PROBABILITY MEASURES ON REAL SEPARABLE BANACH SPACES

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ABSTRACT

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Two fundamental problems are considered in this thesis, they can be described as follows. In Chapter I the problem of characterizing the characteristic functions of probability measures is examined. When the probability measures are defined on a real Banach space with a Schauder basis we obtain general results which are applied to various sequence spaces.

In Chapter II we introduce the notion of covariance form for a Gaussian probability measure. We obtain several representation theorems for the covariance form when the Gaussian measures are defined on real separable Banach space. We then apply them to sequence spaces and spaces of continuous functions.

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CHAPTER 0

INTRODUCTION

In the development of the classical theory of probability, the concept of characteristic function has played a powerful and central role. As a consequence this concept has been extended, initally by Kolmogorov, to the study of probability measures on infinite dimensional linear spaces, the aim being to duplicate the results of probability theory on finite dimensional spaces. The impetus behind this is, the fact that much of the study of stochastic processes is equivalent to the study of measures on suitably chosen infinite dimensional linear function spaces.

In this thesis we shall consider two fundamental problems concerning characteristic functions, they can be described in the following general terms.

1. <u>The General Bochner Problem</u>.

It is well known, see for example [4], that the characteristic functions of probability distributions can be characterized as the continuous positive definite functions on the real line. The general Bochner problem is to find analogous characterizations for the characteristic functions of probability measures on infinite dimensional spaces.

In chapter one we shall consider initially the problem of determining sufficient conditions for a given function to be a characteristic function. Some general theorems are derived in this direction when the spaces under consideration are Banach spaces with Schauder bases. Our general theorems allow us to derive the Bochner theorems of Gross [6], Sazonov [18] and to extend the theorems of A. de Acosta [2].

A general Bochner theorem is obtained utilizing the concept of λ-families of measures, first introduced in Kuelbs and Mandrekar [9], [10]. The results of [9] and [10] are extended to Orlicz spaces and the hypotheses are weakened. As an illustration of the power of the techniques developed in chapter one we conclude with a derivation of the Bochner theorem of Kuelbs [8].

The methods of chapter one differ substantially from those that have hitherto been employed in the solution of the Bochner problem. In all previous papers mentioned, the methods employed center on establishing the existence of a probability measure by showing that it is a limit of a compact set of probability measures. We shall establish the existence of our measure by first finding a measure on too large a space and then finding conditions for its support to be suitable.

2. Representation Theorems for the Characteristic Functions of Gaussian Measures.

In [11], Kumar and Mandrekar have shown that the only possible limiting distributions of normalized sums of independent identically distributed Banach space valued random variables are the so called stable distributions. That is to say, only the stable distributions have non empty domains of attraction. The most important stable distributions are those that are Gaussian and it is these that are studied in chapter two. An attempt to characterize the domain of attraction of a Gaussian distribution should begin by obtaining a representation of its characteristic function. Using a result of X. Fernique [11] we define the concept of the covariance function for Gaussian distributions, extending the concept first introduced by Vakhania [19]. A general representation for the characteristic function of a Gaussian distribution is obtained which includes as special cases the results of Vakhania [19], A. de Acosta [2] and Kuelbs and Mandrekar [10].

CHAPTER 1

BOCHNER THEOREMS ON BANACH SPACES WITH SCHAUDER BASIS

§1. Preliminaries.

Let X and Y be real vector spaces in duality with respect to some bilinear form $\langle\cdot,\cdot\rangle$ on $X \times Y$. For any $y_1,\ldots,y_n$ in Y and any Borel set B in the n-dimensional Euclidean space R^n, a sub-set of X of the form $\{x \in X : (\langle x,y_1\rangle,\ldots,\langle x,y_n\rangle) \in B\}$ is called a cylinder set in X based in the finite subspace generated by $y_1,\ldots,y_n$. The class of all cylinder sets in X forms an algebra, and the class of those based on a fixed finite subspace of Y forms a σ-algebra. We shall denote by $\mathcal{B}(X,Y)$ the smallest σ-algebra containing the algebra of cylinder sets in X.

(1.1) Definition. Let μ be a finite measure on $(X, \mathcal{B}(X,Y))$. Then the complex valued function $\hat{\mu}$ defined on Y by

$$\hat{\mu}(y) = \int_X \exp\{i\langle x,y\rangle\}d\mu(x) \quad \text{for all} \quad y \in Y$$

is called the characteristic function of μ. #

We shall mainly be interested in the case where X is some separable Banach space and Y is the topological dual of X. In this case the σ-algebra $\mathcal{B}(X,Y)$ has some important properties summarized in the following lemma.

(1.2) Lemma. Let E be a separable Banach space and let E' be its topological dual. Then

(i) $\mathcal{B}(E,E')$ is the smallest σ-algebra containing the (norm) open sets in E.

(ii) If $\hat{\nu} = \hat{\mu}$ on E', where μ and ν are finite measures on $(E, \mathcal{B}(E,E'))$, then $\nu = \mu$.

(iii) Every finite positive measure μ on $(E, \mathcal{B}(E,E'))$ is <u>tight</u> in the sense that if $\epsilon > 0$ there exists a (norm) compact set K in E such that $\mu(E \setminus K) < \epsilon$. #

We shall not prove these statements, proofs of (i) and (ii) can be found in Ito and Nisio [7], proposition 2.2, and a proof of (iii) can be found in Parthasarathy [14], theorem 3.2.

In this chapter we shall consider the question of when a function on E' is the characteristic function of a finite positive measure on $(E, \mathcal{B}(E,E'))$. We shall first demonstrate some algebraic and topological properties of characteristic functions.

(1.3) <u>Lemma</u>. Let E be a separable Banach space with topological dual E' and let μ be a finite positive measure on $(E, \mathcal{B}(E,E'))$. Then the characteristic function $\hat{\mu}$ satisfies the following properties:

(i) Let $y_1,\ldots,y_n$ be any finite subset of E' and let $\alpha_1,\ldots,\alpha_n$ be any finite subset of the complex numbers, then

$$\sum_{j,k}^{n} \alpha_j\bar{\alpha}_k\hat{\mu}(y_j - y_k) \geq 0 .$$

(ii) $\hat{\mu}$ is $\tau_s(E',E)$ continuous at the origin.

<u>Proof</u>. (i) can be proved by direct computation, observing that

$$\sum_{j,k}^{n} \alpha_j\bar{\alpha}_k\hat{\mu}(y_j - y_k) = \int_E \Big| \sum_{j=1}^{n} \alpha_j \exp\{i\langle x,y_j\rangle\}\Big|^2 d\mu(x)$$

and the fact that μ is a positive measure.

(ii) Since by lemma (1.2) part (iii) the measure μ is tight there exists, given $\varepsilon > 0$, a norm compact subset K such that $\mu(E\setminus K) < \varepsilon$. Let $\{y_\alpha\}$ be a net converging to 0 in the $\tau_s(E',E)$ topology. Then

$$\lim_\alpha |1 - \hat{\mu}(y_\alpha)| \leq 2\varepsilon + \lim_\alpha \int_K |\exp\{i\langle x,y_\alpha\rangle\} - 1| d\mu(x) = 2\varepsilon .$$

Hence since ε is arbitrary the result follows. #

The property (1) that $\hat{\mu}$ satisfies, is of special interest.

(1.4) Definition. Let ϕ be a complex valued function on E'. If for every choice of $y_1,\ldots,y_n$ on E' and every choice of complex numbers $c_1,\ldots,c_n$ we have that

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c}_k \phi(y_j - y_k) \geq 0$$

then we say that ϕ is Positive Definite. #

With this terminology, lemma 1.3 can be summarized as: every characteristic function on E' is positive definite and $\tau_s(E',E)$ continuous at the origin.

Bochner has shown, [4], that if E is of finite dimension, then every complex valued function on E' that is positive definite and continuous at the origin is necessarily a characteristic function. It is well known, Prohorov [16], that this cannot be true for all Banach spaces of infinite dimension. The following question naturally occurs. If E is a Banach space of infinite dimension then does there exist a topology $\mathcal{S}$ on E' such that a complex valued function on E' is a characteristic function if and only if it is $\mathcal{S}$-continuous at the origin and positive definite? Clearly if such an $\mathcal{S}$ exists then $\mathcal{S}$ is coarser than $\tau_s(E',E)$.

In [13], Mūstari has determined the spaces for which the above question may be answered in the affirmative. Most spaces of interest do <u>not</u> have such an $\mathcal{S}$. As a consequence, characteristic functions on E' cannot in general be characterized as the complex valued positive definite functions on E' that are continuous at the origin in some special topology.

