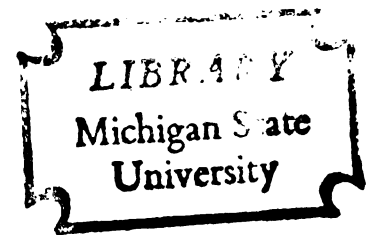




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NORMAL LIGHT INTERIOR FUNCTIONS
DEFINED IN THE UNIT DISK

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
JOHN HENRY MATHEWS
1969



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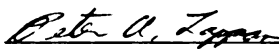
NORMAL LIGHT INTERIOR FUNCTIONS
DEFINED IN THE UNIT DISK

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John Henry Mathews

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ABSTRACT

NORMAL LIGHT INTERIOR FUNCTIONS DEFINED IN THE UNIT DISK

By

John Henry Mathews

Let f be a light interior function from the unit disk into the Riemann sphere. Then f can be factored $f = g \circ h$ where h is a homeomorphism and g is a meromorphic function. Although this factorization is not unique it is shown that there is a unique factorization type. Conditions are established to determine the normality of f ; and it is shown that boundedness is not sufficient for a light interior function to be normal.

Several examples are presented which show that the classical theorems of Fatou, Koebe, Lindelöf and Riesz cannot be extended for even bounded normal light interior functions in the unit disk. For example, there exists a bounded normal light interior function in the unit disk for which the total outer angular cluster set is one point.

Conditions are established to determine when some of the classical theorems will hold for light interior functions. It is shown that several theorems hold for pseudo-meromorphic functions. For example, Koebe's theorem and Lindelöf's theorem remain true for normal pseudo-meromorphic functions.

John Henry Mathews

Let f be a light interior function in the unit disk with factorization $f = g \circ h$ where h is a homeomorphism of the unit disk onto the unit disk and g is a non-constant meromorphic function in the unit disk. Then the asymptotic behavior of f is shown to be closely related to the asymptotic behavior of its component factors g and h .

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DEFINED IN THE UNIT DISK

By

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I. INTRODUCTION

Lehto and Virtanen [15] defined the concept of a normal meromorphic function as follows: If g is meromorphic in a simply connected domain G , then g is normal if and only if the family $\{g(S(z))\}$, where $S(z)$ denotes any arbitrary one-one conformal mapping of G onto G , is normal in the sense of Montel. Meromorphic normal functions defined in the unit disk were found by Lappan [14] to be precisely those which are uniformly continuous with respect to the non-Euclidean hyperbolic metric in the unit disk and the chordal metric on the Riemann sphere. We will say that a function f mapping the unit disk D into the Riemann sphere W is a normal function in D if and only if f is uniformly continuous with respect to the non-Euclidean hyperbolic metric in D and the chordal metric in W .

Väisälä proved [21, Theorem 2, p. 17] that if one uses the Lehto-Virtanen definition of normal then there are no non-constant normal meromorphic functions in the finite complex plane Ω . Yosida [23, p. 227] has defined the concept of a normal meromorphic function in Ω as follows: If g is a meromorphic function in Ω , then g is normal if and only if the family $\{g(a + z) : a \in \Omega\}$ is normal in the sense of Montel. Recently, Gauthier [10, p. 560] has proved, using Lappan's technique, that Yosida's definition is equivalent to the following: A meromorphic function g in Ω is

normal if and only if it is uniformly continuous with respect to the Euclidean metric in Ω and the chordal metric in W . This definition does not exclude all non-constant meromorphic functions and includes, for example, elliptic functions, periodic functions and rational functions in Ω [23, p. 227]. We will define the concept of a normal function in Ω as follows: A function f mapping the finite complex plane Ω into the Riemann sphere W is a normal function in Ω if and only if f is uniformly continuous with respect to the Euclidean metric in Ω and the chordal metric in W .

Normal meromorphic functions in D have been investigated by Lehto and Virtanen [15], Bagemihl [2], Bagemihl and Seidel [3], [4], Lappan [13] and others. The question has been posed: To what extent do the results depend upon the fact that the functions are meromorphic? In this dissertation we investigate the behavior of normal light interior functions. A function f mapping a domain G into the Riemann sphere W is said to be light if for every point $w \in f(G)$ the set $f^{-1}(w)$ is totally disconnected, and f is said to be interior if for each open set $U \subset G$ the set $f(U)$ is open in W [22].

The following definitions and conventions will be used. We shall denote by C the unit circle and by D the open unit disk in the finite complex plane Ω . Let W denote the Riemann sphere, and let $\chi(w_1, w_2)$ represent the chordal distance between the points $w_1, w_2 \in W$. If A and B are sets in W then $\chi(A, B)$ denotes the chordal distance between the sets A and B . In the unit disk, let $\rho(z_1, z_2)$ denote the non-Euclidean hyperbolic distance between

the points $z_1, z_2 \in D$ [5, Chapter 2], [12, Chapter 15];

$$\rho(z_1, z_2) = \frac{1}{2} \ln((1+u)/(1-u)) = \tanh^{-1}(u)$$

where

$$u = |(z_1 - z_2)/(1 - \overline{z_1} z_2)|,$$

alternately

$$\rho(z_1, z_2) = \inf_{\Gamma} \int_{\Gamma} \frac{|dz|}{1-|z|^2}$$

where Γ ranges over all paths joining z_1 to z_2 . If $\{z_n\}$ and $\{z'_n\}$ are two sequences of points in D with $\rho(z_n, z'_n) \rightarrow 0$, we shall say that $\{z_n\}$ is close to $\{z'_n\}$, or that $\{z_n\}$ and $\{z'_n\}$ are close sequences.

Let f be a function from D into W and let $e^{i\theta}$ be a point of C . We define the cluster set $C(f, \theta)$ of f at $e^{i\theta}$ as follows: $C(f, \theta)$ is the set of points $w \in W$ for which there exists a sequence $\{z_n\}$ of points in D with $z_n \rightarrow e^{i\theta}$ and $f(z_n) \rightarrow w$. Furthermore, the total cluster set $C(f)$ of f is given by $C(f) = \bigcup_{\theta} C(f, \theta)$, where the union is taken over all θ ($0 \leq \theta < 2\pi$). If S is a subset of D and $e^{i\theta} \in [\overline{S} \cap C]$, where $\overline{S}$ is the closure of S , we define the cluster set $C_S(f, \theta)$ of f at $e^{i\theta}$ relative to S as follows: $C_S(f, \theta)$ is the set of points $w \in W$ such that there exists a sequence $\{z_n\}$ of points in S with $z_n \rightarrow e^{i\theta}$ and $f(z_n) \rightarrow w$.

By a Stolz domain Δ at $e^{i\theta}$ we mean a set of the form

$$\{z \in D: -\pi/2 < \phi_1 < \arg(1 - z/e^{i\theta}) < \phi_2 < \pi/2\},$$

and by a terminal Stolz domain at $e^{i\theta}$ we mean a set of the form

$$\Delta \cap \{z: |z - e^{i\theta}| < \epsilon\} \quad (0 < \epsilon < 1).$$

The function f is said to have a Fatou point at $e^{i\theta}$ with Fatou value c , or angular limit c , if $f(z) \rightarrow c$ as $z \rightarrow e^{i\theta}$ from within each Stolz domain Δ at $e^{i\theta}$. The outer angular cluster set $C_A(f, \theta)$ of f at $e^{i\theta}$ is defined as follows:

$$C_A(f, \theta) = \bigcup_{\Delta} C_{\Delta}(f, \theta)$$

where the union is taken over all Stolz domains Δ at $e^{i\theta}$. Using this notation we see that $e^{i\theta}$ is a Fatou point with Fatou value c if and only if $C_A(f, \theta) = \{c\}$.

A simple continuous curve $\Gamma: z(t) \quad (0 \leq t < 1)$ contained in D is called a boundary path if $|z(t)| \rightarrow 1$ as $t \rightarrow 1$. The end of a boundary path Γ is the set $\bar{\Gamma} \cap C$. If a boundary path Γ ends at $e^{i\theta}$ then Γ is said to be a Jordan arc at $e^{i\theta}$. A boundary path $\Gamma: z(t) \quad (0 \leq t < 1)$ is an asymptotic path of f for the value c provided $f(z(t)) \rightarrow c$ as $t \rightarrow 1$. The point c is called an asymptotic value, or asymptotic limit, of f if there exists an asymptotic path of f for the value c , and c is said to be a point asymptotic limit of f if there exists an asymptotic path of f for the value c whose end consists of a single point.

Let A be an open subarc of C , possibly C itself. A Koebe sequence of arcs, relative to A , is a sequence of Jordan arcs $\{J_n\}$ in D such that: (a) for every $\epsilon > 0$,

$$J_n \subset \{z \in D: |z - a| < \epsilon, \text{ for some } a \in A\}$$

for all but finitely many n , and (b) every open sector Δ of D subtending an arc of C that lies strictly interior to A has the property that, for all but finitely many n , the arc J_n contains a subarc L_n lying wholly in Δ except for its two end points which lie on distinct sides of Δ .

If f is a function in D and if $c \in W$, we say that f has the limit c along the sequence of arcs $\{J_n\}$, provided that, for every $\epsilon > 0$,

$$\chi(c, f(J_n)) < \epsilon$$

for all but finitely many n . We will write $f(J_n) \rightarrow c$. When $\{J_n\}$ is a Koebe sequence of arcs we will call c a Koebe limit.

