ABSTRACT

ON THE OPTIMAL SAMPLED-DATA TRACKING PROBLEM

By

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The optimal sampled-data tracking problem is formulated and solved using an efficient computational algorithm. The optimization is performed on the number of samples, the sampling instants sequence, and on the order of polynomial approximation to the control law over each sampling interval. This sampled-data control is parameterized by specifying the parameters and order polynomial approximation over each sampling interval, the number of samples, and the length of each sampling interval. Comparisons are made on both control performance and sampling efficiency for control laws with different order approximations and with both periodic and optimal aperiodic sampling criteria. These results form a basis for analyzing the performance advantages and costs for using higher order control approximations and optimal aperiodic sampling criterion.

Sampled-data controllability and observability are defined for the case where both the number of sampling times and the lengths of sampling intervals are free and considered control variables. The sampled-data system is proved to be observable (controllable) if and only if the continuous time system is observable (controllable).

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A sufficient condition on the sampling time sequence is stated which guarantees the preservation of controllability and observability when the continuous measurements and controls are replaced by sampled ones.

The infinite-time sampled-data regulator problem is formulated for the case where both the number of sampling times and the lengths of sampling intervals are considered control variables.

The existence of an optimal closed-loop sampled-data control law is proved for the cases where the number of samples are both finite and infinite. Computational algorithms for calculating the optimal control are also proposed.

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To my parents

I-Ching and Su-Yuen Ma

and my wife

Linda Chung-Fan Ma

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CHAPTER I

INTRODUCTION

Periodic sampling criteria have often been used in industrial control to simplify the design and analysis. Aperiodic sampling criteria have become quite practical in both design and control with the introduction of computers. Therefore, numerous [13 - 20] aperiodic sampling criteria have been studied in an effort to improve the system performance and sampling efficiency relative to a periodic sampling criterion. Improved control performance with reduced computer memory and communication requirement makes aperiodic sampling criteria particularly useful for numerical control applications.

The optimal sampled-data tracking problem originated from the research on the development of optimal programmed control for machine tools [8]. In a computer-aided-manufacturing (CAM) system of the future, a large central computer system will compute and store the programmed control for each part. The programmed control would be stored and then transmitted at the proper time to the mini-computer or controller that monitors and controls a particular machine tool. Immense data storage and communication facilities are required to accurately specify the cutter path for each part and each machine tool. Since a major commitment in computer and communication hardware is required to handle machine tool control and since the computer and communication system must also handle material handling, scheduling and inventory control, the programmed control for each part should be specified with as little information as possible.

Therefore, the optimal programmed control for a machine tool should be designed to not only produce excellent quality parts but also minimize the information-handling requirement. Since the control is parameterized by specifying the polynomial approximation over each sampling interval and the length of each sampling interval, this minimization will be accomplished by selecting both the best control approximation parameters on each sampling interval and the optimal sampling intervals sequence. The additional flexibility provided by selecting the order of the polynomial approximation in each sampling interval and the flexibility of selecting the length of each sampling interval and the number of sampling intervals promise to permit great reduction in data transmission and storage required to obtain a particular tolerance level and surface finish quality.

This optimal sampled-data control problem was first formulated [6, 7] in an effort to obtain sampling criteria that provide better performance than any periodic or arbitrary aperiodic sampling criteria. Necessary conditions were derived in both papers but were never used to obtain an efficient computational algorithm for the optimal solution. A sequential unconstrained minimization technique (SUMT) has been used with some success in the special case where the continuous time problem can be transformed to an equivalent discrete time one [9].

An efficient computational algorithm was developed for this optimal sampled-data control problem for the special case where the optimal control sequence can be determined as a unique function of the particular sampling intervals sequence chosen. For this special case, the performance index can be determined as a function of this sampling intervals sequence. The optimal sampling intervals sequence can be found by minimizing this derived performance index. The optimal sampled-data control law is then specified by the optimal control sequence which results upon the substitution of the optimal sampling intervals sequence.

This algorithm was applied to compute the optimal sampled-data control law for the regulator problem with constrained [9], state-dependent [10], and adaptive [11] sampling criteria. The excellent performance obtained with very few control changes indicates that the computer memory and system - computer communication required to store and transmit the control can be significantly reduced if the sampling intervals are determined optimally rather than specified apriori.

With the same concept of optimal sampling for control, the optimal sampled-data tracking (and servo*) problem is investigated in this thesis. Instead of assuming a step (sample and hold) control approximation, the control approximation is assumed to be of polynomial form over each sampling interval. The order of the control approximation is varied from zero to two.

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* By convention, if the plant's outputs are to follow a class of desired trajectories, the problem is referred to as a servo problem; on the other hand, if the desired trajectories is a particular function of time, it is called a tracking problem.

Necessary conditions are obtained and are used to derive the control sequence as a function of the sampling intervals sequence. The optimal control law and a derived performance index are then proved to exist and to be unique for any sequence of sampling intervals. The existence of an optimal sampling interval sequence is finally proved.

An algorithmic procedure for computing the optimal sampled-data control is proposed. This algorithmic procedure extends the previous procedure [9] by not just searching over a sequence of sampling intervals for a particular number of samples but also searching over the number of samples required. The sub-algorithm for searching over the sampling intervals, developed by Schlueter [9], is implemented with both a gradient and a non-gradient algorithm. The computational results show that the non-gradient Powell algorithm [33] is more efficient than a Fletcher-Powell gradient algorithm [32], i.e. the computational effort and the number of iterations required to obtain convergence are less.

A cost of implementation is adjoined to the performance index for the first time because the optimal sampled-data control was proved to be the optimal continuous-time if a cost of implementation was neglected. An optimal continuous-time control is in general sub-optimal if a cost of implementation is added. After a search of the literature, a particular form for this cost of implementation is adopted. The computational results show that augmenting the performance index with a cost of implementation not only makes the design problem more reasonable but also improves the convergence of the computational algorithm.

A comparison of performance of an optimal sampled-data control law with periodic, optimal aperiodic, and adaptive sampling criteria was made. A comparison of performance was also made for sampled-data control laws with a zero, first and second order control approximation. Comparisons were also made for various sampling criteria - control approximation combinations to determine which combination needs the fewest parameters to specify a control with a given level of performance. These comparisons of control approximations and sampling criteria were carried out on three different systems and for different trajectories. These results form a basis for analyzing the performance advantages and costs for using higher order control approximations and optimal aperiodic sampling criteria.

Sampled-data controllability and observability are defined for the case where both the number of sampling times and the length of each sampling interval are free and considered to be control variables. The sampled-data system is proved to be observable (controllable) if the continuous time system is observable (controllable). Moreover, it is proved that if the system is observable (controllable), it can be observed (controlled) in q sampling intervals, where q is the order of the minimal polynomial of the plant. A test is proposed to determine whether controllability or observability is preserved for a particular sequence of sampling intervals. The test depends only upon the eigenvalues of the plant and the sampling intervals chosen. Some preliminary results are derived to indicate the condition which must be satisfied for a system which is observable (controllable) to be unobservable (uncontrollable) for a particular sampling intervals sequence.

The infinite-time sampled-data regulator problem is formulated for the case where both the number of sampling times and the lengths of the sampling intervals are considered control parameters. The existence of an optimal closed-loop sampled-data control law is proved for the cases where the number of samples are both finite and infinite. Computational algorithms for calculating the optimal control are proposed for both the case of finite and infinite number of samples.

CHAPTER II

PROBLEM FORMULATION

Consider the linear dynamic system

$$\underline{\dot{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t) \tag{1}$$

$$\underline{y}(t) = \underline{C}\underline{x}(t) \tag{2}$$

$$\underline{x}(t_o) = \underline{\xi}$$

where $\underline{x}(t) \in R^n$, $\underline{u}(t) \in R^r$, $\underline{y}(t) \in R^m$ and $\underline{A}$, $\underline{B}$, $\underline{C}$, are compatible matrices. Initial time t_o and terminal time t_N are both assumed fixed.

The design objective is to maintain the output trajectory $\underline{y}(t)$ as close as possible to a desired trajectory $\underline{z}(t)$ with minimum control effort along with minimum communication requirement. Besides, this cost functional should penalize the system for error or excessive control inputs continuously in time, not only at sampling instants.

To achieve this objective, a performance index of the form

$$S = J + C \tag{3a}$$

is chosen where the control performance is measured by

$$J = \frac{1}{2} \langle \underline{y}(t_N) - \underline{z}(t_N), \underline{F}(\underline{y}(t_N) - \underline{z}(t_N)) \rangle$$
$$+ \frac{1}{2} \int_{t_o}^{t_N} [\langle \underline{y}(t) - \underline{z}(t), \underline{Q}(\underline{y}(t) - \underline{z}(t)) \rangle + \langle \underline{u}(t), \underline{R}\underline{u}(t) \rangle] dt \tag{3b}$$

and the cost of implementation is assumed to be measured by

$$C = \sum_{i=0}^{N-1} \alpha e^{-\beta T_i} \tag{3c}$$

where $\underline{F}$ $\underline{Q}$ are positive semi-definite symmetric matrices not both identically zero and $\underline{R}$ is positive-definite, symmetric. These matrices are respectively the "weighting factor" for the end point error, error energy $\underline{y}(t) - \underline{z}(t)$ and control energy.

A cost for implementation is adjoined and represents the economic costs for implementing and operating a sampled-data control law. This cost for implementation can be considered to represent the cost for transmitting and storing the optimal sampled-data control law. It is similar in form to the costs for sampling used in the analytic derivation of adaptive sampling rules [13] and the optimal periodic sampling rate for a feedback control problem [56].

The sampled-data control law is a polynomial approximation of the true optimal control and is constrained to be piecewise polynomial of the order up to two. The order of polynomial approximation is determined by the tradeoff between the control performance and the amount of information to be transmitted. The control is assumed to have the form

$$\underline{u}(t) = \sum_{j=0}^{k} \underline{u}_{ji}(t - t_i)^j \qquad t \in [t_i, t_{i+1}) \tag{4}$$

where $k = 0,1,2$ represents a step, ramp and parabolic control approximation respectively.

The N sampling instants $\{t_{i+1}\}_{i=0}^{N-2}$ are chosen such that the sampling intervals $t_{i+1} - t_i = T_i$ satisfy

$$0 < T_{i\ min} \leq T_i \leq T_{i\ max} \tag{5a}$$

$$\underline{g}(T_0, T_1, \ldots, T_{N-1}) = \underset{\sim}{0}_v \tag{5b}$$

and N satisfies

$$N_{min} \leq N \leq N_{max} \tag{5c}$$

These sampling constraints (5a, 5b, 5c) can specify an optimal periodic sampling criterion if

$$\underline{g}(T_0, T_1, \ldots, T_{N-1}) = T_i - \left(\frac{t_N - t_0}{N}\right) = 0 \qquad i = 0,1,\ldots,N-1$$

On the other hand, a suboptimal periodic sampling criterion can be specified by fixing $N = N_{min} = N_{max}$.

Similarly, a suboptimal aperiodic sampling criterion can be specified by choosing N as

$$N_{min} = N_{max}$$

$$\underline{g}(T_0, T_1, \ldots, T_{N-1}) = \sum_{i=0}^{N-1} T_i - (t_N - t_0) = 0 \tag{5d}$$

Finally, the optimal aperiodic sampling criterion has to satisfy (5a), (5c) and (5d).

The T_{imax}'s, T_{imin}'s, N_{min} and N_{max} all come from the hardware limitation.

The optimal sampled-data tracking problem with polynomial control approximation over constrained sampling intervals can be stated formally as follows:

Given the linear dynamic system (1), (2) with polynomial control approximation (4); determine the optimal control sampling intervals sequence, and the number of sampling intervals

$$\{\underline{u}_i\}_{i=0}^{N-1},\ \underline{T}' = (T_0,\ T_1, \ldots, T_{N-1}) \quad \text{and} \quad N$$

that minimizes the cost functional (3) and satisfies (5a), (5b), and (5c) where $\underline{u}_i' = (\underline{u}_{0i}', \ldots, \underline{u}_{ki}')$.

CHAPTER III

PROBLEM SOLUTION

This tracking problem cannot be solved directly because the admissible controls are constrained to be piecewise polynomial. Nevertheless, the constrained problem can be transformed into an equivalent unconstrained one by integrating the differential equation (1) and cost functional (3) over each sampling interval $[t_{i+1}, t_i)$ separately, substituting output equation (2) and finally invoking the control constraints (4).

The resulting discrete state equation becomes (derived in Appendix A, B)

$$\underline{x}_{i+1} = \underline{\phi}_i \underline{x}_i + \underline{D}_i \underline{u}_i \qquad (7)$$

$$\underline{y}_i = \underline{C}\ \underline{x}_i$$

where

$$\underline{x}_i = \underline{x}(t_i)$$

$$\underline{\phi}_i = \underline{\phi}(T_i) = e^{\underline{A}T_i}$$

$$\underline{D}_i = \underline{D}(T_i) = (\underline{D}_{0i}, \ldots, \underline{D}_{ki})$$

and

$$\underline{D}_{ki} = \underline{D}_{ki}(T_i) = \int_0^{T_i} (T_i - t)^k \, e^{\underline{A}t}\underline{B} \, dt \qquad k = 0,1,2$$

The cost functional in discrete form is

$$S = J_0 + \frac{1}{2} \underline{x}_N'\underline{F} \, \underline{x}_N - \underline{h}_N\underline{x}_N + \frac{1}{2} \sum_{i=0}^{N-1} (\underline{x}_i'\underline{Q}_i\underline{x}_i + 2\underline{x}_i'M_iu_i + \underline{u}_i'\underline{R}_i\underline{u}_i$$

$$- 2\underline{h}_i\underline{x}_i - 2\underline{q}_i\underline{u}_i) + \sum_{i=0}^{N-1} \alpha e^{-\beta T_i} \qquad (8)$$

where

$$t_i = \sum_{j=0}^{i-1} T_j$$

$$\hat{\underline{Q}} = \underline{C}'\underline{Q} \, \underline{C}$$

$$\hat{\underline{F}} = \underline{C}'\underline{F} \, \underline{C}$$

$$\underline{h}_N = \underline{z}'(t_N)\underline{F} \, \underline{C}$$

$$\hat{\underline{R}}(t) = \begin{bmatrix} \underline{R} \; . \; . \; . \; . \; . \; . \; . & \underline{R}t^{2k-3} & \underline{R}t^{2k-2} \\ . & . & \\ . & . & \underline{R}t^{2k-1} \\ . & . & \\ . \; . \; . \; . \; . \; . \; . \; . & \underline{R}t^{2k-1} & \underline{R}t^{2k} \end{bmatrix}$$

$$J_0 = \frac{1}{2}[\underline{z}'(t_N)\underline{F} \, \underline{z}(t_N) + \int_{t_0}^{t_N} \underline{z}'(t)\underline{Q} \, \underline{z}(t)dt]$$

$$\underline{Q}_i = \underline{Q}(T_i) = \int_0^{T_i} e^{\underline{A}'t} \, \hat{\underline{Q}} \, e^{\underline{A}t} \, dt$$

$$\underline{M}_i = \underline{M}(T_i) = \int_0^{T_i} e^{\underline{A}'t}\hat{\underline{Q}}\underline{D}(t)dt$$

$$\underline{R}_i = \underline{R}(T_i) = \int_0^{T_i} [\hat{\underline{R}}(t) + \underline{D}'(t)\hat{\underline{Q}}\ \underline{D}(t)]dt$$

$$\underline{h}_i = \underline{h}(T_i) = \int_0^{T_i} \underline{z}'(t_i + t)\underline{Q}\ \underline{C}\ e^{\underline{A}t}\ dt$$

$$\underline{g}_i = \underline{g}(T_i) = \int_0^{T_i} \underline{z}'(t_i + t)\underline{Q}\ \underline{C}\ \underline{D}(t)dt$$

$$\underline{D}(t) = [\underline{D}_0(t),\ldots,\underline{D}_k(t)]$$

$$\underline{D}_k(t) = \int_0^t (t - x)^k\ e^{\underline{A}x}\ \underline{B}\ dx$$

$$\sum_{j=0}^{N-1} T_j = t_N \quad \text{fixed}$$

Even though $\underline{Q}$ and $\underline{R}$ are constant, $\underline{Q}_i$, $\underline{M}_i$, $\underline{R}_i$ are in general, time varying. $\underline{\phi}_i$ is nonsingular because it s a fundamental matrix [29]. $\underline{Q}_i$ ($\underline{R}_i$) is positive semidefinite (definite) symmetric since $\underline{Q}$ ($\underline{R}$) is positive semidefinite (definite) symmetric.

The discrete time problem becomes:

Given the sampled-data system (7) with specified initial condition, determine the control and sampling intervals sequence $\{\underline{u}_i\}_{i=0}^{N-1}$, $\underline{T}' = (T_0,T_1,\ldots,T_{N-1})$ and N that minimizes the cost functional (8) subject to the constraints (5a), (5b), (5c).

The following theorems establish both the existence of an optimal solution and the structure for the computational algorithm.

For any specified $\underline{T}$ and N satisfying the sampling constraints (5a), (5b), (5c), the existence of an optimal control and an optimal state sequence are guaranteed if it satisfies the following Kuhn-Tucker conditions.

THEOREM 1 (Kuhn-Tucker Necessary Condition)

If the sampling constraint $\underline{T} \in [\underline{a}, \underline{b}]$ holds, then an optimal solution $\underline{u}_i(\underline{T}) = \underline{u}_i$ and $\underline{x}_{i+1}(\underline{T}) = \underline{x}_{i+1}$ exist if and only if there exists vectors $\underline{p}_i$ such that

$$\underline{x}_0 = \underline{\xi}$$

$$\underline{x}_{i+1} = \underline{\phi}_i \underline{x}_i + \underline{D}_i \underline{u}_i$$

$$\underline{p}_N = \underline{\hat{F}}\, \underline{x}_N - \underline{h}_N$$

$$\underline{p}_i = \underline{Q}_i \underline{x}_i + \underline{M}_i \underline{u}_i + \underline{\phi}_i' \underline{p}_{i+1} - \underline{h}_i'$$

$$\underline{R}_i \underline{u}_i + \underline{M}_i' \underline{x}_i + \underline{D}_i' \underline{p}_{i+1} - \underline{g}_i' = \underline{0}$$

for

$$i = 0,1,\ldots,N-1$$

Proof

The Kuhn-Tucker necessary condition for the quadratic programming problem is stated in Appendix C and the above conditions are established in Appendix D.

THEOREM 2

For each $\underline{T}$ satisfying $\underline{T} \in [\underline{a}, \underline{b}]$, there exists an unique control law and trajectory sequence. The control law is

$$\underline{u}_i = -(\underline{R}_i^{-1} \underline{M}_i' + \underline{S}_i^{-1} \underline{D}_i' \underline{K}_{i+1} \underline{\Theta}_i) \underline{x}_i + \underline{S}_i^{-1} (\underline{g}_i' - \underline{D}_i' \underline{k}_{i+1}) \qquad (9)$$

the cost functional is $S(\underline{T},N,k) = J(\underline{T},N,k) + C(\underline{T},N,k)$ where

$$J(\underline{T},N,k) = J_0 + \frac{1}{2}\,\underline{x}_0'\underline{K}_o\underline{x}_0 + \underline{k}_0'\underline{x}_0$$

$$- \frac{1}{2}\sum_{i=0}^{N-1} (\underline{g}_i' - \underline{D}_i'\underline{k}_{i+1})'\underline{S}_i^{-1}(\underline{g}_i' - \underline{D}_i'\underline{k}_{i+1}) \qquad (10)$$

$$C(\underline{T},N,k) = \sum_{i=0}^{N-1} \alpha e^{-\beta T_i}$$

$$\underline{S}_i = (\underline{R}_i + \underline{D}_i'\underline{K}_{i+1}\underline{D}_i)$$

$$\underline{\Theta}_i = \underline{\phi}_i - \underline{D}_i\underline{R}_i^{-1}\underline{M}_i'$$

$$\underline{G}_i' = \underline{M}_i\underline{R}_i^{-1} + \underline{\Theta}_i'\underline{K}_{i+1}\underline{D}_i\underline{S}_i^{-1} \qquad (11)$$

and $\underline{K}_i$, $\underline{k}_i$ satisfy

$$\underline{K}_i = (\underline{Q}_i - \underline{M}_i\underline{R}_i^{-1}\underline{M}_i') + \underline{\Theta}_i'\underline{K}_{i+1}[\underline{I} - \underline{D}_i\underline{S}_i^{-1}\underline{D}_i'\underline{K}_{i+1}]\,\underline{\Theta}_i \qquad (12)$$

$$\underline{k}_i = \underline{\Theta}_i'(\underline{I} - \underline{D}_i\underline{S}_i^{-1}\underline{D}_i'\underline{K}_{i+1})'\underline{k}_{i+1} + \underline{G}_i'\underline{g}_i' - \underline{h}_i' \qquad (13)$$

with terminal conditions

$$\underline{K}_N = \underline{C}'\underline{F}\,\underline{C} \quad \text{and} \quad \underline{k}_N' = -\underline{z}'(t_N)\underline{F}\,\underline{C} \qquad (14)$$

Proof

The existence and uniqueness are proved in Appendix G and the derivation of (9), (10), (11), (12), (13), (14) are in Appendix E and F.

THEOREM 3

If the sampling constraints are satisfied, then there exists an optimal sampling intervals sequence $\underline{T}^*$.

Proof

There exists an unique optimal control and trajectory sequence $\{\underline{u}_i(\underline{T})\}_{i=0}^{N-1}$, $\{x_{i+1}(\underline{T})\}_{i=0}^{N-1}$, for each $\underline{T}$ satisfying (5a), (5b), (5c). The cost functional is obviously a continuous function of $\underline{T}$ since $\underline{\phi}_i$, $\underline{D}_i$, $\underline{Q}_i$, $\underline{M}_i$, $\underline{R}_i$, $\underline{g}_i$, $\underline{h}_i$, $\underline{K}_i$, $\underline{k}_i$ are continuous matrix functions of $\underline{T}$. Therefore, the cost functional $S(\underline{T})$ is continuous on a compact set and an optimal solution $\underline{T}^*$ for this derived problem exists. Q.E.D.

Thus, there exists a solution

$$\{\underline{u}_i(\underline{T}^*)\}_{i=0}^{N-1} , \{\underline{x}_i(\underline{T}^*)\}_{i=0}^{N-1} \quad \text{and} \quad \underline{T}^*$$

for the optimal linear tracking problem with constrained sampling times.

This control law is open loop and pre-programmed since the derived cost function and thus $\underline{T}^*$ depends on initial state $\underline{x}_0$ and the entire trajectory $\underline{z}(t)$, $t \epsilon [t_0, t_N]$.

This theorem shows that the solution $\underline{T}^*$ to the derived optimization problem (minimize $S(\underline{T},N,k)$ over the set $[\underline{a},\underline{b}]$) can be used to determine the optimal control and trajectory from the state equation (7) and the control law (9) after matrices $\underline{K}_i(\underline{T}^*)$ and $G_i(\underline{T}^*)$ have been computed for $i = N, N - 1, \ldots, 1,0$ from (11) to (14) assuming N and k are specified.

CHAPTER IV

COMPUTATIONAL ALGORITHM

The derived minimization problem

$$\min_{N,\underline{T}\in\Omega} S(\underline{T},N,k)$$

$$\Omega = \left\{ N,\underline{T};\ \text{satisfying} \quad \begin{array}{lr} 0 < T_{imin} \leq T_i \leq T_{imax} & \text{(5a)} \\ N_{min} \leq N \leq N_{max} & \text{(5b)} \\ \underline{g}(T_0,T_1,\ldots,T_{N-1}) = \underline{0}_v & \text{(5c)} \end{array} \right.$$

can be solved for any given particular values of both N and k using a sequential unconstrained minimization technique (SUMT). The convergence of this algorithm was proved in [34].

Another level of optimization can be performed to determine both the optimal number of sampling times N^* and the optimal order of control approximation k^*. This optimization could be performed by searching the optimal system performance $S(\underline{T}^*,N,k)$, which results from solving this derived minimization problem over each N and k satisfying

$$N_{min} \leq N \leq N_{max}$$

$$k = 0,1,2$$

This level of optimization over N and k can be performed using an integer programming algorithm. This generalized algorithm now has three levels of optimization

(1) determine $\{\underline{u}^*_i(\underline{T},N,k)\}_{i=0}^{N-1}$ by solving the Kuhn-Tucker necessary conditions for any $(\underline{T},N,k)$ and determine the derived performance index $S(\underline{T},N,k)$.

(2) determine the optimal sampling intervals sequence $\underline{T}^*(N,k)$ for any N and K using the SUMT algorithm and determine the performance $S(\underline{T}^*,N,k)$.

(3) determine the optimal number of sampling intervals N^* and k^* that minimize $S(\underline{T}^*,N,k)$ using an integer programming algorithm and determine the optimal sampled-data control law specified by

$$\{\underline{u}^*_i(\underline{T}^*,N^*,k^*)\}_{i=0}^{N-1} \;,\; \underline{T}^*(N^*,k^*),\; N^*,\; k^*$$

and the performance $S(\underline{T}^*,N^*,k^*)$.

Although such an algorithm could be implemented, no effort was made to optimize over either N or k in this research. However, extensive evaluation of system performance $S(\underline{T}^*,N,k)$ is performed for different values of N and k.

This generalized algorithm has several advantages over other possible procedures:

(1) the optimization over integers and real variables are separated.

(2) the search dimension on the real variables is reduced from $N(n + kr + 1)$ to N and the Nn equality constraints (7) are eliminated by solving for $\underline{u}^*_i(\underline{T},N,k)$ using the Kuhn-Tucker necessary conditions.

The SUMT algorithm, used to determine $\underline{T}^*$ in this generalized algorithm, has never been tested for the case where an equality constraint

$$\underline{g}(T_0, T_1, \ldots, T_{N-1}) = \underline{0}_v$$

was imposed on the sampling intervals. The SUMT algorithm can be used in this case if an appropriate penalty function is used. However, convergence may be slow and the cost of computation may be high. The form of the equality constraints imposed are quite simple and therefore the v equations can be uniquely solved for v variables as follows. The v variables

$$\underline{\tilde{T}}' = (T_{i_1}, T_{i_2}, \ldots, T_{i_v})$$

can be expressed in terms of the $N-v$ variables

$$\underline{\hat{T}}' = (T_{i_{v+1}}, T_{i_{v+2}}, \ldots, T_{i_N})$$

such that

$$\underline{\tilde{T}} = \underline{f}(\underline{\hat{T}}) .$$

The derived performance index becomes

$$S(\underline{\hat{T}},N,K) = S(\underline{\hat{T}}, \underline{f}(\underline{\hat{T}}),N,k)$$

where variables T_{i_ℓ} are considered to belong to a new set $[a_{i_\ell}, b_{i_\ell}]$ which guarantees that

$$T_{min} \leq T_i \leq T_{max}$$

is satisfied for $i = 0,1,\ldots,N-1$.

The vector $\underline{\hat{T}}$ can now be used to represent the $N-1$ free sampling intervals

$$\underline{\hat{T}}' = [T_0, T_1, \ldots, T_{N-2}]$$

if the sampling constraint specifies the terminal time

$$\underline{g}(T_0,\ldots,T_{N-1}) = \sum_{i=0}^{N-1} T_i - (t_f - t_0) = 0 \ .$$

This vector $\underline{\hat{T}}$ can also represent the sampling period T

$$\underline{\hat{T}} = T = \frac{t_f - t_0}{N}$$

if the sampling constraint specifies periodic sampling

$$\underline{g}(T_0,\ldots,T_{N-1}) = \begin{cases} T_i - \dfrac{t_f - t_0}{N} = 0 \\ i = 0,1,\ldots,N-2 \end{cases}$$

Thus the vector $\underline{\hat{T}}$ can represent either the first $N-1$ sampling intervals if an aperiodic sampling criterion is used or the sampling period if a periodic sampling criterion is used. This vector is defined in order to simplify the notation in Chapter V. The proper interpretation of $\underline{\hat{T}}$ is always clear from the context where it is used.

The SUMT algorithm can solve this reduced derived minimization problem with less computational effort because the search dimension is reduced from N to N-v. A penalty function $P(\underline{T})$ is adjoined to form an augmented cost functional

$$L(\hat{\underline{T}},N,K,V) = S(\hat{\underline{T}},N,k) + VP(\hat{\underline{T}})$$

$$P(\hat{\underline{T}}) = \sum_{\ell=v+1}^{N} [\min(0,b_{i\ell}-T_{i\ell})](b_{i\ell}-T_{i\ell})$$

$$+ [\min(0,T_{i\ell}-\alpha_{i\ell})](T_{i\ell}-\alpha_{i\ell})$$

which incorporates the sampling interval constraint

$$\hat{\underline{T}} \in [\underline{a},\ \underline{b}]$$

where

$$\underline{a}' = [a_{i_{v+1}}, a_{i_{v+2}}, \ldots, a_{i_N}]$$

$$\underline{b}' = [b_{i_{v+1}}, b_{i_{v+2}}, \ldots, b_{i_N}]$$

Since the penalty for violating the latter is proportional to V (V > 0), the minimization of $L(\hat{\underline{T}},N,k,V)$ for monotonically increasing sequence $\{V_\rho\}$ results in a sequence $\{\underline{T}\}_{\rho=1}^{\infty}$ that converges [9] to the optimal $\hat{\underline{T}}^*$.

The computational effort required by the SUMT algorithm developed by Schlueter [9] is quite large because the gradient of the performance index must be computed every time the performance index is evaluated. Therefore, a non-gradient algorithm and a gradient algorithm are used to solve the same problem in order to determine whether the non-gradient algorithm will require less computational effort.

The Fletcher-Powell gradient search algorithm used by Schlueter to determine $\underline{T}^*$ requires $(N + 1)$ evaluations of the performance index at every iteration to compute the gradient and evaluate the performance index. The Powell algorithm requires only one evaluation of the performance index because the gradient is not required. The following example problem was solved using both a non-gradient Powell search algorithm and the Fletcher-Powell algorithm and the numbers of evaluations of the performance index are compared.

<u>EXAMPLE</u> 4-1.

Given the system

$$\dot{x}(t) = u(t) \qquad\qquad x(0) = 1$$

with the cost functional

$$J = x^2(t_f) + \frac{1}{2}\int_0^{t_f} (x^2(t) + u^2(t))dt$$

where the control satisfies the constraints

$$u(t) = u_i \qquad\qquad t \in [t_i, t_{i+1})$$

$$0 < t_{i+1} - t_i < \infty$$

for $i = 0,1$ and $t_f = t_2$ is free .

The Powell and Fletcher-Powell algorithms both converge as shown in Table 4-1.

TABLE 4-1. Convergence of Powell and Fletcher-Powell Algorithm

	POWELL	FLETCHER-POWELL
1	0.54064	0.54064
6	0.53340	0.52309
11	0.52799	0.52050
14	0.52174	0.52019 (converge)
20	0.52020 (converge)	

The results indicate the Powell algorithm needs a few more iterations for convergence, but the computational effort is much less since the Powell algorithm requires only 20 evaluations of the performance index while the Fletcher-Powell algorithm requires 42 evaluations to obtain both the performance index and the gradient at each iteration. Therefore, the Powell algorithm is used in all the computational work which follows.