In sections 2 and 3 we shall consider the problem of finding a topology $\mathcal{Z}$ on E' such that a complex valued positive definite function that is z-continuous at the origin is necessarily a characteristic function. In order for our results to be of interest we shall want $\mathcal{Z}$ to be as fine as possible and compatible with the algebraic structure of E'. Dudley has shown in [3] that $\sigma(E',E)$ continuous positive definite functions are necessarily characteristic functions and hence we should naturally seek topologies, $\mathcal{Z}$, finer than $\sigma(E',E)$ and, of course, coarser than $\tau_s(E',E)$, that is to say, topologies of the dual pair (E',E) coarser than $\tau_s(E',E)$.

The following theorem will prove to be extremely useful.

(1.5) <u>Theorem</u>. Let X be a real linear space with algebraic dual X^*. If ϕ is a complex valued function on X then ϕ is the characteristic function of a positive measure on $(X^*, \mathcal{B}(X^*,X))$ if and only if ϕ is positive definite and continuous on finite dimensional subspaces of X. #

This theorem is well known, a reference for it is Dudley [3], theorem 1.4. As a consequence, if E is a separable Banach space with dual E' and if ϕ is a positive definite function on E' continuous in <u>any</u> topology of the dual pair (E',E), then there exists a finite positive measure μ on $(E'^*, \mathcal{B}(E'^*,E'))$ such that $\phi = \hat{\mu}$.

The linear space E'^* contains an identification of E as a proper subspace, so that it is clear that our aim should be to find conditions for μ to give all its mass to E. In the following sections we shall consider the measure space $(E'^*, \mathcal{B}(E'^*,E'),\mu)$ and establish estimates for the support of μ in terms of properties of $\hat{\mu}$.

§2. Topologies on Vector Spaces.

In this chapter the topologies to be considered on vector spaces are those determined by families of semi-norms. We know, [17], theorem 3, p. 15, that if Γ is a family of semi-norms on the vector space E, then there is a coarsest topology on E compatible with the algebraic structure in which every semi-norm in Γ is continuous. We call this topology, the topology determined by Γ. Under this topology, E is a locally convex topological vector space and a base of closed neighborhoods of the origin is formed by the sets $\{x \in E : \sup_{1 \le j \le n} p_j(x) \le \varepsilon\}$ where $\varepsilon > 0$ and $p_j \in \Gamma$.

We shall mainly be concerned with families of semi-norms satisfying the following property.

(2.1) Definition. The family of semi-norms, Γ, on a vector space E, is said to be sequentially dominated, if given any countable subfamily Δ, there exists $p \in \Gamma$ such that for all $q \in \Delta$ there exists $0 < c(q) < \infty$ satisfying

$$q(x) \le c(q)p(x) \quad \text{for all} \quad x \in E \ . \qquad \#$$

Clearly if Γ is sequentially dominated and Δ is any countable subfamily then there exists $p \in \Gamma$ such that the topology on E determined by Δ is coarser than that determined by p. This observation allows us to establish a useful property of such families.

(2.2) Lemma. Let ψ be a real valued function on the vector space E that is continuous at the origin in the topology determined by a sequentially dominated family of semi-norms Γ. Then there exists a semi-norm $p \in \Gamma$ such that ψ is continuous at the origin in the topology on E determined by the single semi-norm p.

Proof. For all $n \in Z^+$ there exists a finite family $\Gamma_n \subset \Gamma$ such that $\{x: |\psi(x) - \psi(0)| < n^{-1}\}$ contains a neighborhood of 0 in the topology on E determined by Γ_n. Hence ψ is continuous at the origin in the topology determined by $\bigcup_{n=1}^{\infty} \Gamma_n$. Since Γ is sequentially dominated there exists $p \in \Gamma$ such that the topology determined by $\bigcup_{n=1}^{\infty} \Gamma_n$ is coarser than that determined by p and hence ψ is continuous at the origin in the topology determined by p. #

We now introduce the notion of Gaussian summability of a semi-norm. It is well known that there exists a probability space $(\Omega,\mathcal{F},P)$ on which we may define a sequence of independent random variables $\{X(j) : j \in Z^+\}$ which are all normally distributed with mean zero and unit variance.

(2.3) Definition. Let F be a vector space. Let $B = \{e_j : j \in Z^+\}$ be a subset of F. Then a semi-norm p on F is said to be Gaussian summable of order k with respect to B and $\lambda \in \ell^+$ if

$$\sup_{N\geq 1} E\, p^k\{\sum_{j=1}^{N} \lambda(j)X(j)e_j\} < \infty$$

where E is the expected value taken with respect to P. # It should be observed that $p\{\sum_{j=1}^{N} \lambda(j)X(j)e_j\}$ is necessarily a random variable on $(\Omega,\mathcal{F},P)$ since the map $(y_1,\ldots,y_N) \to p(\sum_{j=1}^{N} y_j e_j)$ is a continuous map on R^n into R with the usual topologies.

(2.4) Lemma. Let $\{E, (\cdot,\cdot)\}$ be an inner product space and let $p(x) = (x,x)^{\frac{1}{2}}$ for all $x \in E$. Then p is Gaussian summable of order 2 with respect to the countable set $B = \{e_j : j \in Z^+\}$ and

$\lambda \in \ell^+$ such that $\sum_{j=1}^{\infty} \lambda^2(j)(e_j, e_j) < \infty$.

Proof.

$$p^2\left(\sum_{j=1}^{N} \lambda(j)X(j)e_j\right) = \sum_{j=1}^{N} \sum_{k=1}^{N} \lambda(j)\lambda(k)X(j)X(k)(e_j, e_k)$$

and

$$E\, p^2\left(\sum_{j=1}^{N} \lambda(j)X(j)e_j\right) = \sum_{j=1}^{N} \lambda^2(j)(e_j, e_j)$$

and hence

$$\sup_{N\geq 1} E\, p^2\left(\sum_{j=1}^{N} \lambda(j)X(j)e_j\right) \leq \sum_{j=1}^{\infty} \lambda^2(j)(e_j, e_j) < \infty . \qquad \#$$

(2.5) Lemma. Let E be a vector space. Let $B = \{e_j : j \in Z^+\}$ be a subset of E. Let μ be a probability measure on $(E^*, \mathcal{B}(E^*, E))$ such that $\hat{\mu}$ is continuous at the origin in the topology on E defined by a single semi-norm p. If p Gaussian summable of order k with respect to B and λ $(\in \ell^+)$ then

$$\mu\{x^* : \sum_{j=1}^{\infty} \lambda^2(j) |\langle e_j, x^*\rangle|^2 < \infty\} = 1.$$

Proof. From the assumption of Gaussian summability,

$$\text{(1)} \qquad E p^k\left(\sum_{j=1}^{N} \lambda(j)X(j)e_j\right) \leq M$$

for some M and all N. From the continuity of $\hat{\mu}$, if ε is a positive number then there is a C such that

$$\text{(2)} \qquad 1 - \text{Real}\, \hat{\mu}(x) < \varepsilon + C\, p^k(x) \quad \text{for all } x.$$

For a $t > 0$ and a positive integer n, set

$$X = X_{n,t} = t \sum_{j=1}^{n} \lambda_j X_j e_j$$

and notice that by (1) and (2), with $K = CM\epsilon$ and all $|t| \le K^{1/k}$

(3) $$E(1 - \text{Real } \hat{\mu}(X)) \le 2\epsilon .$$

But

$$E\hat{\mu}(X) = E\int_{E^*} \exp\{i\langle X,x^*\rangle\}d\mu(x^*) = \int_{E^*} E\{\exp i\langle X,x^*\rangle\}d\mu(x)$$

by the Fubini theorem. For a fixed x^*, $\langle X,x^*\rangle$ is a Gaussian random variable with mean zero and variance

(4) $$\sigma^2 = t^2\sigma_n^2(x^*) = t^2 \sum_{j=1}^{n} \lambda^2|\langle e_j,x^*\rangle|^2 .$$

Thus $E\{\exp i\langle X,x^*\rangle\} = \exp\{-\frac{1}{2}\sigma^2\}$ and so (3) can be rewritten as

(5) $$\int_{E^*} \exp\{-\tfrac{1}{2} t^2\sigma_n^2(x^*)\}d\mu(x^*) \ge 1 - 2\epsilon .$$

Take the limit of the left hand side, first for $n \to \infty$, and then for $t \to 0$. By the Lebesgue dominated convergence theorem the limit is the integral of the limit of the integrand, which is 1 on the set $A = \{x^* : \lim_{n\to\infty} \sigma_n^2(x^*) < \infty\}$ and zero outside of A. It follows then that

$$\mu(A) \ge 1 - 2\epsilon .$$

Since ϵ was arbitrary, $\mu(A) = 1$, which is the assertion of the theorem. #

We may now combine lemma (2.5) with the concept of a sequentially dominated family of semi-norms to obtain the main result of this section.

