TANGENTIAL BOUNDARY BEHAVIOR IN THE UNIT DISK AND THE EXCEPTIONAL SETS OF FUNCTIONS AND THEIR FRACTIONAL INTEGRALS

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ABSTRACT

TANGENTIAL BOUNDARY BEHAVIOR IN THE UNIT DISK AND THE EXCEPTIONAL SETS OF FUNCTIONS AND THEIR FRACTIONAL INTEGRALS

By

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Let $\phi(x)$ be a non-negative, non-decreasing function in $(0,1)$ which is integrable there. Define $\Phi(n) = \int_{1-\frac{1}{n}}^{1} \phi(x)dx$. This definition is due to P. B. Kennedy [9] who uses it to generalize some results concerning the function $\phi(x) = (1-x)^{\alpha-1}$, $0 < \alpha \leq 1$, and the corresponding discrete function $\Phi(n) = n^{-\alpha}$. A similar generalization is made for the case $\phi(x) = (1-x)^{\alpha-1}$, $\alpha > 1$. We note that the factor $n^{-\alpha}$ is present in the definition of the integral of fractional order f_α as given by Weyl:

$$f(x) = \sum_{-\infty}^{\infty} a_n e^{inx}, \quad f_\alpha(x) = k_\alpha \sum n^{-\alpha} a_n e^{inx}$$

where k_α is a constant depending on α alone. We observe further that $\phi(x) = \sum_{n=0}^{\infty} \gamma_n x^n$ where

$$\gamma_n = \frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{n!} \cong n^{-\alpha} \quad \text{so that}$$

$$(*) \qquad \phi(x) \cong \sum_{n=0}^{\infty} \Phi(n)x^n.$$

We consider the class of all general ϕ satisfying the property (*), and we define the Φ-fractional integral of f as $f_\Phi(x) = \sum_{n=1}^{\infty} \Phi(n)a_n e^{inx}$. We proceed to show that certain results

for fractional integrals hold true for the generalized case. In addition, several results concerning function $f(x) = \Sigma\, c_n e^{inx}$ whose coefficients satisfy the condition $\Sigma\, n^{\alpha} |c_n|^2 < \infty$ are shown to extend to those f for which $\Sigma \frac{|c_n|^2}{\Phi(n)} < \infty$. Most results are given in terms of exceptional sets whose h-measure is zero, the general ϕ being sometimes defined in terms of h by the equation $\phi(x) = h(\frac{1}{1-x})$. Certain other classes of functions are discussed which are defined in terms of h alone. Finally we discuss the behavior of elements of several function classes which are defined in the unit disk D as the functions take on functions near the boundary, the values being restricted to the set

$$R[\tau,\theta] = \{z \in D\colon 1 - |z| \geq \tau(|e^{i\theta} - |z||)\}$$

where τ is an increasing function for which $\tau(0) = 0$.

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Joseph Thomas Matti

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TO CAROLINE

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INTRODUCTION

This paper deals basically with three principal ideas: measure functions, fractional integrals, and tangential limits. Each of these topics will be discussed in a general sense, i.e. the discussion will not be in terms of specific functions, but rather in terms of classes of functions possessing certain properties. Each of these topics deals then with a generalization of a concept which is itself a generalization of a more basic property. The measure functions $h(r)$ defined below extend the concepts of Hausdorff dimension and capacities which include Lebesgue measure as a special case. Similarly our generalized fractional integrals extend the ordinary fractional integrals which are defined by Riemann and Weyl and which reduce to ordinary integration and differentiation in the integer cases. Likewise our τ-tangential limits have as examples the usual tangential limits which in turn are generalized Stolz angles or radial limits. We now discuss each of these ideas in greater detail.

1. Throughout, let $h(r)$ be a real-valued, non-decreasing, continuous function defined for $r \geq 0$ and satisfying the conditions: $h(0) = 0$, $h(r) > 0$ if $r > 0$, $h(\infty) > 1$. For any plane set E and any $0 < \rho < \infty$ let $h_\rho^*(E) = \inf \sum_{\nu=1}^{\infty} h(r_\nu)$, the infimum taken over all countable systems of open circular disks, with radii $0 < r_\nu < \rho$, covering E. Then $h^*(E) = \lim_{\rho \to 0} h_\rho^*(E)$ is the h-measure of E.

Similarly let E be an arbitrary bounded set, let $\{C_\nu\}_1^\infty$, be a family of circles with radii $\{r_\nu\}_1^\infty$, covering E. Let $M_h(E) = \inf \sum_1^\infty h(r_\nu)$, the infimum taken over all such coverings. If in addition $h(r)/r^2$ is non-increasing we call h a measure function, though in fact the resulting M are not generally additive and hence are not measures in the sense of Carathéodory. However they do determine the same exceptional sets as a class of completely additive set functions introduced by Hausdorff; however this latter class is not well-suited for application to the theory of functions. These general h have been previously discussed by Rung [11] in the first form, by Carleson [4] in the latter. Similar set functions are defined by Aronszahn-Smith [1].

Next let $\mu(x)$ be a distribution concentrated on $E \subset [0,2\pi]$ in the sense that $\int_0^{2\pi} d\mu = \int_E d\mu$ (= 1 say). Then $V_h = \sup_{x\in[0,2\pi]} \int_0^{2\pi} h(\frac{1}{|x-t|})d\mu(t)$ is the h-potential of E. If V_h is finite, then E is said to be of positive h-capacity, otherwise E is said to be of zero h-capacity. Equivalently, let μ be a distribution spread on the unit circle $C = \{|z| = 1\}$, concentrated on $E \subset C$, and let $V = \int_0^{2\pi} h\left(\frac{1}{|e^{it}-re^{ix}|}\right)d\mu(t)$. If there exists a μ such that V is bounded uniformly in x as $r \to 1$, then E is of positive h-capacity, it is of zero h-capacity otherwise.