The Powell computational algorithm was first tested on problems where the cost of implementation was neglected as in Example 4 -1. The computational algorithms often did not converge or converged to local minima rather than global minima. This lack of convergence is not always caused by the round off error or by the failure of the computational algorithms to converge, but can be attributed to the fact that the sampling constraints which require the sampling intervals to be positive were never imposed in the optimization algorithm. The following theorem, which states that the sampling intervals will tend to approach zero if the cost of implementation is zero and the number of sampling times is unbounded, provides an indication that some difficulty with convergence might be observed if a cost of implementation is omitted.

THEOREM 4-1

The optimal sampled-data control for the regulator problem is the optimal continuous-time control if the cost of implementation is negligible and the number of samples t is unbounded.

Proof

The optimal sampled-data solution to the regulator problem for every $\underline{T}$ and N has been shown to be an optimal approximation to the optimal continuous-time solution for the appropriate Hilbert space norm [57]. Since a sampled-data control

$$\underline{u}(t) = \underline{u}(t_i) \qquad\qquad t \in [t_i,\ t_{i+1})$$

is a restricted class of controls, the control performance for the optimal continuous-time control is less than or equal to the performance of an optimal sampled-data control for any $\underline{T}$ and N. However, the optimal periodic sampled-data control with period

$$T = \frac{t_f - t_0}{N}$$

has been shown to converge to the optimal continuous-time control [1] as N approaches infinity. Therefore, since the optimal continuous-time control has the minimum value of control performance of all optimal sampled-data control laws specified by $\underline{T}$ and N, the optimal continuous-time control is the optimal sampled-data control for the special case where the cost of implementation is negligible and the number of sampling times is unbounded.

Q.E.D.

If a cost of implementation is included or if N_{max} is bounded, the continuous time control law will not be the optimal sampled-data control law. If N_{max} is unbounded but the cost of implementation is omitted some elements of the optimal sampling intervals sequences will always be very small. Thus, the computational algorithm, in searching for the optimal sampling intervals sequence, would often select a negative sampling interval which caused the algorithms to diverge. This convergence difficulty could be overcome by either adjoining a penalty term on the performance index which penalizes negative sampling intervals or by including the sampling constraints which restrict the sampling intervals to have non-negative values. The first approach is taken because there are penalties on performance in actual engineering design which prevent the sampling intervals from becoming too small. This penalty, which is the economic cost for implementing and operating a sampled-data control law, has been overlooked in previous work on optimal sampling criteria [9 - 11].

The design of sampling criteria always includes a tradeoff between control performance and economic cost which usually occurs after the continuous time control law is designed [2, 16]. The inclusion of economic cost permits the design of a sampling criterion and control law together in one step using a single performance index.

The economic cost should represent the cost for implementing and operating the hardware to

1) measure and collect the data about the states of the system.
2) transmit this data to the controller.
3) estimate the state and compute the control.
4) transmit the control back to the system.
5) actuate the control.

The cost of implementation will be negligible if the cost of computing, storing, transmitting data and implementing a continuous-time control law is low. In this case, the continuous-time system is optimal. However, in most cases the cost of implementing a continuous-time control will be high and thus the cost of implementation (COI) must be included.

From this perspective, the optimal continuous-time control is a special case of the optimal sampled-data control problem. Thus, the sampled-data control problem formulation is more general and should be used as the basis for design. The formulation of the continuous-time problem implicitly assumes that the cost of implementation is negligible. The omission of a cost of implementation term in the performance index when it is not negligible is just as severe in terms of overall system performance as omitting any other significant term in the performance index.

The inclusion of a cost of implementation is an important contribution to the art of optimal design since many optimal control laws are often criticized for being overly costly or impractical.

Thus, a cost of implementation term in the performance index may prove to be an effective approach toward making optimal design techniques more consistent with present engineering practice.

The literature on a proper form for a COI for control laws is sparse. Research is presently under way [51] to develop models for implementation cost. The best intuitive models for COI presently available were found in the literature on adaptive sampling and optimal aperiodic sampled-data control law [13, 56]. This form for the COI is adopted for this study.

The effects of including a COI are illustrated in the following example.

EXAMPLE 4-2

Consider the linear system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 15 \end{bmatrix} u(t)$$

with cost functional

$$J = \frac{1}{2} \underline{x}'(t_f) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x}(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (\underline{x}'(t) \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \underline{x}(t)$$

$$+ u^2(t))dt \ .$$

The initial time $t_0 = 0$ and the terminal time $t_f = 20$ are specified and the initial state is

$$\underline{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The control is piecewise constant and changes only at the sampling time t_1 such that

$$u(t) = u(t_i) = u_i \qquad t \in [t_i, t_{i+1}) \quad \text{for} \quad i = 0,1$$

Augmenting the above performance index with a cost of implementation

$$C(\underline{T},N) = COI = \sum_{i=0}^{1} 0.1e^{-10T_i}$$

has a dramatic effect on the convergence of the Powell algorithm.

The computational results, shown in Table 4-2, indicates that the inclusion of a COI term not only causes the algorithm to converge when it did not without COI but also suggests that optimal solution with COI may be global.

The lack of convergence and divergence problems exhibited in the computational results, obtained when a cost of implementation was omitted, indicate

(1) the changes in control performance for changes in the sampling times is often small near the optimal.

(2) there can be several local minima for the derived control performance $J(\underline{T},N)$.

(3) the contraints that require the sampling intervals must be included if the cost of implementation is omitted in order to prevent divergence of the computational algorithm.

TABLE 4-2. Computational Results for the Cases with and without COI in using a Powell Algorithm

Table 4-2A (without COI)

Iteration(ρ)	$\underline{T}_\rho$	$J(\underline{T}_\rho)$
1	(6.67, 6.67, 6.66)	2.4254
5	(2.33, 5.34, 12.33)	1.2563
10	(0.67, 4.23, 15.1)	0.8456
11	(-1.24, 6.53, 14.71)	diverge

Table 4-2B (with COI)

Iteration(ρ)	$\underline{T}_\rho$	$J(\underline{T}_\rho)$
1	(6.67, 6.67, 6.66)	2.4255
11	(1.334, 5.336, 13.33)	0.93807
21	(0.7801, 0.7801, 18.44)	0.77126
31	(0.0606, 0.7006, 19.24)	1.0512
41	(0.4592, 0.8437, 18.70)	0.7188
51	(0.2068, 1.4363, 18.36)	0.67649
61	(0.2506, 1.5365, 18.21)	0.65686
71	(0.2301, 2.5941, 17.17)	0.64038
81	(0.231, 2.4757, 17.29)	0.64077
91	(0.28748, 2.6862, 17.026)	0.63469 (converge)

TABLE 4-2 (continued)

Table 4-2C Performance with COI and an aperiodic initial

Iteration (ρ)	$\underline{T}_\rho$	$J(\underline{T}_\rho)$
1	(0.05, 0.5, 19.45)	1.2143
11	(0.6401, 2.1, 17.29)	0.69897
21	(0.25121, 1.7759, 17.97)	0.64734
31	(0.29498, 1.896, 17.81)	0.64088
41	(0.42699, 1.7907, 17.782)	0.65835
46	(0.28754, 2.6831, 17.03)	0.63469 (converge)

The high rate of convergence and apparent global convergence of the algorithm, when a cost of implementation is included, indicates the inclusion of a cost of implementation

(1) makes for a better formulation of the control design problem since the minimal value of the performance is more clearly defined.

(2) prevents the algorithm from diverging by penalyzing small positive or negative values for the sampling intervals. The algorithm no longer selects negative values for the sampling intervals which previously had caused it to diverge.

Although Powell algorithm works poorly with more than ten independent variables, it is quite satisfactory with the three examples computed , since the cost functional for the optimal sampling starts leveling off before N (number of free sampling intervals) reaches five.

The optimal sampling intervals sequence becomes periodic if the cost of implementation is much larger than the control performance cost so that

$$S(\underline{T},N,k) \cong COI = \sum_{i=0}^{N-1} \alpha e^{-\beta T_i}$$

This cost of implementation becomes large if the cost per sample α or the number of sampling times becomes large. A heuristic proof that optimal aperiodic sampling is periodic when the cost of implementation is very large is included below.

The optimal sampling interval sequence $\underline{T}^*$ can be obtained by solving the necessary conditions for the problem

$$\min_{\underline{T}}\{S(\underline{T},N,k) = \sum_{i=0}^{N-1} \alpha e^{-\beta T_i}\}$$

subject to the condition

$$\underline{g}(T_0,T_1,\ldots,T_{N-1}) = \sum_{i=0}^{N-1} T_i - (t_f - t_0) = 0 \ .$$

The necessary conditions become

$$\frac{\partial}{\partial T_i}\left[S(\underline{T},N,k) + \lambda\, \underline{g}(\underline{T})\right]\Big|_{\underline{T}^*} = -\alpha\beta e^{-\beta T_i^*} + \lambda = 0$$

$$i = 0,1,\ldots,N-1 \ .$$

These conditions can be solved to obtain

$$T_i^* = T = \frac{t_f - t_0}{N}$$

and thus the optimal aperiodic sampling criterion is periodic when the cost of implementation is high. This result fits intuition and provides justification for the particular form of the cost of implementation chosen.

CHAPTER V

COMPUTATIONAL RESULTS

5.1 Introduction

The performance of the optimal periodic and optimal aperiodic sampled-data control law will be compared in this chapter. Performance of a sampled-data control law can be measured in several ways. Control performance $J(\hat{\underline{T}},N,k)$ defined by (3b), is the performance of the control law in meeting its objectives and has been the standard measure of performance. If this measure of performance were used exclusively, the continuous-time control law $(\underline{T} = \underline{0}, N = \infty)$ would always be optimal as proved in Chapter IV.

A second performance measure, system performance $S(\hat{\underline{T}},N,k)$ defined by (3a), includes both the control performance $J(\hat{\underline{T}},N,k)$ and the cost of implementation $C(\hat{\underline{T}},N,k)$. This cost for implementation should include the hardware and software costs for measuring the outputs, transmitting this data from the plant to the computer, computing the control law and state estimates, transmitting the control back to the plant, and actuating the control commands.

These two performance measures can be used to compare the relative control performance and system performance for different sampling criteria $(\hat{\underline{T}},N)$ or different control approximations (k) specified by $(\hat{\underline{T}},N,k)$.

A third measure of performance is the sampling index $I(J_0,k,\underline{T})$ which is defined as the number of sampling intervals required for a particular sampling criterion $(N,\underline{T})$ and control law approximation (k) required to obtain a control performance value J_0. The sampling index can also be based on system performance rather than control performance. These three measures of performance will be used to

(1) compare the control performance, system performance and information required for differencecontrol approximations in section 5.2.

(2) compare the control performance, system performance and sampling efficiency of optimal aperiodic and periodic sampled-data control laws in section 5.3.

(3) compare the performance of an optimal control law which is sampled adaptively using different adaptive sampling schemes and the performance of the optimal aperiodic sampled-data control law, in section 5.4.

(4) compare the control performance and sampling efficiency for both an unstable and a stable system with ramp and parabolic desired trajectory in section 5.5.

This study is made to illustrate a design procedure which designs both the control law and sampling criterion together by performing the tradeoff between control performance and cost of implementation in a single step. This procedure is used to evaluate different order control approximations and compare optimal periodic and optimal aperiodic sampled-data control laws. This study is intended to provide the basis for understanding the design procedure, but is not intended as an indication of performance tradeoffs for any particular application.

The system chosen for investigation was selected because it has been used extensively [13-20] for the evaluation of sampling criteria in the literature on adaptive sampling. Therefore, it provides a basis for comparing periodic and adaptive sampling criteria on an optimal control law with the optimal aperiodic sampled-data control law. This system is also chosen because it is unstable without feedback and therefore provides a good basis for comparing performance of the optimal control law implemented with different sampling criteria.

This example problem will be used to compare the performances of different control approximations in section 5.2 and optimal aperiodic, periodic and adaptive optimal sampled-data control laws in sections 5.3 and 5.4.

EXAMPLE 1

Consider the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [10 \quad 100]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with cost functional

$$J = \frac{1}{2}(y(t_N) - z(t_N))^2 + \frac{1}{2}\int_{t_0}^{t_N} [(y(t) - z(t))^2 + .02\, u^2(t)]dt + \sum_{i=0}^{N-1} \alpha e^{-\beta T_i}$$

where t_0 is zero, $t_N = 20$, and the desired trajectory and initial conditions are given below

$$z(t) = 0 \qquad t \geq 0$$

$$\underline{x}(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Matrices $\underline{F}$ and $\underline{Q}$ are set as 1 while $\underline{R}$ is set as 0.02 since Athans [24] suggested that in order to obtain satisfactory tracking performance, the weighting coefficient on the error energy should be at least 50 times greater than that on the control energy.

α and β are chosen as 0.1 and 10 respectively because:

(1) small intervals below 0.1 second will be penalized heavily.

(2) it is common practice in the design of adaptive sampling criteria [13] that $\alpha\beta = 1$. Therefore, the same practice is used here.

The design objective is to

(1) keep the output as close as possible to the desired trajectory.

(2) minimise the control energy expenditure.

(3) minimize the information to be transmitted.

5.2 Comparison of Control Approximations

The control performance, system performance and information required to specified a control will be compared for an optimal zero, first, and second order control approximation. These comparisons will be performed for a control law with both periodic and aperiodic sampling .

The performance indices $J(\hat{\underline{T}},N,k)$, $S(\hat{\underline{T}},N,k)$ and $I(J_0,k,\hat{\underline{T}})$ do not compare the relative performance of the system with different order control approximations. Therefore, the performance of the zero, and the first order control approximation will be normalized by dividing this performance value with N samples, $(J(\hat{\underline{T}},N,k)$ for $k = 0,1)$ by the performance for the second order $(k = 2)$ control approximation with N samples. The normalization can be based on either the control performance measures

$$R_J(N,k) = \frac{J(\hat{\underline{T}}^*,N,k)}{J(\hat{\underline{T}}^*,N,2)}$$

or the system performance measure

$$R_S(N,k) = \frac{S(\hat{\underline{T}}^*,N,k)}{S(\hat{\underline{T}}^*,N,2)}$$

If the sampling criterion is periodic the optimal sampling sequence $\underline{T}^*$ is specified as

$$\hat{\underline{T}}^* = T = \frac{t_f - t_0}{N}$$

and thus the vector $\hat{\underline{T}}^*$ is identical in the numerator and denominator of these performance ratios. However, for optimal aperiodic sampling, the optimal sampling interval sequence

$$\hat{\underline{T}}^* = [T_0^*, T_1^*, \ldots, T_{N-2}^*]$$

are determined by optimizing $S(\underline{T},N,k)$ for some specified number of samples N and a particular control approximation k. Therefore, the optimal sampling sequence in the numerator and denominator of these performance ratios are not identical and depend on the order of the control approximation (k) specified.

The number of samples, $I(J_0,k,\hat{\underline{T}}^*)$, is not a proper measure of performance for comparing different control approximations. The number of data words required to transmit a particular control approximation is a more significant measure of control approximation performance. Thus, an information index is defined as

$$\hat{I}_J(J_0,k,\hat{\underline{T}}) = (k + 2)I_J(J_0,k,\hat{\underline{T}})$$

where $(k + 2)$ represents the number of parameters required to specify the control approximation and the length of each sampling interval. This information index is the number of data words required by a control approximation k to obtain a control performance value J_0. This information index can be based on either control performance or system performance. The normalization of this information index can be performed based on either the control performance

$$E_J(N,k) = \frac{\hat{I}_J(J_0,k,\hat{\underline{T}}^*)}{4N}$$

or the system performance

$$E_S(N,k) = \frac{\hat{I}_S(S_0,k,\hat{\underline{T}}^*)}{4N}$$

where $\hat{I}_J(\cdot)$ and $\hat{I}_S(\cdot)$ are the number of data words required to achieve the same value of control performance J_0 or system performance S_0 obtained by parabolic control approximation with $4N$ data words. This information ratio index thus compares the number of data words used by a step or ramp control approximation with the number used by a parabolic control approximation.

5-2-1 Periodic Sampling

The effects of the order of control approximation on control performance and information requirements can be observed for a periodic sampling criterion in Fig. 1 and 2. The two figures indicate the values of control performance over two separate ranges of N (i.e. 2 to 14 and 14 to 49) in order to provide better resolutions for comparison of the control approximations of order zero, one and two.

The value of the control performance decreases monotonically to the value which could be obtained with the optimal continuous time control law. The ratios of the control performance for step and ramp control approximation

$$R_J(N,k) = \frac{J(T,N,k)}{J(T,N,2)} \qquad k = 0,1$$

are shown in Table 5-1.

TABLE 5-1. Control Performance Ratio for Periodic Sampling

	N	2	4	8	14	19	24	20	34	39	44	49
STEP	$R_J(N,0)$	3.98	4.19	4.47	4.82	5.50	5.20	5.32	5.41	5.44	5.46	5.49
RAMP	$R_J(N,1)$	1.76	1.79	1.86	1.92	1.95	1.95	1.97	1.99	1.98	1.98	1.98

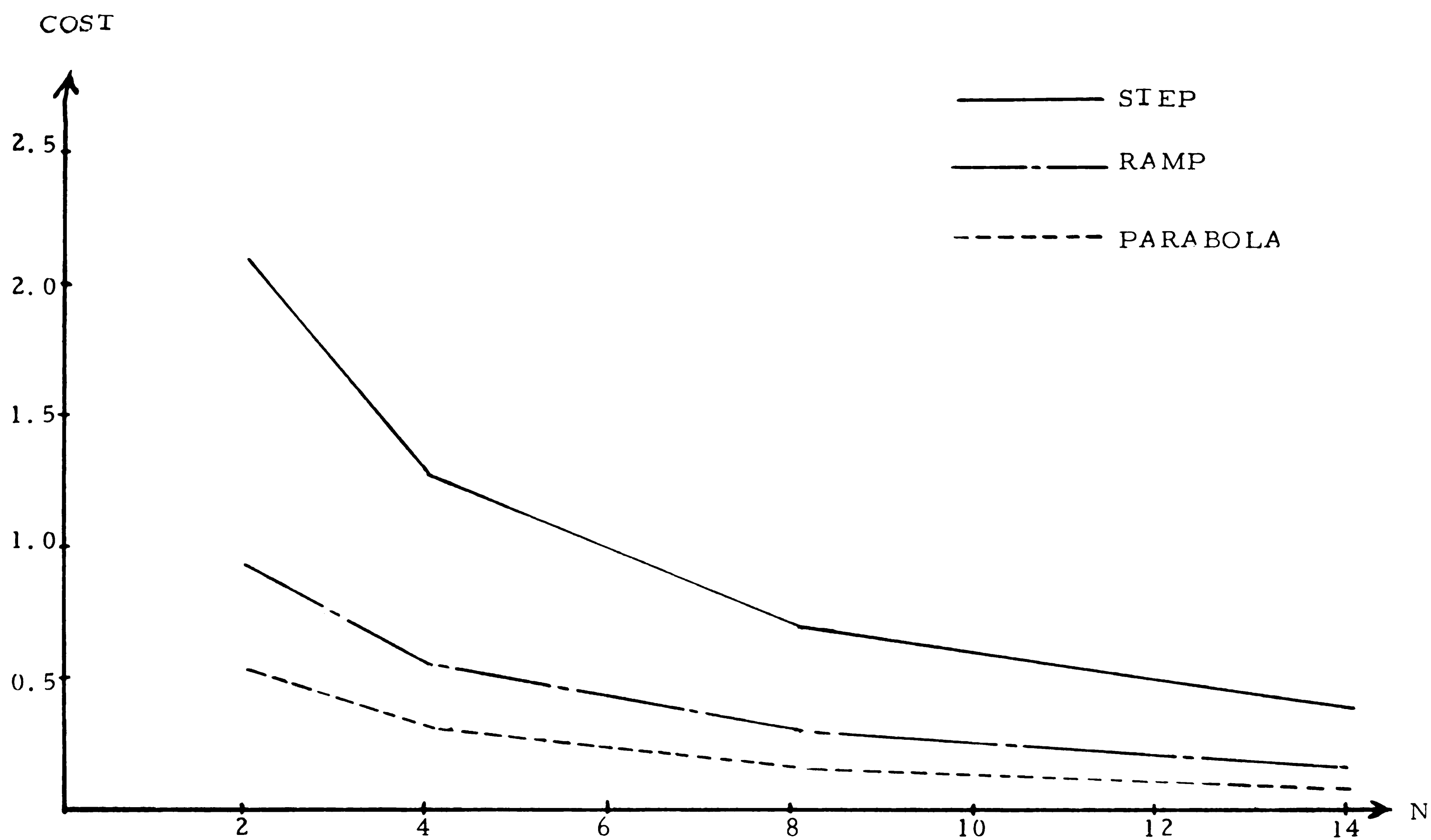


Fig. 1 Control Performance for Periodic Sampling

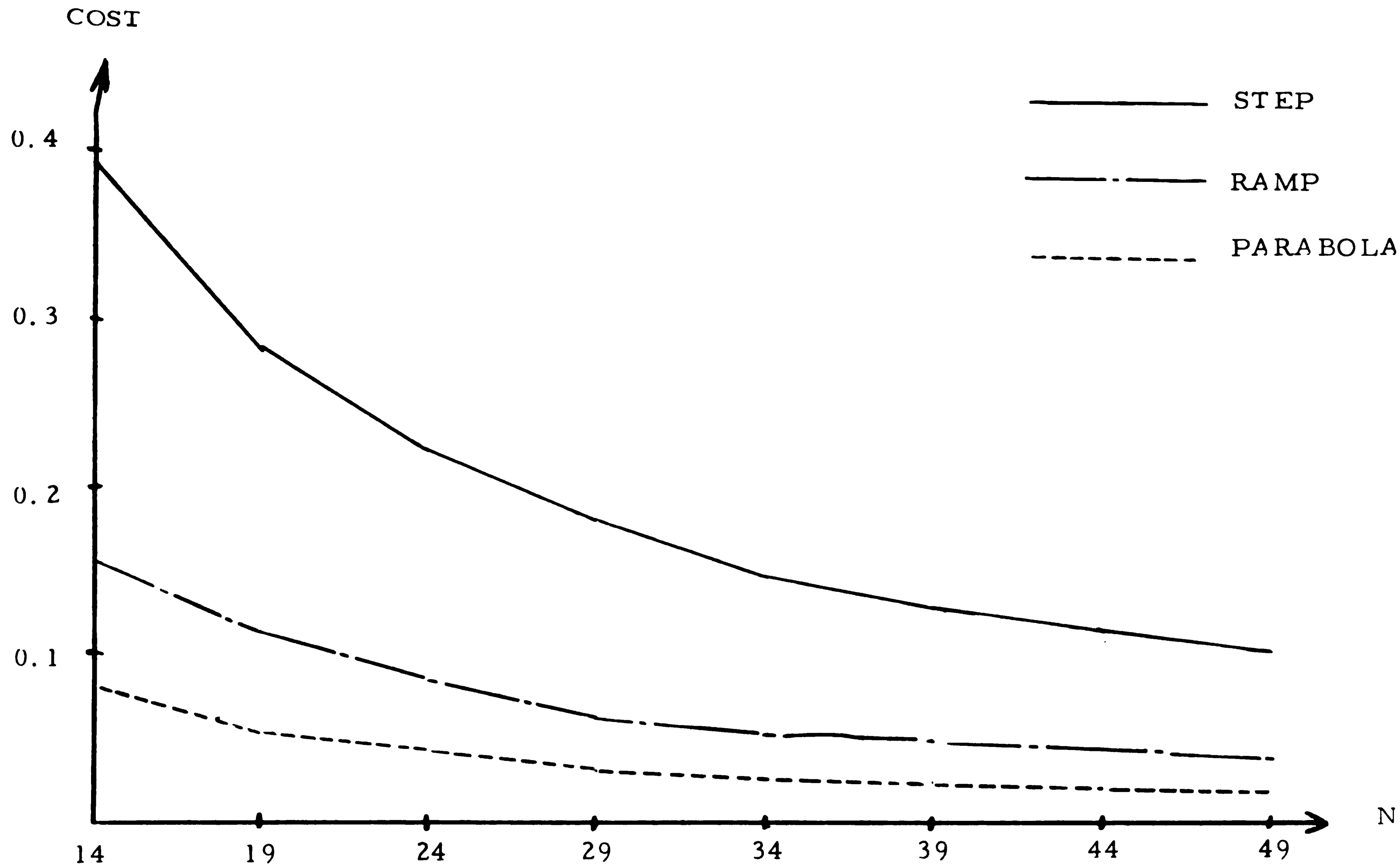


Fig. 2 Control Performance for Periodic Sampling

These results indicate a very significant improvement in control performance can be obtained by using higher order control approximation. The control performance ratio increases as N increases which indicates the performance improvement per additional sample is greater for step and ramp than for a parabolic control approximation.

The information ratio index for step and ramp control approximation based on control performance is shown in Table 5-2.

TABLE 5-2. Information Ratio for Periodic Sampling

	N	2	4	8	14	19	24	29
STEP	$E_J(N,0)$	2.75	2.35	2.2	*	*	*	*
RAMP	$E_J(N,1)$	1.65	1.43	1.31	1.32	1.31	1.30	1.29

* indicates that value could not be computed from data shown on figure

The information ratio index indicates the parabolic control approximation can achieve the same performance as a lower order control approximation with significant fewer specifying parameters. This information ratio index decreases as N increases indicating the importance of each sample or data word decreases faster for lower order control approximation than the higher order control approximation. The performance ratio and information ratio index indicate very significant improvement in both performance and information requirements are possible by using a higher order control approximation.

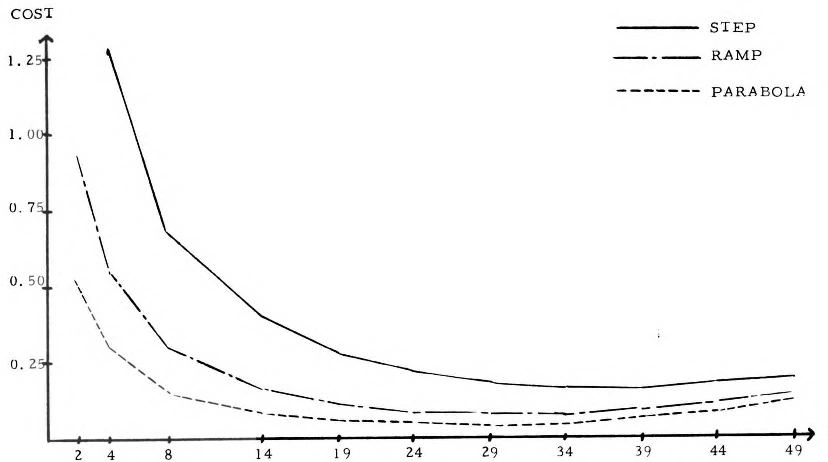


Fig. 3 System Performance for Periodic Sampling

The system performance for the three control approximations is plotted in Figure 3. These results indicate that the system performance decreases for small N and has approximately the same value as obtained for control performance. The system performance, however, levels off and increases as N becomes large. This increase in system performance can be attributed to an increase in the cost of implementation.

$$COI = N\alpha \exp(-\beta \frac{t_N - t_0}{N})$$

as N becomes large. Therefore, the system performance curves have a parabolic shape since the decrease in control performance as N increases (due to better approximation to the optimal continuous-time control) is eventually offset by the increase in the cost of implementation.

The minimum system performance index $S(N_k^*, k, \underline{T}^*)$ occurs at value N_k^* which decreases as k increases. This value N_k^* specifies the optimal sampling rate $N_k^*/(t_f - t_0)$ for that control approximation. Thus, the optimal sampling rate also decreases as the order of the control approximation increases. Since the cost of implementation is proportional to N_k^*, the parabolic control approximation not only improves the control performance and requires fewer control changes and less data to specify the control, but also has a lower cost of implementation using this particular form for the cost of implementation.

The increase in system performance as N becomes large is reflected in the performance ratio table $R_S(N,k)$ and the information ratio table $E_S(N,k)$ shown in Table 5-3 and 5-4 respectively. The performance ratio does not continue to increase as N gets large but levels off and begins to decrease. Thus, the control performance advantage of higher order of control approximations is no longer as significant for large N because the cost of implementation is so high. The information ratio $E_S(N,k)$ decreases more rapidly than $E_J(N,k)$ indicating the effect of implementation costs.

TABLE 5-3. System Performance Ratio for Periodic Sampling

	N	2	4	8	14	19	24	29	34	39	44	49
STEP	$R_S(N,0)$	3.98	4.19	4.47	4.82	5	5.10	4.88	4.69	3.07	2.24	1.73
RAMP	$R_S(N,1)$	1.76	1.79	1.86	1.92	1.95	1.95	1.89	1.69	1.46	1.27	1.16

TABLE 5-4 Information Ratio for Periodic Sampling

	N	2	4	8	14
STEP	$E_S(N,0)$	2.75	2.35	2.13	*
RAMP	$E_S(N,1)$	1.65	1.43	1.31	1.30

In summary, the value of the performance ratio and information ratio indicate that a significant improvement in performance can be obtained using higher order control approximations if each control approximation is constrained to have the same number of specifying parameters. Moreover, if the desired value of control performance is specified, the number of data-words of information required to specify a control with that performance value decreases significantly as the order of control approximation increases. The only disadvantage of using a higher order control approximation is the increased computational effort required to compute the optimal control when the effective dimension of the control vector $(k + 1)r$ increases with k. This cost for computation should be included in the cost of implementation if a proper measure of the performance of different order control approximations is to be made.

5-2-2 Optimal Aperiodic Sampling

The effects of the order of the control approximation on system performance and information requirement will now be investigated for the case of optimal aperiodic sampling. The control performance and system performance for all three control approximations ($k = 0,1,2$) are plotted in Figure 4. The J-axis is on a logarithmic scale in order to include a wide range of variation. The N-axis varies from 1 to 6 since all curves level off beyond $N = 3$ as also shown in Table 5-5.