II. NORMALCY AND THE STOÏLOW FACTORIZATION

1. Stoïlow factorization

Let f be a light interior function from the unit disk D into the complex plane Ω . Stoïlow [20, p. 121] has shown that f has the representation $f = g \circ h$ where h is a homeomorphism of D onto a Riemann surface R and g is a non-constant analytic function defined on R . Church [8, p. 86] pointed out that this result can be extended to light interior functions which map D into the Riemann sphere W provided g is allowed to be meromorphic. In view of the uniformization theorem [1, p. 181] there exists a conformal mapping of R onto either D or Ω . Therefore, if f is a light interior function from D into W then f has a Stoïlow factorization $f = g \circ h$ where h is a homeomorphism of D onto D (or Ω) and g is a non-constant meromorphic function in D (or Ω). Conversely, if h is a homeomorphism of D onto D (or Ω) and g is a non-constant meromorphic function in D (or Ω) then the function $f = g \circ h$ is light interior.

2. Uniqueness of the Stoïlow factorization

DEFINITION 1. Let h be a homeomorphism of D onto D (or Ω). If h is uniformly continuous with respect to the non-Euclidean hyperbolic metric in its domain D and the non-Euclidean

hyperbolic metric in its range D (or the Euclidean parabolic metric in its range Ω), then we shall say that h is HUC (or PUC).

DEFINITION 2. Let f be a light interior function in D with Stoilow factorization $f = g \circ h$ where h is a homeomorphism of D onto D (or Ω) and g is a non-constant meromorphic function in D (or Ω). If h is HUC (or PUC) then f has a type I factorization; otherwise f has a type II factorization.

THEOREM 1. If f is a light interior function in D then f has a unique factorization type,

Proof. Case i. The light interior function f has a Stoilow factorization $f = g \circ h$ with h a homeomorphism of D onto D . Suppose f also has the Stoilow factorization $f = G \circ H$ where H is a homeomorphism of D onto D (or Ω). Then as pointed out by Church [8, p. 88] $h \circ H^{-1}$ is a conformal homeomorphism, hence from Liouville's theorem $h \circ H^{-1}$ must be a conformal homeomorphism of D onto D . In view of Pick's theorem [12, Theorem 15.1.3, p. 239] both $h \circ H^{-1}$ and $h^{-1} \circ H$ are HUC. Since the composition of two uniformly continuous functions is uniformly continuous, it follows that h is HUC if and only if H is HUC.

Case ii. The case when f has a Stoilow factorization $f = g \circ h$ with h a homeomorphism of D onto Ω is handled similarly, and the proof of the theorem is complete.

There is an abundance of HUC homeomorphisms, for example, every conformal homeomorphism of D onto D is HUC. The existence of a homeomorphism of D onto Ω which is PUC is established in the following theorem.

THEOREM 2. There exists a homeomorphism h of D onto Ω which is PUC.

Proof. Define the mapping h in D by

$$h(z) = z \rho(0, z).$$

Then it is easy to verify that h is a homeomorphism of D onto Ω . Let $A_n = \{z: n \leq \rho(0, z) \leq n+1\}$. Let $n \geq 3$ be fixed but arbitrary; the proof will be complete if we can find a constant K , independent of n , such that $|h(z_1) - h(z_2)| \leq K \rho(z_1, z_2)$ for each pair of points $z_1, z_2 \in A_n$ with $\rho(z_1, z_2) < 1$.

Let $z_1, z_2 \in A_n$, with $\rho(z_1, z_2) < 1$, where $n \geq 3$ is arbitrary but fixed. We may assume that $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ with $r_1 \leq r_2$. Then we have the following inequalities

$$\begin{aligned} |h(z_1) - h(z_2)| &= |z_1 \rho(0, z_1) - z_2 \rho(0, z_2)| \\ &\leq |z_1 \rho(0, z_1) - r_1 e^{i\theta_2} \rho(0, r_1)| \\ &\quad + |r_1 e^{i\theta_2} \rho(0, r_1) - r_1 e^{i\theta_2} \rho(0, r_2)| \\ &\quad + |r_1 e^{i\theta_2} \rho(0, r_2) - z_2 \rho(0, z_2)| \\ &\leq 2\pi |\theta_2 - \theta_1| \rho(0, r_1) + \rho(z_1, z_2) \\ &\quad + |r_2 - r_1| \rho(0, r_2) \\ &= P_1 + P_2 + P_3. \end{aligned}$$

Consider the first term P_1 . Let Γ be non-Euclidean geodesic joining $r_1 e^{i\theta_1}$ to $r_1 e^{i\theta_2}$. Let R be a real number ($0 < R < 1$) for which $\rho(0, R) = n-1$. Then

$$\begin{aligned} \frac{|\theta_2 - \theta_1| R}{1 - R^2} &\leq \int_{\Gamma} \frac{|dz|}{1 - |z|^2} \\ &= \rho(r_1 e^{i\theta_1}, r_1 e^{i\theta_2}) \leq \rho(z_1, z_2). \end{aligned}$$

Also, $\rho(0, r_1) \leq 2 \rho(0, R)$ so that we obtain

$$\begin{aligned} |\theta_2 - \theta_1| \rho(0, r_1) &\leq ((1 - R^2)/R) \rho(z_1, z_2) 2 \rho(0, R) \\ &= (\rho(0, R) (1 - R^2)/R) 2 \rho(z_1, z_2) \\ &\leq 2 \rho(z_1, z_2). \end{aligned}$$

Consider the third term P_3 . We observe that

$$\frac{|r_2 - r_1| r_1}{1 - r_1^2} \leq \int_{r_1}^{r_2} \frac{dx}{1 - x^2} = \rho(r_1, r_2) \leq \rho(z_1, z_2).$$

Also, $\rho(0, r_2) \leq 2 \rho(0, r_1)$ so that we obtain

$$\begin{aligned} |r_2 - r_1| \rho(0, r_2) &\leq ((1 - r_1^2)/r_1) \rho(z_1, z_2) 2 \rho(0, r_1) \\ &= (\rho(0, r_1) (1 - r_1^2)/r_1) 2 \rho(z_1, z_2) \\ &\leq 2 \rho(z_1, z_2). \end{aligned}$$

Finally, combining the estimates for P_1 , P_2 and P_3 we obtain $|h(z_1) - h(z_2)| \leq (4\pi + 3)\rho(z_1, z_2)$. We choose $K = (4\pi + 3)$ and the proof is complete.

3. Necessary conditions for both f and g normal

Noshiro [18, p. 154] (or Yosida [23, p. 227]) has divided the class of normal meromorphic functions in D (or Ω) into two kinds which are defined as follows: A normal meromorphic function g in D (or Ω) is of the first kind if the normal family $\{g(\frac{a-z}{1-\bar{a}z}) : a \in D\}$ (or the normal family $\{g(a+z) : a \in \Omega\}$) admits no constant limit; otherwise g is of the second kind.

THEOREM 3. Let f be a normal light interior function in D with Stoilow factorization $f = g \circ h$ where h is a homeomorphism of D onto D (or Ω) and g is a non-constant meromorphic function in D (or Ω). If g is a normal meromorphic function in D (or Ω), then h is normal. Furthermore, if g is a normal meromorphic function of the first kind in D (or Ω), then h is HUC (or PUC).

Proof. Case i. The normal light interior function f has a Stoilow factorization $f = g \circ h$ with h a homeomorphism of D onto D . If h is not normal there exist close sequences $\{z_n\}$ and $\{z'_n\}$ such that $h(z_n) \rightarrow e^{i\alpha}$ and $h(z'_n) \rightarrow e^{i\beta}$ with $0 < \beta - \alpha < 2\pi$ [14]. For each integer n , let J_n be the non-Euclidean geodesic joining z_n to z'_n . Then $\{h(J_n)\}$ is a sequence of Jordan arcs such that for every $\epsilon > 0$,

$$h(J_n) \subset \{z \in D : 1 - \epsilon < |z| < 1\}$$

for all but finitely many n , and the end points of $h(J_n)$ tend to $e^{i\alpha}$ and $e^{i\beta}$. Choosing a subsequence of $\{h(J_n)\}$ if necessary, we may assume that there exists a Koebe sequence of arcs $\{L_n\}$ relative to either the open arc (α, β) or the open

arc $(\beta, \alpha + 2\pi)$ such that $L_n \subset h(J_n)$, and that there exists a constant c such that $f(z_n) \rightarrow c$.

From the normality of f we have $f(J_n) \rightarrow c$, and it follows that $g(L_n) \rightarrow c$. By a theorem of Bagemihl and Seidel [4, Theorem 1, p. 10], $g \equiv c$ in violation of our hypothesis. Therefore h is normal and the proof of the first part is complete.

Now assume that g is a normal meromorphic function of the first kind in D . If h is not HUC there exists close sequences $\{z_n\}$ and $\{z'_n\}$ and a $\delta > 0$ such that $\rho(h(z_n), h(z'_n)) \geq \delta$, and there exists a constant c such that $f(z_n) \rightarrow c$.

Let

$$S_n(z) = \frac{h(z_n) - z}{1 - \overline{h(z_n)} z}$$

and let $G_n(z) = g(S_n(z))$. Then the normal family $\{G_n\}$ has a subsequence which converges uniformly on each compact subset of D to a meromorphic function G [15, p. 53]. Let J_n be the non-Euclidean geodesic joining z_n to z'_n and let $L_n = h(J_n)$. Then $d(L_n) = d(S_n^{-1}(L_n)) \geq \delta$, where $d(E)$ is the hyperbolic diameter of the set $E \subset D$. From the normality of f we have $f(J_n) \rightarrow c$, so that $g(L_n) \rightarrow c$, and hence $G_n(S_n^{-1}(L_n)) \rightarrow c$. For r ($0 \leq r \leq \delta$) fixed, there exists a point $Z_n \in S_n^{-1}(L_n)$ such that $\rho(0, Z_n) = r$. Let Z_0 be a cluster point of the sequence $\{Z_n\}$ on the circle $\{z: \rho(0, z) = r\}$.

