\Big)$ and there exists some constant c_2 such that $|\tilde{V}(y,t)| \leq c_2$. Suppose that there exists $a > 0$, such that

$$W(y,t) = \begin{cases} \tilde{V}(y,t), & y \neq 0, \\ \\ a, & y = 0. \end{cases}$$

Hence $\exists\ \delta > 0$, s.t. when $|y| < \delta$, $\dfrac{1}{|y|^2}|V(x,t)| \geq a > 0$. Thus $|V(x,t)| \geq a \cdot |y|^2 = a|x|^{-2} \notin L^2(B_R^c)$. This contradiction implies that $a = 0$, i.e. $W \equiv 0$ when $y = 0$. Thus there exists a constant c_3, not depending on t, such that $|W(y,t)| = |\sum c_j y^j| \leq c_3 |y|$. Therefore,

$$|V(x,t)| \leq c_3 |y|^3 = c_3 |x|^{-3} \in L^2(B_R^c).$$

By the Dominant Convergence Theorem, $V(x,t) \to 0$ strongly in $L^2(B_R^c)$ as $t \to \infty$.

(iii) If $n \geq 5$, we already have $|V(x,t)| \leq C_1 |x|^{-\frac{n}{2}}$ for any fixed $t > 0$ by (3.15). Define $Y(x) = \dfrac{C_2}{|x|^{n-2}}$. Let $Z(x,t) = Y(x) - V(x,t)$, for a $R_1 > 0$, we can find a constant $C_2 > 0$ such that $Z(x,t)\Big|_{|x|=R_1} > 0$ and we claim that

$$Z(x,t) \geq 0 \quad \text{in } \Big\{x \in \mathbb{R}^n; |x| \geq R_1\Big\}.$$

If not, there exists $x_0 \in \mathbb{R}^n$, $|x_0| > R_1$ and $\alpha > 0$ such that $Z(x_0,t) = -\alpha < 0$. Therefore, there exists $R_2 > R_1$, s.t. $\Big(Z(x,t) + \dfrac{\alpha}{2}\Big)\Big|_{|x|=R_2} > 0$. Applying the maximum principle to $Z(x,t) + \frac{\alpha}{2}$ on $\Big\{x \in \mathbb{R}^n; R_1 \leq |x| \leq R_2\Big\}$ yields a contradiction! Therefore,

$$Z(x,t) \geq 0 \quad \text{in } \Big\{x \in \mathbb{R}^n; |x| \geq R_1\Big\}.$$

i.e.$V(x,t) \leq Y(x)$ in $\left\{x \in \mathbb{R}^n; |x| \geq R_1\right\}$ and $Y(x) \in L^2(\mathbb{R}^n \backslash B_1)$. Therefore when $n \geq 5$,

$$V(x,t) \to 0 \quad \text{strongly}, \quad \text{as } t \to \infty, \quad \text{when } x > R_1$$

by the Dominant Convergence Theorem.

Combining all the results, for any compact $\tilde{\Omega} \supset\supset \Omega$,

$$F(x,t) \to \bar{F}(x) \quad \text{strongly in } L^2(\tilde{\Omega}^c), \quad \text{as}\, t \to \infty.$$

$\square$

3.2.3 Possible dynamics for magnetization m

Note that the gradient flow (3.13) on $\mathbb{H} = L^2(\mathbb{R}^n; \mathbb{R}^n)$ determines a (nonunique) time-dependent vector function $\mathbf{m}(t) = \mathbf{m}(x,t)$ with the property

$$\begin{cases} \operatorname{div}(\dot{F}(t) + F(t) + \mathbf{m}(t)\chi_\Omega) = 0 \quad (t > 0), \\ \mathbf{m}(x,t) \in \partial\Phi(F(x,t) + H(x)) \qquad a.e.\, x \in \Omega, \ \ t > 0. \end{cases} \tag{3.18}$$

The first condition asserts that $F_{\mathbf{m}(t)} = -\dot{F}(t) - F(t)$.

From the general theory of gradient flow [10], we have

$$\int_0^\infty \|\dot{F}(t)\|_{\mathbb{H}}^2 \, dt < \infty,$$

which implies that, along a subsequence $t_k \to \infty$, one has $\dot{F}(t_k) \to 0$ in $\mathbb{H}$. Hence, along a further subsequence of $\{t_k\}$, we have $\mathbf{m}(t_k) \rightharpoonup \bar{\mathbf{m}}$; for this $\bar{\mathbf{m}}$ we have

$$\operatorname{div}(\bar{F} + \bar{\mathbf{m}}\chi_\Omega) = 0.$$

If, in addition, one has that $F(t) \to \bar{F}$ strongly in $L^2(\mathbb{R}^n;\mathbb{R}^n)$ then it would follow that $\bar{\mathbf{m}}(x) \in \partial\Phi(\bar{F}(x) + H(x))$ for a.e. $x \in \Omega$ and thus $\bar{\mathbf{m}}(x)$ would be a generalized minimizer for energy $I(\mathbf{m})$.

Therefore, in some sense, the system (3.18) defines a reasonable evolution process for the functional $I(\mathbf{m})$.

3.2.4 Study of a special case

We investigate a special case of the gradient flow in the soft case in $\mathbb{R}^3$. Assume $\varphi \equiv 0$ and thus $\Phi(\xi) = |\xi|$. Let H be a constant and Ω be a ball in $\mathbb{R}^3$. We study the gradient flow (3.13) with initial datum $F_0 \in \mathbb{V}$ that equals a constant vector parallel to H; that is, $F_0 = \alpha H$ in Ω, where $\alpha \in \mathbb{R}$ is a constant.

We are trying to find the solution $F(t)$ such that $F(x,t) = \eta(t)$ on Ω and a corresponding vector function $\mathbf{m}(t) = \mathbf{m}(x,t)$ of (3.18) is also constant in Ω; that is, $\mathbf{m}(x,t) = k(t) \in \mathbb{R}^3$ for $x \in \Omega$. Therefore $P_{\mathbb{V}}(k(t)\chi_\Omega)(x,t) = -\dot{F}(x,t) - F(x,t)$ for $x \in \mathbb{R}^3$.

Let us recall from Section 2.2 that for unit ball $\Omega = B_1(0) \in \mathbb{R}^n$ and any constant $k \in \mathbb{R}^n$, we have

$$P_{\mathbb{V}}(k\chi_\Omega) = \begin{cases} \frac{1}{n}k & \text{in } \Omega, \\ \nabla\Big(\dfrac{1}{n}\dfrac{k\cdot x}{|x|^n}\Big) & \text{in } \Omega^c. \end{cases}$$

Therefore, in our case, since

$$\partial\Phi(\xi) = \begin{cases} \xi/|\xi| & (\xi \neq 0), \\ \\ \overline{B_1(0)} & (\xi = 0), \end{cases}$$

the gradient flow implies

$$\dot{\eta}(t) + \eta(t) + \frac{1}{3}K(\eta(t)) \ni 0 \quad (t > 0), \tag{3.19}$$

where $K(\eta)$ is the set-valued function defined by

$$K(\eta) = \begin{cases} \dfrac{\eta + H}{|\eta + H|}, & \eta \neq -H, \\ \\ \overline{B_1(0)}, & \eta = -H. \end{cases}$$

Once we solve $\eta(t)$ from (3.19), let $k(t) = K(\eta(t))$ and solve $f(x,t)$ for all $x \in \Omega^c$ by

$$\begin{cases} \dot{f}(x,t) + f(x,t) + \nabla\left(\frac{k(t)\cdot x}{3|x|^3}\right) = 0 \quad (t > 0), \\ \\ f(x,0) = F_0(x). \end{cases}$$

Then the function $F(x,t) = \eta(t)\chi_\Omega(x) + f(x,t)\chi_{\Omega^c}(x)$ will be the solution to the gradient flow (3.13). Consequently, the gradient flow (3.13) becomes equivalent to the 3-D gradient flow (3.19) on $\mathbb{R}^3$. Define

$$E(\eta) = \frac{1}{2}|\eta|^2 + \frac{1}{3}|\eta + H| \quad (\eta \in \mathbb{R}^3).$$

Then the problem (3.19) becomes the gradient flow of $E(\eta)$ on $\eta \in \mathbb{R}^3$.

We now study the solution $\eta(t)$ to (3.19) in difference cases.

Case 1. $|H| \leq 1/3$. (In this case, notice that the minimizer of $E(\eta)$ is $\bar{\eta} = -H$.)

I. If $\alpha = -1$, i.e. $\eta(0) = -H$, then $\eta(t) \equiv -H$. Next, consider that $\alpha \neq -1$. The gradient flow is given by

$$\dot{\eta} + \eta + \frac{1}{3}\frac{\eta + H}{|\eta + H|} = 0,$$
$$\eta(0) = \alpha H.$$

Let P be a vector such that $P \perp H$ and we denote $\eta(t) \cdot P = h(t)$ and we dot product both sides by P, then

$$h'(t) + h(t) + \frac{1}{3}\frac{h(t)}{|\eta + H|} = 0,$$
$$h(0) = 0.$$

We can conclude that $h(t) \equiv 0$, which implies that there exists $g(t)$, such that $\eta(t) = g(t) \cdot H$. Consequently,

$$\begin{cases} g'(t) + g(t) + \dfrac{1}{3|H|}\dfrac{g(t)+1}{|g(t)+1|} = 0, \\ g(0) = \alpha. \end{cases} \tag{3.20}$$

Let $g(t) + 1 = r(t)$, then the equation above reads

$$r'(t) + r(t) + \frac{1}{3|H|}\frac{r(t)}{|r(t)|} = 1,$$
$$r(0) = \alpha + 1.$$

II. If $\alpha > -1$, then $r(0) > 0$. Thus $r(t) > 0$ in $0 \leq t < \bar{t}$, for some $\bar{t} > 0$. Hence the equation becomes $r'(t) + r(t) + \dfrac{1}{3|H|} = 1$, then $r(t) = 1 - \dfrac{1}{3|H|} + \Big(\alpha + \dfrac{1}{3|H|}\Big)e^{-t}$, and let $r(\bar{t}) = 0$, then we have $\bar{t} = \ln\Big[\dfrac{3\alpha|H| + 1}{-3|H| + 1}\Big]$.

III. $\alpha < -1$, i.e. $r(0) < 0$. Thus $r(t) < 0$ in $0 \leq t < \bar{t}$, for some $\bar{t} > 0$. Thus the equation becomes $r'(t) + r(t) - \dfrac{1}{3|H|} = 1$, then $r(t) = 1 + \dfrac{1}{3|H|} + \Big(\alpha - \dfrac{1}{3|H|}\Big)e^{-t}$, and let $r(\bar{t}) = 0$, then we have $\bar{t} = \ln\Big[\dfrac{3\alpha|H| - 1}{-3|H| - 1}\Big]$, accordingly.

Combing all the cases when $|H| \leq \dfrac{1}{3}$, there exist $\bar{t} < \infty$ such that $r(\bar{t}) = 0$, which is equivalent to say $\eta(\bar{t}) = -H \equiv \bar{\eta}$, shown in Figure 3.2.

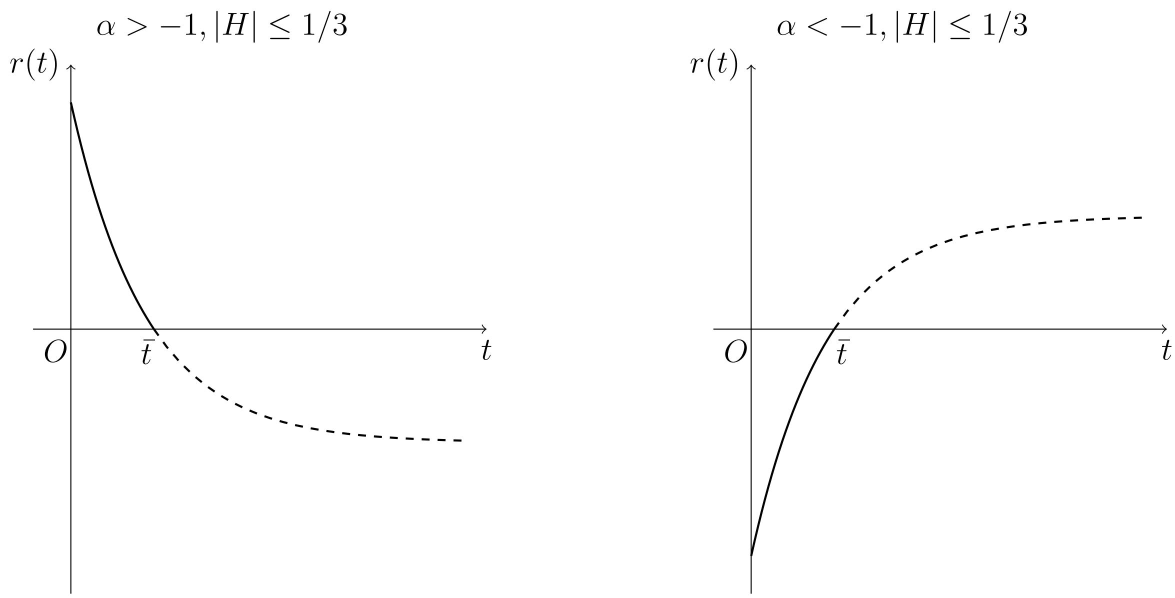

Figure 3.2: The graphs for the case of $|H| \leq \frac{1}{3}$.

Case 2. $|H| \geq 1/3$. (In this case, notice that the minimizer of $E(\eta)$ is $\bar{\eta} = -\frac{H}{3|H|}$.)

Now we still consider problem (3.20).

I. $\alpha > -\dfrac{1}{3|H|}$, i.e. $g(0) > -\dfrac{1}{3|H|}$. Therefore $g(0) + 1 > -\dfrac{1}{3|H|} + 1 > 0$, which implies that $g(t)+1 > 0$ in $0 \leq t < \bar{t}$, for some $\bar{t} > 0$. Now the equation becomes: $g'(t)+g(t)+\dfrac{1}{3|H|} = 0$. Hence $g(t) = -\dfrac{1}{3|H|} + \Big(\alpha + \dfrac{1}{3|H|}\Big)e^{-t}$. It is easy to see that $g(t)$ is decreasing and

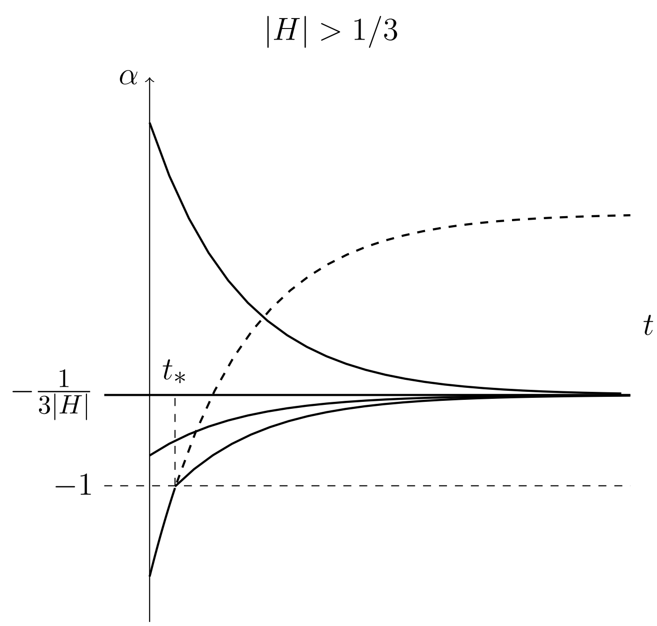

Figure 3.3: The graphs for the case of $|H| > \frac{1}{3}$.

$g(t) \to -\dfrac{1}{3|A|}$ as $t \to \infty$.

II. $-1 < \alpha < -\dfrac{1}{3|H|}$, i.e. $-1 < g(0) < -\dfrac{1}{3|H|}$. Accordingly, $g(0) + 1 > -1 + 1 = 0$, which implies that $g(t) + 1 > 0$ in $0 \leq t < \bar{t}$, for some $\bar{t} > 0$. Now the equation becomes: $g'(t) + g(t) + \dfrac{1}{3|H|} = 0$ and the solution is $g(t) = -\dfrac{1}{3|H|} + \Big(\alpha + \dfrac{1}{3|H|}\Big)e^{-t}$. It is easy to see that $g(t)$ is increasing and $g(t) \to -\dfrac{1}{3|H|}$ as $t \to \infty$.

III. $\alpha < -1$, i.e. $g(0) < -1$. In this case, $g(0) + 1 < -1 + 1 = 0$, which implies that $g(t)+1 < 0$ in $0 \leq t < t_*$, for some $t_* > 0$. Now the equation becomes $g'(t)+g(t)-\dfrac{1}{3|H|} = 0$, when $0 \leq t < t_*$. The and the solution is $g(t) = \dfrac{1}{3|H|} + \Big(\alpha - \dfrac{1}{3|H|}\Big)e^{-t}$, which is increasing. And t_* can be solved by setting $g(t_*) + 1 = 0$ and $\bar{t} = \ln \dfrac{1 - 3|H|\alpha}{1 + 3|H|}$, which indicates that $g(t)$ should satisfies that

$$\begin{cases} g'(t) + g(t) - \dfrac{1}{3|H|} = 0, \quad \text{with } g(0) = \alpha, \qquad 0 \leq t < t_* \\[2ex] g'(t) + g(t) + \dfrac{1}{3|H|} = 0, \quad \text{with } g(t_*) = -1, \quad t \geq t_*. \end{cases}$$

with

$$g(t) = \begin{cases} \dfrac{1}{3|H|} + \Big(\alpha - \dfrac{1}{3|H|}\Big)e^{-t}, & 0 \leq t < t_* \\ -\dfrac{1}{3|H|} + \Big(-1 + \dfrac{1}{3|H|}\Big)e^{-\left(t - \frac{1-3|H|\alpha}{1+3|A|}\right)}, & t \geq t_*. \end{cases}$$

We can also see that $g(t)$ is increasing and $g(t) \to -\dfrac{1}{3|H|}$ as $t \to \infty$.

Combining all the cases when $|H| > \dfrac{1}{3}$, We can also see that $g(t) \to -\dfrac{1}{3|H|}$ as $t \to \infty$ but never equals $-\dfrac{1}{3|H|}$, shown in Figure 3.3.

Chapter 4

A Minimization Problem on Annulus

In this chapter, we assume $n \geq 3$ is an integer and $\Omega = \{x \in \mathbb{R}^n \mid a < |x| < 1\}$ is a spherical shell in $\mathbb{R}^n$. We study the reduced micromagnetic energy:

$$I(\mathbf{m}) = -\int_\Omega \lambda m_1(x)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |F_{\mathbf{m}}(x)|^2\,dx$$

for certain constants $\lambda \geq 0$.

Related to the minimization of this functional $I(\mathbf{m})$, we study the boundary value problem of the Eikonal equation:

$$\begin{cases} |\nabla\psi(s,t)| = t^{n-2} & \text{in } a^2 < s^2 + t^2 < 1,\ t > 0, \\ \psi(s,t) = 0 & \text{on } s^2 + t^2 = a^2,\ t \geq 0, \\ \psi(s,t) = \frac{n\lambda}{n-1} t^{n-1} & \text{on } s^2 + t^2 = 1,\ t \geq 0, \\ \psi(s,t) = 0 & \text{on } t = 0,\ a \leq |s| \leq 1, \end{cases} \tag{4.1}$$

where $0 < a < 1$, $\lambda \geq 0$ and $n \geq 3$ are given numbers. The problem (4.1) is also related to the problem of finding a divergence-free unit-length vector function $G \in L^2(\Omega; \mathbb{S}^{n-1})$ satisfying

$$G \cdot \nu = 0 \ \text{ on } |x| = a, \ \ G \cdot \nu = n\lambda x_1 \ \text{ on } |x| = 1, \tag{4.2}$$

where ν denotes the outer unit normal on $\partial\Omega$. For example, if $\psi(s,t)$ is a solution to (4.1), then the vector function $G(x)$ defined by

$$G(x) = |x'|^{2-n} \left(\psi_t(x_1, |x'|)\mathbf{e}_1 - \psi_s(x_1, |x'|) \frac{x'}{|x'|} \right) \tag{4.3}$$

is a divergence-free unit-length vector function satisfying (4.2) in the distributional sense, where $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$ and $x' = x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$. Let

$$w^\lambda(x) = \lambda x_1\, \chi_{\{|x|\le 1\}} + \frac{\lambda x_1}{|x|^n}\, \chi_{\{|x|>1\}}, \quad F^\lambda = -\nabla w^\lambda.$$

Then, all such unit-length vector functions G can be described by

$$\operatorname{div}(F^\lambda + G\chi_\Omega) = 0 \tag{4.4}$$

in the sense of distributions on $\mathbb{R}^n$.