In the case $h(r) = r^\alpha$, $0 < \alpha < 1$, these potentials have been discussed by duPlessis [5] and Salem-Zygmund [12]. Frostman [6] and Carleson [4, p. 15] give an account of the relation between these capacities and the earlier definitions.

2. Let $f(x)$ be integrable in an interval (a,b). Let $F_1(x)$ denote the integral of $f(t)$ over (a,x), $F_\alpha(x)$ the integral of $F_{\alpha-1}(t)$ over (a,x), $\alpha = 2,3,\dots$. By induction it can be shown that

$$(*) \qquad F_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt,\ x \in [a,b]$$

and $\Gamma(\alpha) = (\alpha-1)!$

Extend $\Gamma(\alpha)$ to all values $\alpha > 0$ by letting it be the Euler Gamma function. Then define the integral of fractional order α of f to be the function F_α defined by (*) with the noted change. We thus have a definition of a fractional integral, this one defined by Riemann and Liouville, which coincides in the case where α is a positive integer with the αth integral of f. The fact that F_α is not necessarily periodic even if f is makes this definition unsatisfactory in the theory of periodic functions. This gives reason to consider a second definition of fractional integral as proposed by Weyl, a definition better suited to trigonometrical series: Let f be periodic and integrable,

$$f \sim \sum_{-\infty}^{\infty} c_n e^{2\pi inx},\ c_0 = 0.$$

Define $f_\alpha(x) = \sum_{-\infty}^{\infty} c_n \frac{e^{2\pi inx}}{(2\pi in)^\alpha} = \int_0^1 f(t)\, \Psi_\alpha(x-t)dt$ where $\Psi_\alpha(x)$ has the complex Fourier coefficients $\gamma_n^{(\alpha)} = (2\pi in)^{-\alpha}$, $\gamma_0 = 0$. A more detailed discussion of these concepts is given by Zygmund [14, p. 222]. Also duPlessis [5] notes that for $f \in L^q$ that the Riemann-Liouville version and the Weyl version of the fractional integral differ by only a bounded function.

Derivatives of fractional order may be defined in a like fashion; we use the Cauchy Integral Formula in the first case, while in the second the method is quite the same as with fractional integration. In particular, we note that $f^{\alpha}(x) = K_{\alpha} \sum n^{\alpha} c_n e^{2\pi inx}$ denotes the fractional derivative of order α, $\alpha > 0$, where K_{α} is a constant depending on α alone.

Heywood [8] obtains certain results concerning the function $\phi(x) = (1-x)^{-\gamma}$, $\gamma < 1$. P.B. Kennedy [9] has abstracted basic properties of this function to obtain generalizations of Heywood's results. Letting $\phi(x)$ be a non-negative and non-decreasing function in $(0,1)$, with $\phi(x) \in L(0,1)$, we have the case $0 \le \gamma < 1$ of the Heywood function. If there exists an integer $p \ge 1$ such that $\phi'(x), \phi''(x), \ldots, \phi^{(p-1)}(x)$ are all absolutely continuous for $0 \le x \le 1$ and vanish at $x = 1$, and are such that $\phi^{(p)}(x)$ has constant sign, and $|\phi^{(p)}(x)|$ is non-decreasing in $(0,1)$ wherever $\phi^{(p)}(x)$ exists, then we have properties satisfied by the Heywood function in the case where $\gamma < 0$. If ϕ satisfies the first stated conditions we then define

$$\Phi(n) = \int_{1-\frac{1}{n}}^{1} \phi(x)dx .$$

In the second case we define

$$\Phi(n) = n^{-p}|\phi^{(p-1)}(1 - \frac{1}{n})|.$$

Here we note (see Salem-Zygmund [12]) that

$$[1 - re^{i(x-t)}]^{-\alpha} = 1 + \alpha re^{i(x-t)} + \ldots + \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!} r^n e^{ni(x-t)} + \ldots$$

in which we note that the coefficients

$$\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!} \cong \frac{1}{\Gamma(\alpha)} n^{\alpha-1} \quad \text{by Stirling's formula.}$$

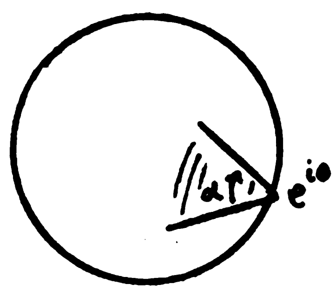
It is the observation that in the case $\phi(x) = (1-x)^{-\alpha}$ we have $\Phi(n) = n^{\alpha-1}$ (ignoring constants which depend on α alone) which leads us to consider the class of functions ϕ for which the following holds: Let $\phi(x) = \Sigma\, a_k x^k$ and $a_k = A[\Phi(k) + O(j(k))]$ where $j(k) \to 0$ and A is a constant depending on ϕ alone. Consequently $\phi(x) \cong A\, \Sigma\, \Phi(n)\, x^n$. This leads us to make the following definition: Let $f(e^{ix}) = \Sigma\, c_k\, e^{ikx}$, let $\phi(x)$ possess the properties described above; then

$$f_{\Phi}(x) = \sum_{k=-\infty}^{\infty} \Phi(k) c_k e^{ikx} = C \int_{-\pi}^{\pi} \phi(e^{i(x-t)}) f(t)\,dt$$

is called the ϕ-fractional integral of f. Here C is a constant depending only on ϕ, $\Phi(-n)$ is taken to be $-\Phi(n)$ and $\phi(e^{i(x-t)})$ is the natural extension of ϕ to complex values. The case $\phi(x) = (1-x)^{\alpha-1}$, $0 < \alpha < 1$, gives the ordinary fractional integral, while positive powers of $(1-x)$ give the ordinary fractional derivative. Fractional integrals and derivatives have been discussed by duPlessis [5], Kinney [10] and Rung [11].