(2.6) <u>Theorem</u>. Let E be a vector space. Let $B = \{e_j : j \in Z^+\}$ be a subset of E. Let μ be a probability measure on $(E^*, \mathcal{B}(E^*,E))$ such that $\hat{\mu}$ is continuous at the origin of E in the topology determined by a sequentially dominated family of semi-norms Γ. Then there exists $p \in \Gamma$ such that if p is Gaussian summable of order k with respect to B and $\lambda \in e^+$ then

$$\mu\{x^* : \sum_{j=1}^{\infty} \lambda^2(j) |<e_j, x^*>|^2 < \infty\} = 1.$$

<u>Proof</u>. The proof follows directly from lemmas (2.2) and (2.5). #

We may now apply theorem (2.6) to obtain Bochner's theorem on some sequence spaces.

§3. Bochner's Theorem on Sequence Spaces.

Let Λ and ℓ be as in the appendix and let $\{u_j \in j \in Z^+\} = B$ be the canonical basis for Λ. We shall derive a Bochner theorem for positive definite functions on Λ and extend it to spaces of type p.

(3.1) Definition. A real finite bilinear form Ψ on $\Lambda \times \Lambda$ is said to be of trace class k if

(i) Ψ is symmetric. That is to say, for all $x,y \in \Lambda$ we have $\Psi(x,y) = \Psi(y,x)$.

(ii) Ψ is positive definite. That is to say, for all $x \in \Lambda\setminus\{0\}$, $\Psi(x,x) > 0$.

(iii) $\sum_{j=1}^{\infty} \Psi^{k/2}(u_j,u_j) < \infty$. #

We may define for any positive definite bilinear form Ψ a semi-norm p_Ψ by $p_\Psi(x) = \Psi^{\frac{1}{2}}(x,x)$ for $x \in \Lambda$. Let τ_k be the topology on Λ determined by the family of semi-norms $\{p_\Psi : \Psi$ is of trace class $k\}$.

(3.2) Lemma. Let $k \leq 2$. Then $\{p_\Psi : \Psi$ is of trace class $k\}$ is a sequentially dominated family.

Proof. Let $\{\Psi_n : n \in Z^+\}$ be a countable set of bilinear forms of trace class k. There exists $c \in \ell$ such that

$$\sum_{n=1}^{\infty} c^{k/2}(n) \sum_{j=1}^{\infty} \Psi_n^{k/2}(u_j,u_j) < \infty .$$

Let $q = p_\Psi$ where

$$\Psi(x,y) = \sum_{n=1}^{\infty} c(n)\Psi_n(x,y)$$

Since $\sum_{n=1}^{\infty} c(n)\Psi_n(u_j,u_j) \leq \{ \sum_{n=1}^{\infty} c^{k/2}(n)\Psi_n^{k/2}(u_j,u_j)\}^{2/k}$ for all j we have that Ψ is of trace class k for all n.

$$p_{\Psi_n}(x) \le \frac{q(x)}{c^{\frac{1}{2}}(n)} . \qquad \#$$

We may now apply theorem (2.6) to obtain:

(3.3) Lemma. Let $k \le 2$. Let μ be a probability measure on $(\ell, \mathcal{B}(\ell,\Lambda))$ such that $\hat{\mu}$ is τ_k continuous at the origin. Then $\mu(\ell_k) = 1$.

Proof. By lemma (3.2) and theorem (2.6) there exists p_Ψ such that if p_Ψ is Gaussian summable with respect to $\lambda \in \ell^+$ and B then

$$\mu\{x \in \ell : \sum_{j=1}^{\infty} \lambda^2(j)x^2(j) < \infty\} = 1.$$

By lemma (2.4) we then have $\mu(A) = 1$ where

$$A = \{x \in \ell : \sum_{j=1}^{\infty} \{t(j)\}^{k/2-1}x^2(j) < \infty\}$$

and where $t(j) = \Psi(u_j, u_j)$ for all $j \in Z^+$.

If we now show that $A \subset \ell_k$ then the result follows. Let $k > 2$ and $r = 2/k > 1$ and $r' = 2(2-k)^{-1}$. Then $1/r + 1/r' = 1$ and by Hölder's inequality

$$\sum_{j=1}^{\infty} |x(j)|^k \le \{\sum_{j=1}^{\infty} |x(j)t^{\frac{1}{2}(k/2-1)}(j)|^{kr}\}^{1/r}\{\sum_{j=1}^{\infty} |t(j)^{-\frac{1}{2}(k/2-1)kr'}\}^{1/r'}$$

$$\le \{\sum_{j=1}^{\infty} x^2(j)t^{k/2-1}(j)\}^{1/r}\{\sum_{j=1}^{\infty} t^{k/2}(j)\} .$$

Hence if $x \in A$ then $\sum_{j=1}^{\infty} |x(j)|^k < \infty$ that is to say $x \in \ell_k$.

If $k = 2$ then $A = \ell_k$ since $t(j) = 1$ for all j. #

(3.4) Corollary. Let ϕ be a positive definite function on Λ such that $\phi(0) = 1$ and ϕ is τ_k continuous at the origin. Then there exists a unique probability measure μ on $(\ell_k\ \mathcal{B}(\ell_k,\Lambda))$ such that $\hat{\mu} = \phi$ on Λ.

Proof. The proof follows directly from theorem 1.5 and lemma (3.3). #

We may now extend this result to spaces that generalize ℓ_p spaces.

(3.5) Definition. A Banach space X with a Schauder basis $\{e_j : j \in Z^+\}$ and coordinate functional $\{e_j' : j \in Z^+\}$ is said to be of type p if for all $x \in X$ and $n \in Z^+$ we have that

$$\|\sum_{j=1}^{n} \langle x, e_j' \rangle e_j\|^p \leq \sum_{j=1}^{n} |\langle x, e_j' \rangle|^p . \qquad \#$$

This definition is less restrictive than that of A. de Acosta [ ], the motivation behind our definition is the fact, initially observed in [ ], that such spaces contain isomorphs of ℓ_p. More precisely we have:

(3.6) Lemma. Let X be a Banach space of type p. Then there exists a continuous linear map S from ℓ_p into X.

Proof. Define S for all $x \in \ell_p$ by $Sx = \sum_{j=1}^{\infty} x(j) e_j$. S is clearly a continuous linear map of ℓ_p into X. #

(3.7) Definition. Let $\Lambda(X)$ be the linear subspace of X' spanned by the coordinate functionals and let $\tau_k(X)$ be the topology on $\Lambda(X)$ determined by the bilinear functionals on $\Lambda(X) \times \Lambda(X)$ of trace class k. #

Clearly the map ${}^tS : (\Lambda, \tau_k) \to (\Lambda(X), \tau_k(X))$ is continuous, allowing us to prove the main theorem of this section.

(3.8) Theorem. Let X be a Banach space of type $p \leq 2$. Let ϕ be a positive definite function on $\Lambda(X)$ with $\phi(0) = 1$. If ϕ is $\tau_p(X)$ continuous at the origin then there exists a probability measure ν on $(X, \mathcal{B}(X,X'))$ such that $\hat{\nu} = \phi$ on $\Lambda(X)$.

<u>Proof</u>. The function $\phi \circ {}^tS$ on Λ is positive definite and τ_p continuous at the origin. By corollary (3.4) there exists a probability measure μ on $(\ell_p, \mathcal{B}(\ell_p,\Lambda))$ such that $\hat{\mu} = \phi \circ {}^tS$ on Λ. It can be trivially verified that if $\nu = \mu \circ ({}^tS)^{-1}$ then $\hat{\nu} = \phi$ on Λ. #

§4. Probability Measures of λ Type.

Let E be a Banach space with Schauder basis $\{e_j : j \in Z^+\}$ and coordinate functionals $\{e_j' : j \in Z^+\}$. If ϕ is a positive definite function on E', continuous in some topology compatible with the algebraic structure and $\phi(0) = 1$ then we have by theorem 1.5 that there exists a unique probability measure μ_ϕ on $(E'^*, \mathcal{B}(E'^*,E'))$ such that $\hat{\mu}_\phi = \phi$. In this section we shall consider the problem of representing ϕ by a measure on E by putting certain conditions on μ_ϕ.

(4.1) Lemma. Let E and ϕ be as above. Then there exists a unique probability measure ν on $(E, \mathcal{B}(E,E'))$ such that $\phi = \hat{\nu}$ if and only if

(i) ϕ is continuous at the origin in the topology of uniform convergence on compacta.

