ON THE GENERAL SELF-EQUILIBRATED END LOADING OF A SOLID LINEARLY ELASTIC CYLINDER

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ABSTRACT

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By

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The problem studied is that of a semi-infinite cylinder with the long sides free from stress and a self-equilibrated, but not necessarily axisymmetric, load applied to the finite end. The method of biorthogonal vectorial eigenfunctions is applied to a formulation of the problem, within the classical theory of elasticity, using the stress equations of equilibrium and the Beltrami-Michell equations of compatibility. The axisymmetric case is shown to unify the classical work on axisymmetric self-equilibrated torsion end loading, with the more recent work on torsionless self-equilibrated end loading. The first few eigenvalues are calculated for various indices of theta dependence, and the numerical solution of two illustrative problems is presented.

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James Louis Klemm

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To the memory of my father,

Paul O. Klemm

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CHAPTER I: INTRODUCTION TO THE PROBLEM

Section 1.1: Origin and Scope of the Problem

The problem studied is that of the deformation of a semi-infinite elastic solid cylinder due to a self-equilibrated end load. The long sides are assumed to be free from stress. All analysis is undertaken within the classical theory of elasticity.

The problem arises naturally in either of the following two ways: as a difference problem for the general loading of long cylinders or as an example in the general discussion of St. Venant's principle.

Solutions are known which can treat the side loading of a long cylinder (see, for example, Lur'e [24, Chapter 7]). These solutions carry with them, however, specific end conditions with control only over the resultant force and resultant moment applied to the ends. The difference problem for a semi-infinite cylinder, between general loading problems and the class of problems amenable to treatment using these solutions, is a self-equilibrated end loading problem. In the review of the literature that follows it will be seen that a number of people have studied this difference problem for axisymmetric loadings, but that much remains for study in the non-symmetric case.

St. Venant encountered this difficulty in his Treatise on Torsion [28], since he could satisfy the end conditions, using his solutions for the torsion of long rods, only up to the prescribed resultant force and the prescribed resultant moment. He asserted, therefore, that this failure to satisfy the end conditions exactly had little effect apart from the ends of the rods and this assertion, along with several variations of it on the part of others, became known as St. Venant's principle. As contributions to the study of the principle made by Thomson and Tait [34], Bousinesque [5], and others seemed to give the principle some validity, it became widely accepted, if only out of necessity.

In 1945, however, von Mises [38], raised the question as to whether these more general assertions being justified by appealing to "St. Venant's principle" were indeed true, and produced counterexamples to prove that for certain kinds of loadings, the effect away from the area of loading was much greater than was widely believed. Furthermore, he proposed, as sufficient, a criterion for assuring the rapid decay of stress away from the region of loading. Since the publication of his paper the study of problems connected with the statement of such a principle, the range of validity of a given principle, and the study of the St. Venant boundary region (the region of the solid where the distribution of the loading is not negligible) have been among the major areas of interest in the theory of elasticity.

Of the papers that followed that of von Mises, Sternberg [31], who first proved the correctness of von Mises' formulation of the principle, Erin [11], Boley [4], Dzhanelidze [10], Donnel [9], Toupin [35], Keller [19], Knowles [20], and Roseman [27] have presented papers which attempt to give a more precise statement of the conditions governing the rate of decay of stress and estimates of the orders of magnitude for the resulting stress within the body.

The difference problem, for the case of self-equilibrated stresses, is related to these studies in the following way: The stresses are known to decay as one moves away from the vicinity of a self-equilibrated loading, and their entire effect is confined to the St. Venant boundary region. Thus the difference problem is a St. Venant boundary region problem. The investigations mentioned above provide estimates for the effective width of this region and give bounds on the stresses that such a loading provides in the interior of the body, often for quite general geometries, but within restricted geometries it is sometimes possible to solve the St. Venant's problem, itself, and obtain the exact stresses that a given self-equilibrated loading provides at each point of the body.

The formulation and solution of the St. Venant boundary region problem for the end loading of a semi-infinite cylinder is presented in this thesis and numerical results are reported for two particular problems as an indication of the rate of convergence of the series of eigenfunctions which constitute the solutions.

When displacement end conditions (or displacement conditions in conjunction with stress end conditions) are prescribed, one faces difficulties of two sorts: First, the conditions for the existence of decaying solutions for these kinds of boundary conditions are known only for the end loading of a semi-infinite two-dimensional strip (see Gusein-Zade [15, 16]) and these conditions are quite different from the corresponding conditions for stresses. Second, when displacement end conditions are prescribed in conjunction with stress conditions, one is known to be liable to the generation of stress singularities (see Williams [42]).

Although these difficulties have not yet been resolved, this thesis does present a way of formulating two of the fundamental mixed problems in terms of stress-type variables and provides a solution in terms of a series of eigenfunctions, whose coefficients may be chosen directly by a biorthogonality relation derived in this study.

Section 1.2: Progress on the Class of End Loading Problems for a Cylinder

Problems of axisymmetric end loadings of cylinders have been studied by a variety of methods: Purser (see Love [23, Section 226B]) obtained the eigenfunction expansion governing the axisymmetric torsion end loading problem. Prokopov (see Lur'e [24, Chapter 7]), studied the solution of axisymmetric end loading problem in a displacement formulation using the Papkovich-Neuber formulation and calculating the first few eigenvalues, along with the values of some useful parameters. Valov [37] used a Fourier-Bessel expansion with the Papkovich-Neuber potentials to study two mixed problems, obtaining an infinite system of linear equations, whose regularity he also investigated. Horvay and Mirabal [17] applied the calculus of variations to the study of axisymmetric normal and shear end loading.

Alekandrov and Solov'ev [2] reduced the Papkovich-Neuber formulation of three-dimensional axisymmetric problems into problems of a complex variable, and Chemeris [8] to integral equations. Mitra [25] studied the transversely isotropic cylinder of finite length using a single second order Love-type strain function.

Little and Childs [22] obtained a vector biorthogonality from Love's strain function by use of the calculus of residues. Swan [32] integrated the displacement equations to obtain the governing eigenfunctions for the case when the sides of the cylinder were free from stress, but expanded the solution obtained in terms of a complete set of orthonormal functions and lost computational efficiency due to the slow convergence of series of Bessel functions.

Approximate methods include work by Warren and Roark [39], who employed an aggregate of exact solutions obtained from a displacement formulation and chose the coefficients for two torsionless stress - stress end loadings by a least squares technique.

The list of papers presented above is not complete, but does indicate the scope of interest and the variety of methods which have been applied to this class of problems when restricted to being axisymmetric . Relatively few papers, however, report on methods of attacking the solution of non-axisymmetric three dimensional problems. Muki [26] (in Japanese) extended Sneddon's axisymmetric solution for a half-plane to non-symmetric loadings, and Alexandrov and Solov'ev [3] have extended their formulation of axisymmetric problems in terms of complex variables to that of non-axisymmetric ones.

Flügge and Kelkar [12] have proposed a way of solving the general displacement loading of a semi-infinite cylinder, but the method is presented in detail only for the axisymmetric case. The paper derives the following generalized vector orthogonality relation for the axisymmetrically loaded solid cylinder:

$$(\lambda_m-\lambda_n)\int_0^a \vec{y}_n^{\,T}\bar{S}\,\vec{y}_m dr = -\frac{\lambda_m-\lambda_n}{\lambda_m\lambda_n}\mu_1 h_{2n}(0)h_{2m}(0)$$

where λ_m and λ_n are, respectively, the $m^{\underline{th}}$ and $n^{\underline{th}}$ eigenvalues with corresponding eigenvectors $\vec{y}_m$ and $\vec{y}_n$, $\bar{S}$ is a weighting matrix, μ_1 a constant and h_m, h_n are functions whose form is known but which depend on the eigenfunction chosen. The coefficients desired are the constants d_n occurring in an expansion of the form

$$\vec{y}_o = \sum_{n=1}^{\infty} d_n\vec{y}_n(r)$$

but the orthogonality relation cannot be used directly to isolate the $m^{\underline{th}}$ coefficient since the boundary data involves all of the eigenfunctions, simultaneously, and the right hand side remains valid only when the functions, h, have a prescribed relationship to which eigenfunction is used. No numerical work was included, (not even to the extent of tabulating eigenvalues) and there is no indication that the authors were aware of the computational difficulties, including the possible necessity of solving a truncation of an infinite system of equations, frequently associated with this class of problems.

Finally, D.R. Childs [7] claims to have obtained a general solution for three dimensional problems of elasticity in terms of a single biharmonic function. The reviewer, B.E. Gatewood [13], suggests that the method, at best, could possibly apply to some axisymmetric problems. The author presents no example or argument to indicate that his solution is sufficiently general to be of use in the study of boundary value problems.

Section 1.3: The Equations of Compatibility

The three equations of equilibrium for an elastic body may be written as three equations involving the three unknown displacements in the coordinate directions or as three equations in the six independent stresses. It is therefore necessary that the six small strains or the six independent stresses satisfy three equations, in addition to the equations of equilibrium, to assure that they give rise to a single-valued displacement field for a simply-connected body. This information is, in each case, carried in a set of six partial differential equations, known as the St. Venant equations of compatibility for the small strains case, and the Beltrami-Michell equations of compatibility for the stress case. An outline of a way of deriving the latter set of equations from the former may be found in any text on elasticity.

The fact that there are obtained six equations of compatibility in the general case, when it is known that the system carries the information of three independent equations has apparently been responsible for delaying the study of the stress formulation of three dimensional problems of elasticity. The dependence of the system has been studied by Southwell [30], Washizu [40] and Grycz [14], obtaining, at best, a system of three conditions which must be satisfied on the boundary of the region and three different conditions which must be satisfied in the interior. It is only in the case of "plane" problems, when the number of independent stresses reduces to three, the number of equations of equilibrium to two and the number of compatibility equations to one, that the stress formulation has widely been used.

Section 1.4: Method of Analysis

The present work follows that of Smith [29], Johnson and Little [18], and Little and Childs [22] in its use of biorthogonal vectorial eigenfunction expansions. (The relationship between the eigenfunction solution and that obtained by a Fourier-Laplace transform is discussed, for the case of a semi-infinite strip, by Buckwald [6].) In particular, it follows the approach of Johnson and Little in its formulation of the problem, but is somewhat more complex in its details due to the increase in the number of variables.

Two fundamental mixed boundary value problems are studied, the first with prescribed normal stress and prescribed tangential displacements applied to the end, the second with prescribed normal displacements and shear stresses. Both have the long sides free from stress.

Variables are introduced, for each of these two cases, which are of the order of stresses with regard to differentiation; carry the information of the displacement boundary conditions; allow each problem to be stated, independently, in terms of a three-vector second order partial differential equation; and integrate certain of the Beltrami-Michell equations. As might be expected by the number of conditions that these variables were required to satisfy, to find them was a major step in the solution of this problem.

The vector partial differential equations are reduced to ordinary differential equations by a separation of variables. (The validity of separation of variables for the equations of elasticity in this geometry was discussed by Tsay [36].) It is found that a generalized biorthogonality, of the kind of Langer [21], can be found which would permit direct calculation of the generalized Fourier coefficients of the eigenfunction expansion, if it could be assured that the problem under consideration was both nonsingular and involved a decaying solution.

A six component vector is formed consisting of the variables of the two mixed problems described above, and it is shown that this vector satisfies a first order partial differential equation, whose constituent equations come from the equations of equilibrium, the equations defining the new variables introduced, and equations obtained by integrating certain of the Beltrami-Michell equations. A generalized biorthogonality relation is obtained for this six-vector and through use of the first order vector partial differential equation, it is rendered useful even in higher orders of theta dependence.

Finally, numerical work for certain trial total stress problems is performed by substituting the boundary data into the six-vector, using the generalized biorthogonality, and solving a finite system of linear equations, obtained from an infinite system of linear equations by truncation.

CHAPTER II: THE ANALYTICAL FORMULATION

Section 2.1: The First Fundamental Mixed Boundary Value Problem

The problem studied here, and throughout this study, is the equilibrium of a semi-infinite, linearly elastic, solid cylinder. The radius is taken to be one and the cylinder is to be described with a polar cylindrical coordinate system, with the positive z-axis along the axis of symmetry of the cylinder and the finite face along the plane $z = 0$. The displacements in the r, θ, and z directions will be denoted by u, v, and w, respectively.

The first fundamental mixed problem denotes the following mixed boundary value problem: The long side, $r = 1$ is free from stress and on the end $z = 0$, the normal stress σ_{zz}, and the two tangential displacements u and v are prescribed. Furthermore, it is assumed that these end conditions have been prescribed in such a manner that the resulting end loading is self-equilibrating.

The equations governing the equilibrium of an elastic body may be written in polar cylindrical coordinates as follows:

<u>Equations of Equilibrium</u> (3 equations, 6 unknowns)

1.1) $\partial\sigma_{zz}/\partial z + L\,\tau_{rz} + 1/r\;\partial\tau_{\theta z}/\partial\theta = 0$

1.2) $L\,\sigma_{rr} - \sigma_{\theta\theta}/r + 1/r\;\partial\tau_{r\theta}/\partial\theta + \partial\tau_{rz}/\partial z = 0$

1.3) $1/r\;\partial\sigma_{\theta\theta}/\partial\theta + \partial\tau_{rz}/\partial z + L\,\tau_{r\theta} + \tau_{r\theta}/r = 0$

where

1.4) $L = \partial/\partial r + 1/r$

<u>The Beltrami-Michell Equation of Compatibility</u>

(6 equations, 6 unknowns)

1.5) $\nabla^2\sigma_{zz} + \frac{1}{1+\nu}\,\partial^2 K/\partial z^2 = 0$

1.6) $\nabla^2\sigma_{rr} + \frac{2}{r^2}(\sigma_{\theta\theta} - \sigma_{rr}) - 4/r^2\,\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{1+\nu}\,\frac{\partial^2 K}{\partial r^2} = 0$

1.7) $\nabla^2\sigma_{\theta\theta} - 2/r^2\,(\sigma_{\theta\theta} - \sigma_{rr}) + 4/r^2\;\partial\tau_{r\theta}/\partial\theta$

$$+ \frac{1}{1+\nu}\left(1/r\,\frac{\partial K}{\partial r} + 1/r^2\,\frac{\partial^2 K}{\partial\theta^2}\right) = 0$$

1.8) $\nabla^2\tau_{r\theta} - 2/r^2\,\partial/\partial\theta\,(\sigma_{\theta\theta} - \sigma_{rr}) - 4/r^2\,\tau_{r\theta}$

$$+ \frac{1}{1+\nu}\,\frac{\partial}{\partial r}\left(1/r\,\frac{\partial K}{\partial\theta}\right) = 0$$

1.9) $(\nabla^2 - 1/r^2)\tau_{rz} - 2/r^2\,\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{1}{1+\nu}\,\frac{\partial^2 K}{\partial r\partial z} = 0$

1.10) $$(\nabla^2 - 1/r^2)\tau_{\theta z} + 2/r^2 \frac{\partial\tau_{rz}}{\partial\theta} + \frac{1}{1+\nu}\, 1/r\, \frac{\partial^2 K}{\partial\theta\partial z} = 0$$

where ν is Poisson's ratio,

1.11) $$K = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} = R + \sigma_{zz}$$

represents the first invariant of the stress tensor and

1.12) $$\nabla^2 = \partial^2/\partial r^2 + 1/r\, \partial/\partial r + 1/r^2\, \partial^2/\partial\theta^2 + \partial^2/\partial z^2$$

represents the Laplacian operator in this coordinate system.

The definition of small strains is as follows

1.13) $$\epsilon_{rr} = \frac{\partial u}{\partial r}\ , \quad \epsilon_{\theta\theta} = \frac{u}{r} + 1/r\, \frac{\partial v}{\partial\theta}\ , \quad \epsilon_{zz} = \frac{\partial w}{\partial z}$$

1.14) $$\epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\ , \quad \epsilon_{\theta z} = \frac{1}{2}\left(1/r\, \frac{\partial w}{\partial\theta} + \frac{\partial v}{\partial z}\right)$$

$$\epsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + 1/r\, \frac{\partial u}{\partial\theta}\right)$$

with u, v, and w, being the displacements in the r, θ, and z directions, respectively. Hooke's law may be written

1.15) $$\epsilon_{rr} = \frac{1}{E}\left[\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right]\ , \quad \epsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})\right]$$

$$\epsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})\right]$$

1.16) $$\epsilon_{rz} = \frac{1+\nu}{E}\tau_{rz}\ , \quad \epsilon_{\theta z} = \frac{1+\nu}{E}\tau_{\theta z}\ , \quad \epsilon_{r\theta} = \frac{1+\nu}{E}\tau_{r\theta}$$

where E is Young's modulus.

Two more useful equations are obtained as follows: Adding equations 1.5), 1.6), and 1.7) yields:

1.17) $$\nabla^2 K = 0$$

and using this equation with the sum of 1.6) and 1.7) gives

1.18) $$\nabla^2 R - \frac{1}{1+\nu} \frac{\partial^2}{\partial z^2} (R + \sigma_{zz}) = 0$$

From equations 1.13) and 1.15) one obtains

1.19) $$Lu + 1/r \frac{\partial v}{\partial \theta} = \frac{1}{E} [(1-\nu)R - 2\nu \sigma_{zz}]$$

Define a new variable, T, by

1.20) $$L \tau_{\theta z} - 1/r \frac{\partial \tau_{rz}}{\partial \theta} + \frac{\partial T}{\partial z} = 0$$

Subject to the boundary condition $\lim_{z \to \infty} T = 0$. Now

$$\frac{2(1+\nu)}{E} \tau_{\theta z} = 1/r \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} , \quad \frac{2(1+\nu)}{E} 1/r \frac{\partial \tau_{rz}}{\partial \theta} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

so

$$\frac{2(1+\nu)}{E} L \tau_{\theta z} = 1/r \frac{\partial^2 w}{\partial \theta \partial r} + \frac{\partial}{\partial z} Lv \; ; \quad \frac{2(1+\nu)}{E} 1/r \frac{\partial \tau_{rz}}{\partial \theta} = \frac{\partial}{\partial z} (1/r \frac{\partial u}{\partial \theta})$$

$$+ 1/r \frac{\partial^2 w}{\partial \theta \partial r}$$

Subtracting gives

$$\frac{2(1+\nu)}{E}\left(1/r\,\frac{\partial\tau_{rz}}{\partial\theta} - L\,\tau_{\theta z}\right) = \frac{\partial}{\partial z}\left(1/r\,\frac{\partial u}{\partial\theta} - Lv\right)$$

or

1.21) $$\frac{2(1+\nu)}{E}\,T = 1/r\,\frac{\partial u}{\partial\theta} - Lv$$

Equations 1.19) and 1.21) show that the displacements u and v are specified up to rigid displacements by the variables R, σ_{zz} and T. Therefore, a formulation is sought of the fundamental mixed boundary problem, with normal stress, σ_{zz} and tangential displacements u and v prescribed in terms of the three variables R, σ_{zz}, and T.