In some cases, it is more convenient to speak of a Ψ/Φ-fractional integral defined as $f_{\Psi/\Phi}(e^{ix}) = \Sigma \frac{\Psi(n)}{\Phi(n)} c_n e^{inx}$ where ψ fits the description of the ϕ discussed above. Also, in the event that the function ψ does not enter into the discussion, we may choose to define $f_{\Psi}(e^{ix}) = \Sigma\, \Psi(k) c_k e^{ikx}$ without mention of ψ, allowing even the possibility that for a given $\Psi(n)$ no suitable ψ exists. This would permit for example the definition of $f'(x)$ by $\Sigma\, nc_n e^{inx}$ even though the natural choice $\psi(x) = (1-x)^{-2}$ fails to be $L(0,1)$.

3. The idea of examining the behaviour of a function defined in the unit disk D as it approaches the boundary is at least as old as Abel. His concept of summability and the theorem of Fatou are probably the best-known examples of radial limits. A condition which is slightly more general than the radial limit at a point $e^{i\theta}$ on the unit circle is the concept of a Stolz angle, $S(e^{i\theta},\alpha)$, at $e^{i\theta}$ and of opening α. In this case the function is permitted to assume non-radial values as it nears the boundary, but only values interior to $S(e^{i\theta},\alpha)$.



$S(e^{i\theta},\alpha)$

Now let $\tau(r)$ be a real-valued function, increasing with r; and with $\tau(0) = 0$. Define the set $R[\tau,\nu]$ as follows: $R[\tau,\nu] = \{z \in D\colon 1 - |z| \ge \tau(|z - e^{i\nu}|)\}$. If $z_j \to e^{i\nu}$, $z_j \in R[\tau,\nu]$, $j = 1,2,\ldots$ implies that $\lim_{j\to\infty} f(z_j)$ exists, then we say that $\tau\text{-}\lim_{z\to e^{i\nu}} f(z) = \lim_{j\to\infty} f(z_j)$ or that the τ-tangential limit exists at $e^{i\nu}$. More briefly we say that the τ-limit exists at $e^{i\nu}$ if $\lim_{z\to e^{i\nu}} f(z)$ exists whenever z is confined to $R[\tau,\nu]$.

In the case where $\tau(r) = kr$, $R[\tau,\nu]$ is just the usual Stolz angle. If $\tau(r) = r^{\gamma}$, $\gamma > 1$, then $R[\tau,\nu]$ is the ordinary tangential limit, a parabolic approach whose order of tangency is $\gamma - 1$. Tangential limits of this latter type have been previously discussed by Kinney [10], Cargo [3] and in a sense by Rung [11].

In Chapter I we discuss some properties concerning h-measure and the associated exceptional sets. Our first result corresponds to the following theorem of Salem-Zygmund [12] concerning the points of divergence of a Fourier series whose coefficients satisfy a certain order of convergence:

<u>Theorem</u>. If $0 < \alpha \leq 1$ and if the series $\sum n^{\alpha}(a_n^2 + b_n^2)$ converges, then the set of points of divergence of the Fourier series $\frac{a_o}{2} + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx)$ is a set of $(1-\alpha)$-capacity zero if $\alpha \neq 1$ and of logarithmic capacity zero if $\alpha = 1$. In the course of this proof, necessary and sufficient conditions are established for a set E to have positive $(1-\alpha)$-capacity. We extend these results from capacities to the general h-measures. By choosing $\phi(x) = h(\frac{1}{1-x})$ and $\Phi(n) = \int_{1-\frac{1}{n}}^{1} \phi(x)dx$ we have the result stated

above, with the requirement that $\Sigma(a_n^2 + b_n^2)\cdot 1/\Phi(n) < \infty$, by taking $h(r) = r^{1-\alpha}$, $\phi(r) = (1-r)^{\alpha-1}$, $\Phi(n) = n^{-\alpha}$. This idea of requiring a level of convergence beyond mere square summability in order to obtain broader results is also discussed by Kinney [10] and Carleson [4].

At this point we offer some examples which show the signifigance of the presence of the factor $1/\Phi(n)$. Consider the following example:

Let $\phi(t) = \Sigma\, a_n \sin b_n t$, $\phi'(t) = \Sigma\, a_n b_n \cos b_n t$. Say, for example, $a_n = 2^{-n}$, $b_n = 3^n$, then $\Sigma\, a_n < \infty$ so that $\phi(t)$ converges for every t, while $\phi'(t)$ converges almost nowhere. Geometrically we are considering the behavior near the boundary of D of the expression $g(t) = \Sigma\, a_n r^n e^{int}$. Then unless the coefficients converge rather rapidly, the function $g(t)$ may behave badly as $r \to 1$ on a rather large set of t, giving rise to such undesirable properties as the image of the function taken along a radius failing to have a tangent line. Basically then the rate at which a series converges determines its smoothness near the boundary; our requirement that the series converge with the factor $1/\Phi(n)$ present is then equivalent to the function not behaving too badly near the boundary. For the example $\phi(t)$ cited above we observe that the order of convergence determines a path defined in terms of a suitable Lipschitz condition, within which the oscillation of ϕ is contained. For example, in the case $a_n = 2^{-n}$, $b_n = 3^n$, the function ϕ oscillates within a path p determined by a function which satisfies a Lip $\dfrac{\log 3}{\log 2}$ condition, while it will pass outside any path p' interior to p.