Suppose ψ is a Lipschitz solution to (4.1) and let $\mathbf{m} = G(x)$ be defined by (4.3); so (4.4) holds. Since $F^\lambda = -\lambda\mathbf{e}_1$ in Ω, for all $F \in L^2(\mathbb{R}^n;\mathbb{R}^n)$ with $\operatorname{curl} F = 0$, we have $|F + \lambda\mathbf{e}_1| = |F - F^\lambda| \ge G \cdot (F - F^\lambda)$ and hence

$$J^*(F) - J^*(F^\lambda) \ge \int_{\mathbb{R}^n} F^\lambda \cdot (F - F^\lambda)\, dx + \int_\Omega G \cdot (F - F^\lambda)\, dx$$

$$= \int_{\mathbb{R}^n} (F^\lambda + G\chi_\Omega) \cdot (F - F^\lambda)\, dx = 0.$$

Therefore F^λ is the unique minimizer of J^*. For $\mathbf{m} = G(x)$, by (4.4), we have that $F_{\mathbf{m}} = -F^\lambda$ and that $\int_{\mathbb{R}^n} F^\lambda \cdot F^\lambda dx = -\int_\Omega F^\lambda \cdot G dx$, and hence

$$I(\mathbf{m}) = \int_\Omega F^\lambda(x) \cdot G(x)\, dx + \frac{1}{2}\int_{\mathbb{R}^n} |F^\lambda(x)|^2\, dx = -\frac{1}{2}\int_{\mathbb{R}^n} |F^\lambda|^2\, dx = -J^*(F^\lambda).$$

Consequently, $\bar{\mathbf{m}} = G(x)$ is a minimizer of I.

Some existence results on the Lipschitz solutions to the boundary value problem for general first-order differential equations have been given in [20], but to apply such an existence theorem to our problem, one would need to find a Lipschitz function ψ^0 satisfying, in addition to the required boundary conditions, the inequality $|\nabla\psi^0(s,t)| \le t^{n-2}$; the construction of such a function ψ^0 is equally difficult as that of a solution ψ of (4.1). Even such a function ψ^0 is known to exist, the general existence theorem would only assert the existence of infinitely many Lipschitz solutions to (4.1) without specifying the structures of any such solutions.

Our main idea is to use the characteristics method to construct the local solutions near the boundaries and then glue them together with certain trivial solutions away from the boundaries. As we shall see in Theorems 4.4.6 and 4.4.9 below, the solutions constructed this way do not have infinitely many oscillations, which would be otherwise expected by the general existence theorems of [20].

4.1 The characteristics method

The rest of the chapter is devoted to the construction of the Lipschitz solutions $\psi(s,t)$ of (4.1) that are even in s. For this purpose, let

$$\omega = \{z = (s,t) \mid s > 0,\, t > 0,\, a^2 < s^2 + t^2 < 1\}.$$

We state our main result as the following theorem.

Theorem 4.1.1. *The following problem has a Lipschitz solution $\psi(s,t)$ on $\bar{\omega}$ if and only if* *the following condition holds.*

$$0 \le \lambda \le \frac{1}{n}(1 - a^{n-1}), \tag{4.5}$$

$$\begin{cases} |\nabla \psi(s,t)| = t^{n-2} & \textit{in } \omega, \\ \psi(s,t) = 0 & \textit{on } |z| = a \textit{ with } s > 0, t > 0, \\ \psi(s,t) = 0 & \textit{on } t = 0,\ s \in [a,1], \\ \psi(s,t) = \frac{n\lambda}{n-1} t^{n-1} & \textit{on } |z| = 1 \textit{ with } s > 0, t > 0. \end{cases} \tag{4.6}$$

Proof of the necessity of (4.5). Suppose ψ is a Lipschitz solution on $\bar{\omega}$. Then

$$\begin{aligned} \frac{n}{n-1}\lambda = \int_a^1 \psi_t(0,t)\,dt &\le \int_a^1 |\nabla \psi(0,t)|\,dt \\ &\le \int_a^1 t^{n-2}\,dt = \frac{1}{n-1}(1 - a^{n-1}). \end{aligned}$$

This proves that $\lambda \le \frac{1}{n}(1 - a^{n-1})$. $\square$

The proof of the sufficiency of (4.5) is the main purpose of this thesis; we state this sufficiency part in the following theorem, including some application to certain minimizers of the functional $I(\mathbf{m})$.

Theorem 4.1.2. *If* $0 \le \lambda \le \frac{1}{n}(1 - a^{n-1})$*, then the problem* (4.6) *has infinitely many Lipschitz solutions* $\psi(s,t)$*, constructed in Theorems 4.4.6 and 4.4.9 below. In this case, when* $n \ge 3$ *is integer and* Ω *is defined as above, the minimizers* $\bar{\mathbf{m}} = G(x)$ *of* $I(\mathbf{m})$ *given by* (4.3) *with the constructed solution* ψ *will be the constant* $\pm\mathbf{e}_1$ *in certain subdomains* $\Omega_0 = \{(x_1, x') \in \Omega \mid (|x_1|, |x'|) \in \mathcal{Z}_0\}$ *away from the boundary* $\partial\Omega$.

Some existence results on the Lipschitz solutions to the boundary value problem for general first-order differential equations have been given in [20], but to apply such an existence theorem to our problem, one would need to find a Lipschitz function ψ^0 on $\bar{\omega}$ satisfying, in addition to the required boundary conditions, the inequality $|\nabla\psi^0(s,t)| \leq t^{n-2}$ in ω. Construction of such a function ψ^0 is equally difficult as that of a solution ψ of (4.6).

Our main idea is to use the characteristics method to construct the local solutions near the two quarter-circles of the boundary of ω and then to glue them together with certain trivial solutions away from the boundaries. We write the equation $|\nabla\psi| = t^{n-2}$ as

$$F(s,t,\psi,\psi_s,\psi_t) = 0,$$

where $F(s,t,z,p,q) = \frac{1}{2}(p^2+q^2-t^{2n-4})$. The characteristics ODEs for this first-order PDE are given by (see [28])

$$\begin{aligned} &\frac{ds}{d\tau} = p, \quad \frac{dt}{d\tau} = q, \quad \frac{dz}{d\tau} = t^{2n-4}, \\ &\frac{dp}{d\tau} = 0, \quad \frac{dq}{d\tau} = (n-2)t^{2n-5}. \end{aligned} \tag{4.7}$$

We solve the system (4.7) on $\tau \geq 0$ with given initial data

$$(s,t,z,p,q)|_{\tau=0} = (\alpha(\theta),\beta(\theta),\gamma(\theta),f(\theta),g(\theta)), \tag{4.8}$$

where $\alpha(\theta),\beta(\theta),\gamma(\theta),f(\theta)$ and $g(\theta)$ depend on a parameter θ in an interval I. Assume the functions $\alpha(\theta),\beta(\theta),\gamma(\theta),f(\theta)$ and $g(\theta)$ are smooth in I and satisfy the *characteristics strip conditions*:

$$\begin{cases} f(\theta)^2+g(\theta)^2 = \beta(\theta)^{2n-4}, \\ f(\theta)\alpha'(\theta)+g(\theta)\beta'(\theta) = \gamma'(\theta) \end{cases} \quad \forall\ \theta \in I. \tag{4.9}$$

For each $\theta \in I$, the smooth solutions to (4.7)-(4.8) will be denoted by

$$s = S(\tau, \theta),\ t = T(\tau, \theta),\ z = Z(\tau, \theta),\ p = P(\tau, \theta),\ q = Q(\tau, \theta).$$

We easily solve $s = S(\tau, \theta)$ and $p = P(\tau, \theta)$ to have

$$P(\tau, \theta) = f(\theta), \quad S(\tau, \theta) = \alpha(\theta) + f(\theta)\tau \quad \forall\ \tau \geq 0,\ \theta \in I. \tag{4.10}$$

Solving (t, q) in (4.7) we have that, for each $\theta \in I$, the unique smooth solution $t = T(\tau, \theta)$ exists on a maximal interval $[0, \tau_M(\theta))$ and satisfies

$$\begin{cases} t'^2 = t^{2n-4} - f(\theta)^2, \quad 0 < \tau < \tau_M(\theta), \\ t(0) = \beta(\theta), \quad t'(0) = g(\theta), \end{cases} \tag{4.11}$$

and hence the solution $Q(\tau, \theta) = T_\tau(\tau, \theta)$ satisfies

$$f(\theta)^2 + Q(\tau, \theta)^2 = T(\tau, \theta)^{2n-4} \quad \forall\, \theta \in I,\ \tau \in [0, \tau_M(\theta)). \tag{4.12}$$

After we solve $T(\tau, \theta)$, we easily obtain $z = Z(\tau, \theta)$ by integration:

$$Z(\tau, \theta) = \gamma(\theta) + \int_0^\tau T(\eta, \theta)^{2n-4} d\eta \quad \forall\, \theta \in I,\ \tau \in [0, \tau_M(\theta)). \tag{4.13}$$

Using $T^{2n-4} = f^2 + Q^2$ and $T_\tau = Q$, an integration by parts yields that

$$Z(\tau, \theta) = \gamma(\theta) + f(\theta)^2\tau + \int_0^\tau T_\eta(\eta, \theta) Q(\eta, \theta)\, d\eta$$

$$= \gamma(\theta) + f(\theta)^2\tau + [T(\eta,\theta)Q(\eta,\theta)]_0^\tau - \int_0^\tau T(\eta,\theta)Q_\eta(\eta,\theta)\,d\eta.$$

Plugging in $Q_\eta(\eta,\theta) = (n-2)T(\eta,\theta)^{2n-5}$ and rearranging terms, we have

$$\begin{aligned} Z(\tau,\theta) &= \gamma(\theta) + \frac{1}{n-1}[f(\theta)^2\tau + T(\tau,\theta)Q(\tau,\theta) - \beta(\theta)g(\theta)] \\ &= \left(\gamma - \frac{f\alpha + g\beta}{n-1}\right) + \frac{S(\tau,\theta)f(\theta) + T(\tau,\theta)Q(\tau,\theta)}{n-1}. \end{aligned} \tag{4.14}$$

We remark that when $n = 3$ the equations for (t, q) become *linear* and the system (4.7) can be solvable in an explicit form; however, the subsequent calculations are complicated and too specific. As we see later, the case $n > 3$ presents some different features from the case $n = 3$.

4.1.1 The maximal existence time $\tau_M(\theta)$

In view of the two subsequent cases to be considered, we make the following assumption:

$$\alpha(\theta) > 0, \quad \beta(\theta) > 0, \quad f(\theta) \neq 0 \quad \forall\, \theta \in I. \tag{4.15}$$

For a given $\theta \in I$ we find the number $\tau_M = \tau_M(\theta)$ according to the sign of $g(\theta)$.

Case (i): Assume $g(\theta) \geq 0$. In this case, the equation in (4.11) becomes

$$\frac{dt}{d\tau} = \sqrt{t^{2n-4} - f(\theta)^2} \quad \text{on } \tau \in (0, \tau_M).$$

The solution $t = T(\tau, \theta)$ is increasing in τ and satisfies $T(\tau, \theta) > \beta(\theta)$ on $(0, \tau_M)$; moreover, since $(0, \tau_M)$ is maximal interval of existence for $T(\tau, \theta)$, we must have

$$\lim_{\tau \to \tau_M^-} T(\tau, \theta) = \infty.$$

Given any $\tau \in [0, \tau_M)$ and $t \geq \beta(\theta)$, it follows that $t = T(\tau, \theta)$ if and only if

$$\tau = A(t, \theta) := \int_{\beta(\theta)}^{t} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}}. \tag{4.16}$$

Therefore, we obtain

$$\tau_M = \tau_M(\theta) = \int_{\beta(\theta)}^{+\infty} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}} \quad \text{if } g(\theta) \geq 0.$$

Note that $\tau_M = +\infty$ if $n = 3$ and $\tau_M < +\infty$ if $n > 3$ since $f(\theta) \neq 0$. In this case, $t = T(\tau, \theta)$ is the inverse function of $\tau = A(t, \theta)$, and $Q(\tau, \theta) > 0$ for all $0 < \tau < \tau_M(\theta)$.

Case (ii): Assume $g(\theta) < 0$. In this case, we have that $T_\tau(\tau, \theta) = Q(\tau, \theta) < 0$ and thus $T(\tau, \theta)$ is decreasing in τ on some interval $\tau \in [0, \tau_m)$, where $\tau_m = \tau_m(\theta) > 0$ is a number such that $Q(\tau_m^-, \theta) = T_\tau(\tau_m^-, \theta) = 0$. By (4.12), we obtain

$$\lim_{\tau \to \tau_m^-} T(\tau, \theta) = |f(\theta)|^{\frac{1}{n-2}}.$$

Clearly, the functions $t = T(\tau, \theta)$ and $q = Q(\tau, \theta)$ satisfy

$$\frac{dt}{d\tau} = q = -\sqrt{t^{2n-4} - f(\theta)^2} \quad \forall\, \tau \in (0, \tau_m). \tag{4.17}$$

It follows that $|f(\theta)|^{\frac{1}{n-2}} < T(\tau,\theta) \leq \beta(\theta)$ for $0 \leq \tau < \tau_m$. Moreover, given any $0 \leq \tau < \tau_m(\theta)$ and $|f(\theta)|^{\frac{1}{n-2}} < t \leq \beta(\theta)$, it follows that $t = T(\tau,\theta)$ if and only if

$$\tau = -A(t,\theta) = \int_t^{\beta(\theta)} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}}. \tag{4.18}$$

Letting $\tau \to \tau_m^-$, we see that

$$\tau_m = \tau_m(\theta) = \int_{|f(\theta)|^{\frac{1}{n-2}}}^{\beta(\theta)} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}}. \tag{4.19}$$

Note that $0 < \tau_m < +\infty$ since $f(\theta) \neq 0$. We now solve $T(\tau,\theta)$ and $Q(\tau,\theta)$ for $\tau > \tau_m$. As in the first case, using $T(\tau_m,\theta) = |f(\theta)|^{\frac{1}{n-2}}$, solutions $t = T(\tau,\theta)$ and $q = Q(\tau,\theta)$ satisfy

$$\frac{dt}{d\tau} = q = \sqrt{t^{2n-4} - f(\theta)^2} \quad \forall\, \tau \in (\tau_m, \tau_M).$$

Hence $t = T(\tau,\theta)$ is increasing in τ and satisfies $T(\tau,\theta) > |f(\theta)|^{\frac{1}{n-2}}$ on (τ_m,τ_M). The maximal existence time τ_M of $T(\tau,\theta)$ must satisfy

$$\lim_{\tau \to \tau_M^-} T(\tau,\theta) = \infty.$$

Moreover, given $\tau \in [\tau_m,\tau_M)$ and $t \geq |f(\theta)|^{\frac{1}{n-2}}$, it follows that $t = T(\tau,\theta)$ if and only if

$$\tau - \tau_m = \int_{|f(\theta)|^{\frac{1}{n-2}}}^{t} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}},$$

that is,

$$\tau = 2\tau_m(\theta) + A(\tau,\theta) = 2\tau_m(\theta) + \int_{\beta(\theta)}^{t} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}}. \tag{4.20}$$

Letting $\tau \to \tau_M^-$, we obtain

$$\tau_M(\theta) = 2\tau_m(\theta) + \int_{\beta(\theta)}^{+\infty} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}} \quad \text{if } g(\theta) < 0.$$

Again, $\tau_M = +\infty$ if $n = 3$ and $\tau_M < \infty$ if $n > 3$. In this case, it is easily shown that the solution $T(\tau, \theta)$ so constructed is smooth on $[0, \tau_M(\theta))$ by verifying $T_\tau(\tau_m^-, \theta) = T_\tau(\tau_m^+, \theta)$. (Some related computation is given below.) Furthermore, $Q(\tau, \theta) < 0$ for $0 < \tau < \tau_m(\theta)$ and $Q(\tau, \theta) > 0$ for $\tau_m(\theta) < \tau < \tau_M(\theta)$.

4.1.2 Inverting the characteristics map

Let $\tau_M(\theta)$ be defined as above, and define

$$\mathcal{D} = \{(\tau, \theta) \mid \theta \in I, \;\; 0 \leq \tau < \tau_M(\theta)\}. \tag{4.21}$$

Define the *characteristic map*

$$(S(\tau, \theta), T(\tau, \theta)) \colon \mathcal{D} \to \mathbb{R}^2,$$

and consider the curve $\Gamma = \{(\alpha(\theta), \beta(\theta)) \mid \theta \in I\}$. We would like to find a subdomain $\mathcal{Z}$ of ω with $\Gamma \subset \partial\mathcal{Z}$ and a subdomain $\mathcal{Y}$ of $\mathcal{D}$ such that for each $(s, t) \in \mathcal{Z}$ there exists a unique $(\tau, \theta) = (\eta(s, t), \xi(s, t))$ in $\mathcal{Y}$ satisfying

$$(s, t) = (S(\tau, \theta), T(\tau, \theta));$$

that is, the map $(S,T)\colon \mathcal{Y} \to \mathcal{Z}$ is *bijective.* Let $(\tau,\theta) = (\eta(s,t),\xi(s,t))\colon \mathcal{Z} \to \mathcal{Y}$ be its inverse map. Of course, a standard method would be to study the Jacobian of the map $(s,t) = (S(\tau,\theta),T(\tau,\theta))$. However, we use different (but, eventually, equivalent) methods depending on the specific parametrization of curve Γ.

4.1.3 Construction of the local solutions

Once we obtain the inverse map (η,ξ) of the map (S,T), we define a local solution ψ by

$$\psi(s,t) = Z(\eta(s,t),\xi(s,t)) \quad \forall\,(s,t) \in \mathcal{Z},$$

where $Z(\tau,\theta)$ is defined by (4.13) above. By continuity we may also extend ψ to some of the boundary points of $\mathcal{Z}$. Note that by (4.14) the solution ψ on $\mathcal{Z}$ can be computed as

$$\psi(s,t) = \left[\gamma(\theta) - \frac{f(\theta)\alpha(\theta) + g(\theta)\beta(\theta)}{n-1} + \frac{sf(\theta) \pm t\sqrt{t^{2n-4} - f(\theta)^2}}{n-1}\right]_{\theta=\xi(s,t)} \tag{4.22}$$

with the choice of "$\pm$" the same as the sign of $Q(\eta(s,t),\xi(s,t))$.

In the next two sections, we carry out these constructions near the inner circle $|z| = a$ and near the outer circle $|z| = 1$ separately.

4.2 Construction near the inner quarter-circle

In this case, we choose the interval $I = (0,\pi/2)$ and define

$$\alpha(\theta) = a\cos\theta, \quad \beta(\theta) = a\sin\theta, \quad \gamma(\theta) = 0 \quad \forall\,\theta \in I.$$

We fulfill the strip conditions (4.9) by selecting

$$f(\theta) = (a\sin\theta)^{n-2}\cos\theta, \quad g(\theta) = (a\sin\theta)^{n-2}\sin\theta.$$

Hence the condition (4.15) holds and $g(\theta) > 0$ for all $\theta \in I$ and the domain $\mathcal{D}$ in (4.21) becomes

$$\mathcal{D} = \left\{ (\tau,\theta) \;\middle|\; \theta \in (0,\pi/2),\ 0 \le \tau < \int_{\beta(\theta)}^{+\infty} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}} \right\}.$$

4.2.1 The characteristics solutions

The function $t = T(\tau,\theta)$ is determined uniquely by

$$\tau = \int_{\beta(\theta)}^{T(\tau,\theta)} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}} \quad \forall\ (\tau,\theta) \in \mathcal{D}.$$

Also $Q(\tau,\theta) = T_\tau(\tau,\theta) = \sqrt{T(\tau,\theta)^{2n-4} - f(\theta)^2} \ge 0$ and a change of variables yields that

$$Z(\tau,\theta) = \int_{\beta(\theta)}^{T(\tau,\theta)} \frac{y^{2n-4}dy}{\sqrt{y^{2n-4} - f(\theta)^2}} \quad \forall\ (\tau,\theta) \in \mathcal{D}.$$

Let

$$\mathcal{R} := \{(t,\theta) \mid \theta \in I,\ t \ge \beta(\theta)\}$$

and

$$U(t,\theta) = \int_{\beta(\theta)}^{t} \frac{y^{2n-4}dy}{\sqrt{y^{2n-4} - f(\theta)^2}} \quad \forall\ (t,\theta) \in \mathcal{R}. \tag{4.23}$$

Then

$$Z(\tau,\theta) = U(T(\tau,\theta),\theta) \quad \forall\ (\tau,\theta) \in \mathcal{D}.$$

As above, let

$$A(t,\theta) = \int_{\beta(\theta)}^{t} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}} \quad \forall\ (t,\theta) \in \mathcal{R}.$$

Then, for $(\tau,\theta) \in \mathcal{D}$ and $(t,\theta) \in \mathcal{R}$, it follows that $t = T(\tau,\theta)$ if and only if $\tau = A(t,\theta)$.