Differentiating the first equation of equilibrium (1.1), and substituting the result in the Beltrami-Michell equation, equation 1.9), gives

$$-\frac{\partial}{\partial r}\left(\frac{\partial\sigma}{\partial z}\right) - 1/r\,\frac{\partial^2\tau_{\theta z}}{\partial\theta\partial r} - \frac{1}{r^2}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\tau_{rz}}{\partial\theta^2} + \frac{\partial^2\tau_{rz}}{\partial z^2}$$

$$+ \frac{1}{1+\nu}\frac{\partial^2}{\partial r\partial z}(R + \sigma_{zz}) = 0$$

or

$$\frac{1}{1+\nu}\frac{\partial^2}{\partial r\partial z}(R - \nu\sigma_{zz}) + 1/r\,\frac{\partial}{\partial\theta}\left(-L\,\tau_{\theta z} + 1/r\,\frac{\partial\tau_{rz}}{\partial\theta}\right) + \frac{\partial^2\tau_{rz}}{\partial z^2} = 0$$

Applying equation (1.20), the definition of T, integrating with respect to z, and using as the boundary condition that the solution is zero at infinity, yields

1.22) $$\frac{1}{1+\nu}\frac{\partial}{\partial r}(R - \nu\sigma_{zz}) + 1/r\,\frac{\partial T}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} = 0$$

or

1.22') $$\frac{\partial \sigma_{zz}}{\partial r} = \frac{1+\nu}{\nu r}\frac{\partial T}{\partial \theta} + \frac{1+\nu}{\nu}\frac{\partial \tau_{rz}}{\partial z} + 1/\nu\,\frac{\partial R}{\partial r}\ .$$

Now operating on the first equation of equilibrium, equation 1.7), with operator $1/r\,\frac{\partial}{\partial \theta}$, and substituting the result in the Beltrami-Michell equation, equation 1.10) gives

$$- 1/r\ \partial/\partial\theta\ \left(\frac{\partial \sigma_{zz}}{\partial z}\right) - \partial/\partial r\ \left(1/r\ \frac{\partial \tau_{rz}}{\partial \theta}\right) + \partial/\partial r\ L\ \tau_{\theta z} + \frac{\partial^2 \tau_{\theta z}}{\partial z^2}$$

$$+ \frac{1}{1+\nu}\ 1/r\ \frac{\partial^2}{\partial\theta\partial z}(R + \sigma_{zz}) = 0\ .$$

Again using the definition of T, equation 1.20), and integrating as above gives

1.23) $$\frac{\partial T}{\partial r} - \frac{1}{1+\nu}\ 1/r\ \frac{\partial}{\partial \theta}(R - \nu\sigma_{zz}) - \frac{\partial \tau_{\theta z}}{\partial z} = 0$$

or

1.23') $$1/r\ \frac{\partial \sigma_{zz}}{\partial \theta} = -\frac{1+\nu}{\nu}\frac{\partial T}{\partial r} + \frac{1+\nu}{\nu}\frac{\partial \tau_{\theta z}}{\partial z} + \frac{1}{\nu}\frac{1}{r}\frac{\partial R}{\partial \theta}\ .$$

Operating on equation 1.23) with L and on equation 1.22) with $1/r\,\frac{\partial}{\partial \theta}$ and adding the result yields

$$L\,\frac{\partial T}{\partial r} - L\,\frac{\partial \tau_{\theta z}}{\partial z} + 1/r^2\,\frac{\partial^2 T}{\partial \theta^2} + 1/r\,\frac{\partial^2 \tau_{rz}}{\partial\theta\partial z} = 0$$

Applying the definition of T, equation 1.20), one finds that T is harmonic, i.e.,

1.24) $$\nabla^2 T = 0$$

At this stage a system of three second order differential equations, equations 1.18), 1.5), and 1.24), have been obtained describing this fundamental mixed problem. Only three boundary conditions (finiteness at $r = 0$ will have the effect of an additional three boundary conditions) are needed for the solution of this problem.

Having four first order equations in the five fundamental variables, R, σ_{zz}, T, τ_{rz}, and $\tau_{\theta z}$, if the procedure of Johnson and Little [18] is to be followed, one must fill out this set of first order equations to a set of six. What would first suggest itself would be to attempt to apply the program established above to the other Beltrami-Michell equations. The usefulness of the second and third equations of equilibrium and the Beltrami-Michell equations 1.6), 1.7) and 1.8) (along with the limitations to their use) will be seen in the consideration of boundary conditions. This leaves the first equation of equilibrium, equation 1.1), the Beltrami-Michell equation 1.5), and the derived first order equations 1.20), 1.22) and 1.23). These first order equations do not extract further information from equation 1.5) indicating, perhaps, that the information is independent from that carried by equations 1.1), 1.9) and 1.10) and their derived equations.

The missing information is obtained directly from consideration of the fundamental mixed problem dual to the one now begin studied.

Writing equations 1.18), 1.5) and 1.24) as a single vector partial differential equation gives

1.25) $$L\, \partial/\partial r\, \{\vec{f}\} + 1/r^2\, \partial^2/\partial\theta^2\, \{\vec{f}\} + [U]\, \frac{\partial^2}{\partial z^2} \{\vec{f}\} = 0$$

where

1.25a) $$\{\bar{f}\} = \begin{Bmatrix} R \\ \sigma_{zz} \\ T \end{Bmatrix}$$, 1.25b) $$[U] = \frac{1}{1+\nu} \begin{bmatrix} \nu & -1 & 0 \\ 1 & 2+\nu & 0 \\ 0 & 0 & 1+\nu \end{bmatrix}$$

From the separation of variables

1.26) $$\{\vec{f}(r,\theta,z)\} = \vec{\varphi}(r)\Theta(\theta)Z(z)$$

one obtains

1.26a) $$\{\vec{f}\} = \sum_{m=0}^{\infty} \vec{\varphi}_m(r) e^{im\theta}\, e^{-\alpha_m z}$$

where m is an integer, α is an undetermined parameter, and $\{\vec{\varphi}_m\}$ satisfies

1.26b) $$L\, d/dr\, \{\vec{\varphi}_m\} - \frac{m^2}{r^2} \{\vec{\varphi}_m\} + \alpha_m^2 [U]\{\vec{\varphi}_m\} = 0$$

Substituting 1.26a) into 1.25) one obtains the form of the solution for the variables R, σ_{zz}, and T.

Thus

1.27) $$R = \sum_{m=0}^{\infty} R_m(r) e^{im\theta} e^{-\alpha_m z}$$

$$\sigma_{zz} = \sum_{m=0}^{\infty} \sigma_{zzm}(r) e^{im\theta} e^{-\alpha_m z}$$

$$T = \sum_{m=0}^{\infty} T_m(r) e^{im\theta} e^{-\alpha_m z}$$

where R_m, σ_{zzm}, and T_m, are components of the vector $\{\vec{\varphi}_m\}$, and

1.28a) $$R_m = -[(A_m + 2(1+\nu)B_m)J_m(\alpha_m r) + B_m \cdot \alpha_m r \cdot J_{m+1}(\alpha_m r)]$$

1.28b) $$\sigma_{zzm} = A_m J_m(\alpha_m r) + B_m \cdot \alpha_m r \cdot J_{m+1}(\alpha_m r)$$

1.28c) $$T_m = -i\, C_m J_m(\alpha_m r) .$$

Boundary Conditions

The vanishing of stress along the sides of the cylinder gives rise to three boundary equations for the determination of the unknown constants in the equation 1.28). All three of these equations may be obtained by use of the equations of equilibrium in conjunction with certain of the Beltrami-Michell equations.

1^{o}) First boundary condition: $\tau_{rz} = 0$ along $r = 1$.

Because of the requirement that the solution should decay to zero as z tends to infinity, it suffices to require that

$$\left.\frac{\partial \tau_{rz}}{\partial z}\right|_{r=1} = 0$$

which, by equation 1.22), is assured by

1.29) $$\left.\frac{1}{1+\nu}\left(\frac{\partial R}{\partial r} - \nu \frac{\partial \sigma_{zz}}{\partial r}\right) + 1/r \frac{\partial T}{\partial \theta}\right|_{r=1} = 0$$

or, for the eigenfunction associated with the index m, of theta dependence,

1.29a) $$\frac{1}{1+\nu}\left(\frac{dR_m}{dr} - \nu \frac{d\sigma_{zzm}}{dr}\right) + im\, T_m = 0$$

Using equations 1.27) gives

1.29b) $$[m\, J_m(\alpha_m) - \alpha_m J_{m+1}(\alpha_m)]A_m$$

$$+ [(\alpha_m^2+2m)J_m(\alpha_m) - (m+2)\alpha_m J_{m+1}(\alpha_m)]B_m - m\, J_m(\alpha_m)C_m = 0$$

2^o) Second boundary condition: $\sigma_{rr} = 0$ along $r = 1$.

Operating on the second equation of equilibrium, equation 1.2), by $-\partial/\partial r + 1/r$ and the third equation of equilbrium, equation 1.3), by $1/r\ \partial/\partial\theta$ and adding, gives

$$-\frac{\partial^2 \sigma_{rr}}{\partial r^2} + 1/r \frac{\partial \sigma_{\theta\theta}}{\partial r} + \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta}) + 4/r^2 \frac{\partial \tau_{\theta\theta}}{\partial \theta} + 1/r^2 \frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2}$$

$$+ \partial/\partial z \left(1/r \frac{\partial \tau_{\theta z}}{\partial \theta} - \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r}\right) = 0$$

Adding this to the Beltrami-Michell equation, equation 1.6), yields

$$1/r \frac{\partial R}{\partial r} + 1/r^2 \frac{\partial^2 R}{\partial \theta^2} + \frac{\partial^2 \sigma_{rr}}{\partial z^2} + \frac{\partial}{\partial z} \left(1/r \frac{\partial \tau_{\theta z}}{\partial \theta} - \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r}\right)$$

$$+ \frac{1}{1+\nu} \frac{\partial^2}{\partial r^2} (R + \sigma_{zz}) = 0$$

Using equation 1.17) gives

$$- \frac{\partial^2 \sigma_{rr}}{\partial z^2} = \frac{1}{1+\nu} [1/r\ \partial/\partial r + 1/r^2\ \partial^2/\partial\theta^2](\nu R - \sigma_{zz})$$

$$+ \frac{\partial}{\partial z} \left(1/r \frac{\partial \tau_{\theta z}}{\partial \theta} - \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r}\right) - \frac{1}{1+\nu} \frac{\partial^2 K}{\partial z^2}$$

Using equation 1.1) to eliminate $\frac{\partial \tau_{rz}}{\partial r}$, equation 1.23) to eliminate $1/r \frac{\partial \tau_{\theta z}}{\partial \theta}$, and the first boundary condition, one obtains the second boundary condition

1.30) $$\frac{1}{1+\nu} [1/r\ \partial/\partial r + 1/r^2\ \partial^2/\partial\theta^2](\sigma_{zz} - \nu R)$$

$$+ \frac{1}{1+\nu} \frac{\partial^2}{\partial z^2} (R - \nu\sigma_{zz})$$

$$- 2/r \frac{\partial^2 T}{\partial\theta\partial r} + \frac{2}{1+\nu} 1/r^2 \frac{\partial^2}{\partial\theta^2} (R - \nu\sigma_{zz}) \Big|_{r = 1} = 0$$

or

1.30a) $$\frac{1}{1+\nu} [1/r\ d/dr - m^2/r^2](\sigma_{zzm} - \nu R_m) + \frac{\alpha m^2}{1+\nu} (R_m - \nu\sigma_{zzm})$$

$$- \frac{2im}{r} \frac{dT_m}{dr} - \frac{2m^2}{(1+\nu)r^2} (R_m - \nu\sigma_{zzm}) \Big|_{r = 1} = 0$$

This condition may be written as follows:

1.30b)
$$[(m(m+1) - \alpha_m^2)J_m(\alpha_m) - \alpha_m J_{m+1}(\alpha_m)]A_m$$
$$+ [(-\alpha_m^2 - 2\nu m(m-1) + 4m^2)J_m(\alpha_m) + (m(m-1) - 2\nu - \alpha_m^2)\alpha_m J_{m+1}(\alpha)]B_m$$
$$+ [-2m^2 J_m(\alpha_m) + 2m\alpha_m J_{m+1}(\alpha_m)]C_m = 0$$

3^o) Third boundary condition: $\tau_{r\theta} = 0$ on $r = 1$.

Operating on the third equation of equilibrium, equation 1.3) by the operator $-\frac{\partial}{\partial r} + 1/r$ and the second equation of equilibrium, equation 1.2) by $1/r\ \partial/\partial\theta$ and adding, one obtains

$$-L \frac{\partial\tau_{r\theta}}{\partial r} - \frac{1}{r^2}\frac{\partial^2\tau_{r\theta}}{\partial\theta^2} - \frac{1}{r}\frac{\partial^2 R}{\partial r\partial\theta} + \frac{4\tau_{r\theta}}{r^2} + \frac{\partial}{\partial z}\left(-L\,\tau_{\theta z} + \frac{2}{r}\tau_{\theta z} - \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}\right)$$
$$- 1/r^2 \frac{\partial\sigma_{rr}}{\partial\theta} + \frac{3}{r^2}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} = 0$$

Adding this to equation 1.8), gives

$$\frac{\partial^2\tau_{r\theta}}{\partial z^2} - \frac{1}{r}\frac{\partial^2 R}{\partial r\partial\theta} + 1/r^2\frac{\partial^2 R}{\partial\theta} + \frac{\partial}{\partial z}\left(-L\,\tau_{\theta z} + \frac{2\tau_{\theta z}}{r} - 1/r\frac{\partial\tau_{rz}}{\partial\theta}\right)$$
$$+ \frac{1}{1+\nu}\frac{\partial}{\partial r}\left(1/r\frac{\partial}{\partial\theta}K\right) = 0$$

so using equation 1.20) to eliminate $1/r\,\frac{\partial\tau_{rz}}{\partial\theta}$, gives

$$\frac{\partial^2\tau_{r\theta}}{\partial z^2} = \frac{-1}{1+\nu}\frac{\partial}{\partial r}\left[1/r\frac{\partial}{\partial\theta}(\sigma_{zz} - \nu R)\right] + \frac{\partial^2 T}{\partial z^2} + 2\frac{\partial}{\partial r}\left[\frac{\partial\tau_{\theta z}}{\partial z}\right]$$

and by use of equation 1.23), this may be written as follows:

$$\frac{\partial^2 \tau_{r\theta}}{\partial z^2} = \frac{-1}{1+\nu}\,[\partial/\partial r(1/r\;\partial/\partial\theta(\sigma_{zz} - \nu R))] + \frac{\partial^2 T}{\partial z^2}$$

$$+ 2\,\frac{\partial}{\partial r}\,[\frac{\partial T}{\partial r} - \frac{1}{1+\nu}\,1/r\;\partial/\partial\theta\;(R - \nu\sigma_{zz})]$$

Using equation 1.24) the boundary condition reduces to

1.31)
$$\frac{1}{1+\nu}\frac{\partial}{\partial r}\,[1/r\;\partial/\partial\theta(\sigma_{zz} - \nu R)] + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{2}{r^2}\frac{\partial^2 T}{\partial\theta^2} + \frac{\partial^2 T}{\partial z^2}$$

$$+ \frac{2}{1+\nu}\frac{\partial}{\partial r}\,[1/r\;\partial/\partial\theta\;(R - \nu\sigma_{zz})]\Big|_{r=1} = 0$$

or

1.31a)
$$\frac{2}{r}\frac{d}{dr}\,(T_m) - \frac{2m^2}{r^2}\,T_m + \alpha_m^2 T_m + \frac{im}{1+\nu}\,[1/r\,\frac{d}{dr}((2-\nu)R_m + (1-2\nu)\sigma_{zzm})$$

$$- \frac{im}{1+\nu}\,\frac{1}{r^2}((2-\nu)R_m + (1-2\nu)\sigma_{zzm})]\Big|_{r=1} = 0$$

The final form becomes,

1.31b)
$$[-m(m-1)J_m(\alpha_m) + m\,\alpha_m\,J_{m+1}(\alpha_m)]A_m$$

$$+ [(-\alpha_m^2 m + 2\nu m(m-1) - 4m(m-1))J_m(\alpha_m)$$

$$+ (m^2 + 5m - 2\nu m)\alpha_m\,J_{m+1}(\alpha_m)]B_m$$

$$+ [(2m(m-1) - \alpha_m^2)J_m(\alpha_m) + 2\alpha_m\,J_{m+1}(\alpha_m)]C_m = 0$$

Alternatively, the last two boundary conditions can be obtained from the definition of small strains in polar co-ordinates and Hooke's law as follows:

$2^{o'}$) Consider

$$\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{E}[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})]$$

$$= \frac{1}{E}[-(1+\nu)\sigma_{rr} + (R - \nu\sigma_{zz})]$$

which may be written

$$\sigma_{rr} = \frac{1}{1+\nu}(R - \nu\sigma_{zz}) - \frac{E}{1+\nu}[\frac{\partial u}{\partial r} + 1/r\,\frac{\partial v}{\partial \theta}]$$

$$\frac{\partial \sigma_{rr}}{\partial z} = \frac{1}{1+\nu}\frac{\partial}{\partial z}(R - \nu\sigma_{zz}) - \frac{E}{1+\nu}[1/r(\frac{2(1+\nu)}{E}\tau_{rz} - \frac{\partial w}{\partial r})$$

$$+ 1/r\,\frac{\partial}{\partial \theta}(\frac{2(1+\nu)}{E}\tau_{\theta z} - 1/r\,\frac{\partial w}{\partial \theta})]$$

$$\frac{\partial^2 \sigma_{rr}}{\partial z^2} = \frac{1}{1+\nu}\frac{\partial^2}{\partial z^2}(R - \nu\sigma_{zz}) - \frac{2}{r}\frac{\partial \tau_{rz}}{\partial z} - \frac{2}{r}\frac{\partial}{\partial \theta}(\frac{\partial \tau_{\theta z}}{\partial z})$$

$$+ \frac{1}{1+\nu}1/r\,\frac{\partial}{\partial r}(\sigma_{zz} - \nu R) + \frac{1}{1+\nu}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\sigma_{zz} - \nu R)$$

and proceeding as in 2^o) one obtains equation 1.30).