We note here that the boundary results obtained may be quite good metrically yet quite bad topologically, i.e. the points of discontinuity may form a too large set. To illustrate we consider the following example of a lacunary series which is due to M. Weiss [13]:

Suppose $\Sigma|a_n| = \infty$, $\Sigma\, a_n^2 < \infty$ (or even $\Sigma\, a_n^2\, n^\alpha < \infty$), $b_n > q\, b_{n-1}$ where $q \geq 2$. Let $\phi(z) = \Sigma\, a_n z^{b_n}$. Then $\phi(z)$ converges a.e. on the unit circle C. However, if γ denotes an arbitrary complex number, and α,β are any two real numbers, then for $e^{i\theta} \in C$, $\alpha \leq \theta \leq \beta$, $\phi(z) = \gamma$ for non-denumerably many θ in $[\alpha,\beta]$, i.e. within any $[\alpha,\beta]$, every number γ is attained by ϕ on a dense subset of $[\alpha,\beta]$. Beyer [2] obtains a similar result for Hausdorff dimension and Rademacher functions.

Following the results which corresponds to the stated Salem-Zygmund results, we move to a discussion of equi-summability and equi-convergence of series and integrals of certain functions satisfying Lipschitz conditions. Also we discuss the class T_α defined by Carleson [4]: let $|\omega(z)| \leq 1$, then

$$\omega \in T_\alpha \Leftrightarrow k \int_0^1 \frac{dr}{(1-r)^\alpha} \int_0^r rdr \int_0^{2\pi} |\omega'(re^{i\theta})|^2 d\theta < \infty \Leftrightarrow$$

$$\Leftrightarrow \Sigma |a_n|^2 n^\alpha < \infty \quad \text{for} \quad \omega = \Sigma\, a_n z^n.$$

In each case, we obtain an h analogue of the capacity results.

As an example of a function in his class T_α, Carleson offers the class B_α which is that subclass of the class of Blaschke products which satisfies the convergence condition $\Sigma(1 - |a_n|)^{1-\alpha} < \infty$. He proceeds to generalize a result of Frostman [7] concerning radial limits, giving a necessary and sufficient condition, in terms of Borel series, under which the $(1-\alpha)$-capacity of the set where the radial limits fail to exist is zero. We modify this result slightly (Carleson's theorem is already expressed in terms of general h-measure) to make it adaptable to a general tangential limit argument which we use later.

We conclude the section with a result on our generalized fractional integral. duPlessis has shown that if $f \in L^q$, then the fractional integral $f_{\alpha/q}$, $0 < \alpha < 1$, is finite except on a set of zero β-capacity, where $\beta > 1-\alpha$ if $q > 2$ and $\beta = 1-\alpha$ if $1 \le q \le 2$. Letting $\phi(r) = h(\frac{1}{1-r})$ where h is a measure function, we show this result holds true for a generalized fractional integral having an exceptional set given in terms of h-measure.

In Chapter II we are mainly concerned with tangential limits of certain classes of functions defined in the unit circle. We first discuss the class B_h of all Blaschke products which satisfy the condition that $\Sigma\, h(1 - |a_\nu|) < \infty$. For the previously mentioned example B_α of Frostman (he uses $h(r) = r^\alpha$, $\Sigma(1 - |a_n|)^\alpha < \infty$), there is shown to hold a local property, namely that for a given θ, the convergence of the series $\Sigma \dfrac{1 - |a_n|}{|e^{i\theta}-a_n|}$ implies the existence of a radial limit there. The theorem is extended to ordinary tangential limits by Cargo [3] who shows that for $\gamma \geq 1$, the convergence of $\Sigma \dfrac{1 - |a_n|}{|e^{i\theta}-a_n|^\gamma}$ implies that at θ there is a tangential approach whose order of tangency is $\gamma - 1$. We make the natural extension of this local result to the series $\Sigma \dfrac{1 - |a_n|}{\tau(|e^{i\theta}-a_n|)}$ and τ-tangential limits. Frostman shows that his local condition holds (and hence radial limits exist) except possibly on a set of α-capacity zero $0 < \alpha < 1$; Cargo's result is expressed in terms of $\alpha\gamma$-capacity where it is assumed $\alpha\gamma < 1$. Similarly our result is seen to hold off a set whose $h\circ\tau$-measure is zero (where $h\circ\tau$ is assumed to be a measure function). Finally this same result is also shown to be true in a different way using the previously stated result of Carleson concerning Borel series.

The class $S_\alpha = \{f(z) = \Sigma c_n z^n : \Sigma n^\alpha |c_n|^2 < \infty\}$ has been discussed by Kinney [10] and is noted to be similar in nature to the Fourier series discussed by Salem-Zygmund [12] and to the class T_α discussed by Carleson [4]. Kinney establishes the existence of tangential limits for all functions in this class, as well as for their fractional integrals and fractional derivatives of certain orders. Since, as noted by Carleson, $B_\alpha \subset T_\alpha$, this includes the result of Cargo stated above. We here show that these results can be extended to the class $S_\Phi = \{f(z) = \Sigma c_n z^n : \Sigma \dfrac{|c_n|^2}{\Phi^2(n)} < \infty\}$ where $\Phi(n)$ is defined in terms of ϕ satisfying the earlier properties

(p. 4). Then for certain τ, τ-tangential limits exist except possibly on a set of h-measure zero for each $f \in S_\Phi$ along with its generalized integral $f_{\Psi/\Phi}$.