4.2.2 Inverting the characteristics map

Define $B(t,\theta) = S(A(t,\theta),\theta)$, that is,

$$B(t,\theta) = \alpha(\theta) + f(\theta) \int_{\beta(\theta)}^{t} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}} \quad \forall\ (t,\theta) \in \mathcal{R}.$$

A direct computation yields that

$$B_\theta(t,\theta) = -\frac{a}{\sin\theta} + f'(\theta) \int_{\beta(\theta)}^{t} \frac{y^{2n-4}\, dy}{(y^{2n-4} - f(\theta)^2)^{3/2}} \quad \forall\ \theta \in I. \tag{4.24}$$

Lemma 4.2.1. *Let $U(t,\theta)$ and $B(t,\theta)$ be defined as above. Then*

$$U_\theta(t,\theta) = B_\theta(t,\theta) f(\theta) \quad \forall\ (t,\theta) \in \mathcal{R}. \tag{4.25}$$

Proof. By (4.23),

$$U_\theta(t,\theta) = -\frac{\beta' \beta^{2n-4}}{g} + ff' \int_{\beta}^{t} \frac{y^{2n-4}\, dy}{(y^{2n-4} - f^2)^{3/2}}.$$

Since

$$ff' \int_{\beta}^{t} \frac{y^{2n-4}\, dy}{(y^{2n-4} - f^2)^{3/2}} = f(\theta) B_\theta(t,\theta) + \frac{af(\theta)}{\sin\theta},$$

consequently, it follows that

$$U_\theta(t,\theta) = -\frac{\beta' \beta^{2n-4}}{g} + \frac{af(\theta)}{\sin\theta} + B_\theta(t,\theta)f(\theta) = B_\theta(t,\theta)f(\theta),$$

resulting from the identity $\frac{\beta'\beta^{2n-4}}{g} = \frac{af(\theta)}{\sin\theta}$. $\square$

Note that

$$f'(\theta) = a(a\sin\theta)^{n-3}[(n-2)-(n-1)\sin^2\theta].$$

So, for $\hat{\theta} = \arcsin(\sqrt{(n-2)/(n-1)})$, it follows that

$$f'(\theta) > 0 \quad \forall\, \theta \in (0,\hat{\theta}), \quad f'(\theta) < 0 \quad \forall\, \theta \in (\hat{\theta}, \pi/2).$$

Hence, by (4.24),

$$B_\theta(t,\theta) < 0 \quad \forall\, \theta \in [\hat{\theta}, \pi/2),\ t \geq \beta(\theta).$$

Lemma 4.2.2. *For all $t > 0$,*

$$\lim_{\theta\to 0^+} B_\theta(t,\theta) = \begin{cases} a\int_1^\infty \frac{d\eta}{\sqrt{\eta^{2n-4}-1}} & (n>3), \\ +\infty & (n=3). \end{cases} \tag{4.26}$$

Proof. By a change of variables and integration by parts, we have

$$B_\theta(t,\theta) = -\frac{a(n-1)}{n-2}\sin\theta + \frac{1}{n-2}\left(\int_{\beta/k}^{t/k} \frac{f' f^{\frac{3-n}{n-2}}\, d\eta}{\sqrt{\eta^{2n-4}-1}} - \frac{tf'}{\sqrt{t^{2n-4}-f^2}}\right),$$

where $k = f^{\frac{1}{n-2}}$. Since $\beta/k = 1/(\cos\theta)^{\frac{1}{n-2}} \to 1$ and $t/k \to \infty$ as $\theta \to 0^+$, it follows that

$$\lim_{\theta\to 0^+} \int_{\beta/k}^{t/k} \frac{d\eta}{\sqrt{\eta^{2n-4}-1}} = \begin{cases} \int_1^\infty \frac{d\eta}{\sqrt{\eta^{2n-4}-1}} & (n > 3), \\ +\infty & (n = 3). \end{cases}$$

Note that

$$\lim_{\theta\to 0^+} f'(\theta) = \begin{cases} a & (n = 3), \\ 0 & (n > 3) \end{cases} \quad \text{and} \quad \lim_{\theta\to 0^+} f'(\theta) f(\theta)^{\frac{3-n}{n-2}} = a(n-2).$$

Combining these limits, we have (4.26). □

For each $t > 0$, let

$$\theta_0(t) = \begin{cases} \arcsin\frac{t}{a} & 0 < t < a, \\ \pi/2 & t \geq a. \end{cases}$$

Then $B_\theta(t, \theta_0(t)) = -a^2/t < 0$ if $0 < t < a\sin\hat{\theta}$, and $B_\theta(t,\theta) < 0$ for all $\theta \in [\hat{\theta}, \pi/2)$ if $t \geq a\sin\hat{\theta}$. Hence the following quantity is well-defined:

$$\theta_*(t) = \inf\{\theta \in (0, \theta_0(t)) \mid B_\theta(t, \theta') \leq 0 \;\; \forall\, \theta < \theta' < \theta_0(t)\}. \tag{4.27}$$

Clearly $\theta_*(t) \leq \hat{\theta}$ for all $t > 0$ and, by (4.26), $\theta_*(t) > 0$ for all $t > 0$. Furthermore,

$$B_\theta(t, \theta_*(t)) = 0, \quad B_\theta(t,\theta) \leq 0 \quad \forall\, \theta_*(t) \leq \theta < \theta_0(t). \tag{4.28}$$

Lemma 4.2.3. *For each $t > 0$, the function $B(t,\theta)$ is one-to-one on the interval $\theta \in$*

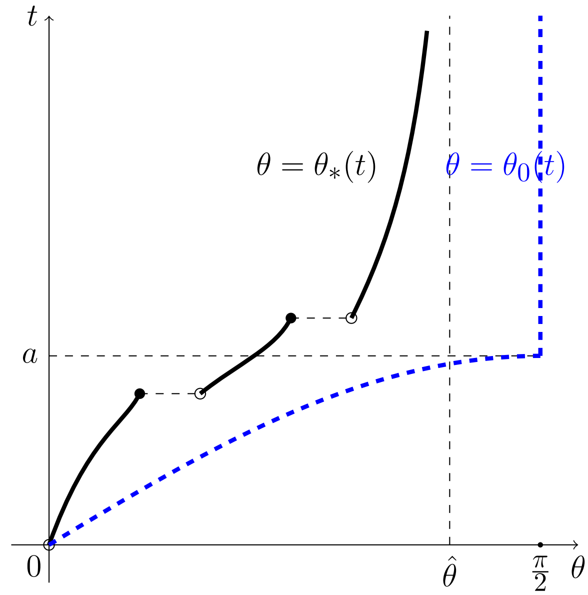

Figure 4.1: The function $\theta = \theta_*(t)$ defined by (4.27) is strictly increasing and left-continuous.

$[\theta_*(t), \theta_0(t))$. *Moreover, the function* $\theta_*(t)$ *is strictly increasing and left-continuous on* $t > 0$ *and* $\theta_*(0^+) = 0$.

Proof. Let $a, b \in [\theta_*(t), \theta_0(t))$ be such that $B(t, a) = B(t, b)$. We show $a = b$. If $a < b$ then $B_\theta(t, \theta) = 0$ for all $\theta \in (a, b)$, which is impossible by the formula of $B_\theta(t, \theta)$ given above. Therefore $B(t, \theta)$ is one-to-one on $[\theta_*(t), \theta_0(t))$. To show θ_* is strictly increasing, let $0 < t < t'$ and suppose, for the contrary, $\theta_*(t) \geq \theta_*(t')$; then $\theta_*(t') \leq \theta_*(t) \leq \hat{\theta}$ and $B_\theta(t', \theta_*(t)) \leq 0$. Note that

$$B_{\theta t}(t, \theta) = \frac{f'(\theta) t^{2n-4}}{(t^{2n-4} - f(\theta)^2)^{3/2}} > 0 \quad \forall\, 0 < \theta \leq \hat{\theta},\ t \geq \beta(\theta).$$

We have that $B_\theta(t', \theta_*(t)) > B_\theta(t, \theta_*(t)) = 0$, which gives a contradiction. To show the left-continuity of θ_*, given $t > 0$, let $l = \theta_*(t^-)$; then $0 < l \leq \theta_*(t)$ and $B_\theta(t, l) = 0$. Given each $\theta' \in (l, \theta_0(t))$, for all $t' < t$ sufficiently closed to t, we have $\theta_*(t') < l < \theta' < \theta_0(t)$ and hence $B_\theta(t', \theta') \leq 0$. Taking $t' \to t^-$ yields that $B_\theta(t, \theta') \leq 0$. By definition, $l \geq \theta_*(t)$; so $l = \theta_*(t)$.

This proves that θ_* is left-continuous on $t > 0$. Furthermore, from $\beta(\theta_*(t)) < \beta(\theta_0(t)) \le t$, it follows that $\theta_*(0^+) = 0$. $\square$

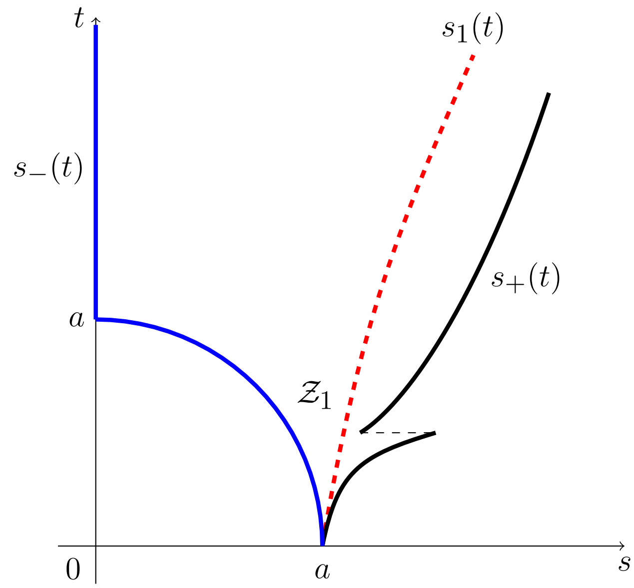

Figure 4.2: The domain $\mathcal{Z}_1$ and the smooth increasing function $s = s_1(t)$ determined in Lemma 4.2.7.

Define (see Figure 4.2)

$$s_+(t) = B(t, \theta_*(t)), \;\; s_-(t) = \sqrt{(a^2 - t^2)^+} \quad \forall\, t > 0,$$

and

$$\mathcal{Z}_1 = \{(s,t) : \; t > 0, \;\; s_-(t) < s < s_+(t)\}.$$

In general, $s_+(t)$ is left-continuous but may not be continuous on $(0, \infty)$. Note that

$$s_-(t) = \lim_{\theta \to \theta_0(t)^-} B(t, \theta) \quad \forall\, t > 0. \tag{4.29}$$

Lemma 4.2.4. *There exists a unique function $\theta = \xi(s,t)$ on $\mathcal{Z}_1$ such that*

$$s = B(t, \xi(s,t)) \quad \forall\ (s,t) \in \mathcal{Z}_1.$$

Moreover, the function $\theta = \xi(s,t)$ is continuous on $\mathcal{Z}_1$ and is differentiable at every point (s_0, t_0) of $\mathcal{Z}_1$ where $B_\theta(t_0, \xi(s_0,t_0)) \neq 0$ and, at any such point (s,t), we have

$$\xi_t = -B_t(t,\xi)/B_\theta(t,\xi), \quad \xi_s = 1/B_\theta(t,\xi).$$

Proof. Let $(s,t) \in \mathcal{Z}_1$; then $t > 0$ and $s_-(t) < s < s_+(t)$. Hence $B(t, \theta_0(t)^-) < s < B(t, \theta_*(t))$. Since $B(t,\theta)$ is one-to-one on $\theta \in [\theta_*(t), \theta_0(t))$, there exists a unique $\theta = \xi(s,t)$ such that

$$\theta_*(t) < \xi(s,t) < \theta_0(t), \quad B(t, \xi(s,t)) = s.$$

We now prove the continuity of $\xi(s,t)$; namely, for all $(s_0,t_0) \in \mathcal{Z}_1$, it follows that

$$\lim_i \xi(s_i, t_i) = \xi(s_0, t_0)$$

for all sequences (s_i, t_i) in $\mathcal{Z}_1$ converging to (s_0, t_0). Let any convergent subsequence of $\xi(s_i,t_i) \to l$. Note that

$$\theta_*(t_i) \le \xi(s_i,t_i) \le \theta_0(t_i), \quad s_i = B(t_i, \xi(s_i,t_i)).$$

Since $\theta_*(t_0) = \theta_*(t_0^-) \le \theta_*(t_0^+) < \theta_0(t_0)$, it follows that $\theta_*(t_0) \le l < \theta_0(t_0)$ and $s_0 = B(t_0, l)$. Hence, by definition, $l = \xi(s_0,t_0)$. This proves the continuity of ξ.

The differentiability of $\xi(s,t)$ at point $(s_0,t_0) \in \mathcal{Z}_1$ where $B_\theta(t_0, \xi(s_0,t_0)) \neq 0$ follows from the continuity of ξ by the implicit function theorem. $\square$

4.2.3 Construction of the solution on $\mathcal{Z}_1$

Let $\xi(s,t)$ be the function defined above and the function $U(t,\theta)$ be defined by (4.23).

Theorem 4.2.5. *Define*

$$\psi^1(s,t) = U(t,\xi(s,t)) \quad \forall\,(s,t) \in \mathcal{Z}_1.$$

Then ψ^1 is differentiable in $\mathcal{Z}_1$ with $\psi^1_s(s,t) > 0$ and satisfies the Eikonal equation $|\nabla\psi^1(s,t)| = t^{n-2}$ in $\mathcal{Z}_1$. Furthermore, ψ^1 can be extended continuously to the curve $s = s_-(t)$ for all $t > 0$ such that

$$\psi^1(s_-(t),t) = \frac{1}{n-1}(t^{n-1} - a^{n-1})^+ \quad \forall\,t > 0. \tag{4.30}$$

Proof. The function ψ^1 is clearly continuous in $\mathcal{Z}_1$. At each point $(s,t) \in \mathcal{Z}_1$ where $B_\theta(t,\xi(s,t)) \neq 0$, ψ^1 is differentiable, and by (4.25),

$$\psi^1_s(s,t) = U_\theta(t,\xi)\xi_s(s,t) = \frac{U_\theta(t,\xi)}{B_\theta(t,\xi)} = f(\xi(s,t)) > 0.$$

On the other hand, with $\theta = \xi(s,t)$,

$$\psi^1_t(s,t) = U_t(t,\theta) + U_\theta(t,\theta)\xi_t(s,t) = U_t(t,\theta) + f(\theta)B_\theta(t,\theta)\xi_t(s,t)$$

$$= U_t(t,\theta) - f(\theta)B_t(t,\theta) = U_t(t,\theta) - f(\theta)^2 A_t(t,\theta)$$

$$= \frac{t^{2n-4}}{\sqrt{t^{2n-4} - f(\theta)^2}} - \frac{f(\theta)^2}{\sqrt{t^{2n-4} - f(\theta)^2}}$$

and hence

$$\psi_t^1(s,t) = \sqrt{t^{2n-4} - f(\xi(s,t))^2}.$$

The formulas of ψ_s^1 and ψ_t^1 also show that ψ^1 is differentiable at every point of $\mathcal{Z}_1$, with $\psi_s^1(s,t) = f(\xi(s,t)) > 0$, and satisfies the equation $|\nabla\psi^1| = t^{n-2}$. Finally, we extend ψ^1 to the curve $s = s_-(t)$ by letting

$$\psi^1(s_-(t),t) = \lim_{s\to(s_-(t))^+} \psi^1(s,t) \quad \forall\, t > 0.$$

We show

$$\psi^1(s_-(t),t) = \frac{1}{n-1}(t^{n-1} - a^{n-1})^+ \quad \forall\, t > 0.$$

Fix $t > 0$. By (4.29), we have

$$\lim_{s\to(s_-(t))^+} \xi(s,t) = \theta_0(t).$$

Therefore

$$\lim_{s\to(s_-(t))^+} \psi^1(s,t) = \lim_{s\to(s_-(t))^+} U(t,\xi(s,t))$$

$$= U(t,\theta_0(t)) = \int_{\beta(\theta_0(t))}^{t} \frac{y^{2n-4}\,dy}{\sqrt{y^{2n-4} - f(\theta_0(t))^2}}$$

$$= \frac{1}{n-1}(t^{n-1} - a^{n-1})^+.$$

□

Lemma 4.2.6. *For all $(s,t) \in \mathcal{Z}_1 \cap \{s > 0,\ s^2 + t^2 = 1\}$, we have*

$$\psi^1(s,t) > \frac{1-a^{n-1}}{n-1}t^{n-1}. \tag{4.31}$$

Proof. Let $(s,t) \in \mathcal{Z}_1$ be such that $s > 0,\ s^2 + t^2 = 1$. Then $\psi^1(s,t) = U(t, h(t))$, where $h(t) = \xi(\sqrt{1-t^2}, t)$. We first show that

$$h(t) < \arcsin t.$$

To this end, let $\tilde{\theta} = \arcsin t$; so $\alpha(\tilde{\theta}) = a\sqrt{1-t^2}$ and $\beta(\tilde{\theta}) = at$. We then have

$$B(t,\tilde{\theta}) = \alpha(\tilde{\theta}) + f(\tilde{\theta}) \int_{\beta(\tilde{\theta})}^{t} (y^{2n-4} - f(\tilde{\theta})^2)^{-1/2} dy$$

$$< a\sqrt{1-t^2} + f(\tilde{\theta}) \int_{at}^{t} (\beta(\tilde{\theta})^{2n-4} - f(\tilde{\theta})^2)^{-1/2} dy$$

$$= a\sqrt{1-t^2} + \frac{f(\tilde{\theta})}{g(\tilde{\theta})}(t - at) = \sqrt{1-t^2} = B(t, h(t)).$$

Since $B_\theta(t,\theta) \le 0$ for all $\theta \in (\theta_*(t), \theta_0(t))$, we derive $\tilde{\theta} > h(t)$. Finally, by the formula of $U(t,\theta)$, we have

$$\psi^1(s,t) = U(t,h(t)) \ge \int_{\beta(h(t))}^{t} y^{n-2}\, dy = \frac{t^{n-1} - \beta(h(t))^{n-1}}{n-1}$$

$$> \frac{t^{n-1} - \beta(\tilde{\theta})^{n-1}}{n-1} = \frac{1-a^{n-1}}{n-1} t^{n-1}.$$

$\square$

Lemma 4.2.7. *There exists a smooth* increasing *function* $s_1(t) \in (s_-(t), s_+(t))$ *such that, for all* $t > 0$,

$$\psi^1(s,t) < \frac{t^{n-1}}{n-1} \quad \forall\ s_-(t) < s < s_1(t),$$

$$\psi^1(s,t) > \frac{t^{n-1}}{n-1} \quad \forall\ s_1(t) < s < s_+(t).$$

Proof. Let $K(s,t) = \psi^1(s,t) - \frac{t^{n-1}}{n-1}$. Then $K_s(s,t) = \psi^1_s(s,t) = f(\xi(s,t)) > 0$ and so, for each $t > 0$, $K(s,t)$ is strictly increasing in $s \in [s_-(t), s_+(t)]$. We will show that

$$K(s_-(t),t) < 0, \quad K(s_+(t),t) > 0 \quad \forall\, t > 0. \tag{4.32}$$

This will prove the existence of $s_1(t) \in (s_-(t), s_+(t))$ such that

$$K(s_1(t),t) = 0.$$

Moreover, since $K_s(s_1(t),t) > 0$, by the implicit function theorem, the function $s_1(t)$ is also differentiable in $t > 0$, with

$$s_1'(t) = \frac{t^{n-2} - \psi^1_t}{\psi^1_s} > 0 \quad \forall\, t > 0,$$

which completes the proof. To prove (4.32), first of all, note that $K_-(t) = K(s_-(t),t) = U(t,\theta_0(t)) - \frac{t^{n-1}}{n-1}$. If $0 < t < a$, then $U(t,\theta_0(t)) = 0$; if $t \geq a$,

$$U(t,\theta_0(t)) = U(t,\pi/2) = \int_a^t y^{n-2}\, dy = \frac{1}{n-1}(t^{n-1} - a^{n-1}).$$

So $K_-(t) < 0$ for all $t > 0$. We now show that

$$K_+(t) = K(s_+(t),t) = U(t,\theta_*(t)) - \frac{t^{n-1}}{n-1} > 0 \quad \forall\, t > 0.$$

Given $t > 0$, we have $B_\theta(t, \theta_*(t)) = 0$ and hence $\theta_*(t) \in (0, \hat{\theta})$. We will actually show that

$$U(t,\theta) > \frac{t^{n-1}}{n-1} \quad \text{whenever } B_\theta(t,\theta) = 0. \tag{4.33}$$

Note that, by (4.14),

$$(n-1)U(t,\theta) = f(\theta)^2 A(t,\theta) + tQ(A(t,\theta),\theta) - \beta(\theta)g(\theta)$$

$$= f(\theta)^2 A(t,\theta) + t\sqrt{t^{2n-4} - f(\theta)^2} - \beta(\theta)g(\theta).$$