Choosing a subsequence of $\{G_n\}$ if necessary, we can assume that $Z_n \rightarrow Z_0$ and $G_n(Z_n) \rightarrow c$. A familiar argument (see e.g. [6, p. 179]) in the theory of continuous convergence shows that $G(Z_0) = c$. Since r ($0 \leq r \leq \delta$) was arbitrary, 0 is a limit

point of values for which G assumes c and hence $G \equiv c$ in violation of our hypothesis. Therefore h is HUC and the proof of the first case is complete.

Case ii. The normal light interior function f has a Stoilow factorization $f = g \circ h$ with h a homeomorphism of D onto Ω . In this case h is always normal and when g is a normal meromorphic function of the first kind in Ω the proof is handled similarly to Case i; and the proof of the theorem is complete.

4. Sufficient conditions for f normal

Every bounded holomorphic function is normal [3], but the following result shows that boundedness is not sufficient for a light interior function to be normal.

THEOREM 4. If a homeomorphism h of D onto D is not HUC, then there exists a Blaschke product B in D such that the bounded light interior function $f = B \circ h$ is not normal in D .

Proof. If h is not HUC there exist close sequences $\{z_n\}$ and $\{z'_n\}$ and a $\delta > 0$ such that $\rho(h(z_n), h(z'_n)) \geq \delta$. Let $h(z_n) = w_n$ and $h(z'_n) = w'_n$. Since h is uniformly continuous on compact subsets we necessarily have that $|z_n| \rightarrow 1$, $|z'_n| \rightarrow 1$, $|w_n| \rightarrow 1$, and $|w'_n| \rightarrow 1$. Hence, choosing a subsequence of $\{w_n\}$ if necessary, we may assume that $\{w_n\}$ is a Blaschke sequence, i.e. $\sum_{n=1}^{\infty} (1 - |w_n|) < \infty$. We now construct a Blaschke subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and a corresponding subsequence $\{w'_{n_k}\}$ of $\{w'_n\}$.

Let $w_{n_1} = w_1$ and $w'_{n_1} = w'_1$, and let $r_1 = \min\{|w_{n_1}|, |w'_{n_1}|\}$ and $R_1 = \max\{|w_{n_1}|, |w'_{n_1}|\}$. We can find an integer $n_2 > 1$ such that for $r_2 = \min\{|w_{n_2}|, |w'_{n_2}|\}$ we have $\rho(R_1, r_2) \geq \tanh^{-1}(1-1/2^2)$. Let $R_2 = \max\{|w_{n_2}|, |w'_{n_2}|\}$. We proceed inductively to obtain subsequences $\{w_{n_k}\}$ and $\{w'_{n_k}\}$ of $\{w_n\}$ and $\{w'_n\}$, respectively, such that $\rho(R_{k-1}, r_k) \geq \tanh^{-1}(1-1/k^2)$ for each integer $k \geq 2$, where $r_k = \min\{|w_{n_k}|, |w'_{n_k}|\}$ and $R_k = \max\{|w_{n_k}|, |w'_{n_k}|\}$.

It follows easily that

$$\rho(w_{n_k}, w'_{n_j}) \geq \begin{cases} \tanh^{-1}(1-1/(k+1)^2) & (1 \leq k < j) \\ \tanh^{-1}(1-1/k^2) & (1 \leq j < k), \end{cases}$$

and hence

$$\left| \frac{w_{n_k} - w'_{n_j}}{1 - \overline{w_{n_k}} w'_{n_j}} \right| \geq \begin{cases} 1 - 1/(k+1)^2 & (1 \leq k < j) \\ 1 - 1/k^2 & (1 \leq j < k). \end{cases}$$

Recall that $\rho(w_{n_k}, w'_{n_k}) \geq \delta > 0$ ($k = 1, 2, \dots$) so that

$$\left| \frac{w_{n_k} - w'_{n_k}}{1 - \overline{w_{n_k}} w'_{n_k}} \right| \geq \tanh^{-1} \delta > 0 \quad (k = 1, 2, \dots).$$

$$\text{Set } B(z) = \prod_{k=1}^{\infty} \frac{|w_{n_k}| (w_{n_k} - z)}{w_{n_k} (1 - \overline{w_{n_k}} z)}.$$

Consider $B(w'_{n_j})$ for $j \geq 1$,

$$\begin{aligned}
|B(w'_{n_j})| &= \prod_{k=1}^{j-1} \left| \frac{w_{n_k} - w'_{n_j}}{1 - \overline{w_{n_k}} w'_{n_j}} \right| \cdot \prod_{k=j+1}^{\infty} \left| \frac{w_{n_k} - w'_{n_j}}{1 - \overline{w_{n_k}} w'_{n_j}} \right| \\
&\geq (\tanh^{-1} \delta) \prod_{k=1}^{j-1} (1 - 1/(k+1)^2) \prod_{k=j+1}^{\infty} (1 - 1/k^2) \\
&= (\tanh^{-1} \delta) \prod_{k=2}^{\infty} (1 - 1/k^2) = \frac{\tanh^{-1} \delta}{2} > 0.
\end{aligned}$$

Let $f = B \circ h$. By assumption $\{z_{n_k}\}$ and $\{z'_{n_k}\}$ are necessarily close sequences with

$$\lim f(z_{n_k}) = \lim B(h(z_{n_k})) = \lim B(w_{n_k}) = 0$$

$$\text{and } \lim f(z'_{n_k}) = \lim B(h(z'_{n_k})) = \lim B(w'_{n_k}) \neq 0.$$

By a theorem of Lappan [14, Theorem 2, p. 156], f is not normal and the proof is complete.

Let f be a light interior function in D with Stoilow factorization $f = g \circ h$ where h is a homeomorphism of D onto D and g is a non-constant meromorphic function in D . The previous theorem suggests that the normality of g does not insure the normality of f . An even stronger statement is the following result.

THEOREM 5. There exists a homeomorphism h of D onto D with the property: If g is a non-constant normal meromorphic function in D , which has two distinct asymptotic limits, then the light interior function $f = g \circ h$ is not normal in D .

Proof. Construct a sequence $\{R_n\}$ of real numbers $0 = R_1 < R_2 < \dots < R_n < \dots < 1$ for which $\rho(R_n, R_{n+1}) = 1/n$. Define

the mapping h in D by

$$h(z) = h(re^{i\theta}) = r \exp(i\theta + 2\pi i(r - R_n)/(R_{n+1} - R_n))$$

for $R_n \leq r < R_{n+1}$ ($n = 1, 2, \dots$). It is easy to verify that h is a homeomorphism of D onto D .

Since g has two distinct asymptotic limits, a theorem of Lehto and Virtanen [15, Theorem 2, p. 53] implies that g has two distinct radial limits. Let τ_α and τ_β be the radii which terminate at the points $e^{i\alpha}$ and $e^{i\beta}$, respectively, for which $g(re^{i\alpha}) \rightarrow a$ and $g(re^{i\beta}) \rightarrow b$ with $b \neq a$.

Now the radii of D are mapped onto spirals by h^{-1} . Let $h^{-1}(\tau_\alpha) \cap [R_n, R_{n+1}) = z_n$ and $h^{-1}(\tau_\beta) \cap [R_n, R_{n+1}) = z'_n$. Then $\rho(z_n, z'_n) \leq \rho(R_n, R_{n+1}) = 1/n$ with $f(z_n) = g(h(z_n)) \rightarrow a$ and $f(z'_n) = g(h(z'_n)) \rightarrow b$. Hence, by a theorem of Lappan [14], f is not normal and the theorem is proved.

Since a bounded holomorphic function in D is normal and possesses uncountably many distinct radial limits [9] we obtain the following corollary.

COROLLARY. There exists a homeomorphism h of D onto D with the property: If g is a non-constant bounded holomorphic function in D , then the bounded light interior function $f = g \circ h$ is not normal in D .

We now determine conditions on h and g which insure the normality of f . Since the composition of two uniformly continuous functions is uniformly continuous the first result in this direction is obvious.

THEOREM 6. Let h be a homeomorphism of D onto D
(or Ω) which is HUC (or PUC). If g is a non-constant
normal meromorphic function in D (or Ω), then the light
interior function $f = g \circ h$ is normal in D . Furthermore, if
both h and h^{-1} are HUC, then g is normal in D if and only
if f is normal in D .

DEFINITION 3. Let h be a homeomorphism of D onto D .
Define the set $F(h)$ as follows: $e^{i\theta} \in F(h)$ if there exist
close sequences $\{z_n\}$ and $\{z'_n\}$ and a $\delta > 0$ for which
 $\rho(h(z_n), h(z'_n)) \geq \delta$ and $h(z_n) \rightarrow e^{i\theta}$.

THEOREM 7. Let h be a normal homeomorphism of D onto D .
If g is a non-constant normal meromorphic function in D which
is continuous on $D \cup F(h)$, then the light interior function
 $f = g \circ h$ is normal in D .