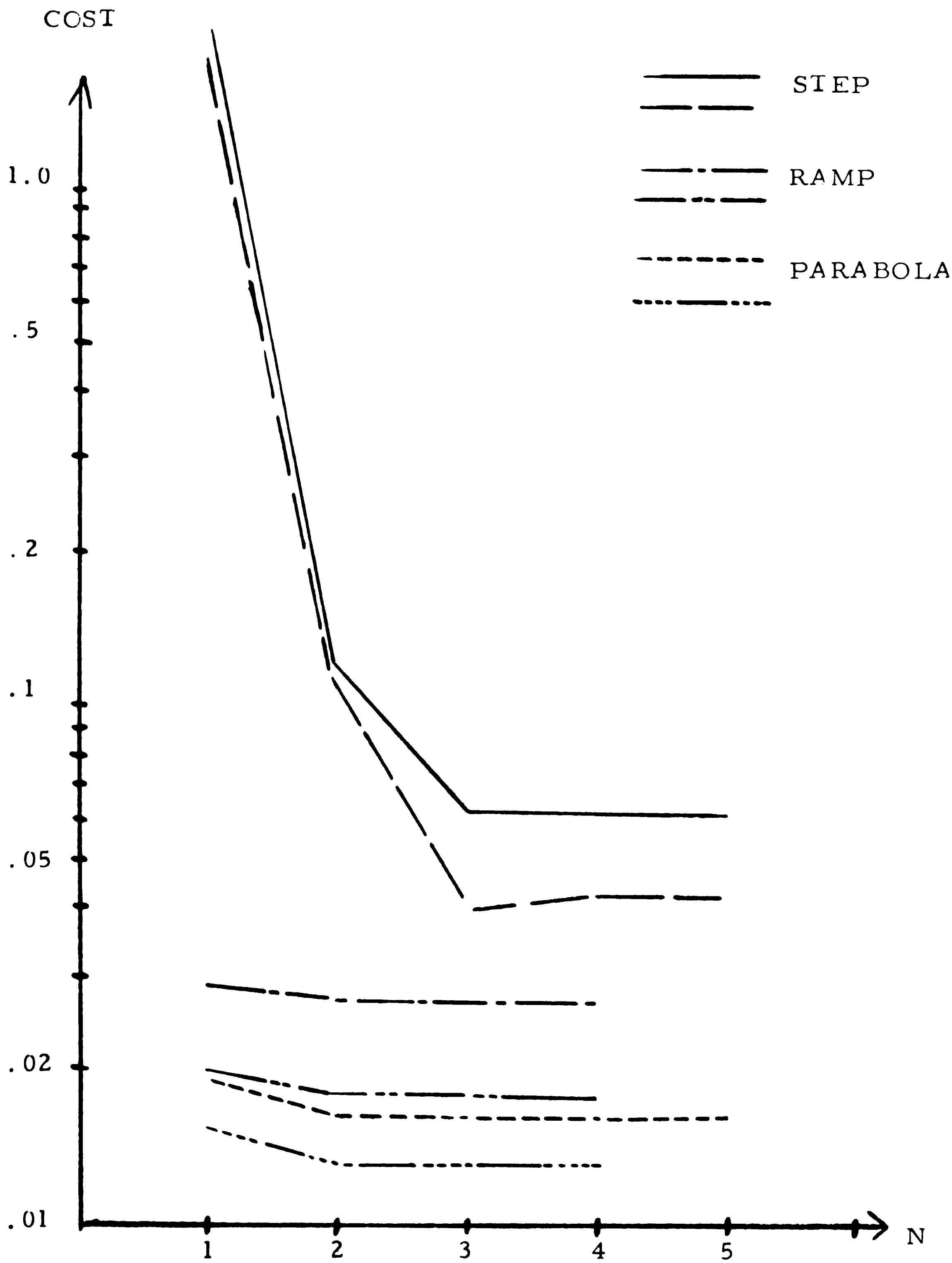


Fig. 4 System and Control Performance for Optimal Aperiodic Sampling

TABLE 5-5. $(J^*(N + 1) - J^*(N))/J^*(N)$

Control \ N	1	2	3	4	5
STEP	89%	47%	.31%	.13%	0%
RAMP	.94%	0%	0%	0%	0%
PARABOLA	7.86%	.11%	0%	0%	0%

TABLE 5-6. System Performance Ratio for Optimal Aperiodic Sampling

Control \ N	1	2	3	4	5
STEP $R_S(N,0)$	60.6	6.8	3.6	3.6	3.6
RAMP $R_S(N,1)$	1.47	4.3	2.3	2.3	2.3

The results indicate the system performance levels off very quickly as the number of sampling intervals increases. Moreover, the system performance level for the lower order control approximation will never even approach the level of system performance obtained for the parabolic control approximation because

(1) the control performance decreases so slowly as N increases as shown in Table 5-5.

(2) the cost of implementation increases as N increases.

The performance advantage of higher order control approximationsis also indicated by noting that the system performance ratio $R_S(N,k)$ is extremely large when N is small. These values for $R_S(N,k)$ for OAS are much larger than were obtained for PS. This dramatic improvement in performance obtained by selecting both the control sequence and the sampling intervals sequence combination optimally occurs because the control approximation parameters and performance index have been shown to depend on the sampling intervals sequence chosen and therefore the performance index is decreased significantly as the sampling intervals sequence approaches the optimal. The performance also decreases more rapidly with increase in the order of the control approximation as the sampling intervals sequence approaches the optimal.

The optimal number of sampling times N_k^* for OAS for zero, first and second order control approximations are 3, 2 and 2 respectively while the optimal number of sampling times for periodic sampling are close to 39, 34 and 29 respectively. Thus, the selection of both the sampling intervals and the control approximation coefficients over each interval also significantly reduces the number of sampling intervals and thus the information required to transmit the control. A more complete comparison of the performance of optimal aperiodic (OAS) and periodic (PS) sampled-data controls will be performed for zero, first and second order control approximations in the next section.

5.3 Comparison of Optimal Aperiodic and Periodic Sampling Criteria

A comparison of optimal periodic and aperiodic criteria will now be performed using control performance, system performance, and sampling index as performance indices. The optimal control performance and system performance for both periodic and optimal aperiodic sampling criteria are plotted together for zero, first and second order control approximations in Figures 5, 6 and 7 respectively. A logarithmic scale is used to cover the large range of performance index values.

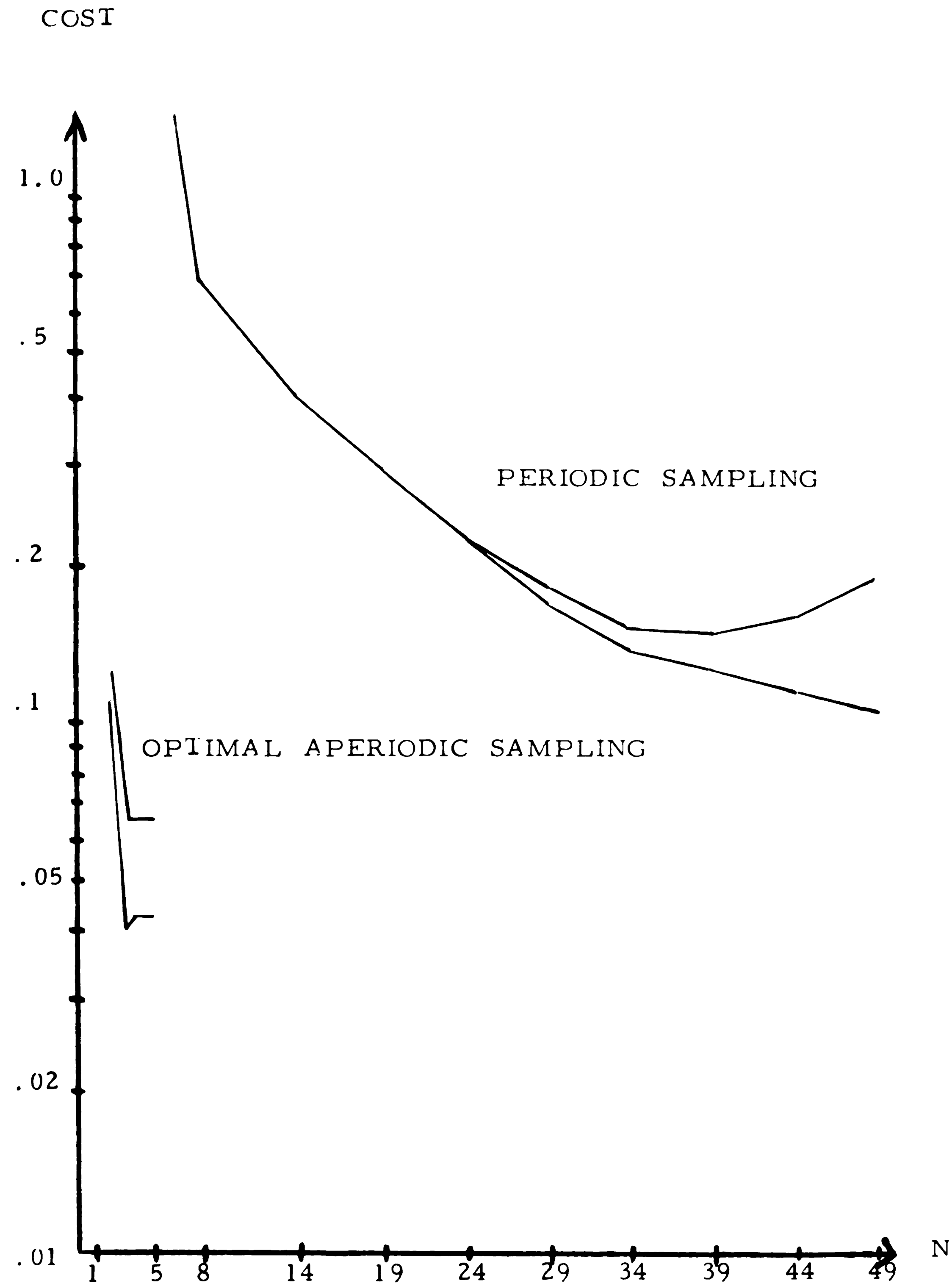


Fig. 5 System and Control Performance for PS and OAS with the Step Control Approximation

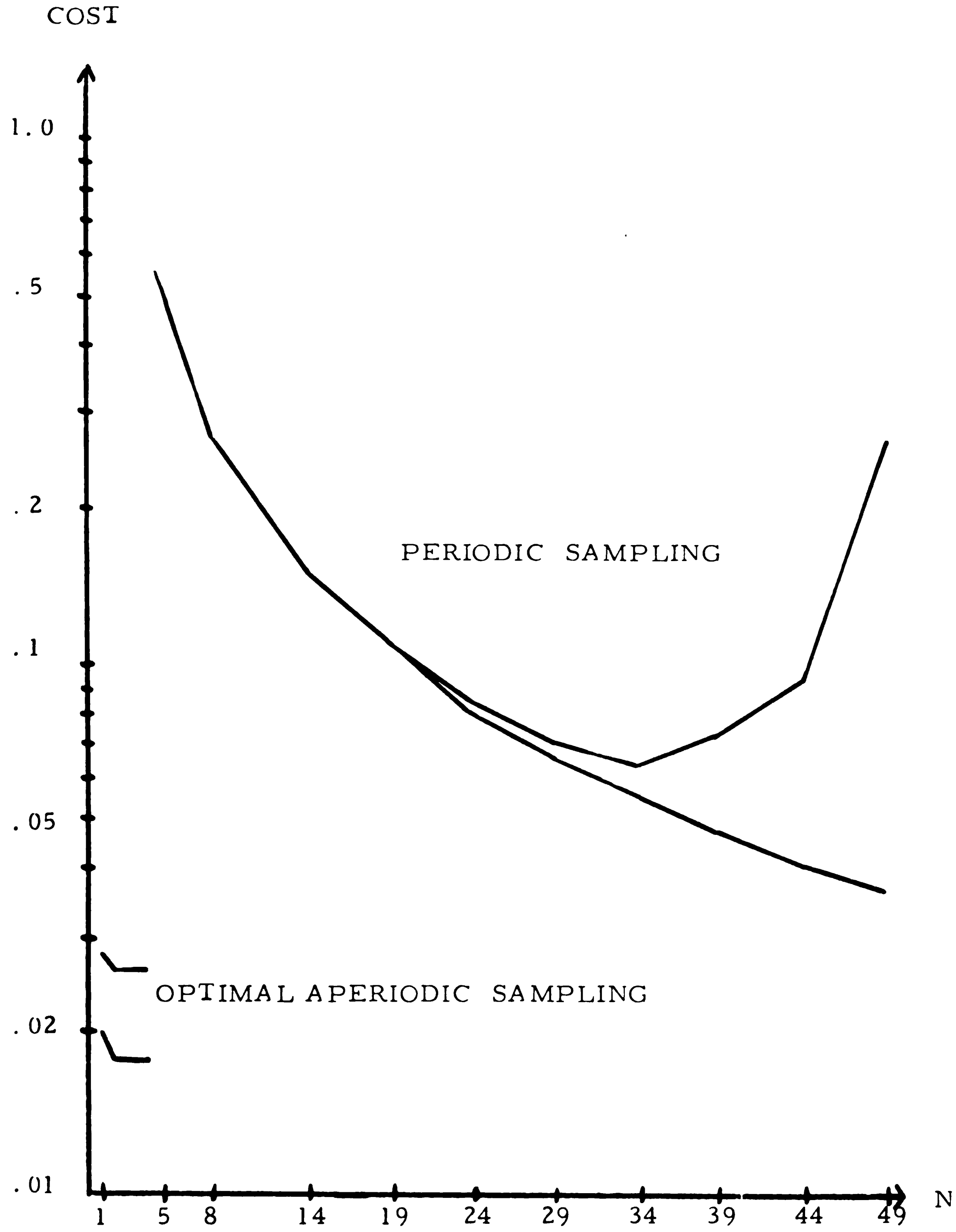


Fig. 6 System and Control Performance for PS and OAS with the Ramp Control Approximation

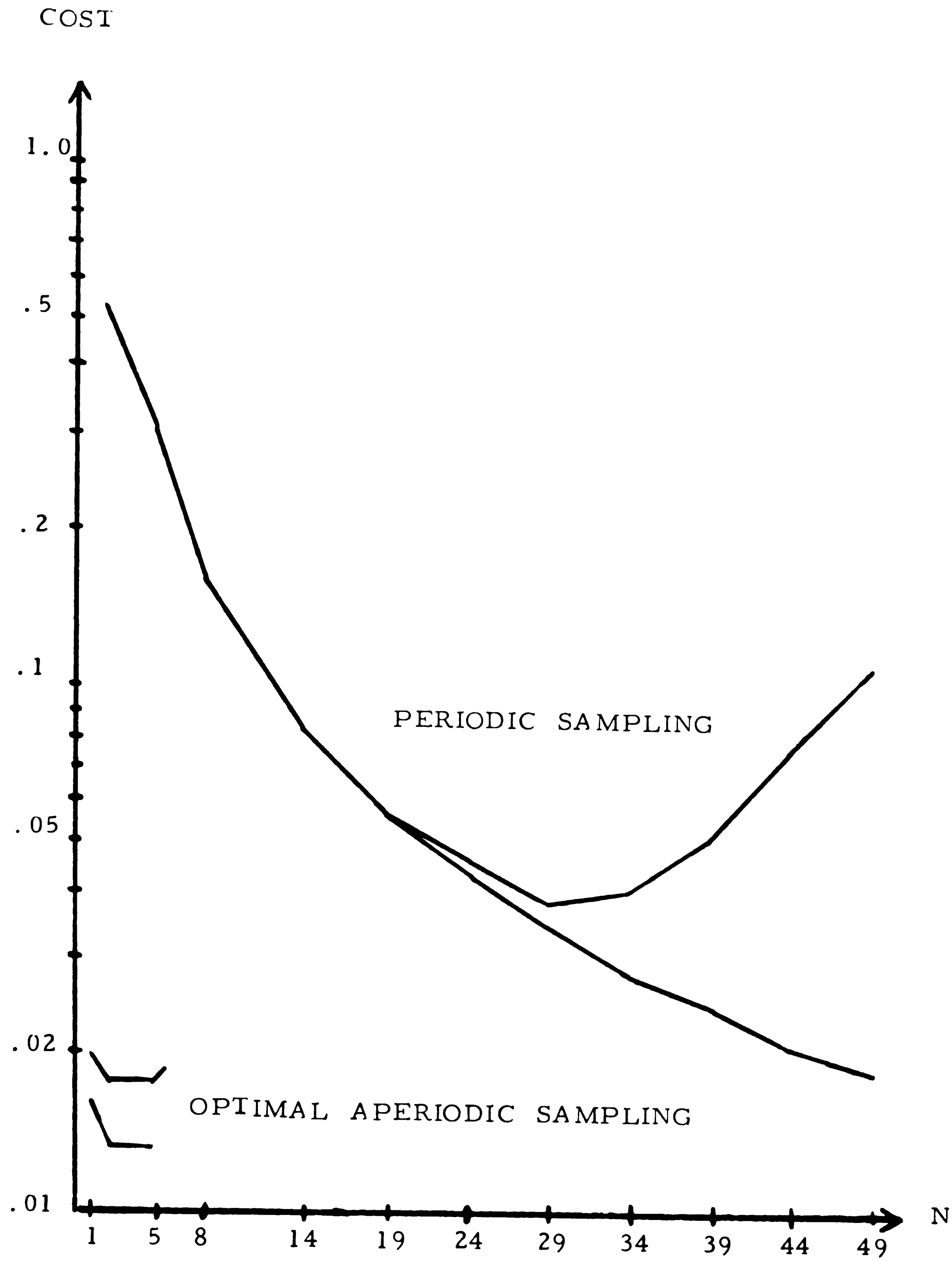


Fig. 7 System and Control Performance for PS and OAS with the Parabolic Control Approximation

For each case, the clear superiority of OAS over PS is shown for each control approximation. The control performance for 49 periodic control changes does not achieve the control performance obtained with two control changes made over optimally chosen sampling intervals. Thus, OAS requires one control change for every 25 periodic control changes to obtain the same performance regardless of the order of the control approximation used. Moreover, the performance index depends very significantly on the sampling intervals sequence chosen since both the control approximation parameters over each sampling interval and the length of the sampling interval both depend on the sampling intervals sequence chosen.

The performance advantages of OAS over PS for the different control approximations can be determined by computing the control performance ratio

$$R_J^*(N,k) = \frac{J(\frac{t_N - t_0}{N}, N, k)}{J(\underline{T}^*, N, k)}$$

or the system performance ratio

$$R_S^*(N,k) = \frac{S(\frac{t_N - t_0}{N}, N, k)}{S(\underline{T}^*, N, k)}$$

which is the optimal performance value of PS divided by the optimal performance value of OAS with the same control approximation (k) and the same number of sampling intervals. The values of $R_J^*(N,k)$ and $R_S^*(N,k)$ are shown in Table 5-7 and 5-8 for $N = 2$ and 4 and for zero, first and second order control approximations.

Since optimal performance for OAS occurs for $N_k^* < 4$, for each k, the computation is limited to small values of N.

TABLE 5-7. Control Performance Ratio between OAS and PS

Control	N	2	4
STEP	$R_J^*(N,1)$	19.3	31.7
RAMP	$R_J^*(N,2)$	50.5	30
PARABOLA	$R_J^*(N,3)$	40.8	23.3

TABLE 5-8. System Performance Ratio between OAS and PS

Control	N	2	4
STEP	$R_S^*(N,1)$	17.5	20
RAMP	$R_S^*(N,2)$	33.1	19.3
PARABOLA	$R_S^*(N,3)$	30	17

The performance ratios are quite large for each control approximation and again confirms the significant improvements in performance possibly by using OAS rather than PS. The performance improvement ratio for the ramp control approximation are the largest indicating the parabolic term u_{2i} in the control approximation does not contribute to control performance improvement as significnatly as does the linear term u_{1i}.

The decrease in this performance ratio as N increases indicates that OAS's performance advantage over PS decreases as N increases. The system performance achieved by OAS can never be reached using PS since the improvement of control performance is offset by the increase in COI as the number of samples increases.

The ratio of the optimal system performance of OAS is about 2:3:7 achieved at $N_k^* = 2$, 2 and 4 for parabolic, ramp, and step control approximations whereas the ratio of PS is close to 4:7:16 at $N_k^* = 29$, 34, and 39. The difference in optimal system performance ratio is less than for PS because the performance for OAS is so good that the order of control approximation has less effect.

5.4 Comparisons of OAS and Adaptive Sampling Rules

Numerous studies have been made to develop sampling criteria that outperform a periodic sampling criterion [13-20]. Additional studies have been made to compare the performance of periodic and adaptive sampling criteria for a simple feedback control system with a specified non-optimal control law.

An optimal aperiodic sampled-data control law with second order control approximation is sampled by a sample and hold mechanism triggered by different adaptive criteria. The performance of this adaptively sampled optimal control law is compared with the performance of an optimal aperiodic sampled-data control law with zero order control approximation. The study is performed to compare the performance of adaptive and optimal aperiodic sampling criteria on an optimal control law. This comparison is not perfect because the control law that is sampled adaptively is not the continuous time control.

All the presently known adaptive sampling criteria depend only on the input signal variations and do not directly depend on the system dynamics or the performance indices. Therefore, applying those criteria to an optimal control input can be expected to provide poor performance compared to the optimal aperiodic sampled-data control law. This conjecture is supported by the following simulation results.

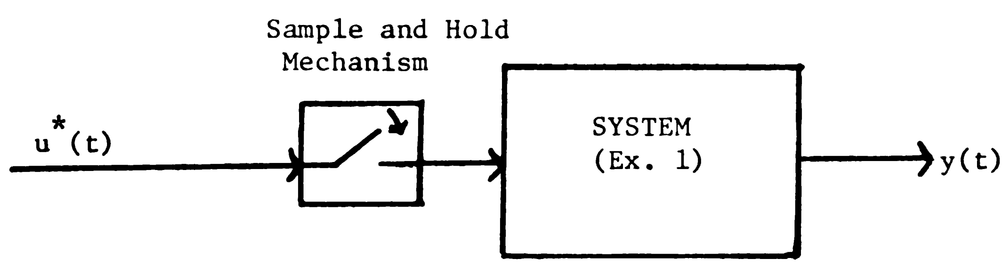
The comparison of optimal aperiodic and adaptive sampling criteria will be compared based on the performance of the system given in Example 1. The optimal control law, which is to be sampled adaptively, is determined by computing the optimal aperiodic sampled-data control law with three control (N = 3) changes and second order control approximation (k = 2). This control law has the form

$$u^*(t) = \begin{cases} 1.7172 - 26.974 + 78.461\, t^2 & 0 \le t < 0.30376 \\ -0.085785 + 0.37588(t - 0.30376) & 0.30376 \le t < 1.10653 \\ \qquad - 0.37389(t - 0.30376)^2 & \\ .0032652 - .0091914(t - 1.10653) & 1.10653 \le t < 2.36193 \\ \qquad + .0057176(t - 1.10653)^2 & \\ -5.8153 \times 10^{-7} & 2.36193 \le t \le 20 \\ + 1.1827 \times 10^{-7}(t - 2.36193) & \\ - 5.5011 \times 10^{-9}(t - 2.36193)^2 & \end{cases}$$

This optimal control is then sampled by a sample and hold mechanism triggered by the following adaptive sampling rules developed by Hsia [13], Dorf [20], Gupta [14] and Mitchell [17].

Hsia	$T_i = \dfrac{\sqrt{\alpha\beta}}{\lvert \dot{u}_i \rvert}$	$\alpha = 0.1$	$\beta = 10$
Dorf	$T_i = \dfrac{\sqrt{2/3\alpha\beta}}{\sqrt{\lvert \dot{u}_i \rvert}}$	$\alpha = 0.1$	$\beta = 10$
Gupta	$T_i = \begin{cases} -2.303\ a & a < 0 \\ 0.5\ \ \ a & a \geq 0 \end{cases}$		$a = \dfrac{\dot{u}_i}{\ddot{u}_i}$
Mitchell	$T_i = \dfrac{-\dot{u}_i}{\ddot{u}_i} \pm \sqrt{(\dot{u}_i/\ddot{u}_i)^2 + 2R/\ddot{u}_i}$	$R = 0.1$	

The variable $\dot{u}_i$ and $\ddot{u}_i$ in this table represent the first and second derivatives of the control $u(t)$ at $t = t_i$.



The value of control performance for each of these adaptive sampled-data control laws and the number of samples required are then recorded. The value of the performance index computed for the optimal aperiodic sampled-data control law with four control changes (N = 4) and zero order control approximation (k = 0) is also determined. These values of the performance index and the resultant number of samples required are tabulated in Table 5-9 for both the optimal aperiodic and adaptively sampled-data control laws.

TABLE 5-9. Performance of Different Sampling Criteria

	Number of Sampling	Cost
Hsia	5	614469.53
Dorf	4	13239277.91
Gupta	∞	$\cong$ 0.0178
Mitchell	880,000	$\cong$ 0.0178
Optimal	4	0.06382

The optimal aperiodic sampled-data control with zero order control approximation outperforms all of the adaptively sampled optimal control. This optimal aperiodic sampled-data control law, specified by an optimal control sequence-optimal sampling intervals sequence combination, had approximately the same level of control performance as the optimal control law sampled using Gupta's and Mitchell's criteria, but with significantly fewer control changes. This optimal aperiodic sampled-data control had significantly better control performance than the optimal control laws adaptively sampled by Dorf's and Hsia's criteria. This comparison is based on approximately the same number of control changes.

The values of costs for Gupta and Mitchell's criteria are obtained by the observation that they both sample almost continuously on the optimal control. Therefore, the control performance is approximately the value obtained using the optimal control.

The large values of control performance obtained using Hsia and Dorf's criteria can be explained by the fact that they fail to sample the small variations of the optimal control in the final period which is the longest. Since this system is highly unstable, the control performance will deteriorate if the sampling mechanism is not triggered sufficiently often.

The results indicate that the selection of a sampling rule for even an optimal control law can have disastrous results if the rule is not selected properly or if the sampling rate for periodic sampling is not high enough. Moreover, it is obvious that by selecting the optimal control sequence and sampling intervals sequence combination, excellent control performance can be obtained with very few sampling instants. Since the optimal control sequence depends on the sampling intervals sequence chosen for this optimal aperiodic sampled-data control, the sampling instants can be viewed as tuned to the system dynamics, optimal sampled-data control law, the performance index, the trajectory and the initial conditions.

5.5 Performance, Different Systems with Different Inputs

The control performance, system performance and sampling efficiency will be compared using both periodic and optimal aperiodic sampling for different systems with different desired trajectories. A stable and an unstable system will be tested with both ramp and parabolic inputs.

5.5.1 Performance of a Stable System

The following example uses a system with both eigenvalues negative. This system makes a good model of a closed-loop system and thus the tracking performance can be compared for both periodic and optimal aperiodic sampling since the control energy required to perform the regulation function is negligible.

EXAMPLE 2

Consider the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -.5 & -1.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ .5 \end{bmatrix} u(t)$$

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with cost functional

$$J = \frac{1}{2}(y(10) - z(10))^2 + \frac{1}{2}\int_0^{10}[(y(t) - z(t))^2 + .02u^2(t)]dt$$

$$+ \sum_{i=0}^{N-1} \alpha e^{-\beta T_i} \qquad \alpha = 0.1 \qquad \beta = 10$$

and initial condition

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

<u>Case I</u>. Ramp Trajectory $\qquad z(t) = 0.1t$

This Type 0 (plant has no poles at the origin) system will follow a ramp trajectory with a monotonically increasing error. Therefore, one might expect a rather large performance index value regardless of the sampling criteria or control approximation used. The optimal control performance, plotted in Fig. 8 and 9 for PS

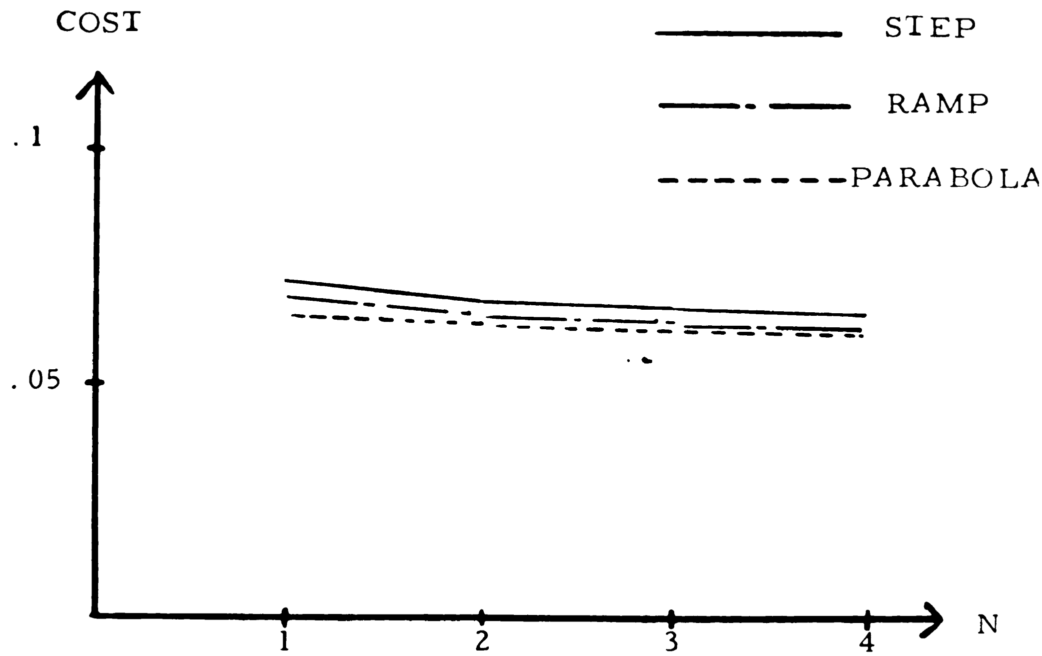


Fig. 8 System Performance of PS for the system of example 2 in Tracking a ramp trajectory

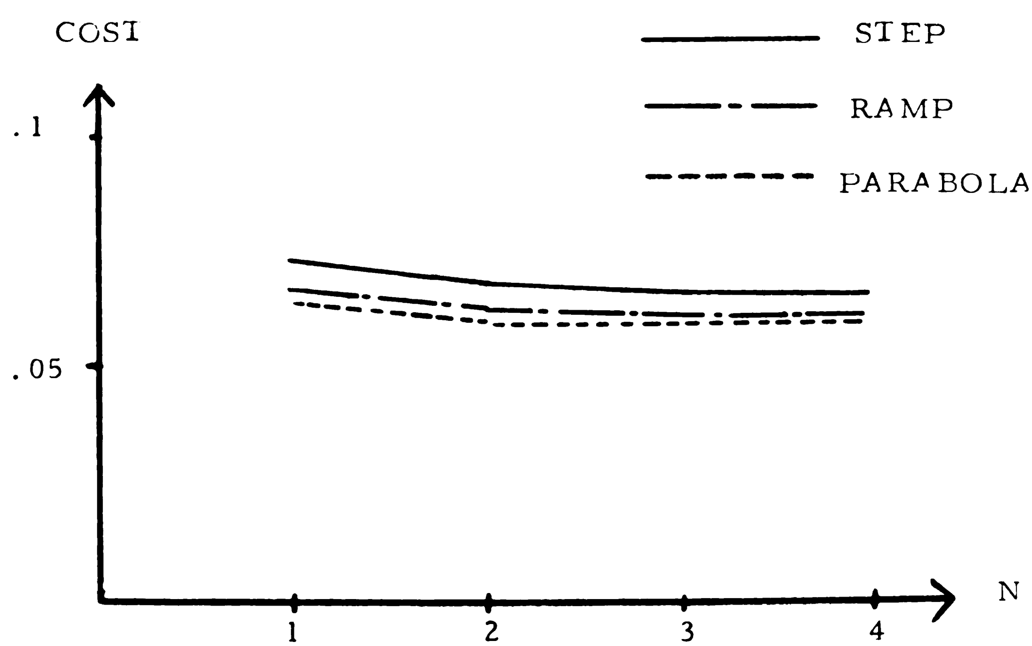


Fig. 9 System Performance of OAS for the system of example 2 in Tracking a ramp trajectory

and OAS sampling criteria, are rather large as expected. Although the performance index value decreases as the order of control approximation increases, the relative improvement in performance is insignificant. Moreover, the decrease in control performance is also very small as the number of sampling times increases.