From the definition of $A(t,\theta)$, integration by parts yields that

$$A(t,\theta) = \frac{t}{\sqrt{t^{2n-4} - f^2}} - \frac{\beta}{g} + (n-2)\int_\beta^t \frac{y^{2n-4}\,dy}{(y^{2n-4} - f^2)^{3/2}}.$$

Assume, at point (t,θ), $B_\theta(t,\theta) = 0$; so $0 < \theta < \hat{\theta}$. By (4.24), we have

$$\int_\beta^t \frac{y^{2n-4}\,dy}{(y^{2n-4} - f^2)^{3/2}} = \frac{a^2}{\beta f'}.$$

Hence

$$A(t,\theta) = \frac{t}{\sqrt{t^{2n-4} - f^2}} - \frac{\beta}{g} + \frac{(n-2)a^2}{\beta f'}.$$

Therefore,

$$(n-1)U(t,\theta) = \frac{tf^2}{\sqrt{t^{2n-4} - f^2}} - \frac{f^2\beta}{g} + \frac{(n-2)a^2 f^2}{\beta f'} + t\sqrt{t^{2n-4} - f^2} - \beta g$$

$$= \frac{t^{2n-3}}{\sqrt{t^{2n-4} - f^2}} - \frac{\beta^{2n-3}}{g} + \frac{(n-2)a^2 f^2}{\beta f'} > t^{n-1} - \frac{\beta^{2n-3}}{g} + \frac{(n-2)a^2 f^2}{\beta f'}.$$

Clearly

$$0 < \frac{f'}{f} = (n-2)\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} < (n-2)\frac{\cos\theta}{\sin\theta}.$$

Hence

$$\frac{(n-2)a^2 f^2}{\beta f'} > \frac{(n-2)a^2 f}{\beta}\frac{\sin\theta}{(n-2)\cos\theta} = \frac{\beta^{2n-3}}{g},$$

from which it follows that $(n-1)U(t,\theta) > t^{n-1}$. $\square$

4.3 Construction near the outer quarter-circle

In this case, again let $I = (0, \pi/2)$ but define

$$\alpha(\theta) = \cos\theta, \quad \beta(\theta) = \sin\theta, \quad \gamma(\theta) = \frac{n\lambda}{n-1}(\sin\theta)^{n-1} \quad \forall\, \theta \in I.$$

4.3.1 Characteristics strip conditions

To select the functions $f(\theta)$ and $g(\theta)$ to fulfill the strip condition

$$f^2(\theta) + g^2(\theta) = (\sin\theta)^{2n-4}, \tag{4.34}$$

$$-f(\theta)\sin\theta + g(\theta)\cos\theta = \gamma'(\theta) = n\lambda(\sin\theta)^{n-2}\cos\theta, \tag{4.35}$$

we define

$$f(\theta) = (\sin\theta)^{n-2}\cos\varphi, \quad g(\theta) = (\sin\theta)^{n-2}\sin\varphi,$$

where $\varphi = \varphi(\theta)$ is a function of θ to be selected below. In view of (4.35), we have

$$\sin(\varphi - \theta) = n\lambda\cos\theta.$$

This condition alone does not determine $\sin\varphi$ and $\cos\varphi$ uniquely. We require the characteristic curve go inside the disc $s^2 + t^2 < 1$ for small $\tau > 0$. To this end, let $\rho(\tau,\theta) = S^2(\tau,\theta) + T^2(\tau,\theta)$. We require that

$$\frac{d\rho}{d\tau}(0^+,\theta) = 2\alpha(\theta)f(\theta) + 2\beta(\theta)g(\theta) = 2(\sin\theta)^{n-2}\cos(\varphi-\theta) < 0$$

and so that

$$\cos(\varphi-\theta) = -\sqrt{1-(n\lambda\cos\theta)^2}.$$

In this way, f and g are uniquely determined if we set, for all $\theta \in I$,

$$\sin\varphi = n\lambda\cos^2\theta - \sin\theta\sqrt{1-(n\lambda\cos\theta)^2},$$

$$\cos\varphi = -n\lambda\cos\theta\sin\theta - \cos\theta\sqrt{1-(n\lambda\cos\theta)^2}.$$

Lemma 4.3.1. *It follows that*

$$1 \le \varphi'(\theta) < 2, \quad \varphi''(\theta) \ge 0 \quad \forall\, \theta \in I.$$

Proof. Differentiating $\sin(\varphi-\theta) = n\lambda\cos\theta$ twice, we have

$$\cos(\varphi-\theta)(\varphi'-1) = -n\lambda\sin\theta,$$

$$-\sin(\varphi-\theta)(\varphi'-1)^2 + \cos(\varphi-\theta)\varphi'' = -n\lambda\cos\theta = -\sin(\varphi-\theta).$$

Hence

$$\varphi' = 1 - \frac{n\lambda\sin\theta}{\cos(\varphi-\theta)} = 1 + \frac{n\lambda\sin\theta}{\sqrt{1-(n\lambda\cos\theta)^2}},$$

$$\cos(\varphi - \theta)\varphi'' = \sin(\varphi - \theta)\varphi'(\varphi' - 2).$$

The first equation implies $1 \leq \varphi' < 2$ since $0 \leq (n\lambda)^2 < 1$; so, from the second equation, it follows that $\varphi'' \geq 0$, due to the inequalities $\sin(\varphi - \theta) \geq 0$ and $\cos(\varphi - \theta) < 0$. $\square$

Note that $f(\theta) < 0$ but $g(\theta)$ changes signs on I. We solve $g(\theta) = 0$ to obtain $\theta^* = \arctan(n\lambda) \in [0, \pi/4)$ such that

$$g(\theta) > 0 \quad \forall\, \theta \in (0, \theta^*), \quad g(\theta) < 0 \quad \forall\, \theta \in (\theta^*, \pi/2).$$

Let

$$t^* = \sin\theta^* = \frac{n\lambda}{\sqrt{1 + (n\lambda)^2}}, \quad s^* = \cos\theta^* = \frac{1}{\sqrt{1 + (n\lambda)^2}}.$$

Lemma 4.3.2. *There exists a (unique) number $\bar\theta \in (\theta^*, \pi/2)$ such that*

$$f'(\theta) < 0 \ \forall\theta \in (0, \bar\theta); \quad f'(\theta) > 0, \ f''(\theta) > 0 \ \forall\theta \in (\bar\theta, \pi/2).$$

Proof. We easily have $f'(\theta) = (\sin\theta)^{n-3}h(\theta)$, where

$$h(\theta) = (n-2)\cos\theta\cos\varphi - (\sin\theta\sin\varphi)\varphi'.$$

So $f'(\theta^*) = (2-n)(t^*)^{n-2}s^* < 0$ and $f'(\pi/2) = 1 + n\lambda > 0$. It is easy to check, by Lemma 4.3.1, that

$$h'(\theta) = -(n-2)(\sin\theta\cos\varphi + \varphi'\cos\theta\sin\varphi) - \varphi'\cos\theta\sin\varphi$$

$$-(\sin\theta\cos\varphi)(\varphi')^2 - (\sin\theta\sin\varphi)\varphi'' > 0 \quad \forall\, \theta \in (\theta^*, \pi/2);$$

hence h is strictly increasing on $(\theta^*, \pi/2)$. Therefore, h and f' have a unique zero $\bar\theta \in$

$(\theta^*, \pi/2)$, that is,

$$f'(\theta) < 0 \quad \forall\ \theta \in [\theta^*, \bar{\theta}), \quad f'(\theta) > 0 \quad \forall\ \theta \in (\bar{\theta}, \pi/2).$$

From this we also obtain that $f''(\theta) > 0$ for all $\theta \in (\bar{\theta}, \pi/2)$. Finally, it is easy to see that $f'(\theta) < 0$ for all $\theta \in (0, \theta^*]$. This completes the proof. □

As above, let $S(\tau,\theta), T(\tau,\theta), P(\tau,\theta), Q(\tau,\theta)$ and $Z(\tau,\theta)$ be the characteristic solutions defined on the domain

$$\mathcal{D} = \{(\tau,\theta) \mid \theta \in (0,\pi/2),\ \ 0 \le \tau < \tau_M(\theta)\}.$$

Here $\tau_M(\theta) = +\infty$ if $n = 3$, and if $n > 3$, $\tau_M(\theta)$ is defined by

$$\tau_M(\theta) = \begin{cases} \int_{\beta(\theta)}^{+\infty} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}} & \forall\ \theta \in (0, \theta^*], \\ 2\tau_m(\theta) + \int_{\beta(\theta)}^{+\infty} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}} & \forall\ \theta \in (\theta^*, \pi/2), \end{cases}$$

where

$$\tau_m(\theta) = \int_{|f(\theta)|^{\frac{1}{n-2}}}^{\beta(\theta)} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}} \quad \forall\ \theta \in (\theta^*, \pi/2).$$

4.3.2 Inverting the characteristic map

Unlike the case of the inner circle, we solve τ from $s = S(\tau,\theta)$ to have

$$\tau = C(s,\theta) = \frac{s - \alpha(\theta)}{f(\theta)} \quad \forall\ 0 < \theta < \pi/2.$$

Consider the function

$$F(s,\theta) = T(C(s,\theta),\theta) \tag{4.36}$$

defined for all (s,θ) with $s > 0$ and $0 < C(s,\theta) < \tau_M(\theta)$; that is, in the set

$$\mathcal{S} = \{(s,\theta) \mid 0 < \theta < \pi/2,\ \tilde{l}(\theta) < s < \alpha(\theta)\},$$

where $\tilde{l}(\theta) = \max\{0,\ l(\theta)\}$ with

$$l(\theta) = \begin{cases} -\infty & \text{if } n = 3, \\ \alpha(\theta) + f(\theta)\tau_M(\theta) & \text{if } n > 3. \end{cases}$$

Let

$$l_1(\theta) = \alpha(\theta) + f(\theta)\tau_m(\theta) \quad \forall\, \theta \in (\theta^*, \pi/2).$$

Note that $0 < C(s,\theta) < \frac{\tilde{l}(\theta)-\alpha(\theta)}{f(\theta)}$ for all $(s,\theta) \in \mathcal{S}$. Let

$$\mathcal{D}_0 = \{(\tau,\theta) \mid 0 < \theta < \pi/2,\ \ 0 < \tau < \tau_0(\theta)\},$$

where

$$\tau_0(\theta) = \frac{\tilde{l}(\theta) - \alpha(\theta)}{f(\theta)} = \min\left\{-\frac{\alpha(\theta)}{f(\theta)},\ \tau_M(\theta)\right\}.$$

Proposition 4.3.3. *(a) If $n > 3$, then $l(0^+) = 1$ and there exists a (unique) number $\hat{\theta} \in (0,\pi/2)$ such that*

$$l(\theta) > 0,\ \ l'(\theta) < 0 \quad \forall\, \theta \in (0,\hat{\theta});\quad l(\theta) \le 0 \quad \forall\, \theta \in (\hat{\theta},\pi/2).$$

(b) For all $n \geq 3$, it follows that $l_1((\theta^*)^+) = s^*$ and there exists a (unique) number $\hat{\theta}_1 \in (\theta^*, \pi/2)$ such that

$$l_1(\theta) > 0,\ \ l_1'(\theta) < 0 \quad \forall\, \theta \in (\theta^*, \hat{\theta}_1); \quad l_1(\theta) \leq 0 \quad \forall\, \theta \in (\hat{\theta}_1, \pi/2).$$

Proof. Let $k = |f(\theta)|^{\frac{1}{n-2}}$. We have

$$\int_\beta^{+\infty} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}} = k^{3-n} \int_{\frac{\beta}{k}}^{\infty} \frac{d\eta}{\sqrt{\eta^{2n-4} - 1}} \quad \forall\, \theta \in (0, \pi/2),$$

$$\tau_m(\theta) = k^{3-n} \int_1^{\frac{\beta}{k}} \frac{d\eta}{\sqrt{\eta^{2n-4} - 1}} \quad \forall\, \theta \in (\theta^*, \pi/2).$$

1. We first prove part (a). In this case, $n > 3$ and $l = \alpha + f\tau_M$ can be written as

$$l(\theta) = \begin{cases} \alpha - k \int_{\frac{\beta}{k}}^{\infty} \frac{d\eta}{\sqrt{\eta^{2n-4} - 1}} & \forall\, \theta \in (0, \theta^*], \\ \alpha - k \int_{\frac{\beta}{k}}^{\infty} \frac{d\eta}{\sqrt{\eta^{2n-4} - 1}} - 2k \int_1^{\frac{\beta}{k}} \frac{d\eta}{\sqrt{\eta^{2n-4} - 1}} & \forall\, \theta \in (\theta^*, \pi/2). \end{cases}$$

It then follows that $l(0^+) = 1$ and $l((\pi/2)^-) = 0$. In order to find $l'(\theta)$, we use the elementary identities

$$\frac{k'}{k} = \frac{f'}{(n-2)f}, \quad \left(\frac{\beta}{k}\right)' = \left(\frac{\beta}{k}\right) \frac{\varphi' \sin\varphi}{(n-2)\cos\varphi} \tag{4.37}$$

to obtain

$$l'(\theta) = \begin{cases} \alpha' - \frac{\varphi'\beta}{n-2} - k' \left(\int_{\frac{\beta}{k}}^{\infty} \frac{d\eta}{\sqrt{\eta^{2n-4} - 1}} \right) & \forall\, \theta \in (0, \theta^*), \\ \alpha' - \frac{\varphi'\beta}{n-2} - k' \left(\int_{\frac{\beta}{k}}^{\infty} \frac{d\eta}{\sqrt{\eta^{2n-4} - 1}} + 2 \int_1^{\frac{\beta}{k}} \frac{d\eta}{\sqrt{\eta^{2n-4} - 1}} \right) & \forall\, \theta \in (\theta^*, \pi/2). \end{cases} \tag{4.38}$$

From this, we see that l' exists at θ^* and also $l'((\pi/2)^-) = +\infty$. In either case of the formula (4.38), the term in the parenthesis equals $\frac{\alpha - l}{k}$ and so we simplify (4.38) to obtain that

$$\begin{aligned} l'(\theta) &= \left[\alpha' - \frac{\varphi'\beta}{n-2} - \frac{f'\alpha}{(n-2)f}\right] + \frac{f'l}{(n-2)f} \\ &= -\frac{1}{\sin\theta} - \frac{n\lambda\varphi'}{(n-2[n\lambda\sin\theta + \sqrt{1-(n\lambda\cos\theta)^2}]} + \frac{f'l}{(n-2)f}. \end{aligned} \tag{4.39}$$

Since $l((\pi/2)^-) = 0$, there exists a $\theta' < \pi/2$ closed to $\pi/2$ such that $l(\theta') < 0$. Let $\bar{\theta}$ be determined in Lemma 4.3.2. Then, by (4.38) and (4.37), it follows that $l'(\theta) < 0$ for all $\theta \in (0, \bar{\theta}]$, and by (4.39), it follows that $l'(\theta) < 0$ whenever $l(\theta) \geq 0$ and $\theta \in [\bar{\theta}, \pi/2)$.

2. We proceed in two cases.

Case 1: $l(\bar{\theta}) \leq 0$. In this case, since $l(0^+) = 1 > 0$, there exists a number $\hat{\theta} \in (0, \bar{\theta}]$ such that $l(\hat{\theta}) = 0$. We show that this $\hat{\theta}$ satisfies the conclusion of the lemma. Clearly $l(\theta) > 0, l'(\theta) < 0$ for all $\theta \in (0, \hat{\theta})$. To show $l(\theta) \leq 0$ for all $\theta \in (\hat{\theta}, \pi/2)$, suppose otherwise, for some $d \in (\hat{\theta}, \pi/2)$, $l(d) > 0$. Then the maximum of l on $[\hat{\theta}, d]$ must attain at some $c \in (\hat{\theta}, d]$, where $l(c) > 0$ and $l'(c) \geq 0$, and so $c \in (\bar{\theta}, \pi/2)$; this is a contradiction to (4.39).

Case 2: $l(\bar{\theta}) > 0$. In this case, there exists a number $\hat{\theta} \in (\bar{\theta}, \theta')$ such that $l(\hat{\theta}) = 0$. We show that this $\hat{\theta}$ satisfies the conclusion of the lemma. We first show $l(\theta) \leq 0$ on $\theta \in (\hat{\theta}, \pi/2)$. Suppose otherwise, for some $d \in (\hat{\theta}, \pi/2)$, $l(d) > 0$. Then the maximum of l on $[\hat{\theta}, d]$ is positive and attains at some point $c \in (\hat{\theta}, d)$ with $l(c) > 0$ and $l_1'(c) = 0$; this is a contradiction to (4.39). We now show that $l(\theta) > 0, l'(\theta) < 0$ for all $\theta \in (0, \hat{\theta})$. It suffices to show $l'(\theta) < 0$ for all $\theta \in (0, \hat{\theta})$. Suppose otherwise $l'(e) \geq 0$ for some $e \in (0, \hat{\theta})$. Then $e \in (\bar{\theta}, \hat{\theta})$. The maximum of l on $[e, \hat{\theta}]$ must attain at some $f \in (e, \hat{\theta}]$. At this point f we must have $l(f) \geq 0$ and $l'(f) \geq 0$; this is again a contradiction to (4.39).

3. To prove the part (b), note that, similar to $l(\theta)$, we have

$$l_1(\theta) = \alpha - k \int_1^{\frac{\beta}{k}} \frac{d\eta}{\sqrt{\eta^{2n-4} - 1}} \quad \forall\, \theta \in (\theta^*, \pi/2).$$

It follows that $l_1((\theta^*)^+) = s^*$ and $l_1((\pi/2)^-) = 0$; moreover,

$$l_1'(\theta) = \alpha' - \frac{\varphi'\beta}{n-2} - k' \int_1^{\frac{\beta}{k}} \frac{d\eta}{\sqrt{\eta^{2n-4} - 1}} \quad \forall\, \theta \in (\theta^*, \pi/2). \tag{4.40}$$

We simplify (4.40) to obtain that

$$\begin{aligned} l_1'(\theta) &= \alpha' - \frac{\varphi'\beta}{n-2} - \frac{f'\alpha}{(n-2)f} - \frac{f' l_1}{(n-2)f} \\ &= -\frac{1}{\sin\theta} - \frac{n\lambda\varphi'}{(n-2)[n\lambda\sin\theta + \sqrt{1-(n\lambda\cos\theta)^2}]} - \frac{f' l_1}{(n-2)f}. \end{aligned} \tag{4.41}$$

So $l_1'((\pi/2)^-) = +\infty$. Furthermore, by (4.40) and (4.37), it follows that $l_1'(\theta) < 0$ for all $\theta \in (\theta^*, \bar{\theta}]$, and by (4.41), it follows that $l_1'(\theta) < 0$ whenever $l_1(\theta) \geq 0$ and $\theta \in [\bar{\theta}, \pi/2)$. Therefore, in a completely analogous way to the proof of part (b), we can prove part (b). $\square$

Let

$$\tau_1(\theta) = \min\{\tau_0(\theta),\ \tau_m(\theta)\} = \min\left\{-\frac{\alpha(\theta)}{f(\theta)},\ \tau_m(\theta)\right\}$$

and consider the following subsets of $\mathcal{D}_0$:

$$\mathcal{D}_1 = \{(\tau, \theta) \mid 0 < \theta < \theta^*,\ 0 < \tau < \tau_0(\theta)\},$$

$$\mathcal{D}_2 = \{(\tau, \theta) \mid \theta^* < \theta < \pi/2,\ \tau_1(\theta) < \tau < \tau_0(\theta)\},$$

$$\mathcal{D}_3 = \{(\tau, \theta) \mid \theta^* < \theta < \pi/2,\ 0 < \tau < \tau_1(\theta)\}.$$

Define $\mathcal{R} = \{(t, \theta) \mid \theta \in (0, \pi/2),\ t \geq |f(\theta)|^{\frac{1}{n-2}}\}$ and

$$A(t, \theta) = \int_{\beta(\theta)}^{t} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}} \quad \forall\ (t, \theta) \in \mathcal{R}.$$

By (4.16), (4.18) and (4.20), it follows that

$$\begin{aligned} \tau &= A(T(\tau, \theta), \theta) \quad \forall\ (\tau, \theta) \in \mathcal{D}_1, \\ \tau &= A(T(\tau, \theta), \theta) + 2\tau_m(\theta) \quad \forall\ (\tau, \theta) \in \mathcal{D}_2, \\ \tau &= -A(T(\tau, \theta), \theta) \quad \forall\ (\tau, \theta) \in \mathcal{D}_3. \end{aligned} \tag{4.42}$$