Proof. If f is not normal there exist close sequences
 $\{z_n\}$ and $\{z'_n\}$ such that $f(z_n) \rightarrow a$ and $f(z'_n) \rightarrow b$ with
 $b \neq a$ [14]. It follows from the normality of g that $\{h(z_n)\}$
and $\{h(z'_n)\}$ are not close. Choosing a subsequence of $\{z_n\}$
and a corresponding subsequence of $\{z'_n\}$ if necessary, we may
assume that $h(z_n) \rightarrow e^{i\theta}$ and $h(z'_n) \rightarrow e^{i\theta}$ with $e^{i\theta} \in F(h)$.
But g is continuous on $D \cup F(h)$ and hence

$$b = \lim f(z'_n) = \lim g(h(z'_n)) = \lim g(h(z_n)) = \lim f(z_n) = a$$

which is a contradiction. Therefore f is normal and the proof
is complete.

III. FAILURE OF THE CLASSICAL THEOREMS

We now investigate the boundary behavior of normal light interior functions and show that the classical theorems cannot be extended even for bounded normal light interior functions.

1. Fatou's theorem and Koebe's theorem

Fatou's theorem [9] states that a bounded holomorphic function in D possesses radial limits at almost every point of C . The following result shows that a bounded normal light interior function need not possess any point asymptotic limits.

Koebe's theorem [19] states that a non-constant bounded holomorphic function in D possesses no Koebe limits. The following result shows that a bounded normal light interior function can possess uncountably many distinct Koebe limits relative to C .

THEOREM 8. There exists a homeomorphism h of D onto D with the property: If g is a non-constant normal meromorphic function in D , then the light interior function $f = g \circ h$ is normal and possesses no point asymptotic limits. Furthermore, if g possesses a point asymptotic limit, then f possesses a Koebe limit relative to C .

Since a bounded holomorphic function in D is normal and possesses uncountably many distinct radial limits we obtain the following corollary.

COROLLARY. There exists a homeomorphism h of D onto D
with the property: If g is a non-constant bounded holomorphic
function in D , then the bounded light interior function $f = g \circ h$
is normal and possesses no point asymptotic limits. Furthermore,
 f possesses uncountably many distinct Koebe limits relative to C .

Before proving Theorem 8 we establish the following lemma.

LEMMA 1. There exists a homeomorphism h of D onto D
such that the radii of D are mapped onto spirals and h is HUC.

Proof. Construct a sequence $\{R_n\}$ of real numbers,

$$0 = R_0 < R_1 < R_2 < \dots < R_n < \dots < 1$$

for which $\rho(R_n, R_{n+1}) = 1/(1 - R_n^2)$ ($n = 0, 1, \dots$). Define the mapping Φ_1 of the interval $[0, R_2)$ onto $[0, R_1)$ by $\Phi_1(r) = (rR_1)/R_2$. And define the mapping Φ_n ($n = 2, 3, \dots$) of the interval $[R_n, R_{n+1})$ onto $[R_{n-1}, R_n)$ as follows: $\Phi_n(r)$ is to be the solution of the equation

$$\rho(R_{n-1}, \Phi_n(r)) / \rho(R_{n-1}, R_n) = \rho(R_n, r) / \rho(R_n, R_{n+1}).$$

A straightforward calculation shows that if $R_n \leq r_1 \leq r_2 < R_{n+1}$ then $\rho(\Phi_n(r_1), \Phi_n(r_2)) \leq \rho(r_1, r_2)$.

Define the mapping $\Psi_1(r)$ of the interval $[0, R_2)$ onto $[0, 2\pi)$ by $\Psi_1(r) = 2\pi \rho(0, r) / \rho(0, R_2)$; and define the mapping Ψ_n ($n = 2, 3, \dots$) of the interval $[R_n, R_{n+1})$ onto $[0, 2\pi)$ by $\Psi_n(r) = 2\pi \rho(R_n, r) / \rho(R_n, R_{n+1})$.

Define the mapping h in D by

$$h(z) = h(re^{i\theta}) = \Phi_n(r) \exp(i\theta + i\Psi_n(r))$$

for $0 \leq r < R_2$ ($n = 1$), and $R_n \leq r < R_{n+1}$ ($n = 2, 3, \dots$). It is easy to verify that h is a homeomorphism of D onto D and that the radii of D are mapped onto spirals.

Let $A_n = \{z: R_n \leq |z| < R_{n+1}\}$. Let $n \geq 2$ be fixed but arbitrary; the proof will be complete if we can find a constant K , independent of n , such that $\rho(h(z_1), h(z_2)) < K \rho(z_1, z_2)$ for each pair of points $z_1, z_2 \in A_n$ with $\rho(z_1, z_2) < 1$. Let $z_1, z_2 \in A_n$, with $\rho(z_1, z_2) < 1$, where $n \geq 2$ is arbitrary but fixed. We may assume that $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ with $r_1 \leq r_2$. Then we have the following inequalities

$$\begin{aligned} \rho(h(z_1), h(z_2)) &= \rho(h(r_1 \exp(i\theta_1)), h(r_2 \exp(i\theta_2))) \\ &\leq \rho(\Phi_n(r_1) \exp(i\theta_1 + i\Psi_n(r_1)), \Phi_n(r_1) \exp(i\theta_2 + i\Psi_n(r_1))) \\ &\quad + \rho(\Phi_n(r_1) \exp(i\theta_2 + i\Psi_n(r_1)), \Phi_n(r_1) \exp(i\theta_2 + i\Psi_n(r_2))) \\ &\quad + \rho(\Phi_n(r_1) \exp(i\theta_2 + i\Psi_n(r_2)), \Phi_n(r_2) \exp(i\theta_2 + i\Psi_n(r_2))) \\ &= P_1 + P_2 + P_3. \end{aligned}$$

Consider the first term P_1 . From the fact that

$\Phi_n(r_1) \leq r_1$ we obtain

$$\begin{aligned} P_1 &= \rho(\Phi_n(r_1) \exp(i\theta_1), \Phi_n(r_1) \exp(i\theta_2)) \leq \rho(r_1 \exp(i\theta_1), r_1 \exp(i\theta_2)) \\ &\leq \rho(z_1, z_2). \end{aligned}$$

Consider the second term P_2 . From the facts that

$\Phi_n(r_1) \leq R_n$ and $\rho(R_n, R_{n+1}) = 1/(1 - R_n^2)$ we obtain

$$\begin{aligned}
P_2 &\leq \frac{\Psi_n(r_2)}{\Psi_n(r_1)} \int \frac{\Phi_n(r_1) d\theta}{1 - \Phi_n(r_1)^2} \\
&\leq \frac{2\pi}{(1 - R_n^2)\rho(R_n, R_{n+1})} [\rho(R_n, r_2) - \rho(R_n, r_1)] \\
&= 2\pi \rho(r_1, r_2) \leq 2\pi \rho(z_1, z_2).
\end{aligned}$$

Consider the third term P_3 . From the fact that $\rho(\Phi_n(r_1), \Phi_n(r_2)) \leq \rho(r_1, r_2)$ we obtain

$$P_3 = \rho(\Phi_n(r_1), \Phi_n(r_2)) \leq \rho(r_1, r_2) \leq \rho(z_1, z_2).$$

Finally, combining the estimates for P_1 , P_2 and P_3 we obtain $\rho(h(z_1), h(z_2)) \leq (2 + 2\pi)\rho(z_1, z_2)$. We choose $K = (2 + 2\pi)$ and the proof is complete.

Proof of Theorem 8. Let h be the homeomorphism of Lemma 1. Let g be a non-constant normal meromorphic function in D . Then by Theorem 6, the light interior function $f = g \circ h$ is normal. If f has a point asymptotic limit c along a boundary path Γ , then it is easy to verify that $h(\Gamma)$ is a spiral asymptotic path of g for the value c . Construct a Koebe sequence of arcs $\{J_n\}$ in D by letting the J_n be the consecutive turns of the spiral $h(\Gamma)$. Then $g(J_n) \rightarrow c$, and by Theorem 1 of [4], $g \equiv c$ in violation of our hypothesis. Therefore f has no point asymptotic limits.

If g has a point asymptotic limit c along a boundary path Λ , then $h^{-1}(\Lambda)$ is a spiral asymptotic path of f for the

value c . Construct a Koebe sequence of arcs $\{J_n\}$ in D by letting the J_n be the consecutive turns of the spiral $h^{-1}(\Lambda)$. Then f has the Koebe limit c relative to C and the proof is complete.

2. Lindelöf's theorem

Lindelöf's theorem [9] states that if a bounded holomorphic function f in D possesses the point asymptotic limit c at $e^{i\theta}$, then f possesses the angular limit c at $e^{i\theta}$. Consequently, a bounded holomorphic function can possess only one point asymptotic limit at $e^{i\theta}$. The first result shows that a bounded normal light interior function can possess point asymptotic limits at almost every point of C and possess no radial limits. The second result shows that a bounded normal light interior function can possess uncountably many distinct point asymptotic limits at the point $z = 1$.

THEOREM 9. There exists a bounded normal light interior function f in D which possesses point asymptotic limits at almost every point of C but which possesses no radial limits.

Before proving the theorem we establish the following lemma.

LEMMA 2. There exists a homeomorphism h of $\bar{D}$ onto $\bar{D}$ with the following properties: (a) the radius τ_0 at $z = 1$ is mapped onto an arc Γ_0 , where Γ_0 is a Jordan arc lying in $D \cup \{1\}$ internally tangent to C at $z = 1$, with one end point 0 , (b) if Γ_θ denotes the image of Γ_0 under a rotation through an angle θ about the origin, then the radius τ_θ at $e^{i\theta}$ is mapped onto Γ_θ , (c) the restriction of h to C is the identity and (d) h is HUC in D .