(ii) For all $\Delta > 0$

$$\lim_{n\to\infty} \sup_{m>n} \mu_\phi\{x^* \in E'^* : \|\sum_{j=n+1}^{m} \langle e_j', x^*\rangle e_j\| > \Delta\} = 0 .$$

Proof. Suppose that conditions (i) and (ii) hold. Let $F = \{x^* \in E'^* : \sum_{j=1}^{\infty} \langle e_j', x^*\rangle e_j \in E\}$. Since E is complete we have that

$$F = \{x^* \in E'^* : \lim_{n\to\infty} \sup_{m>n} \|\sum_{j=n+1}^{m} \langle e_j', x^*\rangle e_j\| = 0\}$$

$$= \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n+1}^{\infty} \{x^* \in E'^* : \|\sum_{j=n+1}^{m} \langle e_j', x^*\rangle e_j\| \le k^{-1}\} .$$

As a consequence $F \in \mathcal{B}(E'^*,E')$ and moreover

$$\mu_\phi(E'^* - F) \le \sum_{k=1}^{\infty} \lim_{n\to\infty} \sup_{m>n} \mu_\phi\{x^* \in E'^* : \|\sum_{j=n+1}^{m} \langle e_j', x^*\rangle e_j\| > k^{-1}\} = 0.$$

That is to say, $\mu_\phi(F) = 1$.

We may now consider the measurable transformation $\psi : (F, \mathcal{B}(F,E')) \to (E, \mathcal{B}(E,E'))$ given by $\psi(x^*) = \sum_{j=1}^{\infty} \langle e_j', x^* \rangle e_j$ for all $x^* \in F$. By the transformation theorem we have that for all $y \in E'$ and $n \geq 0$

$$\phi({}^t\pi_n y) = \hat{\nu}({}^b\pi_n y)$$

where $\hat{\nu} = \mu_\phi \circ \psi^{-1}$.

Since ${}^t\pi_n y$ converges uniformly to y on compacta, we have by condition (i) of the lemma and condition (ii) of lemma 1. that $\phi(y) = \hat{\nu}(y)$ for all $y \in E'$.

Conversely suppose that there exists ν on $(E, \mathcal{B}(E,E'))$ such that $\hat{\nu} = \phi$. Let us define the canonical map $q : E \to E'^*$ by

$$\langle x,y \rangle = \langle y,q(x) \rangle \text{ for all } x \in E,\ y \in E'.$$

Clearly $q : (E, \mathcal{B}(E,E')) \to (E'^*, \mathcal{B}(E'^*,E'))$ is a measurable transform and by the transformation theorem and the uniqueness of μ_ϕ, we have that $\nu \circ q^{-1} = \mu_\phi$. Since $q^{-1}(F) = E$ we have that $\mu_\phi(F) = 1$ and this clearly implies that condition (ii) holds. #

Condition (ii) of lemma 4.1 suggests that we introduce the following concept of λ-measure.

(4.2) Definition. Let E be a Banach space with Schauder basis $\{e_j : j \in Z^+\}$ and coordinate functionals $\{e_j' : j \in Z^+\}$. Let $\lambda \in \ell^+$ and let P be a probability measure on $(E'^*, \mathcal{B}(E'^*,E'))$. If there exists a real valued strictly increasing function on $[0,\infty)$ with $h(0) = 0$ such that

$$\limsup_{n\to\infty\ m>n} P\{x^* \in E'^* : \|\sum_{j=n+1}^{m} \langle e'_j, x^*\rangle e_j\| \geq h(\delta)\}$$

$$\leq \limsup_{n\to\infty\ m>n} P\{x^* \in E'^* : \sum_{j=n+1}^{m} \lambda(j)|\langle e'_j, x^*\rangle|^2 \geq \delta\}$$

then we say that P is a λ-measure or P is of λ-type.

If ϕ is positive definite on E' and μ_ϕ is a λ-measure then condition (ii) of lemma (4.1) will hold if for all $\delta > 0$

$$\limsup_{n\to\infty\ m>n} \mu_\phi\{x^* \in E'^* : \sum_{j=n+1}^{m} \lambda(j)\langle e'_j, x^*\rangle^2 \geq \delta\} = 0 .$$

This latter condition can be shown to hold if ϕ has some continuity properties. The continuity of ϕ will be defined with respect to a topology on E' determined by a family of bilinear forms.

(4.3) Definition. Let E be as in lemma 4.1. Let Q be a family of bilinear forms on $E \times E'$ such that for all $\Psi \in Q$

(i) Ψ is symmetric and positive definite.

(ii) For all $y \in E'$: $\sum_{j=1}^{\infty} \Psi^{\frac{1}{2}}(e'_j, e'_j)|\langle e_j, y\rangle| < \infty$.

We define $\tau(Q)$ to be the topology on E' determined by the family of semi-norms $\{p_\Psi : \Psi \in Q\}$. #

The condition (ii) implies that ${}^t\pi_n(y)$ converges to y in the $\tau(Q)$-topology.

(4.4) Theorem. Let E be a Banach space with a Schauder basis $\{e_j : j \in Z^+\}$ and coordinate functionals $\{e'_j : j \in Z^+\}$. Let ϕ be a positive definite function on E' satisfying the following conditions

(i) ϕ is $\tau(Q)$ continuous at the origin, $\phi(0) = 1$.

(ii) The measure μ_ϕ on $(E'^*, \mathcal{B}(E'^*, E'))$ is a λ measure when $\lambda \in \ell^+$ and for all $\Psi \in Q$,

$$\sum_{j=1}^{\infty} \lambda(j)\Psi(e_j',e_j') < \infty .$$

Then there exists a unique probability measure ν such that $\hat{\nu} = \phi$.

Proof. Clearly by lemma (4.1) and definition (4.2) we need only show that for all $\delta > 0$

$$\lim_{n\to\infty} \sup_{m>n} \mu_\phi\{x^* \in E'^* : \sum_{j=n+1}^{m} \lambda(j)|<e_j',x^*>|^2 \geq \delta\} = 0 .$$

Let $\epsilon > 0$ be arbitrary. Since ϕ is $\tau(Q)$ continuous there exist $\{\Psi_j : 1 \leq j \leq p\} \subset Q$ and $C < \infty$ such that for all $y \in E'$

$$1 - \text{Real } \phi(y) \leq \epsilon + C \sup_{1\leq j\leq p} \Psi_j(y,y).$$

Let $\{X(k) : k \in Z^+\}$ be independent identically distributed standard normal random variables and let $y = \sum_{k=n+1}^{m} \lambda^{\frac{1}{2}}(k)X(k)e_k'$. Then

$$1 - \text{Real } \phi(y) \leq \epsilon + C \sup_{1\leq j\leq p} \sum_{k=n+1}^{m} \sum_{\ell=n+1}^{m} \lambda^{\frac{1}{2}}(k)\lambda^{\frac{1}{2}}(\ell)X(k)X(\ell)\Psi_j(e_k',e_\ell') .$$

On taking expected values we obtain

$$\int_{E'^*} 1 - \exp\{-\tfrac{1}{2} \sum_{k=n+1}^{m} \lambda(k)|<e_k',x^*>|^2\}d\mu_\phi(x^*)$$

$$\leq \epsilon + C \sup_{1\leq j\leq p} \sum_{k=n+1}^{m} \lambda(k)\Psi_j(e_k',e_k') .$$

Hence by the Markov inequality

$$\mu_\phi\{x^* \in E'^* : \sum_{S=n+1}^{m} \lambda(S)<e_S',x^*>^2 \geq \delta\} \leq \frac{\epsilon + \sup_{1\leq j\leq p} \sum_{S=n+1}^{m} \lambda(S)\Psi_j(e_S',e_S')}{(1 - \exp(-\tfrac{1}{2}\delta))} .$$

By condition (ii) of the theorem and noting that ϵ was arbitrary we obtain the required result. #

We shall now apply lemma (4.4) to the case where E is an Orlicz sequence space. Let α and β be complementary functions in the sense of Young and let ℓ_α and ℓ_β be the Orlicz spaces defined in the appendix such that $\ell_\alpha' = \ell_\beta$.

(4.5) Definition. Let $\{u_j : j \in Z\}$ be the canonical basis for ℓ_α and ℓ_β. Let Q_α be the set of all symmetric, positive definite, bilinear forms Ψ on $\ell_\beta \times \ell_\beta$ such that $\sum_{j=1}^{\infty} \alpha(\Psi^{\frac{1}{2}}(u_j,u_j)) < \infty$. Let $\tau_\alpha = \tau(Q_\alpha)$. #

(4.6) Theorem. Let ℓ_α be an Orlicz space with topological dual ℓ_β. Let ϕ be a complex valued function on ℓ_β such that

(i) $\phi(0) = 1$ and ϕ is positive definite,

(ii) ϕ is τ_α-continuous at the origin, and

(iii) μ_ϕ is a λ-measure where $\lambda \in \ell^+$ and $\sum_{j=1}^{\infty} \alpha(|t^{\frac{1}{2}}(j)|) < \infty$ implies that $\sum_{j=1}^{\infty} \lambda(j)|t(j)| < \infty$.

Then there exists a unique probability measure ν on $(\ell_\alpha, \mathcal{B}(\ell_\alpha,\ell_\beta))$ such that $\hat{\nu} = \phi$.

