CERTAIN SUBCLASS OF INFINITELY DIVISIBLE PROBABILITY MEASURES ON BANACH SPACES

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ABSTRACT

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By

Arunod Kumar

In this thesis stable probability measures and self-decomposible probability measures on a real separable Banach space are considered. Semi-stable probability measures on a real separable Hilbert space are also considered.

In Chapter 2 stable probability measures on a real separable Banach space are defined and several characterizations of these measures are established using a generalization of the convergence types theorem. These results are used to identify stable probability measures as limit laws of certain normed sums of independent, identically distributed Banach space valued random variables. These limit laws possess a Lévy-Khinchine representation that can be characterized on certain Orlicz spaces in terms of the representing Lévy-Khinchine measure.

In Chapter 3 self-decomposible probability measures on a real separable Banach space are characterized as a limit law of certain uniformly infinitesimal sequence of independent Banach space valued random variables. These limit laws are infinitely divisible and possess a Lévy-Khinchine representation that can be characterized on certain Orlicz spaces in terms of the representing Lévy-Khinchine measure.

Finally in Chapter 4, semi-stable laws on a real separable Hilbert space are considered. These laws are also limit laws. Lévy-Khinchine representation of a symmetric semi-stable law has also been obtained.

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Arunod Kumar

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TO MY PARENTS

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0. INTRODUCTION

In [5], Donsker proved a much stronger theorem than the classical theorem on convergence to normal ([21], p. 280) by studying the problem as the convergence to a limit law on a function space. This opened up interest in the study of probability laws on abstract spaces ([19], [25], [26]).

Motivated by the classical technique of Lévy [22], LeCam [19] introduced the concept of the characteristic functional of a finite measure on a linear space and pointed out that the limit laws can be easily handled almost in the classical way if the analogues of the celebrated Lévy continuity theorem and Bochner theorem can be made available. Unfortunately, such a situation was hard to come by, as can be seen in the work of Getoor [8]. In [9], L. Gross introduced the idea of the extension of characteristic functions and gave complete analytic analogue of the Lévy continuity theorem on a separable Hilbert space. Extensions of the work of Gross can be obtained for certain subclasses of Banach spaces as shown by J. Kuelbs and V. Mandrekar in [17], but the situation here from the point of view of the use in limit laws is very unsatisfactory. However in some spaces J. Kuelbs and V. Mandrekar in [16] have been able to obtain complete solution to the so-called Central Limit Problem [16] and gave the form of the limit laws in terms of their Lévy-Khinchine representation. On general Banach spaces, however, the general Central Limit Problem is difficult to handle as can be seen from the work of LeCam [20].

In this thesis, we look at limit laws associated with the normed sums of Banach-space valued random variables borrowing ideas in the scalar case from Loève ([21], §23, p. 319). In spite of the unavailability of the Lévy continuity theorem we have been able to obtain complete generalization of Theorem 23.3A and its corollary ([21], p. 323) for the probability measures on Banach spaces. These generalizations can also be regarded as the extensions of the work of Jajte [14] on stable probability measures on separable Hilbert space. In fact, on Banach spaces for which complete solution to the general central limit problem is available [16], we are able to give even analogues of Theorems 23.3B and Theorem 23.4B ([21], p. 324 and 327). These extensions also include and considerably generalize work of Jajte [14] on stable laws on separable Hilbert spaces.

In order to show the usefulness of the Lévy Continuity Theorem, we treat in Chapter IV the case of semi-stable laws on a real separable Hilbert space. Here we use, of course, the work of Gross [9]. The main technique in other chapters is a combination of methods of functional analysis and characteristic functionals.

We start in the next chapter with the setting up of the preliminary ideas and notations. Chapter II is devoted to the study of stable laws, Chapter III studies the self-decomposible laws and Chapter IV the semi-stable laws. Chapter V contains remarks leading to some unsolved problems, solution of which could be used to connect earlier work of Chapters II, III and IV to the convergence of stochastic processes.

CHAPTER I

BASIC CONCEPTS AND RELATED KNOWN RESULTS

1.0 Introduction

In this chapter we present for the sake of completeness and easy reference some standard known results, concepts and definitions. For further details the reader is referred to [1] and [23].

We shall denote by E a real separable Banach space with norm $\|\cdot\|$ and by R the space of all real numbers with the usual topology. The elements of E will be denoted by $x,y,z,\ldots$ and of R by $a,b,c,\ldots$, etc. R^+ and R^- will stand for the set of all positive and all negative real numbers, respectively. R^n will stand for the n-dimensional Euclidean space. E^* will denote the (topological) dual of E. $\mathcal{B}(E)$ will denote the σ-field generated by the open subsets of E and $\mathfrak{M}$ the space of all probability measures defined on $\mathcal{B}(E)$. For $\mu \in \mathfrak{M}$ and $a \in R$, $a \neq 0$, $T_a\mu$ is defined to be an element in $\mathfrak{M}$, by

$$T_a\mu(B) = \mu(a^{-1}B) \quad \text{for every} \quad B \in \mathcal{B}(E);$$

where $a^{-1}B = \{a^{-1}x\colon x \in B\}$, and for $a = 0$ we define $T_a\mu = \delta_0$, where for each $B \in \mathcal{B}(E)$,

$$\delta_x(B) = 1 \quad \text{if} \quad x \in B$$
$$= 0 \quad \text{otherwise.}$$

We shall call δ_x the probability measure degenerate at x.

1.1 Basic Definitions and Results

1.1.1 Definition. (a) For μ and ν in $\mathcal{M}$, we shall denote by $\mu * \nu$ an element of $\mathcal{M}$ defined by

$$\mu * \nu(B) = \int_E \mu(B-x)d\nu(x) \quad \text{for every} \quad B \in \mathcal{B}(E),$$

where $B-x = \{y-x: y \in B\}$. We shall call $\mu * \nu$ the convolution of μ and ν.

(b) A probability measure μ on $\mathcal{B}(E)$ will be called infinitely divisible (i.d.) if for each positive integer n there exists an element λ_n in $\mathcal{M}$ and an x_n in E such that

$$\mu = (\lambda_n)^{*n} * \delta_{x_n},$$

where $(\lambda_n)^{*n}$ denotes λ_n convoluted n times with itself.

1.1.2 Definition. (a) A sequence $\{\mu_n\}$ in $\mathcal{M}$ is said to converge weakly to μ in $\mathcal{M}$, if for every bounded continuous real-valued function f on E, $\int_E f d\mu_n \to \int_E f d\mu$. We shall denote this convergence by $\mu_n \Rightarrow \mu$.

(b) A sequence $\{\mu_n\}$ in $\mathcal{M}$ is said to be weakly sequentionally compact (for short, compact or tight) if for every $\varepsilon > 0$ there exists a compact set K^ε in E such that $\mu_n(K^\varepsilon) > 1-\varepsilon$ for all n.

(c) A sequence $\{\mu_n\}$ in $\mathcal{M}$ is said to be shift compact, if there exists a sequence $\{x_n\}$ in E such that $\{\mu_n * \delta_{x_n}\}$ is compact.

The following theorem will be used repeatedly.

1.1.3 Theorem ([23], pp. 58, 59, 153). (a) Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ be three sequences in $\mathcal{M}$ such that $\lambda_n = \mu_n * \nu_n$ for each n. Then (i) if the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are compact, then so is the sequence $\{\nu_n\}$;

(ii) If the sequence $\{\lambda_n\}$ is compact, then μ_n and ν_n are shift compact.

(b) Let μ_n be a compact sequence in $\mathcal{M}$ and $\hat{\mu}_n(y) \to \varphi(y)$ for each $y \in E^*$, where $\hat{\mu}_n$ is as defined in 1.1.7. Then $\mu_n \Rightarrow \mu$, for some μ in $\mathcal{M}$. Furthermore, $\hat{\mu}(y) = \varphi(y)$ for each $y \in E^*$.

1.1.4 Definition. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let X be a random variable on Ω. Then X is said to be distributed as ν if $\nu = P \circ X^{-1}$.

We now recall some ideas on measures on linear spaces. The following definitions are due to I. Segal and are taken here from [9].

1.1.5 Definition. A weak distribution on a topological linear space L is an equivalence class of linear mappings F from the (topological) dual space L^* to the class of real-valued random variables on a probability space (depending on F) where two such mappings F_1 and F_2 are equivalent if for every finite set of vectors $y_1, \ldots, y_k$ in L^* the sets $\{F_i(y_1), \ldots, F_i(y_k)\}$ have the same distribution in k-space for $i = 1,2$.

In a finite dimensional space a weak distribution coincides with the notion of a measure, that is, if L is finite dimensional then for any given weak distribution there exists a unique Borel probability measure on the Borel subsets of L such that the identity map on L^* is a representative of the given weak distribution ([11], p. 372).

1.1.6 Definition. A measure μ on a locally convex topological linear space L is defined to be Gaussian if for every continuous linear functional y on L, y(x) has a Gaussian distribution. The Gaussian measure μ is said to have mean zero if y(x) has mean zero for each y.

1.1.7 Definition. The characteristic functional (ch.f. or Fourier transform) of a proability measure μ on the Borel subsets of a linear topological space L is a function $\hat{\mu}$ on L^* (the topological dual of L) given by

$$\hat{\mu}(y) = \int_L \exp\{i(y,x)\}d\mu(x) \quad , \quad \text{for each} \quad y \in L^* .$$

1.1.8 Remark. One special example of a weak distribution on a real separable Hilbert space H is the canonical normal distribution (with variance parameter one). This weak distribution is that unique weak distribution which assigns to each vector y in H^* a normally distributed random variable with mean zero and variance $\|y\|^2$. It follows from the preceeding property that the canonical normal distribution carries orthogonal vectors into independent random variables ([9], p. 4). It is known that some of the theory of integration with respect to a measure can also be carried out with respect to a weak distribution on H. For details we refer the reader to [11] and the bibliography given there. We need the following definition which is taken from ([30], p. 190). The following preliminaries are essential for the work in Chapter IV.

1.1.9 Definition. An operator from a real separable Hilbert space H into H, which is linear, symmetric, non-negative definite, compact and having finite trace is called an S-operator.

If T is an S-operator on H, then it is well known that T has the representation

$$(1.1.10) \qquad Tx = \sum_{n=1}^{\infty} \lambda_n (x,e_n)e_n ,$$

where $\{e_n\}$ is an orthonormal subset of H, $\lambda_n \geq 0$, and $\sum_{n=1}^{\infty} \lambda_n < \infty$.

The S-operator T on a real separable Hilbert space H has a representation as an infinite symmetric, non-negative-definite matrix $T = \{t_{ij}\}$ where by non-negative-definite it is meant that $\sum_{i,k=1}^{n} t_{ik}x_i x_k \geq 0$ for any integer n and any $(x_1,\ldots,x_n) \in R^n$. Furthermore, if $t_{ik} = (Te_i,e_k)$, where $\{e_i\}$ is a complete orthonormal system in H, then

$$\sum_{i=1}^{\infty} t_{ii} < \infty .$$

From the representation in (1.1.10) it is easy to verify that $(Tcx, cx)^{\frac{1}{2}} = |c|(tx,x)^{\frac{1}{2}}$ for any real number c and $(T(x+y),x+y)^{\frac{1}{2}} \leq (Tx,x)^{\frac{1}{2}} + (Ty,y)^{\frac{1}{2}}$. Thus $(Tx,x)^{\frac{1}{2}}$ is a semi-norm on H. Let Σ be the class of all S-operators on H.

1.1.11 <u>Definition of τ-topology</u>. The τ-topology on H is the smallest locally convex topology generated by the family of semi-norms $p_T(x) = (Tx,x)^{\frac{1}{2}}$ on H as T varies through Σ ([29], p. 172).

1.1.12 <u>Definition</u>. Let H_1, H_2 be Hilbert spaces with orthonormal systems $\{e_n\}$, $\{f_n\}$ respectively. Then a continuous linear operator A from H_1 into H_2 is called Hilbert-Schmidt operator if there exists an orthonormal system $\{g_n\}$ in H_1 such that $\sum_{n=1}^{\infty} \|Ag_n\|_{H_2}^2 < \infty$ ([7], p. 34).

1.1.13 Remark. If T is an S-operator on H then T possesses a unique non-negative, symmetric square root, which we denote by $T^{\frac{1}{2}}$ ([28], Theorem, p. 265). Now using the fact (see [7], Theorem 4, p. 39) that the square roots of S-operators are Hilbert-Schmidt operators one can easily show that the topology τ on H is the weakest topology on H for which all Hilbert-Schmidt operators are continuous from τ to strong topology on H. Thus a basic open neighborhood of x_0 is $\{x: \|A(x-x_0)\| < \epsilon\}$ whenever A is a Hilbert-Schmidt operator on H. Therefore our definition of τ-topology coincides with that of L. Gross ([9], p. 5).

1.1.14 Definition. A tame function on a real Hilbert space H is a function of the form $f(x) = \Phi(Px)$ where P is a finite dimensional orthogonal projection on H and Φ is a Baire function on the finite dimensional space PH.