Proof. Let $\{R_n\}$ be the sequence of real numbers constructed in Lemma 1. Define the mapping Φ_2 of the interval $[0, R_3)$ onto $[0, R_2)$ by $\Phi_2(r) = (rR_2)/R_3$. And define the mapping Φ_n ($n = 3, 4, \dots$) of the interval $[R_n, R_{n+1})$ onto $[R_{n-1}, R_n)$ as in Lemma 1. Define the mapping Ψ_2 of the interval $[0, R_3)$ onto $[0, 1)$ by $\Psi_2(r) = \rho(0, r)/\rho(0, R_3)$; and define the mapping Ψ_n ($n = 3, 4, \dots$) of the interval $[R_n, R_{n+1})$ onto $[0, 1)$ by $\Psi_n = \rho(R_n, r)/\rho(R_n, R_{n+1})$.

Let $C^* = \{z: \operatorname{Im} z > 0, |z - \frac{1}{2}| = \frac{1}{2}\}$ and let $C_n = \{z: |z| = R_n\}$. Let $C^* \cap C_n = w_n$ and let $\alpha_n = \arg(w_n)$ ($n = 1, 2, \dots$) and $\alpha_0 = 0$. Define the mapping h in $\bar{D}$ by

$$h(z) = h(re^{i\theta}) = \Phi_n(r) \exp(i\theta + i\alpha_{n-2} + i(\alpha_{n-1} - \alpha_{n-2})\Psi_n(r))$$

for $0 \leq r < R_3$ ($n = 2$), and $R_n \leq r < R_{n+1}$ ($n = 3, 4, \dots$); and $h(e^{i\theta}) = e^{i\theta}$. It is easy to verify that h is homeomorphism of $\bar{D}$ onto $\bar{D}$. By reasoning similar to that in Lemma 1 it is easy to verify that h is HUC in D . Setting $\Gamma_0 = h(\tau_0)$ it follows that h possesses all the desired properties and the proof is complete.

Proof of Theorem 9. Let h be the homeomorphism and Γ_0 be the Jordan arc of Lemma 2. By a theorem of Lohwater and Piranian [16, Theorem 9, p. 15], there exists a bounded holomorphic function g in D which does not approach a limit as z approaches $e^{i\theta}$ along Γ_θ ($0 \leq \theta < 2\pi$). Hence the bounded light interior function $f = g \circ h$ possesses no radial limits. By Theorem 6, f is normal.

Since g is bounded, g possesses radial limits at almost every point of C . Let τ_θ be the radius terminating at $e^{i\theta}$. It follows easily that f has point asymptotic limits at almost every point of C along the paths $h^{-1}(\tau_\theta)$; and the proof is complete.

THEOREM 10. There exists a bounded normal light interior function f in D which possesses a continuum of distinct point asymptotic limits at the point $z = 1$.

Proof. Define the mapping f in D by

$$f(z) = f(x + iy) = x + iy/(1 - x^2)^{\frac{1}{2}}.$$

It is easy to verify that f is a homeomorphism of D onto the unit square $Q = \{z = x + iy: \max\{|x|, |y|\} < 1\}$.

Let $z, z' \in D$, $z = x + iy$, $z' = x' + iy'$ with $z \neq z'$. From the fact that $|y| < (1 - x^2)^{\frac{1}{2}}$ we obtain

$$\begin{aligned} \left| \frac{f(z') - f(z)}{z' - z} \right| &= \left| \frac{x' + iy'/(1 - x'^2)^{\frac{1}{2}} - x - iy/(1 - x^2)^{\frac{1}{2}}}{x' + iy' - x - iy} \right| \\ &\leq 1 + \frac{1}{(1 - x'^2)^{\frac{1}{2}}} \cdot \left| \frac{y' - y}{x' + iy' - x - iy} \right| \\ &\quad + \frac{|y|}{[(1 - x'^2)(1 - x^2)]^{\frac{1}{2}}} \cdot \left| \frac{(1 - x^2)^{\frac{1}{2}} - (1 - x'^2)^{\frac{1}{2}}}{x' + iy' - x - iy} \right| \\ &< 1 + \frac{1}{(1 - x'^2)^{\frac{1}{2}}} \\ &\quad + \frac{1}{(1 - x'^2)^{\frac{1}{2}}} \cdot \frac{2}{(1 - x'^2)^{\frac{1}{2}} + (1 - x^2)^{\frac{1}{2}}} . \end{aligned}$$

And it follows that

$$M(f(z)) = \limsup_{z' \rightarrow z} \left| \frac{f(z') - f(z)}{z' - z} \right| < \frac{3}{1 - |z|^2}.$$

And we obtain $\rho^*(f(z)) = \frac{M(f(z))}{1 + |f(z)|^2} < \frac{3}{1 - |z|^2}$. By a theorem of Lappan [14, Theorem 3, p. 156], f is normal.

For α ($-1 < \alpha < 1$) fixed, the Jordan arc $z(t) = t + i\alpha(1 - t^2)^{\frac{1}{2}}$ ($0 \leq t < 1$) in D is a point asymptotic path at $z = 1$ for the value $1 + i\alpha$. Therefore f has a continuum of distinct point asymptotic limits at the point $z = 1$ and the proof is complete.

3. Riesz's theorem

Riesz's theorem [9] states that if f is a non-constant bounded holomorphic function in D and c is a fixed value in Ω , then the set of points on C for which $C_A(f, \theta) = \{c\}$ cannot have positive measure on C . The following theorem shows that there exists a bounded normal light interior function f in D with $C_A(f, \theta) = \{1\}$ for every $\theta \in [0, 2\pi)$.

THEOREM 11. There exists a bounded normal light interior function f in D such that $C_A(f, \theta) = \{1\}$ for every $\theta \in [0, 2\pi)$.

Proof. Let h be the homeomorphism of D onto the unit square Q which was constructed in Theorem 10, i.e.

$$h(z) = h(x + iy) = x + iy/(1 - x^2)^{\frac{1}{2}}.$$

Define the mapping H in Q by

$$H(z) = H(x + iy) = x(1 - y^2)^{\frac{1}{2}} + iy.$$

It is easy to verify that H is a homeomorphism of Q onto D , and that H is uniformly continuous with respect to the chordal metric in both Q and D . Since h is normal it follows easily that the homeomorphism

$$H(h(z)) = x[(1 - x^2 - y^2)/(1 - x^2)]^{\frac{1}{2}} + iy/(1 - x^2)^{\frac{1}{2}}$$

is a normal homeomorphism of D onto D .

It is easy to verify that $C(H \circ h, \theta) = \{i\}$ for all $\theta \in (0, \pi)$ and that $C(H \circ h, \theta) = \{-i\}$ for all $\theta \in (\pi, 2\pi)$.

For α ($-\infty < \alpha < \infty$) fixed, the segment

$$z(t) = t + i\alpha(1 - t)$$

in D terminating at the point $z = 1$ is mapped onto

$$H(h(z(t))) = t[1 - \alpha^2(1 - t)/(1 + t)]^{\frac{1}{2}} + i\alpha[(1 - t)/(1 + t)]^{\frac{1}{2}},$$

so that $H(h(z(t))) \rightarrow 1$ as $t \rightarrow 1$. Hence, it is easy to verify that $C_A(H \circ h, 0) = \{1\}$ and similarly that $C_A(H \circ h, \pi) = \{-1\}$. Therefore $C_A(H \circ h, \theta) \in \{1, -1, i, -i\}$ for $\theta \in [0, 2\pi)$.

Let $g(z) = z^4$, then g is continuous on $D \cup C$ and by Theorem 7 the light interior function $f = g \circ h$ is normal in D . But g was chosen so that $C_A(f, \theta) = \{1\}$ for every $\theta \in [0, 2\pi)$ and the theorem is proved.

The previous theorem shows that the total outer angular cluster set of a light interior function can be a single point. If we consider the total cluster set instead of the total outer angular cluster set we are able to establish the following result.

THEOREM 12. Let f be a light interior function in D which omits the value ∞ . If $C(f) = \{c\}$, then $c = \infty$.

Proof. Suppose $C(f) = \{c\}$ and $c \neq \infty$ then we have two cases:

Case i. $f = g \circ h$ where h is a homeomorphism of D onto D , and g is a non-constant holomorphic function in D .

Then $C(g) = \{c\}$, hence by a theorem of Lusin and Privalof [19, p. 72], $g \equiv c$ which is a contradiction.

Case ii. $f = g \circ h$ where h is a homeomorphism of D onto Ω , and g is a non-constant holomorphic function in Ω . Now g holomorphic in Ω and $g(z) \rightarrow c \neq \infty$ as $z \rightarrow \infty$ implies g is bounded in Ω . By Liouville's theorem, $g \equiv c$ which is a contradiction.

Therefore we cannot have $C(f) = \{c\}$ for $c \neq \infty$ and the theorem is established.

A homeomorphism h of D onto Ω is a normal light interior function in D which omits the value ∞ and $C(f) = \{\infty\}$. Therefore the hypothesis $c \neq \infty$ was necessary for the previous theorem. The previous theorem is sharp as shown by the following result.

THEOREM 13. There exists a bounded normal light interior function f in D such that $C(f, \theta) = \{0\}$ for every $\theta \in (0, 2\pi)$.