In the case when $\alpha(\sqrt{\cdot})$ is a convex function on $[0,\infty)$ we have (i), (ii) and (iii) holding for every characteristic function of a probability measure on $(\ell_\alpha, \mathcal{B}(\ell_\alpha,\ell_\beta))$.

Proof. If ϕ satisfies (i), (ii) and (iii) then by theorem (4.4) there exists ν such that $\hat{\nu} = \phi$.

Conversely, let us suppose that $\phi = \hat{\nu}$ where ν is a probability measure on $(\ell_\alpha, \mathcal{B}(\ell_\alpha,\ell_\beta))$.

Let $K_n = \{x \in \ell_\alpha : \sum_{j=1}^{\infty} \alpha(|x(j)|) < n\}$ for all n. Since $\bigcup_{n=1}^{\infty} K_n = \ell_\alpha$ given $\epsilon > 0$, there exists n such that $\nu(K_n) \geq 1 - \epsilon$.

If Ψ is defined by

$$\Psi(z,y) = \int_{K_n} \langle x,z\rangle\langle x,y\rangle d\nu(x) \quad \text{for all} \quad z,y \in \ell_\beta$$

we have that $|1 - \phi(z)| \leq \frac{1}{2}\Psi(z,z) + \epsilon$. Since $\alpha(\sqrt{\cdot})$ is convex we may use Jensen inequality to show that Ψ is in Q_α, and hence since ϵ was arbitrary ϕ is τ_α-continuous at 0, and condition (ii) of the theorem follows.

By lemma (4.1) condition (ii) we have that μ_ϕ is a λ-measure for all $\lambda \in \ell^+$. Hence condition (iii) of the theorem will hold by choosing λ such that $\sum_{j=1}^{\infty} \beta(\lambda^{\frac{1}{2}}(j)) < \infty$. #

If $\alpha(x) = x^p$ we obtain the following result of Mandrekar and Kuelbs.

(4.7) <u>Corollary</u>. If $2 \leq p < \infty$ and ϕ is defined on ℓ_p', then ϕ is the characteristic function of a probability measure on ℓ_p if and only if

(i) ϕ is positive definite, $\phi(0) = 1$,

(ii) ϕ is τ_p-continuous,

(iii) μ_ϕ is a λ-measure for some $\lambda \in (\ell'_{p/2})^+$.

<u>Proof</u>. The sufficiency of conditions (i), (ii) and (iii) follows by the first part of Theorem 4.6 and the fact that $\lambda \in (\ell'_{p/2})^+$ and $\sum_{j=1}^{\infty} |t(j)|^{p/2} < \infty$ imply that $\sum_{j=1}^{\infty} \lambda(j)|t(j)| < \infty$. The necessity follows since $\alpha(\sqrt{x}) = x^{p/2}$ is a convex function for $p \geq 2$.

§5. Banach Spaces with Schauder Basis and Accessible Norm.

For the purpose of this section E will be a real Banach space with Schauder basis $\{e_j : j \in Z^+\}$ and coordinate functionals $\{e_j' : j \in Z^+\}$.

(5.1) Definition. Let Φ be a real continuous positive definite function on E and let $c : (0,\infty) \to (0,\infty)$ be any strictly increasing function. Then if for all $x \in E$, $\Phi(0) - \Phi(x) \geq c(\|x\|)$ we say, following Kuelbs, that the norm of E is accessible with respect to Φ.

Since Φ is positive definite and continuous there exists a positive measure P_Φ defined on $\mathcal{B}(E^*,E)$ such that $\Phi(x) = \int_{E^*} \exp i\langle x,x^*\rangle dP_\Phi(x^*)$.

(5.2) Theorem. Let E and Φ be as above and let the norm of E be accessible with respect to Φ. Then a complex valued function ϕ on E' is the Fourier-Stieltjes transform of a unique probability measure on $(E, \mathcal{B}(E))$ if and only if

(i) ϕ is positive definite, $\phi(0) = 1$

(ii) ϕ is $\tau_s(E',E)$ continuous.

(iii) $\lim\limits_{n\to\infty} \sup\limits_{m>n} \int_{E^*} 1 - \phi({}^t\pi_m(x^*) - {}^t\pi_n(x^*))dP_\Phi(x^*) = 0.$

Proof. Suppose that conditions (i), (ii) and (iii) hold for ϕ. By theorem (1.5) there exists a μ_ϕ on E'^* such that for all $x' \in E'$

$$\phi(x') = \int_{E'^*} \exp\{i\langle x',x^*\rangle\}d\mu_\phi(x^*) .$$

If $J(m,n)$ denotes the integral in condition (iii) we then have that

$$J(m,n) = \int_{E'} \int_{E'^*} 1 - \exp\{i \sum_{j=n+1}^{m} \langle e_j, x'\rangle\langle e_j', x^*\rangle\} d\mu_\phi(x^*) dP_\Phi(x').$$

By Fubini's theorem we may change the order of integration above, noting that the required measurability of the integrand is trivally satisfied. Hence $J(m,n) = \int_{E'^*} \Phi(0) - \Phi(\sum_{j=n+1}^{m} \langle e_j', x^*\rangle e_j) d\mu_\phi(x^*)$.

$$\mu_\phi\{x^* : \|\sum_{j=n+1}^{m} \langle e_j', x^*\rangle e_j\| > \epsilon\} \leq \mu_\phi\{x^* : \Phi(0) - \Phi(\sum_{j=n+1}^{m} \langle e_j', x^*\rangle e_j') > c(\epsilon)\}$$

$$\leq \frac{1}{c(\epsilon)} J(m,n).$$

By condition (iii) we now have that for all $\epsilon > 0$

$$\lim_{n\to\infty} \sup \mu_\phi\{x^* : \|\sum_{j=n+1}^{m} \langle e_j', x^*\rangle e_j\| > \epsilon\} = 0 .$$

Hence by lemma (4.1) the result follows.

As a simple application of this theorem we shall give a proof that positive definite functions on ℓ_2 that are τ_1-continuous are necessarily Fourier-Stieltjes transforms. Suppose $\phi : \ell_2 \to C$ is positive definite, $\phi(0) = 1$, and ϕ is τ_1-continuous. For any $\epsilon > 0$ there exists a symmetric, positive definite bilinear form Ψ on $\ell_2 \times \ell_2$ such that for $x \in \ell_2$

$$|1 - \phi(x)| < \epsilon + \Psi(x,x) .$$

We let $(u_n : n \in Z^+)$ be the usual complete orthonormal basis for ℓ_2. Let $\Phi(x) = \exp\{-\frac{1}{2}\|x\|^2\}$, then Φ is a real valued continuous positive definite function on ℓ_2 and the norm of ℓ_2 is clearly accessible with respect to Φ. Moreover if P_Φ is the corresponding additive measure on $\ell_2' = \ell_2$ we have that $\int_{\ell_2} \langle e_j, x'\rangle\langle e_k, x'\rangle dP_\Phi(x') = \delta_j^k$. Hence

$$\int_{\ell_2} 1 - \phi\Big(\sum_{j=n+1}^{m} \langle e_j, x\rangle e_j\Big)\, dP_\Phi(x') \le e + \sum_{j=n+1}^{m} \Psi(j,j) \ .$$

Now $\sum_{j=1}^{\infty} \Psi(j,j) < \infty$ since Ψ is of trace class 1 and hence

$$\lim_{n\to\infty} \sup_{m>n} \int_{\ell_2} 1 - \phi\Big(\sum_{j=n+1}^{m} \langle e_j, x\rangle e_j\Big)\, dP_\Phi(x') \le \varepsilon \ .$$

But ε was arbitrary and the conditions of the theorem are satisfied. Hence the function ϕ is a Fourier-Stieltjes transform.

We may similarly obtain the results of §3 by careful choice of Φ, we shall not, however, include the details here.

CHAPTER II

GAUSSIAN MEASURES ON BANACH SPACES

§1. Introduction.

There are many possible equivalent definitions of Gaussian measures on vector spaces; in this chapter we will use that of X. Fernique [5].

(1.1) Definition. Let E be a real vector space and let $\mathcal{B}$ be a σ-algebra of subsets of E. We say that $\mathcal{B}$ is compatible with the algebraic structure of E if

(i) The map $(x,y) \to x + y$ of $(E \times E, \mathcal{B} \times \mathcal{B})$ into $(E,\mathcal{B})$ is measurable.

(ii) The map $(x,\lambda) \to \lambda x$ of $(E \times R, \mathcal{B} \times \mathcal{B}(R))$ into $(E,\mathcal{B})$ is measurable.