For such a function we have $f(x) = \Psi((x,e_1),\ldots,(x,e_k))$ where $e_1,\ldots,e_k$ is an orthonormal basis of PH and Ψ is a Baire function of k real variables. If F is a representative of a weak distribution then the random variable $f^{\sim} = \Psi(F(e_1),\ldots,F(e_k))$ depends only on the function f and the mapping F while integration properties of $f^{\sim}$ such as the integral of $f^{\sim}$, the distribution of $f^{\sim}$, convergence in probability of sequences $f_n^{\sim}$, etc. depend only on f and the f_n's and on the weak distribution of which F is a representative. Let us denote by $\mathcal{F}$ the directed set of finite dimensional orthogonal projections on H directed under inclusion of the ranges. For a given continuous function f on H and a given weak distribution one may consider whether the net

$(f \circ P)^\sim$ of the random variables where P ranges over directed set $\mathcal{F}$, converges in probability as P approaches the identity through $\mathcal{F}$. If so then the limit which we shall denote by $f^\sim$ is a random variable whose integration properties are completely determined by the function f and the weak distribution. In [10] and [11] classes of continuous functions are described for which the limit defining the random variable $f^\sim$ exists when the weak distribution in question is the canonical normal distribution, and some explicit evaluations are also given.

1.1.15 Remark. Let H be a real separable Hilbert space, and let $\{P_j\}$ be any sequence of finite dimensional projections converging strongly to the identity operator. If a complex-valued function f on H is uniformly continuous in the topology τ then $\lim_{j\to\infty} \text{ in prop } (f \circ P_j)^\sim$ exists with respect to the canonical normal distribution. It will be called extension of f and will be denoted by $f^\sim$ ([9], Theorem, p. 5).

1.2 Orlicz Spaces and Associated Hilbert Space

We shall now recall definition of Orlicz spaces and some results from [16] for use in later chapters (cf. §2.3 and §3.3).

Let Γ be a function satisfying the following conditions

$$
(1.2.1)\quad \begin{cases}
\text{(a)} & \Gamma \text{ is defined on } [0,\infty) \text{ into } [0,\infty), \\
\text{(b)} & \Gamma(0) = 0,\ \Gamma(s) > 0 \text{ for } s > 0, \\
\text{(c)} & \Gamma \text{ is convex and strictly increasing on } [0,\infty) \\
\text{(d)} & \Gamma(2s) \le M\,\Gamma(s) \text{ for all } s \in [0,\infty), \text{ where } M \text{ is a} \\
 & \text{positive constant independent of } s.
\end{cases}
$$

Then it follows that

$$\Gamma(s) = \int_0^s \rho(x)\,dx$$

where $\rho(0) = 0$ and $\rho(s)$ is non-decreasing on $[0,\infty)$. We assume without loss of generality that $\rho(s)$ is left-continuous.

Let $\varphi(x)$ be the inverse function of ρ as defined in ([32], p. 76). We now define

$$\Lambda(s) = \int_0^s \varphi(x)\,dx \tag{1.2.2}$$

Then Γ and Λ are complementary in the sense of Young ([32], p. 77). By S_Γ^* we mean all real sequences $\{x_i\}$ such that

$$\sum_1^\infty \Gamma(|x_i|) < \infty \,. \tag{1.2.3}$$

Similarly, S_Λ^* is all real sequences $\{x_i\}$ such that

$$\sum_1^\infty \Lambda(|x_i|) < \infty \,.$$

If $x = \{x_i\}$ is a sequence we define

$$\|x\|_\Gamma = \sup_y \{\sum_1^\infty |x_i y_i| \;:\; \sum_{i=1}^\infty \Lambda(|y_i|) \le 1\}$$

and

$$\|x\|_\Lambda = \sup_y \{\sum_1^\infty |x_i y_i| \;:\; \sum_{i=1}^\infty \Gamma(|y_i|) \le 1\} \,.$$

1.2.4 Definition. The Orlicz space S_Γ (S_Λ) is the collection of all real sequences such that $\|x\|_\Gamma$ ($\|x\|_\Lambda$) is finite.

1.2.5 Remarks. (a) Let α be a function satisfying (1.2.1) and $\Gamma(s) = \alpha(s^2)$. Then clearly Γ satisfies (a), (b), (c) in (1.2.1) and since

$$\Gamma(2s) = \alpha(4s^2) \le M\,\alpha(2s^2) \le M^2(s^2) = M^2\Gamma(s) \quad ,$$

Γ satisfies (1.2.1), hence we know ([32], Corollary, p. 81) that S_Γ^* (and, therefore E_α) contains the same sequences as S_Γ. Further, it is known that S_Γ is a real separable Banach space in the norm $\|x\|_\Gamma$ and since $\Gamma(2s) \le M^2\Gamma(s)$ for $s \ge 0$ we also have ([32], Lemma α, p. 83) that $\{p_n\} \subseteq S_\Gamma$ converges to $p \in S_\Gamma$ in norm provided

$$\lim_n \sum_{i=1}^{\infty} \Gamma(|x_{i,n}-x_i|) = \lim_n \sum_{i=1}^{\infty} \alpha[(x_{i,n}-x_i)^2] = 0 \quad ,$$

where $x_{i,n}, x_i$ mean the i^{th} elements of p_n, p respectively.

(b) By Theorem 6.2 of [16], $(E_\alpha, \|\cdot\|_\Gamma)$ is a Banach space with a basis $\{b_n\}$ where b_n is the vector with one as the n^{th} coordinate and other coordinates zero.

Let α be a function satisfying (1.2.1) and assume that $\alpha_c(\cdot)$, the complementary function of $\alpha(\cdot)$ in the sense of Young ([32], p. 77), satisfies (1.2.1). Notice that if $E_\alpha = \ell_2$ then a natural choice for the function α is $\alpha(s) \equiv s$. Hence $\alpha_c(s) = 0$ on $[0,1]$ but $\alpha_c(s) = \infty$ for $s > 1$. Thus $\alpha_c(\cdot)$ does not satisfy (1.2.1) when $E_\alpha = \ell_2$ and this is a special case which is easily handled. Following [16], we shall denote by E_α either the Hilbert space ℓ_2 or an E_α space where α and α_c satisfy (1.2.1).

In terms of the notations we have used in this section, E_α is equivalent (isometrically isomorphic) to the Orlicz space S_Γ where $\Gamma(s) = \alpha(s^2)$. We will let S_α, S_{α_c} denote the Orlicz spaces given by $\alpha(\cdot)$ and $\alpha_c(\cdot)$, respectively. Then the dual space of S_α can be identified as S_{α_c} and since $\alpha(\cdot)$ also satisfies (1.2.1), except when $E_\alpha = \ell_2$, it follows that the dual of S_{α_c} is S_α ([32], p. 150).

For each $\lambda = (\lambda_1, \lambda_2, \ldots)$ in the positive cone of S_{α_c}, we define the space H_λ as all sequences $x = (x_1, x_2, \ldots)$ such that $\sum_1^\infty \lambda_i x_i^2 < \infty$. Then H_λ is a Hilbert space with $\|x\|_\lambda^2 = \sum_1^\infty \lambda_i x_i^2$ and the inner product $(x,y) = \sum_1^\infty \lambda_i x_i y_i$. In the special case $E_\alpha = \ell_2$ we have $S_{\alpha_c} = \ell_\infty$ and for simplicity we take $\lambda = (1,1,\ldots)$. Then $H_\lambda = \ell_2$ and we shall assume without loss of generality that $\alpha(s) \equiv s$. The following lemma is proved in [16].

1.2.6 Lemma. E_α is a Borel subset of H_λ for each λ in the positive cone of S_{α_c}. Furthermore, every Borel subset of E_α is a Borel subset of H_λ.

1.2.7 Definition. A linear operator from E_α^* into E_α is an α-operator if the matrix of the operator, $\{t_{ij}\}$, is symmetric, non-negative definite with $\sum_{i=1}^\infty \alpha(t_{ii}) < \infty$.

The proof of the following lemma is included in ([12], p. 42).

1.2.8 Lemma. Let T be an infinite dimensional matrix $\{t_{ij}\}$ such that T is symmetric, non-negative-definite and $\sum_{i=1}^\infty \alpha(t_{ii}) < \infty$. Then T is an α-operator on E_α^* into E_α.

We know (Lemma 1.2.6) that E_α is a Borel subset of H_λ and the $\|\cdot\|_\Gamma$-topology is stronger than $\|\cdot\|_\lambda$-topology on E_α. Hence it follows that H_λ^* is a subset of E_α^*. We now identify H_λ^* by H_λ and state the following lemma whose proof is in ([12], p. 43).

1.2.9 Lemma. Every α-operator T on E_α^* is a trace class operator on H_λ.

CHAPTER II

STABLE PROBABILITY MEASURES ON BANACH SPACES

2.0 Introduction

In this chapter, we consider stable probability measures (laws) on a real separable Banach space. Using a generalization of the convergence types theorem ([3], p. 174) we establish several characterizations of stable probability measures and deduce as corollaries extensions to Banach space of known results on stable laws ([3], p. 199, [14], p. 64, [21], p. 327). These results allow us to identify stable probability measures on Banach space as the limit laws of certain normed sums of independent, identically distributed Banach space valued random variables. Finally, we characterize stable probability measures on certain Orlicz spaces in terms of their Lévy-Khinchine representation given in [16].

In Section 2.1, following the preliminaries, the convergence type theorem is established. Section 2.2 presents the characterizations of stable laws and final section characterizes the Lévy-Khinchine representation of the stable laws on Orlicz spaces.

The lemmas in Section 2.2 are suggested by some recent work of Jajte [14]. The proofs in [14] in the Hilbert case treated there are incomplete and the main theorem in ([14], p. 64) which is extended here to certain Orlicz spaces, contains a lacuna ([14], p. 70).

2.1 Preliminary Results for Stable Laws

In this section we present basic results needed in this chapter. Throughout this chapter E will denote a real separable Banach space with norm $\|\cdot\|$.

2.1.1 Lemma. Let $\{\mu_n\}$ and μ be probability measures on $\mathcal{B}(E)$ and $\{a_n\}$, $a \in R$. Then $\mu_n \Rightarrow \mu$ and $a_n \to a$ implies $T_{a_n}\mu_n \Rightarrow T_a\mu$.

Proof of the lemma is immediate from ([1], p. 34).

Before we prove the main theorem of this section, we need the following lemma.

2.1.2 Lemma. Let $\hat{\mu}(\cdot)$ be the ch.f. of a probability measure on $\mathcal{B}(E)$ such that for some $\delta > 0$, $|\hat{\mu}(y)| = 1$ whenever $\|y\|_{E^*} \leq \delta$. Then $\mu = \delta_x$ for some $x \in E$.

Proof:[1] Let $\Delta = \{y \in E^*: \|y\|_{E^*} \leq \delta\}$. Consider the random variable $(\cdot,y)$ defined on E for each fixed y in E^*. Then by ([21], p. 202), $(\cdot,y)$ is degenerate say at $\theta(y)$. Hence $\hat{\mu}(y) = e^{i\theta(y)}$.

Let $\Delta_y = \{x: (x,y) = \theta(y)\}$. Then Δ_y is closed and $\mu(\Delta_y) = 1$, for every y in E^*. Consequently, the support c_μ of μ (see [23], p. 27) is contained in $\bigcap_{y \in E^*} \Delta_y$.

Suppose there exist two points x_1 and x_2 in c_μ. Then $(x_1,y) = (x_2,y)$ for every $y \in E^*$. Hence $x_1 = x_2$. Thus the support of μ contains only one point. This completes the proof of the lemma.

2.1.3 Theorem (Convergence of types theorem). Let $\{\mu_n\}$ and μ be probability measures on $\mathcal{B}(E)$ such that $\mu_n \Rightarrow \mu$, and

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[1] I thank Professor J.F. Hannan for pointing out this proof.

and for positive constants a_n's and a sequence $\{x_n\}$ in E $T_{a_n}\mu_n * \delta_{x_n} \Rightarrow \mu'$ where μ and μ' are non-degenerate probability measures on $\mathcal{B}(E)$. Then there exist an $a \in R$ and an $x \in E$ such that $\mu' = T_a\mu * \delta_x$, $a_n \to a$ and $\|x_n - x\| \to 0$ as $n \to \infty$

Proof: Suppose $\overline{\lim}\, a_n = \infty$. Then there exists a subsequence $\{m\} \subseteq \{n\}$ such that $a_m \to \infty$. Let $c_m = a_m^{-1}$. Then $\mu_m = \{T_{c_m}(T_{a_m}\mu_m * \delta_{x_m})\} * \delta_{-c_m x_m} \Rightarrow \mu$. Since $T_{a_m}\mu_m * \delta_{x_m} \Rightarrow \mu'$, therefore by Lemma 2.1.1 $T_{c_m}(T_{a_m}\mu_m * \delta_{x_m}) \Rightarrow \delta_0$. Hence by Theorem 1.1.3(a), $\{\delta_{-c_m x_m}\}$ being compact converges to δ_{x_o} for some x_o belonging to E by ([1], p. 37). Hence μ is degenerate, contradicting the hypothesis. Hence, $\overline{\lim}\, a_n < \infty$.