Proof. Let h be the homeomorphism of Theorem 11, i.e.

$$h(z) = h(x + iy) = x[(1 - x^2 - y^2)/(1 - x^2)]^{\frac{1}{2}} + iy/(1 - x^2)^{\frac{1}{2}}.$$

Then h is a normal homeomorphism of D onto D for which

$$C(h, \theta) \subset [\pi/2, 3\pi/2] \quad \text{for } \theta \in (0, 2\pi).$$

There exists a conformal mapping S_1 of D onto the square $Q = \{z: 0 < x < 2, -1 < y < 1\}$, such that the arc $[\pi/2, 3\pi/2]$ of C is mapped onto the side $L = \{z \in \bar{Q}: x = 0\}$ of $\bar{Q}$. The mapping $S_2(z) = x + ixy/2$ is a homeomorphism of Q onto the triangle $T = \{z: 0 < x < 2, |y| < x/2\}$ such that the side L of $\bar{Q}$ is collapsed to the point 0 . The function $S_2 \circ S_1$ is continuous on $C \cup D$, and by Theorem 7 the bounded light interior function $f = S_2 \circ S_1 \circ h$ is normal in D . By construction we have $C(f, \theta) = \{0\}$ for every $\theta \in (0, 2\pi)$ and the proof is complete.

IV. GENERALIZATIONS AND APPLICATIONS TO K-PM FUNCTIONS

1. Quasiconformal functions and pseudo-meromorphic functions

We now investigate the behavior of a light interior function f with Stoilow factorization $f = g \circ h$ when h is a quasiconformal homeomorphism of D onto D and g is a non-constant meromorphic function in D . Let Q be a simply connected region in Ω bounded by a Jordan curve, and let z_1, z_2, z_3, z_4 be four distinct boundary points of Q , which lie in this order on the positively oriented boundary curve. We call such a configuration a quadrilateral, and denote it by $Q(z_1, z_2, z_3, z_4)$. An orientation preserving homeomorphism of the plane transforms quadrilaterals into quadrilaterals. Map the region Q conformally onto a rectangle $R: 0 < u < 1, 0 < v < t$, in the $w = u + iv$ plane, in such a manner that z_1, z_2, z_3, z_4 correspond to the vertices $w = 0, 1, 1 + it, it$ respectively. We call the positive number t the modulus of the quadrilateral $Q(z_1, z_2, z_3, z_4)$, and denote it by $\text{mod}Q(z_1, z_2, z_3, z_4)$.

A homeomorphism h of D onto D is called K-quasiconformal or simply K-QC, if i) h preserves orientation of the plane, and ii) for any quadrilateral $Q(z_1, z_2, z_3, z_4)$ contained in D together with its boundary,

$$\text{mod } h(Q(z_1, z_2, z_3, z_4)) \leq K \text{mod}Q(z_1, z_2, z_3, z_4),$$

where K is a constant, $K \geq 1$. If f is a light interior function in D with Stoilow factorization $f = g \circ h$ with h a K -QC homeomorphism of D onto D and g a non-constant meromorphic function in D , then we shall call f a K -pseudo-meromorphic function, or simply K -PM.

2. Normality

If f is a K -pseudo-meromorphic function in D with Stoilow factorization $f = g \circ h$, then we show that f is normal in D if and only if g is normal in D . This result was proved by Väisälä [21, Theorem 5, p. 20] whose proof is considerably different.

THEOREM 14. If h is a K -QC homeomorphism of D onto D , then both h and h^{-1} are HUC.

THEOREM 15. Let f be a K -pseudo-meromorphic function in D with Stoilow factorization $f = g \circ h$ where h is a K -QC homeomorphism of D onto D and g is a non-constant meromorphic function in D . Then f is normal in D if and only if g is normal in D .

Proof of Theorem 14. Since h is K -QC, by a theorem of Mori [17] h^{-1} is also K -QC. Hersch and Pfluger [11] have shown that if h is K -QC then $\rho(h(z), h(z')) \leq \Psi_K(\rho(z, z'))$ where Ψ_K is continuous and strictly increasing and defined for all $x \geq 0$ with $\Psi_K(0) = 0$. It follows easily that h is HUC. Similarly h^{-1} is HUC and the theorem is proved.

Proof of Theorem 15. From Theorem 14 both h and h^{-1} are HUC. By Theorem 6, f is normal in D if and only if g is normal in D and the theorem is proved.

3. Preservation of Stolz domains

Let h be a homeomorphism of D onto D . If for every $e^{i\theta} \in C$ and every Stolz domain Δ at $e^{i\theta}$ the image of some terminal Stolz domain of Δ is contained in a Stolz domain, then we shall say that h weakly preserves Stolz domains.

If h weakly preserves Stolz domains then we note that h has radial limits everywhere, and that $C_A(h, \theta) = C_{\tau_\theta}(h, \theta)$ for every $\theta \in C$, where τ_θ is the radius at $e^{i\theta}$. The following result is a generalization of Lindelöf's theorem [9].

THEOREM 16. Let f be a light interior function in D with Stoilow factorization $f = g \circ h$ where h is a homeomorphism of D onto D for which both h and h^{-1} weakly preserve Stolz domains and g is a non-constant normal meromorphic function in D . If f has the point asymptotic limit c at $e^{i\theta}$, then f has the angular limit c at $e^{i\theta}$.

Before we prove the theorem we establish the following lemma.

LEMMA 3. If h is a homeomorphism of D onto D for which both h and h^{-1} weakly preserve Stolz domains, then h can be extended to a homeomorphism of $\bar{D}$ onto $\bar{D}$.

Proof. Suppose h cannot be extended to be continuous in $\bar{D}$. Then there exists a point $e^{i\theta} \in C$ such that $C(h, \theta) = [\alpha_1, \alpha_2]$, with $0 < \alpha_2 - \alpha_1 \leq 2\pi$. There exist two radii τ_1 and τ_2 terminating at $e^{i\phi_1}$ and $e^{i\phi_2}$, respectively, with $\alpha_1 < \phi_1 < \phi_2 < \alpha_2$ such that $h^{-1}(re^{i\phi_1}) \rightarrow e^{i\theta}$ and $h^{-1}(re^{i\phi_2}) \rightarrow e^{i\theta}$. Since h has only one non-tangential limit at $e^{i\theta}$ either $h^{-1}(\tau_1)$ or $h^{-1}(\tau_2)$ is a non-tangential boundary path. But then h^{-1} does not weakly preserve Stolz domains which

contradicts our hypothesis. Therefore h can be extended to be continuous in $\bar{D}$ and similarly h^{-1} can be extended to be continuous in $\bar{D}$. By considering $h \circ h^{-1}$ and $h^{-1} \circ h$ it is easy to see that h can be extended to a homeomorphism of $\bar{D}$ onto $\bar{D}$; and the lemma is proved.

Proof of Theorem 16. By Lemma 3 h can be extended to a homeomorphism of $\bar{D}$ onto $\bar{D}$. Let f have the point asymptotic limit c along an asymptotic path Γ terminating at $e^{i\theta}$. Then $h(\Gamma)$ is an asymptotic path terminating at $h(e^{i\theta})$ along which g has the asymptotic limit c . By a theorem of Lehto and Virtanen [15, Theorem 2, p. 53], g has the angular limit c at $h(e^{i\theta})$. Since h weakly preserves Stolz domains it follows easily that f has the angular limit c at $e^{i\theta}$ and the proof is complete.

The function f in Theorem 9 shows that h merely being a homeomorphism of $\bar{D}$ onto $\bar{D}$ is not sufficient for the previous theorem. By a theorem of Mori [17, Theorem 6, p. 69], if h is a K-QC homeomorphism of D onto D , then both h and h^{-1} weakly preserve Stolz domains. Thus we obtain the following result which was first proved by Väisälä [21, Theorem 8, p. 22].

COROLLARY. Let f be a normal K-pseudo-meromorphic function in D . If f has the point asymptotic limit c at $e^{i\theta}$, then f has the angular limit c at $e^{i\theta}$.

4. Preservation of Koebe arcs

Let h be a homeomorphism of D onto D . If for every Koebe sequence of arcs $\{J_n\}$ in D the sequence $\{h(J_n)\}$ is

a Koebe sequence of arcs in D , then we shall say that h preserves Koebe sequences of arcs. The following result is a generalization of Koebe's theorem.

THEOREM 17. Let f be a light interior function in D with Stoilow factorization $f = g \circ h$ where h is a homeomorphism of D onto D which preserves Koebe sequences of arcs and g is a non-constant normal meromorphic function in D . Then f possesses no Koebe limits.

Proof. Suppose $f(J_n) \rightarrow c$ along the Koebe sequence of arcs $\{J_n\}$. Then $g(h(J_n)) \rightarrow c$ along the Koebe sequence of arcs $\{h(J_n)\}$. By a theorem of Bagemihl and Seidel [4], $g \equiv c$ which contradicts our hypothesis; and the theorem is proved.

COROLLARY. Let f be a non-constant normal K -pseudo-meromorphic function in D . Then f possesses no Koebe limits.

Before we prove the corollary we establish the following lemma.

LEMMA 4. Let h be a homeomorphism of D onto D . Then h can be extended to a homeomorphism of $\bar{D}$ onto $\bar{D}$ if and only if Koebe sequences of arcs are preserved by both h and h^{-1} .