Let $\mathcal{S}_k = \{(s, \theta) \in \mathcal{S} \mid (C(s, \theta), \theta) \in \mathcal{D}_k\}$, $k = 1, 2, 3$, be subdomains of $\mathcal{S}$; namely,

$$\mathcal{S}_1 = \{(s, \theta) \mid 0 < \theta < \theta^*,\ \tilde{l}(\theta) < s < \alpha(\theta)\},$$

$$\mathcal{S}_2 = \{(s, \theta) \mid \theta^* < \theta < \pi/2,\ \tilde{l}(\theta) < s < \tilde{l}_1(\theta)\},$$

$$\mathcal{S}_3 = \{(s, \theta) \mid \theta^* < \theta < \pi/2,\ \tilde{l}_1(\theta) < s < \alpha(\theta)\},$$

where $\tilde{l}_1(\theta) = \max\{0, l_1(\theta)\}$. Taking $\tau = C(s, \theta)$ in (4.42) yields that

$$\begin{aligned} C(s, \theta) &= A(F(s, \theta), \theta) \quad \forall\ (s, \theta) \in \mathcal{S}_1, \\ C(s, \theta) &= A(F(s, \theta), \theta) + 2\tau_m(\theta) \quad \forall\ (s, \theta) \in \mathcal{S}_2, \\ C(s, \theta) &= -A(F(s, \theta), \theta) \quad \forall\ (s, \theta) \in \mathcal{S}_3. \end{aligned}$$

Differentiating with respect to θ yields that

$$F_\theta(s,\theta) = \begin{cases} \sqrt{F^{2n-4} - f^2}(C_\theta - A_\theta(F,\theta)) & \text{if } (s,\theta) \in \mathcal{S}_1, \\ \sqrt{F^{2n-4} - f^2}(C_\theta - A_\theta(F,\theta) - 2\tau_m') & \text{if } (s,\theta) \in \mathcal{S}_2, \\ -\sqrt{F^{2n-4} - f^2}(C_\theta + A_\theta(F,\theta)) & \text{if } (s,\theta) \in \mathcal{S}_3. \end{cases} \tag{4.43}$$

In order to find $F_\theta(s,\theta)$, we need to derive the formula for $A_\theta(t,\theta)$. Assume $(t,\theta) \in \mathcal{R}$ with $\theta \neq \theta^*$ and $t > |f(\theta)|^{\frac{1}{n-2}}$. Let $k(\theta) = |f(\theta)|^{\frac{1}{n-2}}$. Then from

$$A(t,\theta) = k(\theta)^{3-n} \int_{\beta(\theta)/k(\theta)}^{t/k(\theta)} \frac{d\eta}{\sqrt{\eta^{2n-4} - 1}} \quad \forall\ (t,\theta) \in \mathcal{R},$$

it follows that

$$\begin{aligned} A_\theta(t,\theta) =& (3-n)k^{2-n}k' \int_{\beta/k}^{t/k} \frac{d\eta}{\sqrt{\eta^{2n-4} - 1}} \\ &+ k^{3-n} \left[\frac{(t/k)'}{\sqrt{(t/k)^{2n-4} - 1}} - \frac{(\beta/k)'}{\sqrt{(\beta/k)^{2n-4} - 1}} \right], \end{aligned}$$

which, by (4.37), simplifies to

$$A_\theta(t,\theta) = -\frac{\operatorname{sgn}(g)\varphi'\beta}{(n-2)f} - \frac{(n-3)f'}{(n-2)f}A - \frac{tf'}{(n-2)f\sqrt{t^{2n-4} - f^2}}. \tag{4.44}$$

Also we have

$$\tau_m'(\theta) = -\frac{n-3}{n-2}\frac{f'}{f}\tau_m - \frac{\beta\varphi'}{(n-2)f} \quad \forall\ \theta \in (\theta^*, \pi/2). \tag{4.45}$$

Proposition 4.3.4. *It follows that*

$$F_\theta(s,\theta) = \begin{cases} \sqrt{F^{2n-4} - f^2}\, L(s,\theta) + \frac{f'F}{(n-2)f} & \text{if } (s,\theta) \in \mathcal{S}_{1,2}, \\ -\sqrt{F^{2n-4} - f^2}\, L(s,\theta) + \frac{f'F}{(n-2)f} & \text{if } (s,\theta) \in \mathcal{S}_3, \end{cases} \tag{4.46}$$

$$F_{\theta s}(s,\theta) = (n-2)F^{2n-5}\frac{L(s,\theta)}{f(\theta)} \quad \forall\ (s,\theta) \in \mathcal{S}, \tag{4.47}$$

where

$$L(s,\theta) = \frac{\beta}{f}\left(1 + \frac{\varphi'}{n-2}\right) - \frac{f'}{(n-2)f}C(s,\theta). \tag{4.48}$$

Moreover, $L(s,\theta) < 0$ *for all* $s \in [0,1]$ *and* $\theta \in (0,\pi/2)$*; therefore,* $F_{\theta s}(s,\theta) > 0$ *for all* $(s,\theta) \in \mathcal{S}$.

Proof. If $\theta \neq \theta^*$, then two formulas in (4.46) follow from (4.43), (4.44) and (4.45). For $\theta = \theta^*$, the formula follows by continuity. Formula (4.47) follows from (4.46). We only need to prove $L(s,\theta) < 0$. Using the identities $f' = (n-2)\frac{f\alpha}{\beta} - g\varphi'$ and $\alpha^2 + \beta^2 = 1$, we compute that

$$(n-2)f(\theta)^2 L(s,\theta) = (n-2)\beta f + \beta f\varphi' - f'fC$$

$$= (n-2)\beta f + \beta f\varphi' + f'\alpha - sf'$$

$$= (n-2)\frac{f}{\beta}(1 - \alpha s) + (\beta f - g\alpha + sg)\varphi' < 0$$

for all $s \in [0,1]$ and $\theta \in (0,\pi/2)$, thanks to the fact that $f < 0$, $\varphi' > 0$ and the elementary calculation using (4.35),

$$\beta f - g\alpha + sg = (\sin\theta)^{n-2}(s\sin\varphi - n\lambda\cos\theta)$$

$$= (\sin\theta)^{n-2}[sn\lambda\cos^2\theta - s\sin\theta\sqrt{1-(n\lambda\cos\theta)^2} - n\lambda\cos\theta]$$

$$< (\sin\theta)^{n-2}(n\lambda\cos^2\theta - n\lambda\cos\theta) < 0.$$

□

Let $\tilde{Z}(s,\theta) = Z(C(s,\theta),\theta)$. Then, after a change of variables,

$$\tilde{Z}(s,\theta) = \gamma(\theta) + \frac{1}{f(\theta)}\int_{\cos\theta}^{s} F(y,\theta)^{2n-4}\,dy \quad \forall\ (s,\theta)\in\mathcal{S}. \tag{4.49}$$

This function serves in the same role as does the function $U(t,\theta)$ used above. For example, we have the following result.

Lemma 4.3.5. *Let $\tilde{Q}(s,\theta) = Q(C(s,\theta),\theta)$. Then*

$$\tilde{Z}_\theta(s,\theta) = \tilde{Q}(s,\theta)F_\theta(s,\theta) \quad \forall\ (s,\theta)\in\mathcal{S}.$$

Proof. A direct proof by brutal calculations seems too complicated and getting nowhere; instead, for fixed θ, we consider function $\rho(s) = \tilde{Z}_\theta(s,\theta) - \tilde{Q}(s,\theta)F_\theta(s,\theta)$ on interval $s \in (l(\theta),\cos\theta)$ and show that $\rho'(s) = 0$ and $\rho((\cos\theta)^-) = 0$. This proves that $\rho(s) = 0$ and finishes the proof. By (4.49), we have

$$\tilde{Z}_\theta(s,\theta) = \gamma' - \frac{f'}{f^2}\int_{\cos\theta}^{s} F(y,\theta)^{2n-4}\,dy + \frac{\beta^{2n-4}\sin\theta}{f}$$

$$+\frac{1}{f}\int_{\cos\theta}^{s}(2n-4)F(y,\theta)^{2n-5}F_\theta(y,\theta)\,dy.$$

From (4.51), in terms of function $\tilde{Q}(s,\theta) = Q(C(s,\theta),\theta)$, we have

$$F_\theta(s,\theta) = \tilde{Q}(s,\theta)L(s,\theta) + \frac{f'F}{(n-2)f} \quad \forall\ (s,\theta) \in \mathcal{S}. \tag{4.50}$$

Therefore

$$\lim_{s\to(\cos\theta)^-} \rho(s) = \gamma' + \frac{\beta^{2n-4}\sin\theta}{f} - g\left[\frac{\beta g}{f}(1+\frac{\varphi'}{n-2}) + \frac{\beta f'}{(n-2)f}\right] = 0.$$

We also compute that

$$\rho'(s) = -\frac{F^{2n-4}f'}{f^2} + \frac{(2n-4)F^{2n-5}F_\theta}{f} - \tilde{Q}_s F_\theta - \tilde{Q}F_{\theta s}.$$

Using (4.50) and

$$\tilde{Q}_s = Q_\tau C_s = \frac{(n-2)F^{2n-5}}{f}, \quad F_{\theta s} = \frac{(n-2)F^{2n-4}L}{f},$$

we have $\rho'(s) = 0$. This completes the proof. □

Let $\theta_0(s) = \arccos(s)$. By Proposition 4.3.3, we see that domains $\mathcal{S}$ and $\mathcal{S}_3$ can be written as

$$\mathcal{S} = \{(s,\theta) : 0 < s < 1,\ \tilde{\theta}(s) < \theta < \theta_0(s)\},$$

$$\mathcal{S}_3 = \{(s,\theta) : 0 < s < s^*,\ \tilde{\theta}_1(s) < \theta < \theta_0(s)\},$$

where $\tilde{\theta}(s) = 0$ if $n = 3$, and $\tilde{\theta}(s)$ (if $n > 3$) and $\tilde{\theta}_1(s)$ are the inverse functions of $l(\theta)$ (if $n > 3$) on $(0,\hat{\theta})$ and $l_1(\theta)$ on $(0,\hat{\theta}_1)$, respectively. Note that $\mathcal{S}$ has the property that every vertical or horizontal line-segment belongs to $\mathcal{S}$ whenever the endpoints belong to $\mathcal{S}$.

In what follows, for each fixed $s \in (0,1)$, we study $F(s,\theta)$ as a function of θ defined on interval $(\tilde{\theta}(s), \theta_0(s))$. We first have the following result.

Lemma 4.3.6. *It follows that*

$$\lim_{\theta \to (\theta_0(s))^-} F_\theta(s,\theta) = \frac{\sqrt{1-(n\lambda s)^2}}{n\lambda s\sqrt{1-s^2} + s\sqrt{1-(n\lambda s)^2}} \quad \forall\ s \in (0,1), \tag{4.51}$$

$$\lim_{\theta \to (\tilde{\theta}(s))^+} F_\theta(s,\theta) = -\infty \quad \forall\ s \in (0,1), \tag{4.52}$$

$$F_\theta(s, \tilde{\theta}_1(s)) = \frac{1}{2-n} f'(\tilde{\theta}_1(s))|f(\tilde{\theta}_1(s))|^{\frac{3-n}{n-2}} \quad \forall\ s \in (0, s^*). \tag{4.53}$$

Proof. 1. We first prove (4.51). Note that, as $\theta \to (\theta_0(s))^-$, it follows that

$$C(s,\theta) \to 0, \quad F(s,\theta) = T(C(s,\theta),\theta) \to \beta(\theta_0(s)) = \sqrt{1-s^2}.$$

If $\theta < \theta_0(s)$ and is sufficiently closed to $\theta_0(s)$, we have $(s,\theta) \in \mathcal{S}_{1,3}$ and hence, by (4.46),

$$\lim_{\theta \to (\theta_0(s))^-} F_\theta(s,\theta) = \left[\frac{g(\theta)\sin\theta}{f(\theta)} + \frac{g\varphi'\sin\theta}{(n-2)f} + \frac{\sin\theta f'}{(n-2)f}\right]_{\theta=\theta_0(s)}$$

$$= \left[\frac{g(\theta)\sin\theta}{f(\theta)} + \cos\theta\right]_{\theta=\theta_0(s)} = \frac{\sqrt{1-(n\lambda s)^2}}{n\lambda s\sqrt{1-s^2} + s\sqrt{1-(n\lambda s)^2}}.$$

2. To prove (4.52), we first assume $n > 3$. In this case, $l(\tilde{\theta}(s)) = s$ and so $C(s,\tilde{\theta}(s)) = \tau_M(\tilde{\theta}(s))$; hence,

$$\lim_{\theta \to (\tilde{\theta}(s))^+} F(s,\theta) = \lim_{\theta \to (\tilde{\theta}(s))^+} T(C(s,\theta),\theta) = +\infty.$$

Thus, with L defined by (4.48), we have

$$\lim_{\theta \to (\tilde{\theta}(s))^+} \left[L(s,\theta) + \frac{f'}{(n-2)f} \frac{F}{\sqrt{F^{2n-4} - f^2}} \right] = L(s, \tilde{\theta}(s)) < 0.$$

From this, (4.52) follows by (4.46).

3. Now assume $n = 3$. Then $\tilde{\theta}(s) = 0$. First, assume $\theta^* > 0$ (so $\lambda \neq 0$). Then, for all $0 < \theta < \theta^*$, we have that $C(s,\theta) = A(F(s,\theta), \theta)$ and hence

$$s - \alpha(\theta) = -k \int_{\beta/k}^{F/k} \frac{d\eta}{\sqrt{\eta^2 - 1}},$$

where $k = |f(\theta)|$. Since $k(\theta) \to 0$ and $\beta(\theta)/k(\theta) \to \mu = \frac{1}{\sqrt{1-9\lambda^2}} > 1$ as $\theta \to 0^+$, from the above equation, we obtain that

$$\frac{F(s,\theta)}{k(\theta)} \geq \frac{(1-s)M}{k(\theta)^2}$$

as $\theta \to 0^+$, where $M > 0$ is a constant. Hence $F(s,\theta) \to +\infty$ as $\theta \to 0^+$. Note that, by (4.46),

$$\lim_{\theta \to 0^+} F_\theta(s,\theta) = \lim_{\theta \to 0^+} \frac{\sqrt{F^2 - f^2}}{f^2} \left[f^2 L(s,\theta) + f' f \frac{F}{\sqrt{F^2 - f^2}} \right]. \tag{4.54}$$

By (4.48), we have $f^2 L(s,\theta) = \beta f(1 + \varphi') - (s - \alpha) f'$. Since $f'(0) = -\sqrt{1 - 9\lambda^2} < 0$, we have

$$\lim_{\theta \to 0^+} \left[f^2 L(s,\theta) + f' f \frac{F}{\sqrt{F^2 - f^2}} \right] = (1-s) f'(0) < 0.$$

Consequently, (4.52) follows from (4.54). Now assume $\theta^* = 0$ (so $\lambda = 0$). In this case,

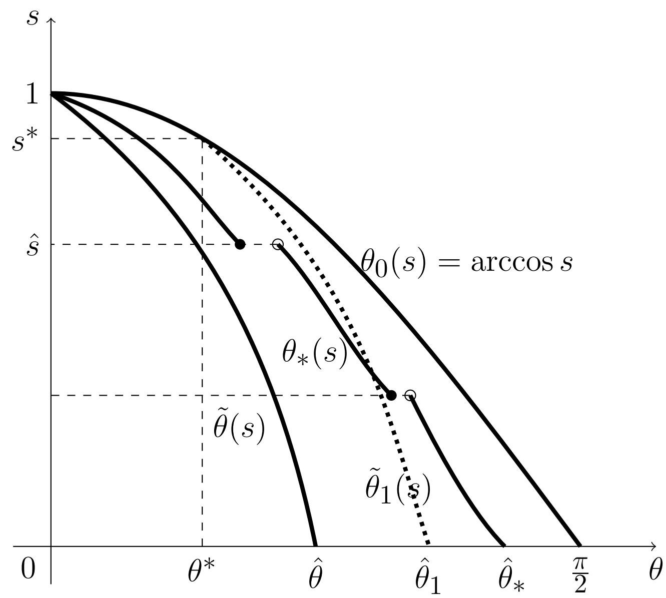

Figure 4.3: The domain $\mathcal{S}$ is between the two smooth curves $\theta_0(s)$ and $\tilde{\theta}(s)$, while $\mathcal{S}_1$ is the part with $0 < \theta < \theta^*$ (empty if $\theta^* = 0$), $\mathcal{S}_2$ is the part bounded by $\tilde{\theta}$ and $\tilde{\theta}_1$ with $\theta^* < \theta < \hat{\theta}_1$, and $\mathcal{S}_3$ is the part between $\tilde{\theta}_1$ and θ_0 with $\hat{\theta}_1 < \theta < \pi/2$. The number $\hat{s}$ is determined in Lemma 4.3.8.

$C(s,\theta) = A(F(s,\theta),\theta) + 2\tau_m(\theta)$, which gives

$$s - \alpha(\theta) = -k \int_{\beta/k}^{F/k} \frac{d\eta}{\sqrt{\eta^2 - 1}} - 2k \int_1^{\beta/k} \frac{d\eta}{\sqrt{\eta^2 - 1}}.$$

where $k = |f(\theta)|$. As above, we still have that $F(s,\theta) \to +\infty$ as $\theta \to 0^+$ and, again, that (4.52) follows from (4.54).

4. Finally (4.53) is immediate from (4.46). This completes the proof. $\square$

For each $s \in (0,1)$, define (see Figure 4.3)

$$\theta_*(s) = \inf\{\theta \in (\tilde{\theta}(s), \theta_0(s)) : \ F_\theta(s,\theta') \geq 0 \ \ \forall\, \theta' \in (\theta, \theta_0(s))\}.$$

The well-definedness of θ_* follows from Lemma 4.3.6; moreover, for all $s \in (0,1)$, it also follows that $\tilde{\theta}(s) < \theta_*(s) < \theta_0(s)$ and

$$F_\theta(s, \theta_*(s)) = 0, \quad F_\theta(s, \theta) \geq 0 \quad \forall\, \theta_*(s) \leq \theta < \theta_0(s).$$

Furthermore, there exists a sequence $\theta_i' \to \theta_*(s)^-$ such that $F_\theta(s, \theta_i') < 0$, which shows that $F_{\theta\theta}(s, \theta_*(s)) \geq 0$.