$3^{o'}$) In a similar manner, one obtains

$$\frac{2(1+\nu)}{E}\tau_{r\theta} = ((\frac{\partial}{\partial r} - 1/r)v + 1/r\frac{\partial u}{\partial \theta})$$

$$\frac{2(1+\nu)}{E}\frac{\partial \tau_{r\theta}}{\partial z} = \frac{2(1+\nu)}{E}(\frac{\partial \tau_{\theta z}}{\partial r} - \frac{\tau_{\theta z}}{r}) - \frac{2}{r}\frac{\partial^2 w}{\partial \theta \partial r} + \frac{2}{r^2}\frac{\partial w}{\partial \theta}$$

$$+ \frac{2(1+\nu)}{E} 1/r \frac{\partial \tau_{rz}}{\partial \theta}$$

Differentiation yields

$$\frac{\partial^2 \tau_{r\theta}}{\partial z^2} = \frac{\partial}{\partial z}(\frac{\partial \tau_{\theta z}}{\partial r} - \frac{\tau_{\theta z}}{r} + 1/r\frac{\partial \tau_{rz}}{\partial \theta}) - \frac{1}{1+\nu}\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{zz} - \nu R))$$

which reduces to the condition 1.31) by the same argument as in 3^o).

All three equations of equilibrium have now been used, and the Beltrami-Michell, equations 1.9) and 1.10), have been integrated. It may be seen that the equations of equilibrium reduce 1.6) and 1.8) to a simpler form, but leave the second z-derivatives of σ_{rr} and $\tau_{r\theta}$, respectively, so they become appropriate for specifying boundary conditions, but will not be used further. This leaves only equations 1.5) and 1.7) to consider. Equation 1.7) will not be used directly, but only in its sum with equation 1.6). This focuses attention on the use of equations 1.5), 1.7) and (or) 1.18) for the further information needed to complete the study of the end loading. Consideration of this case, however, will be completed first.

The Transcendental Equation

Equations 1.29b), 1.30b), and 1.31b) give three homogeneous equations for the coefficients A_m, B_m, and C_m. The necessary and sufficient condition that these three equations should have a non-trivial solution is that their determinant should vanish, which gives rise, after appropriate simplification, to the transcendental equation for α_m

$$
\begin{aligned}
1.32)\quad & (\gamma_m^6 - 2\alpha_m^4 m(m-1) + 4(1-\nu)\alpha_m^2 m^2(m-1) + \alpha_m^2 m^2(m^2-1))J_m^3(\alpha_m) \\
& + (-2\alpha_m^5(m+1) - 4(1-\nu)\gamma_m^3 m(m-1) + 4\alpha_m^3 m^2(m-1) \\
& \qquad - 8(1-\nu)\alpha_m m^2(m^2-1) - 2\gamma_m m^3(m^2-1))J_m^2(\alpha_m)J_{m+1}(\alpha_m) \\
& + (\alpha_m^6 - 2(1-\nu)\alpha_m^4 - 2\alpha_m^4 m(m-1) + 4\alpha_m^4 m \\
& \qquad + 12(1-\nu)\alpha_m^2 m(m^2-1) + \alpha_m^2 m^2(m^2-1))J_m(\alpha_m)J_{m+1}^2(\gamma_m) \\
& + (-2\alpha_m^5 - 4(1-\nu)\alpha_m^3(m^2-1))J_{m+1}^3(\alpha_m) = 0
\end{aligned}
$$

It is interesting to note that if one sets $m = 0$ in this expression, corresponding to axisymmetric loading, equation 1.32) reduces to

$$
1.33)\quad \alpha_o^3(\alpha_o J_o(\alpha_o) - 2J_1(\alpha_o))(\alpha_o^2 J_o^2(\alpha_o) + \alpha_o^2 J_1^2(\alpha_o) - 2(1-\nu)J_1^2(\alpha_o)) = 0
$$

Now the first factor gives simply, the real roots of J_2. These were shown by F. Purser to govern the decay of stress for the axisymmetric torsion problem, and the results may be found in the classic treatise of Love [23]. The other factor, providing complex roots, has been shown by Little and Childs [22] and others to govern the decay of axisymmetric torsionless end loading.

Considering eauation 1.32) to be an equation of the form

$$f(\alpha_n) = 0$$

and expanding f into a power series, one sees that the coefficients of the expansion are all real, hence the complex roots occur in conjugate pairs. Furthermore, for m an even integer, the series is even and for m an odd integer, the series is odd, so that the negative of any root is again a root. Only roots with positive real parts, however, give rise to decaying solutions and are therefore relative to the problems being considered.

The eigenvalues for m from 0 to 4 are found in Tables 1 through 5.

Other Stress Variables

The equations developed so far determine the form of other variables of interest, as follows:

Applying equations 1.28) to equation 1.22) one obtains

$$1.34) \qquad \tau_{rzmj} = -[(\alpha_{mj} r) \cdot B_{mj} + \frac{m A_{mj} + 2m B_{mj} - m C_{mj}}{\alpha_{mj} r}] J_m(\alpha_{mj} r)$$

$$+ [A_{mj} + (m+2) B_{mj}] J_{m+1}(\alpha_{mj} r)$$

where the last subscript indicates that one is dealing with the j^{th} eigenvalue, α_{mj}. Similarly, from equation 1.23) one obtains

$$1.35) \qquad \tau_{\theta zmj} = -im[A_{mj} + 2B_{mj} - C_{mj}] \frac{J_m(\alpha_{mj} r)}{\alpha_{mj} r} - i(m\, B_{mj} + C_{mj}) J_{m+1}(\alpha_{mj} r)$$

These two equations, along with equations 1.28) provide the possibility of determining the coefficients A_{mj}, B_{mj}, and C_{mj}, by a least squares technique after the manner of Warren and Roark [39] or Thompson [33] and provide an alternative method of obtaining numerical results.

Section 2.2: The Adjoint Problem

The adjoint differential equation to equation 1.26b) is

$$2.1) \qquad \partial/\partial r\; L^*\{\vec{\Psi}_m\} - \frac{m^2}{r^2}\{\vec{\Psi}_m\} + \bar{\alpha}_m^2[U]^+\{\vec{\Psi}_m\} = 0$$

where

$$+ = \text{complex conjugate transpose}$$

$$2.2) \qquad - = \text{complex conjugate}$$

$$L^* = \partial/\partial r - 1/r$$

The equation can be written as follows

2.3)

$$\frac{d}{dr} L^* \begin{Bmatrix} \Psi_m^{(1)} \\ \Psi_m^{(2)} \\ \Psi_m^{(3)} \end{Bmatrix} - \frac{m^2}{r^2} \begin{Bmatrix} \Psi_m^{(1)} \\ \Psi_m^{(2)} \\ \Psi_m^{(3)} \end{Bmatrix} + \frac{\bar{\alpha}_m^2}{1+\nu} \begin{bmatrix} \nu & 1 & 0 \\ -1 & 2+\nu & 0 \\ 0 & 0 & 1+\nu \end{bmatrix} \begin{Bmatrix} \Psi_m^{(1)} \\ \Psi_m^{(2)} \\ \Psi_m^{(3)} \end{Bmatrix} = 0$$

Solving the equations above, yields the form of the adjoint function to be as follows:

2.4a) $$\Psi_m^{(1)} = A_m^* \cdot (\bar{\alpha}_m r) \cdot J_m(\bar{\alpha}_m r) + B_m^* \cdot (\bar{\alpha}_m r)^2 \cdot J_{m+1}(\bar{\alpha}_m r)$$

2.4b) $$\Psi_m^{(2)} = [A_m^* - 2(1+\nu)B_m^*] \cdot (\bar{\alpha}_m r) \cdot J_m(\bar{\alpha}_m r) + B_m^* \cdot (\bar{\alpha}_m r)^2 \cdot J_{m+1}(\bar{\alpha}_m r)$$

2.4c) $$\Psi_m^{(3)} = i\ C_m^* \cdot (\bar{\alpha}_m r) \cdot J_m(\bar{\alpha}_m r)$$

At this stage α_m is an undetermined parameter. It is required, however, that the α_m's associated with the adjoint functions come from the same series as in the previous section. This will be verified later. One proceeds now, for a fixed index, m, of theta dependence, to form

2.5) $$I_{jk}^{(m)} = (\alpha_{mj}^2 - \alpha_{mk}^2) \int_0^1 \{\vec{\Psi}_{mk}\}^+ [U] \{\vec{\varphi}_{mj}\} dr$$

$$= -\int_0^1 [\bar{\alpha}_{mk}^2 [U]^+ \{\vec{\Psi}_{mk}\}]^+ \{\vec{\varphi}_{mj}\} dr + \int_0^1 \{\vec{\Psi}_{mk}\}^+ \alpha_{mj}^2 [U] \{\varphi_{mj}\} dr$$

Using the differential equations 2.1) and 1.26b), respectively, and integrating by parts, yields

$$2.6)\quad I_{jk}^{(m)} = -\left[\{\vec{\Psi}_{mk}\}^{+}\cdot\frac{\partial}{\partial r}\{\vec{\varphi}_{mj}\} + 1/r\{\vec{\Psi}_{mk}\}^{+}\cdot\{\vec{\varphi}_{mj}\} - \left[\frac{\partial}{\partial r}\{\vec{\Psi}_{mk}\}^{+}\right]\cdot\{\vec{\varphi}_{mj}\}\right]_{r=0}^{r=1}$$

From the boundary conditions 1.29), 1.30), and 1.31) one obtains, for $m \neq 1$, that for a given eigenvalue α_{mj} of m,

$$2.7)\quad \frac{\partial R_{mj}}{\partial r} = -\frac{imT_{mj}}{(1-\nu)} - \frac{im\,\nu\alpha_{mj}^{2}T_{mj}}{(1-m^{2})(1-\nu)} - \frac{\nu\,\alpha_{mj}^{2}R_{mj}}{(1-\nu^{2})(1-m^{2})} + \frac{\nu^{2}\alpha_{mj}^{2}\,\sigma_{zzmj}}{(1-\nu^{2})(1-m^{2})}$$

$$2.8)\quad \frac{\partial\sigma_{zzmj}}{\partial r} = -\frac{im\nu}{1-\nu}T_{mj} - \frac{im\,\alpha_{mj}^{2}}{(1-m^{2})(1-\nu)}T_{mj} - \frac{\alpha_{mj}^{2}\,R_{mj}}{(1-\nu^{2})(1-m^{2})} + \frac{\nu\,\alpha_{mj}^{2}\,\sigma_{zzmj}}{(1-\nu^{2})(1-m^{2})}$$

$$2.9)\quad \frac{\partial T_{mj}}{\partial r} = \frac{-\alpha_{mj}^{2}\,T_{mj}}{2(1-m^{2})} + \frac{im(2-\nu)}{2(1+\nu)}R_{mj} + \frac{im(1-2\nu)}{2(1+\nu)}\sigma_{zzmj} + \frac{im\,\alpha_{mj}^{2}\,R_{mj}}{2(1+\nu)(1-m^{2})} - \frac{im\,\nu\,\alpha_{mj}^{2}}{2(1+\nu)(1-m^{2})}\sigma_{zzmj}$$

Substituting 2.7), 2.8), and 2.9) into 2.6) one finds that

$$-I_{jk}^{(m)} = \Bigg[\Bigg(\frac{-\nu\,\alpha_{mj}^2}{(1-\nu^2)(1-m^2)}\,\bar{\Psi}_{mk}^{(1)} - \frac{\alpha_{mj}^2}{(1-\nu^2)(1-m^2)}\,\bar{\Psi}_{mk}^{(2)}$$

$$+ \frac{im}{2}\,\frac{(2-\nu)}{(1+\nu)}\,\bar{\Psi}_{mk}^{(3)} + \frac{im\,\alpha_{mj}^2}{2(1+\nu)(1-m^2)}\,\bar{\Psi}_{mk}^{(3)} + \bar{\Psi}_{mk}^{(1)} - \frac{\partial\bar{\Psi}_{mk}^{(1)}}{\partial r}\Bigg) R_{mj}$$

$$+ \Bigg(\frac{\nu^2\,\alpha_{mj}^2}{(1-\nu^2)(1-m^2)}\,\bar{\Psi}_{mk}^{(1)} + \frac{\nu\,\alpha_{mj}^2}{(1-\nu^2)(1-m^2)}\,\bar{\Psi}_{mk}^{(2)} - \frac{im\,\nu\,\alpha_{mj}^2}{2(1+\nu)(1-m^2)}\,\bar{\Psi}_{mk}^{(3)}$$

$$+ \frac{im}{2}\,\frac{(1-2\nu)}{(1+\nu)}\,\bar{\Psi}_{mk}^{(3)} + \bar{\Psi}_{mk}^{(2)} - \frac{\partial\bar{\Psi}_{mk}^{(2)}}{\partial r}\Bigg)\sigma_{zzmj}$$

$$+ \Bigg(\frac{-im}{1-\nu}\,\bar{\Psi}_{mk}^{(1)} - \frac{im\,\nu\,\alpha_{mj}^2}{(1-m^2)(1-\nu)}\,\bar{\Psi}_{mk}^{(1)} - \frac{im\nu}{1-\nu}\,\bar{\Psi}_{mk}^{(2)}$$

$$- \frac{im\,\alpha_{mj}^2}{(1-m^2)(1-\nu)}\,\bar{\Psi}_{mk}^{(2)} - \frac{\alpha_{mj}^2}{2(1-m^2)}\,\bar{\Psi}_{mk}^{(3)}$$

$$+ \bar{\Psi}_{mk}^{(3)} - \frac{\partial\bar{\Psi}_{mk}^{(3)}}{\partial r}\Bigg) T_{mj}\Bigg]_{r=0}^{r=1}$$

The adjoint boundary conditions are taken as follows:

2.10) $$\bar{\Psi}_{mk}^{(1)} - \frac{\partial\bar{\Psi}_{mk}^{(1)}}{\partial r} - \frac{im}{2}\,\frac{(2-\nu)}{(1+\nu)}\,\bar{\Psi}_{mk}^{(3)} - \frac{\nu\,\alpha_{mk}^2}{(1-\nu^2)(1-m^2)}\,\bar{\Psi}_{mk}^{(1)}$$

$$\frac{-\alpha_{mk}^2}{(1-m^2)(1-\nu^2)}\,\bar{\Psi}_{mk}^{(2)} + \frac{im\,\alpha_{mk}^2}{2(1+\nu)(1-m^2)}\,\bar{\Psi}_{mk}^{(3)}\Bigg|_{r=1} = 0$$

$$2.11)\quad \bar{\Psi}^{(2)}_{mk} - \frac{\partial\bar{\Psi}^{(2)}_{mk}}{\partial r} + \frac{im}{2}\,\frac{(1-2\nu)}{(1+\nu)}\,\bar{\Psi}^{(3)}_{mk} + \frac{\nu^2\,\alpha^2_{mk}}{(1-\nu^2)(1-m^2)}\,\bar{\Psi}^{(1)}_{mk}$$

$$+ \frac{\nu\,\alpha^2_{mk}}{(1-\nu^2)(1-m^2)}\,\bar{\Psi}^{(2)}_{mk} - \frac{im\,\nu\,\alpha^2_{mk}}{2(1+\nu)(1-m^2)}\,\bar{\Psi}^{(3)}_{mk}\Bigg|_{r=1} = 0$$

$$2.12)\quad \frac{-im}{1-\nu}\,\bar{\Psi}^{(1)}_{mk} - \frac{im\nu}{1-\nu}\,\bar{\Psi}^{(2)}_{mk} + \bar{\Psi}^{(3)}_{mk} - \frac{\partial\bar{\Psi}^{(3)}_{mk}}{\partial r}$$

$$- \frac{im\,\nu\,\alpha^2_{mk}}{(1-m^2)(1-\nu)}\,\bar{\Psi}^{(1)}_{mk} - \frac{im\,\alpha^2_{mk}}{(1-m^2)(1-\nu)}\,\bar{\Psi}^{(2)}_{mk} - \frac{\alpha^2_{mk}}{2(1-m^2)}\,\bar{\Psi}^{(3)}_{mk}\Bigg|_{r=1} = 0$$

As a result of this choice of boundary conditions, equation 2.6) takes the form

$$-I^{(m)}_{jk} = \frac{(\alpha^2_{mj} - \alpha^2_{mk})}{(1 - m^2)}\left[\frac{1}{1+\nu}\left(\frac{-\nu}{1-\nu}\,\bar{\Psi}^{(1)}_{mk} - \frac{1}{1-\nu}\,\bar{\Psi}^{(2)}_{mk} + \frac{im}{2}\,\bar{\Psi}^{(3)}_{mk}\right) R_{mj}\right.$$

$$+ \frac{1}{1+\nu}\left(\frac{\nu^2}{1-\nu}\,\bar{\Psi}^{(1)}_{mk} + \frac{\nu}{1-\nu}\,\bar{\Psi}^{(2)}_{mk} - \frac{im\,\nu}{2}\,\bar{\Psi}^{(3)}_{mk}\right)\sigma_{zzmj}$$

$$+ \left(\frac{-im\,\nu}{1-\nu}\,\bar{\Psi}^{(1)}_{mk} - \frac{im}{1-\nu}\,\bar{\Psi}^{(2)}_{mk} - \frac{1}{2}\,\bar{\Psi}^{(3)}_{mk}\right)\cdot T_{mj}\Bigg|_{r=1}$$