Suppose now $\{a_m\}$ and $\{a_\ell\}$ are two subsequences of $\{a_n\}$, such that $a_m \to a$, $a_\ell \to a'$, where $a \neq a'$. We note that neither a nor a' can be zero, since μ' is non-degenerate. Also, we have

$$\mu_m = \{T_{c_m}(T_{a_m}\mu_m * \delta_{x_m})\} * \delta_{-x_m c_m} \Rightarrow \mu \quad \text{and}$$

$$\mu_\ell = \{T_{c_\ell}(T_{a_\ell}\mu_\ell * \delta_{x_\ell})\} * \delta_{-x_\ell c_\ell} \Rightarrow \mu .$$

Now by Lemma 2.1.1 and the hypothesis we get

$$\mu = T_c\mu' * \delta_{x_1} = T_{c'}\mu' * \delta_{x_2}$$

where $x_1 = \lim_{m\to\infty} - x_m c_m$, and $x_2 = \lim_{\ell\to\infty} - x_\ell c_\ell$.

Therefore

$$(2.1.4) \qquad |\hat{\mu}'(ay)| = |\hat{\mu}'(a'y)| \quad \text{for every } y \in E^*.$$

Without loss of generality we can assume $b = \frac{a}{a'} < 1$. Hence, by iteration $|\hat{\mu}'(y)| = |\hat{\mu}'(by)| = \ldots = |\hat{\mu}'(b^n u)|$. Letting $n \to \infty$ we get $|\hat{\mu}'(y)| = 1$, for every $y \in E^*$. Hence by Lemma 2.1.2 μ' is degenerate, contradicting the hypothesis. This proves that $a_n \to a$ and $0 < a < \infty$.

Now it follows that $T_{a_n}\mu_n \Rightarrow T_a\mu$ by Lemma 2.1.1, and from hypothesis $T_{a_n}\mu_n * \delta_{x_n} \Rightarrow \mu'$. Therefore, by Theorem 1.1.3(a) $\{\delta_{x_n}\}$ is compact and hence by ([1], p. 37) x_n converges to some x in E. Hence, $T_{a_n}\mu_n * \delta_{x_n} \Rightarrow T_a\mu * \delta_x$. Thus $\mu' = T_a\mu * \delta_x$, which completes the proof.

2.2 Stable Probability Measure on a Banach Space

We define a stable probability measure on a real separable Banach space following Loève ([21], p. 326). (See also ([14], p. 64).)

2.2.1 Definition. Let μ be a probability measure on the Borel subsets $\mathcal{B}(E)$ of a real separable Banach space E. We say that μ is a stable probability measure if for each pair of positive real numbers a and b there exist a positive real number c and an $x \in E$, such that

$$T_a\mu * T_b\mu = T_c\mu * \delta_x \quad . \tag{2.2.2}$$

Our main effort in this section will be to prove various characterizations of the stable probability measures which will be useful in studying stable probability measures as limit laws of the sums of independent random variables. For this we need the following lemmas.

2.2.3 Lemma. If μ is a stable probability measure on $\mathcal{B}(E)$, then there exists a sequence $\{a_n\}$ of positive numbers and a sequence $\{x_n\}$ of elements of E such that $T_{a_n}\mu^{*n} * \delta_{x_n} \Rightarrow \mu$.

Proof: We shall prove the lemma by showing that for each n, there exist a_n and x_n such that $\mu = \delta_{x_n} * T_{a_n}\mu^{*n}$.

For $n = 1$, take $x_1 = 0$, and $a_1 = 1$. Suppose that we have $x_1,\ldots,x_{m-1}$ and $a_1,a_2,\ldots,a_{m-1}$ such that

$$\mu = \delta_{x_i} * T_{a_i}\mu^{*i} \quad \text{for} \quad i = 1,2,\ldots,m-1 .$$

Then $\mu^{*m-1} = T_{a_{m-1}^{-1}}(\mu * \delta_{-x_{m-1}})$. Hence,

$$\mu^{*m} = T_{a_{m-1}^{-1}}\mu * \delta_{-x_{m-1}a_{m-1}^{-1}} * \mu .$$

Now we use the fact that μ is stable to conclude that

$$\mu^{*m} = T_{c_m}\mu * \delta_{x-a_{m-1}^{-1}x_{m-1}} \quad \text{for some} \quad c_m > 0 \text{ and } x \in E.$$

Consequently,

$$\mu = T_{c_m^{-1}}\mu^{*m} * \delta_{c_m^{-1}(a_{m-1}^{-1}x_{m-1} - x)} .$$

Define, $a_m = c_m^{-1}$, $x_m = c_m^{-1}(a_{m-1}^{-1}x_{m-1} - x)$. Thus we have shown by induction that $\mu = \delta_{x_m} * T_{a_m}\mu^{*m}$ for every m. This completes the proof of the lemma.

2.2.4 Lemma. If for some sequence of positive real numbers $\{a_n\}$ and a sequence $\{x_n\}$ of elements of the space E, we have

$$\nu = \lim_{n\to\infty} (\delta_{x_n} * T_{a_n}\mu^{*n}) ,$$

where ν is non-degenerate, then $a_n \to 0$, $\dfrac{a_n}{a_{n+1}} \to 1$ as $n \to \infty$.

Proof: Suppose $a_n \not\to 0$. Then there exists a subsequence $\{a_m\}$ of $\{a_n\}$ such that $a_m^{-1} \to a < \infty$. Therefore by Lemma 2.1.1

$$\delta_{y_m} * \mu^{*m} = T_{a_m^{-1}}(\delta_{x_m} * T_{a_m}\mu^{*m}) \to T_a \nu$$

where $y_m = \frac{x_m}{a_m}$. This implies $|\hat{\mu}(h)|^m \to |\hat{\nu}(ah)|$ for every $h \in E^*$. Since ν is continuous at the origin, therefore $|\hat{\nu}(ah)| > 0$ for those h with $\|h\| < \delta$ for some $\delta > 0$.

Hence $|\hat{\mu}(h)| = 1$ on $\|h\| < \delta$ and hence by Lemma 2.1.2, μ is degenerate. Consequently, ν is degenerate which contradicts the assumption and hence $a_n \to 0$.

Suppose $\frac{a_n}{a_{n+1}} \not\to 1$. Then there exists a subsequence $\{m\}$ of $\{n\}$ such that $\frac{a_m}{a_{m+1}} \to a$ where $a \neq 1$.

If $a = \infty$. Then $c_m = \frac{a_{m+1}}{a_m} \to 0$, and

$$\delta_{x_{m+1}} * T_{a_{m+1}}\mu^{*m} = \{T_{c_m}(\delta_{x_m} * T_{a_m}\mu^{*m})\} * \delta_{x_{m+1}-c_m x_m} \, .$$

$$(2.2.5) \qquad \widehat{\delta_{x_{m+1}} * T_{a_{m+1}}\mu^{*m+1}}(y) = \frac{e^{i(x_{m+1},h)}(\hat{\mu}(a_{m+1}h))^{m+1}}{\hat{\mu}(a_{m+1}h)} \to \hat{\nu}(h)$$

because for every h, $\hat{\mu}(a_{m+1}h) \to 1$ and $e^{i(x_{m+1},h)}(\hat{\mu}(a_{m+1}h))^{m+1} \to \hat{\nu}(h)$. Since $T_{c_m}(\delta_{x_m} * T_{a_m}\mu^{*m}) \to \delta_0$ by Lemma 2.1.1, we conclude that $|\hat{\nu}(h)| = 1$ for every $h \in E^*$. Hence by Lemma 2.1.2, ν is degenerate which contradicts the assumption.

Now suppose $a < \infty$. Then $d_m = \frac{a_m}{a_{m+1}} \to a$, and

$$\delta_{x_m} * T_{a_m}\mu^{*m+1} = \{T_{d_m}(\delta_{x_{m+1}} * T_{a_{m+1}}\mu^{m+1})\} * \delta_{x_m - d_m x_{m+1}}$$

$$\widehat{\delta_{x_m} * T_{a_m}\mu^{*m+1}}(y) = \frac{e^{i(x_m,h)} * [\hat{\mu}(a_m h)]^m}{\hat{\mu}(a_m h)} \to \hat{\nu}(h)$$

by the reasoning similar to one following (2.2.5). But $T_{d_m}(\delta_{x_{m+1}} * T_{a_{m+1}}\mu^{*m+1}) \to T_a\nu$, hence $|\hat{\nu}(h)| = |\hat{\nu}(ah)|$ for every $h \in E^*$. Without loss of generality we can assume $a < 1$. Hence by the same argument as in (2.1.4) we conclude that ν is degenerate which contradicts the hypothesis. Hence $\frac{a_n}{a_{n+1}} \to 1$.

2.2.6 Lemma. Let for every positive integer n, $x_n \in E$, $a_n \in R^+$, and

$$\delta_{x_n} * T_{a_n}\mu^{*n} \Rightarrow \nu$$

where μ and ν are probability measures on $\mathcal{B}(E)$. Then there exists an $r > 0$ and a function z of two variables defined for every pair of non-negative numbers a and b with values in E, such that

$$\hat{\nu}(ah)\hat{\nu}(bh) = e^{i(z(a,b),h)}\hat{\nu}((a^r + b^r)^{1/r}h) \quad \text{for every} \quad h \in E^*.$$

Hence in particular, ν is stable.

Proof: If ν is degenerate, then there is nothing to prove. So assume ν is non-degenerate. Now by Lemma 2.2.4, for any arbitrary pair of positive numbers a and b, there exist subsequences $\{a_{n_k}\}$ and $\{a_{m_k}\}$ of $\{a_n\}$ such that

$$\omega_k = \frac{a_{n_k}}{a_{m_k}} \to \frac{b}{a} \qquad \text{(Loève [21], p. 323)} .$$

Suppose $\overline{\lim} \dfrac{a_{n_k}}{a_{n_k + m_k}} = s = \infty$. Then the sequence $\{c_k\}$ where $c_k = \dfrac{a_{n_k + m_k}}{a_{n_k}}$, will have a subsequence $\{c_{k'}\}$ such that $c_{k'} \to 0$.

$$(2.2.7) \quad \delta_{x_{n_{k'}+m_{k'}}} * T_{a_{n_{k'}+m_{k'}}} \mu^{*n_{k'}+m_{k'}} = \{T_{c_{k'}}(\delta_{x_{k'}} * T_{a_{n_{k'}}} \mu^{*n_{k'}})\} * \{T_{c_{k'} \cdot \omega_{k'}}(\delta_{x_{m_{k'}}} * T_{a_{m_{k'}}} \mu^{*m_{k'}})\} * \delta_{z_{k'}}$$

for suitable $z_{k'}$. By the hypothesis we conclude that $\delta_{x_{n_{k'}+m_{k'}}} * T_{a_{n_{k'}+n_{k'}}} \mu^{*m_{k'}+n_{k'}} \to \nu$. The terms in the parenthesis of (2.2.7) converge weakly to δ_0. Hence, $|\hat{\nu}(y)| = 1$ for every $y \in E^*$. Thus by Lemma 2.1.2, ν is degenerate, which is a contradiction. Hence $s < \infty$.

Suppose there exists two subsequences $\{c_{k'}^{-1}\}$, $\{c_{k''}^{-1}\}$ of $\{c_k^{-1}\}$ converging to b' and b'' respectively where $b' \neq b''$. Making use of the following equality

$$(2.2.8) \quad [T_a(\delta_{x_{n_k}} * T_{a_{n_k}} \mu^{*n_k})] * [T_{a\omega_k}(\delta_{x_{m_k}} * T_{a_{m_k}} \mu^{*m_k})]$$
$$= T_{aa_{n_k}} \mu^{*n_k+m_k} * \delta_{z_k} = [T_{ac_k^{-1}}(\delta_{x_{n_k+m_k}} * T_{a_{n_k+m_k}} \mu^{*n_k+m_k})] * \delta_{z_{k'}}$$

for suitable z_k and z_k' we conclude that

$$|\hat{\nu}(ah)|\cdot|\hat{\nu}(bh)| = |\hat{\nu}(xb'h)| = |\hat{\nu}(xb''h)| \quad \text{for every} \quad h \in E^* .$$

Since $b' \neq b''$ and both are finite, we can conclude by the same reasoning as in (2.1.4), that ν is degenerate. This contradicts the assumption. Hence, $s = \lim \dfrac{a_{n_k}}{a_{n_k + m_k}}$.

Now we make use of equation (2.2.8) and Theorem 1.1.3 to conclude that there exists a function $z(a,b)$ which is the limit of $z_k' = a\, x_{n_k} + a \dfrac{a_{n_k}}{a_{m_k}} x_{m_k} - a \dfrac{a_{n_k}}{a_{n_k + m_k}} \cdot x_{m_k + n_k}$ and satisfies the equation

$$\hat{\nu}(ah)\hat{\nu}(bh) = e^{i(z(a,b),h)}\hat{\nu}(ash) \text{ for every } h \text{ in } E^* .$$

Define a function $g(.,.)$ on $[0,\infty) \times [0,\infty)$ as follows

$$g(x,y) = x.s,\ x,y > 0;\ g(x,0) = x,\ x \geq 0;\ g(0,y) = y,\ y \geq 0;$$

then the equality

(2.2.9) $\hat{\nu}(xh)\hat{\nu}(yh) = e^{i(z(x,y),h)}\nu(g(x,y).h)$ holds for all h in E^*,

and for all $x,y \geq 0$. We shall prove in this part that g is the only function which satisfies (2.2.9).