The difference in performance for OAS and PS sampling criteria is also slight. Thus the control performance is apparently dominated by the large output error and the large control energy requirements which result from requiring a Type 0 system to follow a ramp trajectory.

The optimal sampling intervals sequence are shown below for different values of N and k.

TABLE 5-10. The Optimal Sampling Intervals Sequence in Tracking a Ramp Trajectory

N \ k	Step (0)	Ramp (1)	Parabolic (2)
1	4.7189 5.2811	8.5 1.5	9 1
2	3.2807 2.6595 4.0598	4.2885 4.1087 .16028	3.9074 3.5406 2.5520
3	3.7024 2.7380 2.8871 0.6726	1.5848 4.8031 2.5885 1.0238	2.9422 2.6346 2.499 1.9242
4	2.9613 2.3063 2.0169 1.9763 0.7297	1.4271 3.6 2.1168 1.8497 1.0063	2.2155 3.0819 2.0005 1.9892 1.7318

The optimal sampling intervals sequence depends on the shape of the trajectory to be followed, the order of the control approximation, and the performance index used. Since the terminal error is weighted, the error at the terminal time should be small and therefore the last sampling interval $[t_{N-1}, t_N]$ should be short. The results indicate the last interval is generally the shortest in the sequence. The lengths of the other sampling intervals in the sequence depend on how well the control approximation can represent the desired trajectory to be followed since in this case the shape of the desired trajectory z(t) and control u(t) should be nearly identical a short time after the input is applied. The higher order control approximation (k = 1,2) can accurately represent the ramp trajectory and thus for $N > 2$ the sampling intervals sequence is close to periodic as the order of control approximation increases. For $N = 2$, the initial sampling interval increases as k increases because the control approximation can better represent the continuous-time optimal control over this interval as k increases. Thus, since the control over the initial interval is more accurate the length of that interval increases and the length of the final interval is reduced.

<u>Case II</u>. Parabolic Trajectory $\qquad z(t) = 0.1\ t^2$

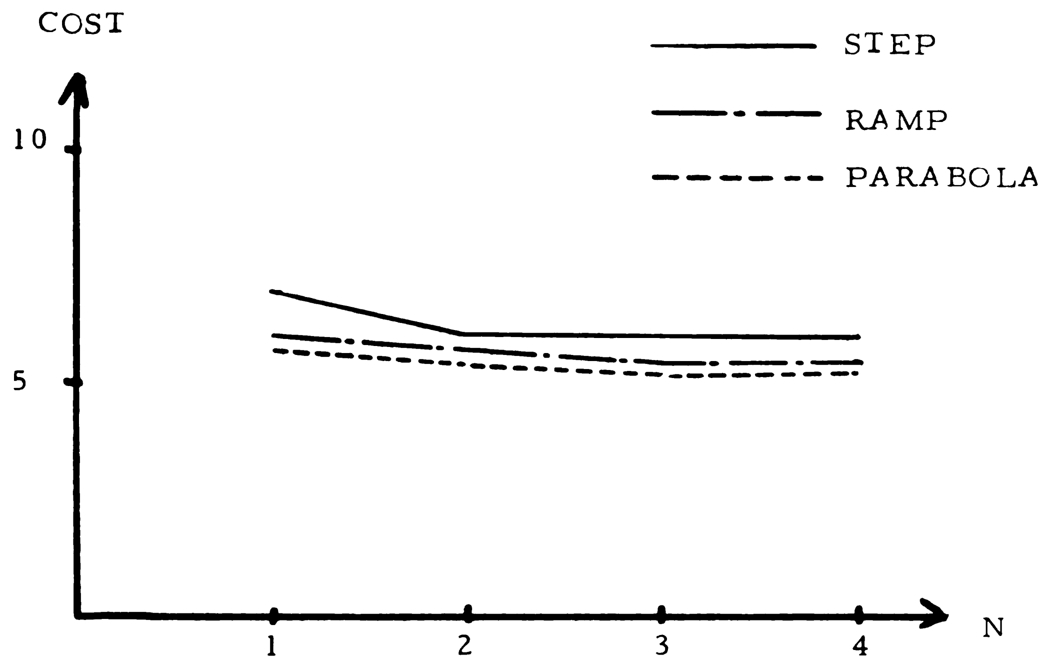


Fig. 10 System Performance of PS for the system of example 2 in Tracking a parabolic trajectory

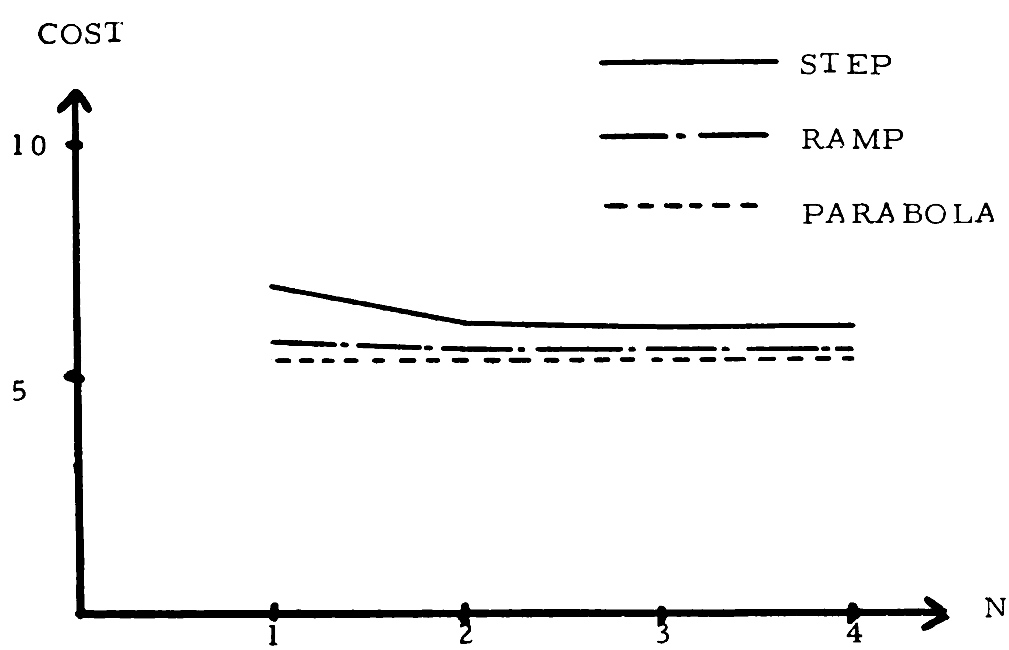


Fig. 11 System Performance of OAS for the system of exmaple 2 in Tracking a parabolic trajectory

The optimal control performance for PS and OAS are plotted in Fig. 10 and 11 respectively. The control performance value is large and again does not change greatly for changes in sampling criteria (N,T) or control approximation (k). These changes are however considerably greater than observed for the ramp trajectory input. This result might be expected since the parabolic trajectory is more difficult to follow than the ramp trajectory.

The control performance is always lower for OAS than PS and decreases with increase in either N or k as expected. The sampling intervals sequence as a function of N and k are shown in Table 5-11.

TABLE 5-11. Optimal Sampling Intervals Sequence in Tracking a Parabolic Trajectory

N \ k	Step (0)	Ramp (1)	Parabola (2)
1	5.0932	8.0559	9
	4.9068	1.9441	1
2	2.9891	4.8577	8.7762
	3.2119	3.9979	0.6149
	3.7990	1.1444	0.6089
3	2.1042	3.9095	8.9975
	2.2253	2.4563	0.3340
	2.2527	2.5670	0.3340
	3.4179	1.0671	0.3340
4	0.8979	3.0226	8.1
	1.8	2.0322	0.6916
	2.0	2.0019	0.4814
	2.0	1.9737	0.3618
	3.3021	0.9696	0.3651

The second order control approximation can approximate the parabolic change in the desired trajectory very well and therefore the initial sampling interval T_0 is always large for this control approximation. The error near the end of the control interval is heavily weighted in the performance index and therefore the number of sampling times at the end of the control interval increases as N increases in order to minimize the terminal error.

The initial sampling interval T_0 for the optimal aperiodic sampled-data control laws with lower order control approximations are much smaller than for the second order approximation because these lower order control approximations cannot approximate the parabolic trajectory as well over the initial interval. This can be observed especially on the zero order control approximation because many more sampling instants occur near the initial part of the control interval as N increases.

5-5-2. Performance of an Unstable System

A non-minimum phase system with the same gain characteristics as the previous example is now considered. Since this system is unstable, control energy must now be expended to perform both regulation and tracking functions.

EXAMPLE 3

Consider the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -.5 & 1.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ .5 \end{bmatrix} u(t)$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with cost functional

$$J = \frac{1}{2}(y(10) - z(10))^2 + \frac{1}{2}\int_0^{10}[(y(t) - z(t))^2 + .02u^2(t)]dt$$

$$+ \sum_{i=0}^{N-1} \alpha e^{-\beta T_i}$$

$$\alpha = 0.1 \qquad \beta = 10$$

Case I. Ramp Trajectory $z(t) = 0.1\ t$

The system performance for OAS and PS sampling criteria are plotted in Fig. 12 and 13 respectively for a ramp trajectory. The performance curves for zero, first and second order control approximations are shown in both figures.

The system performance decreases significantly as the number of samples increases for both PS and OAS criteria. The performance for the second order control approximation is almost identical for OAS and PS. However, for lower order control approximation the performance of the OAS is significantly better than for PS. Apparently the optimal sampled-data control with second order control approximation so closely approximates the optimal continuous time control that the selection of sampling intervals does not greatly affect the performance.

The system performance ratio $R_S(N,k)$ are given in Table 5-12 and 5-13 for PS and OAS. These performance ratios decrease as N increases for both zero and first order control approximations and for both sampling criteria. Thus, the performance advantage of the parabolic control approximation decreases as the number of samples increases.

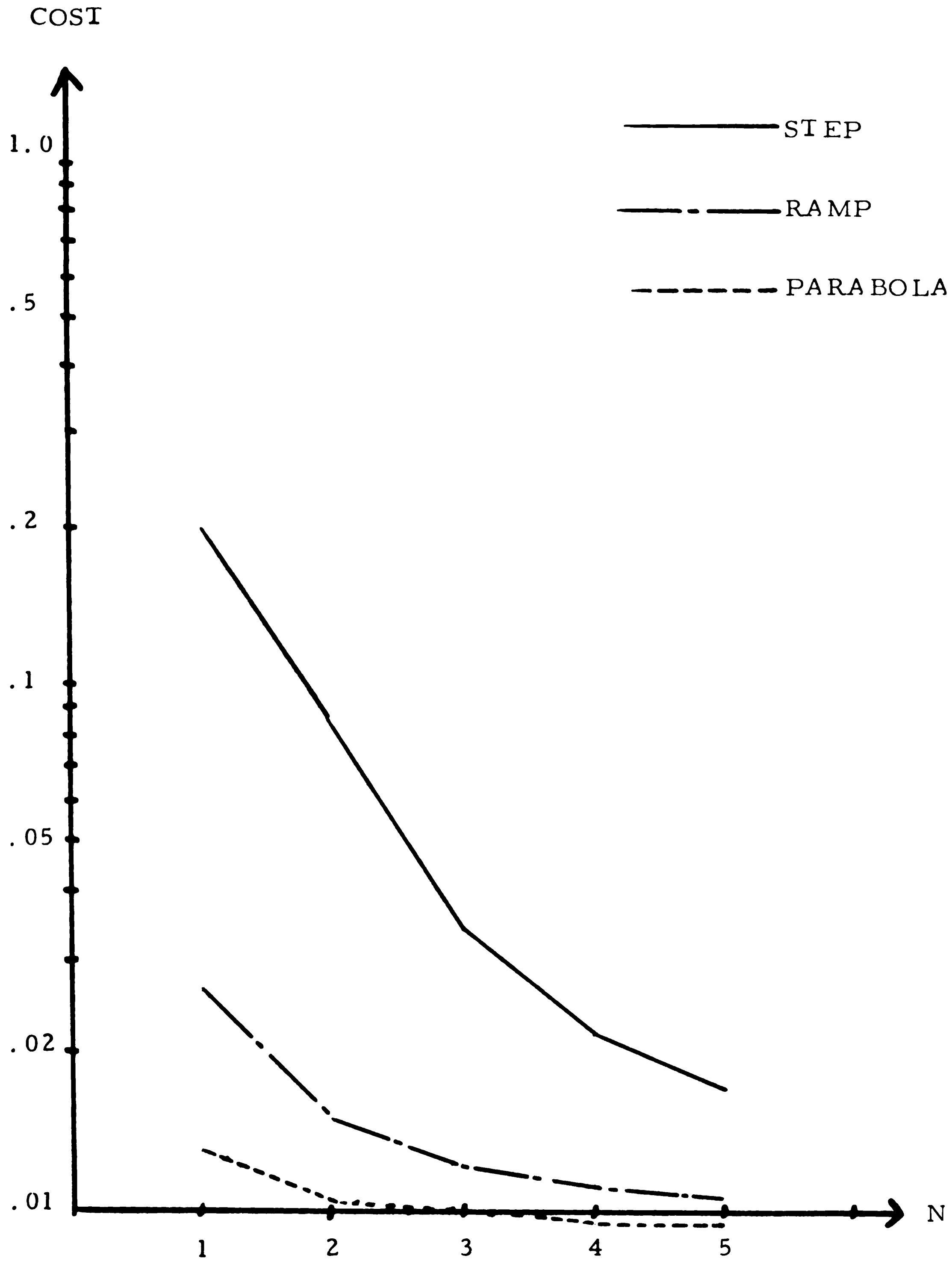


Fig. 12 System Performance of PS for the system of example in tracking a ramp trajectory

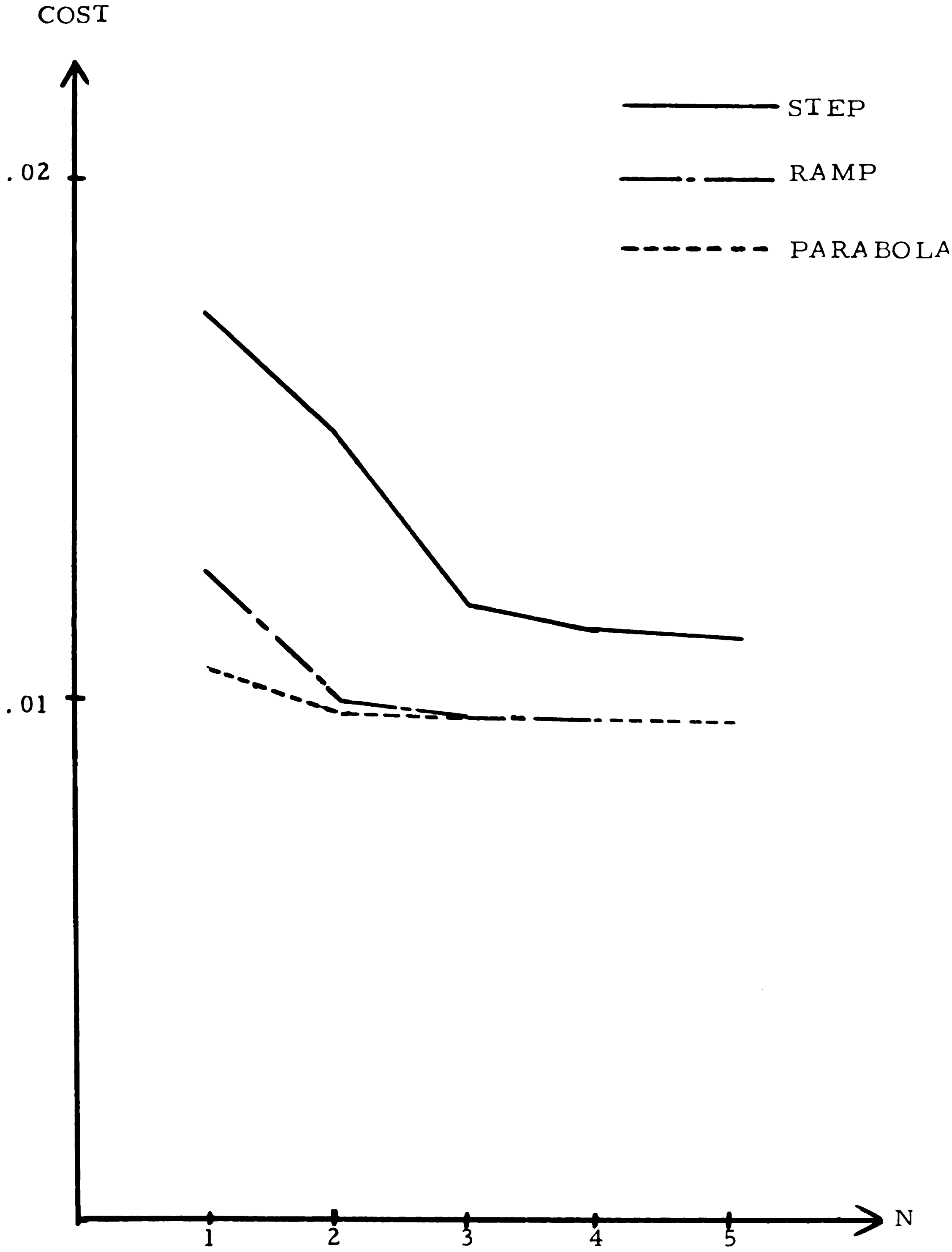


Fig. 13 System Performance of OAS for the system of example 3 in tracking a ramp trajectory

The performance ratio for PS is much higher than for OAS because the optimal sampled-data control with OAS and with any control approximation is so close to the optimal continuous time control that the improvement due to additional terms in the control approximation is less than for the control law with periodic sampling.

TABLE 5-12. System Performance Ratio for Periodic Sampling

	N	1	2	3	4	5
Step	$R_S(N,0)$	16.61	8.17	3.34	2.08	1.59
Ramp	$R_S(N,1)$	2.17	2.34	1.14	1.07	1.02

TABLE 5-13. System Performance Ratio for Optimal Aperiodic Sampling

	N	1	2	3	4	5
Step	$R_S(N,0)$	1.69	1.69	1.19	1.15	1.14
Ramp	$R_S(N,1)$	1.20	1.11	1	1	1

The optimal sampling intervals sequence for different control approximation and different number of samplings are shown below.

TABLE 5-14. Optimal Sampling Intervals Sequence in Tracking a Ramp Trajectory

N \ k	Step (0)	Ramp (1)	Parabolic (2)
1	.58262	1	1.7529
	9.4174	9	8.2471
2	0.57325	0.97197	1.5114
	3.7592	6.9998	6.7511
	5.6672	2.0284	1.7375
3	0.64978	0.82625	1.9130
	2.3205	5.0144	3.1614
	2.9347	2.8353	2.8392
	4.1042	1.3241	2.0864
4	0.65095	1.0101	1.9701
	1.7476	3.5440	2.1187
	1.9167	2.2670	2.0333
	2.0385	1.9722	2.0014
	3.6477	1.2067	1.8764
5	0.68421	0.96478	1.5909
	1.4281	2.8156	2.2513
	1.1949	1.7785	1.6850
	1.6309	1.6672	1.6685
	1.5873	1.5893	1.6605
	3,4747	1.1746	1.1439

The approximation to the optimal continuous-time control should be excellent over the initial segment of the control interval in order to adequately regulate the unstable system and to track the ramp trajectory. Therefore, the initial interval $[t_0, t_1)$ was samll for all N and k. Moreover, in general, the length of this interval increased as N and k increased. The performance index penalizes terminal error and therefore the lengths of the terminal interval $[t_{N-1}, t_N)$ is also small. The number of samples in the middle of the control interval increase as the number of samples increase. The sampling intervals in the middle of the control interval becomes closer to periodic as both N and k increase indicating the regulation and tracking tasks require constant control effort for this particular system.

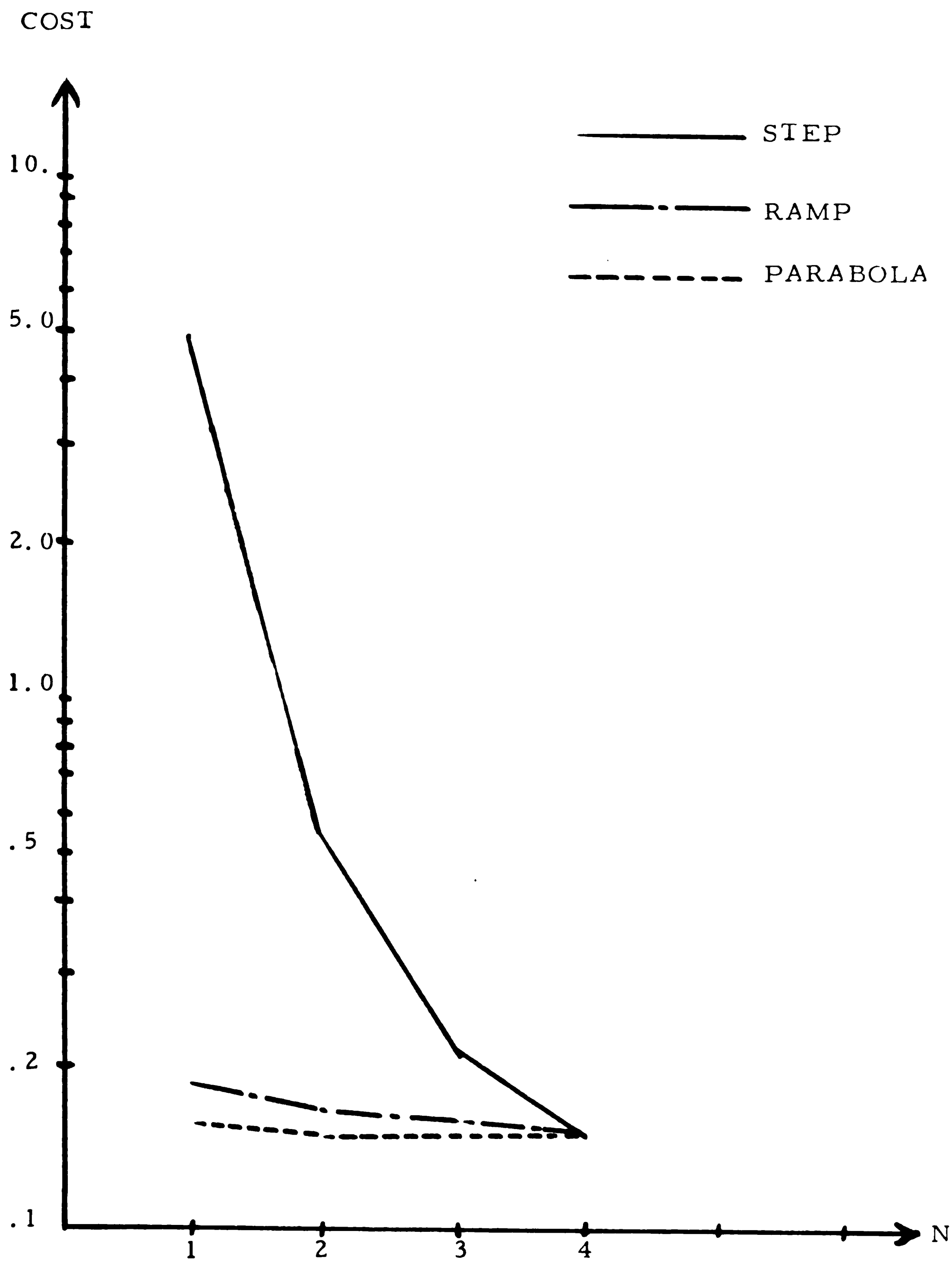


Fig. 14 System Performance of PS for the system of example 3 in tracking a parabolic trajectory

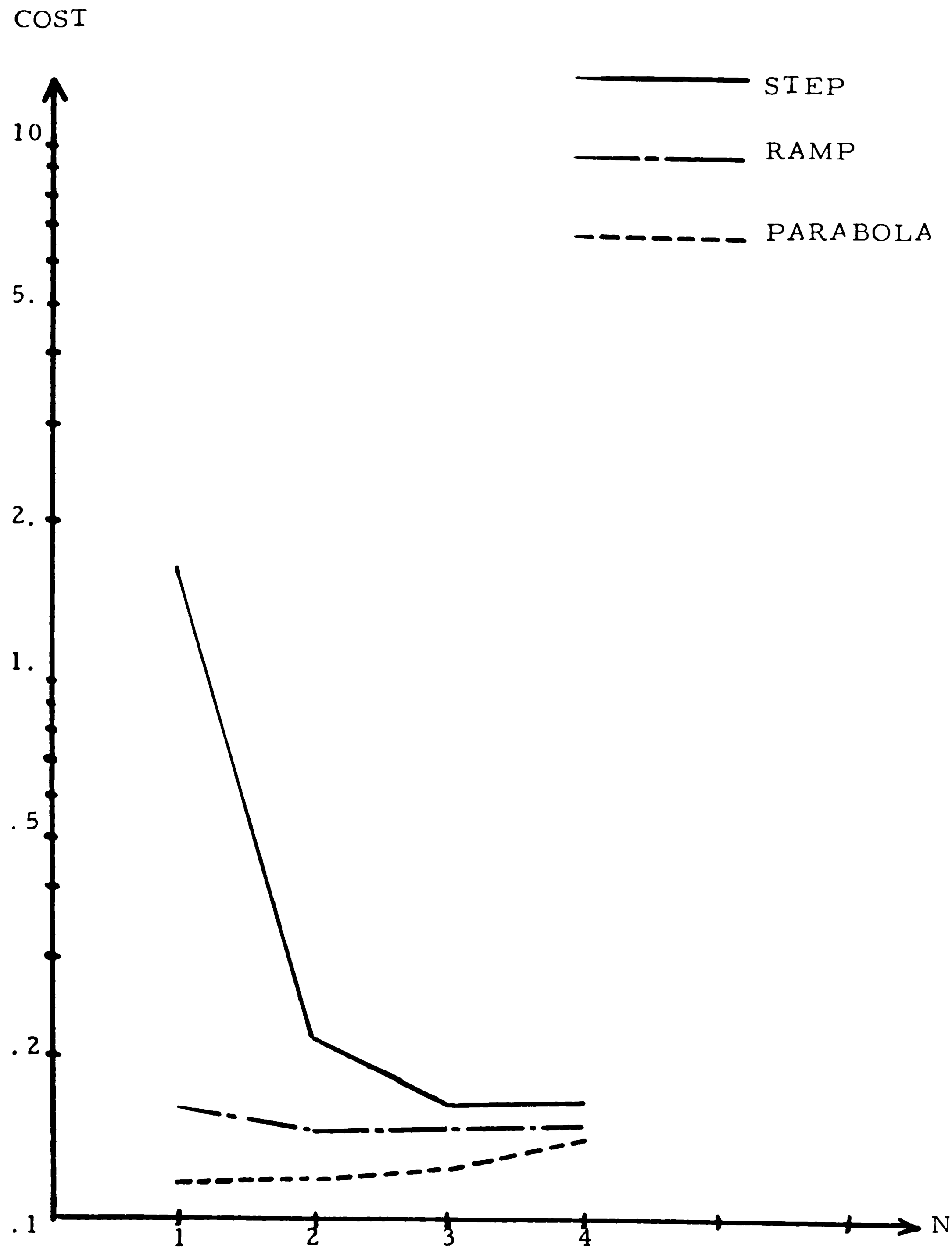


Fig. 15 System Performance of OAS for the system of example 3 in tracking a parabolic trajectory

Case II. Parabolic Trajectory $z(t) = 0.1t^2$

The system performance for OAS and PS criteria are plotted in Fig. 14 and 15 respectively for a parabolic trajectory. The performance curves for the zero, first and second order control approximations are also shown in both figures.

The system performance generally decreases significantly as either the order of control approximation increases or as the number of sampling times increases. However, for the case of parabolic control approximation and OAS, the increased number of small sampling times in the initial period increases the COI without improving the performance enough to offset it and therefore the system performance increases with the number of sampling times. Thus, for this case, excellent system performance was achieved using very few optimal aperiodic sampling times and a high order of control approximation.

The system performance ratios $R_S(N,k)$ are given in Table 5-15and 5-16 for PS and OAS. These performance ratios decrease as N increases for both the zero and first order control approximation. This result implies the performance advantage of the second order control approximation is much greater when the number of sampling times is small. The ratio is larger for PS than for OAS because the control law with OAS closely approximates the continuous-time optimal control so that increasing the order of the control approximation does not significantly increase system perfromance.

TABLE 5-15. System Performance Ratio for Periodic Sampling

	N	1	2	3	4
Step	$R_S(N,0)$	32.46	3.78	1.5	1.28
Ramp	$R_S(N,1)$	1.26	1.14	1.07	1

TABLE 5-16. System Performance Ratio for Optimal Aperiodic Sampling

	N	1	2	3	4
Step	$R_S(N,0)$	12.82	1.81	1.33	1.23
Ramp	$R_S(N,1)$	1.39	1.20	1.16	1.08

The optimal sampling intervals sequence is shown below for different values of N and k.

TABLE 5.17. Optimal Sampling Intervals Sequence in Tracking a Parabolic Trajectory

N \ k	Step (0)	Ramp (1)	Parabolic (2)
1	0.2	2.095	9
	9.8	7.905	1
2	1.9083	3.7552	9.317
	4.0514	5.6	0.3417
	4.0403	0.6448	0.3413
3	0.52507	2.6451	8.8372
	4.8085	3.5664	0.4136
	1.6811	2.6352	0.3752
	2.9881	1.1533	0.3743
4	1.1063	2.3539	6.8133
	1.9981	2.0519	1.3542
	1.9946	1.9996	0.7699
	1.8982	2.0013	0.512
	3.0028	1.5933	0.5236

The optimal sampling interval sequences are chosen based on the order of the control approximation and the trajectory to be followed. The second order control approximation can accurately follow the parabolic trajectory and thus the first sampling interval is large. As the number of sampling intervals increases, the length of the initial sampling interval decreases and the length of the other sampling intervals increase. The zero and first order control approximations cannot follow the parabolic trajectory accurately over any interval. Thus, the length of the initial interval decreases significantly as the order of the control approximation decreases. The last sampling interval, where the rate of change of trajectory is the largest, tends to be the smallest of the sampling intervals for the first and second order control approximations. The first interval is the smallest for the zero order approximation because the sampling intervals have to be chosen to provide effective control because the approximation to the parabolic trajectory is so poor. Thus, the sampling times for lower order control approximations must be used to maintain tracking accuracy much more than for higher order control approximations.