A biorthogonality relation of the following form is therefore obtained

$$2.13)\quad (\alpha^2_{mj} - \alpha^2_{mk})\left\{\int_0^1 \{\vec{\Psi}_{mk}\}^{+}[U]\{\vec{\varphi}_{mj}\}dr\right.$$

$$- \frac{1}{(1-m^2)}\left[\frac{1}{1+\nu}\left(\frac{\nu}{1-\nu}\,\bar{\Psi}^{(1)}_{mk} + \frac{1}{1-\nu}\,\bar{\Psi}^{(2)}_{mk} - \frac{im}{2}\,\bar{\Psi}^{(3)}_{mk}\right) R_{mj}\right.$$

$$+ \frac{1}{1+\nu}\left(\frac{-\nu^2}{1-\nu}\,\bar{\Psi}^{(1)}_{mk} - \frac{\nu}{1-\nu}\,\bar{\Psi}^{(2)}_{mk} + \frac{im\,\nu}{2}\,\bar{\Psi}^{(3)}_{mk}\right)\sigma_{zzmj}$$

$$\left.\left. + \left(\frac{im\,\nu}{1-\nu}\,\bar{\Psi}^{(1)}_{mk} + \frac{im}{1-\nu}\,\bar{\Psi}^{(2)}_{mk} + \frac{1}{2}\,\bar{\Psi}^{(3)}_{mk}\right)T_{mj}\right]_{r=1}\right\} = 0$$

It remains to show that the boundary conditions 2.10), 2.11), and 2.12) determine the same transcendental equation as 1.32). Substituting the form of the adjoint functions into the boundary conditions one obtains three simultaneous linear equations in the unknown coefficients $\vec{A}^*_{mk}$, $\vec{B}^*_{mk}$, and $\vec{C}^*_{mk}$:

2.14) $$\left[\left(-m - \frac{\alpha^2_{mk}}{(1-\nu)(1-m^2)}\right) \alpha_{mk} J_m(\alpha_{mk}) + \alpha^2_{mk} J_{m+1}(\alpha_{mk})\right] \vec{A}^*_{mk}$$

$$+ \left[\left(-\alpha_{mk} + \frac{2}{(1-\nu)(1-m^2)}\right) \alpha^2_{mk} J_m(\alpha_{mk}) + \left(m - \frac{\alpha^2_{mk}}{(1-\nu^2)(1-m^2)}\right) \alpha^2_{mk} J_{m+1}(\alpha_{mk})\right] \vec{B}^*_{mk}$$

$$+ \left[\left(2-\nu + \frac{\alpha^2_{mk}}{1-m^2}\right) \frac{m\,\alpha_{mk}}{2(1+\nu)} J_m(\alpha_{mk})\right] \vec{C}^*_{mk} = 0$$

2.15) $$\left[\left(-m + \frac{\nu\,\alpha_{mk}}{(1-\nu)(1-m^2)}\right) \alpha_{mk} J_m(\alpha_{mk}) + \alpha^2_{mk} J_{m+1}(\alpha_{mk})\right] \vec{A}^*_{mk}$$

$$+ \left[\left(-\alpha^2_{mk} + 2(1+\nu)m - \frac{2\nu\,\alpha^2_{mk}}{(1-\nu)(1-m^2)}\right) \alpha_{mk} J_m(\alpha_{mk})\right.$$

$$\left. + \left(m - 2(1+\nu) + \frac{\nu\,\alpha^2_{mk}}{(1-\nu)(1-m^2)}\right) \alpha^2_{mk} J_{m+1}(\alpha_{mk})\right] \vec{B}^*_{mk}$$

$$+ \left[\left(1-2\nu - \frac{\nu\,\alpha^2_{mk}}{1-m^2}\right) \frac{m\,\alpha_{mk}}{2(1+\nu)} J_m(\alpha_{mk})\right] \vec{C}^*_{mk} = 0$$

2.16) $$\left[\frac{1+\nu}{1-\nu}\left(1 + \frac{\alpha^2_{mk}}{1-m^2}\right) m\,\alpha_{mk} J_m(\alpha_{mk})\right] \vec{A}^*_{mk}$$

$$+ \left[\frac{2(1+\nu)}{1-\nu}\left(-\nu - \frac{\alpha^2_{mk}}{1-m^2}\right) m\,\alpha_{mk} J_m(\alpha_{mk})\right.$$

$$\left. + \frac{1+\nu}{1-\nu}\left(1 + \frac{\alpha^2_{mk}}{1-m^2}\right) m\,\alpha^2_{mk} J_{m+1}(\alpha_{mk})\right] \vec{B}^*_{mk}$$

$$+ \left[\left(-m - \frac{\alpha^2_{mk}}{2(1-m^2)}\right) \alpha_{mk} J_m(\alpha_{mk}) + \alpha^2_{mk} J_{m+1}(\alpha_{mk})\right] \vec{C}^*_{mk} = 0$$

Proceeding as before, to obtain the transcendental equation determining the α_{mk}'s by setting the determinant of the coefficients equal to zero, one obtains again the equation 1.32).

For $m = 1$, as observed above, the derivation given does not hold beyond equation 2.6). This presents no difficulty, but the subsequent equations must be modified, as follows:

$$2.7_1) \qquad \frac{\partial R_{1j}}{\partial r} = \nu \frac{\partial \sigma_{zz1j}}{\partial r} + R_{1j} - \nu \sigma_{zz1j}$$

$$2.8_1) \qquad T_{1j} = \frac{i}{1+\nu} R_{1j} - \frac{i\nu}{1+\nu} \sigma_{zz1j}$$

$$2.9_1) \qquad \frac{\partial T_{1j}}{\partial r} = \frac{i(2-\alpha_{1j}^2)}{2(1+\nu)} R_{1j} + \frac{i(1-2\nu-\nu^2+\nu\alpha_{1j}^2)}{2(1+\nu)} \sigma_{zz1j} - \frac{i(1-\nu)}{2} \frac{\partial \sigma_{zz1j}}{\partial r}$$

Substituting $2.7_1)$, $2.8_1)$, and $2.9_1)$ into 2.6), yields

$$-I_{jk}^{(1)} = \Bigg[\left(\nu \bar{\Psi}_{1k}^{(1)} + \bar{\Psi}_{1k}^{(2)} - \frac{i(1-\nu)}{2} \bar{\Psi}_{1k}^{(3)}\right) \frac{\partial \sigma_{zz1j}}{\partial r}$$
$$+ \left(2\bar{\Psi}_{1k}^{(1)} + \frac{i(4-\alpha_{1j}^2)}{2(1+\nu)} \bar{\Psi}_{1k}^{(3)} - \frac{\partial \bar{\Psi}_{1k}^{(1)}}{\partial r} + \frac{1}{1+\nu} \frac{\partial \bar{\Psi}_{1k}^{(3)}}{\partial r}\right) R_{1j}$$
$$+ \left(-\nu \bar{\Psi}_{1k}^{(1)} + \bar{\Psi}_{1k}^{(2)} + \frac{i(1-4\nu-\nu^2+\nu\alpha_{1j}^2)}{2(1+\nu)} \bar{\Psi}_{1k}^{(3)}\right.$$
$$\left. - \frac{\partial \bar{\Psi}_{1k}^{(2)}}{\partial r} + \frac{i\nu}{1+\nu} \frac{\partial \bar{\Psi}_{1k}^{(3)}}{\partial r}\right) \sigma_{zz1j}\Bigg]_{r=1}$$

Adjoint boundary conditions for the case $m = 1$ are as follows:

$$2.10_1) \qquad \nu\,\bar{\Psi}^{(1)}_{1k} + \bar{\Psi}^{(2)}_{1k} - \frac{i(1-\nu)}{2}\,\bar{\Psi}^{(3)}_{1k}\Bigg|_{r=1} = 0$$

$$2.11_1) \qquad 2\bar{\Psi}^{(1)}_{1k} + \frac{i(4-\alpha^2_{1k})}{2(1+\nu)}\,\bar{\Psi}^{(3)}_{1k} - \frac{\partial\bar{\Psi}^{(1)}_{1k}}{\partial r} - \frac{i}{1+\nu}\,\frac{\partial\bar{\Psi}^{(3)}_{1k}}{\partial r}\Bigg|_{r=1} = 0$$

$$2.12_1) \qquad -\nu\,\bar{\Psi}^{(1)}_{1k} + \bar{\Psi}^{(2)}_{1k} + \frac{i(1-4\nu-\nu^2+\nu\,\alpha^2_{1k})}{2(1+\nu)}\,\bar{\Psi}^{(3)}_{1k}$$

$$- \frac{\partial\bar{\Psi}^{(2)}_{1k}}{\partial r} + \frac{i\nu}{1+\nu}\,\frac{\partial\bar{\Psi}^{(3)}_{1k}}{\partial r} = 0$$

Equation 2.6) for $m = 1$ takes the form

$$-I^{(1)}_{jk} = \frac{i(\alpha^2_{1j} - \alpha^2_{1k})}{2(1+\nu)}\left[-\bar{\Psi}^{(3)}_{1k}\cdot R_{1j} + -\nu\,\bar{\Psi}^{(3)}_{1k}\cdot\sigma_{zz1j}\right]_{r=1}$$

and the biorthogonality relation for $m = 1$ takes the form

$$2.13_1) \qquad (\alpha^2_{1j} - \alpha^2_{1k})\left\{\int_0^1\{\vec{\Psi}_{1k}\}^+[U]\{\vec{\varphi}_{1j}\}dr\right.$$

$$\left. - \frac{i}{2(1+\nu)}\left[-\bar{\Psi}^{(3)}_{1k}\cdot R_{1j} + \nu\,\bar{\Psi}^{(3)}_{1k}\cdot\sigma_{zz1j}\right]_{r=1}\right\} = 0$$

Substituting the form of the adjoint functions from equation 2.4), with $m = 1$, into the boundary expressions 2.10_1), 2.11_1) and 2.12_1), one obtains three linear equations in the unknowns $\vec{A}^*_{1k}$, $\vec{B}^*_{1k}$, and $\vec{C}^*_{1k}$:

$$2.14_1) \qquad (1+\nu)\alpha_{1k}J_1(\alpha_{1k})\vec{A}^*_{1k} + (1+\nu)[-2\alpha_{1k}J_1(\alpha_{1k}) + \alpha^2_{1k}J_2(\alpha_{1k})]\vec{B}^*_{1k}$$

$$- \frac{1-\nu}{2}\,\alpha_{1k}J_1(\alpha_{1k})\vec{C}^*_{1k} = 0$$

$$2.15_1) \quad \alpha_{1k}^2 J_2(\alpha_{1k})\overline{A}_{1k}^* + [-\alpha_{1k}^3 J_1(\alpha_{1k}) + 2\alpha_{1k}^2 J_2(\alpha_{1k})]\overline{B}_{1k}^*$$

$$\frac{-\alpha_{1k}^3}{2(1+\nu)} J_1(\alpha_{1k}) + \frac{\alpha_{1k}^2}{1+\nu} J_2(\alpha_{1k}) \quad \overline{C}_{1k}^* = 0$$

$$2.16_1) \quad [-(1+\nu)\alpha_{1k} J_1(\alpha_{1k}) + \alpha_{1k}^2 J_2(\alpha_{1k})]\overline{A}_{1k}^*$$

$$+ [(-\alpha_{1k}^3 + 2(1+\nu)\alpha_{1k})J_1(\alpha_{1k}) - (1+3\nu)\alpha_{1k}^2 J_2(\alpha_{1k})]\overline{B}_{1k}^*$$

$$+ \left[\left(\frac{(1-\nu)\alpha_{1k}}{2} + \frac{\nu\,\alpha_{1k}^3}{2(1+\nu)}\right) J_1(\alpha_{1k}) - \frac{\nu\,\alpha_{1k}^2}{(1+\nu)} J_2(\alpha_{1k})\right] \overline{C}_{1k}^* = 0$$

Taking the determinant of this expression one obtains the expression resulting from equation 1.32), when that equation is specialized for $m = 1$.

Section 2.3: The Second Mixed Boundary Value Problem

In this problem the normal displacement, w, is specified along with the two shearing tractions, τ_{rz} and $\tau_{\theta z}$. The long side, $r = 1$, is again free of stress and sufficient restrictions on the loading of the end, $z = 0$, are assumed so that the entire solution decays to zero as z tends to infinity. Several possible ways of indirectly representing this displacement present themselves, but the deciding criterion for their usefullness is their ability to extract further information from the Beltrami-Michell equations. The consideration of these equations shall therefore precede the discussion of the relation of the variable to displacements.

Let the variable Q be defined by the equation

$$3.1)\qquad \frac{\partial Q}{\partial z} = \frac{\partial K}{\partial r} + i/r\,\frac{\partial K}{\partial \theta}$$

Subject to the boundary condition $\lim_{z\to\infty} Q = 0$. Then

$$L\,\frac{\partial Q}{\partial z} - i/r\,\frac{\partial^2 Q}{\partial\theta\partial z} = -\frac{\partial^2 K}{\partial z^2}$$

or

$$3.2)\qquad LQ - i/r\,\frac{\partial Q}{\partial \theta} + \frac{\partial K}{\partial z} = 0$$

From equation 1.9), one therefore obtains

$$3.3)\qquad (\nabla^2 - 1/r^2)\tau_{rz} - 2/r^2\,\frac{\partial \tau_{\theta z}}{\partial\theta}$$

$$+\frac{1}{1+\nu}\left[-\partial/\partial r\,LQ + i/r\,\frac{\partial^2 Q}{\partial\theta\partial r} - i/r^2\,\frac{\partial Q}{\partial\theta}\right] = 0$$

and from 1.10)

$$3.4)\qquad (\nabla^2 - 1/r^2)\tau_{\theta z} + \frac{2}{r^2}\,\frac{\partial \tau_{rz}}{\partial\theta}$$

$$+\frac{1}{1+\nu}\left[-1/r\,\frac{\partial^2 Q}{\partial\theta\partial r} - 1/r^2\,\frac{\partial Q}{\partial\theta} + i/r^2\,\frac{\partial^2 Q}{\partial\theta^2}\right] = 0$$

Operating on equation 1.17) with the operator

$$\partial/\partial r + i/r\,\partial/\partial\theta$$

provides, using equation 3.1),

$$\partial/\partial r\,L\left(\frac{\partial Q}{\partial z}\right) + 1/r^2\,\partial^2/\partial\theta^2\left(\frac{\partial Q}{\partial z}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial Q}{\partial z}\right)$$

$$+\frac{2}{r}\,L\left(\frac{\partial Q}{\partial z}\right) + \frac{2}{r}\,\frac{\partial^2 K}{\partial z^2} = 0$$

Integrating this equation with respect to z, and using equation 3.2) gives

3.5) $$(\nabla^2 + \frac{2i}{r}\frac{\partial}{\partial\theta} - 1/r^2)\,Q = 0$$

Now the equations 3.3), 3.4), and 3.5) may be written in vector form, as follows:

3.6) $$\frac{\partial}{\partial r} L\,\{\vec{\eta}\} + [V_1]\,1/r^2\,\frac{\partial^2}{\partial\theta^2}\{\vec{\eta}\} + [V_2]\,\frac{\partial^2}{\partial z^2}\{\vec{\eta}\} + [V_3]\,\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}\{\vec{\eta}\} + [V_4]\,\frac{1}{r^2}\frac{\partial}{\partial\theta}\{\vec{\eta}\} = 0$$

where

3.7a) $$\{\vec{\eta}\} = \begin{Bmatrix} \tau_{rz} \\ \tau_{\theta z} \\ Q \end{Bmatrix}$$, 3.7b) $$[V_1] = \begin{bmatrix} 1 & 0 & 1/1+\nu \\ 0 & 1 & i/1+\nu \\ 0 & 0 & 1 \end{bmatrix}$$

3.7c) $$[V_2] = \begin{bmatrix} 1 & 0 & 1/1+\nu \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$, 3.7d) $$[V_3] = \begin{bmatrix} 0 & 0 & i/1+\nu \\ 0 & 0 & -1/1+\nu \\ 0 & 0 & 0 \end{bmatrix}$$

3.7e) $$[V_4] = \begin{bmatrix} 0 & -2 & i/1+\nu \\ 2 & 0 & -i/1+\nu \\ 0 & 0 & 2i \end{bmatrix}$$

Assuming a solution of the form

3.8) $$\{\vec{\eta}\} = \{\vec{\eta}_m(r)\}e^{im\theta}e^{-\alpha_m z}$$

where $\{\vec{\eta}_m\}$ satisfies

$$3.9)\quad d/dr\, L\,\{\vec{\eta}_m\} - [V_1]\frac{m^2}{r^2}\{\vec{\eta}_m\} + \alpha_m^2[V_2]\{\vec{\eta}_m\} + \frac{im}{r}[V_3]\frac{\partial}{\partial r}\{\vec{\eta}_m\} + \frac{im}{r^2}[V_4]\{\vec{\eta}_m\} = 0$$

Solving the system 3.9), gives

$$3.10a)\quad \tau_{rzm} = (D_m \cdot \alpha_m r - \frac{mE_m - 2mF_m}{\alpha_m r})J_m(\alpha_m r) + (E_m + F_m)J_{m+1}(\alpha_m r)$$

$$3.10b)\quad \tau_{\theta zm} = i\{(-m\,E_m + 2m\,F_m)\frac{J_m(\alpha_m r)}{\alpha_m r} + F_m J_{m+1}(\alpha_m r)\}$$

$$3.10c)\quad Q_m = 2(1+\nu)\, D_m J_{m+1}(\alpha_m r)$$

It may be noted that the solving of systems 3.9) is facilitated if one first solves for the variable Q_m, then for the combination $\tilde{\tau}_m = \tau_{rzm} + i\,\tau_{\theta zm}$ and finally, for the variables τ_{rzm} and $\tau_{\theta zm}$.