Suppose not. Then there exist g_1 and g_2 satisfying (2.2.9) and for some x_0 and y_0, $g_1(x_0,y_0) < g_2(x_0,y_0)$. Let $u = \dfrac{g_1(x_0,y_0)}{g_2(x_0,y_0)}$. Then $u < 1$.

$$e^{i(z(x_0,y_0),h/g_2(x_0,y_0))}\hat{\nu}(uh) = e^{i(z(x_0,y_0)h/g_2(x_0,y_0))}\hat{\nu}(h).$$

Thus $|\hat{\nu}(uh)| = |\hat{\nu}(h)|$. Hence the same argument as in (2.1.4) yields that ν is degenerate. This contradicts the assumption. Hence the uniqueness of g has been proved.

The function g so defined is continuous. Let $x_n \to x$; $y_n \to y$. Then we shall prove that $t = \overline{\lim}\, g(x_n,y_n)$ is finite. Suppose not, then there exists a subsequence $\{n'\}$ of $\{n\}$ such that $g^{-1}(x_{n'},y_{n'}) \to 0$. Therefore $e_{n'} = x_{n'}g^{-1}(x_{n'},y_{n'})$ and $k_{n'} = y_{n'}g^{-1}(x_{n'},y_{n'}) \to 0$. Hence from (2.2.9) we get

$$|\hat{\nu}(e_{n'}.h)|\cdot|\hat{\nu}(k_{n'}.h)| = |\hat{\nu}(h)| \quad \text{for every } h \text{ in } E^*.$$

By letting $n \to \infty$, in view of Lemma 2.1.2 we get ν is degenerate. Thus $t < \infty$.

To conclude that $t = \lim g(x_n,y_n)$, we shall show that no two distinct subsequences of $g(x_n,y_n)$ can converge to two different limits. If not, let $t' = \lim g(x_{n'},y_{n'})$ and $t'' = \lim g(x_{n''},y_{n''})$, where $t' \neq t''$. Consequently, from (2.2.9) we get $|\hat{\nu}(t'h)| = |\hat{\nu}(t''h)|$ for every h in E^*. Since $t' \neq t''$, therefore from the same reasoning as in (2.1.4) we conclude that ν is degenerate. Hence, $t = \lim g(x_n,y_n)$. Now we make use of (2.2.9) again to conclude that

$$\hat{\nu}(xh).\hat{\nu}(yh) = e^{i(z(x,y),h)}\hat{\nu}(t,h) \quad \text{for every } h \text{ in } E^* .$$

Since g is unique, therefore $t = g(x,y)$. Thus g is continuous.

It can be verified that the function g satisfies the hypothesis of Theorem 4.1 of ([2], p. 632). Hence by ([2], p. 632)

$$g(x,y) = (x^r + y^r)^{1/r} \quad \text{for some } 0 < r < \infty, \text{ and}$$

$$\hat{\nu}(ah)\hat{\nu}(bh) = e^{i(z(a,b),h)}\hat{\nu}((a^r + b^r)^{1/r}h) \quad \text{for every } h \in E^*$$

where a,b are positive real numbers. This completes the proof.

2.2.10 Theorem (Characterizations of stable probability measures). Let E be a real separable Banach space and μ be a probability measure on $\mathcal{B}(E)$. Then the following are equivalent.

(a) μ is stable.

(b) There exists a sequence a_n of positive real numbers and $\{x_n\} \subseteq E$ such that $\delta_{x_n} * T_{a_n}\mu^{*n} \Rightarrow \mu$.

(c) For each integer n, there exists a $y_n \in E$ and $c_n > 0$ such that $\mu^{*n} = \delta_{y_n} * T_{c_n}\mu$.

Proof: The equivalence of (a) and (b) follows from Lemma 2.2.3 and Lemma 2.2.6. We note that for each n, (c) implies,

$$\mu = T_{c_n^{-1}}(\mu^{*n} * \delta_{-y_n}) = T_{c_n^{-1}}\mu^{*n} * \delta_{-c_n^{-1}y_n} .$$

Hence (c) implies (b). Also if μ is degenerate clearly (b) implies (c). Assume that (b) holds and μ is not degenerate. Then for every $k = 1,2,\ldots$

$$\delta_{x_{nk}} * T_{a_{nk}}\mu^{*nk} = \mu_{nk} \Rightarrow \mu \quad \text{where} \quad \mu_n = \delta_{x_n} * T_{a_n}\mu^{*n} .$$

Hence,

$$\frac{(\delta_{x_n} * T_{a_n}\mu^{*n}) *\ldots* (\delta_{x_n} * T_{a_n}\mu^{*n})}{k \text{ factors}} = \delta_{k.x_n} * T_{a_n}\mu^{*nk}$$

$$= (T_{\frac{a_n}{a_{nk}}}\mu_{nk}) * \delta_{kx_n - \frac{a_n}{a_{nk}}x_{nk}}$$

$$= (T_{d_{n_k}}\mu_{nk}) * \delta_{c_{n_k}}$$

where $d_{n_k} = \frac{a_n}{a_{nk}}$ and $c_{n_k} = kx_n - \frac{a_n}{a_{nk}} x_{nk}$. Let $n \to \infty$ above. Then, $T_{d_{n_k}} \mu_{nk} * \delta_{z_{n_k}} \Rightarrow \mu^{*k}$. Since μ is not degenerate, therefore by Theorem 2.1.3 we conclude that $d_{n_k} \to d_k$, $z_{n_k} \to z \in E$, as $n \to \infty$, and $\mu^{*k} = T_{d_k}\mu * \delta_z$. Thus (b) $\Rightarrow$ (c) which completes the proof of the theorem.

2.2.11 <u>Corollary</u> (Proposition 9.25 of ([3], p. 199)). Let $X_1, X_2, \ldots$ be identically distributed, non-degenerate, independent, random variables taking values in E. Then $\mu = \lim_n \mu_{S_n}$ where $S_n = \frac{X_1 + \ldots + X_n}{A_n} - y_n$, for some sequence A_n of positive real numbers and $\{y_n\} \subset E$ iff for each non-negative integer n, there exists a $c_n > 0$ and $z_n \in E$ such that $\mu^{*n} = \delta_{z_n} * T_{c_n}\mu$.

The proof follows from the equivalence of (b) and (c) in Theorem 2.2.10. In particular this shows that limit laws of the normed sums given in (b) of Theorem 2.2.10 are infinitely divisible.

2.2.12 <u>Corollary</u> ([21], p. 327). Class of stable probability measures on $\mathcal{B}(E)$ coincides with the limit laws of normed sums of independent and identically distributed random variables taking values in E.

The proof follows from the equivalence of (a) and (b). The following corollary is now obvious.

2.2.13 <u>Corollary</u> ([14], p. 64). Every stable law on a real separable Banach space is infinitely divisible.

Corollaries 2.2.11, 2.2.12, and 2.2.13 relate stable probability measures to a certain subclass of infinitely divisible measures. Recently, J. Kuelbs and V. Mandrekar [16] have obtained Lévy-Khinchine representation for infinitely divisible measures on certain Orlicz spaces extending the work of S.R.S. Vardhan ([31], p. 227) on Hilbert space. In the next section we obtain a characterization of stable probability measures as a subclass of of these infinitely divisible measures in terms of the Lévy-Khinchine representation of their ch.f.'s. This result will generalize the recent work of Jajte ([14], p. 64) to these Orlicz spaces.

2.3 Lévy-Khinchine Representation of Stable Measures on Certain Orlicz Spaces

The Lévy-Khinchine representation for the characteristic functional of a stable probability measure on Hilbert spaces has been studied by Jajte in [14]. In this section we shall obtain similar representation for stable probability measures on certain Orlicz spaces. We remark that the proof of the main theorem in [14] is incomplete and contains a locuna which can be corrected (Cf. Lemma 2.3.6).

We recall for easy reference some notation and results on Orlicz spaces [16] from Chapter I. The function α used in this section will have the following properties.

(2.3.1)

(a) α is defined on $[0,\infty)$ into $[0,\infty)$,

(b) $\alpha(0) = 0$, $\alpha(s) > 0$ for $s > 0$,

(c) α is convex and strictly increasing on $[0,\infty)$

(d) $\alpha(2s) \leq M\alpha(s)$ for all $s \in [0,\infty)$, where M is a finite positive constant independent of s,

(e) $\int_{-\infty}^{\infty} \alpha(u^2)d\nu(u) \leq c\alpha\left[\int_{-\infty}^{\infty} u^2 d\nu(u)\right]$ for all Gaussian measures ν on $(-\infty,\infty)$ with mean zero, where c is a constant.

2.3.2 <u>Definition</u>. The space of real sequence $x = (x_1, x_2, x_3, \ldots)$ satisfying $\sum_{i=1}^{\infty} \alpha(x_i^2) < \infty$ is denoted by E_α.

The Orlicz space S_Γ given by $\Gamma(t) = \alpha(t^2)$, $t \in [0,\infty)$, is isomorphically isometric to E_α. Throughout this section we use this identification for E_α, [16].

Let α_c be the function complementary to α in the sense of Young ([32], p. 77) and s_{α_c} be the Orlicz space corresponding to α_c ([32], p. 79). Then for each λ in the positive cone of S_{α_c} (except when $E_\alpha = \ell_2$), whose norm is less than or equal to one half, define; $\|x\|_\lambda^2 = \sum_{i=1}^{\infty} \lambda_i x_i^2$ and if $E_\alpha = \ell_2$, then $\|x\|_\lambda^2 = \Sigma x_i^2$. The space of sequences with property that $\|x\|_\lambda < \infty$, will be denoted by H_λ. Obviously $E_\alpha \subseteq H_\lambda$ by Young's inequality ([32], p. 77). In fact, H_λ is a Hilbert space containing E_α as its measurable subset [16].

2.3.3 <u>Theorem</u>. Let μ be a probability measure on the Orlicz space E_α, where α satisfies (2.3.1). Then μ is stable on E_α iff either

$$(2.3.4) \qquad \hat{\mu}(y) = \exp[i(x_0, y) - \tfrac{1}{2}(Ty, y)] \quad \text{for all } y \in E_\alpha^*,$$

where $x_0 \in E_\alpha$ and T is an α-operator (i.e. μ is a Gaussian measure) OR

$$(2.3.5) \qquad \hat{\mu}(y) = \exp[i(x_0, h) + \int_U (e^{i(x,y)} - 1 - \frac{i(x,y)}{1+\|x\|_\lambda^2}) dF(x) + \int_{E_\alpha - U} (e^{i(x,y)} - 1 - \frac{i(x,y)}{1+\|x\|_\Gamma^2}) dF(x)]$$

where $x_0 \in E_\alpha$, $\|x\|_\Gamma$ is the norm of x in S_Γ, $U = \{x \in E_\alpha : \sum_{i=1}^{\infty} \alpha(x_i^2) \leq 1\}$, F is a σ-finite measure on E_α, finite on on the complement of every neighborhood of zero in E_α and such that $\sum_{i=1}^{\infty} \alpha(\int_U x_i^2 dF(x)) < \infty$, and there exists a r $(0 < r < 2)$ such that $T_a F = a^r F$ for every positive real a (Stable probability measure of index r).

Proof: Let (2.3.4) hold. Then

$$\hat{\mu}(ay).\hat{\mu}(by) = \exp[i(x_0,y)(a+b) - \frac{1}{2}(Ty,y)(a^2+b^2)]$$

for every a and b positive and therefore

$$\hat{\mu}(ay).\hat{\mu}(by) = \hat{\mu}((a^2+b^2)^{\frac{1}{2}}y) \exp[i(x_0,y)((a+b) - (a^2+b^2)^{\frac{1}{2}})] .$$

Hence by ([13], p. 37), $T_a\mu * T_b\mu = T_c\mu * \delta_x$ where $c = (a^2+b^2)^{\frac{1}{2}}$ and $x = ((a+b) - (a^2+b^2)^{\frac{1}{2}})x_0 \in E_\alpha$. Consequently, μ is stable. If (2.3.5) holds, then by ([16], p. 71), μ is infinitely divisible on E_α. Hence, there exists a sequence of finite measure F_n on E_α such that $F_n \uparrow F$ and a sequence $\{x_n\} \subset E_\alpha$, such that $e(F_n) * \delta_{x_n} \Rightarrow \mu$ on E_α. We can regard F_n's and μ as measures on H_λ (see Lemma 1.2.6).