CHAPTER VI

SAMPLED-DATA CONTROLLABILITY AND OBSERVABILITY

Controllability and observability were originally developed as purely mathematical concepts. However, they were soon found to be related to the possibility of achieving a desired degree of control and obtaining the desired information about the system.

Controllability assures that the optimal control law designed for a linear system using a quadratic performance index will be asymptotically stable. Observability assures the Kalman filter will be asymptotically stable. Moreover, controllability and observability are also important in the realm of mathematical modeling. Although a state space model is desired for analytic design of the control law, one often starts with an input-output model obtained experimentally. The minimal realization which does not introduce any phenomena that cannot be accounted for by an input-output description of the system, is intimately related to the concepts of controllability and observability. Thus, controllability and observability are important concepts in the areas of control, estimation, and identification of dynamical systesm.

Controllability and observability will be investigated for sampled-data control systems where the continuous-time plant is known but the actuators and sensors are not specified and must be designed as part of the control law. For this case, the number of sampling times and the lengths of sampling intervals are design parameters or control variables for the system.

Definitions of controllability and observability have been recently proposed by Troch [41] for the case where the number of sampling intervals is specified but the lengths of the sampling intervals are free and considered control variables. However, there never existed definitions that considered both the number of sampling times and the length of each sampling interval as control variables.

Therefore, extended definitions of controllability and observability are proposed. Under these extended definitions, any system which is either controllable or observable when the control and measurements are continuous functions of time is shown to be controllable or observable when controls are changed and measurements are made only at the sampling times. Since controllability and observability should be only a property of the dynamic system being controlled and not a property of the hardware used to implement this control, the number of sampling times should be as much a control parameter as the lengths of the sampling intervals and the control levels over each sampling interval. Under this extended definition the actuators and sensors must be viewed as part of the control law being implemented rather than part of the system to be controlled. This point of view is required because the number of sampling times and the lengths of the sampling intervals are control parameters or variables.

Sufficient conditions were derived by Troch [41] which guaranteed that an observable system would be sampled-data observable over q sampling times where q is the order of the minimal polynomial. The conditions derived for observability were never extended to controllability. Moreover, the conditions were quite restrictive and did not indicate the conditions under which a system could not be observed on q sampling intervals.

Necessary and sufficient conditions for the controllability and observability of sampled-data system are derived. These theorems state that a sampled-data system is controllable (observable) if and only if the continuous-time system is controllable (observable) and the sampling time sequence is such that a certain matrix is non-singular. This nonsingularity of this matrix can be used as a test for controllability or observability of a sampled-data system. This trst is used to determine conditions on the sampling times for which an observable and controllable continuous time system will not be observable and controllable on a sequence of sampling times. Finally, conditions on the sampling times are derived for guaranteeing that a system which is controllable and observable with continuous measurements and controls will be controllable and observable with sampled measurements and controls.

The sampled-data control problem is now formulated in order to provide an appropriate framework for defining sampled-data controllability and observability.

Consider the linear system

$$\underline{\dot{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{u}(t) \qquad \underline{x}(t_0) = \underline{\xi} \tag{15}$$

$$\underline{y}(t) = \underline{C}\ \underline{x}(t) \tag{16}$$

where $\underline{x}(t)$ is the n-dimensional state vector, $\underline{u}(t)$ is the r-dimensional control, and $\underline{y}(t)$ is the m-dimensional output vector and $\underline{A}$, $\underline{B}$, $\underline{C}$ are compatible time-invariant matrices.

The sensor provides measurements

$$\underline{y}(t_{h+i}) = \underline{C}\ \underline{x}(t_{h+i}) \tag{17}$$

at the sampling times $\{t_{h+i}\}_{i=0}^{N}$ that are not specified but are constrained to satisfy

$$0 < T_{min} \leq t_{h+i+1} - t_{h+i} = T_i \leq T_{max} \tag{18}$$

The control actuator is also assumed to be a sampled-data device and therefore the control $\underline{u}(t)$ is sampled-data of the form

$$\underline{u}(t) = \underline{u}(t_{h+i}) = \underline{u}_{h+i} \qquad t \in [t_{h+i},\ t_{h+i+1}) \tag{19}$$

for $i = 0,1,\ldots,N-1$. This control is assumed specified by knowing the control sequence $\{\underline{u}_{h+i}\}_{i=0}^{N-1}$, the sampling intervals sequence $\{t_{h+i}\}_{i=0}^{N-1}$, and the number of sampling times N.

This system can be represented by a set of difference equations if the state differential equation is integrated over each sampling interval $[t_{h+i},\ t_{h+i+1})$ separately. The difference equations have the form

$$\underline{x}_{h+i+1} = \underline{\phi}_{h+i}\underline{x}_{h+i} + \underline{D}_{h+i}\underline{u}_{h+i} \qquad i = 0,1,\ldots,N-1 \qquad (20)$$

where
$$\underline{x}_{h+i} = \underline{x}(t_{h+i})$$

$$\underline{\phi}_{h+i} = \underline{\phi}(T_i) = e^{\underline{A}T_i}$$

$$\underline{D}_{h+i} = \underline{D}(T_i) = \int_0^{T_i} \underline{\phi}(t)\underline{B}dt \ .$$

This representation does not indicate clearly that the sampling times are control variables, but imbeds these variables in the matrices $\underline{\phi}_{h+i}$ and $\underline{D}_{h+i}$. Moreover, the state $\underline{x}_{h+i}$ is the state at the sampling time specified by knowing the control $\{N, \{t_{h+i}\}_{i=0}^{N}, \{\underline{u}_i\}_{i=0}^{N-1}\}$. This representation of the sampled-data system is used for notational convenience. The dependence of $\underline{\phi}_{h+i}$ and $\underline{D}_{h+i}$ on $\{t_{h+i}\}_{i=0}^{N}$ must always be considered in this development.

6-1. Observability

Definition

The system (15) is said to be sampled-data observable at t_h if there exists a finite N and a sequence $\{t_{h+i}\}_{i=0}^{N-1}$ such that any initial state $\underline{x}(t_h)$ can be determined from the knowledge of $\{\underline{y}(t_{h+i})\}_{i=0}^{N-1}$ and $\{\underline{u}(t_{h+i})\}_{i=0}^{N-1}$.

It should be noted that N is arbitrary but finite and the sequence $\{t_{h+i}\}_{i=0}^{N-1}$ is not specified but is contrained to satisfy

$$t_h < t_{h+1} < \ldots < t_{h+N-1}$$

Although a system may be sampled-data observable, it may not be observable if $N = p$ measurements are used. Therefore, p-sampled-data observability is defined as follows.

Definition

The system (15) is said to be p-sampled-data observable at t_h if there exists a sequence $\{t_{h+i}\}_{i=0}^{p-1}$ such that any state $\underline{x}(t_h)$ can be determined from the knowledge of $\{\underline{y}(t_{h+i})\}_{i=0}^{p-1}$ and $\{\underline{u}(t_{h+i})\}_{i=0}^{p-1}$.

A system which is sampled-data observable is p-sampled-data observable for all p greater than some $N = N_o$ where N_o is the minimum number of sampling intervals required to determine the initial state $\underline{x}_h$. The following theorem is obtained by extending a theorem on discrete-time observability (Appendix H) to obtain a necessary and sufficient condition for p-sampled-data observability.

THEOREM 6-1 (Necessary and Sufficient Condition on p-Sampled-Data Observability)

The system (15) is p-sampled-data observable at t_h if and only if there exists a finite time sequence $\{t_{h+i}\}_{i=0}^{p-1}$ such that

$$\underline{Q}_o = [\underline{C}' \,|\, \underline{\Phi}'_{h+1,h}\, \underline{C}' \,|\, \cdots \,|\, \underline{\Phi}'_{h+p-1,h}\, \underline{C}']$$

has rank n where

$$\underline{\Phi}_{j,k} = \underline{\Phi}(t_j, t_k) \; .$$

For the case of periodic sampling, p-sampled-data observability is assured if a sampling period T can be found such that

$$\underline{Q}_o = [\underline{C}' \mid \underline{\phi}'\ \underline{C}' \mid \cdots \mid (\underline{\phi}')^{p-1}\underline{C}']$$

has rank n where

$$\underline{\phi}_i = \underline{\phi}(T) \qquad i = 0,1,\ldots,p-1$$

This theorem can be used to test for the minimum number of sampling times N_o required for observability. The test could be performed by determining the minimum integer p that satisfies

$$p \geq n/m$$

for which a sequence $\{t_{h+i}\}_{i=0}^{p-1}$ can be found such that $\underline{Q}_o$ has rank n. Although N_o can be the smallest integer which satisfies this inequality, the actual value of N_o is often larger than this minimum number because some of the measurements are redundant.

The condition of this theorem can also be used to determine whether a system is observable on a particular sequence of $N = p$ sampling times. However, a stronger condition can be found by properly decomposing the condition found in Theorem 6-1 into a condition on the plant (15) and a condition on the sampling times for the case where $p = q$, the order of the minimal polynomial for the system. This strong condition for p-sampled-data observability is stated in the following theorem.

<u>THEOREM</u> 6-2 (Necessary and Sufficient Condition on q-Sampled-Data Observability)

A system (15) which has a system matrix $\underline{A}$ with minimal polynomial of degree q;

$$m(\lambda) = (\lambda - \lambda_1)^{r_1}, \ldots, (\lambda - \lambda_k)^{r_k} \quad \text{with} \quad \sum_{i=1}^{k} r_i = q$$

is q-sampled-data observable if and only if

(i) the system (15) is observable in Kalman's sense [36]

(ii) there exists a sequence $\{t_{h+i}\}_{i=0}^{q-1}$ such that the $q \times q$ matrix

$$\underline{X}_o = \left[\begin{array}{cccc}
e^{\lambda_1 t_h} & e^{\lambda_1 t_{h+1}} & \cdots & e^{\lambda_1 t_{h+q-1}} \\
t_h e^{\lambda_1 t_h} & t_{h+1} e^{\lambda_1 t_{h+1}} & \cdots & t_{h+q-1} e^{\lambda_1 t_{h+q-1}} \\
\vdots & & & \vdots \\
\frac{t_h^{(r_1-1)}}{(r_1-1)!} e^{\lambda_1 t_h} & & \cdots & \frac{t_{h+q-1}^{(r_1-1)}}{(r_1-1)!} e^{\lambda_1 t_{h+q-1}} \\
\hline
e^{\lambda_2 t_h} & & \cdots & e^{\lambda_2 t_{h+q-1}} \\
\vdots & & & \vdots \\
\frac{t_h^{(r_2-1)}}{(r_2-1)!} e^{\lambda_2 t_h} & & \cdots & \frac{t_{h+q-1}^{(r_2-1)}}{(r_2-1)} e^{\lambda_2 t_{h+q-1}} \\
\hline
\vdots & & & \vdots \\
\hline
\vdots & & & \vdots \\
\frac{t_h^{(r_k-1)}}{(r_k-1)!} e^{\lambda_k t_h} & & \cdots & \frac{t_{h+q-1}^{(r_k-1)}}{(r_k-1)!} e^{\lambda_k t_{h+q-1}}
\end{array}\right]$$

is nonsingular.

Proof

The parameters h and t_h are assumed to be zero without loss of generality. The condition will be proved for the special case where all the eigenvalues are distinct. The general case follows directly. This condition will be proved by examining the condition

$$\text{Rank }(\underline{Q}_o) = n$$

where

$$\underline{Q}_o = [\underline{C}' \,|\, e^{\underline{A}' t_1}\underline{C}' \,|\, e^{\underline{A}' t_2}\underline{C}' \,|\, \ldots \,|\, e^{\underline{A}' t_{q-1}}\underline{C}'] \;.$$

Since the fundamental matrix can be expressed as

$$e^{\underline{A}t} = \sum_{k=0}^{q-1} \alpha_k(t)\, \underline{A}^k$$

$\underline{Q}_o$ can be expressed as

$$\underline{Q}_o = [\underline{C}' \;\; \underline{A}'\underline{C}' \;\; (\underline{A}')^2\underline{C}' \;\; . \; . \; . \;\; (\underline{A}')^{q-1}\underline{C}'] \begin{bmatrix} \underline{I} & \alpha_o(t_1)\underline{I} \cdot \cdot & \alpha_o(t_{q-1})\underline{I} \\ 0 & \cdot & \cdot \\ \vdots & \vdots & \vdots \\ 0 & \alpha_{q-1}(t_1)\underline{I}. & \alpha_{q-1}(t_{q-1})\underline{I} \end{bmatrix}$$

It is known that the powers of $\underline{A}$ and the constituent matrices $\underline{Z}_i$ of $\underline{A}$ are related by

$$\begin{bmatrix} \underline{I} \\ \underline{A} \\ \cdot \\ \cdot \\ \cdot \\ \underline{A}^{q-1} \end{bmatrix} = \underline{V} \begin{bmatrix} \underline{Z}_0 \\ \underline{Z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underline{Z}_{q-1} \end{bmatrix}$$

where $\underline{V}$ is a generalized Vandermonde matrix [54]. Since $\underline{V}$ is nonsingular

$$\begin{bmatrix} \underline{Z}_0 \\ \cdot \\ \cdot \\ \cdot \\ \underline{Z}_{q-1} \end{bmatrix} = \underline{V}^{-1} \begin{bmatrix} \underline{I} \\ \cdot \\ \cdot \\ \cdot \\ \underline{A}^{q-1} \end{bmatrix}$$

Now letting $\underline{V}^{-1} = \underline{W} = \{w_{ik}\}$, it can be shown that

$$\underline{Z}_{i-1} = \sum_{j=1}^{q} w_{i,j+1}\underline{A}^j$$

Since the fundamental matrix $e^{\underline{A}t}$ can also be expressed in terms of the constituent matrices

$$e^{\underline{A}t} = \sum_{i=1}^{q} e^{\lambda_i t} \underline{Z}_{i-1}$$

and since each constituent matrix can be expressed as a matrix polynomial function of $\underline{A}$,

$$e^{\underline{A}t} = \sum_{j=0}^{q-1} \alpha_j(t)\ \underline{A}^j = \sum_{i=1}^{q} e^{\lambda_i t}\ \underline{Z}_{i-1} = \sum_{i=1}^{q} e^{\lambda_i t} \left(\sum_{j=0}^{q-1} w_{i,j+1}\underline{A}^j\right)$$

the $\alpha_j(t)$ functions become

$$\alpha_j(t) = \sum_{i=1}^{q} e^{\lambda_i t}\ w_{i,j+1} \quad j = 0,1,\ldots,q-1$$

The observability matrix $\underline{Q}_o$ can now be expressed as

$$\underline{Q}_o = [\underline{C}' \ \underline{A}'\underline{C}' \ \dots \ (\underline{A}')^{q-1}\underline{C}'] \begin{bmatrix} \alpha_o(0)\underline{I} & \dots & \alpha_o(t_{q-1})\underline{I} \\ \alpha_1(0)\underline{I} & \dots & \vdots \\ \vdots & & \\ \alpha_{q-1}(0)\underline{I} & \dots & \alpha_{q-1}(t_{q-1})\underline{I} \end{bmatrix}$$

$$= [\underline{C}' \ \underline{A}'\underline{C}' \dots (\underline{A}')^{q-1}\underline{C}'] \cdot \hat{\underline{W}} \cdot \hat{\underline{X}}_o$$

where $\underline{I}$ is an m dimensional identity matrix and

$$\hat{\underline{X}}_o = \begin{bmatrix} \underline{I} & e^{\lambda_1 t_1}\underline{I} & \dots & e^{\lambda_1 t_{q-1}}\underline{I} \\ & & & \\ \underline{I} & e^{\lambda_q t_1}\underline{I} & \dots & e^{\lambda_q t_{q-1}}\underline{I} \end{bmatrix}$$

$$\hat{\underline{W}} = \begin{bmatrix} w_{11}\,\underline{I} & w_{21}\,\underline{I} & \dots & w_{q\,1}\,\underline{I} \\ \vdots & & & \\ w_{1q}\,\underline{I} & & \dots & w_{qq}\,\underline{I} \end{bmatrix}$$

The matrix $\hat{\underline{W}}$ is nonsingular if and only if $\underline{W}$ is nonsingular since

$$\text{Det}\ \hat{\underline{W}} = (\text{Det}\ \underline{W}')^m$$

The matrix $\hat{\underline{X}}_o$ is nonsingular if and only if $\underline{X}_o$ is nonsingular since

$$\text{Det}\ \hat{\underline{X}}_o = (\text{Det}\ \underline{X}_o)^m$$

Now, the system is q-sampled data observable if and only if matrix $\underline{Q}_o$ has rank n. Matrix $\underline{Q}_o$ has rank n if and only if matrix

$$[\underline{C}' \ \underline{A}'\underline{C}' \ . \ . \ . \ \underline{A}'^{q-1}\underline{C}']$$

has rank n and there exists a sequence $\{t_{h+i}\}_{i=0}^{q-1}$ such that matrix $\hat{\underline{X}}_o$ is nonsingular. Therefore, since the above matrix has full row rank if and only if the system is observable and since $\hat{\underline{X}}_o$ is nonsingular if and only if $\underline{X}_o$ is nonsingular, the system is q-sampled-data observable if and only if condition (i) and (ii) are satisfied.

Q.E.D.

This theorem clearly states a system is observable with sampled measurements $\{\underline{y}(t_{h+i})\}_{i=0}^{q-1}$ if and only if it is observable with continuous measurements $\underline{y}(t)$; $t \in [t_0, t_N)$. Moreover, the theorem places a necessary and sufficient condition on the q sampling times which must be satisfied if the system is to be observable on a particular sampling time sequence. Finally it should be noted that p is constrainted to be the order of the minimal polynomial for the theorem to hold. However, if a sequence of q sampling times can be found for which the system is observable, the system will be observable for some sequence $\{t_{h+i}\}_{i=0}^{p-1}$ for each $p > q$ since there is a guarantee that n independent measurements can be found by selecting only q sampling times. Thus, if the system is q-sampled-data observable it is p-sampled-data observable for all $p > q$. The following theorem, stated and proved by Troch

[41], provides a sufficient condition on the sampling times which guarantees that a system which is observable with continuous measurements will be observable with sampled measurements.

THEOREM 6-3 (Sufficient Condition for q-Sampled-Data Observability)

A system (15) which is observable with continuous measurements is q-sampled-data observable.

(i) for all sampling intervals sequence $\{t_{h+i}\}_{i=0}^{q-1}$ if all of the eigenvalues of $\underline{A}$ are real.

(ii) for all sampling intervals sequences $\{t_{h+i}\}_{i=0}^{q-1}$ such that

$$t_{h+i} - t_h < \frac{\pi}{\omega_{\ell max}} \qquad i = 1,2,\ldots,q-1$$

where $\omega_{\ell max}$ is the greatest imaginary part of the eigenvalues of $\underline{A}$.

Proof

The proof follows immediately if $\underline{X}_o$ can be proved nonsingular over the set of sampling intervals sequence specified in case (i) and (ii) respectively. The functions $\alpha_k(t)$ form a Chebyshev system [41] over $[t_h, \infty)$ if all eigenvalues are real and over $[t_h, t_h + \frac{\pi}{\omega_{\ell max}})$ if some eigenvalues are complex. Since the functions $\alpha_k(t)$ are Chebyshev over these respective intervals for the two cases, the functions $e^{\lambda_i t}$, $te^{\lambda_i t}, \ldots, \frac{t^{\lambda_i - 1}}{(\gamma_i - 1)!} e^{\lambda_i t}$, $i = 1,2,\ldots,k$ are also linearly independent over the same respective intervals for the two cases. Thus $\underline{X}_o$ is nonsingular over the intervals specified for the respective cases and the theorem is proved. Q.E.D.

This theorem clearly indicates that a system which is observable with continuous measurements is observable with any sequence of q sampled measurements as long as all of the eigenvalues are real. If some eigenvalues are complex, the q sampling times must all be selected in an interval of length $\pi/\omega_{\ell_{max}}$ to insure that observability will be preserved using sampled measurements instead of continuous measurements. Since $\omega_{\ell_{max}}$ is the largest imaginary part of the complex eigenvalues of $\underline{A}$, this constraint implies sampling must occur at a rate at least q times faster than the Nyquist rate $(T = \pi/\omega_{\ell_{max}})$ in order to insure all q sampling times occur in a $\pi/\omega_{\ell_{max}}$ interval. This constraint is restrictive for some applications and since it is only a sufficient condition, less restrictive conditions are investigated in Section 6.3.

The result of this theorem also guarantees that there will always exist q sampling times for which the system is observable and therefore the following theorem can be established.

<u>THEOREM</u> 6-4 (Sufficient Condition for Sampled-Data Observability)

If the system (15) is observable with continuous measurements, it is sampled-data observable.

<u>Proof</u>

From Theorem 6.3 it has been established that there always exists q sampling times $\{t_{h+i}\}_{i=0}^{q-1}$ such that a system which is observable with continuous measurements will be observable with q sampling measurements. The system is always p-sampled-data observable for any $p > q$ if it is q-sampled-data observable. Thus, there always exists an N and a sampling times sequence $\{t_{h+i}\}_{i=0}^{N-1}$ to make the system sampled-data observable. Q.E.D.

In the previous definitions of observability for sampled-data system [39, 41], either the number of sampling intervals and the length of each sampling interval are both specified or the number of sampling intervals is specified and the lengths of sampling intervals are considered control parameters. In both of these definitions, a system which is observable with continuous measurements may not be observable with sampled measurements. This extended definition, where both the number of sampling intervals and the lengths of sampling intervals are control parameters, permits the preservation of observability when the outputs are no longer measured continuously but are sampled. This preservation of observability under the imposition of sampling requires a system designer to view the sensor and its sampling intervals sequence specified by

$$\{t_{h+i}\}_{i=0}^{N-1} \quad \text{and} \quad N \tag{*}$$

to be part of the control law rather than part of the plant being controlled. This perspective is required since the sampling intervals sequence are control parameters in this extended definition of observability.

Some explicit conditions on the sampling times $\{t_{n+i}\}_{i=0}^{q-1}$ for which $\underline{X}_o$ is singular will be investigated in Section 6-3 after a similar matrix condition is derived for controllability. In postponing this development, the similarity of conditions for sampled-data controllability and observability will be emphasized and no duplication of discussion is required.

6-2. Controllability

Definition

The system (15) is said to be sampled-data controllable at t_h if for every initial state $\underline{x}_h$, there exists a finite N and a control

$$\underline{u}(t) = \underline{u}_{h+i} \quad t \in [t_{h+i}, t_{h+i+1}) \quad i = 0,1,\ldots,N-1$$

defined by the control sequence $\{u_{h+i}\}_{i=0}^{N-1}$, the sampling time sequence $\{t_{h+i}\}_{i=0}^{N}$ and N, such that $\underline{x}_{h+N} = \underline{0}$.

As stated above, N is finite but arbitrary and the sequence $\{t_{h+i}\}_{i=0}^{N-1}$ is not constrainted in any way except

$$t_h < t_{h+1} < \ldots < t_{h+N} .$$

Although a system may be sampled-data controllable, it may not be controllable if only $N = p$ piecewise constant controls are used. Therefore, p-sampled-data controllability is defined as follows.

Definition

The system (15) is said to be p-sampled-data controllable at t_h if for every initial state $\underline{x}_h$, there exists a control

$$\underline{u}(t) = \underline{u}_{h+i} \quad t \in [t_{h+i}, t_{h+i+1}) \quad i = 0,1,\ldots,p-1$$

defined by specifying both the control sequence $\{\underline{u}_{h+i}\}_{i=0}^{p-1}$, and sampling time sequence $\{t_{h+i}\}_{i=0}^{p}$, such that $\underline{x}_{h+p} = \underline{0}$.

A system which is sampled-data controllable is p-sampled data controllable for all p greater than some $N = N_c$ where N_c is the minimum number of sampling intervals required to return $\underline{x}_h$ to the origin. The following theorem which is obtained by extending a theorem on discrete-time controllability (Appendix H) states a necessary and sufficient condition for p-sampled-data controllability.

THEOREM 6-5 (Necessary and Sufficient Condition on p-Sampled-Data Controllability)

The system (15) is p-sampled-data controllable at t_h if and only if there exists a finite time sequence $\{t_{h+i}\}_{i=0}^{p-1}$ such that

$$\underline{Q}_c = [\underline{D}_{h+p-1} \mid \underline{\phi}_{h+p,h+p-1}\underline{D}_{h+p-2} \mid \cdots \mid \underline{\phi}_{h+p,h+1}\underline{D}_h]$$

has rank n where

$$\underline{\phi}_{N,i} = \underline{\phi}_{N-1} \cdot \underline{\phi}_{N-2} \cdots \underline{\phi}_i \; .$$

For the periodic sampling case, p-sampled-data controllability is assured if a sampling period T can be found such that

$$\underline{Q}_c = [\underline{D} \; \underline{\phi}\underline{D} \; \underline{\phi}^2\underline{D} \; \ldots \; \underline{\phi}^{p-1}\underline{D}]$$

has rank n where

$$\underline{\phi}_i = \underline{\phi}(T) \quad \text{and} \quad \underline{D}_i = \underline{D}(T) \qquad i = 0,1,\ldots,p-1$$

This theorem provides a condition which can be used to test for the minimum number of sampling times N_c required to assure sampled-data controllability. The test could be performed by finding the minimum integer p satisfying

$$p \geq n/r$$

where r is the dimension of control $\underline{u}$, for which the matrix $\underline{Q}_c$ has rank n. Although N_c can be the smallest integer which satisfies this inequality, the actual value of N_c is often larger than this minimum number because some of the controls are redundant.

The conditions of this theorem can also be used to determine whether a system is controllable on a particular sequence of N = p sampling times. However, a stronger condition can be formed by properly decomposing the condition found in Theorem 6-5 into a condition on the plant (1) and a condition on the sampling times. This strong condition which holds only for the case where p is the order of the minimal polynomial of matrix $\underline{A}$, is stated in the following theorem.

<u>THEOREM</u> 6-6 (Necessary and Sufficient Condition for q-Sampled-Data Controllability)

A system (15) which has system matrix $\underline{A}$ with minimal polynomial of order q

$$m(\lambda) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2}\ldots(\lambda - \lambda_k)^{r_k}$$

with
$$\sum_{j=1}^{K} r_j = q$$

is q-sampled-data controllable if and only if

(i) the system (15) is controllable with continuous controls

(ii) there exists a sampling intervals sequence $\{t_{h+i}\}_{i=0}^{q}$ such that $q \times q$ matrix

$$
\underline{Y} = \left[\begin{array}{cccc}
\int_{t_h}^{t_{h+1}} e^{-\lambda_1 \zeta} d\zeta & \int_{t_{h+1}}^{t_{h+2}} e^{-\lambda_1 \zeta} d\zeta & \cdots & \int_{t_{h+q-1}}^{t_{h+q}} e^{-\lambda_1 \zeta} d\zeta \\
\int_{t_h}^{t_{h+1}} (-\zeta) e^{-\lambda_1 \zeta} d\zeta & \int_{t_{h+1}}^{t_{h+2}} (-\zeta) e^{-\lambda_1 \zeta} d\zeta & \cdots & \int_{t_{h+q-1}}^{t_{h+q}} (-\zeta) e^{-\lambda_1 \zeta} d\zeta \\
\vdots & & & \\
\int_{t_h}^{t_{h+1}} \frac{(-\zeta)^{r_1-1}}{(r_1-1)!} e^{-\lambda_1 \zeta} d\zeta & \int_{t_{h+1}}^{t_{h+2}} \frac{(-\zeta)^{r_1-1}}{(r_1-1)!} e^{-\lambda_1 \zeta} d\zeta & \cdots & \int_{t_{h+q-1}}^{t_{h+q}} \frac{(-\zeta)^{r_1-1}}{(r_1-1)!} e^{-\lambda_1 \zeta} d\zeta \\
\hline
\int_{t_h}^{t_{h+1}} e^{-\lambda_2 \zeta} d\zeta & & & \int_{t_{h+q-1}}^{t_{h+q}} e^{-\lambda_2 \zeta} d\zeta \\
\vdots & & & \\
\int_{t_h}^{t_{h+1}} \frac{(-\zeta)^{r_2-1}}{(r_2-1)!} e^{-\lambda_2 \zeta} d\zeta & & & \int_{t_{h+q-1}}^{t_{h+q}} \frac{(-\zeta)^{r_2-1}}{(r_2-1)!} e^{-\lambda_2 \zeta} d\zeta \\
\hline
\vdots & & & \vdots \\
\hline
\int_{t_h}^{t_{h+1}} e^{-\lambda_k \zeta} d\zeta & & & \int_{t_{h+q-1}}^{t_{h+q}} e^{-\lambda_k \zeta} d\zeta \\
\vdots & & & \vdots \\
\int_{t_h}^{t_{h+1}} \frac{(-\zeta)^{r_k-1}}{(r_k-1)!} e^{-\lambda_k \zeta} d\zeta & & & \int_{t_{h+q-1}}^{t_{h+q}} \frac{(-\zeta)^{r_k-1}}{(r_k-1)!} e^{-\lambda_k \zeta} d\zeta
\end{array}\right]
$$

is nonsingular.