The Boundary Conditions

One of the boundary conditions can be stated directly:

$$3.11)\quad \tau_{rzm} = 0 \quad \text{on} \quad r = 1$$

The other boundary conditions are derived from the boundary conditions 1.30) and 1.31) of the previous problem by differentiating through with respect to z, using the differential equations 3.2), and 1.1) to eliminate the variable R and σ_{zz}, and equations 3.3), 3.4), and 3.5) to eliminate the resulting second order differentiations with respect to r. After substituting

3.8) in the resulting expression, one obtains

$$3.12)\quad \frac{1}{1+\nu}\,(-\alpha_m^2 - m\nu(m-1)-m)\frac{\partial Q_m}{\partial r} + \frac{1}{1+\nu}\,(-\alpha_m^2\nu - m\alpha_m^2 - \nu m(m^2-1)$$

$$- m(m+1))Q_m + (\alpha_m^2 + m^2)\frac{\partial \tau_{rzm}}{\partial r} - im\,\frac{\partial \tau_{\theta zm}}{\partial r}$$

$$- i(\alpha_m^2 m - m(m^2-1))\tau_{\theta zm}\Big|_{r=1} = 0$$

$$3.13)\quad \frac{i}{1+\nu}\left[(m^2 - \nu m(m-1))\frac{\partial Q_m}{\partial r} + (-(1-\nu)m\,\alpha_m^2 + (1-\nu)m^3 + m^2 + m\nu)Q_m\right]$$

$$- im\,\frac{\partial \tau_{rzm}}{\partial r} + (\alpha_m^2 - m^2)\,\frac{\partial \tau_{\theta zm}}{\partial r} - \alpha_m^2\,\tau_{\theta zm}\Big|_{r=1} = 0$$

Substituting equations 3.10) into these three expressions provides three linear equations in the variables D_m, E_m, and F_m:

$$3.14)\quad \alpha_m^2 J_m(\alpha_m)D_m + (-mJ_m(\alpha_m) + \alpha_m J_{m+1}(\alpha_m))E_m$$

$$+ (-2m\,J_m(\alpha_m) + \alpha_m\,J_{m+1}(\alpha_m))F_m = 0$$

$$3.15)\quad [(\alpha_m^2(m-1) - 2\nu m(m-1) - 2m + m^2(m+1))\alpha_m J_m(\alpha_m)$$

$$+ (-\alpha_m^2 + 2(1-\nu) - m^2)\alpha_m^2 J_{m+1}(\alpha_m)]D_m$$

$$+ [(\alpha_m^2 - m(m-1))\alpha_m J_m(\alpha_m) - \alpha_m^2 J_{m+1}(\alpha_m)]E_m$$

$$+ [(\alpha_m^2 - 3m(m-1))\alpha_m J_m(\alpha_m) + (2m-1)\alpha_m^2 J_{m+1}(\alpha_m)]\,F_m = 0$$

$$3.16)\quad [(1-2\nu)\alpha_m m(m-1)J_m(\alpha_m) - (1-2\nu)\alpha_m^2 m\, J_{m+1}(\alpha_m)]D_m$$

$$+ \left[\left(-\alpha_m m(m-1) + \frac{m^2(m^2-1)}{\alpha_m}\right) J_m(\alpha_m) + (\alpha_m^2 m - m^2(m^2-1))J_{m+1}(\alpha_m)\right] E_m$$

$$+ \left[\left(\alpha_m^3 - 3\alpha_m m(m-1) + \frac{2m^2(m^2-1)}{\alpha_m}\right) J_m(\alpha_m)\right.$$

$$\left. + (-2\alpha_m^2 + \alpha_m^2 m - m(m^2-1))J_{m+1}(\alpha_m)\right] F_m = 0$$

As before, the vanishing of the determinant of this system gives the values of α_m for the system to have a non-trivial solution for D_m, E_m, and F_m, and again, the resulting equation for α_m is that of equation 1.32).

<u>The representation of the displacement</u>

From equations 1.14) and 1.15), one may note

$$L \frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{2(1+\nu)}{E}\left(L\,\tau_{rz} + 1/r\,\frac{\partial \tau_{\theta z}}{\partial \theta}\right) - \frac{\partial}{\partial z}\left(Lu + \frac{1}{r}\frac{\partial v}{\partial \theta}\right)$$

and by 1.19), one has

$$L \frac{\partial w}{\partial r} + 1/r^2\,\frac{\partial^2 w}{\partial \theta^2} = \frac{2(1+\nu)}{E}\left(L\,\tau_{rz} + 1/r\,\frac{\partial \tau_{\theta z}}{\partial \theta}\right) - \frac{1}{E}\left((1-\nu)\frac{\partial R}{\partial z} - 2\nu\,\frac{\partial \sigma_{zz}}{\partial z}\right)$$

so

$$E\left(L \frac{\partial w}{\partial r} + 1/r^2\,\frac{\partial^2 w}{\partial \theta^2}\right) = (1+\nu)\left(L\,\tau_{rz} + 1/r\,\frac{\partial \tau_{\theta z}}{\partial \theta}\right) + (1-\nu)\left(LQ - i/r\,\frac{\partial Q}{\partial \theta}\right)$$

Section 2.4: The Adjoint Problem II

The adjoint equation of equation 3.9) for a given index, m, of θ dependence is the following:

4.1) $$L^* \frac{\partial}{\partial r}\{\vec{\zeta}_m\} - \frac{m^2}{r^2}[V_1]^+\{\vec{\zeta}_m\} + \bar{\alpha}_m^2[V_2]^+\{\vec{\zeta}_m\}$$

$$+ \frac{im}{r}\frac{\partial}{\partial r}[V_3]^+\{\vec{\zeta}_m\} - \frac{im}{r^2}[V_4]^+\{\vec{\zeta}_m\} = 0$$

Again, the computation is facilitated if one first solves for

$$\tilde{\zeta}_m = \zeta_m^{(1)} + i\,\zeta_m^{(2)}$$

then for $\zeta_m^{(1)}$, $\zeta_m^{(2)}$, and, finally $\zeta_m^{(3)}$.

Solving the system 4.1), one finds

4.2a) $$\bar{\zeta}_m^{(1)} = [2(1+\nu)\alpha_m r\, J_{m+1}(\alpha_m r)]\vec{D}_m^*$$

$$+ [-4(1+\nu)m\, J_m(\alpha_m r) + 2(1+\nu)\alpha_m r\, J_{m+1}(\alpha_m r)]\vec{E}_m^*$$

4.2b) $$\bar{\zeta}_m^{(2)} = i\{[-2(1+\nu)\alpha_m r\, J_{m+1}(\alpha_m r)]\vec{D}_m^*$$

$$+ [-4(1+\nu)m\, J_m(\alpha_m r) + 2(1+\nu)\alpha_m r\, J_{m+1}(\alpha_m r)]\vec{E}_m^*\}$$

4.2c) $$\bar{\zeta}_m^{(3)} = [(\alpha_m r)^2 J_m(\alpha_m r)]\vec{D}_m^* + [((\alpha_m r)^2 + rm)J_m(\alpha_m r)]\vec{E}_m^*$$

$$+ [(\alpha_m r)J_{m+1}(\alpha_m r)]\vec{F}_m^*$$

Letting

4.3) $$I_{jk}^{(m)} = (\alpha_{mj}^2 - \alpha_{mk}^2)\int_0^1 \{\vec{\zeta}_{mk}\}[V_2]\{\vec{\eta}_{mj}\}dr$$

and using the differential equations 4.1) and 3.9) and integrating by parts, one finds

4.4) $$I_{jk}^{(m)} = \left[\{\vec{\zeta}_{mk}\}^{+} L\{\vec{\eta}_{mj}\} + \frac{im}{r} \{\vec{\zeta}_{mk}\}^{+}[V_3]\{\vec{\eta}_{mj}\} - \left[\frac{\partial}{\partial r} \{\vec{\zeta}_{mk}\}\right]^{+} \cdot \{\vec{\eta}_{mj}\} \right]_{r=1}^{r=0}$$

where the boundary conditions on the $\vec{\eta}$'s are given by equations 3.11), 3.12), and 3.13).

For $m \neq 0$, 1, one has from the boundary conditions of equation 3.9)

4.5) $$\tau_{rzmj} = 0$$

4.6) $$[\alpha_{mj}^4 + m^2(m^2-1)]\frac{\partial \tau_{rzmj}}{\partial r} = \frac{1}{1+\nu}[(\alpha_{mj}^4 + m^2(m^2-1)) - (1-\nu)m(m^2-1) \times$$
$$\times (\alpha_{mj}^2 - m(m-1))]\frac{\partial Q_{mj}}{\partial r} + \frac{1}{1+\nu}[(m+1)(m-\nu(m-1))(\alpha_{mj}^4 + m^2(m^2-1)$$
$$- \alpha_{mj}^2(1-\nu)m(m^2-1)(2m+1)]Q_{mj}$$
$$+ i[m(\alpha_{mj}^4 + m^2(m^2-1)) - 2\alpha_{mj}^2 m(m^2-1)]\frac{\partial \tau_{\theta zmj}}{\partial r} = 0$$

4.7) $$[\alpha_{mj}^4 + m^2(m^2-1)]\tau_{\theta zmj} = \frac{i}{1+\nu}[(1-\nu)m(m-1)(\alpha_{mj}^2 + m(m+1))]\frac{\partial Q_{mj}}{\partial r}$$
$$+ \frac{i}{1+\nu}[-(1-\nu)m(\alpha_{mj}^4 - m^2(m^2-1)) + (1-\nu)m^2(m^2-1)]Q_{mj}$$
$$+ [\alpha_{mj}^4 - m^2(m^2-1)]\frac{\partial \tau_{\theta zmj}}{\partial r}$$

Because of the form of the boundary conditions, the treatment of the biorthogonality is not as straightforward as it has been found to be in the previous cases and the algebra associated with these results becomes quite formidable. It is convenient, however, to consider the following quantity:

$$
\begin{aligned}
4.8)\quad & [\alpha_{mj}^4 + m^2(m^2-1)]I_{jk}^{(m)} \\
&= \Bigg[\frac{1}{1+\nu}((\alpha_{mj}^4+m^2(m^2-1)) - \alpha_{mj}^2(1-\nu)m(m^2-1)+(1-\nu)m^2(m-1)(m^2-1))\bar{\zeta}_{mk}^{(1)} \\
&\quad + \frac{i}{1+\nu}(\alpha_{mj}^2(1-\nu)m(m-1) + (1-\nu)m^2(m^2-1))\left(\bar{\zeta}_{mk}^{(2)} - \frac{\partial\bar{\zeta}_{mk}^{(2)}}{\partial r}\right) \\
&\quad + (\alpha_{mj}^4 + m^2(m^2-1))\bar{\zeta}_{mk}^{(3)}\Bigg]\frac{\partial Q_{mj}}{\partial r} \\
&+ \Bigg[\frac{1}{1+\nu}((m+1)(m-\nu(m-1))(\alpha_{mj}^2+m^2(m^2-1))-\alpha_{mj}^2(1-\nu)m(m^2-1)(2m+1))\bar{\zeta}_{mk}^{(1)} \\
&\quad + \frac{i}{1+\nu}(-(1-\nu)m(\alpha_{mj}^4-m^2(m^2-1)) + (1-\nu)m^2(m^2-1))\left(\bar{\zeta}_{mk}^{(2)} - \frac{\partial\bar{\zeta}_{mk}^{(2)}}{\partial r}\right) \\
&\quad + (\alpha_{mj}^4+m^2(m^2-1))\left(\frac{-m}{1+\nu}\bar{\zeta}_{mk}^{(1)} - \frac{im}{1+\nu}\bar{\zeta}_{mk}^{(2)} + \bar{\zeta}_{mk}^{(3)} - \frac{\partial\bar{\zeta}_{mk}^{(3)}}{\partial r}\right)\Bigg]Q_{mj} \\
&+ i\Bigg[(m(\alpha_{mj}^4 + m^2(m^2-1)) - 2\alpha_{mj}^2 m(m-1))\bar{\zeta}_{mk}^{(1)} \\
&\quad - i(\alpha_{mj}^4 + m^2(m^2-1))\bar{\zeta}_{mk}^{(2)} \\
&\quad - i((\alpha_{mj}^4+m^2(m^2-1))-2m^2(m^2-1))\left(\bar{\zeta}_{mk}^{(2)} - \frac{\partial\bar{\zeta}_{mk}^{(2)}}{\partial r}\right)\Bigg]\frac{\partial\tau_{\theta zmj}}{\partial r}\Bigg]_{r=0}^{r=1}
\end{aligned}
$$

Taking as boundary conditions

$$4.9)\quad \frac{1}{1+\nu}[(\alpha^4_{mk}+m^2(m^2-1))-\alpha^2_{mk}(1-\nu)m(m^2-1)+(1-\nu)m^2(m-1)(m^2-1)]\bar{\zeta}^{(1)}_{mk}$$

$$+\frac{i}{1+\nu}[\alpha^2_{mk}(1-\nu)m(m-1)+(1-\nu)m^2(m^2-1)]\left(\bar{\zeta}^{(2)}_{mk}-\frac{\partial\bar{\zeta}^{(2)}_{mk}}{\partial r}\right)$$

$$+[\alpha^4_{mk}+m^2(m^2-1)]\bar{\zeta}^{(3)}_{mk}=0$$

$$4.10)\quad \frac{1}{1+\nu}[(m+1)(m-\nu(m-1))(\alpha^4_{mk}+m^2(m^2-1))$$

$$-\alpha^2_{mk}(1-\nu)m(m^2-1)(2m-1)]\bar{\zeta}^{(1)}_{mk}$$

$$+\frac{i}{1+\nu}[-(1-\nu)m(\alpha^4_{mk}+m^2(m^2-1))+(1-\nu)m^2(2m+1)(m^2-1)]\left(\bar{\zeta}^{(2)}_{mk}-\frac{\partial\bar{\zeta}^{(2)}_{mk}}{\partial r}\right)$$

$$+[\alpha^4_{mk}+m^2(m^2-1)]\left[\frac{-m}{1+\nu}\bar{\zeta}^{(1)}_{mk}-\frac{im}{1+\nu}\bar{\zeta}^{(2)}_{mk}+\bar{\zeta}^{(3)}_{mk}-\frac{\partial\bar{\zeta}^{(3)}_{mk}}{\partial r}\right]=0$$

$$4.11)\quad [m(\alpha^4_{mk}+m^2(m^2-1))-2\alpha^2_{mk}m(m^2-1)]\bar{\zeta}^{(1)}_{mk}$$

$$-i[\alpha^4_{mk}+m^2(m^2-1)]\bar{\zeta}^{(2)}_{mk}$$

$$-i[(\alpha^4_{mk}+m^2(m^2-1))-2m^2(m^2-1)]\left[\bar{\zeta}^{(2)}_{mk}-\frac{\partial\bar{\zeta}^{(2)}_{mk}}{\partial r}\right]=0$$

These boundary conditions give

$$4.12) \quad [\alpha_{mj}^4 + m^2(m^2-1)]I_{jk}^{(m)} =$$

$$= (\alpha_{mj}^2 - \alpha_{mk}^2)\Bigg\{\Bigg[\frac{1}{1+\nu}(\alpha_{mj}^2 + \alpha_{mk}^2) - (1-\nu)m(m^2-1))\bar{\zeta}_{mk}^{(1)}$$

$$+ \frac{i}{1+\nu}(1-\nu)m(m-1)\left(\bar{\zeta}_{mk}^{(2)} - \frac{\partial\bar{\zeta}_{mk}^{(2)}}{\partial r}\right) + (\alpha_{mj}^2 + \alpha_{mk}^2)\bar{\zeta}_{mk}^{(3)}\Bigg] \frac{\partial Q_{mj}}{\partial r}$$

$$+ \Bigg[\frac{1}{1+\nu}((\alpha_{mj}^2+\alpha_{mk}^2)(m+1)(m-\nu(m-1)) - (1-\nu)m(m^2-1)(2m+1))\bar{\zeta}_{mk}^{(1)}$$

$$+ \frac{i}{1+\nu}\left(-(1-\nu)m(\alpha_{mj}^2 + \alpha_{mk}^2)\left(\bar{\zeta}_{mk}^{(2)} - \frac{\partial\bar{\zeta}_{mk}^{(2)}}{\partial r}\right)\right)$$

$$(\alpha_{mj}^2 + \alpha_{mk}^2)\left(\bar{\zeta}_{mk}^{(3)} - \frac{m}{1+\nu}\bar{\zeta}_{mk}^{(1)} - \frac{im}{1+\nu}\bar{\zeta}_{mk}^{(2)} - \frac{\partial\bar{\zeta}_{mk}^{(3)}}{\partial r}\right)\Bigg] Q_{mj}$$

$$+ \Bigg[i(m(\alpha_{mj}^2 + \alpha_{mk}^2) - 2m(m^2-1))\bar{\zeta}_{mk}^{(1)} + (\alpha_{mj}^2 + \alpha_{mk}^2)\bar{\zeta}_{mk}^{(2)}$$

$$+ (\alpha_{mj}^2 + \alpha_{mk}^2)\left(\bar{\zeta}_{mk}^{(2)} - \frac{\partial\bar{\zeta}_{mk}^{(2)}}{\partial r}\right)\Bigg] \frac{\partial\tau_{\theta z}}{\partial r}\Bigg|_{r=1}\Bigg\}$$

and one obtains a biorthogonality relation of the form

$$4.13) \quad (\alpha_{mj}^2 - \alpha_{mk}^2)\int_0^1[\alpha_{mj}^4 + m^2(m^2-1)]\{\zeta_{mk}\}^+[V_2]\{\vec{\eta}_{mj}\}dr$$

$$- [\alpha_{mj}^4 + m^2(m^2-1)]I_{jk}^{(m)} = 0$$

It may appear that equation 4.13) presents a biorthogonality relation that is completely unusable. A similar problem, however, will be seen to arise in the treatment in the next section, and there a means of rendering such an expression useful for the solution of specific boundary value problems will be developed. As the numerical work of this present study will be confined to that development, these methods will not be applied here to equation 4.13).