Our final results are related to the function class A^s which is discussed by Rung [11]. A function f belongs to the class A^s, $s > -1$, if the following integral is finite: $\int_D\int |f'(z)|^2(1 - |z|)^s dxdy < \infty$. This condition is shown to be equivalent, where $f(z) = \Sigma a_n z^n$, to the convergence of the series $\Sigma \frac{|a_n|^2}{n^{s-1}}$. We define the class $A(\Psi,\phi)$ according to the convergence of the integral $\int_D\int |f_\Psi(z)|^2 \phi(|z|)dxdy < \infty$, where Ψ, ϕ have the properties given earlier. The convergence of the integral is shown to be equivalent to the convergence of the series $\Sigma \Psi^2(n)\Phi(n)|a_n|^2$. These equivalences and some related properties are then used to prove some results on tangential limits.

CHAPTER I

Some Properties Concerning h-Measure and the Associated Exceptional Sets

1. On the convergence of certain Fourier Series

Salem and Zygmund [12] established the following concerning the points of divergence of the Fourier series of a particular class:

Theorem: If $0 < \alpha \leq 1$ and if the series $\Sigma\, n^{\alpha}(a_n^2 + b_n^2)$ converges, then the set of points of divergence of the Fourier series

$$\frac{a_o}{2} + \Sigma_1^{\infty}\, a_n \cos nx + b_n \sin nx$$

is of $(1-\alpha)$-capacity zero if $\alpha \neq 1$ and of logarithmic capacity zero if $\alpha = 1$.

Similar classes have been discussed by Kinney [10] and Carleson [4].

a. Necessary and sufficient conditions for convergence.

Letting $h(x)$ denote a measure function having the property that $h(\frac{1}{1-x})$ is non-negative, increasing and integrable in $(0,1)$, we define $\phi(x) = h(\frac{1}{1-x})$ and

$$\Phi(n) = \int_{1-\frac{1}{n}}^{1} \phi(x)dx.$$

We assume also that $h(ab) \cong h(a)h(b)$ and $\phi(x) \cong \Sigma\, \Phi(n)x^n$. Then we can show that the following extension holds:

Theorem 1

If $\Sigma \frac{1}{\Phi(n)} (a_n^2 + b_n^2)$ converges, then the Fourier series $a_o/2 + \Sigma_1^\infty a_n \cos nx + b_n \sin nx$ converges except possibly on a set whose h-measure is zero.

Choosing $h(r) = r^{1-\alpha}$ we have the Salem-Zygmund result noted above in the non-lograithmic case. We first must determine an equivalent condition in terms of Fourier series to a set having positive h-measure.

Lemma 1. In order that a set E have positive h-measure it is necessary and sufficient that there exist a positive distribution μ concentrated on E such that if the Fourier-Stieltjes series of $d\mu(x)$ is

$$d\mu \sim \frac{1}{2\pi} + \Sigma\, a_n \cos nx + b_n \sin nx$$

then the series $\Sigma(a_n \cos nx + b_n \sin nx)\Phi(n)$ is the Fourier series of a bounded function.

Proof.

Suppose E has positive h-measure. Then there exists a positive distribution μ with $d\mu \sim \frac{1}{2\pi} + \Sigma\, a_n \cos nx + b_n \sin nx$ concentrated on E such that

$$V = \int_o^{2\pi} h\left(\frac{1}{|e^{it} - re^{ix}|}\right) d\mu(t) = \int_o^{2\pi} h\left(\frac{1}{|1-re^{i(x-t)}|}\right) d\mu(t)$$

is bounded uniformly in x as $r \to 1$. By our assumption $\phi(re^{i\theta}) \cong \Sigma\, \Phi(n) r^n e^{in\theta}$. Then let

$$U = \int_o^{2\pi} h\left(\frac{1}{1-re^{i(x-t)}}\right) d\mu(t) = \int_o^{2\pi} \phi(re^{i(x-t)})\, d\mu(t)$$

where we are using the natural extensions of our functions to complex values.

$$U = \int \Sigma\, \Phi(n) r^n e^{in(x-t)} d\mu(t) =$$

$$\int (\Sigma\, \Phi(n) r^n [\cos n(x-t) + i \sin n(x-t)]) (\frac{1}{2\pi} + \Sigma\, a_n \cos nt + b_n \sin nt) dt$$

which has real part

$$\Sigma\, \Phi(n) r^n (a_n \cos nx + b_n \sin nx)$$

and this is uniformly bounded in x as $r \to 1$ due to the finiteness of V.

Conversely the sufficiency is established by noting that the above argument can be used to show the real part of U is bounded as $r \to 1^-$ for some positive distribution μ concentrated on E. For every $r < 1$ and every choice of argument β,

$$1 - re^{i\beta} = |1 - re^{i\beta}| \cdot e^{i\theta} \quad \text{with} \quad -\pi/2 < \theta < \pi/2.$$

Then taking real parts

$$\mathrm{Re}[\phi(1 - re^{i\beta})] = \mathrm{Re}[\phi(|1 - re^{i\beta}| \cdot e^{i\theta}]$$

where the right hand expression is bounded by the integrability of ϕ. It follows that the boundedness of the real part of U implies the boundedness of V. This completes the proof of Lemma 1.