Lemma 4.3.7. *For each $s \in (0,1)$, function $F(s,\theta)$ is one-to-one on the closed interval $\theta \in [\theta_*(s), \theta_0(s))$. Moreover, the function $\theta_*(s)$ is strictly decreasing, right-continuous on $(0,1)$, and satisfies that $\theta_*(1^-) = 0$.*

Proof. The proof is similar to that of Lemma 4.2.3. Let $a, b \in [\theta_*(s), \theta_0(s))$ be such that $F(s,a) = F(s,b)$. We show $a = b$. If $a < b$ then $F_\theta(s,\theta) = 0$ for all $\theta \in (a,b)$, which is impossible by the formula of $F_\theta(s,\theta)$ given above. Therefore $F(s,\theta)$ is one-to-one on $[\theta_*(s), \theta_0(s))$. To show θ_* is strictly decreasing in $(0,1)$, let $0 < s < s' < 1$. Suppose, for the contrary, $\theta_*(s) \leq \theta_*(s')$; then $\tilde{\theta}(s) < \theta_*(s) \leq \theta_*(s') < \theta_0(s') < \theta_0(s)$ and so $F_\theta(s, \theta_*(s')) \geq 0$. Because the line-segment connecting points $(s, \theta_*(s')$ and $(s', \theta_*(s'))$ belongs to $\mathcal{S}$ and because $F_{\theta s} > 0$ on $\mathcal{S}$, we have $F_\theta(s, \theta_*(s')) < F_\theta(s', \theta_*(s')) = 0$, which gives a contradiction. To show the right-continuity of θ_*, given $s \in (0,1)$, let $l = \theta_*(s^+)$; then $\tilde{\theta}(s) \leq l \leq \theta_*(s)$ and $F_\theta(s,l) = 0$, which implies $l > \tilde{\theta}(s)$. Given each $\theta' \in (l, \theta_0(s))$, for all $s' > s$ sufficiently closed to s, we have $\theta_*(s') < l < \theta' < \theta_0(s')$ and hence $F_\theta(s', \theta') \geq 0$. Taking $s' \to s^+$ yields that $F_\theta(s, \theta') \geq 0$. By definition, $l \geq \theta_*(s)$; so $l = \theta_*(s)$. This proves that θ_* is right-continuous on $(0,1)$. Finally, from $0 < \theta_*(s) < \theta_0(s)$, we have that $\theta_*(1^-) = 0$. $\square$

Lemma 4.3.8. *There exists a number $\hat{s} \in (0,1)$ such that*

$$\tilde{Z}(s, \theta_*(s)) > \frac{1}{n-1} F(s, \theta_*(s))^{n-1} \quad \forall\, \hat{s} \leq s < 1. \tag{4.55}$$

Proof. From the definition of $\tilde{Z}(s,\theta)$, by (4.13), we have

$$(n-1)\tilde{Z}(s,\theta) = (n-1)\gamma - \beta g + C(s,\theta)f^2 + F(s,\theta)\tilde{Q}(s,\theta).$$

Since $F_\theta = \tilde{Q}L + \frac{f'F}{(n-2)f}$ and $F_\theta(s,\theta_*(s)) = 0$, it follows that

$$L(s,\theta_*(s)) = -\frac{f'(\theta_*(s))F(s,\theta_*(s))}{(n-2)f(\theta_*(s))\tilde{Q}(s,\theta_*(s))} \quad \forall\ s \in (0,1).$$

Substitution into the definition of $L(s,\theta)$ yields

$$C(s,\theta_*(s)) = \left[\frac{F(s,\theta)}{\tilde{Q}(s,\theta)} + \frac{(n-2)\beta\varphi')}{f'}\right]_{\theta=\theta_*(s)}.$$

Simplifying, we obtain that

$$\begin{aligned}(n-1)\tilde{Z}(s,\theta_*(s)) = &\left[(n-1)\gamma - \beta g + \frac{(n-2+\varphi')\beta f^2}{f'}\right]_{\theta=\theta_*(s)} \\ &+ \frac{F(s,\theta_*(s))^{2n-3}}{\tilde{Q}(s,\theta_*(s))}.\end{aligned} \tag{4.56}$$

First, let $s' \in (0,1)$ such that $\theta_*(s) \in (0,\bar{\theta})$ for all $s \in [s',1)$, where $\bar{\theta} \in (\theta^*,\pi/2)$ is determined in Lemma 4.3.2. Hence $f'(\theta_*(s)) < 0$ and thus $0 < \tilde{Q}(s,\theta_*(s)) < F(s,\theta_*(s))^{n-2}$ for all $s \in [s',1)$. Therefore

$$\frac{F(s,\theta_*(s))^{2n-3}}{\tilde{Q}(s,\theta_*(s))} > F(s,\theta_*(s))^{n-1} \quad \forall\ s' \le s < 1.$$

We now show that there exists a number $\theta' \in (0, \bar{\theta})$ such that

$$(n-1)\gamma - \beta g + \frac{(n-2+\varphi')\beta f^2}{f'} > 0 \quad \forall\, 0 < \theta < \theta'.$$

This is proved by computing that

$$(n-1)\gamma - \beta g + \frac{(n-2+\varphi')\beta f^2}{f'}$$

$$= (\sin\theta)^{n-1}\left[1 - \sin\varphi + \frac{(n-2+\varphi')\sin\theta\cos\varphi}{\left(n-2-\frac{\varphi'\sin\theta\sin\varphi}{\cos\theta\cos\varphi}\right)\cos\theta}\right],$$

and noticing that

$$\lim_{\theta\to 0^+}\left[1 - \sin\varphi + \frac{(n-2+\varphi')\sin\theta\cos\varphi}{\left(n-2-\frac{\varphi'\sin\theta\sin\varphi}{\cos\theta\cos\varphi}\right)\cos\theta}\right] = 1 - n\lambda > 0.$$

Finally, let $\hat{s} \in [s', 1)$ be such that $\theta_*(s) \in (0, \theta')$ for all $s \in [\hat{s}, 1)$. Then, for this $\hat{s}$, (4.55) follows from (4.56). □

Lemma 4.3.9. *Let $t_*(s) = F(s, \theta_*(s))$ for all $s \in (0,1)$. Then the function $t_*(s)$ is right-continuous on $(0,1)$ with $t_*(1^-) = 0$ and $0 < t_*(0^+) < 1$.*

Proof. Since $\theta_*(s)$ is right-continuous and $F(s,\theta)$ is continuous, it follows that $t_*(s)$ is right-continuous. Also, since $0 < F(s, \theta_*(s)) \le \sqrt{1-s^2}$ for all $0 < s < 1$, we easily see that $t_*(1^-) = 0$. Let $\hat{\theta}_* = \theta_*(0^+)$. We consider the following cases:

Case (a): $0 < \hat{\theta}_* < \pi/2$. Given any $\hat{\theta}_* < \theta < \pi/2$, we have $F(s, \theta_*(s)) < F(s, \hat{\theta}_*) < F(s, \theta) < \sqrt{1-s^2}$ for all sufficiently small s, and hence, letting $s \to 0^+$,

$$t_*(0^+) \le F(0^+, \hat{\theta}_*) \le F(0^+, \theta) \le 1$$

Suppose, for the contrary, that $t_*(0^+) = 1$. Then $F(0^+, \theta) = 1$ for all $\hat{\theta}_* < \theta < \pi/2$. For $s > 0$ sufficiently small and $\theta < \pi/2$ sufficiently close to $\pi/2$, we have $(s, \theta) \in \mathcal{S}_3$ and hence

$$\frac{s - \cos\theta}{f(\theta)} = \int_{F(s,\theta)}^{\sin\theta} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}}.$$

Let $s \to 0^+$ and we have

$$-\frac{\cos\theta}{f(\theta)} = \int_1^{\sin\theta} \frac{dy}{\sqrt{y^{2n-4} - f(\theta)^2}}.$$

for all $\theta < \pi/2$ sufficiently close to $\pi/2$; this is impossible as seen by taking the limits as $\theta \to \pi/2$. Therefore $0 < t_*(0^+) < 1$.

Case (b): $\hat{\theta}_* = \pi/2$. In this case, $(s, \theta_*(s)) \in \mathcal{S}_3$ for all sufficiently small $s > 0$. Hence, from $F_\theta(s, \theta_*(s)) = 0$, by (4.46), it follows that

$$(n-2) f(\theta_*(s))\, L(s, \theta_*(s)) = f'(\theta_*(s)) \frac{t_*(s)}{\sqrt{t_*(s)^{2n-4} - f(\theta_*(s))^2}}. \tag{4.57}$$

Note that

$$fL = \beta(1 + \frac{\varphi'}{n-2}) - \frac{f'C}{n-2}, \quad C = \int_F^\beta \frac{dy}{\sqrt{y^{2n-4} - f^2}}.$$

So (4.57) implies

$$[\beta(n - 2 + \varphi')]\Big|_{\theta=\theta_*(s)} = \left[\frac{f' t_*(s)}{\sqrt{t_*(s)^{2n-4} - f^2}} + \int_{t_*(s)}^{\beta} \frac{f'\, dy}{\sqrt{y^{2n-4} - f^2}} \right]_{\theta=\theta_*(s)}$$

Taking limit as $s \to 0^+$, since $\theta_*(0^+) = \pi/2$ and $f'((\pi/2)^-) = \varphi'((\pi/2)^-) = 1 + n\lambda$, we

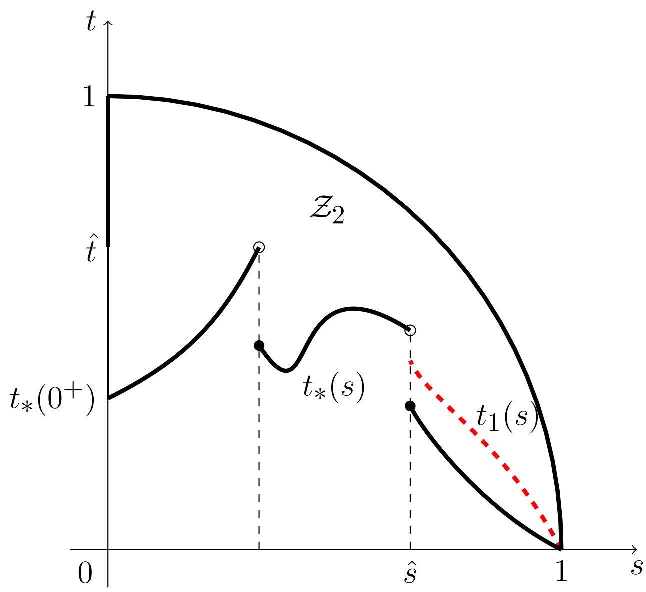

Figure 4.4: The function $t_*(s) = F(s, \theta_*(s))$ and the domain $\mathcal{Z}_2$. The number $\hat{t}$ is determined in Lemma 4.3.11 and the smooth function $t_1(s)$ is determined in Lemma 4.3.13.

have that $\hat{y} = t_*(0^+)$ satisfies

$$n - 2 + (1 + n\lambda) = (1 + n\lambda)\left(\hat{y}^{3-n} + \int_{\hat{y}}^{1} y^{2-n}\, dy\right).$$

From this, we solve for $\hat{y}$ to obtain

$$t_*(0^+) = \begin{cases} e^{-\frac{1}{1+3\lambda}} & \text{if } n = 3, \\ \left(\frac{1+n\lambda}{n-2+n\lambda}\right)^{\frac{1}{n-3}} & \text{if } n > 3. \end{cases} \tag{4.58}$$

Therefore $0 < t_*(0^+) < 1$. $\square$

Lemma 4.3.10. *(See Figure 4.4.) Let*

$$\mathcal{Z}_2 = \{(s, t) \mid 0 < s < 1,\ t_*(s) < t < \sqrt{1 - s^2}\}.$$

Then, for each $(s,t) \in \mathcal{Z}_2$*, there exists a unique number* $\theta = \xi(s,t) \in (\theta_*(s), \theta_0(s))$ *such that*

$$t = F(s, \xi(s,t)) \quad \forall\ (s,t) \in \mathcal{Z}_2.$$

The function $\theta = \xi(s,t)$ *is continuous in* $\mathcal{Z}_2$ *and is differentiable at every point* (s_0, t_0) *of* $\mathcal{Z}_2$ *where* $F_\theta(s_0, \xi(s_0, t_0)) \neq 0$*. Moreover, at any such point, we have*

$$\xi_s = -F_s(s,\xi)/F_\theta(s,\xi), \quad \xi_t = 1/F_\theta(s,\xi).$$

Proof. For each $(s,t) \in \mathcal{Z}_2$, since $F(s,\theta)$ is one-to-one on $\theta \in [\theta_*(s), \theta_0(s))$, there exists a unique number $\theta = \xi(s,t) \in (\theta_*(s), \theta_0(s))$ such that

$$t = F(s, \xi(s,t)) \quad \forall\ (s,t) \in \mathcal{Z}_2.$$

We first show the continuity of $\xi(s,t)$; namely, for all $(s_0, t_0) \in \mathcal{Z}_2$, it follows that

$$\lim_i \xi(s_i, t_i) = \xi(s_0, t_0)$$

for all sequences $(s_i, t_i) \to (s_0, t_0)$ in $\mathcal{Z}_2$. Let a subsequence of $\xi(s_i, t_i) \to l$. Note that

$$\theta_*(s_i) \leq \xi(s_i, t_i) \leq \theta_0(s_i), \quad t_i = F(s_i, \xi(s_i, t_i)).$$

Since $\theta_*(s_0) = \theta_*(s_0^+) \leq \theta_*(s_0^-)$, it follows that $\theta_*(s_0) \leq l < \theta_0(s_0)$ and $t_0 = F(s_0, l)$. Hence, by definition, $l = \xi(s_0, t_0)$. This proves the continuity of ξ.

The differentiability of $\xi(s,t)$ at every point (s_0, t_0) of $\mathcal{Z}_2$ where $F_\theta(s_0, \xi(s_0, t_0)) \neq 0$ follows from the continuity of ξ and the implicit function theorem. $\square$

Lemma 4.3.11. *There exists a number* $\hat{t} \in [t_*(0^+), 1)$ *such that*

$$\lim_{\substack{(s,t)\to(0,t_0) \\ (s,t)\in\mathcal{Z}_2}} \xi(s,t) = \pi/2 \quad \forall\, \hat{t} \le t_0 < 1,$$

$$\lim_{\substack{(s,t)\to(0,t_0) \\ (s,t)\in\mathcal{Z}_2}} \tilde{Q}(s,\xi(s,t)) = -t_0^{n-2} \quad \forall\, \hat{t} \le t_0 \le 1.$$

Proof. 1. Assume the first limit is proved. Then we have $(s,\xi(s,t)) \in \mathcal{S}_3$ for all sufficiently small $s > 0$ and $\hat{t} \le t < 1$ and hence, for all such (s,t),

$$\tilde{Q}(s,\xi(s,t)) = -\sqrt{t^{2n-4} - f(\xi(s,t))^2},$$

from which the second limit follows.

2. We now prove that there exists a number $\hat{t}$ such that the first limit holds. Let $\xi(s_i,t_i) \to \theta'$ along a sequence $(s_i,t_i) \to (0,t_0)$ in $\mathcal{Z}_2$. Since $\theta_*(s) < \xi(s,t) < \theta_0(s)$ for all $(s,t) \in \mathcal{Z}_2$, it follows that

$$0 < \theta_*(0^+) \le \theta' \le \pi/2.$$

If $\hat{\theta}_* = \theta_*(0^+) = \pi/2$ (the *Case* (b) in the proof of Lemma 4.3.9), then $\theta' = \pi/2$ and, in this case, the number $\hat{t}$ can be chosen to be $\hat{t} = t_*(0^+)$.

3. Now assume $\hat{\theta}_* < \pi/2$. First, let $\mu_1 \in (\hat{\theta}_*, \pi/2)$ be such that $|f(\theta)|^2 < 1/2$ for all $\theta \in [\mu_1, \pi/2)$ and such that $(s,\theta) \in \mathcal{S}_3$ for all $0 < s < s'$ and $\mu_1 \le \theta < \theta_0(s)$, where $s' > 0$ is a small number. We claim that there exists a number $0 < \mu_2 < 1$ such that

$$\forall\, \hat{\theta}_* < \theta < \pi/2, \;\; F(0^+,\theta) \ge \mu_2 \;\Rightarrow\; \theta \ge \mu_1.$$

If not, then there exist numbers $\theta_i \in (\hat{\theta}_*, \pi/2)$ such that

$$F(0^+, \theta_i) \geq 1 - \frac{1}{i}, \quad \theta_i < \mu_1 \quad \forall\, i = 1, 2, \cdots .$$

Assume $\theta_i \to \mu_3 \leq \mu_1$ as $i \to \infty$. Then $F(0^+, \theta) = 1$ for all $\theta \in (\mu_3, \pi/2)$. This is impossible, as proved in the *Case* (a) of the proof of Lemma 4.3.9. Let $0 < \mu_4 < 1$ be a number such that

$$\mu_4^{2n-4} > 2/3, \quad \sqrt{6}(1 - \mu_4) < \inf_{\theta \in [\hat{\theta}_*, \pi/2)} \frac{-\cos\theta}{f(\theta)}. \tag{4.59}$$

Finally, let $\hat{t} = \max\{t_*(0^+),\ \mu_2,\ \mu_4\}$. Then $t_*(0^+) \leq \hat{t} < 1$. We claim that $\theta' = \pi/2$ for all $\hat{t} \leq t_0 < 1$. Suppose, for the contrary, that $\hat{\theta}_* \leq \theta' < \pi/2$. Then $t_0 = F(0^+, \theta') \geq \mu_2$; so $\theta' \geq \mu_1$. Hence $|f(\theta')|^2 < 1/2$ and $(s, \theta') \in \mathcal{S}_3$ for all sufficiently small $s > 0$. So

$$\frac{s - \cos\theta'}{f(\theta')} = \int_{F(s,\theta')}^{\sin\theta'} \frac{dy}{\sqrt{y^{2n-4} - f(\theta')^2}}.$$

Let $s \to 0^+$ and, by (4.59), we arrive at a desired contradiction:

$$\frac{-\cos\theta'}{f(\theta')} = \int_{t_0}^{\sin\theta'} \frac{dy}{\sqrt{y^{2n-4} - f(\theta')^2}}$$

$$< \int_{t_0}^{1} \frac{dy}{\sqrt{y^{2n-4} - f(\theta')^2}} \leq \sqrt{6}(1 - \mu_4) < \inf_{\theta \in [\hat{\theta}_*, \pi/2)} \frac{-\cos\theta}{f(\theta)}.$$

$\square$

4.3.3 Construction of the solution on $\mathcal{Z}_2$

Let $\xi(s, t)$ be the function on $\mathcal{Z}_2$ defined above and $\tilde{Z}(s, \theta)$ be defined by (4.49).

Theorem 4.3.12. *Define*

$$\psi^2(s,t) = \tilde{Z}(s, \xi(s,t)) \quad \forall\ (s,t) \in \mathcal{Z}_2.$$

Then ψ^2 is differentiable in $\mathcal{Z}_2$, with $\psi^2_s(s,t) < 0$, and satisfies the Eikonal equation $|\nabla\psi^2(s,t)| = t^{n-2}$ at every point $(s,t) \in \mathcal{Z}_2$. Moreover, $\psi^2(s,t)$ can be continuously extended to the curve $s = \sqrt{1-t^2}$, $0 < t \le 1$ and to the line segment $s = 0$, $\hat{t} \le t \le 1$ to satisfy

$$\psi^2(s,t) = \begin{cases} \frac{n\lambda}{n-1} t^{n-1} & \text{on } s = \sqrt{1-t^2},\ 0 < t \le 1, \\ \frac{1}{n-1}(1 + n\lambda - t^{n-1}) & \text{on } s = 0,\ \hat{t} \le t \le 1. \end{cases} \tag{4.60}$$

Proof. Since ξ is continuous on $\mathcal{Z}_2$, we easily see that ψ^2 is continuous on $\mathcal{Z}_2$. As ξ is differentiable at every point (s,t) of $\mathcal{Z}_2$ with $F_\theta(s,\xi(s,t)) \neq 0$, we see that ψ^2 is differentiable at every point (s,t) of $\mathcal{Z}_2$ with $F_\theta(s,\xi(s,t)) \neq 0$. Moreover, at all such points, using $\xi_s = -F_s(s,\xi)/F_\theta(s,\xi)$, $\xi_t = 1/F_\theta(s,\xi)$ and $\tilde{Z}_\theta = \tilde{Q}F_\theta$, we arrive at the formula

$$\psi^2_s(s,t) = f(\xi(s,t)) < 0, \quad \psi^2_t(s,t) = \tilde{Q}(s,\xi(s,t)). \tag{4.61}$$

Therefore, by the continuity of ξ, ψ^2_s and ψ^2_t can be extended continuously to every point $(s,t) \in \mathcal{Z}_2$. This proves that ψ^2 is differentiable at every point of $\mathcal{Z}_2$. By (4.61),

$$(\psi^2_s)^2 + (\psi^2_t)^2 = f^2 + \tilde{Q}^2 = F(s,\xi)^{2n-4} = t^{2n-4} \quad \forall\ (s,t) \in \mathcal{Z}_2.$$

We now prove the continuity of ψ^2 under the extension (4.60). First, it is easily seen that

$$\lim_{(s,t)\to(s_0,t_0)} \xi(s,t) = \arcsin t_0 \quad \forall\, 0 < t_0 < 1,\ s_0 = \sqrt{1-t_0^2}.$$

Hence, by (4.49),

$$\lim_{(s,t)\to(s_0,t_0)} \psi^2(s,t) = \lim_{(s,t)\to(s_0,t_0)} \tilde{Z}(s,\xi(s,t)) = \gamma(\arcsin t_0) = \frac{n\lambda}{n-1} t_0^{n-1}.$$

Next, by Lemma 4.3.11 and the formula (4.22), we obtain that

$$\lim_{(s,t)\to(s_0,t_0)} \psi^2(s,t) = \left[\gamma(\theta) - \frac{f(\theta)\alpha(\theta) + g(\theta)\beta(\theta)}{n-1} - \frac{t_0\sqrt{t_0^{2n-4} - f(\theta)^2}}{n-1}\right]_{\theta=\pi/2}$$

$$= \frac{1}{n-1}(1 + n\lambda - t_0^{n-1}) \quad \forall\, \hat{t} \le t_0 < 1.$$

This completes the proof. $\square$

Lemma 4.3.13. *Let* $t_0(s) = \sqrt{1-s^2}$*. Then there exists a smooth* decreasing *function* $t_1(s)$ *on* $s \in (\hat{s}, 1)$ *such that, for all* $\hat{s} < s < 1$*,*

$$\psi^2(s,t) > \frac{t^{n-1}}{n-1} \quad \forall\, t_*(s) < t < t_1(s),$$

$$\psi^2(s,t) < \frac{t^{n-1}}{n-1} \quad \forall\, t_1(s) < t < t_0(s).$$

Proof. Let

$$K(s,t) = \psi^2(s,t) - \frac{t^{n-1}}{n-1}.$$

Then $K_t(s,t) = \psi_t^2(s,t) - t^{n-2} < 0$ and so, for each fixed $0 < s < 1$, $K(s,t)$ is continuous and strictly decreasing in $t \in [t_*(s), t_0(s)]$. Clearly

$$K(s, t_0(s)) = \frac{(t_0(s))^{n-1}}{n-1}(\lambda n - 1) < 0.$$

On the other hand, by Lemma 4.3.8, we have that $w(s) = K(s, t_*(s)) > 0$ on $s \in (\hat{s}, 1)$. Therefore, there exists a unique $t_1(s) \in (t_*(s), t_0(s))$ for each $s \in (\hat{s}, 1)$ such that $K(s, t_1(s)) = 0$. This function $t_1(s)$ satisfies the requirement of the lemma. Moreover, since $K_t(s, t_1(s)) > 0$, by the implicit function theorem, the function $t_1(s)$ is differentiable in $s \in (\hat{s}, 1)$. Differentiating $K(s, t_1(s)) = 0$ yields that $\psi_s^2 + \psi_t^2 t_1' = t_1^{n-2} t_1'$ and hence

$$t_1'(s) = \frac{\psi_s^2}{t_1^{n-2} - \psi_t^2} = \frac{f(\xi(s, t_1))}{t_1^{n-2} - \tilde{Q}(s, t_1)} < 0 \quad \forall\, \hat{s} < s < 1,$$

which proves that $t_1(s)$ is strictly decreasing on $s \in (\hat{s}, 1)$. □

4.4 Gluing the local solutions: Proof of Theorem 4.1.2

In what follows, we assume

$$0 \leq n\lambda \leq 1 - a^{n-1}.$$

Our goal is to piece together the solutions ψ^1 and ψ^2 constructed above with a suitable trivial solution of the form

$$\eta(t) = \sum_{i=1}^{k} \chi_{[a_{i-1}, a_i]}(t) \frac{(-1)^{i-1}}{n-1}(t^{n-1} - l_i),$$

where k is an integer, $0 = a_0 < a_1 < a_2 < \cdots < a_k < 1$ and $l_i \in \mathbb{R}$, to obtain a solution of (4.6) in Theorem 4.1.2.