Substituting the form of the adjoint functions from equations 4.2) into equations 4.9), 4.10) and 4.11) one obtains three linear equations in the unknowns $\overline{D}^*_{mk}$, $\overline{E}^*_{mk}$, and $\overline{F}^*_{mk}$:

4.14) $$[(\alpha^6_{mk} + 2(1-\nu)\alpha^4_{mk}m(m-1) + \alpha^2_{mk}m^2(m^2-1)(3-2\nu))J_m(\alpha_{mk})$$

$$+ (2\alpha_{mk}(\alpha^4_{mk} + m^2(m^2-1)) - 4(1-\nu)\alpha^3_{mk}m(m^2-1)$$

$$- 4(1-\nu)\alpha_{mk}m^2(m^2-1))J_{m+1}(\alpha_{mk})]\overline{D}^*_{mk}$$

$$+ [(\alpha^2_{mk}(\alpha^4_{mk} + m^2(m^2-1)) - 2(1-\nu)\alpha^4_{mk}m(m-1) + 2(1-\nu)\alpha^2_{mk}m^2(m^2-1)$$

$$+ 4(1-\nu)\alpha^2_{mk}m^2(m-1)^2)J_m(\alpha_{mk})$$

$$+ (2\alpha_{mk}(\alpha^4_{mk} + m^2(m^2-1)) - 2(1-\nu)\alpha^3_{mk}m(m^2-1) - 2(1-\nu)\alpha^3_{mk}m(m-1)^2)$$

$$J_{m+1}(\alpha_{mk})]\ \overline{E}^*_{mk}$$

$$+ [\alpha_{mk}(\alpha^4_{mk} + m^2(m^2-1))J_{m+1}(\alpha_{mk})]\ \overline{E}^*_{mk} = 0$$

$$4.15)\quad [(-2(1-\nu)\alpha_{mk}^{6}m - \alpha_{mk}^{2}(m+1)(\alpha_{mk}^{4}+m^{2}(m^{2}-1))+2(1-\nu)\alpha_{mk}^{2}m^{2}(m+1)(m^{2}-1))J_{m}(\alpha_{mk})$$

$$+ (\alpha_{mk}^{3}(\alpha_{mk}^{4}+m^{2}(m^{2}-1))+4(1-\nu)\alpha_{mk}m(m+1)(\alpha_{mk}^{4}+m^{2}(m^{2}-1))+2\nu\alpha_{mk}(m+1)(\alpha_{mk}^{4}+m^{2}(m^{2}-1))$$

$$- 2(1-\nu)\alpha_{mk}^{3}m(m^{2}-1)(2m+1)-2(1-\nu)\gamma_{mk}m^{2}(m+1)(m^{2}-1)(2m+1))J_{m+1}(\alpha_{mk})]\overline{D}_{mk}^{*}$$

$$+ [(-4(1-\nu)\alpha_{mk}^{4}m(m-1)(2m+1)+2(1-\nu)\alpha_{mk}^{2}m(\alpha_{mk}^{4}+m^{2}(m^{2}-1))$$

$$- \alpha_{mk}^{2}(m+1)(\alpha_{mk}^{4}+m^{2}(m^{2}-1))+2(1-\nu)\alpha_{mk}^{2}m^{2}(2m+1)(m^{2}-1))J_{m}(\alpha_{mk})$$

$$+ (\alpha_{mk}^{3}(\alpha_{mk}^{4}+m^{2}(m^{2}-1))+2\alpha_{mk}(m+1)(m-\nu(m-1))(\alpha_{mk}^{4}+m^{2}(m^{2}-1))$$

$$+ (\alpha_{mk}^{3}(\alpha_{mk}^{4}+m^{2}(m^{2}-1))+2\alpha_{mk}(m+1)(m-\nu(m-1))(\alpha_{mk}^{4}+m^{2}(m^{2}-1))$$

$$+ 2(1-\nu)\alpha_{mk}^{5}m(m-1)-2(1-\nu)\alpha_{mk}^{3}m(m^{2}-1)(2m+1)-2(1-\nu)\alpha_{mk}m^{2}(m^{2}-1)^{2})J_{m+1}(\alpha_{mk})]\overline{E}_{mk}^{*}$$

$$+ [-\alpha_{mk}^{2}(\alpha_{mk}^{4}+m^{2}(m^{2}-1))J_{m}(\alpha_{mk})+\alpha_{mk}(m+1)(\alpha_{mk}^{4}+m(m-1))J_{m+1}(\alpha_{mk})]\overline{F}_{mk}^{*} = 0$$

and

$$4.16)\quad [-\alpha_{mk}^{2}(\alpha_{mk}^{4} - m^{2}(m^{2}-1))J_{m}(\alpha_{mk})$$

$$+ (2\alpha_{mk}^{5}(m+1) - 2m\gamma_{mk}^{3}(m^{2}-1))J_{m+1}(\alpha_{mk})]\overline{D}_{mk}^{*}$$

$$+ [(\alpha_{mk}^{6} - 4\alpha_{mk}^{4}m(m-1) + 3\gamma_{mk}^{2}m^{2}(m^{2}-1))J_{m}(\alpha_{mk})$$

$$+ (2\alpha_{mk}^{5}(m-1) - 2\alpha_{mk}^{3}m(m^{2}-1))J_{m+1}(\alpha_{mk})]\overline{E}_{mk}^{*} = 0$$

Taking the determinant of these equations gives an expression which is equivalent, again, to equation 1.32).

For $m = 0$, one has that the boundary conditions 3.11), 3.12), and 3.13) reduce to

4.5$_o$) $$\tau_{rzoj} = 0$$

4.6$_o$) $$\frac{\partial \tau_{rzoj}}{\partial r} = \frac{1}{1+\nu} \frac{\partial Q_{oj}}{\partial r} + \frac{\nu}{1+\nu} Q_{oj}$$

4.7$_o$) $$\frac{\partial \tau_{\theta zoj}}{\partial r} = \tau_{\theta zoj}$$

and equation 4.4) becomes

4.8$_o$) $$I_{jk}^{(o)} = \left[\left(\frac{1}{1+\nu} \bar{\zeta}_{ok}^{(1)} + \bar{\zeta}_{ok}^{(3)}\right) \frac{\partial Q_{oj}}{\partial r} + \left(2 \bar{\zeta}_{ok}^{(2)} - \frac{\partial \bar{\zeta}_{ok}^{(2)}}{\partial r}\right) \tau_{\theta zoj} + \left(\frac{\nu}{1+\nu} \bar{\zeta}_{ok}^{(1)} + \bar{\zeta}_{ok}^{(3)} - \frac{\partial \bar{\zeta}_{ok}^{(3)}}{\partial r}\right) Q_{oj}\right]_{r=1}$$

Thus the adjoint boundary conditions become

4.9$_o$) $$\frac{1}{1+\nu} \bar{\zeta}_{ok}^{(1)} + \bar{\zeta}_{ok}^{(3)} \Big|_{r=1} = 0$$

4.10$_o$) $$2 \bar{\zeta}_{ok}^{(2)} - \frac{\partial \bar{\zeta}_{ok}^{(2)}}{\partial r} \Big|_{r=1} = 0$$

4.11$_o$) $$\frac{\nu}{1+\nu} \bar{\zeta}_{ok}^{(1)} + \bar{\zeta}_{ok}^{(3)} - \frac{\partial \bar{\zeta}_{ok}^{(3)}}{\partial r} = 0$$

which gives

4.12$_o$) $$I_{jk}^{(o)} = 0$$

hence one has the biorthogonality relation

4.13$_o$) $$(\alpha_{oj}^2 - \alpha_{ok}^2)\int_0^1 \{\vec{\zeta}_{ok}\}^+[V_2]\{\vec{\eta}_{ok}\}dr = 0$$

Substituting the forms of $\bar{\zeta}_{ok}^{(1)}$, $\bar{\zeta}_{ok}^{(2)}$, and $\bar{\zeta}_{ok}^{(3)}$ from equation 4.2) into the boundary conditions 4.9$_o$), 4.10$_o$) and 4.11$_o$) one obtains three linear equations in the unknowns D_{ok}^*, E_{ok}^*, and F_{ok}^*.

4.14$_o$) $$[\alpha_{ok}^2 J_o(\alpha_{ok}) + 2\alpha_{ok} J_1(\alpha_{ok})]\vec{D}_{ok}^* + [\alpha_{ok}^2 J_o(\alpha_{ok}) + 2\alpha_o J_1(\alpha_{ok})]\vec{E}_{ok}^*$$
$$+ \alpha_{ok} J_1(\alpha_{ok})\vec{F}_{ok}^* = 0$$

4.15$_o$) $$[\alpha_{ok}^2 J_o(\alpha_{ok}) - 2\alpha_o J_1(\alpha_{ok})]\vec{D}_{ok}^*$$
$$+ [-\alpha_{ok}^2 J_o(\alpha_{ok}) + 2\alpha_{ok} J_1(\alpha_{ok})]\vec{E}_{ok}^* = 0$$

4.16$_o$) $$[-\alpha_{ok}^2 J_o(\alpha_{ok}) + (\alpha_{ok}^3 + 2\nu\, \alpha_{ok})J_1(\alpha_{ok})]\vec{D}_{ok}^*$$
$$+ [-\alpha_{ok}^2 J_o(\alpha_{ok}) + (\alpha_{ok}^3 + 2\nu\, \alpha_{ok})J_1(\alpha_{ok})]\vec{E}_{ok}^*$$
$$+ [-\alpha_{ok}^2 J_o(\alpha_{ok}) + \alpha_o J_1(\alpha_{ok})]\vec{F}_{ok}^* = 0$$

and the determinant of these three equations gives the same equation as 1.32), when we substitute in that $m = 0$.

Setting $m = 1$, one has

4.5$_1$) $$\tau_{rz1j} = 0$$

4.6$_1$) $$\frac{\partial\tau_{rz1j}}{\partial r} = \frac{1}{1+\nu}\frac{\partial Q_{1j}}{\partial r} + \frac{2}{1+\nu}Q_{1j} + i\,\frac{\partial\tau_{\theta z1j}}{\partial r}$$

4.7$_1$) $$\tau_{\theta z1j} = -i\,\frac{1-\nu}{1+\nu}Q_{1j} + \frac{\partial\tau_{\theta z1j}}{\partial r}$$

Then

4.8$_1$) $$I^{(1)}_{jk} = \Bigg[\left(\frac{1}{1+\nu}\bar{\zeta}^{(1)}_{1k} + \bar{\zeta}^{(3)}_{1k}\right)\frac{\partial Q_{1j}}{\partial r} + \left(i\,\bar{\zeta}^{(1)}_{1k} + \bar{\zeta}^{(2)}_{1k} - \frac{\partial\bar{\zeta}^{(3)}_{1k}}{\partial r}\right)\tau_{\theta z1j}$$
$$+ \left(\frac{1}{1+\nu}\bar{\zeta}^{(1)}_{1k} - i\,\frac{2-\nu}{1+\nu}\bar{\zeta}^{(2)}_{1k} + i\,\frac{1-\nu}{1+\nu}\frac{\partial\bar{\zeta}^{(2)}_{1k}}{\partial r} + \bar{\zeta}^{(3)}_{1k} - \frac{\partial\bar{\zeta}^{(3)}_{1k}}{\partial r}\right)Q_{1j}\Bigg]^{r=1}_{r=o}$$

and the boundary conditions become

4.9$_1$) $$\frac{1}{1+\nu}\bar{\zeta}^{(1)}_{1k} + \bar{\zeta}^{(3)}_{1k} = 0$$

4.10$_1$) $$\bar{\zeta}^{(1)}_{1k} - i\,\bar{\zeta}^{(2)}_{1k} + i\,\frac{\partial\bar{\zeta}^{(2)}_{1k}}{\partial r} = 0$$

4.11$_1$) $$\frac{1}{1+\nu}\bar{\zeta}^{(1)}_{1k} - i\,\frac{2-\nu}{1+\nu}\bar{\zeta}^{(2)}_{1k} + i\,\frac{1-\nu}{1+\nu}\frac{\partial\bar{\zeta}^{(2)}_{1k}}{\partial r} + \bar{\zeta}^{(3)}_{1k} - \frac{\partial\bar{\zeta}^{(3)}_{1k}}{\partial r} = 0$$

These boundary conditions give

4.12$_1$) $$I^{(1)}_{jk} = 0$$

and hence one has the biorthogonality relation

$$4.13_1) \qquad (\alpha_{1j}^2 - \alpha_{1k}^2) \int_0^1 \{\vec{\zeta}_{1k}\}^+ [V_2]\{\vec{\eta}_{1j}\} dr = 0$$

Substituting the forms of $\bar{\zeta}_{1k}^{(1)}$, $\bar{\zeta}_{1k}^{(2)}$, and $\bar{\zeta}_{1k}^{(3)}$ from equation 4.2) into the boundary conditions $4.9_1)$, $4.10_1)$ and $4.11_1)$, one obtains three linear equations in the unknowns $\vec{D}_{1k}^*$, $\vec{E}_{1k}^*$, and $\vec{F}_{1k}^*$.

$$4.14_1) \quad [\alpha_{1k}^2 J_1(\alpha_{1k}) + 2\alpha_1 J_2(\alpha_{1k})]\vec{D}_{1k}^* + [\alpha_{1k}^2 J_1(\alpha_{1k}) + 2\alpha_{1k} J_2(\alpha_{1k})]\vec{E}_{1k}^*$$
$$+ [\alpha_{1k} J_2(\alpha_{1k})]\vec{F}_{1k}^* = 0$$

$$4.15_1) \quad [-2\alpha_{1k}^2 J_1(\alpha_{1k}) + (\alpha_{1k}^3 + 4\nu\, \alpha_{1k}) J_2(\alpha_{1k})]\vec{D}_{1k}^*$$
$$+ [-2\alpha_{1k}^2 J_1(\alpha_{1k}) + (\alpha_{1k}^3 + 4\alpha_{1k}) J_2(\alpha_{1k})]\vec{E}_{1k}^*$$
$$+ [-\alpha_{1k}^2 J_1(\alpha_{1k}) + 2\alpha_{1k} J_2(\alpha_{1k})]\vec{F}_{1k}^* = 0$$

$$4.16_1) \quad [-\alpha_{1k}^2 J_1(\alpha_{1k}) + 4\alpha_{1k} J_2(\alpha_{1k})]\vec{D}_{1k}^*$$
$$+ [\alpha_{1k}^2 J_1(\alpha_{1k})]\vec{E}_{1k}^* = 0$$

Section V: The Six-Vector

In the previous sections a system of six first order equations has been developed consisting of 1.1), 1.20), 1.21), 1.23), 3.1) and 3.2). This system may be written in vector form as

5.1) $$[W_1]\frac{\partial}{\partial r}\{\vec{f}\} + 1/r\ [W_2]\{\vec{f}\} + 1/r\ \frac{\partial}{\partial \theta}\ [W_3]\{\vec{f}\} + \frac{\partial}{\partial z}\ [W_4]\{\vec{f}\} = 0$$

where

5.1a) $$\{\vec{f}\} = \begin{Bmatrix} R \\ \sigma_{zz} \\ T \\ \tau_{rz} \\ \tau_{\theta z} \\ Q \end{Bmatrix}$$,

5.1b) $$[W_1] = \begin{bmatrix} \frac{1}{1+\nu} & \frac{-\nu}{1+\nu} & & & & \\ 1 & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

5.1c) $$[W_2] = \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$,

5.1d) $$[W_3] = \begin{bmatrix} 0 & 0 & 1 & & & \\ i & i & 0 & & & \\ -\frac{1}{1+\nu} & \frac{\nu}{1+\nu} & 0 & & & \\ & & & 0 & 1 & 0 \\ & & & -1 & 0 & 0 \\ & & & 0 & 0 & -i \end{bmatrix}$$

5.1e) $$[W_4] = \begin{bmatrix} & & & 1 & 0 & 0 \\ & & & 0 & 0 & -1 \\ & & & 0 & -1 & 0 \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 1 & 1 & 0 & & & \end{bmatrix}$$

A form of solution provided by a separation of variables is assumed

5.2) $$\{\vec{f}\} = \sum_{m=o}^{\infty} \{\vec{f}_m(r)\}e^{im\theta}e^{-\alpha z}$$

and it is found that $\{\vec{f}_m\}$ satisfies the ordinary differential equation

5.3) $$[W_1]\frac{d}{dr}\{\vec{f}_m\} + \frac{1}{r}[W_2]\{\vec{f}_m\} + \frac{im}{r}[W_3]\{\vec{f}_m\} - \alpha_m[W_4]\{\vec{f}_m\} = 0$$

As the equations of this system were derived by integrating the second order systems of the two previous fundamental problems, it is not necessary to re-solve them, nor is it necessary to re-derive the transcendental equation, as the boundary conditions remain the same as before.

The adjoint equation 5.3) is as follows:

5.4) $$-[W_1]^{+}\frac{\partial}{\partial r}\{\vec{g}_m\} + 1/r\,[W_2]^{+}\{\vec{g}_m\} - \frac{im}{r}[W_3]^{+}\{\vec{g}_m\} - \bar{\alpha}_m[W_4]^{\dagger}\{\vec{g}_m\} = 0$$

It is necessary, however, to solve this system of equations, as no assumption is being made as to a relationship between the adjoint functions of this problem and those of the previous ones. Furthermore, the adjoint boundary conditions will be specified for the convenience of the current problem, so it will be necessary to again check that the choice of eigenvalues so determined will be compatible with those of the previous problem.