Let $(\Omega,\mathcal{F},P)$ be a probability space and X a measurable map from $(\Omega,\mathcal{F})$ into $(E,\mathcal{B})$, we say that X is a random variable with values in E. The law of X is the measure $P \circ X^{-1}$ induced on $(E,\mathcal{B})$ by X. The concept of independence of random variables with values in E may be defined as for real valued random variables with $\mathcal{B}$ replacing $\mathcal{B}(R)$.

We shall say that X is a Gaussian random variable with values in E if the following condition is satisfied:

For all pairs (X_1,X_2) of independent random variables with the same law as X and for all pairs (s,t) of real numbers with $s^2 + t^2 = 1$, the random variables $(sX_1 + tX_2)$ and $(tX_1 - sX_2)$

are independent and have the same law as X.

A measure μ on $(E,\mathcal{B})$ is said to be Gaussian if there exists a probability space $(\Omega,\mathcal{F},P)$ and a Gaussian random variable X with values in E such that μ is the law of X. #

The following result of X. Fernique generalizes a well known result for Gaussian measures on R and is of fundamental importance in this chapter.

(1.2) <u>Theorem</u> (X. Fernique [5]). Let $(E,\mathcal{B})$ be as above and let μ be a Gaussian measure on $(E,\mathcal{B})$. If $\|\cdot\|$ is a $\mathcal{B}$-measurable norm on $(E,\mathcal{B})$ then there exists $\alpha > 0$ such that $\int_E \exp\{\alpha\|x\|^2\}d\mu(x) < \infty$. #

It is clear that for such μ the integral of any power of $\|\cdot\|$ is finite, this is the property that will prove to be most useful. In this chapter we shall extend the results of Vakhania [19] to arbitrary separable Banach spaces and hence we must show that our definition of Gaussian measure is equivalent to the definition of [19].

It is well known, for example [4], theorem 2, p. 526, that a <u>real</u> valued random variable is Gaussian in the sense of definition (1.1) if and only if it has a normal distribution with zero mean. It is this fact that allows us to establish the equivalence of definition (1.1) and the definition of Gaussian measures to be found in [ ].

(1.3) <u>Lemma</u>. Let E be a Banach space. Then

(i) $\mathcal{B}(E,E')$ is a σ-algebra, compatible with the algebraic structure, for which the norm of E is measurable.

(ii) A probability measure μ on $(E, \mathcal{B}(E,E'))$ is Gaussian if and only if for all $x' \in E$ there exists $\sigma(x') \geq 0$ such that x' is a real valued random variable on the probability space $(E, \mathcal{B}(E,E'),\mu)$ with distribution $N(0,\sigma^2(x'))$.

<u>Proof</u>. (i) Since $\mathcal{B}(E,E')$ is generated by sets of the form $\{x : \langle x,x'\rangle < a\}$ in order to prove (i) of definition (1.1) we need only show that $\{(x,y) : \langle x + y,x'\rangle < a\} \in \mathcal{B}(E,E') \times \mathcal{B}(E,E')$. Let $Q = \{r \in R : r$ is rational$\}$. Then we have

$$\{(x,y) : \langle x + y,x'\rangle < a\} = \bigcup_{r\in Q} \{x : \langle x,x'\rangle < r\} \times \{y : \langle y,x'\rangle < a-r\}$$

$$\in \mathcal{B}(E,E') \times \mathcal{B}(E,E') \ .$$

Condition (ii) of definition (1.1) is trivially satisfied.

(ii) It suffices to show that X is a Gaussian random variable with values in E if and only if the real valued random variables $\langle X,x'\rangle$ are Gaussian for all $x' \in E'$.

Suppose that for all $x' \in E'$ the random variable $w \to \langle X(w),x'\rangle$ is Gaussian with values in R. Let X_1 and X_2 be independent and have the same law as X and let (s,t) be real with $s^2 + t^2 = 1$. Then clearly $\langle X_1,x'\rangle$ and $\langle X_2,x'\rangle$ are independent and have the same law as $\langle X,x'\rangle$ for all $x' \in E'$. Hence the real valued random variables $\langle sX_1 + tX_2,x'\rangle$ and $\langle tX_1 - sX_2,x'\rangle$ are independent and have the same law as $\langle X,x'\rangle$ for all $x' \in E'$. Now since $\mathcal{B}(E,E')$ is generated by sets of the form $\{x : \langle x,x'\rangle \in B\}$ where $B \in \mathcal{B}(R)$ we have that $sX_1 + tX_2$ and $tX_1 - sX_2$ are independent with the same law as X.

Conversely suppose that X is Gaussian with values in E. Let X_1 and X_2 be independent random variables with values in E having the same law as X. Then for all (s,t) with $s^2 + t^2 = 1$ and any $x' \in E'$ we have that $(s\langle X_1,x'\rangle + t\langle X_2,x'\rangle)$ and $(t\langle X_1,x'\rangle - s\langle X_2,x'\rangle)$ are independent. Hence by [ ], theorem 2, p. 526 $\langle X_1,x'\rangle$ has a normal distribution and hence $\langle X,x'\rangle$ has a normal distribution. The mean of $\langle X,x'\rangle$ is clearly zero. #

Gaussian measures on finite dimensional spaces are uniquely determined by their covariance matrices. We may extend this result to infinite dimensional spaces by introducing the notion, following Vakhania, of the covariance

(1.4) <u>Definition</u>. Let μ be a Gaussian measure on the Banach space E. For $x',y',\mathcal{X},E'$ we define Ψ_μ by

$$\Psi_\mu(x',y') = \int_E \langle x,x'\rangle\langle x,y'\rangle d\mu(x) \;.$$

Ψ_μ is called the covariance form of μ. The integral exists since $\langle\cdot,x'\rangle$ is a real Gaussian random variable on the probability space $(E, \mathcal{B}(E,E'),\mu)$, moreover it is easily seen that

$$\hat{\mu}(x') = \exp\{-\tfrac{1}{2}\,\Psi_\mu(x',x')\} \quad \text{for all } x' \in E' \;.$$

In this chapter we shall determine properties that a covariance form must have and for some special case we shall characterize such covariance forms by means of operators.

§2. Representation of the Characteristic Function of a Gaussian Measure.

In this section we shall proceed to an operator type representation of $\hat{\mu}$ by means of an initial Lévy-Khinchine type representation.

Let E be a separable Banach space, and let $S = \{x \in E : \|x\| = 1\}$, S will be a complete separable metric space under the induced norm topology. Let $\mathcal{B}(S)$ be the σ-algebra of subsets of S generated by the open sets.

(2.1) Lemma (Lévy-Khinchine type representation of $\hat{\mu}$). Let E be a real separable Banach space and let μ be a Gaussian measure on $(E, \mathcal{B}(E,E'))$ with covariance operator T. Then there exists a unique positive, finite, symmetric measure Γ on $(S, \mathcal{B}(S))$ such that for all $x',y' \in E'$

$$\langle Tx',y'\rangle = \int_S \langle x,x'\rangle\langle x,y'\rangle d\Gamma(x) \quad .$$

Proof. By lemma (1.4) we have that for all $x',y' \in E'$

$$\langle Tx',y'\rangle = \int_E \langle x,x'\rangle\langle x,y'\rangle d\mu(x) .$$

Let λ be the finite measure on $(E, \mathcal{B}(E,E'))$ defined by $d\lambda(x) = \|x\|^2 d\mu(x)$ and let $j : E\ \{0\} \to S$ by the continuous map $j(x) = x/\|x\|$. By the transformation theorem we have that for all $x',y' \in E'$

$$\int_E \langle x,x'\rangle\langle x,y'\rangle d\mu(x) = \int_S \langle x,x'\rangle\langle x,y'\rangle d\Gamma(x)$$

where $\Gamma = \lambda \circ j^{-1}$. The result then follows since Γ is clearly finite and symmetric.

In order to verify that Γ is unique, it suffices to show that if Γ is a finite, symmetric measure on $(S, \mathcal{B}(S))$ such that

(1) $$\int_S |\langle x,x'\rangle|^2 d\Gamma(x) = 0 \quad \text{for all} \quad x' \in E'.$$

There exists $k > 0$ such that for all $b \in R$, $\int_0^\infty (1 - \cos br)r^{-3}dr = k|b|^2$ and hence if (1) holds, $\int_S \int_0^\infty 1 - \cos r\langle x,x'\rangle \frac{drd\Gamma}{r^2}(x) = 0$. If we define the measure Ω by

$$\Omega(A) = \int_{\bigcup_{\lambda>0}} \int_{\lambda A} 1 - \cos\langle rx,x'\rangle r^{-3} dr \, d\Gamma(x)$$

we obtain that

(2) $$\int_E 1 - \exp\{i\langle x,x'\rangle\} d\Omega(x) \quad \text{for all} \quad x' \in E' .$$

By using exactly the method of Parthasarthy [14], p. we see that (2) implies that $\Omega = 0$ and hence that $\Gamma = 0$. #

It is not clear whether $\hat{\mu}$ can be characterized in terms of measures on S, however in §4 we shall see that in some special cases (Orlicz and ℓ_p spaces), such a characterization is possible. We shall now use lemma (2.1) to obtain an operator representation of $\hat{\mu}$ that directly generalizes the results for Hilbert spaces. In the Hilbert space case $\hat{\mu}$ is represented in terms of Hilbert-Schmidt (for definition see [6]) operators, such operators are generalized by the following:

(2.2) Definition. Let X and Y be normed spaces and let $T : X \to Y$ be a linear map. T is said to be absolutely p summing if there exists a constant C such that for any $N \in Z^+$ and $(x_1, \ldots, x_N) \subset E$ we have that

$$\sum_{j=1}^{N} \|Tx_j\|^p \leq C \sup\{ \sum_{j=1}^{N} |\langle x_j, x'\rangle|^p : x' \in X', \|x'\| \leq 1\} . \qquad \#$$

Such operators were first introduced by Pietch [15] who showed that if X and Y are Hilbert spaces then T is absolutely p summing if and only if T is Hilbert-Schmidt.