Proof. Necessity. Assume h is a homeomorphism of $\bar{D}$ onto $\bar{D}$. Let $\{J_n\}$ be a Koebe sequence of arcs in D relative to the open arc (α, β) of C . It is easy to verify that $\{h(J_n)\}$ is a Koebe sequence of arcs relative to the open arc $h((\alpha, \beta))$ of C . Thus h preserves Koebe sequences of arcs, and similarly h^{-1} preserves Koebe sequences of arcs.

Sufficiency. Assume Koebe sequences of arcs are preserved by both h and h^{-1} . Suppose h cannot be extended to be

continuous in $\bar{D}$. Then there exists a point $e^{i\theta} \in \mathbb{C}$ and two sequences $\{z_n\}$ and $\{z'_n\}$ such that $z_n \rightarrow e^{i\theta}$ and $z'_n \rightarrow e^{i\theta}$ and $h(z_n) \rightarrow e^{i\alpha}$ and $h(z'_n) \rightarrow e^{i\beta}$ with $0 < \beta - \alpha < 2\pi$. Let J_n be the Euclidean geodesic joining z_n to z'_n . Then $\{h(J_n)\}$ is a sequence of Jordan arcs in D such that for every $\epsilon > 0$,

$$h(J_n) \subset \{z: 1 - \epsilon < |z| < 1\}$$

for all but finitely many n , and the end points of $h(J_n)$ tend to $e^{i\alpha}$ and $e^{i\beta}$. Choosing a subsequence of $\{h(J_n)\}$ if necessary, we may assume that there exists a Koebe sequence of arcs $\{L_n\}$ relative to either the open arc (α, β) or the open arc $(\beta, \alpha + 2\pi)$ such that $L_n \subset h(J_n)$. But $h^{-1}(L_n) \subset J_n$ so that $\{h^{-1}(L_n)\}$ is not a Koebe sequence of arcs in D , which contradicts our hypothesis that h^{-1} preserves Koebe sequences of arcs.

Therefore h can be extended to be continuous in $\bar{D}$ and similarly h^{-1} can be extended to be continuous in $\bar{D}$. By considering $h \circ h^{-1}$ and $h^{-1} \circ h$ it is easy to see that h can be extended to a homeomorphism of $\bar{D}$ onto $\bar{D}$ and the lemma is proved.

Proof of the Corollary. Since f is a non-constant normal K-pseudo-meromorphic function in D , f has the Stoilow factorization $f = g \circ h$ where h is a K-QC homeomorphism of D onto D and g is a non-constant meromorphic function in D . By Theorem 15 g is a normal meromorphic function in D . By a theorem of Mori [17, Theorem 4, p. 67], if h is a K-QC homeomorphism of D onto D , then h can be extended to a

homeomorphism of $\bar{D}$ onto $\bar{D}$. Thus by Lemma 4 both h and h^{-1} preserve Koebe sequences of arcs. From Theorem 17, f possesses no Koebe limits and the proof is complete.

V. ASYMPTOTIC BEHAVIOR

The asymptotic behavior of a light interior function f in D with Stoilow factorization $f = g \circ h$ is closely related to the asymptotic behavior of its component factors g and h .

DEFINITION 4. Let f be a function in D . We define the set $A(f)$ as follows: $e^{i\theta} \in A(f)$ if there exists an asymptotic path of f in D with end E and $e^{i\theta} \in E$. We also define the set $A_p(f)$ as follows: $e^{i\theta} \in A_p(f)$ if there exists a point asymptotic path of f in D terminating at $e^{i\theta}$.

DEFINITION 5. Let h be a homeomorphism of D onto D . We define the set $B(h)$ as follows: $e^{i\theta} \in B(h)$ if there exists an arc asymptotic path of h in D with end E and $e^{i\theta} \in \text{int } E$, where $\text{int } E$ is the interior of E .

THEOREM 18. Let f be a light interior function in D with Stoilow factorization $f = g \circ h$ where h is a homeomorphism of D onto D and g is a non-constant meromorphic function on D with $A(g)$ dense on C . Then $A(f) \cup B(h)$ is dense in C .

Proof. Suppose $A(f)$ is not dense on C . Let $(\phi_1, \phi_2) \subset C - A(f)$ be arbitrary and let $[\theta_1, \theta_2] \subset (\phi_1, \phi_2)$, with $0 < \theta_2 - \theta_1 < 2\pi$. Let Γ_1 and Γ_2 be Jordan arcs in D at $e^{i\theta_1}$ and $e^{i\theta_2}$, respectively, with $\Gamma_1 \cap \Gamma_2 = \{0\}$. Consider the domain Δ bounded by $\Gamma_1 \cup \Gamma_2$ and the arc $[\theta_1, \theta_2]$ of C .

Then h maps Δ onto a domain R in D .

Case i. $C_{T_1}(h, \theta_1) \cap C_{T_2}(h, \theta_2) \neq \emptyset$ and

$$[\bar{R} \cap C] = C_{T_1}(h, \theta_1) \cup C_{T_2}(h, \theta_2).$$

Let $e^{i\alpha} \in C_{T_1}(h, \theta_1) \cap C_{T_2}(h, \theta_2)$. There exist sequences $\{z_n\}$ and $\{z'_n\}$ in Γ_1 and Γ_2 , respectively, such that $h(z_n) \rightarrow e^{i\alpha}$ and $h(z'_n) \rightarrow e^{i\alpha}$. Let Λ be a Jordan arc at $e^{i\alpha}$ which passes consecutively through the points

$h(z_1), h(z'_1), h(z_2), h(z'_2), \dots$. By a lemma of Collingwood and Cartwright [7, Lemma 1, p. 93], either $[\theta_1, \theta_2] \subset C_\Lambda(h^{-1}, \alpha)$ or $[\theta_2, \theta_1 + 2\pi] \subset C_\Lambda(h^{-1}, \alpha)$. Hence, either $(\theta_1, \theta_2) \subset B(h)$ or $(\theta_2, \theta_1 + 2\pi) \subset B(h)$.

Case ii. $C_{T_1}(h, \theta_1) \cap C_{T_2}(h, \theta_2) \neq \emptyset$ and

$$[\bar{R} \cap C] \supsetneq C_{T_1}(h, \theta_1) \cup C_{T_2}(h, \theta_2).$$

Then $E = [\bar{R} \cap C] - [C_{T_1}(h, \theta_1) \cup C_{T_2}(h, \theta_2)]$ is a non-empty open subarc of C . Let $e^{i\alpha} \in E$ with $e^{i\alpha} \in A(g)$. Then $e^{i\alpha}$ is in the end of an asymptotic path Λ of g . But $C(h^{-1}, \alpha) \subset [\theta_1, \theta_2]$, and hence $h^{-1}(\Lambda)$ is an asymptotic path of f whose end intersects $[\theta_1, \theta_2]$. Thus $[\theta_1, \theta_2] \cap A(f) \neq \emptyset$ in violation of our assumption.

Case iii. $C_{T_1}(h, \theta_1) \cap C_{T_2}(h, \theta_2) = \emptyset$.

Then $E = [\bar{R} \cap C] - [C_{T_1}(h, \theta_1) \cup C_{T_2}(h, \theta_2)]$ is a non-empty open subarc of C and by arguing as in Case ii we arrive at a contradiction.

Therefore, from the above considerations, if

$(\phi_1, \phi_2) \subset C - A(f)$, then $(\phi_1, \phi_2) \cap B(h) \neq \emptyset$. Since (ϕ_1, ϕ_2) was arbitrary it follows that $A(f) \cup B(h)$ is dense on C and the proof is complete.

THEOREM 19. Let f be a light interior function in D with Stoilow factorization $f = g \circ h$ where h is a homeomorphism of D onto D with $A_p(h)$ dense on D , and g is a non-constant meromorphic function in D with $A_p(g)$ dense on C . Then $A_p(f) \cup B(h)$ is dense on C .

Proof. Suppose $B(h)$ is not dense on C . Let

$(\phi_1, \phi_2) \subset C - B(h)$ be arbitrary and let $[\theta_1, \theta_2] \subset (\phi_1, \phi_2)$ with $0 < \theta_2 - \theta_1 < 2\pi$ and $e^{i\theta_1}, e^{i\theta_2} \in A_p(h)$. Let Γ_1 and Γ_2 be two asymptotic paths of h terminating at $e^{i\theta_1}$ and $e^{i\theta_2}$, respectively, with $\Gamma_1 \cap \Gamma_2 = \{0\}$. Consider the domain Δ bounded by $\Gamma_1 \cup \Gamma_2$ and the arc $[\theta_1, \theta_2]$ of C . Then h maps Δ onto a domain R in D .

We have $[\bar{R} \cap C] \neq \{e^{i\gamma}\}$, since $[\theta_1, \theta_2] \cap B(h) = \emptyset$.

Hence, $E = [\bar{R} \cap C] - [C_{\Gamma_1}(h, \theta_1) \cup C_{\Gamma_2}(h, \theta_2)]$ is a non-empty open subarc of C . Since $A_p(g)$ is dense on C there exists a point $e^{i\alpha} \in E$ and an asymptotic path Λ of g terminating at $e^{i\alpha}$. But $C(h^{-1}, \alpha) \subset [\theta_1, \theta_2]$, hence $h^{-1}(\Lambda)$ is an asymptotic path of f whose end intersects $[\theta_1, \theta_2]$. If $h^{-1}(\Lambda)$ ends in a subarc $[\beta_1, \beta_2]$ of $[\theta_1, \theta_2]$, then our assumption that $[\theta_1, \theta_2] \cap B(h) = \emptyset$ is violated. Thus, $h^{-1}(\Lambda)$ ends at a point $e^{i\theta} \in [\theta_1, \theta_2] \cap A_p(f)$.