Solving the system of adjoint equations, one finds that

5.5a) $$\bar{g}_m^{(1)} = [m\,J_m(\alpha_m r)]\vec{\mathcal{B}}_m^{*} + [2(1+\nu)\alpha_m r J_{m+1}(\alpha_m r)]\vec{\mathcal{C}}_m^{*}$$

5.5b) $\bar{g}_m^{(2)} = [-\alpha_m r J_{m+1}(\alpha_m r)]\mathcal{A}_m^* + [-(\alpha_m r)^2 J_m(\alpha_m r) + 2m\,\alpha_m r J_{m+1}(\alpha_m r)]\mathcal{C}_m^*$

5.5c) $\bar{g}_m^{(3)} = i\{[m\,J_m(\alpha_m r) - \alpha_m r J_{m+1}(\alpha_m r)]\vec{\mathcal{B}}_m^*$

$$+ [-2(1+\nu)\alpha_m r J_{m+1}(\alpha_m r)]\vec{\mathcal{C}}_m^*\}$$

5.5d) $\bar{g}_m^{(4)} = [2(1+\nu)\alpha_m r J_m(\alpha_m r)]\vec{\mathcal{C}}_m^*$

5.5e) $\bar{g}_m^{(5)} = i\{[\alpha_m r J_m(\alpha_m r)]\vec{\mathcal{B}}_m^* + [2(1+\nu)\alpha_m r J_m(\alpha_m r)]\vec{\mathcal{C}}_m^*\}$

5.5f) $\bar{g}_m^{(6)} = [\alpha_m r J_m(\alpha_m r)]\vec{\mathcal{A}}_m^* + [-(\alpha_m r)^2 J_{m+1}(\alpha_m r)]\vec{\mathcal{C}}_m^*$

The boundary conditions for the stresses are as follows: The condition $\tau_{rz} = 0$ on $r = 1$ is used directly. The conditions $\sigma_{rr} = 0$ on $r = 1$ and $\tau_{\theta z} = 0$ on $r = 1$ are taken as they are found in equations 1.30) and 1.31) and the r-derivatives are eliminated through the use of the equation 5.1).

The following boundary conditions are thereby obtained

5.6) $\tau_{rzmj}\Big|_{r=1} = 0$

5.7) $\frac{1}{1+\nu}[(1-\nu)m + \nu m^2 + \alpha_{mj}^2]R_{mj} + \frac{1}{1+\nu}[(1-\nu)m - m^2 - \alpha_{mj}^2\nu]\sigma_{zzmj}$

$$+ im\,T_{mj} + 2im\,\alpha_{mj}\tau_{\theta zmj} - \frac{1-\nu}{1+\nu}\alpha_{mj}Q_{mj}\Big|_{r=1} = 0$$

5.8)
$$\frac{-1}{1+\nu}(m\nu + (1-\nu)m^2)R_{mj} + \frac{1}{1+\nu}(m - (1-\nu)m^2)\sigma_{zzmj}$$

$$+ i(\alpha_{mj}^2 - m^2)T_{mj} - 2\,i\,\alpha_m\tau_{\theta zmj} + \frac{1-\nu}{1+\nu}\alpha_{mj}Q_{mj}\Big|_{r=1} = 0$$

Further, these boundary conditions may be expressed in the form

5.9)
$$\tau_{rzmj}\Big|_{r=1} = 0$$

5.10)
$$\alpha_{mj}(m^2-1)\tau_{\theta zmj}\Big|_{r=1} = \frac{i}{2(1+\nu)}[\alpha_{mj}^2 m + \nu m(m^2-1)]R_{mj}$$

$$+ \frac{i}{2(1+\nu)}[-\alpha_{mj}^2\nu m - m(m^2-1)]\sigma_{zzmj} - \frac{\alpha_{mj}^2}{2}T_{mj}\Big|_{r=1}$$

5.11)
$$(1-\nu)\alpha_{mj}(m^2-1)Q_{mj}\Big|_{r=1} = [-\alpha_{mj}^2 + (1-\nu)m(m^2-1)]R_{mj}$$

$$+ [\alpha_{mj}^2\nu + (1-\nu)m(m^2-1)]\sigma_{zzmj}$$

$$+ i(1+\nu)[-\alpha_{mj}^2 m + m(m^2-1)]T_{mj}\Big|_{r=1}$$

Letting

5.12)
$$I_{jk}^{(m)} = (\alpha_{mj} - \alpha_{mk})\int_0^1 \{\vec{g}_{mk}\}^+[W_4]\{\vec{f}_{mj}\}dr$$

expanding, using the differential equations 5.3) and 5.4), and integrating by parts, it is found that:

5.13)
$$I_{jk}^{(m)} = [\{\vec{g}_{mk}\}^+[W_1]\{\vec{f}_{mj}\}]_{r=o}^{r=1}$$

For $m \neq 1$, one obtains, using 5.9), 5.10), and 5.11) in equation 5.13), that

5.14)
$$\alpha_{mj}(m^2-1)I_{jk}^{(m)} = \Bigg[\Bigg(\alpha_{mj}(m^2-1)\left(\frac{1}{1+\nu}\bar{g}_{mk}^{(1)} + \bar{g}_{mk}^{(2)}\right) + \frac{i}{2(1+\nu)}[\alpha_{mj}^2 m+\nu m(m^2-1)]\bar{g}_{mk}^{(5)} + \frac{1}{1-\nu}[-\alpha_{mj}^2+(1-\nu)m(m^2-1)]\bar{g}_{mk}^{(6)}\Bigg) R_{mj}$$
$$+ \Bigg(\alpha_{mj}(m^2-1)\left(\frac{-\nu}{1+\nu}\bar{g}_{mk}^{(1)} + \bar{g}_{mk}^{(2)}\right) + \frac{i}{2(1+\nu)}[-\alpha_{mj}^2\nu m - m(m^2-1)]\bar{g}_{mk}^{(5)} + \frac{1}{1-\nu}[\alpha_{mj}^2\nu + (1-\nu)m(m^2-1)]\bar{g}_{mk}^{(6)}\Bigg)\sigma_{zzmj}$$
$$+ \Bigg(\alpha_{mj}(m^2-1)\bar{g}_{mk}^{(3)} - \frac{\alpha_{mj}^2}{2}\bar{g}_{mk}^{(5)} + i\,\frac{1+\nu}{1-\nu}[-\alpha_{mj}^2 m + m(m^2-1)]\bar{g}_{mk}^{(6)}\Bigg)\cdot T_{mj}\Bigg]_{r=0}^{r=1}$$

Thus the boundary conditions become:

5.15)
$$\alpha_{mk}(m^2-1)\left(\frac{1}{1+\nu}\bar{g}_{mk}^{(1)} + \bar{g}_{mk}^{(2)}\right) + \frac{i}{2(1+\nu)}[\alpha_{mk}^2 m+\nu m(m^2-1)]\bar{g}_{mk}^{(5)} + \frac{1}{1-\nu}[-\alpha_{mk}^2 + (1-\nu)m(m^2-1)]\bar{g}_{mk}^{(6)} = 0$$

5.16)
$$\alpha_{mk}(m^2-1)\left(\frac{-\nu}{1+\nu}\bar{g}_{mk}^{(1)} + \bar{g}_{mk}^{(2)}\right) - \frac{i}{2(1+\nu)}[\alpha_{mk}^2\nu m + m(m^2-1)]\bar{g}_{mk}^{(5)} + \frac{1}{1-\nu}[\alpha_{mk}^2\nu + (1-\nu)m(m^2-1)]\bar{g}_{mk}^{(6)} = 0$$

5.17)
$$\alpha_{mk}(m^2-1)\bar{g}_{mk}^{(3)} - \frac{\alpha_{mk}^2}{2}\bar{g}_{mk}^{(5)} + \frac{i(1+\nu)}{1-\nu}[-\alpha_{mk}^2 m + m(m^2-1)]\bar{g}_{mk}^{(6)} = 0$$

Under these boundary conditions

$$5.18) \quad \alpha_{mj}(m^2-1)I_{jk}^{(m)} = (\alpha_{mj} - \alpha_{mk})\left[\left((m^2-1)\left(\frac{1}{1+\nu}\bar{g}_{mk}^{(1)} + \bar{g}_{mk}^{(2)}\right)\right.\right.$$

$$\left. + \frac{im}{2(1+\nu)}(\alpha_{mj} + \alpha_{mk})\bar{g}_{mk}^{(5)} - \frac{1}{1-\nu}(\alpha_{mj} + \alpha_{mk})\bar{g}_{mk}^{(6)}\right)R_{mj}$$

$$+ \left((m^2-1)\left(\frac{-\nu}{1+\nu}\bar{g}_{mk}^{(1)} + \bar{g}_{mk}^{(2)}\right) + (\alpha_{mj} + \alpha_{mk})\left(-\frac{i\nu m}{2(1+\nu)}\bar{g}_{mk}^{(5)} + \frac{\nu}{1-\nu}\bar{g}_{mk}^{(6)}\right)\right)\sigma_{zzmj}$$

$$\left. + \left((m^2-1)\bar{g}_{mk}^{(3)} - \frac{\alpha_{mj} + \alpha_{mk}}{2}\left(\bar{g}_{mk}^{(5)} - \frac{i(1+\nu)}{1-\nu} m\, \bar{g}_{mk}^{(6)}\right)\right) T_{mn}\right]_{r=1}^{r=0}$$

and one has the biorthogonality condition for $m \neq 1$.

$$5.19) \quad (\alpha_{mj} - \alpha_{mk})\left\{\int_c^1 \{\vec{g}_{mk}\}^+(\alpha_{mj}(m^2-1))[W_4]\{\vec{f}_{mj}\}dr\right.$$

$$+ \left[\left(-(m^2-1)\left(\frac{1}{1+\nu}\bar{g}_{mk}^{(1)} + \bar{g}_{mk}^{(2)}\right) - \frac{im}{2(1+\nu)}\alpha_{mk}\bar{g}_{mk}^{(5)} + \frac{1}{1-\nu}\alpha_{mk}\bar{g}_{mk}^{(6)}\right)R_{mj}\right.$$

$$+ \left(\frac{-im}{2(1+\nu)}\bar{g}_{mk}^{(5)} + \frac{1}{1+\nu}\bar{g}_{mk}^{(6)}\right)\alpha_{mj} R_{mj}$$

$$+ \left((m^2-1)\left(\frac{\nu}{1+\nu}\bar{g}_{mk}^{(1)} - \bar{g}_{mk}^{(2)}\right) + \frac{im\nu\,\alpha_{mk}}{2(1+\nu)}\bar{g}_{mk}^{(5)} - \frac{\nu\,\alpha_{mk}}{1-\nu}\bar{g}_{mk}^{(6)}\right)\sigma_{zzmj}$$

$$+ \left(\frac{im\nu}{2(1+\nu)}\bar{g}_{mk}^{(5)} - \frac{\nu}{1-\nu}\bar{g}_{mk}^{(6)}\right)\alpha_{mj}\,\sigma_{zzmj}$$

$$+ \left(-(m^2-1)\bar{g}_{mk}^{(3)} + 1/2\,\alpha_{mk}\bar{g}_{mk}^{(5)} + \frac{i(1+\nu)}{1-\nu} m\,\bar{g}_{mk}^{(6)}\right) T_{mj}$$

$$\left.\left. + \left(-1/2\,\bar{g}_{mk}^{(5)} + \frac{im(1+\nu)}{1-\nu}\bar{g}_{mk}^{(6)}\right)\alpha_{mj} T_{mj}\right]_{r=1}^{r=0}\right\} = 0$$

This form is analogous to that of equation 4.13) of the previous section, and, for $m > 1$, it is developed to render it usable for total stress end loadings, as follows: Using the differential equation 5.3) one can write

$$5.20)\qquad \int_0^1 \{\vec{g}_{mk}\}^+ \alpha_{mj} (m^2-1)[W_4]\{\vec{f}_{mj}\}dr$$

$$= \int_0^1 \{\vec{g}_{mk}\}^+ (m^2-1)\left\{[W_1]\frac{d}{dr}\{\vec{f}_{mj}\} + \frac{1}{r}[W_2]\{\vec{f}_{mj}\} + \frac{im}{r}[W_3]\{\vec{f}_{mj}\}\right\} dr$$

and since, by equation 1.1)

$$\alpha_{mj}\sigma_{zzmj} = L\,\tau_{rzmj} = \frac{im}{r}\tau_{\theta zmj}$$

equation 5.19) for $m > 1$ takes the form

$$5.21)\qquad (\alpha_{mj} - \alpha_{mk})\left\{\int_0^1 \{\vec{g}_{mk}\}^+ (m^2-1)\left[[W_1]\frac{d}{dr}\{\vec{f}_{mj}\} + \frac{1}{r}[W_2]\{\vec{f}_{mj}\} + \frac{im}{r}[W_3]\{\vec{f}_{mj}\}\right]dr\right.$$

$$- \left[\left((m^2-1)\left(\frac{1}{1+\nu}\bar{g}_{mk}^{(1)} + \bar{g}_{mk}^{(2)}\right) + \frac{im(\alpha_{mj}+\alpha_{mk})}{2(1+\nu)}\bar{g}_{mk}^{(5)} - \frac{\alpha_{mj}+\alpha_{mk}}{1-\nu}\bar{g}_{mk}^{(6)}\right)R_{mj}\right.$$

$$+ \left((m^2-1)\left(\frac{-\nu}{1+\nu}\bar{g}_{mk}^{(1)} + \bar{g}_{mk}^{(2)}\right) - \frac{i\nu m\,\alpha_{mk}}{2(1+\nu)}\bar{g}_{mk}^{(5)} + \frac{\alpha_{mk}\nu}{1-\nu}\bar{g}_{mk}^{(6)}\right)\sigma_{zzmj}$$

$$+ \left(-\frac{im\nu}{2(1+\nu)}\bar{g}_{mk}^{(5)} + \frac{\nu}{1-\nu}\bar{g}_{mk}^{(6)}\right)\left(L\,\tau_{rzmj} + \frac{im}{r}\tau_{\theta zmj}\right)$$

$$+ \left.\left.\left((m^2-1)\bar{g}_{mk}^{(3)} - 1/2(\alpha_{mj}+\alpha_{mk})\bar{g}_{mk}^{(5)} - \frac{im(1+\nu)}{1-\nu}(\alpha_{mj}+\alpha_{mk})\bar{g}_{mk}^{(6)}\right)T_{mj}\right]_{r=1}\right\} = 0$$

Having specified the boundary conditions of the adjoint problem so as to reduce equation 5.14) to a desired form, it is necessary to see that the series of eigenvalues so determined is the same as before. One therefore substitutes the form of the adjoint functions, given in equations 5.5) into the boundary conditions of equations 5.9), 5.10), and 5.11) to obtain three linear equations in the unknowns $\vec{\mathcal{A}}^*_{mk}$, $\vec{\mathcal{B}}^*_{mk}$, and $\vec{\mathcal{C}}^*_{mk}$:

5.22) $$\frac{1}{1-\nu}[(-\alpha^3_{mk}+(1-\nu)\alpha_{mk}m(m^2-1))J_m(\alpha_{mk})-(1-\nu)\alpha^2_{mk}(m^2-1)J_{m+1}(\alpha_{mk})]\vec{\mathcal{A}}^*_{mk}$$

$$+\frac{1}{2(1+\nu)}[(-\alpha^3_{mk}m + (2-\nu)\alpha_{mk}m(m^2-1))J_m(\alpha_{mk})]\vec{\mathcal{B}}^*_{mk}$$

$$+[(-\alpha^3_{mk}(m^2-1) - \alpha^3_{mk}m - \alpha_{mk}\nu m(m^2-1))J_m(\alpha_{mk})$$

$$+\frac{1}{1-\nu}(\alpha^4_{mk}+(1-\nu)\alpha^2_{mk}m(m^2-1)+2(1-\nu)\alpha^2_{mk}(m^2-1))J_{m+1}(\alpha_{mk})]\vec{\mathcal{C}}^*_{mk} = 0$$

5.23) $$\frac{1}{1-\nu}[(\alpha^3_{mk}\nu+(1-\nu)\alpha_{mk}m(m^2-1))J_m(\alpha_{mk})-(1-\nu)\alpha^2_{mk}(m^2-1)J_{m+1}(\alpha_{mk})]\vec{\mathcal{A}}^*_{mk}$$

$$+\frac{1}{2(1+\nu)}[(\alpha^3_{mk}\nu m + (1-2\nu)\alpha_{mk}m(m^2-1))J_m(\alpha_{mk})]\vec{\mathcal{B}}^*_{mk}$$

$$+[(-\alpha^3_{mk}(m^2-1) + \alpha^3_{mk}\nu m + \alpha_{mk}m(m^2-1)J_m(\alpha_{mk})$$

$$+\frac{1}{1-\nu}(-\alpha^4_{mk}\nu+(1-\nu)\alpha^2_{mk}m(m^2-1)-2\nu(1-\nu)\alpha^2_{mk}(m^2-1))J_{m+1}(\alpha_{mk})]\vec{\mathcal{C}}^*_{mk} = 0$$

5.24) $$\frac{1}{1-\nu}[(-\alpha^3_{mk}m + \alpha_{mk}m(m^2-1))J_m(\alpha_{mk})]\vec{\mathcal{A}}^*_{mk}$$

$$+\frac{1}{2(1+\nu)}[(-\alpha^3_{mk} + 2\alpha_{mk}m(m^2-1))J_m(\alpha_{mk}) - 2\alpha^2_{mk}(m^2-1)J_{m+1}(\alpha_{mk})]\vec{\mathcal{B}}^*_{mk}$$

$$+[-\alpha^3_{mk}J_m(\alpha_{mk}) + \frac{1}{1-\nu}(\alpha^4_{mk}m-\alpha^2_{mk}m(m^2-1)-2(1-\nu)\alpha^2_{mk}(m^2-1))J_{m+1}(\alpha_{mk})]\vec{\mathcal{C}}^*_{mk} = 0$$

Setting the determinant of these coefficients equal to zero yields the transcendental equation determining the values of α_{mk}, and this, in turn, reduces to the previous equation, equation 1.32).

For $m = 0$, equations 5.7), 5.8) simplify and may be used to reduce equation 5.13) to a much simpler form, and the biorthogonality relation for $m = 0$ becomes

$$5.19_o) \qquad (\alpha_{oj} - \alpha_{ok})\left\{ \int_o^1 \{\vec{g}_{ok}\}[W_4]\{\vec{\varphi}_{oj}\}dr \right.$$

$$\left. - \left[\frac{1}{1-\nu}(R_{oj} - \nu\,\sigma_{zzoj})\bar{g}^{(6)}_{ok} + 1/2\ \bar{g}^{(5)}_{ok} T_{oj} \right]_{r=1} \right\} = 0$$

Furthermore, the problem and its biorthogonality relation, for $m = 0$, decouples into an axisymmetric torsion problem and an axisymmetric torsionless problem; one part of the biorthogonality determining the single series of constants associated with the axisymmetric torsion problem, the other determining the two series of constants associated with the axisymmetric torsionless problem. The eigenvalues of the former problem are the real eigenvalues; the latter are the complex ones.