(2.3) Theorem. Let E be a real Banach space and μ a Gaussian probability measure on $(E, \mathcal{B}(E,E'))$. There exists an absolutely 2-summing operator, A, on E' into $L_2(S,\Gamma)$ such that $\Psi_\mu(x',y') = \langle A^t A x', y'\rangle$. The transpose map, ${}^tA$, is a map on $L_2(S,\Gamma)$ into E defined by the Bochner integral

$${}^tAf = \int_S x f(x)\, d\Gamma(x) \quad \text{for all} \quad f \in L_2(S,\Gamma) .$$

Proof. Define the bounded linear operator $A : E' \to L_2(S,\Gamma)$ by $Ax' = \langle \cdot, x'\rangle$. If we define B on $L_2(S,\Gamma)$ by

$$Bf = \int_S x f(x)\, d\Gamma(x) \qquad \forall\ f \in L_2(S,\Gamma)$$

then since $\int_S \|xf(x)\|\, d\Gamma(x) = \int_S |f(x)|\, d\Gamma(x) < \infty$ we have that $Bf \in E$. Moreover

$$\langle Bf, x'\rangle = \int_S \langle x, x'\rangle f(x)\, d\Gamma(x) = \langle Ax', f\rangle_{L_2} = \langle {}^tAf, x'\rangle .$$

Hence $B = {}^tA$.

For any $x', y' \in E'$ we have that

$$\Psi(x',y') = \int_E \langle x,x'\rangle\langle x,y'\rangle d\mu(x) = \int_S \langle x,x'\rangle\langle x,y'\rangle d\Gamma(x) =$$

$$\langle Ax', Ay'\rangle_{L_2} = \langle A^t A x', y'\rangle .$$

The proof is now completed by observing that A is absolutely 2-summing as a consequence of corollary 1, p. 187 of Wong [20].

§3. Some Applications to Sequence Spaces.

In this section we shall characterize the covariance operators of Gaussian measures on various sequence spaces, in particular we shall consider the Orlicz space ℓ_α defined in the appendix. The basic theorem, from which we will derive some special cases, is the following:

(3.1) Theorem. (a) The covariance operator of a Gaussian probability measure on ℓ_α is a bounded linear operator $T : \ell_\alpha' \to \ell_\alpha$ satisfying:

(i) T is symmetric and positive.

(ii) $\sum_{j=1}^{\infty} \alpha(<Tu_j, u_j>^{\frac{1}{2}}) < \infty$.

(b) Conversely, if α satisfies condition (2) of the appendix then a bounded linear operator T on ℓ_α' into ℓ_α satisfying conditions (i) and (ii) is the covariance operator of a Gaussian measure on ℓ_α.

Proof. (a) In order to prove (ii) it suffices (see appendix) to show that for all $y \in \ell_\beta^+$

$$(1) \qquad \sum_{j=1}^{\infty} \lambda(j)<Tu_j, u_j>^{\frac{1}{2}} < \infty .$$

For all $d \in Z^+$ the real valued function $x(j)$ defined on the probability space (ℓ_α, μ) (where μ is Gaussian) is a normally distributed random variable with mean zero and variance $\sigma_j^2 = <Tu_j, u_j>$. As a consequence

$$\int_{\ell_\alpha} |x(j)| d\mu = \sqrt{2/\pi}\, \sigma_j = \sqrt{2/\pi}\, <Tu_j, u_j>^{\frac{1}{2}} .$$

Then if $y \in \ell_\beta$

$$\sum_{j=1}^{N} \lambda(j)<Tu_j,u_j>^{\frac{1}{2}} \le \sqrt{\pi/2} \int_{\ell_\alpha} \sum_{d=1}^{N} |x(j)y(j)| d\mu(x)$$

$$\le \sqrt{\pi/2}\ \|y\|_\beta \int_{\ell_\alpha} \|x\|_\alpha d\mu(x)$$

Since the latter term is finite and independent of N the condition 1 holds, imply that $\sum_{j=1}^{\infty} \alpha(<Tu_j,u_j>^{\frac{1}{2}}) < \infty$.

(b) Let $T : \ell_\alpha' \to \ell_\alpha$ be a bounded linear operator satisfying conditions (i) and (ii). If ϕ is defined on Λ by $\phi(x) = \exp\{-\frac{1}{2} <Tx,x>\}$, then, by theorem 1.5, chapter one, there exists a probability measure μ on $(\ell, \mathcal{B}(\ell,\Lambda))$ such that $\hat{\mu}(x) = \phi$ for all $x \in \Lambda$.

For all $j \in Z^+$, $\int_\ell |x(j)|^2 d\mu(x) = <Tu_j,u_j>$ and as a consequence of condition ( ) in the appendix there exists $C < \alpha$ such that

$$\int_\ell \alpha(|x(j)|)d\mu(x) \le C\ \alpha(<Tu_j,u_j>^{\frac{1}{2}}).$$

Utilizing condition (ii) we obtain that

$$\sup_{N\ge 1} \int_\ell \sum_{j=1}^{N} \alpha(|x(j)|) < \infty$$

and by the monotone convergence theorem

$$\int_\ell \sum_{j=1}^{\infty} \alpha(|x(j)|)d\mu(x) < \infty .$$

This implies that $\mu(\ell_\alpha) = 1$. The proof is completed by observing that μ is Gaussian with covariance operator T.

<u>Corollary (A)</u>. A bounded linear operator $T : \ell_p' \to \ell_p$ $(1 \le p < \infty)$ is the covariance operator of a Gaussian probability measure on ℓ_p if and only if

(i) T is symmetric and positive.

(ii) $\sum_{j=1}^{\infty} \langle Tu_j, u_j \rangle^{p/2} < \infty$.

Proof. For $p > 1$ the proof follows directly from theorem (3.1) on taking $\alpha(x) = x^p/p$. For $p = 1$ the proof is essentially of the same form as the proof of theorem (4.1) and will hence be omitted.

Corollary (B). Let E be a Banach space of type p where $1 \leq p < \infty$. A bounded linear operator $T : E' \to E$ is the covariance operator of a Gaussian probability measure on E if

(i) T is symmetric and positive.

(ii) $\sum_{j=1}^{\infty} \langle Te_j', e_j' \rangle^{p/2} < \infty$.

Proof. Let $\lambda \in \ell_p'$. Using Hölder's inequality and the fact that $\langle Te_j', e_k' \rangle \leq \langle Te_j', e_j' \rangle^{\frac{1}{2}} \langle Te_k', e_k' \rangle^{\frac{1}{2}}$ we obtain that

$$\sum_{k=1}^{\infty} \Big| \sum_{j=1}^{\infty} \lambda(j) \langle Te_j', e_k' \rangle \Big|^{p/2} \leq \|\lambda\| \{ \sum_{j=1}^{\infty} \langle Te_j', e_j' \rangle^{p/2} \}^2 < \infty .$$

As a consequence we may define the map $C : \ell_p' \to \ell_p$ by

$$C(\lambda)(k) = \sum_{j=1}^{\infty} \lambda(j) T \langle e_j', e_k' \rangle .$$

Noting that C clearly satisfies conditions (i) and (ii) of corollary (A) and we see that there exists a Gaussian measure μ on ℓ_p with covariance operator C.

If S is the map from ℓ_p into E defined in chapter one we may trivially verify that $\mu \circ S^{-1}$ is a Gaussian measure on E with covariance operator T.

§4. Gaussian Measures on a Space of Continuous Functions.

Let C be the space of real valued continuous functions on $[0,1]$. Under the usual uniform norm, $C[0,1]$ becomes a Banach space with a Schauder basis that we shall exhibit later. As observed by de Acosta [ ], lemma 8.3, Gaussian measures on C are the measures induced by continuous Gaussian processes. Our main representation theorem is essentially that of [ ], theorem 8.4; we repeat it here, with proof, since the proof given in [ ] is incomplete.