Proof

The parameter h is assumed to be zero without loss of generality. The condition will be proved for the special case where all the eigenvalues are distinct. The general case follows directly. This theorem will be proved by examining the condition

$$\text{Rank}\ (\underline{Q}_c) = n$$

where

$$\underline{Q}_c = [\int_0^{T_{q-1}} e^{\underline{A}t}\underline{B}dt \mid e^{\underline{A}T_{q-1}} \int_0^{T_{q-2}} e^{\underline{A}t}\underline{B}dt \mid \ldots \mid e^{\underline{A}(T_{q-1}+T_{q-2}+\ldots+T_1)} \int_0^{T_0} e^{\underline{A}t}\underline{B}dt] .$$

Since

$$e^{\underline{A}(T_{q-1}+T_{q-2}+\ldots+T_{q-i})} \int_0^{T_{q-i-1}} e^{\underline{A}t}\underline{B}dt = e^{\underline{A}t_q} \int_{t_{q-i-1}}^{t_{q-i}} e^{-\underline{A}\zeta}\underline{B}d\zeta$$

and since the fundamental matrix can be expressed as

$$e^{\underline{A}t} = \sum_{k=0}^{q-1} \alpha_k(t)\underline{A}^k$$

$\underline{Q}_c$ can be expressed as

$$\underline{Q}_c = e^{\underline{A}t_q}[\underline{B}\ \underline{AB}\ \underline{A}^2\underline{B}\ \ldots\ \underline{A}^{q-1}\underline{B}] \begin{bmatrix} \int_{t_{q-1}}^{t_q} \alpha_o(-\zeta)d\zeta\underline{I} & \ldots & \int_{t_0}^{t_1} \alpha_o(-\zeta)d\zeta\underline{I} \\ \vdots & & \vdots \\ \int_{t_{q-1}}^{t_q} \alpha_{q-1}(-\zeta)d\zeta\underline{I} & \ldots & \int_{t_0}^{t_1} \alpha_{q-1}(-\zeta)d\zeta\underline{I} \end{bmatrix}$$

However, it is proved in Theorem 6-2 that

$$\alpha_k(t) = \sum_{i=1}^{q} \omega_{i,k+1} e^{\lambda_i t}$$

and therefore $\underline{Q}_c$ becomes

$$\underline{Q}_c = e^{\underline{A}t_q}[\underline{B}\ \underline{A}\ \underline{B}\ \ldots\ \underline{A}^{q-1}\underline{B}] \cdot \underline{\tilde{W}} \cdot \underline{\tilde{Y}}$$

where $\underline{\tilde{W}}$ and $\underline{\tilde{Y}}$ are (qr) square matrices defined by

$$\underline{\tilde{W}} = \begin{bmatrix} w_{11}\underline{I} & w_{21}\underline{I} & \cdots & w_{q1}\underline{I} \\ w_{12}\underline{I} & w_{22}\underline{I} & \cdots & \\ \vdots & & & \\ w_{1q}\underline{I} & & \cdots & w_{qq}\underline{I} \end{bmatrix}$$

and

$$\underline{\tilde{Y}} = \begin{bmatrix} \int_{t_h}^{t_{h+1}} e^{-\lambda_1\zeta} d\zeta\, \underline{I} & \int_{t_{h+1}}^{t_{h+2}} e^{-\lambda_1\zeta} d\zeta\, \underline{I} & \cdots & \int_{t_{h+q-1}}^{t_{h+q}} e^{-\lambda_1\zeta} d\zeta\, \underline{I} \\ \int_{t_h}^{t_{h+1}} e^{-\lambda_2\zeta} d\zeta\, \underline{I} & \int_{t_{h+1}}^{t_{h+2}} e^{-\lambda_2\zeta} d\zeta\, \underline{I} & \cdots & \int_{t_{h+q-1}}^{t_{h+q}} e^{-\lambda_2\zeta} d\zeta\, \underline{I} \\ \vdots & \vdots & & \vdots \\ \int_{t_h}^{t_{h+1}} e^{-\lambda_q\zeta} d\zeta\, \underline{I} & \int_{t_{h+1}}^{t_{h+2}} e^{-\lambda_q\zeta} d\zeta\, \underline{I} & \cdots & \int_{t_{h+q-1}}^{t_{h+q}} e^{-\lambda_q\zeta} d\zeta\, \underline{I} \end{bmatrix}$$

The matrix $e^{\underline{A}t_q}$ is a fundamental matrix and is nonsingular. The matrix $\underline{\tilde{W}}$ is nonsingular if and only if matrix $\underline{W}$ is nonsingular since

$$\text{Det } \underline{\tilde{W}} = (\text{Det } \underline{W}')^r$$

where r is the dimension of the control.

The system is q sampled-data controllable if and only if $\underline{Q}_c$ has rank n. Moreover $\underline{Q}_c$ has rank n if and only if there exists a sampling intervals sequence $\{t_{h+i}\}_{i=0}^{q}$ for which $\underline{Y}$ is nonsingular and the matrix

$$[\underline{B}\ \underline{A}\ \underline{B}\ \underline{A}^2\underline{B}\ \ldots\ \underline{A}^{q-1}\underline{B}]$$

has rank n. Since this matrix is of rank n if and only if the system is controllable with continuous controls and since $\underline{Y}$ is nonsingular if and only if $\underline{Y}$ is nonsingular because

$\underline{Y}$ is nonsingular because

$$\text{Det } \underline{\tilde{Y}} = (\text{Det } \underline{Y})^r$$

the system is q-sampled-data controllable if and only if conditions (i) and (ii) are satisfied. Q.E.D.

The computation of $\underline{Y}$ for controllability is much more difficult than the computation of $\underline{X}_o$ for observability because each element of $\underline{Y}$ is an integral of the similar term in $\underline{X}$. Since these integrals can be evaluated analytically, a matrix condition can be obtained for controllability which is quite similar in form to matrix condition on observability. This condition is derived for the case where the eigenvalues are distinct.

<u>Corollary</u> 6-1

A system (15) which has system matrix $\underline{A}$ with minimal polynomial of order q and with distinct eigenvalues

$$m(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \; . \; . \; . \; (\lambda - \lambda_q)$$

is q-sampled-data controllable if and only if

(i) the system (15) is controllable with continuous controls

(ii) there exists a sampling intervals sequence $\{t_{h+i}\}_{i=0}^{q}$ such that a $(q + 1)$ square matrix

$$\underline{X}_c = \begin{bmatrix} e^{-\lambda_1 t_0} & & & e^{-\lambda_1 t_q} \\ e^{-\lambda_2 t_0} & & & e^{-\lambda_2 t_q} \\ \vdots & & & \\ e^{-\lambda_q t_0} & & & e^{-\lambda_q t_q} \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

is nonsingular. This matrix must be modified by replacing a row

$$[e^{-\lambda_i t_o} e^{-\lambda_i t_1} \; \ldots \; e^{-\lambda_i t_q}]$$

by the $q + 1$ row vector

$$[t_0 \; t_1 \; \ldots \; t_q]$$

if eigenvalue $\lambda_i = 0$.

Proof

Assuming all eigenvalues are non-zero, the matrix $\underline{Q}_c$ can be expressed as

$$\underline{Q}_c = e^{\underline{A}t}[\underline{B} \mid \underline{A}\,\underline{B} \mid \ldots \mid \underline{A}^{q-1}\underline{B}] \cdot \underline{\tilde{W}}\,\underline{\tilde{Y}}$$

where $\underline{\tilde{Y}}$ becomes

$$\underline{\tilde{Y}} = \begin{bmatrix} \dfrac{e^{-\lambda_1 t_{q-1}} - e^{-\lambda_1 t_q}}{\lambda_1}\underline{I} & \cdots & \dfrac{e^{-\lambda_1 t_0} - e^{-\lambda_1 t_1}}{\lambda_1}\underline{I} \\ \vdots & & \vdots \\ \dfrac{e^{-\lambda_q t_{q-1}} - e^{-\lambda_q t_q}}{\lambda_q}\underline{I} & \cdots & \dfrac{e^{-\lambda_q t_0} - e^{-\lambda_q t_1}}{\lambda_q}\underline{I} \end{bmatrix}$$

and $\underline{\tilde{W}}$ is defined in the previous theorem. It has been proved that the matrix $\underline{Q}_c$ has rank n is and only if

$$[\underline{B}\ \underline{A}\ \underline{B}\ \underline{A}^2\underline{B}\ .\ .\ .\ \underline{A}^{q-1}\underline{B}]$$

has rank n and there exists a sequence $\{t_{h+i}\}_{i=0}^{q}$ such that $\underline{\tilde{Y}}$ is nonsingular. Thus since the system is controllable if and only if the matrix $\underline{Q}_c$ has rank n, the theorem is proved if it can be shown that $\underline{\tilde{Y}}$ is nonsingular if and only if $\underline{X}_c$ is nonsingular. The matrix $\underline{\tilde{Y}}$ is nonsingular if and only if

$$\underline{Y} = \begin{bmatrix} \dfrac{e^{-\lambda_1 t_{q-1}} - e^{-\lambda_1 t_q}}{\lambda_1} & \cdots & \dfrac{e^{-\lambda_1 t_0} - e^{-\lambda_1 t_1}}{\lambda_1} \\ \vdots & & \vdots \\ \dfrac{e^{-\lambda_q t_{q-1}} - e^{-\lambda_q t_q}}{\lambda_q} & \cdots & \dfrac{e^{-\lambda_q t_0} - e^{-\lambda_q t_1}}{\lambda_q} \end{bmatrix}$$

is nonsingular because

$$\text{Det}\ \underline{\tilde{Y}} = (\text{Det}\ \underline{Y})^r$$

The matrix $\underline{Y}$ can be expressed as

$$\underline{Y} = \underline{L}\ \underline{M}$$

where

$$\underline{L} = \begin{bmatrix} \frac{1}{\lambda_1} & & & \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_q} \end{bmatrix} \qquad \underline{M} = \begin{bmatrix} (e^{-\lambda_1 t_{q-1}} - e^{-\lambda_1 t_q}) & \cdots & (e^{-\lambda_1 t_0} - e^{-\lambda_1 t_1}) \\ \vdots & & \\ (e^{-\lambda_{q-1} t_{q-1}} - e^{-\lambda_{q-1} t_q}) & \cdots & (e^{-\lambda_q t_0} - e^{-\lambda_{q-1} t_1}) \end{bmatrix}$$

The matrix $\underline{L}$ is nonsingular because all eigenvalues are assumed non-zero and therefore $\underline{Y}$ is nonsingular if and only if $\underline{M}$ is nonsingular. Moreover, $\underline{M}$ is nonsingular if and only if matrix $\underline{P}$ is nonsingular where

$$\underline{P} = \begin{bmatrix} (e^{-\lambda_1 t_0} - e^{-\lambda_1 t_1}) & (e^{-\lambda_1 t_1} - e^{-\lambda_1 t_2}) & \cdots & (e^{-\lambda_1 t_{q-1}} - e^{-\lambda_1 t_q}) & e^{-\lambda_1 t_q} \\ (e^{-\lambda_2 t_0} - e^{-\lambda_2 t_1}) & & & & e^{-\lambda_2 t_q} \\ \vdots & & & & \\ (e^{-\lambda_q t_0} - e^{-\lambda_q t_1}) & (e^{-\lambda_q t_1} - e^{-\lambda_q t_2}) & & (e^{-\lambda_q t_{q-1}} - e^{-\lambda_q t_q}) & e^{-\lambda_q t_q} \\ 0 & 0 & & 0 & 1 \end{bmatrix}$$

Furthermore, $\underline{P}$ can be expressed as

$$\underline{P} = \underline{X}_c \underline{G}$$

where

$$\underline{G} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & \ddots & \\ & & & 1 & \\ & & & -1 & 1 \end{bmatrix}$$

Since $\underline{G}$ is nonsingular $\underline{P}$ is nonsingular if and only if there exists a sequence $\{t_{h+i}\}_{i=0}^{q}$ such that $\underline{X}_c$ is nonsingular. This proves the theorem for the case where all eigenvalues are nonzero. If an eigenvalue is $\lambda_i = 0$, the proof follows identically if the row

$$[e^{-\lambda_i t_h} \quad e^{-\lambda_i t_{h+1}} \quad . \ . \ . \ e^{-\lambda_i t_{h+q}}]$$

in matrix $\underline{X}_c$ is replaced by $q + 1$ row vector

$$[t_h, \ t_{h+1}, \ldots, \ t_{h+q}] \quad . \qquad\qquad \text{Q.E.D.}$$

This theorem states that a system will be controllable with sampled-data controls

$$\{\underline{u}_{h+i}\}_{i=0}^{N-1} \quad \{t_{h+i}\}_{i=0}^{N} \quad \text{and} \quad N$$

if and only if it is controllable with continuous controls $\{\underline{u}(t), \ t \in [t_0, \ t_N]\}$. Moreover, the system is q-sampled-data controllable if and only if there exists a sampling intervals sequence such that $\underline{Y}$ (or $\underline{X}_c$) is nonsingular. The condition on $\underline{Y}$ requires an integration of each term which is inconvenient. The condition on $\underline{X}_c$ does not require integration of each term and provides a condition on the sampling times which is similar to the condition on $\underline{X}_o$ obtained for observability. Although the condition on $\underline{X}_c$ was only stated for the case where eigenvalues are distinct, a matrix $\underline{X}_c$ could be derived for the case where the eigenvalues are not distinct. The derivation of an appropriate form for $\underline{X}_c$ for the case of multiple eigenvalues is a subject for future research.

The condition on $\underline{Y}$ (or $\underline{X}_c$) can be used to test whether a system which is controllable using continuous control will be q-sampled-data controllable on some particular sampling intervals sequence $\{t_{h+i}\}_{i=0}^{q}$. Finally it should be noted p is constrained to be the order of the minimal order of the plant. However, if a sequence of $q + 1$ sampling times can be found for which the system is controllable, the system will be controllable for some sequence $\{t_{h+i}\}_{i=0}^{p}$ for each $p > q$ since there is a guarantee that n independent controls can be found by selecting only $q + 1$ sampling times. Thus, if the system is q-sampled-data controllable it is p-sampled-data controllable for all $p > q$.

The following theorem provides a sufficient condition on the sampling times which guarantees that a system which is controllable with continuous controls will be controllable with sampled controls.

THEOREM 6-7 (Sufficient Condition for q-Sampled-Data Controllability)

A system (15) which is controllable with continuous controls is q-sampled-data controllable.

(i) for all sampling intervals sequence $\{t_{h+i}\}_{i=0}^{q}$ if all of the eigenvalues of $\underline{A}$ are real

(ii) for all sampling intervals sequence $\{t_{h+i}\}_{i=0}^{q}$ such that

$$t_{h+i} - t_h < \pi/\omega_{\ell\max} \quad i = 1,2,\ldots,q$$

where $\omega_{\ell\max}$ is the greatest imaginary part of the complex eigenvalues of $\underline{A}$.

Proof

The proof follows immediately if $\underline{Y}$ can be proved nonsingular over the sets of sampling intervals sequences specified in case (i) and (ii) respectively. The functions $\alpha_k(t)$ form a Chebyshev system [41] over $[t_h,\infty)$ if all eigenvalues are real and over $[t_h, t_h + \pi/\omega_{\ell max})$ if some eigenvalues are complex. Since the function $\alpha_k(t)$ are Chebyshev over these respective intervals for the two cases, the functions

$$\int_{t_{h+i}}^{t_{h+i+1}} \alpha_k(\zeta)d\zeta \qquad k = 1,2,\ldots,q$$

also form a Chebyshev system over the same respective intervals for the two cases. Thus, the functions

$$\int_{t_{h+i}}^{t_{h+i+1}} e^{\lambda_i\zeta}d\zeta \quad \int_{t_{h+i}}^{t_{h+i+1}} \zeta e^{\lambda_i\zeta}d\zeta \quad \ldots \quad \int_{t_{h+i}}^{t_{h+i+1}} \frac{\zeta^{\lambda_i-1}}{(\gamma_i-1)!} e^{\lambda_i\zeta}d\zeta$$

are linearly independent over the same respective intervals for case (i) and (ii). The matrix $\underline{Y}$ is therefore nonsingular over these intervals and the theorem is proved. Q.E.D.

This theorem clearly indicates that a system which is controllable with continuous controls is controllable with a sampled-data control over q sampling intervals as long as all of the eigenvalues of the system matris are real. If some eigenvalues are complex, the $q + 1$ sampling times must all be selected in an interval of length $\pi/\omega_{\ell max}$ to insure that controllability will be preserved using sampled-data controls instead of continuous controls.

Since $\omega_{\ell max}$ is the largest imaginary part of the eigenvalues of $\underline{A}$, this constraint implies sampling must occur at a rate at least q times faster than the Nyquist rate $(T = \pi/\omega_{\ell max})$ in order to insure all q sampling times occur in a $\pi/\omega_{\ell max}$ interval. This constraint is restrictive for some applications and since it is only a sufficient condition, less restrictive conditions are investigated in Section 6.3.

The result of this theorem guarantees that there will always exist $q + 1$ sampling times for which a controllable system will be a q-sampled-data controllable. Therefore, the following theorem can be established.

THEOREM 6-8 (Sufficient Condition for Sampled-Data Controllability)

If the system (15) is controllable with continuous controls, it is sampled-data controllable.

Proof

From Theorem 6.7 it has been established that there always exists $q + 1$ sampling times $\{t_{h+i}\}_{i=0}^{q}$ such that if the system is controllable using continuous controls, it will be controllable using sampled-data controls. The system is always q-sampled-data controllable and is therefore always p-sampled-data controllable for all $p > q$. Thus, there always exists an N and a sampling times sequence $\{t_{h+i}\}_{i=0}^{N}$ to make the system sampled-data controllable.

Q.E.D.

The implications of Theorem 6-8 are quite important. First, if the continuous-time system is completely controllable, then it is sampled-data controllable which implies there exists a control

$$\underline{u}(t) = \underline{u}(t_{h+i}) = \underline{u}_{h+i} \qquad t \in [t_{h+i}, t_{h+i+1})$$

for $i = 0,1,\ldots,N-1$ specified by a finite set of parameters $\{\underline{u}_{h+i}\}_{i=0}^{N-1}$, $\{t_{h+i}\}_{i=0}^{N}$ and N that will for any initial state $\underline{x}_0$ guarantee that $\underline{x}_{h+N}=\underline{0}$. Thus, the controllability of the system does not depend on whether the control is actuated with an analog or sampled-data device. In previous definition of controllability [36, 41], either the number of sampling intervals and the lengths of sampling intervals were specified or the number of the sampling intervals was specified and the lengths of sampling intervals were free and considered control parameters. In both definitions, the sampled-data system could be uncontrollable when the continuous-time system was controllable. The implicit assumption made in these definitions [36, 41] was that the system model included the sampled-data actuator and the sampling intervals sequence. In this definition, the actuator and the sampling intervals sequence are considered part of the control law. This development of sampled-data controllability provides a more general perspective on dynamic system and control.

In the following section, the explicit condition on sampling times for which $\underline{X}_0$ or $\underline{X}_c$ is nonsingular will be investigated.

6-3. Sufficient Condition for the Singularity of $\underline{X}_0$ and $\underline{X}_c$

The sufficient conditions imposed on the sequence of sampling times for the special case where the system matrix has complex eigenvalues may be quite restrictive for some applications where the cost of communicating, storing and processing data are quite high. In these cases, the average sampling rate may be much closer to the Nyquist rate. Sufficient conditions should be established which will guarantee that observability and controllability can be preserved if a sampling process is imposed by design considerations. A rule for selecting sampling times is desired which, if followed, would guarantee the preservation of controllability and observability. Although such a rule is not derived, a rule is suggested by investigating sufficient conditions for the singularity of matrices $\underline{X}_0$ and $\underline{X}_c$. The pattern developed by investigating the conditions sampling intervals sequence must satisfy to make $\underline{X}_0$ and $\underline{X}_c$ singular provide a basis for suggesting a rule for selecting sampling intervals sequence which will preserve controllability and observability.

The following theorems, which are extensions of results by Kalman [36] for periodic sampling, provide a basis for the development of this sampling rule.

<u>THEOREM</u> 6-9

Given a system (15) which is controllable in the Kalman's sense [36], the system is not controllable with a sampled-data control if

$$t_{h+i} = \frac{k\pi}{\omega_\ell} \qquad \begin{array}{l} k = 1,2,\ldots \\ i = 0,1,\ldots,q-1 \end{array}$$

for any ω_ℓ, where ω_ℓ are the imaginary parts of eigenvalues of $\underline{A}$.

<u>Proof</u>

Since the eigenvalues occur in complex conjugate pairs

$$\lambda_\ell = \sigma_\ell + j\omega_\ell$$

$$\lambda_{\ell+1} = \sigma_\ell - j\omega_\ell$$

the appropriate rows of $\underline{X}_c$ for $t_{h+i} = \frac{k\pi}{\omega_\ell}$ have the form

$$[e^{-\lambda_\ell t_h}, e^{-\lambda_\ell t_{h+1}}, \ldots, e^{-\lambda_\ell t_{h+q}}]$$

$$= [e^{-\lambda_{\ell+1} t_h}, e^{-\lambda_{\ell+1} t_{h+1}}, \ldots, e^{-\lambda_{\ell+1} t_{h+q}}]$$

$$= [e^{-\sigma_\ell t_h} \cos \omega_\ell t_h, e^{-\sigma_\ell t_{h+1}} \cos \omega_\ell t_{h+1}, \ldots, e^{-\sigma_\ell t_{h+q}} \cos \omega_\ell t_{h+q}]$$

Since two rows of $\underline{X}_c$ are identical, $\underline{X}_c$ is singular and the theorem is proved. Q.E.D.

THEOREM 6-10

Given a system (15) which is observable in the Kalman's sense [36], the system is not observable with sampled measurements if

$$t_{h+i} = \frac{k\pi}{\omega_\ell} \qquad k = 1,2,\ldots$$

for any ω_ℓ, where ω_ℓ's are the imaginary parts of the complex eigenvalues of $\underline{A}$. The proof of this theorem is identical to the proof of Theorem 6-9 except that $\underline{X}_0$ replaces $\underline{X}_c$.

The results of these two theorems indicate that if the sampling times are all multiples of a basic period π/ω_ℓ for some ℓ, then the system will not be observable or controllable using the sequence. This condition does not imply that the sampling criterion described by this condition be periodic as assumed by Kalman.

General conditions which will describe all sampling intervals sequences for which $\underline{X}_0$ and $\underline{X}_c$ are singular are not derived. However, from a brief study of the following simple cases and the results of Theorem 6-9 and 6-10, a possible set of conditions can be suggested. In the following set of examples, conditions are derived on a set of any two sampling times in the sequences which together could cause $\underline{X}_0$ or $\underline{X}_c$ to be singular. A matrix $\underline{X}$ is used to denote either $\underline{X}_0$ or $\underline{X}_c$.

(i) Assume that $m(\lambda)$ has two complex eigenvalues and thus the matrix $\underline{X}$ has the form

$$\underline{X} = \begin{bmatrix} e^{(\rho+j\omega)t_0} & e^{(\rho+j\omega)t_1} \\ e^{(\rho-j\omega)t_0} & e^{(\rho-j\omega)t_1} \end{bmatrix}$$

This matrix is singular whenever

$$t_1 - t_0 = \frac{k\pi}{\omega} \qquad k = 1,2,...$$

(ii) Assume $m(\lambda)$ is of degree 3 and has eigenvalues $\rho + j\omega$, $\rho - j\omega$, and σ. The 3 by 3 $\underline{X}$ matrix will be singular if any two of the columns of $\underline{X}$ are dependent. Therefore, determine the conditions for which

$$\begin{bmatrix} e^{(\rho+j\omega)t_{i2}} \\ e^{(\rho-j\omega)t_{i2}} \\ e^{\sigma t_{i2}} \end{bmatrix} = c \begin{bmatrix} e^{(\rho+j\omega)t_{i1}} \\ e^{(\rho-j\omega)t_{i1}} \\ e^{\sigma t_{i1}} \end{bmatrix} \tag{21}$$

for some real c. A condition

$$t_{i2} - t_{i1} = \frac{2k\pi}{\omega} \qquad 0 \leq i_1 < i_2 \leq 2$$

is imposed so that

$$\begin{bmatrix} e^{(\rho+j\omega)t_{i2}} \\ e^{(\rho-j\omega)t_{i2}} \\ e^{\sigma t_{i2}} \end{bmatrix} = \begin{bmatrix} e^{\rho(t_{i2}-t_{i1})} \cdot e^{(\rho+j\omega)t_{i1}} \\ e^{\rho(t_{i2}-t_{i1})} \cdot e^{(\rho-j\omega)t_{i1}} \\ e^{\sigma(t_{2i}-t_{i1})} \cdot e^{\sigma t_{i1}} \end{bmatrix}$$

The second condition

$$\rho = \sigma$$

is imposed so that the condition (21) is satisfied for

$$c = e^{\sigma(t_{i2}-t_{i1})} = e^{\rho(t_{i2}-t_{i1})}$$

(iii) Assume $m(\lambda)$ is of degree 5 with two pairs of complex conjugate eigenvalues and one real eigenvalue.

The matrix $\underline{X}$ is singular if any two of the columns of $\underline{X}$ are dependent. Therefore, determine the conditions for which

$$\begin{bmatrix} e^{(\rho_1+j\omega_1)t_{i2}} \\ e^{(\rho_1-j\omega_1)t_{i2}} \\ e^{(\rho_2+j\omega_2)t_{i2}} \\ e^{(\rho_2-j\omega_2)t_{i2}} \\ e^{\sigma t_{i2}} \end{bmatrix} = c \begin{bmatrix} e^{(\rho_1+j\omega_1)t_{i1}} \\ e^{(\rho_1-j\omega_1)t_{i1}} \\ e^{(\rho_2+j\omega_2)t_{i1}} \\ e^{(\rho_2-j\omega_2)t_{i1}} \\ e^{\sigma t_{i1}} \; . \end{bmatrix} \tag{22}$$

The first condition

$$t_{i_2} - t_{i_1} = \frac{2k_1\pi}{\omega_1} = \frac{2k_2\pi}{\omega_2}$$

so that

$$\begin{bmatrix} e^{(\rho_1+j\omega_1)t_{i2}} \\ e^{(\rho_1-j\omega_1)t_{i2}} \\ e^{(\rho_2+j\omega_2)t_{i2}} \\ e^{(\rho_2-j\omega_2)t_{i2}} \\ e^{\sigma t_{i2}} \end{bmatrix} = \begin{bmatrix} e^{\rho_1(t_{i2}-t_{i1})}\,e^{(\rho_1+j\omega_1)t_{i1}} \\ e^{\rho_1(t_{i2}-t_{i1})}\,e^{(\rho_1-j\omega_1)t_{i1}} \\ e^{\rho_2(t_{i2}-t_{i1})}\,e^{(\rho_2+j\omega_2)t_{i1}} \\ e^{\rho_2(t_{i2}-t_{i1})}\,e^{(\rho_2-j\omega_2)t_{i1}} \\ e^{\sigma(t_{i2}-t_{i1})}\,e^{\sigma(t_{i1})} \end{bmatrix}$$

The second condition

$$\rho_2 = \rho_1 = \sigma$$

is imposed so that condition (22) is satisfied with

$$c = e^{\sigma(t_{i2}-t_{i1})} = e^{\rho(t_{i2}-t_{i1})}$$

In summary, Theorem 6-9 and 6-10 indicate that observability and controllability can not be preserved if the $q + 1$ sampling times $\{t_{h+i}\}_{i=0}^{q}$ are chosen to be a multiple of the same period T when

$$T = \pi/\omega_\ell \qquad \ell = 1,2,\ldots,k$$

The results of example (i) - (iii) indicate the observability and controllability may not be preserved if

$$t_{i_2} - t_{i_1} = \frac{2k\pi}{\omega_\ell}$$

for $i_1, i_2 = 0,1,\ldots,q-1$. Thus, these two conditions suggest that a sufficient condition which will preserve controllability and observability is to select all sampling times so that

$$t_{i_2} - t_{i_1} \neq \frac{k\pi}{\omega_\ell} \qquad i_1, i_2 = 0,1,\ldots,q-1$$

for all integers k and all ω_ℓ, $\ell = 1,2,\ldots,q$.

This condition has not been established as a sufficient condition for the singularity of $\underline{X}_c$ and $\underline{X}_o$ and thus is purely a hypothetical condition suggested by the results in this section. The establishment of a sufficient condition for the invertibility of $\underline{X}$ is an important result because it provides guidelines for the system designer. Thus, the derivation of this sufficient condition is an important topic for further research.

CHAPTER VII

THE INFINITE TIME REGULATOR PROBLEM

The infinite-time periodic sampled-data regulator was formulated as an extension of the finite-time problem [2]. The existence of an optimal feedback control and the form of this infinite-time control law were both established formally in a more recent publication [55]. The convergence of the finite-time feedback control law to the infinite-time feedback control law was also formally proven in this latter publication. However, these results were established only for the case of periodic sampling where the length of the sampling period is specified.

The infinite-time sampled-data regulator problem is formulated in this paper for the case where both the number of sampling times and the lengths of the sampling intervals are considered control parameters. The existence of an optimal closed loop sampled-data control law is proved for the cases where the number of samples are both finite and infinite. Computational algorithms for calculating the optimal control are proposed for both the case of finite and infinite number of samples.

7.1 Problem Formulation

Consider the linear system

$$\underline{\dot{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{u}(t) \tag{23}$$

$$\underline{y}(t) = \underline{C}\ \underline{x}(t)$$

with randomly distributed initial state

$$\begin{aligned} \varepsilon\{\underline{x}_0\} &= \underline{\xi}_o \\ \varepsilon\{(\underline{x}_o - \underline{\xi}_o)(\underline{x}_o - \underline{\xi}_0)'\} &= \underline{W} \end{aligned} \tag{24}$$

where $\underline{x}(t)$ is the n-dimensional state vector, $\underline{u}(t)$ is the r-dimensional control, and $\underline{y}(t)$ is the m-dimensional output vector and $\underline{A}$, $\underline{B}$, $\underline{C}$ are compatible time-invariant matrices. The sensor provides measurements

$$\underline{y}(t_i) = \underline{C}\ \underline{x}(t_i) \tag{25}$$

at the sampling times $\{t_i\}_{i=0}^{N}$ that are not specified and are constrained to satisfy

$$0 < T_{min} \leq t_{i+1} - t_i = T_i \leq T_{max} \tag{26}$$

where N is unspecfied and satisfies

$$N_{min} \leq N \leq N_{max} \tag{27}$$

The control actuator is also assumed to be a sampled-data device and therefore the control $\underline{u}(t)$ satisfies

$$\underline{u}(t) = \underline{u}(t_i) = \underline{u}_i \qquad t \in [t_i, t_{i+1}) \tag{28}$$

for $i = 0,1,\ldots,N-1$. This sampled-data control is specified by the control sequence $\{\underline{u}_i\}_{i=0}^{N-1}$, sampling time sequence $\{t_i\}_{i=0}^{N}$ and the number of sampling times N. The initial time $t_o \in (-\infty,\infty)$

and the terminal time $(t_f = t_N = \infty)$ are specified.

The design objective is to minimize the error $\underline{x}(t)$ with minimal control energy and minimal cost for implementing and operating a sampled-data control.

A system performance index is chosen of the form

$$S = J + C \tag{29}$$

where the control performance has the form

$$J = \varepsilon_{\underline{x}_o} \{ \int_{t_o}^{\infty} \frac{1}{2} [\underline{x}'(t) \underline{Q}\, \underline{x}(t) + \underline{u}'(t) \underline{R}\, \underline{u}(t)] dt \} \tag{30}$$

and the cost of implementation has the form

$$C(\underline{T}, N) = \sum_{i=0}^{N-1} \alpha \varepsilon^{-\beta T_i} \tag{31}$$

The matrix $\underline{Q}$ is positive semi-definite symmetric matrix and $\underline{R}$ is a positive definite symmetric matrix.

A cost for implementation is adjoined and represents the economic costs for implementing and operating a sampled-data control law. This cost for implementation can be considered to represent the cost for transmitting and storing the optimal sampled-data control law. It is similar in form to the costs for sampling used in the analytic derivation of adaptive sampling rules [13] and the optimal periodic sampling rate for a feedback control problem [56].

The control problem becomes:

Given the linear system (23, 24) with measurements (25) determine the piecewise constant control (28) specified by the control and sampling times sequence

$$\{\underline{u}_i\}_{i=0}^{N-1} \; ; \; \{t_i\}_{i=0}^{N-1} \; ; \quad \text{and} \quad N$$

that minimizes the performance index (29) satisfies the sampling constraints (26, 27).