For $m = 1$, the boundary conditions on the stress functions can be expressed as

$$5.9_1) \qquad \tau_{rz1j}\Big|_{r=1} = 0$$

$$5.10_1) \qquad R_{1j}\Big|_{r=1} = \nu\,\sigma_{zz1j} - i(1+\nu)T_{1j}\Big|_{r=1}$$

5.11$_1$) $$\tau_{\theta z1j}\Big|_{r=1} = \frac{\alpha_{mj}}{2} T_{1j} - i\,\frac{1-\nu}{2(1+\nu)} Q_{1j}\Big|_{r=1}$$

Setting

5.12$_1$) $$I_{jk}^{(1)} = (\alpha_{1j} - \alpha_{1k}) \int_o^1 \{\vec{g}_{1k}\}^+[W_4]\{\vec{f}_{1j}\}dr$$

expanding, using the differential equations 5.3) and 5.4), and integrating by parts, it is found that

5.13$_1$) $$I_{jk}^{(1)} = \Big[(1+\nu)\bar{g}_{1k}^{(2)}\cdot \sigma_{zz1j} - i(\bar{g}_{1k}^{(1)} + (1+\nu)\bar{g}_{1k}^{(2)} + i\,\bar{g}_{1k}^{(3)} + \frac{i\alpha_{mj}}{2}\bar{g}_{1k}^{(5)})T_{1j} + \left(\frac{-i(1-\nu)}{2(1+\nu)}\bar{g}_{1k}^{(5)} + \bar{g}_{1k}^{(6)}\right)Q_{1j}\Big]_{r=1}$$

One therefore takes as boundary conditions

5.14$_1$) $$\bar{g}_{1k}^{(2)}\Big|_{r=1} = 0$$

5.15$_1$) $$\bar{g}_{1k}^{(1)} + (1+\nu)\bar{g}_{1k}^{(2)} + i\,\bar{g}_{1k}^{(3)} + \frac{i\alpha_{mk}}{2}\bar{g}_{1k}^{(5)} = 0$$

5.16$_1$) $$\frac{-i(1-\nu)}{2(1+\nu)}\bar{g}_{1k}^{(5)} + \bar{g}_{1k}^{(6)} = 0$$

And the biorthogonality relation for $m = 1$ becomes

5.17$_1$) $$(\alpha_{1j} - \alpha_{1k})\left\{\int_o^1 \{\vec{g}_{1k}\}^+[W_1]\{\vec{f}_{1j}\}dr - \bar{g}_{1k}^{(5)}\cdot T_{1j}\Big|_{r=1}\right\} = 0$$

Substituting in the form of the adjoint functions, as given in equations 5.5) with m taken to be equal to 1, one obtains three linear equations in the unknowns $\overline{\mathcal{A}}^*_{1k}$, $\overline{\mathcal{B}}^*_{1k}$, and $\overline{\mathcal{C}}^*_{1k}$.

5.18$_1$) $$[\alpha_{1k}J_2(\alpha_{1k})]\overline{\mathcal{A}}^*_{1k} + [\alpha^2_{1k}J_1(\alpha_{1k}) - 2\alpha_{1k}J_2(\alpha_{1k})]\overline{\mathcal{C}}^*_{1k} = 0$$

5.19$_1$) $$[-2(1+\nu)\alpha_{1k}J_2(\alpha_{1k})]\overline{\mathcal{A}}^*_{1k} + [-\alpha^2_{1k}J_1(\alpha_{1k})+2\alpha_{1k}J_2(\alpha_{1k})]\overline{\mathcal{B}}^*_{1k}$$

$$+ [-4(1+\nu)\alpha^2_{1k}J_1(\alpha_{1k}) + 12(1+\nu)\alpha_{1k}J_2(\alpha_{1k})]\overline{\mathcal{C}}^*_{1k} = 0$$

5.20$_1$) $$[\alpha_{1k}J_1(\alpha_{1k})]\overline{\mathcal{A}}^*_{1k}+[\frac{1-\nu}{2(1+\nu)}\alpha_{1k}J_1(\alpha_{1k})]\overline{\mathcal{B}}^*_{1k}$$

$$+ [(1-\nu)\alpha_{1k}J_1(\alpha_{1k}) - \alpha^2_{1k}J_2(\alpha_{1k})]\overline{\mathcal{C}}^*_{1k} = 0$$

Upon taking the determinant of the coefficients of these equations and setting it equal to zero, one obtains the same result as that obtained when m is set equal to 1 in equation 1.32).

CHAPTER 3

The Formulation of Specific Boundary Value Problems

With displacement boundary conditions on the end of the cylinder, or displacement conditions in conjunction with stress conditions, one is liable to the difficulties mentioned in Chapter 1, namely, the possibility of stress singularities and ignorance of the physical conditions necessary for a decaying solution. Furthermore, the type of problem whose solution is most desired for engineering applications is one with boundary conditions stated entirely in terms of stresses. For these reasons numerical work was not attempted with either of the biorthogonality conditions developed in considering the fundamental mixed problems of Sections 2.1-2.2 or 2.3-2.4. Such a development was thought necessary, however, because the ability of the auxiliary variables to complete the specification of a fundamental mixed problem was felt to be a necessary requirement to assure its usefulness in the construction of the desired 6-vector.

The following problem was considered appropriate for the formulation of numerical methods for the solution of specific boundary value problems. The long side, $r = 1$, is to be free from stress. At the finite end, $z = 0$, the normal stress, σ_{zz}, and two tangential stresses τ_{rz} and $\tau_{\theta z}$ may be prescribed subject only to the requirements that they correspond to a self-equilibrated loading and be free from stress singularities.

A solution is sought which renders the prescribed boundary stresses σ_{zzb}, τ_{rzb}, $\tau\theta_{zb}$ in terms of a single series of constants. That is, one seeks

6.1a) $$\sigma_{zzb} = \left(\sum_m \sum_j a_{mj}\, \sigma_{zzmj}\, e^{im\theta}\, e^{-\alpha_{mj} z}\right)\Big|_{z=0}$$

6.1b) $$\tau_{rzb} = \left(\sum_m \sum_j a_{mj}\, \tau_{rzmj}\, e^{im\theta}\, e^{-\alpha_{mj} z}\right)\Big|_{z=0}$$

6.1c) $$\tau_{\theta zb} = \left(\sum_m \sum_j a_{mj}\, \tau_{\theta zmj}\, e^{im\theta}\, e^{-\alpha_{mj} z}\right)\Big|_{z=0}$$

The numerical solution of a boundary value problem, is accomplished as follows:

One first performs a Fourier analysis to divide the prescribed function into functions of r each associated with a different index of theta dependence. Then one selects the appropriate biorthogonality relation for the 6-vector, equation $5.19_o)$ for $m = 0$, equation $5.17_1)$ for $m = 1$, or equation 5.21) for $m > 1$. Then following the method of Johnson and Little [18], the variables σ_{zzmj}, τ_{rzmj}, $\tau_{\theta zmj}$ are replaced by their prescribed boundary data σ_{zzmb}, τ_{rzmb}, $\tau_{\theta zmb}$. The remaining variables are replaced by formal series

6.4a) $$R_m = \sum_j a_{mj}\, R_{mj}$$

6.4b) $$T_m = \sum_j a_{mj}\, T_{mj}$$

6.5c) $$Q_m = \Sigma\, a_{mj}\, Q_{mj}$$

thus constructing the vector $\{\vec{f}_{mb}\}$, a particular vector of the form of equation 5.1a). One therefore obtains, for each k, a single equation in infinitely many unknowns

$$L_{mk}\{\vec{f}_{mb}\} = a_{mk} N_{mk}$$

where L_{mk} is the $k^{\underline{th}}$ biorthogonality relation for the index m of theta dependence, corresponding to using the adjoint functions of the eigenvalue α_{mk}, and N_{mk} is the $k^{\underline{th}}$ normalization constant. The set of all these equations for $k = 1,2,...$ constitutes a infinite set of linear equations in the infinite number of unknowns $a_{m1}, a_{m2}, ...$ and may be solved to any desired accuracy by truncation.

The functions of r corresponding to theta dependence with $m < 0$ in the Fourier analysis correspond to the parts of the boundary conditions 90° out of phase with respect to the original axis and may be treated by rotating the axis 90°, solving the new problem for $m > 0$, and superposing the solution.

Two trial problems were solved numerically to indicate the rate of convergence of the series expansions. The axisymmetric torsionless problem considered in Little and Childs [22] was programmed using the biorthogonality relations developed in this study and the results have the same accuracy as given there. The results of the present method are found in Tables 6 and 7.

A non-axisymmetric problem with index of theta dependence, $m = 2$ was programmed. An appropriate specification was that σ_{zz2b} and τ_{rz2b} should be entered as real functions and that

$\tau_{\theta z2b}$ should be entered as a purely imaginary function, because the behavior of $\tau_{\theta z}$ is 90° out of phase, with respect to theta, from σ_{zz} and τ_{rz}, as seen in equations 1.1), 1.9), 1.10) and elsewhere. The convergence appears to be somewhat slower than in the axisymmetric case, but of the same order. The results are found in Tables 8, 9, and 10.

Roots of the Transcendental Equation

Table 1

m = 0, $\nu = 0.25$

n	Complex Root*		Real Root+
1	2.6976518	+i1.3673570	5.1356223
2	6.0512222	1.6381471	8.4172441
3	9.2612734	1.8285342	11.619841
4	12.438444	1.9674283	14.795952
5	15.602204	2.0764211	17.959820
6	18.759055	2.1660392	21.116997
7	21.911845	2.2421081	24.270112
8	25.062031	2.3081733	27.420574
9	28.210443	2.3665585	30.569204
10	31.357587	2.4188579	33.716520
11	34.503796	2.4662104	36.862856
12	37.649288	2.5094822	40.008447
13	40.794222	2.5493159	43.153454
14	43.938715	2.5862151	46.297997
15	47.082846	2.6205834	49.442164
16	50.226683	2.6527455	52.586024
17	53.370274	2.6829654	55.729627
18	56.513658	2.7114654	58.873016
19	59.656867	2.7384311	62.016222
20	62.799924	2.7640179	65.159273

* From Little and Childs [22]

\+ From Watson [41]

Table 2

m = 1, $\nu = 0.25$

n	Complex Root		Real Root
1	4.2852175	+i1.49819612	2.7931415
2	7.5952965	1.7398515	6.6948248
3	10.805373	1.9023387	9.9662643
4	13.985446	2.0249445	13.169015
5	17.151977	2.1234422	16.346821
6	20.311147	2.2057668	19.512503
7	23.465843	2.2764824	22.671322
8	26.617603	2.3384588	25.825862
9	29.767330	2.3936165	28.977549
10	32.915586	2.4433066	32.127236

Table 3

m = 2, $\nu = 0.25$

n	Complex Root		Real Root
1	2.1043544	+i0.95922055	4.0882060
2	5.6799579	1.6129679	8.1137406
3	9.0352822	1.8244578	11.425432
4	12.273656	1.9671243	14.650449
5	15.472033	2.0771376	17.842808
6	18.651317	2.1670412	21.018853
7	21.819876	2.2431549	24.185462
8	24.981772	2.3091822	27.346087
9	28.139231	2.3674993	30.502666
10	31.293581	2.4197245	33.656374

Table 4

$m = 3$, $\nu = 0.25$

n	Complex Root		Real Root
1	3.2858613	+i1.1576914	5.2957151
2	6.9912960	1.7106445	9.4615179
3	10.411580	1.8983456	12.826721
4	13.689043	2.0255558	16.083178
5	16.913273	2.1252664	19.296718
6	20.110957	2.2078764	22.488138
7	23.293299	2.2785740	25.666468
8	26.465914	2.3404296	28.836344
9	29.631954	2.3954371	32.000421
10	32.793330	2.4449759	35.160336

Table 5

$m = 4$ $\nu = 0.25$

n	Complex Root		Real Root
1	4.4047845	+i1.3040193	6.4629421
2	8.2535980	1.7961669	10.764020
3	11.743891	1.9646273	14.186596
4	15.064898	2.0791555	17.478662
5	18.318901	2.1701374	20.717023
6	21.538323	2.2464063	23.926895
7	24.737274	2.3122987	27.119542
8	27.923012	2.3703921	30.300877
9	31.099690	2.4223776	33.474347
10	34.269869	2.4694370	36.642109

Table 6

Axisymmetric trial problem: $\sigma_{zz} = 1-2r^2$

number of pairs of eigenvalues used

r	prescribed	1	2	5	10	20
0.0	1.000	1.154	1.048	1.000	.988	.995
.1	.980	1.279	.997	.960	.961	.975
.2	.920	1.616	.866	.884	.905	.915
.3	.820	2.058	.705	.800	.803	.816
.4	.680	2.440	.558	.664	.666	.676
.5	.500	2.565	.429	.472	.486	.496
.6	.280	2.229	.275	.263	.289	.277
.7	.020	1.263	.038	.016	.012	.018
.8	-.280	-.436	-.289	-.287	-.281	-.280
.9	-.620	-2.868	-.605	-.605	-.611	-.616
1.0	-1.000	-5.896	-.684	-.885	-.923	-.978

Table 7

Axisymmetric trial problem: $\tau_{rz}\Big|_{z=o} = 2.4r - 2.6r^2 + 0.2r^5$

number of pairs of eigenvalues used

r	prescribed	1	2	5	10	20
0.0	0.000	0.000	0.000	0.000	0.000	0.000
.1	.237	-.840	.347	.271	.242	.237
.2	.459	-1.541	.622	.471	.461	.461
.3	.650	-1.983	.787	.641	.656	.651
.4	.796	-2.095	.856	.806	.802	.798
.5	.881	-1.863	.873	.905	.891	.883
.6	.894	-1.347	.876	.910	.904	.898
.7	.821	-.680	.858	.837	.837	.825
.8	.654	-.063	.754	.675	.667	.659
.9	.383	.255	.479	.417	.406	.387
1.0	.000	.000	.000	.000	.000	.000

Table 8

m = 2 trial problem: $\sigma_{zz}\Big|_{\substack{z=o\\ \theta=o}} = 3 - 4r^2$

number of triads of eigenvalues used

r	prescribed	1	2	5	10
0.0	0.000	0.000	0.000	0.000	0.000
.1	.260	-.076	.078	.103	.270
.2	.440	-.280	.268	.360	.417
.3	.540	-.550	.456	.565	.540
.4	.560	-.792	.529	.522	.522
.5	.500	-.891	.385	.383	.470
.6	.360	-.734	.090	.270	.306
.7	.140	-.224	-.316	.012	.074
.8	-.160	.708	-.297	-.315	-.226
.9	-.540	2.088	.093	-.615	-.612
1.0	-1.000	3.898	1.119	-1.420	-1.178

Table 9

m = 2 trial problem: $\tau_{rz}\Big|_{\substack{z=o\\ \theta=o}} = 2.4r - 2.6r^3 + 0.2r^5$

number of triads of eigenvalues used

r	prescribed	1	2	5	10
0.0	0.000	0.000	0.000	0.000	0.000
.1	.237	.809	.032	.357	.278
.2	.459	1.512	.182	.530	.458
.3	.650	2.020	.504	.649	.652
.4	.796	2.273	.955	.788	.798
.5	.881	2.250	1.405	.860	.887
.6	.894	1.972	1.682	.912	.906
.7	.822	1.502	1.650	.892	.826
.8	.654	.936	1.269	.618	.667
.9	.383	.392	.640	.313	.366
1.0	.000	.000	.000	.000	.000

Table 10

m = 2 trial problem: $\tau_{\theta z}\Big|_{\substack{z=o\\ \theta=o}} = \frac{16}{9}(r - 4r^2)$

r	prescribed	number of triads of eigenvalues used: 1	2	5	10
0.0	0.000	0.000	0.000	0.000	0.000
.1	.462	.810	.035	.388	.378
.2	.782	1.521	.137	.701	.807
.3	.960	2.047	.332	.951	.952
.4	.996	2.320	.584	1.043	.993
.5	.889	2.303	.772	.864	.891
.6	.640	1.999	.773	.565	.618
.7	.249	1.402	.509	.252	.241
.8	-.284	.593	-.021	-.324	-.327
.9	-.960	-.377	-.736	-1.162	-1.004
1.0	-1.778	-1.398	-1.511	-1.663	-1.720

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SUMMARY

A semi-infinite elastic cylinder is considered with various types of self-equilibrated, but not necessarily axisymmetric, loadings applied to the finite end and with the long sides free from stress.

The paper offers a formulation in terms of stresses and three auxiliary variables of the order of stresses, and the systematic use of the Beltrami-Michell equations of compatibility. Two of the fundamental mixed problems are each reduced to a second order vector partial differential equation. Separation of variables provides a non-self adjoint ordinary differential equation, so the eigenfunctions are non-orthogonal. A generalized biorthogonality condition is derived for each of these mixed problems, which permits the direct determination of the Fourier coefficients of a vector eigenfunction expansion.

Self-equilibrated total-stress end loading problems are here formulated in terms of a six-component vector, involving the three stress variables acting on the finite end of the cylinder and the three auxiliary variables. This vector is shown to satisfy a first order matrix partial differential equation, whose constituent equations and boundary conditions are obtained from the equations of equilibrium, the equations defining the auxiliary variables, and equations obtained by integration of the compatibility equations. A generalized biorthogonality relation

is derived for the six-vector.

Total-stress problems are solved using the biorthogonality relation for the six-vector by replacing the stresses defined on the finite end by their prescribed values, replacing the auxiliary variables by formal series expansions, and by solving numerically the truncation of an infinite system of linear equations.

Numerical work was performed on two trial problems, one an axisymmetric torsionless problem and the other a truly axisymmetric problem. For the axisymmetric torsionless case it was found that ten pairs of eigenvalues matched the stress within a 10% error and twenty pairs matched within 3%. For the non-axisymmetric, ten triads matched all three boundary conditions within a 17% error.

The method of analysis is thought to converge rapidly enough to be practicable for the numerical solution of problems of this type.