At this point we establish the following relation between the functions ϕ and Φ: $\frac{1}{x}\Phi(\frac{1}{x}) \cong \phi(1-x)$ as $x \to 0$. For extending Φ from the discrete case we find

$$\frac{1}{x}\Phi(\frac{1}{x}) = \frac{1}{x}\int_{1-x}^{1} \phi(x)dx = \phi(\xi) \quad \text{where} \quad 1 - x < \xi < 1,$$

and as $x \to 0$, $\xi \to 1-x$ and $\phi(\xi) \to \phi(1-x)$.

Before proving the main theorem we need the following lemma:

Lemma 2. Let $H(x) = \Sigma\, \Phi(k)\cos kx$, $H_n(x) = \Sigma_1^n\, \Phi(k)\cos kx$. Then $|H_n(x)| < A\cdot H(x) + B$ where A, B are positive constants depending on Φ alone.

Proof.

Without loss of generality we assume that $0 < x \le \pi$.

Let x be arbitrary but fixed.

If $n \le \frac{1}{x}$, choose C_x so that $C_x \Phi(1/x) \ge \Phi(1)$.

Then $\sum_{k=1}^{n} \Phi(k)\cos kx \le \sum_{k=1}^{n} \Phi(k) \le n\, \Phi(1) \le \frac{1}{x}\cdot C_x \cdot \Phi(1/x) = 0(\frac{1}{x}\, \Phi(1/x))$

If $n > \frac{1}{x}$, let $S_1 + S_2 = \sum_{1}^{[1/x]} + \sum_{[1/x]+1}^{n} \Phi(k)\cos kx$ and let

$m = [1/x] + 1$

S_1 is covered by the previous case. Apply Abel's formula to S_2 with $u_k = \Phi(k)$, $V_k = \cos kx$ and note that $\Sigma \cos kx = 0(1/x)$ (See Salem-Zygmund [12, p. 27] or Zygmund [14, p. 2])

$$S_2 = \sum_{k=m}^{n-1} [\Phi(k) - \Phi(k+1)]T_k + \Phi(n)T_n$$

where $T_k = \Sigma_m^k \cos kx = 0(1/x)$

$$S_2 = [\Phi(m) - \Phi(n)]0(1/x) + \Phi(n)\cdot 0(1/x) = 0(1/x\ \Phi(1/x))$$

since $m,n > 1/x$ and hence $\Phi(m) \le \Phi(1/x)$ and $\Phi(n) \le \Phi(1/x)$.

We observe that as $x \to 0$, and for proper choice of C_1, that

$$C_1 + H(x) \cong A_2\, \phi(1-x).$$

$$\frac{1}{x}\, \Phi(\frac{1}{x}) \sim \phi(1-x) = h(1/x)$$

$$h(\frac{i}{x}) \cong h(\frac{1}{1-e^{ix}}) = \phi(e^{ix}) = \Sigma\, \Phi(n)e^{inx}$$

$$H(x) = \mathrm{Re}\, \Sigma\, \Phi(n)e^{inx}$$

$$h(i/x) \cong h(i)h(\frac{1}{x}) \sim \phi(1-x)$$

Then since $\frac{1}{x}\Phi(1/x) \cong \phi(1-x)$

$$|H_n(x)| < C_2 \frac{1}{x}\Phi(1/x) \cong C_2\ \phi(1-x) \cong C_2[C_1 + H(x)]$$

so that $|H_n(x)| < A\cdot H(x) + B$.

This proves the lemma and we now proceed to prove the main theorem.

Let S denote a Fourier series $\frac{a_o}{2} + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$ such that $\sum_1^\infty \frac{a_n^2 + b_n^2}{\Phi(n)} < \infty$. Then we wish to show that the set E of points of divergence of S has h-measure zero. For suppose E has positive h-measure. Then there exists a positive distribution μ, concentrated on E, such that if $d\mu \sim \frac{1}{2\pi} + \sum_1^\infty (\alpha_n \cos nx + \beta_n \sin nx)$, then the series $\Sigma(\alpha_n \cos nx + \beta_n \sin nx)\Phi(n)$ is the Fourier series of a bounded function. Choose $\omega(n)$, a positive monotonic function increasing infinitely with n and such that $\Sigma \frac{\omega(n)}{\Phi(n)} (a_n^2 + b_n^2)$ converges. Let $A_n = a_n(\omega(n))^{\frac{1}{2}}$, $B_n = b_n(\omega(n))^{\frac{1}{2}}$. If $x \in E$ then the series $S_n(x) = \sum_1^n (A_k \cos kx + B_k \sin kx)$ is unbounded, otherwise S would converge for an element of E. Let $n(x) \leq n$ denote a positive integer valued measurable function of x. By a well-known argument (see Zygmund [14, p. 253]) the integral $\int_o^{2\pi} S_{n(x)}(x)d\mu$ can be made to increase in absolute value with n for a suitable choice of $n(x)$ for each n. We now show that $\int_o^{2\pi} S_{n(x)}(x)dx$ is bounded, this contradiction proving the theorem.