This problem can not be solved directly due to the constraint on control (16). Therefore, the problem is transformed from a continuous-time one into a discrete-time one by the same technique used in Chapter III.

The sampled-data problem can be transformed into an equivalent discrete-time one by integrating (12) and (18) over each sampling interval $T_i = t_{i+1} - t_i$.

$$\underline{x}_{i+1} = \underline{\Phi}_i \underline{x}_i + \underline{D}_i \underline{u}_i \tag{32}$$

$$S = \varepsilon_{\underline{x}_o} \{ \tfrac{1}{2} \sum_{i=0}^{N-1} (\underline{x}_i' \underline{Q}_i \underline{x}_i + 2\underline{x}_i' \underline{M}_i \underline{u}_i + \underline{u}_i' \underline{R}_i \underline{u}_i) \} + C(\underline{T}, N) \tag{33}$$

where $\underline{x}_i = \underline{x}(t_i)$ and

$$\underline{\Phi}_i = \underline{\Phi}(T_i)$$

$$\underline{D}_i = \underline{D}(T_i) = \int_o^{T_i} \underline{\Phi}(t) \underline{B} \, dt$$

$$\underline{Q}_i = \underline{Q}(T_i) = \int_o^{T_i} \underline{\Phi}'(t) \underline{Q} \, \underline{\Phi}(t) dt$$

$$\underline{M}_i = \underline{M}(T_i) = \int_o^{T_i} \underline{\Phi}'(t) \underline{Q} \, \underline{D}(t) dt$$

$$\underline{R}_i = \underline{R}(T_i) = \int_o^{T_i} [\underline{R} + \underline{D}'(t) \underline{Q} \, \underline{D}(t)] dt$$

The matrices $\underline{M}_i$, $\underline{R}_i$ and $\underline{Q}_i$ are in general time varying even though $\underline{Q}$ and $\underline{R}$ are constant because the sampling intervals are not equal. The matrix $\underline{\Phi}_i$ is nonsingular because it is a fundamental matrix. Moreover, it is easily shown that $\underline{Q}_i(\underline{R}_i)$ is a positive semidefinite (definite) symmetric matrix because $\underline{Q}(\underline{R})$ is a positive semidefinite (definite) symmetric matrix.

The discrete-time problem becomes:

Given the sampled-data system (32, 24) with the measurements (25), determine the control and sampling interval sequences

$$\{\underline{u}_i\}_{i=0}^{N-1} \qquad \underline{T}' = (T_0, T_1, \ldots, T_{N-1}) \quad \text{and} \quad N$$

that minimize the cost function (33) subject to the sampling constraints (26, 27).

7.2 Computational Algorithm

The existence of an optimal control law for the optimal sampled-data regulator problem is now established for both the case where the number of samples is finite and unspecified and for the case where the number of samples is infinite. In both cases, the existence and uniqueness of the control is first established for the case where the number of samples and lengths of sampling intervals are specified. The existence of an optimal sampling interval sequence which defines the optimal sampled-data control law for a specified number of samples is then proved. In order to establish existence of a control for these cases, three separate sets of definitions of controllability and stabilizability are thus required. The definitions are stated for the following three conditions:

(1) where both the number of samples and the lengths of sampling intervals are specified;

(2) where the number of samples is specified but the lengths of sampling intervals are unspecified and considered control parameters;

(3) where both the number of samples and the lengths of sampling intervals are considered control parameters.

The first set of definitions are for the case where both the number of samples and the lengths of sampling intervals are specified.

Definition. A system (23) is said to be controllable on a sampling interval sequence $\{t_i\}_{i=0}^{p}$ if for any initial state $\underline{x}_o$ there exists a control sequence $\{\underline{u}_i\}_{i=0}^{p-1}$ which specifies the sampled-data control (28), such that $\underline{x}_p = \underline{0}$.

Definition. A system (23) is said to be stabilizable on a sequence $\{t_{h+1}\}_{i=0}^{p-1}$ if the part of the system, which can not be controlled by selecting $\{\underline{u}_i\}_{i=0}^{p-1}$, is stable.

The following set of definitions of controllability and stabilizability hold for the case where the number of samples is specified $(N = p)$ but the control sequence $\{\underline{u}_i\}_{i=0}^{p-1}$ and sampling time sequence $\{t_i\}_{i=0}^{p}$ are control parameters.

Definition. The system (23) is p-sampled-data controllable at t_o if for every initial state $\underline{x}_o$ there exists a control sequence $\{\underline{u}_i\}_{i=0}^{p-1}$ and a sampling time sequence $\{t_i\}_{i=0}^{p}$, which specify the sampled-data control (28), such that $\underline{x}_p = \underline{0}$.

Definition. A system is p-sampled-data stabilizable if the part of the system, which can not be controlled by selecting $\{\underline{u}_i\}_{i=0}^{p-1}$ and $\{t_i\}_{i=0}^{p}$ for a sampled-data control (28), is stable.

The final set of definitions hold for the case where both the number of samples and the lengths of sampling intervals are control parameters.

Definition. The system (23) is sampled-data controllable at t_o, if for every initial state $\underline{x}_o$, there exists a finite number of samples N, a control $\{\underline{u}_i\}_{i=0}^{N-1}$, and a sampling time sequence $\{t_i\}_{i=0}^{N}$, which specify the sampled-data control (28), such that $\underline{x}_N = \underline{0}$.

Definition. A system is said to be sampled-data stabilizable if the part of the system which can not be controlled by selecting N, $\{\underline{u}_i\}_{i=0}^{N-1}$, and $\{t_i\}_{i=0}^{N}$, for a sampled-data control (28) is stable.

The existence of an optimal sampled-data control law is first proved for the case where the number of samples is finite and unspecified and then for the case where the number of samples is infinite.

This theoretical development is presented not only to establish the existence of solutions, but also to provide a framework for the computational algorithm which follows development.

7.2.1 The Infinite-Time Problem with a Finite Number of Samples

The existence of an optimal sampled-data control law is proved in the following theorem for the case where the number of samples is specified. This result is proved by first establishing the existence and uniqueness of the closed loop control law for the case where both the number of samples and the lengths of sampling intervals are specified.

Theorem 7.1.

An optimal closed-loop control exists for the infinite-time sampled-data regulator if the system is p-sampled-data controllable or p-sampled-data stabilizable.

Proof: Consider the problem first for the case when $\underline{x}_o$ is specified. A feasible solution $\{\underline{u}_i\}_{i-0}^{p-1}$ exists for each $\{t_{n+i}\}_{i=0}^{p}$ for which the system is controllable or stabilizable. If the system is p-sampled-data controllable there exist sampling interval sequences $\underline{T} \epsilon [\underline{a},\underline{b}]$ for which feasible control sequences $\{\underline{u}_i\}_{i=0}^{p-1}$ exist. Therefore, it follows from Theorems 1 and 3 [22, pp. 137 and 133] respectively that there exist unique optimal control and trajectory sequences

$$\{\underline{u}_i(\underline{T})\}_{i=0}^{p-1} \quad \{\underline{x}_{i+1}(\underline{T})\}_{i=0}^{p-1}$$

for each $\underline{T} \in [\underline{a},\underline{b}]$ for which the system is controllable and stabilizable. For each feasible $\underline{T}$, the necessary conditions [6, 7], can be solved to obtain the control law

$$\underline{u}_i(\underline{T}) = -\underline{G}_i(\underline{T})\underline{x}_i(\underline{T})$$

where the $r \times n$ dimensional feedback gain matrix satisfies

$$\underline{G}_i(\underline{T}) = \underline{R}_i^{-1}\underline{M}_i' + [\underline{R}_i + \underline{D}_i'\underline{K}_{i+1}\underline{D}_i]^{-1}\underline{D}_i'\underline{K}_{i+1}\underline{\Theta}_i$$

The matrixes $\underline{K}_i(\underline{T})$ satisfy the matrix Riccati equation

$$\underline{K}_i = \underline{\Gamma}_i + \underline{\Theta}_i'[\underline{K}_{i+1} - \underline{K}_{i+1}\underline{D}_i[\underline{R}_i + \underline{D}_i'\underline{K}_{i+1}\underline{D}_i]^{-1}\underline{D}_i'\underline{K}_{i+1}]\underline{\Theta}_i$$

for $i = p-1,p-2,\ldots,1,0$ with boundary condition

$$\underline{K}_p = \underline{0}$$

where $\underline{\Theta}_i = \underline{\Phi}_i - \underline{D}_i\underline{R}_i^{-1}\underline{M}_i'$ and $\underline{\Gamma}_i = \underline{Q}_i - \underline{M}_i\underline{R}_i^{-1}\underline{M}_i'$. The associated optimal cost can be expressed as a function $S(\underline{T})$ defined over each $\underline{T} \in [\underline{a},\underline{b}]$. This derived cost function can be expressed as

$$S_o(\underline{T}) = \{S_{\underline{T}} = \frac{1}{2}\,\underline{x}_o\underline{K}_o(\underline{T})\underline{x}_o : \underline{a} \leq \underline{T} \leq \underline{b}\} + C(\underline{T},N)$$

The optimal sampling interval sequence $\underline{T}_o^*$ which minimizes $S_o(\underline{T})$ over this set of feasible $\underline{T}$ specifies an open loop control because $\underline{T}_o^*$ depends directly on the initial state $\underline{x}_o$.

Now letting $\underline{x}_o$ be randomly distributed and letting $\{\underline{u}_i(\underline{x}_o)\}_{i=0}^{p-1}$ be any closed loop control law the system performance (18) becomes

$$S_1(\underline{T}) = E_{\underline{x}_o}\{\frac{1}{2}\,\underline{x}_o\underline{K}_o(\underline{T})\underline{x}_o\} = \frac{1}{2}\,(Tr\{\underline{K}_o\underline{W}\} + \underline{\xi}_o'\underline{K}_o\underline{\xi}_o) + C(\underline{T},N)$$

since the following exchange of operators is valid

$$\min_{\{\underline{u}_i(\underline{x}_o)\}_{i=0}^{p-1}} \{E_{\underline{x}_o}\{\frac{1}{2}\,\underline{x}'_p\underline{Q}\underline{x}_p + \frac{1}{2}\sum_{i=0}^{p-1}(\underline{x}_i'\underline{Q}_i\underline{x}_i + 2\underline{x}_i'\underline{M}_i\underline{u}_i + \underline{u}_i'\underline{R}_i\underline{u}_i)\}\}$$

$$= E_{\underline{x}_o}\{\min_{\{\underline{u}_i(\underline{x}_o)\}_{i=0}^{p-1}} \{\frac{1}{2}\,\underline{x}'_p\underline{Q}\underline{x}_p + \frac{1}{2}\sum_{i=0}^{p-1}(\underline{x}_i'\underline{Q}_i\underline{x}_i + 2\underline{x}_i'\underline{M}_i\underline{u}_i + \underline{u}_i'\underline{R}_i\underline{u}_i)\}\}$$

Since there exist feasible solutions if the system is p-sampled-data controllable or p-sampled-data stabilizable and since the performance index $S_k(\underline{T})$; $k = 0,1$ is non-negative on each feasible sequence of sampling intervals, there exists an infinum $\underline{T}_k^*$. Since the set $\underline{T} \in [\underline{a},\underline{b}]$ is closed and bounded, there exists an optimal solution $\underline{T}_k^*$ for this derived problem. Thus, there exists a solution

$$\{\underline{u}_i(\underline{T}_k^*)\}_{i=0}^{N-1}, \quad \{\underline{x}_i(\underline{T}_k^*)\}_{i=1}^{N}, \quad \text{and} \quad \underline{T}_k^*$$

for the optimal infinite-time sampled-data regulator if the system is p-sampled-data controllable or p-sampled-data stabilizable. The control law is closed loop when the initial state is randomly distributed (k = 1) because the optimal sampling interval sequence $\underline{T}_1^*$ does not depend on the initial state.

An optimal control law will not exist if the system is not p-sampled-data controllable or, p-sampled-data stabilizable for a particular value $N = p$. However, if N is not specified and the system is controllable with continuous controls, it has been proved in Theorem 6.8 that the system will be controllable with sampled-data controls (28) for all $N \geq q$ where q is the order of the minimal polynomial of the system matrix $\underline{A}$. Therefore, an optimal infinite-time sampled-data control exists for all $p \geq q$ if the system is controllable with continuous controls. The maximum number of samples N_{max} should be chosen greater than q in order to insure that an optimal sampled-data control exists for every system for which an optimal continuous-time control law exists.

The algorithm, developed to compute the optimal sampled-data control law for the tracking problem can also be used to compute the control law for this problem with one slight modification. Since the control interval is now infinite, one of the sampling intervals would be infinite. Therefore, this control interval should be selected as large as possible, consistent with the word length of the computer being used to solve this problem.

Since this problem is related to the finite time problem and since extensive computational results have already been obtained for that problem, no effort will be made to present computational results for this case.

7.2.2 The Infinite-Time Problem with an Infinite Number of Samples

The existence of an optimal closed-loop sampled-data control law is now proved for the case where the number of samples is infinite. This result is proved by first establishing the existence of an optimal open loop control and then establishing the existence and uniqueness of a closed loop control law for the case where the number of samples and the lengths of sampling intervals are specified. These two preliminary results are presented in order to indicate the theoretical difficulties in proving the existence and uniqueness of this closed loop control law.

The existence of an optimal open loop sampled-data control law is now established.

THEOREM 7.2

An optimal open loop sampled-data control exists if the system is controllable or stablizable on the sampling time sequence $\{t_i\}_{i=0}^{\infty}$ chosen.

Proof.

If the system is sampled-data controllable or stabilizable for the sampling times sequence chosen, the system can be driven to the origin and therefore the error energy and control energy are both finite. Therefore, there exist feasible solution sequences

$$\{\underline{x}_{i+1}\}_{i=0}^{\infty} \qquad \{\underline{u}_i\}_{i=0}^{\infty}$$

for the sampling interval sequence $\underline{T}$

$$\underline{T}' = (T_0, T_1, \ldots, T_i, \ldots)$$

chosen if the system is controllable or stabilizable on this sequence. Since this set of feasible intervals is compact and since the performance index is bounded above and non-negative on this set, there exists an optimal control sequence $\{\underline{u}_i^*\}_{i=0}^{\infty}$ which minimizes the system performance (33) if the system is controllable or stabilizable on the sequence $\underline{T}$ chosen.

Although an optimal open loop sampled-data control exists for each sampling interval sequence $\underline{T}$ for which the system is controllable or stabilizable, the control is impractical because the entire infinite control sequence and sampling interval sequence must be computed and stored. The cost of implementation would make the system performance high and thus would make the open-loop control suboptimal. Since, the system is assumed observable, a closed loop control law is possible. If the gain of this closed loop control law is time invariant, the cost of implementation will be relatively low and the closed loop control law may be quite practical. Therefore the existence and uniqueness of the closed loop control law is established.

In the previous literature, the infinite-time sampled-data problem was only considered for the case of periodic sampling where the sampling period was specified. The existence and uniqueness of this closed loop control law was established [2] by first assuming the existence of the optimal infinite time control law and then assuming that the extension of the finite-time control converged to the infinite time control law as N and $t_f = t_N$ approached infinity. In a recent paper [55], the form of this infinite-time closed loop control law was established and the existence and uniqueness of the closed loop control was proved. Moreover the finite-time control was proved to converge to the infinite-time control as N approaches infinity. These results have only been established properly for the case of periodic sampling and have never been proved for the case of aperiodic sampling.

The existence and uniqueness of the optimal closed-loop control law for aperiodic sampling is not proved here because

(1) all previous work (except [55]) on the infinite time problem have always assumed that the finite-time feedback control law can be extended to the infinite time case;

(2) the results on the periodic sampled-data control law should be enough to suggest that the finite-time aperiodic sampled-data closed loop control law can also be extended to the infinite-time case; and

(3) the proofs for the aperiodic case are tedious and beyond the scope of this work.

The existence and uniqueness of the extension of the finite time closed loop control law is now proved.

<u>THEOREM</u> 7.3

An optimal closed-loop sampled-data control law exists and is unique if the system is controllable or stabilizable on the sampling time sequence chosen.

Proof.

Consider the case first when the initial state $\underline{x}_o$ is specified. The existence of an optimal open-loop control was proved if the system is controllable or stabilizable on the infinite sampling interval sequence chosen. Moreover, the infinite-time closed loop control was proved to exist for a finite number of samples and thus exists for an infinite number of samples. Moreover, it is assumed that the finite-time control law converges to the optimal infinite time control law as the number of sampling-times in the sequence increases to infinity. Thus, the infinite time control law has the form:

$$\underline{u}_i(\underline{T}) = -\underline{G}_i(\underline{T})\underline{x}_i(\underline{T})$$

where the $r \times n$ dimensional feedback gain matrix satisfies

$$\underline{G}_i(\underline{T}) = \underline{R}_i^{-1}\underline{M}_i' + [\underline{R}_i + \underline{D}_i'\underline{K}_{i+1}\underline{D}_i]^{-1}\underline{D}_i'\underline{K}_{i+1}\underline{\Theta}_i$$

The matrices $\underline{K}_i = \underline{\Gamma}_i + \underline{\Theta}_i'[\underline{K}_{i+1} - \underline{K}_{i+1}\underline{D}_i[\underline{R}_i + \underline{D}_i'\underline{K}_{i+1}\underline{D}_i]^{-1}\underline{D}_i'\underline{K}_{i+1}]\underline{\Theta}_i$

for $i = N-1, N-2, \ldots, 1, 0$ with boundary condition with

$$\underline{K}_N = \underline{0}$$

where $N = \infty$, $\underline{\Theta}_i = \underline{\Phi}_i - \underline{D}_i\underline{R}_i^{-1}\underline{M}_i'$ and $\underline{\Gamma}_i = \underline{Q}_i - \underline{M}_i\underline{R}_i^{-1}\underline{M}_i'$. The associated optimal cost can be expressed as a function $S(\underline{T})$ over the set $\underline{T} \in [\underline{a},\underline{b}]$. This derived cost function has the form

$$S_o(\underline{T}) = \{S_{\underline{T}} = \frac{1}{2}\,\underline{x}_o\underline{K}_o(\underline{T})\underline{x}_o : \underline{a} \le \underline{T} \le \underline{b}\} + C(\underline{T})$$

The optimal sampling interval sequence $\underline{T}_o^*$ which minimizes $S_o(\underline{T})$

over this set of feasible $\underline{T}$ specifies an open loop control because $\underline{T}_0^*$ depends directly on the initial state $\underline{x}_o$.

Now letting $\underline{x}_o$ be randomly distributed and letting $\{\underline{u}_i(\underline{x}_o)\}_{i=0}^{\infty}$ be any closed loop control law, the cost function becomes

$$S_1(\underline{T}) = E_{\underline{x}_o}\{\frac{1}{2}\,\underline{x}_o\underline{K}_o(\underline{T})\underline{x}_o\} = \frac{1}{2}(\mathrm{Tr}\{\underline{K}_o\underline{W}\} + \underline{\xi}_o'\underline{K}_o\underline{\xi}_o) + C(\underline{T})$$

since the following exchange of operators is valid

$$\min_{\{\underline{u}_i(\underline{x}_o)\}_{i=0}^{\infty}} \{E_{\underline{x}_o}\{\frac{1}{2}\sum_{i=0}^{\infty}(\underline{x}_i'\underline{Q}_i\underline{x}_i + 2\underline{x}_i'\underline{M}_i\underline{u}_i + \underline{u}_i'\underline{R}_i\underline{u}_i)\}\}$$

$$= E_{\underline{x}_o}\{\min_{\{\underline{u}_i(\underline{x}_o)\}_{i=0}^{\infty}}\{\frac{1}{2}\sum_{i=0}^{\infty}(\underline{x}_i'\underline{Q}_i\underline{x}_i + 2\underline{x}_i'\underline{M}_i\underline{u}_i + \underline{u}_i'\underline{R}_i\underline{u}_i)\}\}$$

The optimal sampled-data control law obtained by minimizing $S_1(\underline{T})$ over $\underline{T} \in [\underline{a},\underline{b}]$ is closed loop because the infinite sampling interval sequence does not depend on the initial state $\underline{x}_o$. The existence of the control law has been established under the assumption that the extension of the finite-time control law is the optimal infinite time control law for the case where the number of samples and the lengths of sampling intervals are specified.

The uniqueness of the control is proved by establishing the uniqueness of the sequence $\{\underline{K}_i\}_{i=0}^{\infty}$. Two distinct sequences $\{\overline{K}_i\}_{i=0}^{\infty}$ and $\{\hat{\underline{K}}_i\}_{i=0}^{\infty}$ are assumed and are now shown to be identical.

Let all $n \times n$ matrices form a metric space which is complete. From Theorem B [29, pg. 47] this metric space forms a normed linear space with matrix norm

$$\|\underline{A}\|_{\|\underline{x}\|=1} = \frac{\|\underline{A}\ \underline{x}\|}{\|\underline{x}\|}$$

The matrix sequence $\{\overline{\underline{K}}_i\}$ and $\{\hat{\underline{K}}_i\}$ satisfy the Reccah equation and therefore

$$\overline{\underline{K}}_i - \hat{\underline{K}}_i = \underline{\Theta}_i'\overline{\underline{K}}_{i+1}\overline{\underline{\Theta}}_i - \underline{\Theta}_i'\hat{\underline{K}}_{i+1}\hat{\underline{\Theta}}_i$$

where

$$\overline{\underline{\Theta}}_i = [\underline{I} + \underline{D}_i\underline{R}_i^{-1}\underline{D}_i'\overline{\underline{K}}_{i+1}]^{-1}\underline{\Theta}_i$$

$$\hat{\underline{\Theta}}_i = [\underline{I} + \underline{D}_i\underline{R}_i^{-1}\underline{D}_i'\hat{\underline{K}}_{i+1}]\underline{\Theta}_i$$

This matrix difference can be expressed as

$$\overline{\underline{K}}_i - \hat{\underline{K}}_i = \hat{\underline{\Theta}}_i\{[\underline{I} + \hat{\underline{K}}_{i+1}\underline{D}_i\underline{R}_i^{-1}\underline{D}_i'][\overline{\underline{K}}_{i+1} - \hat{\underline{K}}_{i+1}][I + \hat{\underline{K}}_{i+1}\underline{D}_i\underline{R}_i^{-1}\underline{D}_i']^{-1}\}\overline{\underline{\Theta}}_i$$

The term in the parenthesis is just a matrix similar to $\overline{\underline{K}}_{i+1} - \hat{\underline{K}}_{i+1}$ and thus has the same eigenvalues as $\overline{\underline{K}}_{i+1} - \hat{\underline{K}}_{i+1}$ [28]. Since $\overline{\underline{K}}_{i+1} - \hat{\underline{K}}_{i+1}$ is symmetric and $n \times n$, the norm of this matrix on the normed linear space is the maximum absolute value of the eigenvalues of the matrix. Therefore

$$\|[\underline{I} + \hat{\underline{K}}_{i+1}\underline{D}_i\underline{R}_i^{-1}\underline{D}_i][\overline{\underline{K}}_{i+1} - \hat{\underline{K}}_{i+1}][\underline{I} + \hat{\underline{K}}_{i+1}\underline{D}_i\underline{R}_i^{-1}\underline{D}_i']^{-1}\|$$

$$= \|\overline{\underline{K}}_{i+1} - \hat{\underline{K}}_{i+1}\|$$

The norm of this matrix difference in the normed linear space has the form

$$\| \bar{\underline{K}}_i - \hat{\underline{K}}_i \| \leq \|\hat{\underline{\Theta}}_i\| \cdot \|\bar{\underline{K}}_{i+1} - \hat{\underline{K}}_{i+1}\| \cdot \|\bar{\underline{\Theta}}_i\|$$

where the norms

$$\|\hat{\underline{\Theta}}_i\| < 1 \qquad\qquad i = 0,1,\ldots,i,\ldots$$
$$\|\bar{\underline{\Theta}}_i\| < 1$$

if the system is controllable or stabilizable on the sampling interval sequence $\underline{T}$. Then the difference matrix must approach zero as i approaches zero. Thus, there is a unique sequence $\{\underline{K}_i\}_{i=0}^{\infty}$ and a unique control law if the system is controllable or stabilizable on the infinite sampling interval sequence $\underline{T}$.

The existence and uniqueness of the optimal closed loop control law was proved formally for the case of periodic sampling [55]. The finite-time closed loop control law was then proved to be the infinite-time control law as the number of samples approach infinity. Thus, the assumption that the extension of the finite-time control law is the infinite-time control law is valid for the case of periodic sampling. The control law for the case of periodic sampling has also been proved time invariant rather than time varying. The following theorem is stated to establish the form and the existence and uniqueness of the periodic sampled-data control law.

<u>THEOREM</u> 7.4.

An optimal closed loop sampled-data control

$$\underline{u}_i^*(T) = -\underline{G}(T)\underline{x}_i^*(T)$$

where

$$\underline{G}(T) = [[\underline{R}^{-1}\underline{M} + [\underline{R} + \underline{D}\ \underline{K}\ \underline{D}']^{-1}\underline{D}'\underline{K}\ \underline{\Theta}]$$

exists and is unique if the system is controllable or stabilizable on a periodic sampling time sequence with period T. The system performance is

$$S_o(T) = \frac{1}{2}\ \underline{x}_o\underline{K}\ \underline{x}_o + C(T) \tag{34}$$

where $\underline{K}$ satisfies the Riccati equation

$$\underline{K} = \underline{\Gamma} + \underline{\Theta}'[\underline{K} - \underline{K}\ \underline{D}[R + \underline{D}'\underline{K}\ \underline{D}]\underline{D}'\underline{K}]\underline{\Theta}$$

The proof of this theorem is contained in the literature [55] and is not proved here.

The system performance (34) becomes

$$S_1(T) = \frac{1}{2}(\underline{\xi}_o'\underline{K}\ \underline{\xi}_o + \mathrm{Tr}\{\underline{K}\ \underline{W}\}) + C(T)$$

if the initial conditions are randomly distributed as assumed previously. The assumption is made, as pointed out earlier, in order to make the optimal sampling interval sequence $\underline{T}_1^*$ independent of the initial state $\underline{x}_o$. The following theorem establishes the existence of an optimal closed-loop sampled-data control law.

THEOREM 7.5

An optimal closed loop sampled-data control specified by $\{\underline{u}_i^*\}^\infty$ and $\underline{T}^*$ exists if the system is sampled-data controllable or sampled-data stabilizable.

Proof.

In Theorem 7.3, the existence and uniqueness of the optimal control was proved for each $\underline{T} \in [\underline{a},\underline{b}]$ for which the system is either controllable or stabilizable. Since the control is unique a derived performance index $S_1(\underline{T})$ was defined over $\underline{T} \in [\underline{a},\underline{b}]$. Since there exist feasible solution $\{\underline{T}, \{\underline{u}_i\}_{i=0}^{\infty}\}$ and since $S_1(T)$ is non-negative, an infinum exists. Since the set of feasible sampling intervals sequences $\underline{T} \in [\underline{a},\underline{b}]$ is a compactum, an optimal sampling interval sequence $\underline{T}_1^*$ exists. Thus an optimal control sequence $\{\underline{u}_i^*(\underline{T}_1^*)\}_{i=0}^{\infty}$ and trajectory sequence $\{\underline{x}_i(\underline{T}_1)\}_{i=0}^{\infty}$ exists. The control law is closed loop because the optimal sampling interval sequence $\underline{T}_1^*$ does not depend explicitly on the inital state.

Computing the optimal sampling interval sequence is in general impractical due to the high cost of computation storage, and hardware implementation. Thus the optimal sampling interval sequence must be highly structured and must depend on the form of the cost of implementation chosen.

A periodic sampling criterion has been heuristically established as optimal if the cost of implementation has the form

$$C(\underline{T}) = \sum_{i=0}^{\infty} \alpha \varepsilon^{-\beta T_i} \tag{35}$$

It is quite apparent that other structured sampling criteria may be optimal if other forms for the cost of implementation are proposed.

The optimal sampling period T^* for this infinite-time optimal sampled-data regulator problem with cost of implementation (35) can easily be computed using a one dimensional search algorithm, such as Fibonacci search. The use of such an algorithm on several example problems is a subject for future research.

CHAPTER VIII

CONCLUSIONS AND FURTHER INVESTIGATION

The principal contribution of this thesis is the development of a new framework for the design and analysis of sampled-data control systems.

The formulation of the sampled-data control problem is extended by considering both the number of samples and the lengths of sampling intervals as control parameters. A system performance index is proposed which measures not only control performance but also the cost of implementation. The sampled-data control is generalized by assuming polynomial form over each sampling interval.

The controllability and observability of these sampled-data control systems are defined for the case where both the number of sampling times and the lengths of sampling intervals are control variables. It is established that a necessary and sufficient condition for p-sampled-data controllability and observability can be decomposed into a condition on the controllability and observability of the continuous-time system and a condition on the sampling times sequence. A sufficient condition on the sampling time sequence is stated which will guarantee the preservation of controllability and observability when continuous measurements and controls are replaced by sampled measurements and a sampled-data (sample and hold) control. Finally, a system which is controllable with continuous controls and observable with continuous measurements is proved to be controllable with a sampled-data control and observable with sampled measurements if the number of sampling times and the lengths of sampling intervals are chosen properly. Some sufficient conditions are derived which indicate the properties which must be satisfied for a system which is controllable and observable with continuous measurements and controls to be uncontrollable and unobservable with sampled measurements and a sampled-data control. These results on controllability and observability indicate the actuator which implements the control commands and the sensor which makes measurements should be considered part of the control law rather than part of the model of the system to be controlled since the number of samples and the lengths of sampling intervals are shown to be control parameters.

These control problems were formulated in this thesis: the sampled-data tracking problem, the infinite time sampled-data regulator problem. The existence of an optimal sampled-data control law was proved and a computational algorithm was developed for both problems.

The optimal continuous time control law was proved to be a sub-optimal sampled-data control law if the cost for implementation was not negligible and the optimal sampled-data control law if the cost for implementation was negligible. This result indicates this sampled-data formulation should be the general formulation of the optimal control problem because the decision on the form of the control, the form of the measurement system, and the form of the actuator can only be made properly if this formulation is used.

Extensive computation results were obtained to compare the performance of optimal periodic sampled-data control and optimal aperiodic sampled-data control for the tracking problem. Comparisons of performance were made on different examples with different inputs. Moreover the performance of these systems was also evaluated for optimal sampled-data control laws with different order control approximations. The study of both the order of control approximation and the optimal aperiodic sampling were made to determine the possible reduction in information required and possible control performance improvement which can result by properly parameterizing the optimal control.

Topics for further investigation are listed below:

(1) the development of a cost of implementation which more realistically models the cost of utilization of computer and communication hardware, the cost of sensors, actuators, and the cost of computing the control,

(2) the investigation of alternative search algorithms which can outperform Powells algorithm in the computational programs developed,

(3) the extension of controllability and observability of sampled-data systems to the case where the sampled-data control is modeled by a polynomial rather than a sampled and hold mechanism,

(4) the development of computer program which implements the algorithms developed for the infinite time regulator problem,

(5) the formulation and solution of optimal stochastic sampled-data control problem.