Since every bounded and continuous function on H_λ is also bounded and continuous when restricted to E_α by [16], we conclude that $e(F_n) * \delta_{x_n} \Rightarrow \mu$ on H_λ. Hence by ([31], p. 224) we get $\hat{\mu}(y) = \exp[i(x_1,y) + \int_{H_\lambda} (e^{i(x,y)} - 1 - \frac{i(x,y)}{1+\|x\|_\lambda^2}) dF(x)]$ for every $y \in H_\lambda^*$, where $x_1 \in H_\lambda$, $\int_{\|x\|_\lambda \le 1} \|x\|_\lambda^2 dF(x) < \infty$ and F is the σ-finite measure as before. Since $T_a F = a^\lambda F$ on H_λ, therefore by ([14], p. 64) μ is stable on H_λ. Consequently, for every $a,b > 0$ there exists a $c > 0$ and $z \in H_\lambda$ such that $T_a\mu * T_b\mu = T_c\mu * \delta_z$. To prove μ is stable on E_α, it would be enough to show that $z \in E_\alpha$. Denote $z = (z_1, z_2, z_3, \ldots)$.

Define $\mu_n = \mu P_n^{-1}$ where $P_n(x) = (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$ on E_α. Obviously $\mu_n \Rightarrow \mu$ on E_α by argument similar to one in ([17], p. 221). Now it is easy to see that $T_a\mu_n * T_b\mu_n = T_c\mu_n * \delta_{\tau_n}$ where $\tau_n = (z_1, z_2, \ldots, z_n, 0, 0, 0, \ldots)$. We note that $\tau_n \in E_\alpha$ for every n and $\mu_n \Rightarrow \mu$ implies for any real d, $T_d\mu_n \Rightarrow T_d\mu$. Hence by Theorem 1.1.3(a), $\{\tau_n\}$ is compact on E_α. Consequently, $\tau_n \to z_0 \in E_\alpha$ by ([1], p. 37). Hence $\|\tau_n - z_0\|_\lambda \to 0$ by [16] , therefore $z = z_0$. Hence, for all a and $b > 0$ there exists a $c > 0$ and $z_0 \in E_\alpha$ such that $T_a\mu * T_b\mu = T_c\mu * \delta_{z_0}$. This completes the proof of sufficiency.

Suppose μ is stable on E_α. Then by Corollary 2.2.13, μ is infinitely divisible on E_α. Consequently, by [16], $\mu = \nu * \beta$ where β is the Gaussian part of μ on E_α and $\nu = \lim_{n\to\infty} e(F_n) * \delta_{x_n}$ where F_n's are increasing sequence of finite measures on E_α and $x_n \in E_\alpha$ for all n. We can regard F_n's, μ, ν and β as measures on H_λ [16]. Since an α-operator on E_α is also a trace class operator on H_λ (see Lemma 1.2.9), therefore by ([31], p. 226) β is Gaussian on H_λ. Thus $\mu = \nu * \beta$ where β is Gaussian on H_λ, and $\nu = \lim_{n\to\infty} e(F_n) * \delta_{x_n}$ on H_λ. Since μ is stable on H_λ, therefore by ([14], p. 64), $\mu = \beta$ or $\mu = \nu$ where

$$\hat{\nu}(y) = \exp[i(z,y) + \int_{E_\alpha} (e^{i(x,y)} - 1 - \frac{i(x,y)}{1+\|x\|_\lambda^2}) dF(x)]$$

for all $y \in H_\lambda^*$, and where $z \in H_\lambda$, $F = \lim_{n\to\infty} F_n$, $\int_{\|x\|_\lambda \le 1} \|x\|_\lambda^2 dF(x) < \infty$ and for some $0 < r < 2$, $T_a F = a^r F$ for all positive a.

Since $\nu = \lim_{n\to\infty} e(F_n) * \delta_{x_n}$ on E_α, the result follows from ([16], p. 66). This completes the proof of the theorem.

To correct the proof of the theorem in ([14], p. 64) we need the following lemma.

2.3.6 Lemma. Let H be a Hilbert space and F a σ-finite measure on H satisfying for every a and b positive

(2.3.7) $$T_a F + T_b F = T_{(a^\lambda + b^\lambda)^{1/\lambda}} F .$$

Then F satisfies the equation $T_a F = a^\lambda F$ for every positive a.

Proof: Since F is σ-finite, therefore it is enough to prove the above result for finite measure. So assume without loss of generality that F is a finite measure.

Let $B \in \mathcal{B}(H)$ such that $\partial(B)$, the boundary of B, has F measure zero. Then $T_a F(B)$ is a continuous function on $(0,\infty)$, by Lemma 2.1.1 and ([23], p. 40). From (2.3.7) we get

$$F(a^{-1}B) + F(b^{-1}B) = F((a^\lambda + b^\lambda)^{-1/\lambda}B)$$

for all a and $b > 0$. Since the above is true for all a and $b > 0$, therefore, we get

$$F(a^{-1/\lambda}B) + F(b^{-1/\lambda}B) = F((a+b)^{-1/\lambda}B) .$$

Let $F(a^{-1/\lambda}B) = g(a)$. Then g is continuous on $(0,\infty)$ and $g(a) + g(b) = g(a+b)$, for all a and $b > 0$. Therefore $g(a) = c.a$ for $a > 0$, where c is a constant depending on B. Hence $g(a^\lambda) = a^\lambda c$. Thus $F(a^{-1}B) = a^\lambda c$. Let $a = 1$. Then $c = F(B)$. Hence $F(a^{-1}B) = a^\lambda F(B)$. Since the class $\{B: B \in \mathcal{B}(H), F(\partial B) = 0\}$

is a field by ([1], p. 16), therefore by Caratheodory extension theorem

$$F(a^{-1}B) = a^{\lambda}F(B) \quad \text{for every} \quad B \in \mathcal{B}(H).$$

CHAPTER III

SELF-DECOMPOSIBLE PROBABILITY MEASURES ON BANACH SPACES

3.0 Introduction

In this chapter, we consider self-decomposible probability measures (laws) on a real separable Banach space. In Section 3.1, a necessary and sufficient condition for a self-decomposible law to be stable has been obtained in terms of its component. Section 3.2 presents a characterization of a self-decomposible law similar to ([21], p. 323). This result allows us to identify self-decomposible laws as the limit laws of certain normed sums of independent, uniformly infinitesimal Banach space valued random variables. We also show that the self-decomposible laws are subclass of infinitely divisible laws. This result is interesting on general separable Banach spaces since it is not known whether limit laws of uniformly infinitesimal triangular arrays of the Banach space valued random variables are infinitely divisible (see [20]). So in particular, from this result we can say that limit laws of certain subclass of triangular arrays are infinitely divisible. Finally, we characterize self-decomposible laws on certain Orlicz spaces in terms of their Lévy-Khinchine representation given in [16].

3.1 Preliminary Results on Self-Decomposible Laws

In this section we establish some preliminary results used in this chapter. We start with the following proposition.

3.1.1 Proposition. Let μ be a probability measure on $\mathcal{B}(E)$. If there exists a number $c > 0$ and a non-degenerate probability measure μ_c such that $\mu = T_c\mu * \mu_c$, then $c < 1$.

Proof: Since μ_c is non-degenerate on $\mathcal{B}(E)$, we have ([13], p. 37), $c \neq 1$. Now suppose $c > 1$. Then

$$\hat{\mu}(y) = \hat{\mu}(cy)\cdot\hat{\mu}_c(y) \quad \text{for every} \quad y \in E^* .$$

Putting $\frac{y}{c}$ for y above we get,

$$|\hat{\mu}(\tfrac{y}{c})| \leq |\hat{\mu}(y)| .$$

Consequently, $1 \geq |\hat{\mu}(y)| \geq |\hat{\mu}(\frac{y}{c})| \geq |\hat{\mu}(\frac{y}{c^2})| \geq \ldots \geq |\hat{\mu}(\frac{y}{c^n})| \geq \ldots$. Hence $1 \geq |\hat{\mu}(y)| \geq |\hat{\mu}(0)| = 1$. Thus $|\hat{\mu}(y)| = 1$ for every $y \in E^*$. Hence by Lemma 2.1.2 μ is degenerate. Consequently μ_c is degenerate; which contradicts the assumption. Thus $c < 1$.

Now we are ready to define self-decomposible probability measures on $\mathcal{B}(E)$ following Loève ([21], p. 322).

3.1.2 Definition. A probability measure μ on $\mathcal{B}(E)$ is said to be self-decomposible if for each $0 < c < 1$, there exists a probability measure μ_c on $\mathcal{B}(E)$ such that

$$\mu = T_c\mu * \mu_c . \tag{3.1.3}$$

3.1.4 Proposition. If μ is self-decomposible on $\mathcal{B}(E)$, then $\hat{\mu}(y) \neq 0$ for any $y \in E^*$.

Proof: Suppose there exists a $z \in E^*$ such that $\hat{\mu}(z) = 0$. Then the set $S_0 = \{y \in E^* : \hat{\mu}(y) = 0\}$ is non-empty. Let $a = \inf_{y \in S_0} \|y\|_{E^*}$. Then by the continuity of $\hat{\mu}$ it is clear that $a > 0$. Also note that if $\|y\| < a$, then $\hat{\mu}(y) \neq 0$.

There exists a sequence $\{y_n\}$ of elements of S_0 such that $\|y_n\| \to a$. Hence the sequence $\{y_n\}$ is norm bounded. Therefore by Alaoglu's Theorem[2] ([29], p. 202) $\{y_n : n = 1,2,\ldots\}$ is weak* compact. Consequently, there exists a $y_0 \in E^*$ such that for some subsequence $\{y_m\}$ of $\{y_n\}$, $y_m \to y_0$ in weak* sense. Let $2x_0 = y_0$. Then by the dominated convergence theorem $\hat{\mu}(y_m) \to \hat{\mu}(y_0)$. Hence $\hat{\mu}(2x_0) = 0$. Since $\|c2x_0\| < \|2x_0\| \leq a$, therefore $\hat{\mu}(c2x_0) \neq 0$, for every $0 < c < 1$, and hence by (3.1.3) $\hat{\mu}_c(2x_0) = 0$ for every $0 < c < 1$. Therefore by ([13], p. 37),

$$|\hat{\mu}_c(x_0)|^2 \leq 2|1 - \hat{\mu}_c(x_0)| \quad . \tag{3.1.5}$$

Since $\hat{\mu}_c(x_0) = \dfrac{\hat{\mu}(x_0)}{\hat{\mu}(cx_0)} \to 1$ as $c \to 1$, therefore by (3.1.5), $1 \leq 0$, which is a contradiction. Thus $\hat{\mu}(y) \neq 0$ for any $y \in E^*$.

Remark A. From the above proof it follows that if μ is any probability measure on $\mathcal{B}(E)$ and there exists a $z \in E^*$ such that $\hat{\mu}(z) = 0$, then there exists a y_0 such that $\hat{\mu}(y_0) = 0$ and $\|y_0\| = \inf_{y \in S_0} \|y\|$.

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2) I thank Professor P.K. Pathak for suggesting the use of Alaoglu's theorem in this context.

3.1.6 Proposition. Let μ be a non-degenerate probability measure on $\mathcal{B}(E)$. Then μ is self-decomposible with its component μ_c for $0 < c < 1$, given by $\delta_{-x} * T_{(1-c^\lambda)^{1/\lambda}}\mu$, where $x \in E$ and $0 < \lambda \leq 2$, iff μ is stable.

Proof: Let a and b be two positive real numbers. Then by letting $c = \frac{a}{(a^\lambda + b^\lambda)^{1/\lambda}}$, we get

$$\mu = T_{\frac{a}{(a^\lambda+b^\lambda)^{1/\lambda}}}\mu * T_{\frac{b}{(a^\lambda+b^\lambda)^{1/\lambda}}}\mu * \delta_{-x} .$$

Consequently,

$$T_a\mu * T_b\mu = T_{(a^\lambda + b^\lambda)^{1/\lambda}}\mu * \delta_{(a^\lambda + b^\lambda)^{1/\lambda}x} .$$

Hence μ is stable by (2.2.2).

On the other hand if μ is stable, then for every pair of positive numbers a and b, there exists an $x \in E$ by Lemma 2.2.6 and Theorem 2.2.10, such that

$$T_a\mu * T_b\mu = T_{(a^\lambda + b^\lambda)^{1/\lambda}}\mu * \delta_x \quad \text{for some } 0 < \lambda \leq 2 . \tag{3.1.7}$$

Take $b = (1 - c^\lambda)^{1/\lambda}$, where $0 < c < 1$. Then from (3.1.7) we get

$$T_c\mu * T_{(1 - c^\lambda)^{1/\lambda}}\mu = \mu * \delta_x .$$

Hence,

$$\mu_c = \delta_{-x} * T_{(1 - c^\lambda)^{1/\lambda}}\mu .$$

This completes the proof of the proposition.

3.2 Self-Decomposible Laws and Limit Laws

In this section we show that the class of self-decomposible measures coincides with certain limit laws of sums of independent Banach space valued random variables. Let us denote by $\mathcal{N}(E)$ the class of probability measures μ on $\mathcal{B}(E)$ with the property that there exist sequences $\{x_n\} \subset E$ and $\{b_n\} \subset R^+$, and a sequence $\{\mu_n\}$ of probability measures on $\mathcal{B}(E)$ such that

$\sup_{k=1,\ldots,n} \sup_{y \in S} |\hat{\mu}_k(yb_n) - 1| \to 0$ as $n \to \infty$ for every compact set $S \subset E^*$, and $\delta_{x_n} * \prod_{k=1}^{n} * T_{b_n} \mu_k \Rightarrow \mu$.