4.4.1 The preparations

Let ψ^1, ψ^2 be the functions determined above by the characteristics method, with domains $\mathcal{Z}_1$, $\mathcal{Z}_2$, respectively. Let $\mathcal{Z} = \mathcal{Z}_1 \cap \mathcal{Z}_2$.

Let $s = s_1(t)$ be the increasing function determined in Lemma 4.2.7, and let $s = r_1(t)$ be the inverse function of the function $t = t_1(s)$ determined in Lemma 4.3.13. Note that $r_1(t)$ is well-defined, smooth and strictly decreasing on $(0, t_1(\hat{s}^-))$. (See Figure 4.5.)

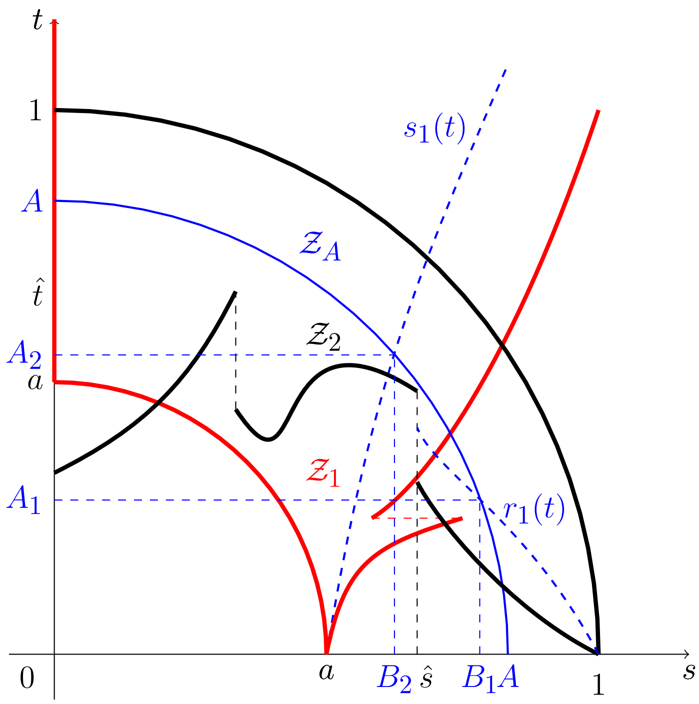

Figure 4.5: The domains $\mathcal{Z}_1, \mathcal{Z}_2$ and $\mathcal{Z}_A$, the curve $s = s_1(t)$ determined in Lemma 4.2.7, and the curve $s = r_1(t)$ that is the inverse function of the function $t = t_1(s)$ determined in Lemma 4.3.13.

Let $\max\{a, \hat{s}, \hat{t}\} < A < 1$ be any given number. We consider the arc $\Gamma_A = \{(s,t) \mid s \geq 0, t \geq 0, s^2 + t^2 = A^2\}$. Then Γ_A intersects the curves $s = r_1(t)$ and $s = s_1(t)$ at unique points (B_1, A_1) and (B_2, A_2), respectively. Let

$$\tilde{t}_1(s) = \begin{cases} \sqrt{A^2 - s^2} & (0 \le s \le B_1), \\ t_1(s) & (B_1 < s < 1), \end{cases}$$
$$\tilde{s}_1(t) = \begin{cases} \sqrt{A^2 - t^2} & (A_2 \le t \le A), \\ s_1(t) & (0 < t < A_2). \end{cases} \tag{4.62}$$

and define the set

$$\mathcal{Z}_A = \{(s,t) \mid 0 < s < 1,\ \tilde{t}_1(s) \le t \le \sqrt{1 - s^2}\}.$$

Note that $A_1 \to 0$ and $A_2 \to \bar{A} \in (0,1)$ as $A \to 1$. Hence, we can choose the number A sufficiently close to 1 so that the following conditions hold:

$$\max\{a, \hat{s}, \hat{t}\} < A < 1, \quad 0 < A_1 < A_2 < A < 1, \quad \mathcal{Z}_A \subset \mathcal{Z}_2. \tag{4.63}$$

4.4.2 Constructions from the function ψ^2

The following result is critical for our construction.

Theorem 4.4.1. *Let A be any number satisfying (4.63) and let $A_0 \in (A, 1)$ be any fixed number. Then there exists an increasing sequence*

$$0 < a_1 < a_2 < \cdots < a_m < \cdots < 1$$

such that, for functions $\eta_i(t) = \frac{(-1)^{i-1}}{n-1}(t^{n-1} - l_i)$, *where numbers* l_i *are defined by* $l_1 = 0$,

$$l_{2j} = 2a_{2j-1}^{n-1} - 2a_{2j-2}^{n-1} + \cdots + 2a_1^{n-1}, \quad l_{2j+1} = 2a_{2j}^{n-1} - l_{2j},$$

there exist functions $r_i(t)$ *on* $[a_{i-1}, a_i]$ *satisfying*

$$b_{i-1} < r_i(t) < \sqrt{1-t^2}, \quad \psi^2(r_i(t), t) = \eta_i(t) \quad \forall\, a_{i-1} < t < a_i, \;\; i = 2, 3, \cdots,$$

where $b_{i-1} = \sqrt{A_0^2 - a_{i-1}^2}$, *and for all* $j = 1, 2, \cdots$,

$$\psi^2(\sqrt{1-t^2}, t) = \eta_{2j}(t) \;\; at\; t = a_{2j}, \;\; \psi^2(b_{2j}, t) = \eta_{2j+1}(t) \;\; at\; t = a_{2j+1}.$$

Proof. (See Figure 4.6.) Let (b_1, a_1) be the intersection point of the curve $s = r_1(t)$ with the arc $s^2 + t^2 = A_0^2$. This defines $a_1 \in (0, A_1)$ and $b_1 = \sqrt{A_0^2 - a_1^2}$.

1. Let $r_1(t)$ be the function defined above. Define $l_1 = 0$ and

$$\eta_1(t) = \frac{1}{n-1}(t^{n-1} - l_1) = \frac{1}{n-1}t^{n-1}.$$

Then

$$(r_1(t), t) \in \mathcal{Z}_A, \quad \psi^2(r_1(t), t) = \eta_1(t) \quad \forall\, 0 < t \leq a_1.$$

Define $l_2 = 2a_1^{n-1}$ and

$$\eta_2(t) = \frac{1}{n-1}(l_2 - t^{n-1}) = \frac{1}{n-1}(2a_1^{n-1} - t^{n-1}).$$

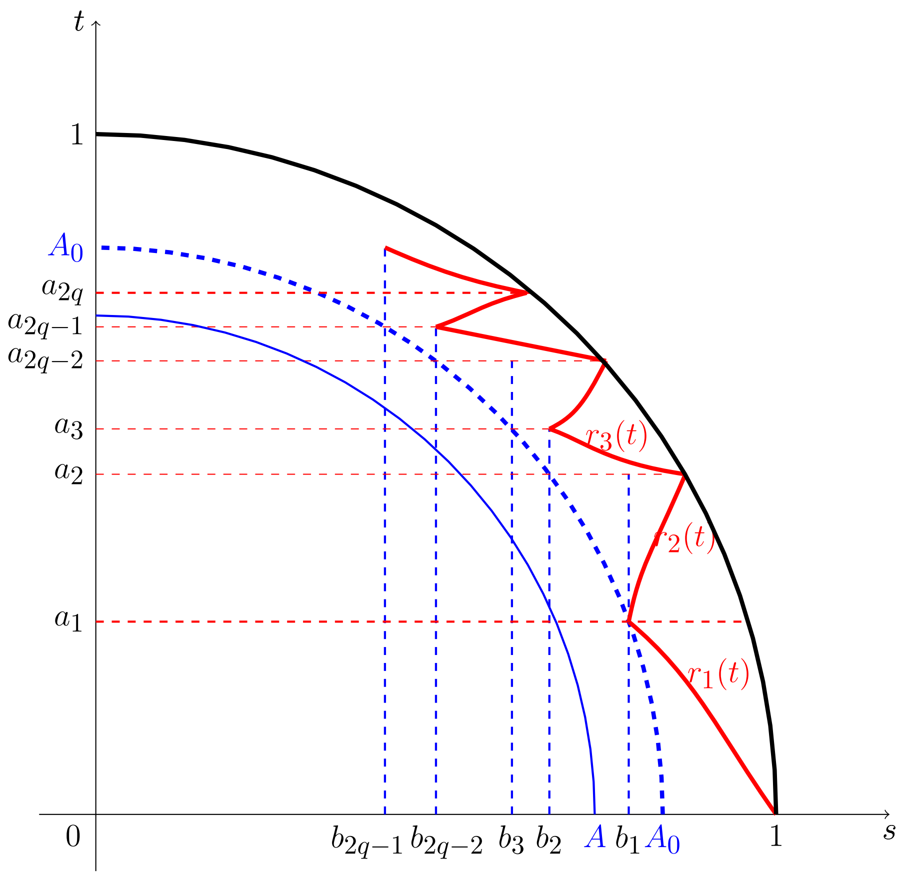

Figure 4.6: A typical construction of sequence $\{a_i\}$ and functions $r_i(t)$ on $[a_{i-1}, a_i]$ in Theorem 4.4.1.

We solve $\eta_2(t) = \frac{n\lambda}{n-1}t^{n-1}$ to obtain the number a_2 given by

$$(1 + n\lambda)a_2^{n-1} = l_2 = 2a_1^{n-1}.$$

So $a_2 > a_1$ and $\frac{n\lambda}{n-1}t^{n-1} < \eta_2(t)$ for all $a_1 < t < a_2$. Now consider the function

$$h(t) = \psi^2(b_1, t) - \eta_2(t) \quad \forall\, a_1 \leq t \leq \sqrt{1 - b_1^2}.$$

Then $h(a_1) = 0$ and $h'(t) = \psi_t^2(b_1, t) + t^{n-2} > 0$ for $a_1 < t < \sqrt{1 - b_1^2}$. Hence $h(t) > 0$ for all $a_1 < t < \sqrt{1 - b_1^2}$; that is, $\eta_2(t) < \psi^2(b_1, t)$ for all $a_1 < t < \sqrt{1 - b_1^2}$. We have thus shown that

$$\psi^2(\sqrt{1-t^2}, t) < \eta_2(t) < \psi^2(b_1, t) \quad \forall\, a_1 < t < \min\{a_2, \sqrt{1-b_1^2}\}. \tag{4.64}$$

Consequently, there exists a continuous function $r_2(t)$ on $a_1 \le t \le \min\{a_2, \sqrt{1-b_1^2}\}$, differentiable in $a_1 < t < \min\{a_2, \sqrt{1-b_1^2}\}$, such that

$$b_1 < r_2(t) < \sqrt{1-t^2}, \quad \psi^2(r_2(t), t) = \eta_2(t) \quad \forall\, a_1 < t < \min\{a_2, \sqrt{1-b_1^2}\}. \tag{4.65}$$

From this we also have that $r_2'(t) > 0$ and hence $r_2(t)$ is strictly increasing in $a_1 < t < \min\{a_2, \sqrt{1-b_1^2}\}$. If $a_2 \ge \sqrt{1-b_1^2}$, then, letting $t \to \sqrt{1-b_1^2}$ in (4.65), we would obtain $b_1 = r_2(\sqrt{1-b_1^2}) > r_2(a_1) = b_1$, a contradiction. Therefore, $a_2 < \sqrt{1-b_1^2}$. Furthermore, by the definition of a_2, we have $r_2(a_2) = \sqrt{1-a_2^2}$.

2. We construct a_{m+1} inductively for $m \ge 2$. Suppose, for some $m \ge 2$, we have defined the numbers $a_2 < \cdots < a_m < 1$ such that, for numbers l_i defined by

$$l_1 = 0, \quad l_{2j} = 2a_{2j-1}^{n-1} - 2a_{2j-2}^{n-1} + \cdots + 2a_1^{n-1}, \quad l_{2j+1} = 2a_{2j}^{n-1} - l_{2j} \tag{4.66}$$

with each $2 \le 2j,\ 2j+1 \le m$ and functions $\eta_i(t) = \frac{(-1)^{i-1}}{n-1}(t^{n-1} - l_i)$, there exist functions $r_i(t)$ on $[a_{i-1}, a_i]$ satisfying

$$\begin{aligned}
&\text{(i)}\ \ b_{i-1} < r_i(t) < \sqrt{1-t^2}, \quad \psi^2(r_i(t), t) = \eta_i(t) \quad \forall\, a_{i-1} < t < a_i,\\
&\qquad \text{where } b_{i-1} = \sqrt{A_0^2 - a_{i-1}^2}, \text{ for each } i = 2, \cdots, m, \text{ and}\\
&\text{(ii)}\ \ \psi^2(\sqrt{1-t^2}, t) = \eta_{2j}(t) \text{ at } t = a_{2j},\ \ \psi^2(b_{2j}, t) = \eta_{2j+1}(t) \text{ at } t = a_{2j+1}\\
&\qquad \text{for each } j \text{ with } 2 \le 2j,\ 2j+1 \le m.
\end{aligned} \tag{4.67}$$

Note that the condition (4.67)(ii) implies

$$l_{2j} = (1+n\lambda)a_{2j}^{n-1}, \quad l_{2j+1} < (1-n\lambda)a_{2j+1}^{n-1} \quad \forall\, 2 \leq 2j,\ 2j+1 \leq m. \tag{4.68}$$

To construct a_{m+1} so that (4.66) and (4.67) hold when m is replaced by $m+1$, we consider the cases of m being even or odd separately.

(a) Assume $m = 2q$ **is even.** Let $l_{2q+1} = 2a_{2q}^{n-1} - l_{2q}$ and

$$\eta_{2q+1}(t) = \frac{1}{n-1}(t^{n-1} - l_{2q+1}) = \frac{1}{n-1}(t^{n-1} - 2a_{2q}^{n-1} + l_{2q}).$$

Then, by (4.68),

$$\frac{n\lambda}{n-1}t^{n-1} = \psi^2(\sqrt{1-t^2}, t) < \eta_{2q+1}(t) \quad \forall\, a_{2q} < t < 1.$$

Consider the function

$$h(t) = \psi^2(b_{2q}, t) - \eta_{2q+1}(t) \quad \forall\, a_{2q} \leq t \leq \sqrt{1-b_{2q}^2}. \tag{4.69}$$

Then $h'(t) = \psi_t^2(b_{2q}, t) - t^{n-2} < 0$ for $a_{2q} < t < \sqrt{1-b_{2q}^2}$. Note that

$$h(a_{2q}) = \psi^2(b_{2q}, a_{2q}) - \eta_{2q}(a_{2q}) = \psi^2(b_{2q}, a_{2q}) - \psi^2(\sqrt{1-a_{2q}^2}, a_{2q}) > 0$$

and

$$h(\sqrt{1-b_{2q}^2}) = \psi^2(b_{2q}, \sqrt{1-b_{2q}^2}) - \eta_{2q+1}(\sqrt{1-b_{2q}^2}) < 0.$$

Therefore, there exists a unique number a_{2q+1} with $a_{2q} < a_{2q+1} < \sqrt{1-b_{2q}^2}$ such that $h(a_{2q+1}) = 0$; hence $h(t) > 0$ in $a_{2q} < t < a_{2q+1}$. So we have

$$\psi^2(\sqrt{1-t^2}, t) < \eta_{2q+1}(t) < \psi^2(b_{2q}, t) \quad \forall\, a_{2q} < t < a_{2q+1}.$$

Consequently, there exists a continuous function $s = r_{2q+1}(t)$ on $a_{2q} \le t \le a_{2q+1}$, differentiable in $a_{2q} < t < a_{2q+1}$, such that

$$b_{2q} < r_{2q+1}(t) < \sqrt{1-t^2}, \quad \psi^2(r_{2q+1}(t), t) = \eta_{2q+1}(t) \quad \forall\, a_{2q} < t < a_{2q+1}. \tag{4.70}$$

Clearly, from $h(a_{2q+1}) = 0$ we have that $\psi^2(b_{2q}, t) = \eta_{2q+1}(t)$ at $t = a_{2q+1}$. Therefore, (4.67) holds with $m = 2q+1$.

(b) Assume $m = 2q - 1 \ge 3$ **is odd.** Let $l_{2q} = 2a_{2q-1}^{n-1} - 2a_{2q-2}^{n-1} + \cdots + 2a_1^{n-1} = 2a_{2q-1}^{n-1} - l_{2q-1}$ and

$$\eta_{2q}(t) = \frac{1}{n-1}(l_{2q} - t^{n-1}).$$

Let a_{2q} be the number determined by $(1+n\lambda)a_{2q}^{n-1} = l_{2q}$. Then, by the second condition of (4.68), we have $a_{2q} > a_{2q-1}$ and

$$\frac{n\lambda}{n-1}t^{n-1} < \eta_{2q}(t) \quad \forall\, a_{2q-1} < t < a_{2q}.$$

Again consider the function

$$h(t) = \psi^2(b_{2q-1}, t) - \eta_{2q}(t) \quad \forall\, a_{2q-1} \le t \le \sqrt{1-b_{2q-1}^2}.$$

Then $h(a_{2q-1}) = 0$, $h'(t) > 0$ and hence $h(t) > 0$ for $a_{2q-1} < t < \sqrt{1 - b_{2q-1}^2}$. We have thus shown that

$$\psi^2(\sqrt{1-t^2}, t) < \eta_{2q}(t) < \psi^2(b_{2q-1}, t) \quad \forall\, a_{2q-1} < t < \min\{a_{2q}, \sqrt{1 - b_{2q-1}^2}\}.$$

So, there exists a continuous function $r_{2q}(t)$ on $a_{2q-1} \leq t \leq \min\{a_{2q}, \sqrt{1 - b_{2q-1}^2}\}$, differentiable in the interior, such that

$$b_{2q-1} < r_{2q}(t) < \sqrt{1-t^2}, \quad \psi^2(r_{2q}(t), t) = \eta_{2q}(t) \tag{4.71}$$

for all $a_{2q-1} < t < \min\{a_{2q}, \sqrt{1 - b_{2q-1}^2}\}$. From this we also have that $r'_{2q}(t) > 0$ and hence $r_{2q}(t)$ is strictly increasing in the interval. If $a_{2q} \geq \sqrt{1 - b_{2q-1}^2}$, then, letting $t \to \sqrt{1 - b_{2q-1}^2}$ in (4.71), we would obtain that

$$b_{2q-1} = r_{2q}(\sqrt{1 - b_{2q-1}^2}) > r_{2q}(a_{2q-1}) = b_{2q-1},$$

a contradiction. Therefore, $a_{2q} < \sqrt{1 - b_{2q-1}^2} < 1$.