Since $\Sigma \frac{1}{\Phi(n)} (A_n^2 + B_n^2) < \infty$, then $\Sigma_1^\infty (A_n \cos nx + B_n \sin nx) \cdot \frac{1}{\Phi^{\frac{1}{2}}(n)}$ is the Fourier series of a function $F \in L^2$.

$$A_k \cos kx + B_k \sin kx = \frac{1}{\pi}\int_o^{2\pi} \Phi^{\frac{1}{2}}(k)F(t)\cos kt \cos kx\, dt +$$
$$+ \frac{1}{\pi}\int_o^{2\pi} \Phi^{\frac{1}{2}}(k)F(t)\sin kt \sin kx\, dt$$

$$= \frac{1}{\pi}\int_o^{2\pi} \Phi^{\frac{1}{2}}(k)F(t)\cos\ k(t-x)dt$$

Let $G_n(x) = \Sigma\, \Phi^{\frac{1}{2}}(k)\ \cos\ kx$

$$S_{n(x)}(x) = \sum_1^{n(x)} (A_k\cos\ kx + B_k\sin\ kx) = \frac{1}{\pi}\int_o^{2\pi} F(t)G_{n(x)}(t-x)dt$$

$$I = \int_o^{2\pi} S_{n(x)}(x)d\mu(x) = \frac{1}{\pi}\int_o^{2\pi}\int_o^{2\pi} F(t)G_{n(x)}(t-x)d\mu(x)dt$$

By Schwarz's inequality:

$$\pi^2 I^2 < \int_o^{2\pi} F^2(t)dt\int_o^{2\pi}\int_o^{2\pi} G_{n(x)}(t-x)G_{n(y)}(t-y)d\mu(x)d\mu(y)$$

Then $\int_o^{2\pi} G_{n(x)}(t-x)G_{n(y)}(t-y)dt = \pi\ H_{n(x,y)}(x-y)$

where $n(x,y) = \min[n(x),n(y)]$, $H_n(x) = \Sigma_1^n\ \Phi(k)\cos\ kx$

Let $A = \int_o^{2\pi} F^2(t)dt$. This gives

$$\pi\ I^2 < A\int_o^{2\pi}\int_o^{2\pi}|H_{n(x,y)}(x-y)|d\mu(x)d\mu(y)$$

$$< A\int_o^{2\pi}\int_o^{2\pi}\{|H_{n(x)}(x-y)| + |H_{n(y)}(x-y)|\}d\mu(x)d\mu(y)$$

$= 2A\int_o^{2\pi}\int_o^{2\pi}|H_{n(y)}(x-y)|d\mu(x)d\mu(y)$. Then since $H(x) = \sum_1^{\infty} \Phi(k)\cos\ kx$,

$$\int_o^{2\pi}|H_{n(y)}(x-y)|d\mu(x) \le A_1\int_o^{2\pi} H(x-y)d\mu(x) + B$$

The Fourier series of $\frac{1}{\pi}\int_o^{2\pi} H(x-y)d\mu(x)$ is $\sum_1^{\infty} \Phi(n)[\alpha_n\cos\ ny + \beta_n\sin\ ny]$ which is by assumption the Fourier series of a bounded function. This demonstrates the boundedness of I, proving the theorem.

b. Analogue of the class T_α

A further equivalence concerning the convergence of the series $\Sigma|a_n|^2 n^\alpha$ is given by Carleson ([4]) who shows that $T_\alpha(\omega)$ $\sum_1^{\infty} |a_n|^2 n^\alpha$ where $|\omega(z)| \le 1$ and $\omega(z) = \Sigma\ a_n z^n$, $|z| < 1$ and we say that $\omega \in T_\alpha$ if the integral

$$T_\alpha(\omega) = k(\omega)\int_0^1 \frac{dr}{(1-r)^\alpha} \int_0^r r\, dr \int_0^{2\pi} |\omega'(re^{i\theta})|^2 d\theta \quad \text{is finite,}$$

where $k(\omega)$ denotes a certain constant $\frac{1}{4} \leq k(\omega) \leq 1$. (By $f \quad g$ is meant that there exist constants m and M such that $mf \leq g \leq Mf$, so that f,g are bounded together). We show here that under the assumption that the limit of $\frac{\Phi(n)\Psi^2(n)}{n} \int_0^1 \frac{r^{2n}}{1-r} h(1-r)dr$ exists as $n \to \infty$, we have the following:

<u>Theorem</u> 2. Let $\omega \in T_{h,\Psi} \equiv \{\omega\colon k(\omega)\int_0^1 \frac{h(1-r)dr}{1-r}\int_0^r rdr\int_0^{2\pi}|\omega_\Psi(re^{i\theta})|^2 d\theta < \infty\}$ where $\omega(z)$ is defined as above. Then $T_{h,\Psi}(\omega) \quad \Sigma|a_n|^2 \cdot 1/\Phi(n)$.

Proof.

$$T_{h,\Psi}(\omega) = K\int_0^1 \frac{h(1-r)}{1-r}\, dr \int_0^r rdr \int_0^{2\pi} |\omega_\Psi(re^{i\theta})|\, d\theta$$

$$= 2\pi K \int_0^1 \frac{h(1-r)}{1-r}\, dr \int_0^r \Sigma|a_n|^2 \Psi^2(n) r^{2n-1} dr$$

$$= \pi K \int_0^1 \frac{h(1-r)}{1-r} \Sigma|a_n|^2 \frac{\Psi^2(n)}{n}\, r^{2n}\, dr$$

$$= \pi K\, \Sigma|a_n|^2 \cdot \frac{1}{\Phi(n)} \quad \frac{\Phi(n)\Psi^2(n)}{n} \int_0^1 r^{2n} \frac{h(1-r)}{1-r}\, dr$$

which with our assumption establishes the result. The case $\Psi(n) = n$, $\Phi(n) = n^{-\alpha}$ is the class discussed by Carleson.