3.2.1 Proposition. If μ is a non-degenerate probability measure on $\mathcal{B}(E)$, and $\mu \in \mathcal{N}(E)$, then $b_n \to 0$, $\frac{b_n}{b_{n+1}} \to 1$.

Proof: If $b_n \not\to 0$, then there exists a subsequence $\{n'\}$ of $\{n\}$ such that $b_{n'}^{-1} \to b$, b finite. By the fact that $\mu \in \mathcal{N}(E)$, we get by letting $y_{n'} = \frac{y}{b_{n'}}$ for $y \in E^*$,

$$\hat{\mu}_k(y) = \hat{\mu}_k(y_{n'} . b_{n'}) \to 1 \quad \text{as} \quad n' \to \infty \quad \text{for each} \quad k.$$

Thus μ_k is degenerate for each k. Hence, μ is degenerate which is a contradiction. Thus $b_n \to 0$.

Suppose now $\frac{b_n}{b_{n+1}} \not\to 1$. Then there exists subsequences $\{n'\}$ of $\{n\}$ such that $\frac{b_{n'}}{b_{n'+1}} \to b$ where $b \neq 1$.

If $b = \infty$, then $c_{n'} = \frac{b_{n'+1}}{b_{n'}} \to 0$, and

$$(3.2.2) \quad \delta_{x_{n'+1}} * \prod_{k=1}^{n'} * T_{b_{n'+1}} \mu_k = T_{c_{n'}} (\delta_{x_{n'}} * \prod_{k=1}^{n'} * T_{b_{n'}} \mu_k) * \delta_{x_{n'+1} - c_{n'} x_{n'}} .$$

Since, $\widehat{\delta_{x_{n'+1}} * \prod_{k=1}^{n'} *T_{b_{n'+1}}\mu_k}(y) = \dfrac{e^{i(x_{n'+1},y)} \prod_{k=1}^{n'+1} \hat{\mu}_k(yb_{n'+1})}{\hat{\mu}_{n'+1}(b_{n'+1},y)} \to \dfrac{\hat{\mu}(y)}{1}$,

and $c_{n'} \to 0$, therefore from (3.2.2) we get $|\hat{\mu}(y)| = 1$ for every $y \in E^*$. Hence, by Lemma 2.1.2 μ is degenerate, which contradicts the assumption. Thus b is finite.

Let $c_{n'} = \dfrac{b_{n'}}{b_{n'+1}}$. Then $c_{n'} \to b$ and

(3.2.3) $$\delta_{x_{n'}} * \prod_{k=1}^{n'+1} * T_{b_{n'}}\mu_k = T_{c_{n'}}(\delta_{x_{n'+1}} * \prod_{k=1}^{n'+1} T_{b_{n'+1}}\mu_k) * \delta_{x_{n'}-c_{n'}x_{n'+1}} .$$

Since $\left|\widehat{\delta_{x_{n'}} * \prod_{k=1}^{n'+1} *T_{b_{n'}}\mu_k}(y)\right| = \left|e^{i(x_{n'},y)} \prod_{k=1}^{n'} \hat{\mu}_k(b_{n'},y)\right| \cdot \left|\hat{\mu}_{n'+1}(b_{n'},y)\right|$

$$\to |\hat{\mu}(y)| \cdot 1 ,$$

because $T_{b_{n'}}\mu_{n'+1} = T_{\frac{b_{n'}}{b_{n'+1}}}(T_{b_{n'+1}}\mu_{n'+1}) \to \delta_0$ by Lemma 2.1.1, and $c_{n'} \to b$, we conclude by (3.2.3), that

$$|\hat{\mu}(by)| = |\hat{\mu}(y)| \quad \text{for every} \quad y \in E^*.$$

Since $b \neq 1$ and is finite, therefore μ is degenerate by Lemma 2.1.2, which contradicts the assumption. Hence, $\dfrac{b_n}{b_{n+1}} \to 1$. This completes the proof of the proposition.

3.2.4 <u>Theorem</u>. If $\mu \in \mathcal{N}(E)$, then $\hat{\mu}(y) \neq 0$ for any $y \in E^*$.

Proof: Let $\nu_n = \delta_{x_n} * \prod_{k=1}^{n} * T_{b_n}\mu_k$. Then

$$(3.2.5)\quad \begin{cases} \hat{\nu}_n(y) = \{e^{i(x_m, \frac{b_n y}{b_m})} \prod_{k=1}^{m} \hat{\mu}_k(\frac{b_n}{b_m} \cdot y\, b_m)\} \cdot \{e^{i(x_n, y) - i(x_m, \frac{b_n}{b_m} y)} \prod_{k=m+1}^{n} \hat{\mu}_k(y\, b_n)\} \\ \\ \nu_n = T_{\frac{b_n}{b_m}} \nu_m * \lambda_{m_n} , \end{cases}$$

where to every integer n there corresponds an integer $m < n$ such that $\frac{b_n}{b_m} \to c \in (0,1)$, m, $n-m \to \infty$ as $n \to \infty$ ([21], p. 323), and λ_{m_n} is the probability measure corresponding to the term in the second bracket on the right hand side of (3.2.5).

Suppose there exists a $z \in E^*$ such that $\hat{\mu}(z) = 0$, then by Remark A, there exists a $y_0 \in E^*$ such that $\hat{\mu}(y_0) = 0$, and $\|y_0\| = \inf_{y \in S_0} \|y\|$, where $S_0 = \{y \in E^* : \hat{\mu}(y) = 0\}$. Hence from (3.2.5) we conclude that $\hat{\lambda}_{m_n}(y_0) \to 0$. Since $\hat{\nu}_n(y_0) \to 0$, $T_{\frac{b_n}{b_m}} \nu_m \to T_c \mu$ (Lemma 2.1.1) and $\|c \cdot \frac{y_0}{2}\| < \|y_0\|$, therefore from (3.2.5)

$$\lim_{n \to \infty} \hat{\lambda}_{m_n}(\frac{y_0}{2}) = \frac{\hat{\mu}(y_0/2)}{\hat{\mu}(c\, y_0/2)} .$$

Now by ([13], p. 37)

$$|\lambda_{m_n}(y_0) - \lambda_{m_n}(y_0/2)|^2 \le 2|1 - \lambda_{m_n}(y_0/2)| .$$

Let $n \to \infty$ above to conclude that

$$|\hat{\mu}(y_0/2)/\hat{\mu}(c\, y_0/2)|^2 \le 2\left|1 - \frac{\hat{\mu}(y_0/2)}{\hat{\mu}(c\, y_0/2)}\right| \quad \text{for every} \quad c \in (0,1).$$

Let $c \to 1$ above. Then $1 \le 0$, which is a contradiction. Hence $\hat{\mu}(z) \neq 0$, which completes the proof of the theorem.

Now we prove the Main Theorems of this section.

3.2.6 <u>Theorem</u>. A probability measure μ belongs to class $\mathcal{N}(E)$ iff it is self-decomposible.

Proof: If μ is degenerate, then $a_n = 0$, $b_n = \frac{1}{n}$, $\mu_n = \mu$ for every n, and we are done. So we can assume that μ is non-degenerate.

Suppose μ is self-decomposible and define probability measures $\nu_k = T_k \mu_c$, for $n = 2,3,\ldots$ where $c = \frac{k-1}{K}$ and μ_c is the component of μ. For $k = 1$, define $\nu_1 = \mu$. Then $\hat{\nu}_k(y) = \frac{\hat{\mu}(ky)}{\hat{\mu}((k-1)y)}$, $k = 1,2,\ldots$. Hence, $\prod_{k=1}^{n} * T_{\frac{1}{n}} \nu_k = \mu$. So take $a_n = 0$, $b_n = \frac{1}{n}$ for $n = 1,2,\ldots$ and note that

$$\sup_{k=1,2,\ldots,n} \sup_{y\in S} \left|\frac{\hat{\mu}(\frac{k}{n} y) - \hat{\mu}(\frac{k-1}{n} y)}{\hat{\mu}(\frac{k-1}{n} y)}\right|^2 \le \frac{\sup_{y\in S} |1 - \hat{\mu}(y/n)|}{\inf_{k=1,\ldots,n} \inf_{y\in S} |\hat{\mu}(\frac{k-1}{n} y)|^2}$$

by ([13], p. 57),

for every compact $S \subseteq E^*$.

Since S is compact, therefore for some $t \ge 0$, $S \subseteq \{y : \|y\| \le t\}$. Because $\|\frac{y}{n}\| \to 0$ uniformly on S as $n \to \infty$, and $\sup_{k=1,\ldots,n} \sup_{y\in S} \|\frac{k-1}{n} . y\| \le \sup_{y\in S} \|y\| \le t$, therefore we conclude by Proposition 3.1.4, that

$$\underline{\lim}_{n} \inf_{k=1,\ldots,n} \inf_{y\in S} |\hat{\mu}(y \tfrac{k-1}{n})| \ge \inf_{\|y\|\le t} |\hat{\mu}(y)| > 0 ,$$

and hence,

$$\sup_{k=1,2,\ldots,n} \sup_{y\in S} |\hat{\nu}_k(\tfrac{y}{n}) - 1| \to 0 .$$

Thus $\mu \in \mathcal{N}(E)$.

Conversely, suppose $\mu \in \mathcal{N}(E)$. Then there exist two sequences $\{x_n\} \subseteq E$ and $\{b_n\} \subseteq R^+$, and a sequence $\{\mu_k\}$ of probability measures such that $\nu_n = \delta_{x_n} * \prod_{k=1}^{n} * T_{b_n} \mu_k \Rightarrow \mu$ and

$\{T_{b_n}\mu_k, k = 1,2,\ldots,n\}$ is a uniformly infinitesimal sequence. Hence, by Proposition 3.2.1 $b_n \to 0$, $\frac{b_n}{b_{n+1}} \to 1$. Consequently, given a c $(0 < c < 1)$, we can correspond to every integer n an integer $m < n$ such that $\frac{b_n}{b_m} \to c$ and $m, n-m \to \infty$ as $n \to \infty$ ([21], p. 323). Note that

$$\nu_n = \frac{T_{b_n}}{b_m} \nu_m * \lambda_{m_n} ,$$

where λ_{m_n} is the probability measure given by

$$\delta_{x_n - b_n x_m} * \prod_{k=m+1}^{n} * T_{b_n}\mu_k .$$

Since $\nu_n \Rightarrow \mu$, therefore by Lemma 2.1.1,

$$\frac{T_{b_n}}{b_m} \nu_m \Rightarrow T_c\mu ,$$

and hence by Theorem 1.1.3, $\{\lambda_{m_n}\}$ is compact. In view of Theorem 3.2.4, $\hat{\lambda}_{m_n}(y) \to \frac{\hat{\mu}(y)}{\hat{\mu}(cy)}$, and hence by Theorem 1.1.3(c),

(3.2.6a) $$\lambda_{m_n} \Rightarrow \mu_c ,$$

where μ_c is given by

$$\hat{\mu}_c(y) = \hat{\mu}(y)/\hat{\mu}(cy), \text{ for every } y \in E^* .$$

Thus $\mu = T_c\mu * \mu_c$. This completes the proof of the theorem.

3.2.7 <u>Theorem</u>. Let μ be a self-dceomposible probability measure on a real separable Banach E. Then μ and for each $(0 < c < 1)$, the associated measure μ_c are (see (3.1.3)) infinitely divisible.