3. Finally we completed the induction process and thus finished the proof. □

Lemma 4.4.2. *Let $\{a_i\}$ be the sequence constructed in Theorem 4.4.1. Then there exists an integer $k \geq 2$ such that $a_{k-1} < A$ and $a_k \geq A$.*

Proof. Suppose, for the contrary, that $a_i < A$ for all $i = 2, 3, \cdots$. Let

$$\Sigma = \{(s,t) \mid a_1 \leq t \leq A,\ \sqrt{A_0^2 - t^2} \leq s \leq \sqrt{1-t^2}\}.$$

Then there exists a number $\epsilon_0 > 0$ such that

$$-\psi_s^2(s,t) \geq \epsilon_0 \quad \forall\ (s,t) \in \Sigma. \tag{4.72}$$

Let $h(t)$ be the function by (4.69). Then, for some $c \in (b_{2q}, \sqrt{1-a_{2q}^2})$ and hence $(c, a_{2q}) \in \Sigma$, we have that

$$h(a_{2q}) = \psi^2(b_{2q}, a_{2q}) - \psi^2(\sqrt{1-a_{2q}^2}, a_{2q}) = \psi_s^2(c, a_{2q})(b_{2q} - \sqrt{1-a_{2q}^2})$$

$$= (-\psi_s^2(c, a_{2q}))(\sqrt{1-a_{2q}^2} - \sqrt{A_0^2 - a_{2q}^2}) \geq \epsilon_0(1 - A_0).$$

Since $h(a_{2q+1}) = 0$, there exists a number $d \in (a_{2q}, a_{2q+1})$ such that $h(a_{2q}) = h'(d)(a_{2q} - a_{2q+1})$. Therefore

$$a_{2q+1} = a_{2q} - \frac{h(a_{2q})}{h'(d)} = a_{2q} + \frac{h(a_{2q})}{-h'(d)} \geq a_{2q} + \frac{\epsilon_0(1-A_0)}{2},$$

since $-h'(d) = |h'(d)| \leq |\psi_t^2(b_{2q}, d)| + d^{n-2} \leq 2d^{n-2} < 2$. Hence $a_{2q+1} - a_{2q} \geq \frac{\epsilon_0(1-A_0)}{2}$ for all $q = 1, 2, \cdots$, which yields

$$A > a_{2q+1} - a_2 > \sum_{j=1}^{q}(a_{2j+1} - a_{2j}) \geq \frac{\epsilon_0(1-A_0)}{2} q \quad \forall\ q = 1, 2, \cdots,$$

a contradiction. □

Let k be the integer determined in Lemma 4.4.2, with $a_{k-1} < A$ and $a_k \geq A$. Fix any *even integer* $K \geq k$. Then $l_K = (1+n\lambda)a_K^{n-1} \geq (1+n\lambda)A^{n-1}$, $\eta_K(t) = \frac{1}{n-1}(l_K - t^{n-1})$ and

$$\eta_{K+1}(t) = \frac{1}{n-1}(t^{n-1} - 2a_K^{n-1} + l_K).$$

Let $\tilde{a}_{K+1}$ be the root of the equation $\eta_{K+1}(t) = \frac{1}{n-1}(1 + n\lambda - t^{n-1})$, which is uniquely determined by

$$\tilde{a}_{K+1}^{n-1} = \frac{1 + n\lambda + 2a_K^{n-1} - l_K}{2} = \frac{1 + n\lambda + (1 - n\lambda)a_K^{n-1}}{2}.$$

Therefore we easily see that $a_K < \tilde{a}_{K+1} < 1$ and

$$\frac{n\lambda}{n-1}t^{n-1} < \eta_{K+1}(t) < \frac{1}{n-1}(1 + n\lambda - t^{n-1}) \quad \forall\, a_K < t < \tilde{a}_{K+1}.$$

Hence, there exists a continuous function $\tilde{r}_{K+1}(t)$ on $[a_K, \tilde{a}_{K+1}]$ such that

$$0 < \tilde{r}_{K+1}(t) < \sqrt{1-t^2}, \quad \psi^2(\tilde{r}_{K+1}(t), t) = \eta_{K+1}(t) \quad \forall\, a_K < t < \tilde{a}_{K+1}.$$

We summarize what we have proved in the following theorem.

Corollary 4.4.3. *Let K be an even number determined above and define*

$$\eta_A(t) = \sum_{i=1}^{K} \chi_{[a_{i-1},a_i]}(t)\eta_i(t) + \chi_{[a_K,\tilde{a}_{K+1}]}(t)\eta_{K+1}(t),$$
$$r_A(t) = \sum_{i=1}^{K} \chi_{[a_{i-1},a_i]}(t)r_i(t) + \chi_{[a_K,\tilde{a}_{K+1}]}(t)\tilde{r}_{K+1}(t),$$

where the numbers $0 = a_0 < a_1 < a_2 < \cdots < a_K < \tilde{a}_{K+1} < 1$ *and the functions* $\eta_i(t)$, $r_i(t)$

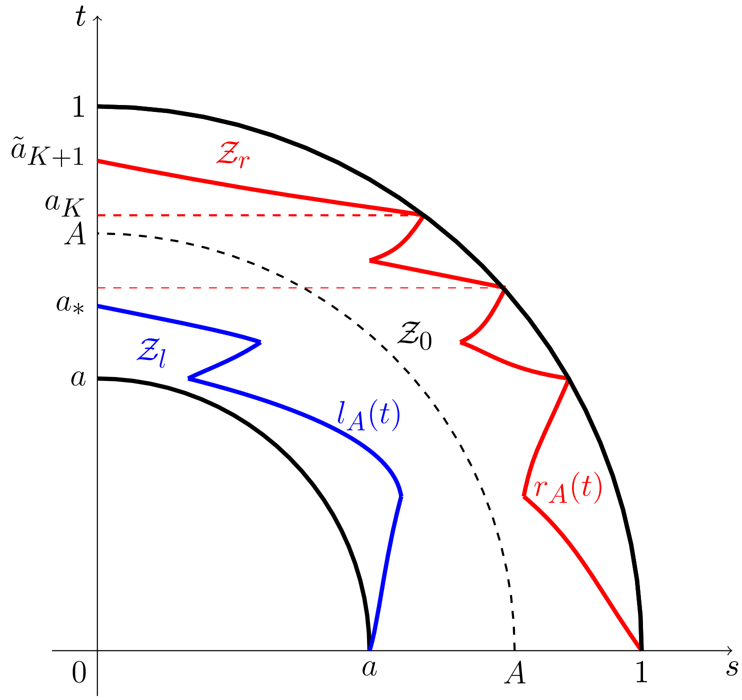

Figure 4.7: **(The case** $0 \le n\lambda < 1 - a^{n-1}$**).** The curve $s = r_A(t)$ determined in Corollary 4.4.3, the curve $s = l_A(t)$ determined Lemma 4.4.4, and the sub-domains divided by $s = l_A(t)$ and $s = r_A(t)$ in the domain ω.

and $\tilde{r}_{K+1}(t)$ *are determined above. Then*

$$(r_A(t), t) \in \mathcal{Z}_A, \quad \psi^2(r_A(t), t) = \eta_A(t) \quad \forall\, t \in (0, \tilde{a}_{K+1}].$$

We now prove Theorem 4.1.2 by constructing the Lipschitz solutions of (4.6) that depend on the choice of the number A; by choosing different A's, we obtain the infinitely many Lipschitz solutions.Let $0 \le n\lambda \le 1 - a^{n-1}$. We proceed with two cases: $0 \le n\lambda < 1 - a^{n-1}$ or $n\lambda = 1 - a^{n-1}$, separately.

4.4.3 The proof in the case $0 \le n\lambda < 1 - a^{n-1}$

In this case, let $\bar{a}$ be the number defined by

$$\bar{a}^{n-1} = \frac{1 + n\lambda + a^{n-1}}{2}.$$

Then $a < \bar{a} < 1$.

By Lemma 4.2.6, the set $\mathcal{S} = \{(s,t) \in \mathcal{Z} \mid \ \psi^1(s,t) = \psi^2(s,t)\}$ does not intersect the circle $\{s^2 + t^2 = 1\}$. Consequently, we select a number A sufficiently close to 1 so that

$$\begin{cases} \text{(i)} \quad \text{Condition (4.63) holds,} \\ \text{(ii)} \quad \bar{a} < A < 1, \\ \text{(iii)} \quad \psi^1(s,t) > \psi^2(s,t) \ \ \forall\, (s,t) \in \mathcal{Z}_A \cap \mathcal{Z}_1. \end{cases} \tag{4.73}$$

For such a number A, let η_A and r_A be the functions determined in Corollary 4.4.3 above. Note that

$$a < \bar{a} < A < \tilde{a}_{K+1}.$$

The following result is crucial to continue our construction.

Lemma 4.4.4. *There exists a (unique) number $a < a_* < A$ such that*

$$\eta_A(a_*) = \psi^1(0, a_*) = \frac{1}{n-1}(a_*^{n-1} - a^{n-1}).$$

Moreover, there exists a continuous function $s = l_A(t)$ on $(0, a_*]$ such that

$$l_A(a_*) = 0, \quad \psi^1(l_A(t), t) = \eta_A(t) \quad \forall\, t \in (0, a_*].$$

Proof. Note that the function $h(t) = \psi^1(0,t) - \eta_A(t)$ is continuous and nondecreasing on $[a,A]$, $h(a) = -\eta_A(a) = -\psi^2(r_A(a),a) < 0$ and

$$h(A) = \psi^1(0,A) - \psi^2(r_A(A),A) > \psi^1(0,A) - \psi^2(0,A) > 0.$$

Hence there exists a unique number $a_* \in (a,A)$ such that $h(a_*) = 0$. For this a_* we have $h(t) \le 0$ for all $a \le t \le a_*$. Furthermore, if $a_* \le A_2$ then $\psi^1(\tilde{s}_1(t),t) = \frac{1}{n-1}t^{n-1} \ge \eta_A(t)$ for all $0 < t \le a_*$, where $\tilde{s}_1(t)$ is defined in (4.62) above; if $a_* > A_2$, then by (4.73)(iii), $\psi^1(\tilde{s}_1(t),t) > \psi^2(\tilde{s}_1(t),t) \ge \psi^2(r_A(t),t) = \eta_A(t)$ for all $A_2 \le t \le a_*$. Therefore we have proved that

$$\psi^1(s_-(t),t) \le \eta_A(t) \le \psi^1(\tilde{s}_1(t),t) \quad \forall\, 0 < t \le a_*,$$

where $s_-(t) = \sqrt{(a^2 - t^2)^+}$. Therefore, there exists a continuous function $s = l_A(t)$ on $(0, a_*]$ such that

$$s_-(t) \le l_A(t) \le \tilde{s}_1(t), \quad \psi^1(l_A(t),t) = \eta_A(t) \quad \forall\, t \in (0, a_*].$$

Finally $l_A(a_*) = 0$ follows from $\psi^1(0,a_*) = \eta_A(a_*)$ by the choice of a_*. □

Lemma 4.4.5. *For all $0 < t \le a_*$, we have $l_A(t) < r_A(t)$.*

Proof. Clearly, $l_A(t) \le \tilde{s}_1(t) \le r_A(t)$. So, if $l_A(t_1) = r_A(t_1)$ for some $t_1 \in (0, a_*]$, then $(l_A(t_1), t_1) \in Z_A$ and, by (4.73)(iii), $\eta_A(t_1) = \psi^1(l_A(t_1),t_1) > \psi^2(r_A(t_1),t_1) = \eta_A(t_1)$, a contradiction. □

Define the functions

$$\tilde{l}_A(t) = \begin{cases} 0 & (a_* < t \le 1) \\ l_A(t) & (0 < t \le a_*) \end{cases}, \quad \tilde{r}_A(t) = \begin{cases} 0 & (\tilde{a}_{K+1} < t \le 1) \\ r_A(t) & (0 < t \le \tilde{a}_{K+1}) \end{cases}$$

and the sets

$$\mathcal{Z}_l = \{(s,t) \in \omega \mid 0 < t \le a_*, \ \ s_-(t) \le s \le l_A(t)\},$$

$$\mathcal{Z}_0 = \{(s,t) \in \omega \mid 0 < t \le \tilde{a}_{K+1}, \ \ \tilde{l}_A(t) \le s \le r_A(t)\},$$

$$\mathcal{Z}_r = \{(s,t) \in \omega \mid 0 < t \le 1, \ \ \tilde{r}_A(t) \le s \le \sqrt{1-t^2}\}.$$

We easily obtain the following result and thus complete the proof of Theorem 4.1.2 in this case.

Theorem 4.4.6. *The function*

$$\psi(s,t) = \begin{cases} \psi^1(s,t) & (s,t) \in \mathcal{Z}_l, \\ \eta_A(t) & (s,t) \in \mathcal{Z}_0, \\ \psi^2(s,t) & (s,t) \in \mathcal{Z}_r \end{cases}$$

is a Lipschitz solution to the problem (4.6)*.*

4.4.4 The proof in the case $n\lambda = 1 - a^{n-1}$

In this case, let A' be a number such that

$$\max\{A_2', \hat{t}\} < A' < 1, \quad \mathcal{Z}_{A'} \subset \mathcal{Z}_2,$$

where $a < A_2' < 1$ is the unique solution of $s_1(t) = \sqrt{1-t^2}$, and $\hat{t}$ is the number determined in Lemma 4.3.11.

We first prove the following result.

Lemma 4.4.7. *There exist number $A'' \in (A', 1)$ and function $s = c(t)$, differentiable in $(A'', 1)$ and continuous on $[A'', 1]$ with $c(1) = 0$, such that*

$$\begin{aligned} &0 < c(t) < \sqrt{1-t^2}, \;\; c'(t) < 0, \\ &\psi^1(s,t) < \psi^2(s,t) \;\; \forall\, A'' \le t < 1, \; 0 \le s < c(t), \\ &\psi^1(s,t) > \psi^2(s,t) \;\; \forall\, A'' \le t < 1, \; c(t) < s \le \sqrt{1-t^2}. \end{aligned} \tag{4.74}$$

Proof. Let $h(s) = \psi^1(s,t) - \psi^2(s,t)$; then $h'(s) = \psi^1_s - \psi^2_s > 0$. Furthermore,

$$h(0) = \frac{t^{n-1} - a^{n-1}}{n-1} - \frac{1 + n\lambda - t^{n-1}}{n-1} = \frac{2t^{n-1} - 2}{n-1} < 0 \quad \forall\, A' \le t < 1,$$

and by Lemma 4.2.6, $h(\sqrt{1-t^2}) > \frac{1-a^{n-1}}{n-1} t^{n-1} - \frac{n\lambda}{n-1} t^{n-1} = 0$. Therefore, there exist a function $s = c(t)$ on $t \in [A', 1)$ with $0 < c(t) < \sqrt{1-t^2}$ such that $\psi^1(c(t),t) = \psi^2(c(t),t)$ and hence the last two conditions of (4.74) hold. Moreover, since $\psi^1 - \psi^2$ is differentiable and $\psi^1_s - \psi^2_s > 0$, by the implicit function theorem, $s = c(t)$ is also differentiable in $(A', 1)$. Differentiating $\psi^1(c(t),t) = \psi^2(c(t),t)$ yields that

$$c'(t) = \frac{\psi^2_t(c(t),t) - \psi^1_t(c(t),t)}{\psi^1_s(c(t),t) - \psi^2_s(c(t),t)}.$$

Note that $\psi^1_s - \psi^2_s > 0$ and $\psi^1_t > 0$. Clearly $c(1^-) = 0$ and so there exists $A'' \in (A', 1)$ such that $\psi^2_t < 0$ near $s = 0$ and $t = 1$. Hence, $c'(t) < 0$ for any $t \in (A'', 1)$. $\square$

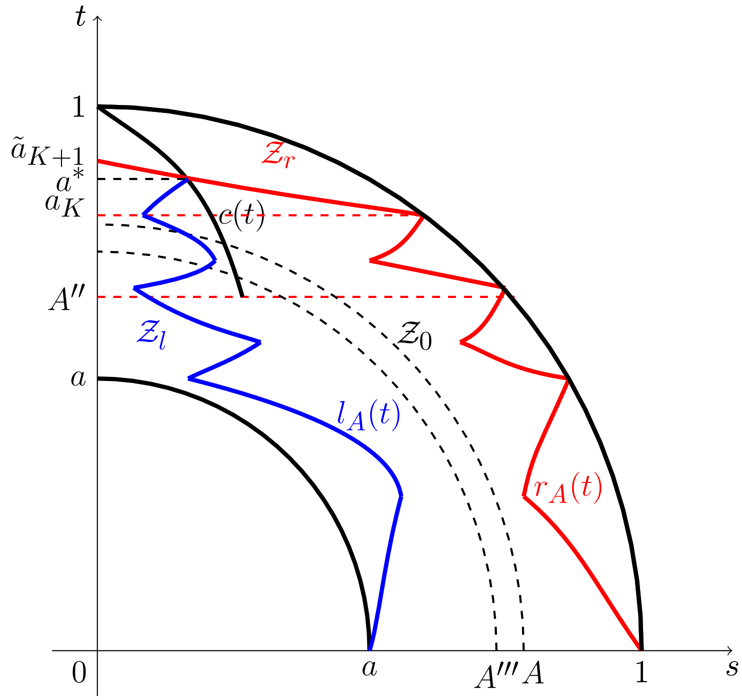

Figure 4.8: **(The case $n\lambda = 1 - a^{n-1}$).** The curve $s = c(t)$ on $[A'', 1]$ determined in Lemma 4.4.7, the curve $s = l_A(t)$ determined in Lemma 4.4.8, and the curve $s = r_A(t)$ intersect at $t = a^*$.

By Lemma 4.2.6, we now select $A''' \in (A'', 1)$ such that

$$\psi^1(s,t) > \psi^2(s,t) \;\; \forall\, (s,t) \in \mathcal{Z}_{A'''} \cap \mathcal{Z}_1 \cap \{0 < t \leq A'''\}. \tag{4.75}$$

We then select a number A sufficiently close to 1 so that

$$\begin{cases} \text{(i)} \quad \text{Condition (4.63) holds,} \\ \text{(ii)} \quad A''' < A < 1. \end{cases} \tag{4.76}$$

For such a number A, let η_A and r_A be the functions determined in Corollary 4.4.3 above.

The following result is crucial to continue our construction.

Lemma 4.4.8. *There exists a (unique) number $a_K < a^* < \tilde{a}_{K+1}$ such that*

$$\eta_A(a^*) = \psi^2(c(a^*), a^*) = \psi^1(c(a^*), a^*).$$

Moreover, there exists a continuous function $s = l_A(t)$ on $(0, a^*]$ such that

$$l_A(a^*) = c(a^*), \quad l_A(t) < r_A(t), \quad \psi^1(l_A(t), t) = \eta_A(t) \quad \forall\, t \in (0, a^*).$$

Proof. Since $c(t)$ is decreasing on $[A'', 1]$ and $r_A(t)$ is increasing on $[a_K, \tilde{a}_{K+1}]$, there must exist a unique $a^* \in (a_K, \tilde{a}_{K+1})$ such that $r_A(a^*) = c(a^*)$. By the selection of $c(t)$ and definition of $\eta_A(t)$, we have that $\eta_A(a^*) = \psi^2(c(a^*), a^*) = \psi^1(c(a^*), a^*)$. If $a < t < a^*$ and we consider $h(t) = \frac{1}{n-1}(t^{n-1} - a^{n-1}) - \eta_A(t)$, then $h'(t) \geq 0$ a.e. on (a, a^*) and $h(a^*) = \psi^1(0, a^*) - \psi^1(c(a^*), a^*) < 0$; hence $h(t) < 0$ for all $t \in (a, a^*)$. This proves that $\psi^1(s_-(t), t) = \frac{1}{n-1}(t^{n-1} - a^{n-1})^+ < \eta_A(t)$ for all $0 < t < a^*$. Let

$$\bar{s}_1(t) = \begin{cases} c(t) & (A'' \leq t < a^*) \\ \sqrt{A^2 - t^2} & (A_2 < t < A'') \\ s_1(t) & (0 < t \leq A_2). \end{cases}$$

Then $\psi^1(\bar{s}_1(t), t) \geq \psi^2(\bar{s}_1(t), t) > \psi^2(r_A(t), t) = \eta_A(t)$. Consequently, for each $t \in (0, a^*)$, there exists a number $s = l_A(t)$ with $s_-(t) < l_A(t) \leq \bar{s}_1(t) < r_A(t)$ such that $\psi^1(l_A(t), t) = \eta_A(t)$. Clearly, $l_A(t)$ is continuous on $(0, a^*]$ and increasing in $[a_K, a^*]$; it also follows that $l_A(a^*) = c(a^*)$. $\square$

Define the functions

$$\bar{l}_A(t) = \begin{cases} c(t) & (a^* < t \le 1) \\ l_A(t) & (0 < t \le a^*) \end{cases}, \quad \bar{r}_A(t) = \begin{cases} c(t) & (a^* < t \le 1) \\ r_A(t) & (0 < t \le a^*) \end{cases}$$

and the sets

$$\mathcal{Z}_l = \{(s,t) \in \omega \mid 0 < t \le 1, \;\; 0 \le s \le \bar{l}_A(t)\},$$

$$\mathcal{Z}_0 = \{(s,t) \in \omega \mid 0 < t \le a^*, \;\; l_A(t) \le s \le r_A(t)\},$$

$$\mathcal{Z}_r = \{(s,t) \in \omega \mid 0 < t \le 1, \;\; \bar{r}_A(t) \le s \le \sqrt{1-t^2}\}.$$

We easily obtain the following result and thus complete the proof of Theorem 4.1.2 in this case.

Theorem 4.4.9. *The function*

$$\psi(s,t) = \begin{cases} \psi^1(s,t) & (s,t) \in \mathcal{Z}_l, \\ \eta_A(t) & (s,t) \in \mathcal{Z}_0, \\ \psi^2(s,t) & (s,t) \in \mathcal{Z}_r \end{cases}$$

is a Lipschitz solution to the problem (4.6).

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