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APPENDICES

A. SAMPLED DATA FORM OF THE SYSTEM EQUATIONS

$$\underline{x}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)$$

has a solution

$$\underline{x}(t) = e^{\underline{A}(t-t_0)}\underline{x}(t_0) + \int_{t_0}^{t} e^{\underline{A}(t-\tau)}\underline{B}\,\underline{u}(\tau)d\tau$$

Therefore the solution over one interval becomes

$$\underline{x}_{i+1} = e^{\underline{A}T_i}\underline{x}_i + \int_{t_i}^{t_{i+1}} e^{\underline{A}(t_{i+1}-\tau)}\underline{B}[\underline{u}_{0i} + \ldots + \underline{u}_{ki}(\tau - t_i)^k]d\tau$$

Changing variables $t_{i+1} - \tau = t$ gives

$$\underline{x}_{i+1} = \underline{\phi}_i\underline{x}_i + \int_0^{T_i} e^{\underline{A}t}\underline{B}dt\;\underline{u}_{0i} + \ldots + \int_0^{T_i} e^{\underline{A}t}\underline{B}(T_i - t)^k dt\;\underline{u}_{ki}$$

which can be put into matrix form as

$$\underline{x}_{i+1} = \underline{\phi}_i\underline{x}_i + [\underline{D}_{0i},\ldots,\underline{D}_{ki}]\begin{bmatrix}\underline{u}_{0i}\\ \cdot\\ \cdot\\ \cdot\\ \underline{u}_{ki}\end{bmatrix} = \underline{\phi}_i\underline{x}_i + \underline{D}_i\underline{u}_i \tag{A1}$$

where $\underline{D}_{ki} = \int_0^{T_i} e^{\underline{A}t}\underline{B}(T_i - t)^k dt$

The variable $k = 0,1,2,$ represents step, ramp and parabolic control approximation respectively.

Similarly, the following equations can be established

$$\underline{x}(t) = \underline{\phi}(t)\underline{x}_i + \underline{D}(t)\underline{u}_i \tag{A2}$$

for $t_i \leq t < t_{i+1}$

where $$\underline{\phi}(t) = e^{\underline{A}(t-t_i)}$$

$$\underline{D}(t) = [\underline{D}_{0i}(t,t_i),\ldots,\underline{D}_{ki}(t,t_i)]$$

$$D_{ki}(t,t_i) = \int_0^{t-t_i} e^{\underline{A}x}\underline{B}(t - t_i - x)^k dx$$

B. DISCRETE FORM OF COST FUNCTIONAL

The continuous form of cost functional is

$$S = \frac{1}{2}\langle \underline{y}(t_N) - \underline{z}(t_N), \underline{F}(\underline{y}(t_N) - \underline{z}(t_N))\rangle + \frac{1}{2}\int_{t_o}^{t_N} [\langle \underline{y}(t) - \underline{z}(t), \underline{Q}(\underline{y}(t)$$

$$- \underline{z}(t))\rangle + \langle \underline{u}(t), \underline{R}\,\underline{u}(t)\rangle]dt + \sum_{i=0}^{N-1} \alpha e^{-\beta T_i} \tag{B1}$$

Since $\underline{y}(t) = \underline{C}x(t)$ and $\underline{F}$ is symmetric

$$\frac{1}{2}\langle \underline{y}(t_N) - \underline{z}(t_N), \underline{F}(\underline{y}(t_N) - \underline{z}(t_N))\rangle = \frac{1}{2}\,\underline{x}_N'\hat{\underline{F}}\,\underline{x}_N + \frac{1}{2}\,\underline{z}'(t_N)\underline{F}\,\underline{z}(t_N) - \underline{h}_N\underline{x}_N \tag{B2}$$

where

$$\hat{\underline{F}} = \underline{C}'\underline{F}\,\underline{C}$$

$$\underline{h}_N = \underline{z}'(t_N)\underline{F}\,\underline{C}$$

$$\underline{x}_N = \underline{x}(t_N)$$

Using (A2) and the fact that

$$\int_{t_o}^{t_N} f(t)dt = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} f(t)dt$$

and

$$\underline{u}(t) = \underline{u}_{0i} +\ldots+ \underline{u}_{ki}(t - t_i)^k = [\underline{I},\ldots,(t - t_i)^k\,\underline{I}] \begin{bmatrix} \underline{u}_{0i} \\ \vdots \\ \underline{u}_{ki} \end{bmatrix} ,$$

it is obvious that

$$\frac{1}{2}\int_{t_o}^{t_N}[<\underline{y}(t) - \underline{z}(t),\underline{Q}(\underline{y}(t) - \underline{z}(t))> + <\underline{u}(t),\underline{R}\ \underline{u}(t)>]dt$$

$$= \sum_{i=0}^{N-1} [\frac{1}{2}\ \underline{x}_i'\underline{Q}_i\underline{x}_i + \underline{x}_i'\underline{M}_i\underline{u}_i + \frac{1}{2}\ \underline{u}_i'\underline{R}_i\underline{u}_i - \underline{h}_i\underline{x}_i - \underline{g}_i\underline{u}_i]$$

$$+ \frac{1}{2}\int_{t_o}^{t_N} \underline{z}'(t)\underline{Q}\ \underline{z}(t)dt \tag{B3}$$

$$\underline{M}_i = \int_o^{T_i} e^{\underline{A}'\tau}\hat{\underline{Q}}\ \underline{D}(\tau)d\tau$$

where $$\underline{Q}_i = \int_o^{T_i} e^{\underline{A}'\tau}\hat{\underline{Q}}\ e^{\underline{A}\tau}d\tau$$

$$\underline{R}_i = \int_o^{T_i}[\underline{D}'(\tau)\hat{\underline{Q}}\ \underline{D}(\tau) + \begin{bmatrix} \underline{R} & \cdots & \underline{R}\tau^{2k-3} & \underline{R}\tau^{2k-2} \\ \cdot & & & \underline{R}\tau^{2k-1} \\ \vdots & & & \\ \cdot & \cdots & \underline{R}\tau^{2k-1} & \underline{R}\tau^{2k} \end{bmatrix}]d\tau$$

$$\hat{\underline{Q}} = \underline{C}'\underline{Q}\ \underline{C}$$

$$\underline{h}_i = \int_o^{T_i}\underline{z}'(t_i + \tau)\underline{Q}\ \underline{C}\ e^{\underline{A}\tau}d\tau$$

$$\underline{g}_i = \int_o^{T_i} \underline{z}'(t_i + \tau)\underline{Q}\ \underline{C}\ \underline{D}(\tau)d\tau$$

Substituting (B2) and (B3) into (B1) gives

$$S = J_0 + \sum_{i=0}^{N-1} \alpha e^{-\beta T_i} + \frac{1}{2}\ \underline{x}_N'\hat{\underline{F}}\ \underline{x}_N - \underline{h}_N\underline{x}_N + \frac{1}{2}\sum_{i=0}^{N-1}(\underline{x}_i'\underline{Q}_i\underline{x}_i + 2\underline{x}_i'\underline{M}_i\underline{u}_i$$

$$+ \underline{u}_i'\underline{R}_i\underline{u}_i - 2\underline{h}_i\underline{x}_i - 2\underline{g}_i\underline{u}_i) \tag{B4}$$

where $$J_0 = \frac{1}{2}\ \underline{z}'(t_N)\underline{F}\ \underline{z}(t_N) + \frac{1}{2}\int_{t_o}^{t_N} \underline{z}'(t)\underline{Q}\ \underline{z}(t)dt$$

C. KUHN-TUCKER NECESSARY CONDITION OF OPTIMALITY FOR QUADRATIC PROGRAMMING PROBLEM

The canonical form of the quadratic programming problem is:

Minimize $\quad \frac{1}{2}\langle \underline{v}, \underline{Qv} \rangle + \langle \underline{d}, \underline{v} \rangle$

subject to the constraints:

$$\underline{R}\ \underline{v} = \underline{c} \quad \text{and} \quad \underline{\alpha} \leq \underline{v} \leq \underline{\beta} \tag{C1}$$

The Kuhn-Tucker necessary conditions for this problem are stated in the following theorem.

<u>THEOREM</u>

If $\underline{\hat{v}}$ is a feasible solution to problem (C1), then $\underline{\hat{v}}$ is an optimal solution if and only if there exists a vector

$$\underline{\psi}' = (\psi', \ldots, \psi^m) \in E^m$$

such that for $i = 1,2,\ldots,n$

$$\langle \underline{\gamma}_i, \underline{\psi} \rangle - \langle \underline{q}_i, \underline{\hat{v}} \rangle - \underline{d}^i = 0 \quad \text{if} \quad \underline{\alpha}^i < \underline{\hat{v}}^i < \underline{\beta}^i$$

$$\langle \underline{\gamma}_i, \underline{\psi} \rangle - \langle \underline{q}_i, \underline{\hat{v}} \rangle - \underline{d}^i \leq 0 \quad \text{if} \quad \underline{\alpha}^i = \underline{\hat{v}}^i$$

$$\langle \underline{\gamma}_i, \underline{\psi} \rangle - \langle \underline{q}_i, \underline{\hat{v}} \rangle - \underline{d}^i \geq 0 \quad \text{if} \quad \underline{\beta}^i = \underline{\hat{v}}^i$$

where for $i = 1,2,\ldots,n$, $\underline{\gamma}_i$ is the ith column of $\underline{R}$, $\underline{q}_i$ is the ith column of $\underline{Q}$, and $\underline{d}^i$ is the ith component of $\underline{d}$.

D. NECESSARY CONDITIONS FOR OPTIMALITY OF THE SAMPLED DATA TRACKING PROBLEM

The sampled-data tracking problem can be put into the canonical form (Appendix C) of the quadratic programming problem as follows.

The cost functional

$$J = J_0 + \sum_{i=0}^{N-1} \alpha e^{-\beta T_i} + \frac{1}{2} \underline{x}_N' \hat{\underline{F}}\, \underline{x}_N - \underline{h}_N \underline{x}_N + \frac{1}{2} \sum_{i=0}^{N-1} (\underline{x}_i' \underline{Q}_i \underline{x}_i + 2\underline{x}_i' \underline{M}_i \underline{u}_i$$

$$+ \underline{u}_i' \underline{R}_i \underline{u}_i - 2\underline{g}_i \underline{u}_i - 2\underline{h}_i \underline{x}_i)$$

can be represented as

$$J = J_0 + \sum_{i=0}^{N-1} \alpha e^{-\beta T_i} + \frac{1}{2} \langle \underline{v}, \underline{Q}\, \underline{v} \rangle + \langle \underline{d}, \underline{v} \rangle \qquad \text{(D1)}$$

The $(n + kr)(N + 1)$* vector $\underline{z}$ has the form

$$\underline{z}' = [\underline{x}_0' \underline{u}_0' \cdots \underline{x}_{N-1}' \underline{u}_{N-1}' \underline{x}_N' \underline{0}]$$

and the $(n + kr)(N + 1)$ square matrix $\underline{Q}$ is

$$\underline{Q} = \begin{bmatrix} \hat{\underline{Q}}_0 & & & & \\ & \hat{\underline{Q}}_1 & & & \\ & & \ddots & & \\ & & & \hat{\underline{Q}}_{N-1} & \\ & & & & \hat{\underline{Q}}_N \end{bmatrix}$$

---

* As the dimension is concerned, k = 1 for step, k = 2 for ramp, k = 3 for parabola.

where the $(n + kr)$ square matrices $\hat{\underline{Q}}_i$, $i = 0,1,\ldots,N$ are defined as follows

$$\hat{\underline{Q}}_i = \begin{bmatrix} \underline{Q}_i & \underline{M}_i \\ \underline{M}_i' & \underline{R}_i \end{bmatrix} \qquad \text{for } i = 0,1,\ldots,N-1$$

$$\hat{\underline{Q}}_N = \begin{bmatrix} \hat{F} & \underline{0} \\ \underline{0} & 0 \end{bmatrix}$$

The $(n + kr)(N + 1)$ vector $\underline{d}$ is

$$\underline{d}' = [-\underline{h}_0, -\underline{g}_0, \ldots, -\underline{h}_{N-1}, -\underline{g}_{N-1}, -\underline{h}_N, \underline{0}]$$

The state equations and the initial condition

$$\underline{x}_0 = \underline{\xi}$$

$$\underline{x}_{i+1} = \underline{\phi}_i \underline{x}_i + \underline{D}_i \underline{u}_i \qquad i = 0,1,\ldots,N-1$$

can be put into the form as

$$\underline{R}\ \underline{v} = \underline{C} \tag{D2}$$

where the $n(N + 1) \times (n + kr)(N + 1)$ matrix $\underline{R}$ is

$$\underline{R} = \begin{bmatrix} \underline{I}\ \underline{0}_{n(kr)} & & & & \\ -\underline{\phi}_o\ -\underline{D}_o & \underline{I}\ \underline{0} & & & \\ & -\underline{\phi}_1 -\underline{D}_1 & \underline{I}\ \underline{0} & & \\ & & & \ddots & \\ & & & -\underline{\phi}_{N-1} -\underline{D}_{N-1} & \underline{I}\ \underline{0} \end{bmatrix}$$

and the $n(N + 1)$ vector $\underline{C}$ is

$$\underline{C}' = [\underline{\xi}\ \underline{0}\ .\ .\ .\ \ \underline{0}]$$

The inequality constraints

$$\underline{\alpha} < \underline{v} < \underline{\beta} \tag{D3}$$

are not restrictive since all the elements in $\underline{\alpha}$ are set as $-\infty$ and in $\underline{\beta}$ as $+\infty$.

From (D1), (D2), (D3) the tracking problem is transformed into a canonical quadratic programming problem as (C1). By using the theorem in Appendix C, if $\hat{\underline{v}}$ is a feasible solution, then $\hat{\underline{v}}$ is an optimal solution if and only if there exists a vector

$$\underline{p}' = [\underline{p}_0',\underline{p}_1',\ldots,\underline{p}_N']$$

where $$\underline{p}_i' = [p_{i1},p_{i2},\ldots,p_{in}]$$

such that for $i = 1,2,\ldots,(N + 1)$

$$<\underline{\gamma}_i,\underline{p}> - <\underline{q}_i,\hat{\underline{v}}> - \underline{d}^i = 0$$

This condition can be stated in matrix form as follows

$$\underline{R}'\underline{p} - \underline{Q}\ \underline{v} - \underline{d} = \underline{0}$$

because $$\alpha^i = -\infty,\ \ \beta^i = \infty\ \ \forall\ i$$

This vector condition can be shown to be equivalent to the following set of conditions.

$$\underline{p}_i = \underline{\phi}_i' p_{i+1} + \underline{Q}_i \underline{x}_i + \underline{M}_i \underline{u}_i - \underline{h}_i' \qquad \text{(D4)}$$

$$\underline{0}_{kr} = \underline{D}_i' \underline{p}_{i+1} + \underline{M}_i' \underline{x}_i + \underline{R}_i \underline{u}_i - \underline{g}_i' \qquad \text{(D5)}$$

for $i = 0,1,\ldots,N-1$ and

$$\underline{p}_N = \underline{\hat{F}}\ \underline{x}_N - \underline{h}_N \qquad \text{(D6)}$$

Thus the necessary conditions for the problem are the equations stated in Theorem 1.

E. DERIVATION OF THE CONTROL LAW

The control law can be solved for uniquely using (D5) to obtain

$$\underline{u}_i = -\underline{R}_i^{-1}(\underline{M}_i'\underline{x}_i + \underline{D}_i'\underline{p}_{i+1} - \underline{g}_i') \tag{E1}$$

since $\underline{R}_i$ is positive definite for all i.

Assuming the (Lagrange) multipliers have the form (verified in [24])

$$\underline{p}_i = \underline{K}_i\underline{x}_i + \underline{k}_i \tag{E2}$$

where $\underline{K}_i$, $\underline{k}_i$ are to be determined.

Using (E1), (E2) eliminates $\underline{u}_i$ from (7) and (D4), followed by rearranging terms with the help of the well known matrix identity

$$(\underline{I}_n + \underline{A}\ \underline{B}')^{-1} = \underline{I}_n - \underline{A}(\underline{I}_r + \underline{B}'\underline{A})^{-1}\underline{B}'$$

firmly yields

$$\underline{x}_{i+1} = \hat{\underline{\Theta}}_i\underline{x}_i + \underline{D}_i\underline{S}_i^{-1}(\underline{g}_i' - \underline{D}_i'\underline{k}_{i+1}) \tag{E3}$$

$$\underline{p}_i = \underline{\Theta}_i'\underline{p}_{i+1} + \underline{\Gamma}_i\underline{x}_i + \underline{M}_i\underline{R}_i^{-1}\underline{g}_i' - \underline{h}_i' \tag{E4}$$

for $i = 0,1,\ldots,N-1$ where

$$\underline{\Theta}_i = \underline{\phi}_i - \underline{D}_i\underline{R}_i^{-1}\underline{M}_i'$$

$$\hat{\underline{\Theta}}_i = (\underline{I} - \underline{D}_i\underline{S}_i^{-1}\underline{D}_i'\underline{K}_{i+1})\underline{\Theta}_i$$

$$\underline{S}_i = (\underline{R}_i + \underline{D}_i'\underline{K}_{i+1}\underline{D}_i)$$

$$\underline{\Gamma}_i = \underline{Q}_i - \underline{M}_i\underline{R}_i^{-1}\underline{M}_i'$$

Comparing (E2) with (E4) and substituting (E3) for $\underline{x}_{i+1}$ give

$$\underline{K}_i\underline{x}_i + \underline{k}_i = (\underline{\Theta}_i'\underline{K}_{i+1}\underline{\hat{\Theta}}_i + \underline{\Gamma}_i)\underline{x}_i + (\underline{G}_i'\underline{g}_i - \underline{h}_i' + \underline{\hat{\Theta}}_i'\underline{k}_{i+1}) \tag{E5}$$

where $\underline{G}_i' = (\underline{M}_i\underline{R}_i^{-1} + \underline{\Theta}_i'\underline{K}_{i+1}\underline{D}_i\underline{S}_i^{-1})$

Since (E5) must hold for <u>any</u> choice of initial state $\underline{\xi}$ and since $\underline{K}_i$, $\underline{k}_i$ does not depend on $\underline{\xi}$, (E5) must be satisfied for all $\underline{x}_i$. This implies

$$\underline{K}_i = \underline{\Theta}_i'\underline{K}_{i+1}\underline{\hat{\Theta}}_i + \underline{\Gamma}_i \tag{E6}$$

$$\underline{k}_i = \underline{G}_i'\underline{g}_i' - \underline{h}_i' + \underline{\hat{\Theta}}_i'\underline{k}_{i+1} \tag{E7}$$

Substituting (E2), (E3) into (E1) and rearranging terms finally obtain

$$\underline{u}_i = -\underline{G}_i\underline{x}_i + \underline{S}_i^{-1}(\underline{g}_i' - \underline{D}_i'\underline{k}_{i+1}) \tag{E8}$$

E. DERIVATION OF THE OPTIMAL COST

The discrete form of cost functional is (from B4)

$$S = J_0 + \sum_{i=0}^{N-1} \alpha e^{-\beta T_i} + \frac{1}{2} \underline{x}_N' \hat{F} \underline{x}_N - \underline{h}_N \underline{x}_N + \sum_{i=0}^{N-1} J_i \qquad \text{(F1)}$$

where

$$J_i = \frac{1}{2}(\underline{x}_i' \underline{Q}_i \underline{x}_i + 2\underline{x}_i' \underline{M}_i \underline{u}_i + \underline{u}_i' \underline{R}_i \underline{u}_i - 2\underline{h}_i \underline{x}_i - 2\underline{g}_i \underline{u}_i)$$

Replacing $\underline{u}_i$ by use of (E1) obtains

$$J_i = \frac{1}{2}(\underline{x}_i' \underline{\Gamma}_i \underline{x}_i + \underline{p}_{i+1}' [\underline{D}_i \underline{R}_i^{-1} \underline{D}_i' \underline{p}_{i+1}]) + \frac{1}{2}(2\underline{x}_i' \underline{M}_i \underline{R}_i^{-1} \underline{g}_i' - 2\underline{h}_i \underline{x}_i - \underline{g}_i \underline{R}_i^{-1} \underline{g}_i') \qquad \text{(F2)}$$

Using (E1), (7) can be rearranged as

$$\underline{D}_i \underline{R}_i^{-1} \underline{D}_i' \underline{p}_{i+1} = -\underline{x}_{i+1} + \underline{\Theta}_i \underline{x}_i + \underline{D}_i \underline{R}_i^{-1} \underline{g}_i' \qquad \text{(F3)}$$

Replacing the bracketed term in (F2) by (F3) substituting $\underline{u}_i$ for using (E1) and using the rearranged (E4)

$$\underline{p}_i' = \underline{p}_{i+1}' \underline{\Theta}_i + \underline{x}_i' \underline{\Gamma}_i + \underline{g}_i \underline{R}_i^{-1} \underline{M}_i' - \underline{h}_i$$

(F2) becomes

$$J_i = \frac{1}{2}[(\underline{p}_i' \underline{x}_i - \underline{p}_{i+1}' \underline{x}_{i+1}) + (\underline{p}_{i+1}' \underline{D}_i \underline{R}_i^{-1} - \underline{g}_i \underline{R}_i^{-1} + \underline{x}_i' \underline{M}_i \underline{R}_i^{-1}) \underline{g}_i' - \underline{h}_i \underline{x}_i]$$

$$= \frac{1}{2} [(\underline{p}_i' \underline{x}_i - \underline{p}_{i+1}' \underline{x}_{i+1}) - \underline{u}_i' \underline{g}_i' - \underline{h}_i \underline{x}_i] \qquad \text{(F4)}$$

Replacing $\underline{u}_i$ with (E8), rearranging (E7) and substituting (E3), the following expression becomes

$$\underline{u}_i'\underline{g}_i' + \underline{h}_i\underline{x}_i = \underline{g}_i\underline{u}_i + \underline{h}_i\underline{x}_i$$

$$= (\hat{\underline{\Theta}}_i\underline{k}_{i+1} - \underline{k}_i)'\underline{x}_i - \underline{g}_i\underline{S}_i^{-1}\underline{D}_i'\underline{k}_{i+1} + \underline{g}_i\underline{S}_i^{-1}\underline{g}_i'$$

$$= (\underline{k}_{i+1}'\underline{x}_{i+1} - \underline{k}_i\underline{x}_i) + (\underline{g}_i' - \underline{D}_i'\underline{k}_{i+1})'\underline{S}_i^{-1}(\underline{g}_i' - \underline{D}_i'\underline{k}_{i+1}) \qquad \text{(F5)}$$

Therefore, putting (F5) back into (F4) obtains

$$J_i = \frac{1}{2}[(\underline{p}_i'\underline{x}_i - \underline{p}_{i+1}'\underline{x}_{i+1}) - (\underline{k}_{i+1}'\underline{x}_{i+1} - \underline{k}_i'\underline{x}_i)$$

$$- (\underline{g}_i' - \underline{D}_i'\underline{k}_{i+1})'\underline{S}_i^{-1}(\underline{g}_i' - \underline{D}_i'\underline{k}_{i+1})] \qquad \text{(F6)}$$

Substituting (F6) into (F1) and using (D6), which implies

$$\underline{K}_N = \hat{\underline{F}} \qquad \text{and} \qquad \underline{k}_N = -\underline{h}_N$$

gives (10) i.e.

$$S = J_0 + \sum_{i=0}^{N-1} \alpha e^{-\beta T_i} + \frac{1}{2}\underline{x}_o'\underline{K}_o\underline{x}_o + \underline{k}_o'\underline{x}_o$$

$$- \frac{1}{2}\sum_{i=0}^{N-1} (\underline{g}_i' - \underline{D}_i'\underline{k}_{i+1})'\underline{S}_i^{-1}(\underline{g}_i' - \underline{D}_i'\underline{k}_{i+1})$$

G. EXISTENCE AND UNIQUENESS OF OPTIMAL CONTROL AND TRAJECTORY FOR EACH SPECIFIED $\underline{T}$

It is shown in Appendix D that a quadratic programming problem is formed by (D1), (D2), (D3). The following two theorems state conditions for the existence and uniqueness of an optimal solution for this problem.

THEOREM

If $\eta^{*}(\underline{Q}) \cap \eta(\underline{R}) = \{0\}$ and the quadratic programming problem (D1, D2, D3) has a feasible solution, then it has a unique optimal solution.

THEOREM

If $<\underline{d},\hat{\underline{v}}> = 0$ for every $\hat{\underline{v}} \in E^n$ satisfying $\underline{Q}\,\hat{\underline{v}} = \underline{0}$ and $\underline{R}\,\underline{v} = \underline{0}$ and if there exists a feasible solution to the quadratic programming problem (D1, D2, D3), then there exists an optimal solution.

It is established that the conditions in the above theorem for existence and uniqueness of an optimal solution are satisfied if $[\frac{\underline{Q}}{\underline{R}}]$ has maximum column rank, and $\underline{Q}$ is positive semi definite. Therefore these two conditions will be established to prove the existence and uniqueness of the optimal solution.

The matrix $\underline{Q}$ is now proved to be positive semi definite. The matrix

$$\underline{Q} = \begin{bmatrix} \hat{\underline{Q}}_o & & & & \\ & \hat{\underline{Q}}_1 & & & \\ & & \ddots & & \\ & & & \hat{\underline{Q}}_{N-1} & \\ & & & & \hat{\underline{Q}}_N \end{bmatrix}$$

$^{*}\eta(\cdot)$ means "the null space of".

is positive semi-definite if $\{\hat{\underline{Q}}_i\}_{i=0}^{N}$ are positive semi-definite. The matrices $\hat{\underline{Q}}_i$; $i = 0,1,\ldots,N-1$ have the form

$$\hat{\underline{Q}}_i = \begin{bmatrix} \underline{Q}_i & \underline{M}_i \\ \underline{M}_i' & \underline{R}_i \end{bmatrix}$$

$$= \begin{bmatrix} \underline{I} & \underline{M}_i\underline{R}_i^{-1} \\ \underline{0} & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{Q}_i - \underline{M}_i\underline{R}_i^{-1}\underline{M}_i' & \underline{0} \\ \underline{0} & \underline{R} \end{bmatrix} \begin{bmatrix} \underline{I} & \underline{0} \\ \underline{R}^{-1}\underline{M}' & \underline{I} \end{bmatrix}$$

$\underline{Q}_i - \underline{M}_i\underline{R}_i^{-1}\underline{M}_i'$ is positive semi-definite from the proof of Lemma 1 of (2, p. 347) and therefore $\hat{\underline{Q}}_i$ are positive semi-definite for $i = 0,1,\ldots,N-1$. The matrix

$$\hat{\underline{Q}}_N = \begin{bmatrix} \hat{\underline{F}} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}$$

is positive semi-definite since $\hat{F}$ is positive semidefinite and therefore $\underline{Q}$ is positive semidefinite which completes one part of the proof.

The $((n + kr)(N + 1) + n(N + 1))$ by $(n + kr)(N + 1)$ matrix $[\frac{\underline{Q}}{\underline{R}}]$ has the form

$$\begin{bmatrix} \underline{Q} \\ \underline{R} \end{bmatrix} = \left[\begin{array}{cccccc} \underline{Q}_o & \underline{M}_o & & & & \\ \underline{M}_o' & \underline{R}_o & & & & \\ & & \underline{Q}_1 & \underline{M}_1 & & \\ & & \underline{M}_1' & \underline{R}_1 & \cdot & \\ & & & & & \cdot \quad \cdot \\ \hline \underline{I} & \underline{0} & & & & \\ -\underline{\phi}_o & -\underline{D}_o & \underline{I} & \underline{0} & & \\ & & -\underline{\phi}_1 & -\underline{D}_1 & \underline{I} & \underline{0} \\ & & & & \cdot & \\ & & & & & \cdot \quad \cdot \end{array}\right]$$

By appropriate column and row operation, it can be transformed into an equivalent matrix.

$$\begin{bmatrix} \underline{Q}^* \\ \underline{R}^* \end{bmatrix} = \left[\begin{array}{cccccc} \underline{0} & \underline{0} & & & & \\ \underline{0} & \underline{R}_o & & & & \\ & & \underline{0} & \underline{0} & & \\ & & \underline{0} & \underline{R}_1 & \ddots & \\ \hline \underline{I} & \underline{0} & & & & \\ \underline{0} & \underline{0} & \underline{I} & \underline{0} & & \\ & & \underline{0} & \underline{0} & \ddots & \end{array}\right]$$

These $\underline{I}$'s and $\underline{R}_i$'s are nonsingular; the matrix $\begin{bmatrix} \underline{Q}^* \\ \underline{R}^* \end{bmatrix}$ has independent columns. Therefore, $\begin{bmatrix} \underline{Q} \\ \underline{R} \end{bmatrix}$ has maximum rank and the second part of the proof is complete. Therefore there exists a unique solution to the tracking problem.

H. SOME PROPERTIES OF MATRIX NORMS

(1) If the norm of a n by n matrix $\underline{A}$ is defined as

$$\|\underline{A}\| = \sup_{\underline{x} \neq 0} \frac{\|\underline{A}\ \underline{x}\|}{\|\underline{x}\|}$$

where $\underline{x}$ is a n-vector, then it satisfies

(a) $\|\underline{A}\| > 0$

(b) $\|\underline{A} + \underline{B}\| \geq \|\underline{A}\| + \|\underline{B}\|$

(c) $\|\underline{A} \cdot \underline{B}\| \leq \|\underline{A}\| \cdot \|\underline{B}\|$

(d) $\|\alpha \underline{A}\| = |\alpha| \cdot \|\underline{A}\|$ $\qquad$ α is a scalar.

and $\|\underline{x}\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$

(2) if $\underline{A}$ is Hermitian $\|\underline{A}\| = \rho(\underline{A})$ where $\rho(\underline{A}) = \max_i |\lambda_i|$ is the spectral radius of $\underline{A}$.

(3) If $\underline{A}$ is positive definite, then $\rho_i(\underline{A}) > 0\ \forall\ i$. This can be expressed as $\underline{A} > 0$.

(4) If $\underline{X} > 0$, $\underline{Y} < 0$ $\quad$ then $\|\underline{X} + \underline{Y}\| < \|\underline{X}\|$

If $\underline{X} > 0$ $\quad$ then $\|(\underline{I} + \underline{X})^{-1}\| < 1$

(5) If $\underline{A} > 0$, $\underline{B} > 0$ $\quad$ then $\underline{A}\ \underline{B} > 0$

(6) If $\underline{C}$ is a triangular (upper or lower) square matrix with $c_{ii} = 0$, then $\|e^{\underline{C}t}\| = 1$.

(7) Two similar square matrices which have the same characteristic polynomial, will have the same values of norm.