Proof: Let $0 < c < 1$. From (3.1.3) it follows by iteration that for each n,

$$\mu = \mu_c * T_c\mu_c * T_{c^2}\mu_c * \ldots * T_{c^{n-1}}\mu_c * T_{c^n}\mu$$

$$= \lambda_{n,c} * T_{c^n}\mu ,$$

where $\lambda_{n,c} = \mu_c * T_c\mu_c * T_{c^2}\mu_c * \ldots * T_{c^{n-1}}\mu_c$. Since $c^n \to 0$ as $n \to \infty$, it follows from Lemma 2.1.1,

$$T_{c^n}\mu \Rightarrow \delta_0 .$$

Consequently by Theorem 1.1.3(a) and (c),

(3.2.8) $$\lambda_{n,c} \Rightarrow \mu \quad \text{as} \quad n \to \infty .$$

Let m be a positive integer. Let

$$\nu_{n,c} = \mu_c * T_{c^m}\mu_c * T_{c^{2m}}\mu_c * \ldots * T_{c^{(n-1)m}}\mu_c .$$

Then

(3.2.9) $$\nu_{n,c} * T_c\nu_{n,c} * T_{c^2}\nu_{n,c} * \ldots * T_{c^{m-1}}\nu_{n,c}$$

$$= \mu_c * T_c\mu_c * \ldots * T_{c^{n\,m-1}}\mu_c ,$$

and the right hand side in (3.2.9) converges weakly to μ as $n \to \infty$ by (3.2.8). Consequently by Theorem 1.1.3(b) $\nu_{n,c}$ is shift compact. Therefore there exists a sequence $x_{n,c}$ in E such that $\nu_{n,c} * \delta_{x_{n,c}}$ is compact. Hence there exists a subsequence $\{\ell^{(c)}\}$ of $\{n\}$ such that $\nu_{\ell,c} * \delta_{x_{\ell,c}} \Rightarrow \lambda_{m,c}$ in $\mathcal{M}$ as $\ell^{(c)} \to \infty$. From (3.2.9) it follows that

$$(3.2.10)\quad (\nu_{\ell,c} * \delta_{x_{\ell,c}}) * T_c(\nu_{\ell,c} * \delta_{x_{\ell,c}}) *\dots*$$

$$T_{c^{m-1}}(\nu_{\ell,c} * \delta_{x_{\ell,c}}) * \delta_{-\frac{c^m-1}{c-1}\cdot x_{\ell,c}}$$

converges weakly to μ as $\ell^{(c)}\to\infty$. Since $\nu_{\ell,c} * \delta_{x_{\ell,c}}$ converges weakly, therefore $\delta_{-\frac{c^m-1}{c-1}x_{\ell,c}}$ is compact. Consequently for some $y_{m,c}$ in E, $-\frac{c^m-1}{c-1}x_{\ell,c}$ converges to $y_{c,m}$ in norm. Hence from (3.2.10) one gets,

$$(3.2.11)\quad \lambda_{m,c} * T_c\lambda_{m,c} * T_{c^2}\lambda_{m,c} *\dots* T_{c^{m-1}}\lambda_{m,c} * \delta_{y_{m,c}} = \mu\ .$$

Let c_k be a sequence in $(0,1)$ converging to 1. Then from (3.2.11)

$$(3.2.12)\quad \lambda_{m,c_k} * T_{c_k}\lambda_{m,c_k} *\dots* T_{c_k^{m-1}}\lambda_{m,c_k} * \delta_{y_{m,c_k}} = \mu \quad \text{for each } k.$$

Consequently, λ_{m,c_k} is shift compact. Hence there exists a subsequence $\{p\}$ of $\{k\}$ and a sequence $z_{m,p}$ in E such that

$$\lambda_{m,c_p} * \delta_{z_{m,p}} \Rightarrow \lambda_m \text{ in } \mathcal{M} \text{ as } p\to\infty\ .$$

Thus from (3.2.12) we get

$$(3.2.13)\quad (\lambda_{m,c_p} * \delta_{z_{m,p}}) * T_{c_p}(\lambda_{m,c_p} * \delta_{z_{m,p}}) *\dots* T_{c_p^{m-1}}(\lambda_{m,c_p} * \delta_{z_{m,p}})$$

$$* \delta_{y_{m,c_p} - m.z_{m,n}} = \mu \quad \text{for each } p.$$

Since $\lambda_{m,c_p} * \delta_{z_{m,p}}$ converges weakly, therefore by Theorem 3.1.3(a) $\delta_{y_{m,c_p} - mz_{m,p}}$ is compact. Hence $y_{m,c_p} - mz_{m,p}$ converges in norm to some element y_m in E. Thus from (3.2.13) we get by letting $p \to \infty$,

$$\lambda_m * \lambda_m * \ldots * \lambda_m * \delta_{y_m} = \mu .$$

Since m was an arbitrary positive integer, therefore μ is infinitely divisible.

To prove that μ_c is infinitely divisible for each $(0 < c < 1)$, consider the Hilbert space H containing E such that every Borel subset of E is a Borel subset of H ([15], p. 355). We can regard μ_c as a measure on H. Then μ_c is i.d. by Theorem 3.2.6, (3.2.6a) and ([23], p. 199). Hence for each m there exists a probability measure ν_m on H such that

$$\nu_m^{*m} = \mu_c \quad \text{on} \quad H .$$

We will be done once we show that ν_m is concentrated on E. Since μ is infinitely divisible on E, therefore there exists a probability measure λ_m on E such that

$$\lambda_m^{*m} = \mu \quad \text{on} \quad E .$$

We can regard λ_m as a measure on H, hence

$$\lambda_m^{*m} = \mu \quad \text{on} \quad H .$$

Since μ is self-decomposible, therefore,

$$\lambda_m^{*m} = T_c\lambda_m^{*m} * \nu_m^{*m} \quad \text{on} \quad H .$$

Hence

$$[\hat{\lambda}_m(y)]^m = [\hat{\lambda}_m(cy)]^m * [\hat{\nu}_m(y)]^m \quad \text{for every} \quad y \in H^*,$$

$$= [\hat{\lambda}_m(cy) \cdot \hat{\nu}_m(y)]^m .$$

Therefore

$$\hat{\lambda}_m(y) = \hat{\lambda}_m(cy)\hat{\nu}_m(y)\, e^{2\pi \frac{n(y)}{m} i} \quad \text{for every} \quad y \in H^*,$$

where $n(y)$ is an integer valued function of y. Note that $e^{2\pi \frac{n(y)}{m}} = \frac{\hat{\lambda}_m(y)}{\hat{\lambda}_m(cy)\hat{\nu}_m(y)}$, since the denominator does not vanish (see Theorem 3.1.4). Therefore $e^{2\pi \frac{n(y)}{m} i}$ is a continuous function of y taking only countably many values. Hence it must be degenerate. Since at $y = 0$, $e^{2\pi \frac{n(y)}{m} i} = 1$, therefore $n(y)$ must take values which are multiples of m. Thus $\hat{\lambda}_m(y) = \hat{\lambda}_m(cy)\hat{\nu}_m(y)$ for each $y \in H^*$. Consequently, $\lambda_m = T_c\lambda_m * \nu_m$ on H.

Since λ_m and $T_c\lambda_m$ are concentrated on E, therefore ν_m is also concentrated on E. This completes the proof of the theorem.

Remark. The above theorems generalize the classical results about the self-decomposible laws (Loève [21], p. 323, Theorem 23.3A and corollary). We note that both these theorems are very easy to handle in the finite-dimensional case because of the powerful Lévy continuity theorem available. However in the case of the Banach space, no complete analogue of the Lévy continuity theorem is available and hence the methods used here are combinations of the methods of characteristic functionals and functional analysis. To bring out the simplicity of the method of characteristic functional in the context of the availability of the full force of the Lévy continuity theorem we treat in

Chapter IV, semi-stable laws whose very definition depends on characteristic functional ([22], p. 92).

In the next section we obtain results for self-decomposible laws analogous to results of §2.3 for stable laws. More specifically we characterize self-decomposible laws in terms of their Lévy-Khinchine representation.

3.3 Lévy-Khinchine Representation of Self-Decomposible Probability Measures on Certain Orlicz Spaces

In view of Theorem 3.2.7, we know that μ is infinitely divisible. Hence on Orlicz spaces it has a Lévy-Khinchine representation given by J. Kuelbs and V. Mandrekar [16].

In Loève ([21], p. 324), self-decomposible laws are characterized in terms of the "Lévy-measure". The purpose of this section is to obtain analogue of Theorem 23.3B of ([21], p. 324) for Orlicz spaces E_α discussed in §1.2. To prove this theorem, we shall first characterize self-decomposible probability measures on Hilbert spaces. The functions α, α_c, Γ, and spaces E_α, S_Γ, H_λ, S_α, S_{α_c}, carry the same meaning as in §2.3.

3.3.1 Theorem. A functional $\varphi(\cdot)$ is a ch.f. of a self-decomposible probability measure on a real separable Hilbert space H iff

$$(3.3.2) \qquad \varphi(y) = \exp[i(x_0,y) - \tfrac{1}{2}(Dy,y) + \int K(x,y)\,dM(x)] \quad \text{for every } y \in H^*,$$

where $x_0 \in H$, $K(x,y) = e^{i(x,y)} - 1 - \frac{i(x,y)}{1+\|x\|^2}$, D is an S-operator, M is a σ-finite measure on $\mathcal{B}(H)$, finite on the complement of every neighborhood of zero in H, $\int_{\|x\|\le 1} \|x\|^2 dM(x) < \infty$ and for each $0 < c < 1$, $M = T_c M + M_c$, where M_c is a measure on $\mathcal{B}(H)$. The representation (3.3.2) of $\varphi(y)$ is unique.

Proof: Suppose φ is a ch.f. of a self-decomposible probability measure on $\mathcal{B}(H)$. Then for each $c \in (0,1)$, there exists a ch.f. φ_c on H, such that

$$\varphi(y) = \varphi(cy) \cdot \varphi_c(y) . \tag{3.3.3}$$

Now by Theorem 3.2.7, φ and φ_c are infinitely divisible. Therefore by ([31], p. 227), $\varphi(y) = \exp[i(x_0,y) - \frac{1}{2}(Dy,y) + \int K(x,y)dM(x)]$ for every $y \in H^*$, where $x_0 \in H$, D is an S-operator, and M is a σ-finite measure on $\mathcal{B}(H)$, finite on the complement of every neighborhood of zero in H and such that $\int_{\|x\|\le 1} \|x\|^2 dM(x) < \infty$. Hence,

$$\varphi(cy) = \exp[i(x_0,cy) - \frac{1}{2} c^2 (Dy,y) + \int K(x,cy)dM(x)].$$

$$K(x,cy) = e^{i(cx,y)} - 1 - \frac{i(cx,y)}{1+\|cx\|^2} + \frac{i(cx,y)}{1+\|cx\|^2} - \frac{i(x,cy)}{1+\|x\|^2} ,$$

$$= K(cx,y) + \frac{i(cx,y)\|x\|^2(1 - c^2)}{(1+\|x\|^2)(1+c^2\|x\|^2)} .$$

Since, $\int \frac{(x,y)\|x\|^2}{(1+\|x\|^2)(1+c^2\|x\|^2)} dM(x) \le \|y\| \int_{\|x\|\le 1} \|x\|^2 + \|y\| \int_{\|x\|>1} dM(x) < \infty$, therefore,

$$\varphi(cy) = \exp[i(\bar{x},y) - \frac{1}{2} c^2 (Dy,y) + \int K(x,y)dM(c^{-1}x)] ,$$

where $(\bar{x},y) = (c\, x_0,y) + \int \frac{c(1-c^2)(x,y)\|x\|^2}{(1+\|x\|^2)(1+c^2\|x\|^2)} dM(x)$. Since, φ_c is infinitely divisible, therefore by ([31], p. 227),

$$\varphi_c(y) = \exp[i(x_0,y) - \frac{1}{2}(D_c y,y) + \int K(x,y)\,dM_c(x)] \quad \text{for every} \quad y \in H^*,$$

where $x_c \in H$, D_c is an S-operator, M_c is a σ-finite measure on $\mathcal{B}(H)$, finite on the complement of every neighborhood of zero in H, and such that $\int_{\|x\|\leq 1} \|x\|^2 dM_c(x) < \infty$.

If we denote $\varphi(y) = [x_0, D, M]$, then we have from (3.3.3)

$$[x_0,D,M] = [\bar{x},c^2D,T_cM]\cdot[x_c,D_c,M_c] = [\bar{x} + x_c,c^2D + D_c,T_cM + M_c] .$$

By the uniqueness of the representation of infinitely divisible ch.f., we have

$$M = T_cM + M_c .$$

To prove sufficiency, we make use of Theorem 4.10 of ([23], p. 181) to conclude that φ is a ch.f. of an infinitely divisible probability measure μ on $\mathcal{B}(H)$, and note that $[x,D,M] = [\bar{x},c^2D,T_cM]\cdot[x_0-\bar{x},(1-c^2)D,M_c]$, where $\bar{x}$ is same as before and $0 < c < 1$.

Now by the one-to-one correspondence between the probability measures on $\mathcal{B}(H)$ and their ch.f., we conclude that

$$\mu = T_c\mu * \mu_c ,$$

where μ, μ_c are the probability measures corresponding to $[x_0,D,M]$, and $[x_0 - \bar{x}, (1-c^2)D,M_c]$ respectively. Uniqueness follows from Theorem 5.10 of ([31], p. 227). This completes the proof of the theorem.

<u>Remark B</u>. The main result of Jajte in [14] can be obtained as a corollary of the above theorem in the following manner.

Let μ be a stable probability measure on $\mathcal{B}(H)$. Then by Proposition 3.1.6 μ is self-decomposible and hence for each $c \in (0,1)$,

$$M_c = T_{(1-c^\lambda)^{1/\lambda}} M .$$

Hence $M = T_c M + T_{(1-c^\lambda)^{1/\lambda}} M.$

Now by the same argument as in Lemma 2.3.6, we get

$$T_c M = c^\lambda M \quad \text{for every} \quad c \in (0,1) .$$

To conclude the remark, note that $T_a M = a^\lambda M$ for every $a \in (0,\infty)$ iff $T_a M = a^\lambda M$ for every $a \in (0,1)$.