Therefore, if $(\phi_1, \phi_2) \subset C - B(h)$, then $(\phi_1, \phi_2) \cap A_p(f) \neq \emptyset$.

Since (ϕ_1, ϕ_2) was arbitrary we have that $A_p(f) \cup B(h)$ is

dense on C and the proof is complete.

COROLLARY. Let f be a light interior function in D with Stoilow factorization $f = g \circ h$ where h is a homeomorphism of D onto D for which both h and h^{-1} are HUC, and g is a non-constant meromorphic function in D with $A_p(g)$ dense on C . Then $A_p(f) \cup B(h)$ is dense on C .

The corollary will follow immediately from the previous theorem when we establish that $A_p(h)$ is dense on C . The proof of the corollary will be complete when we establish Lemma 6.

LEMMA 5. Let h be a homeomorphism of D onto D which is HUC. Then, for any $\delta > 0$ there corresponds a constant $K > 0$ such that $\rho(h(z), h(z')) < K \rho(z, z')$ for every pair of points $z, z' \in D$ with $\rho(z, z') \geq \delta$.

Proof. Suppose that the lemma is false. Then there exist two sequences $\{z_n\}$ and $\{z'_n\}$ such that $\rho(z_n, z'_n) \geq \delta$ and $\rho(h(z_n), h(z'_n)) \geq n \rho(z_n, z'_n)$. Let $1/2N < \delta$ and let Γ_n be the non-Euclidean geodesic joining z_n to z'_n ($n = N, N+1, \dots$). Partition Γ_n by points $z_{n,j}$ ($j = 1, 2, \dots, m_n$) such that $z_n = z_{n,1}$ and $z'_n = z_{n,m_n}$ and $1/2n < \rho(z_{n,j}, z_{n,j+1}) < 1/n$ ($j = 1, 2, \dots, m_n - 1$). There exists an integer k_n ($1 \leq k_n < m_n$) such that $\rho(h(z_{n,k_n}), h(z_{n,k_n+1})) \geq n \rho(z_{n,k_n}, z_{n,k_n+1})$. Then $\{z_{n,k_n}\}$ and $\{z_{n,k_n+1}\}$ are close sequences with $\rho(h(z_{n,k_n}), h(z_{n,k_n+1})) \geq n \rho(z_{n,k_n}, z_{n,k_n+1}) > 1/2$ in violation of our hypothesis that h is HUC. Therefore, the lemma is true.

LEMMA 6. Let h be a homeomorphism of D onto D for which both h and h^{-1} are HUC. Then both h and h^{-1} have radial limits everywhere.

Proof. It will suffice to show that h has radial limits everywhere. Without loss of generality we may assume that $h(0) = 0$, for otherwise we may consider the homeomorphism

$$H(z) = \frac{h(0) - h(z)}{1 - \overline{h(0)} h(z)}$$

where both H and H^{-1} are HUC. From Lemma 5 we can find a smallest integer K for which $\rho(h(z), h(z')) < K \rho(z, z')$ and $\rho(h^{-1}(z), h^{-1}(z')) < K \rho(z, z')$ whenever $\rho(z, z') \geq 1$. Construct a sequence $\{R_n\}$ of real numbers, $0 = R_0 < R_1 < R_2 < \dots < R_n < \dots < 1$, such that $\rho(0, R_n) = n$. Let $A_n = \{z: R_n \leq |z| \leq R_{n+1}\}$. Then $h^{-1}(A_0) \subset \bigcup_{j=0}^N A_j$, for some integer N . Since $h(0) = 0$ we have $1/K \rho(0, z) < \rho(0, h(z)) < K \rho(0, z)$ for all $z \in A_n$ ($n = N+1, N+2, \dots$).

Notation convention. The subscript Kn in R_{Kn} is the integer $j = Kn$, and is not to be confused with double subscripts.

Let θ ($0 \leq \theta < 2\pi$) be fixed but arbitrary. Set $M = \max\{N+1, K^2 + 2\}$. Then

$$\rho(0, R_{Kn}) = Kn = 1/K \rho(0, R_{\frac{2}{K}n} e^{i\theta}) < \rho(0, h(R_{\frac{2}{K}n} e^{i\theta}))$$

($n = M, M+1, \dots$), and we obtain the inequality

$$(1) \quad R_{Kn} < |h(R_{\frac{2}{K}n} e^{i\theta})| \quad \text{and} \quad R_{Kn} < R_{K(n+1)} < |h(R_{\frac{2}{K}(n+1)} e^{i\theta})|$$

($n = M, M+1, \dots$).

Set $\alpha_n = \arg h(R_{\frac{2}{K}n} e^{i\theta})$ ($n = M, M+1, \dots$). Then

$$\rho(R_{Kn} e^{i\alpha_n}, R_{Kn} e^{i\alpha_{n+1}}) \leq \rho(h(R_{Kn}^2 e^{i\beta}), h(R_{K(n+1)}^2 e^{i\beta}))$$

$$< K \rho(R_{Kn}^2, R_{K(n+1)}^2), \text{ and}$$

$$(2) \quad \rho(R_{Kn} e^{i\alpha_n}, R_{Kn} e^{i\alpha_{n+1}}) < K^3$$

$$= K^2 \rho(R_{Kn - K^3}, R_{Kn - K^3})$$

(n = M, M+1, ...).

Let Γ_n be the non-Euclidean geodesic joining $R_{Kn} e^{i\alpha_n}$ to $R_{Kn} e^{i\alpha_{n+1}}$. Then from (1) and (2) it is easy to see that

$$(3) \quad \min \{|z| : z \in \Gamma_n\} \geq R_{Kn - K^3}.$$

Using (3) we obtain the inequality

$$(4) \quad \rho(R_{Kn} e^{i\alpha_n}, R_{Kn} e^{i\alpha_{n+1}}) = \int_{\Gamma_n} \frac{|dz|}{1 - |z|^2}$$

$$> \frac{R_{Kn - K^3}}{1 - R_{Kn - K^3}^2} \cdot |\alpha_{n+1} - \alpha_n|.$$

Using (4) and (2) it follows that

$$|\alpha_{n+1} - \alpha_n| \leq ((1 - R_{Kn - K^3}^2)/R_{Kn - K^3}) \rho(R_{Kn} e^{i\alpha_n}, R_{Kn} e^{i\alpha_{n+1}})$$

$$< K^2 ((1 - R_{Kn - K^3}^2) \rho(R_{Kn - K^3}, R_{Kn - K^3}))/R_{Kn - K^3}$$

$$\leq K^2 (R_{Kn - K^3} - R_{K(n-1) - K^3})/R_{Kn - K^3}$$

Therefore we obtain the inequality

$$(5) \quad \sum_{n=M}^{\infty} |\alpha_{n+1} - \alpha_n| < \sum_{n=M}^{\infty} K^2 (R_{Kn-K}^3 - R_{K(n-1)-K}^3) / R_{Kn-K}^3 \\ < K^2 (1 - R_{KM-K}^3) / R_{KM-K}^3.$$

Now $|h(R_{K^2 n}^2 e^{i\theta})| \rightarrow 1$ and from inequality (5) it follows that $\arg h(R_{K^2 n}^2 e^{i\theta}) \rightarrow \alpha$, hence $h(R_{K^2 n}^2 e^{i\theta}) \rightarrow e^{i\alpha}$. Suppose that $\lim_{r \rightarrow 1} h(re^{i\beta})$ does not exist, then there exists a sequence of points $\{r_n e^{i\beta}\}$ such that $h(r_n e^{i\beta}) \rightarrow e^{i\theta}$ with $\theta \neq \alpha$. Then for each integer n we can choose an integer j_n such that $1 \leq \rho(r_n e^{i\beta}, R_{K^{j_n}}^2 e^{i\beta}) \leq K^2$. But then for n sufficiently large

$$\rho(h(r_n e^{i\beta}), h(R_{K^{j_n}}^2 e^{i\beta})) \geq K \rho(r_n e^{i\beta}, R_{K^{j_n}}^2 e^{i\beta})$$

which contradicts our choice of K from Lemma 5. Therefore

$\lim_{r \rightarrow 1} h(re^{i\beta})$ exists for every β and the proof is complete.

The following result shows that the dense set $A_p(f) \cup B(h)$ of Theorem 19 need not have positive measure.

THEOREM 20. There exists a bounded normal light interior function f in D with Stoilow factorization $f = g \circ h$ where h is a homeomorphism of D onto D with $A_p(h)$ dense on C and g is a non-constant bounded holomorphic function in D with $A_p(g)$ dense on C , but $A_p(f) \cup B(h)$ is of measure zero.

Proof. By a theorem of Beurling and Ahlfors [19, p. 119] for each $K > 1$ there exists a K -quasi-conformal homeomorphism h of $\bar{D}$ onto $\bar{D}$ such that a certain set E of linear measure zero on C is mapped onto a set F of linear measure 2π on C .

By a theorem of Lohwater and Piranian [16, Theorem 4, p. 11], there exists a bounded holomorphic function g that has no radial limits at points of the set $S = C - F$. By a theorem of Lehto and Virtanen [15, Theorem 2, p. 53], g has no point asymptotic limits at points of S . By Theorem 15, $f = g \circ h$ is a normal light interior function. But $A_p(f) \subset E$, hence $\text{meas } A_p(f) = 0$. Since h is a homeomorphism of $\bar{D}$ onto $\bar{D}$ we have $B(h) = \emptyset$ and the proof is complete.

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