(4.1) Theorem. Let P be a Gaussian measure on C. Then there exists a continuous map $K : [0,1]^2 \to R$ such that

(i) $K(s,t) = K(t,s)$ for all $s,t \in [0,1]$.

(ii) K is positive definite, that is to say for all $c_1,\ldots,c_n \in R$ and $s_1,\ldots,s_n \in [0,1]$ we have $\sum_{i,j}^{n} c_i c_j K(s_i,s_j) \geq 0$.

(iii) For all $\nu \in C'$

$$\hat{P}(\nu) = \exp\{-\tfrac{1}{2}\int_0^1\int_0^1 K(s,t)\,d\nu(s)\,d\nu(t)\} .$$

Proof. We define K directly by $\tfrac{1}{2}K(s,t) = \int_C x(t)x(s)\,dP(x)$ for all $s,t \in [0,1]$. K is well defined since $\int_C |x(t)x(s)|\,dP(x) < \int_C \|x\|^2 dP(x) < \infty$ and clearly is continuous. Conditions (i) and (ii) are trivial to verify.

Since $\hat{P}(\nu) = \exp\{-\tfrac{1}{2}\int_C |\langle x, \nu\rangle|^2 dP(x)\}$ and $\langle x,\nu\rangle = \int_0^1 x(s)\,d\nu(s)$ we observe that:

$$-2 \log \hat{P}(\nu) = \int_C\int_0^1\int_0^1 x(s)x(t)\,d\nu(s)\,d\nu(t)\,dP(x) .$$

Since

$$\int_C\int_0^1\int_0^1 |x(s)x(t)|\,d\nu(s)\,d\nu(t)\,dP(x) \leq \{\nu[0,1]\}^2\int_C \|x\|^2 dP(x) < \infty$$

we may use Fubini's theorem to obtain

$$-2 \log \hat{P}(\nu) = \int_0^1 \int_0^1 K(s,t)\, d\nu(s)\, d\nu(t)$$

and the result follows. #

A partial converse of theorem (4.1) is possible.

(4.2) Theorem. Let K be a real continuous function on $[0,1]^2$ satisfying condition (i) and (ii) of theorem 4.2. If for some $\alpha > 2$, K satisfies a Lipshitz condition of order α in each of its variables then the operator T from C' into C defined by $(T\nu)(s) = \int_0^1 K(s,t)\, d\nu(t)$ is the covariance operator of a Gaussian measure on C.

Proof. Let $\{\nu_j : j \in Z^+\}$ be the coordinate functionals corresponding to the Schauder basis for C constructed in [ ], p. 69. The following conditions are satisfied

$$\langle x, \nu_n \rangle = x(b(n)) - \tfrac{1}{2}x(a(n)) - \tfrac{1}{2}x(c(n)) \qquad u > 2$$

where

$$a(n) < b(n) < c(n) \quad \text{and} \quad |c(n) - a(n)| = O(1/n).$$

Let $\phi(\nu) = \exp\{-\tfrac{1}{2} \langle T\nu, \nu \rangle\}$. ϕ is positive definite by condition (ii) of theorem 4.1 and using the fact that K satisfies Lipshitz condition we obtain by direct computation that $\langle T\nu_n, \nu_n \rangle = O(n^{-\alpha})$. Since $\alpha > 2$ we have that T is an operator of trace class 1 and hence by theorem (3.8), chapter one, there exists a measure P on C such that $\hat{P} = \phi$ on the subspace of C' spanned by the coordinate functionals. The result then follows.

APPENDIX

APPENDIX

List of symbols

R	The real numbers.
Z^+	The positive integers.
$\ell = R^{Z^+}$	The space of real valued functions on Z^+.
ℓ^+	The space of positive real valued functions on Z^+.
$\ell_p \quad 0 < p < \infty$	The subset of ℓ consisting of the functions λ with $\sum_{n=1}^{\infty} \lvert\lambda(n)\rvert^p < \infty$.
Λ	The subset of ℓ consisting of the functions that are zero except at a finite number of elements of Z^+.
ℓ_α	The Orlicz space $\{x \in \ell : \sum_{j=1}^{\infty} \alpha(\lvert x(j)\rvert) < \infty\}$.

(1) Vector spaces. Our basic reference throughout this thesis will be Robertson and Robertson [17]. We will only be considering vector spaces over the scalor field of real numbers. If E is such a vector space we will denote its algebraic dual by E^* and define it to be the set of all real linear forms on E. If E has a topology then we shall denote its topological dual by E'.

The concept of duality may be found in [17], p. 31, definitions of the polar topologies may be found on p. 46. Of particular interest are the weak $(\sigma(E,E'))$ topology and the topology of uniform convergence on strongly compact subsets of E. We denote this latter topology $\tau_s(E',E)$ and note that if E is a Banach space then the strongly compact sets are the norm compact sets.

(2) Banach spaces with Schauder basis.

If B is a real Banach space ([1], p. 60), a Schauder basis $\{e_j : j \in Z^+\}$ is a sequence of elements in B such that for each $x \in B$ there is a unique sequence of real numbers $x(j)$ such that $\lim_{n\to\infty} \|x - \sum_{i=1}^{n} x(j)e_j\| = 0$. The mappings $x \to x(j)$ are continuous real linear forms on x and hence there exists a sequence $\{e_j' : j \in Z^+\}$ of elements in B' such that $x(j) = \langle x,e_j'\rangle$. These are called the coordinate functionals of the basis. We shall use π_n for all n to denote the maps given by $\pi_n(x) = \sum_{j=1}^{n} \langle x,e_j'\rangle e_j$ and observe that $\|x - \pi_n(x)\| \to 0$. The transpose map ${}^t\pi_n$ may similarly be defined by ${}^t\pi_n(x') = \sum_{j=1}^{n} \langle e_j,x'\rangle e_j'$.

It is known, [8], §1, that $\pi_n(x)$ converges uniformly to x on norm compact subsets of B and hence ${}^t\pi_n(x')$ converges to x' in the $\tau_s(B',B)$ topology.

(3) Sequence spaces.

Let $u_j \in \ell$ be the map $u_j(k) = 1$ if $j = k$, $u_j(k) = 0$ if $k \neq j$. The set $\{u_j : j \in Z^+\}$ will be called the canonical basis for subspaces of ℓ.

If $p \geq 1$ the space ℓ_p becomes a Banach space with Schauder basis $\{u_j : j \in Z^+\}$ under the norm $\|\lambda\| = (\sum_{j=1}^{\infty} |\lambda(n)|^p)^{1/p}$.

If $p < 1$, ℓ_p can be topologies with the invariant metric

$$d(\lambda_1,\lambda_2) = \sum_{n=1}^{\infty} |\lambda_1(n) - \lambda_2(n)|^p .$$

(4) Orlicz spaces.

We take as our basic reference Zaanen [21].

Definition. If the non-decreasing functions $v = \phi(u)$ and $u = \psi(v)$ are inverse to each other then the functions on $[0,\infty)$

$$\alpha(x) = \int_0^x \phi(x)\,dx \quad \text{and} \quad \beta(x) = \int_0^x \psi(x)\,dx$$

are termed complementary in the sense of Young.

We know that $x,y \le \alpha(x) + \beta(y)$, a most useful inequality.

We shall assume that there exists an M such that $\alpha(2x) \le M\alpha(x)$ and $\beta(2x) \le M\beta(x)$ for all $x > 0$. We define ℓ_α to be

$$\ell_\alpha = \{x \in \ell : \sum_{j=1}^{\infty} \alpha(|x(j)|) < \infty\}.$$

ℓ_α may be normed by

$$\|x\|_\alpha = \sup\{ \sum_{j=1}^{\infty} |x(j)y(j)| : \sum_{j=1}^{\infty} \beta|y(j)| \le 1\} .$$

Under this norm ℓ_α is a Banach space with dual ℓ_β.

$\{u_j : j \in Z^+\}$ will be a Schauder basis for ℓ_α since $\|x - \pi_n x\|_\alpha = \sum_{n\neq 1}^{\infty} \alpha(|x(j)|) \to 0$.

It is sometimes of interest to know when an element $z \in \ell$ is an element of ℓ_α, and the uniform boundedness principle establishes a useful criteria.

<u>Criteria 1</u>. $z \in \ell$ is in ℓ_α if for all $y \in \ell_\beta^+$ the sum $\sum_{j=1}^{\infty} |y(j)z(j)|$ is finite.

The function α is said to satisfy condition (2) if there exists a constant C such that for any Gaussian distribution ν on R

$$\int \alpha(|x|)\,d\nu(x) \le C\, \alpha(\{\int x^2 d\nu(x)\}^{\frac{1}{2}}) .$$

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