3.3.4 Theorem. Let μ be a probability measure on the Orlicz space E_α, where α is as before. Then μ is self-decomposible iff

$$(3.3.5) \quad \hat{\mu}(y) = \exp[i(x_0,y) - \tfrac{1}{2}(Ty,y) + \int_U (e^{i(x,y)} - 1 - \frac{i(x,y)}{1+\|x\|_\lambda^2})\,dM(x)$$

$$+ \int_{E_\alpha - U} (e^{i(x,y)} - 1 - \frac{i(x,y)}{1+\|x\|_\Gamma^2})\,dM(x) \quad \text{for every} \quad y \in E_\alpha^*,$$

where $x_0 \in E_\alpha$, T is an α-operator, $U = \{x \in E_\alpha : \sum_1^\infty \alpha(x_i^2) \le 1\}$, and M is a σ-finite measure on $\mathcal{B}(E_\alpha)$, finite on the complement of every neighborhood of zero in E_α, $\sum_{i=1}^\infty \alpha(\int_U x_i^2 dM(x)) < \infty$ and for each $c \in (0,1)$, $M = T_c M + M_c$, where M_c is a measure on $\mathcal{B}(E_\alpha)$. For each fixed λ, the representation (3.3.5) of $\hat{\mu}(y)$ is unique.

Proof: Suppose μ is self-decomposible on E_α. Then by Theorem 3.2.7, μ is infinitely divisible. Consequently, by [16], $\mu = \nu * \beta$ where β is the Gaussian part of μ and $e(F_n) * \delta_{x_n} \Rightarrow \nu$ where F_n's are increasing sequence of finite measures on E_α and $x_n \in E_\alpha$. Using Theorem 7.2 of [16] , we get that $\hat{\mu}(y)$ has the form given in (3.3.5) except $M = T_c M + M_c$. We can regard F_n's, ν and β as measures on H_λ by [16] . Since an α-operator on E_α is also a trace class operator on H_λ, therefore by ([31], p. 226), β is Gaussian on H_λ. Thus $\mu = \nu * \beta$ on H_λ. Since every bounded and continuous function on H_λ is also bounded and continuous when restricted to E_α by [16] we get $e(F_n) * \delta_{x_n} \Rightarrow \nu$ on H_λ. Since $T_c\mu$ and μ_c are concentrated on E_α, therefore $T_c\mu * \mu_c$ is concentrated on E_α, and hence $\mu = T_c\mu * \mu_c$ on H_λ. Therefore μ is self-decomposible on H_λ. Consequently by Theorem 3.3.1, $M = \lim_{n\to\infty} F_n$ and $M = T_c M + M_c$.

Conversely, suppose (3.3.5) holds. Then μ is infinitely divisible on E_α by [16] . Hence there exists a sequence of finite measures F_n's on E_α such that $F_n \uparrow M$, and a sequence $x_n \in E_\alpha$ such that $e(F_n) * \delta_{x_n} \Rightarrow \nu$ on E_α and $\mu = \nu * \beta$ where β is Gaussian on E_α. By the same reasoning as in the proof of the necessary part of this theorem, we conclude $e(F_n) * \delta_{x_n} \Rightarrow \nu$ on H_λ, and β can be regarded as Gaussian on H_λ. Since $\mu = \nu * \beta$ on H_λ and hence by [16], μ is infinitely divisible on H_λ. Thus by Theorem 3.3.1, μ is self-decomposible on H_λ. Hence for every $c \in (0,1)$,

$$\mu = T_c\mu * \mu_c \text{, on } H_\lambda \text{ .}$$

Since μ and $T_c\mu$ give all its mass to E_α, therefore μ_c will also. Thus μ is self-decomposible on E_α. The uniqueness follows from Theorem 7.2 of [16] .

CHAPTER IV

SEMI-STABLE LAWS ON SEPARABLE HILBERT SPACES

4.0 Introduction

The main purpose of this chapter is to study semi-stable laws which arose classically only through the method of characteristic functions (Lévy [22], p. 95). Thus the results of this section are valid when complete analytic analogues of Lévy continuity theorem and Bochner theorem are available. In case of Hilbert space, L. Gross [9] has established such results. We can therefore use his results to study semi-stable laws on a separable Hilbert space.

Let μ be a probability measure on a real separable Hilbert space. Then its characteristic functional $\hat{\mu}$ is τ-continuous ([9], p. 7) and hence in view of Remark 1.1.15 it has an extension $\hat{\mu}^{\sim}$. We note that if $\hat{\mu}$ is τ-continuous and a is any positive real number, then $(\hat{\mu}^{a})$ is τ-continuous. Hence $(\hat{\mu}^{a})^{\sim}$ is well defined as an extension. But in view of the definition of extension explained in Remark 1.1.15, we obtain $(\hat{\mu}^{\sim})^{a} = (\hat{\mu}^{a})^{\sim}$. This observation will be used in Theorem 4.2.6. For the sake of completeness we recall here the Lévy continuity theorem of Gross ([9], p. 8).

Theorem. Let μ_n be a sequence of probability measures on a real separable Hilbert space H with respective characteristic functionals φ_n. Let φ be a uniformly τ-continuous functional on H such that $\varphi(0) = 1$. If μ_n converges weakly to a measure

μ whose characteristic functional is φ, then φ_n converges to φ on H and $\tilde{\varphi}_n$ converges to $\tilde{\varphi}$ in probability. Conversely if $\tilde{\varphi}_n$ converges to $\tilde{\varphi}$ in probability then μ_n converges weakly to a probability measure μ with characteristic functional φ.

Now we are ready to obtain generalization of Theorems of ([24], p. 780). We also obtain here the Lévy-Khinchine representation of a symmetric semi-stable law on a real separable Hilbert space in the same spirit as Jajte in ([14], p. 64) or alternatively we characterize the symmetric semi-stable laws on a real separable Hilbert space in terms of its Vardhan representation.

4.1 The Main Theorem

4.1.1 Definition. A probability measure μ on a real separable Banach space E is said to be semi-stable if there exist real numbers $a > 0$ and b such that the characteristic functional $\hat{\mu}$ satisfies

$$\hat{\mu}(y) = [\hat{\mu}(by)]^a \quad \text{for every} \quad y \in E^* . \tag{4.1.2}$$

Remark. We can assume $a > 1$, otherwise μ would be degenerate. By the similar argument as in Proposition 3.1.1, it would follow that $|b| < 1$, provided μ is non-degenerate. From now on we shall, without loss of generality, assume $a > 1$, $|b| < 1$.

4.1.3 Theorem. Let μ be a probability measure on $\mathcal{B}(E)$ and for $x > 0$, $[x]$ denote the greatest integer contained in x. If there exist two numbers $a > 0$, and b real such that

$$[T_{b^n}\mu]^{[a^n]} \Rightarrow \mu ,$$

then μ is semi-stable.

Proof: Let $\nu_n = [T_{b^n}\mu]^{[a^n]}$. Then

(4.1.4) $$\nu_{n+1} = [T_{b^{n+1}}\mu]^{[a^{n+1}]} = [T_{b^{n+1}}\mu]^{[a^n]} \cdot [T_{b^{n+1}}\mu]^{[a^{n+1}]-[a^n]} .$$

(4.1.5) $$[\hat{\mu}(b^{n+1}y)]^{[a^{n+1}]-[a^n]} = [\hat{\mu}(b^{n+1}y)]^{a^n(a-1)} \frac{[\hat{\mu}(b^{n+1}y)]^{\theta_n}}{[\hat{\mu}(b^{n+1}y)]^{\theta_{n+1}}} ,$$

where θ_n, θ_{n+1} are fractional parts of a^n, a^{n+1} respectively. Now it is clear from (4.1.5), that as $n \to \infty$,

$$[\hat{\mu}(b^{n+1}y)]^{[a^{n+1}]-[a^n]} \to [\hat{\mu}(by)]^{a-1} .$$

Thus from (4.1.4) we get

$$\hat{\mu}(y) = \hat{\mu}(by) \cdot [\hat{\mu}(by)]^{a-1} = [\hat{\mu}(by)]^a .$$

4.1.6 Main Theorem. Let μ be a semi-stable probability measure on a real separable Hilbert space H. Then $[T_{b^n}\mu]^{[a^n]} \Rightarrow \mu$.

Proof: Let $\nu_n = [T_{b^n}\mu]^{[a^n]}$. Then to show that $\nu_n \Rightarrow \mu$, it is enough by ([9], p. 8) to show that $\hat{\nu}_n^{\sim} \Rightarrow \hat{\mu}^{\sim}$ in probability. Since $\hat{\mu}(y)^{\sim} = [\hat{\mu}(b^n y)]^{[a^n]^{\sim}} \cdot [\hat{\mu}(b^n y)]^{\theta_n^{\sim}}$, therefore

(4.1.7) $$\begin{aligned} |\hat{\nu}_n(y)^{\sim} - \hat{\mu}(y)^{\sim}| &\leq |[\hat{\mu}(b^n y)]^{[a^n]^{\sim}}| \cdot |1 - (\hat{\mu}(b^n y))^{\theta_n^{\sim}}| \\ &\leq |1 - [\hat{\mu}(b^n y)]^{\theta_n^{\sim}}| , \end{aligned}$$

where θ_n is the fractional part of a^n.

Since $T_{b^n}\mu \Rightarrow \delta_0$, therefore $\hat{\mu}(yb^n)^{\sim} \xrightarrow{P} 1$ by ([9], p. 8). Hence

$$\hat{\mu}(yb^n)^{\sim\theta_n} = [\hat{\mu}(yb^n)]^{\theta_n^{\sim}} \xrightarrow{P} 1 .$$

Therefore from (4.1.7), $\hat{\nu}_n^{\sim} \xrightarrow{P} \hat{\mu}^{\sim}$ which completes the proof of the theorem.

Remark. In view of the unavailability of Lévy continuity theorem in the form of Gross for general Banach space ([17], p. 221) the above proof is invalid for Banach spaces.

4.1.8 Theorem. (1) Semi-stable laws on a real separable Hilbert space H are infinitely divisible.

(2) If μ is semi-stable on H, then $[\hat{\mu}(y)]^{\lambda}$ for each $\lambda > 0$ is a characteristic functional of a probability measure on H.

Proof: In view of ([23], p. 181), (1) follows from Theorem 4.1.6 and (2) is obvious from ([23], p. 181).

4.1.9 Theorem. A functional $\varphi(\cdot)$ is a characteristic function of a symmetric-semi-stable probability measure on a real separable Hilbert space H iff

$$(4.1.10) \quad \varphi(y) = \exp[-\tfrac{1}{2}(Dy,y) + \int (\cos(x,y) - 1)\,dM(x)] \quad \text{for every } y \in H^*,$$

where D is an S-operator, M is a σ-finite measure on $\mathcal{B}(H)$, finite on the complement of every neighborhood of zero in H, $\int_{\|x\|\le 1} \|x\|^2 dM(x) < \infty$ and there exist two real numbers $a > 0$, b such that $ab^2 = 1$ and $aT_bM = M$.

Proof: Suppose φ is a characteristic function of a semi-stable law. Then if we denote the Lévy-Khinchine representation of a symmetric infinitely divisible characteristic function by $[0,D,M]$, we get by Theorem 4.1.8

$$\varphi = [0, D, M] .$$

Using (4.1.2) we get

$$[0,D,M] = [0,b^2D,T_bM]^a = [0,b^2aD,aT_bM].$$

Now by the uniqueness of the representation we get

$$ab^2 = 1 \quad \text{and} \quad aT_bM = M .$$

Conversely, suppose there exist two numbers $a > 0$ and $b \ni ab^2 = 1$ and $aT_bM = M$. From ([23], p. 181), it follows that φ is infinitely divisible. Thus,

$$\varphi = [0,D,M] = [0,ab^2D,aT_bM] = [0,b^2D,T_bM]^a .$$

Hence $\varphi(y) = [\varphi(by)]^a$ for every $y \in H^*$.

<u>Remark</u>. In case the symmetric semi-stable law μ is actually stable, then in view of Theorem in ([14], p. 64), we get that $ab^\lambda = 1$ for $0 < \lambda < 2$. Even non-symmetric Gaussian law is not semi-stable.

CHAPTER V

CONCLUDING REMARKS

In this part we state some problems which arise out of the work of previous chapters.

1. The limit theorems proved in §2.2 and §3.2 can be used to obtain certain invariance principles for stochastic processes in the same spirit as D'Acosta [4] used his work on convergence to Gaussian measures. In fact in view of the work of Lamperti [18] it seems that certain limit processes associated with branching processes are symmetric stable processes. So the following two problems raised by Rajput and Cambanis in [27] for Gaussian measure become of interest for stable measures.

2. Given a stable stochastic process with sample paths in a Banach function space, is there a stable measure on the Banach space which is induced by the given stochastic process?

3. Given a stable measure on a Banach function space, is there a stable stochastic process with sample paths in the Banach space which induces the given measure?

Some partial progress has been made on problem 2 and it is hoped that 3 can be handled by similar methods.

4. One can introduce, following Doob's ideas on infinitely divisible processes [6], the idea of self-decomposible process and study its sample path properties. Also problems 2 and 3 can

again be asked about self-decomposible process.

The above problems are under investigation and progress on them will be reported elsewhere.

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