A somewhat different argument is used later (see II, § 3) on a class which is similar to the class

$$T(\Psi,\phi) = \{\omega\colon \int_0^1 \phi(t)dt \int_0^t r\, dr \int_0^{2\pi} |\omega_\Psi(re^{i\theta})|^2 d\theta < \infty\}.$$

Here we choose $\phi(x) = h(\frac{1}{1-x})$ to be increasing, non-negative and $L(0,1)$. Using a theorem due to Kennedy [9] we can show that $\omega \in T(\Psi,\phi)$ is equivalent to the convergence of the series $\Sigma|a_n|^2 \frac{\Psi^2(n)}{n} \Phi(n)$. The later result is true for a more general ϕ, and the details being the same, we defer the proof until that point.

Carleson [4] shows that the convergence of the series $\Sigma(a_n^2 + b_n^2)n^\alpha$ is equivalent to a certain continuity property of the function represented by the trigonometric series. We obtain a similar result for $\Sigma(a_n^2 + b_n^2)\frac{1}{\Phi(n)}$ under the assumptions that $\int_0^\infty \frac{h(r)}{r^2} \sin^2 r \, dr$ exists finitely and $h(\alpha r) \cong h(\alpha)h(r)$. Carleson treats the case $h(t) = t^{1-\alpha}(\log \frac{1}{t})^{-\beta}$, $0 < \alpha < 1$, $\beta > 0$, of the following theorem.

Theorem 3. Let $g(x)$ be a real or complex-valued function defined by $g(x) \sim \frac{a_0}{2} + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$. Then

$$S_h(g) \equiv \int_0^{\frac{1}{2}} \int_0^{2\pi} |g(x+t) - g(x-t)|^2 \frac{h(t)}{t^2} dx \, dt$$

$$= \Sigma(|a_n|^2 + |b_n|^2)A_n$$

where A_n is of the order $1/\Phi(n)$.

Proof.

According to Parseval's relation

$$S_h(g) = 4\pi \, \Sigma(|a_n|^2 + |b_n|^2)\int_0^{\frac{1}{2}} \frac{\sin^2 nt}{t^2} h(t)dt \quad \text{and}$$

$\int_0^{\frac{1}{2}} \frac{\sin^2 nt}{t^2} h(t)dt \cong \frac{n}{h(n)} \int_0^{n/2} \frac{\sin^2 x \, h(x)}{x^2} dx$. The theorem follows by taking limits and noting $\frac{n}{h(n)} \cong \frac{1}{\Phi(n)}$.

c. Some conditions for equiconvergence and equisummability of certain integrals and related series.

A theorem similar to this last result and concerning the equi-summability of the series conjugate to the Fourier series and the corresponding integral may be obtained in the case where $f(x)$ has period 2π and is of a certain Lipschitz class (see Salem-

Zygmund [12] for the case $h(r) = r^{1-\alpha}$).

Theorem 4. Let $\frac{a_o}{2} + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$ be the Fourier series of a continuous function $f(x)$ of period 2π and belonging to lip η, where $\eta(t) = \frac{t}{h(t)}$. Then the difference

$$K_\eta \int_{1-r}^\infty \frac{f(x+t) - f(x-t)}{t^2} h(t)dt - \sum_1^\infty (a_n \sin nx - b_n \cos nx) \frac{r^n}{\Phi(n)}$$

tends uniformly to 0 in x, as $r \to 1^-$. Similarly the difference is bounded uniformly in x if $f \in \text{Lip}\ \eta$.

We shall assume the following properties concerning h: that $\int \frac{1}{th(t)} dt \cong \frac{1}{h(t)}$, that $\int_o^\infty \frac{\sin t\, h(t)}{t^2} dt$ exists finitely and that $\int \frac{h(t)}{t^2} dt \cong \frac{h(t)}{t}$.

Lemma 1. Suppose $g(x) \in \text{Lip}\ \eta$ has period 2π,

$$g(x) \sim \frac{\alpha_o}{2} + \sum_1^\infty \alpha_n \cos nx + \beta_n \sin nx.$$

Let $g(r,x) = \frac{\alpha_o}{2} + \sum_1^\infty (\alpha_n \cos nx + \beta_n \sin nx)r^n$ be the corresponding harmonic function.

Then $g(r,x) - g(x) = O(\eta(1-r))$ as $r \to 1^-$, uniformly in x.

Proof.

Let $P_r(t) = \frac{(1-r)}{2(1-2r \cos t + r^2)}$ denote the Poisson kernel.

Then $P_r(t) < \frac{1}{1-r}$ and also $P_r(t) < \frac{1-r}{4r \sin^2 t/2}$, so that

$$\pi|g(r,x) - g(x)| = |\int_o^\pi [g(x+t) + g(x-t) - 2g(x)]Pr(t)dt|$$

$$\leq \frac{1}{1-r} \int_o^{1-r} O(\eta(t))dt + (1-r) \int_{1-r}^\pi O(\frac{\eta(t)}{t^2})dt$$

$$= O(\eta(1-r)) + O((1-r)\frac{\eta(t)}{t}|_{1-r}^\pi$$

$$= O(\eta(1-r))$$

<u>Lemma</u> 2. Let $g(x)$ and $g(r,x)$ be as in Lemma 1. Then

$$\frac{\partial g(r,x)}{\partial x} = O\left(\frac{\eta(1-r)}{1-r}\right).$$

Proof.

We note first that $\int \frac{1}{th(t)}\,dt \cong \frac{1}{h(t)}$ implies

$$\int \frac{1}{t^2 h(t)}\,dt \cong \frac{1}{t\,h(t)} \quad \text{for} \quad \int \frac{1}{t^2 h(t)}\,dt = \frac{1}{t\,h(t)} + K\int \frac{dt}{t^2 h(t)}.$$

Then $\frac{\partial g(r,x)}{\partial x} = -\frac{1}{\pi}\int_0^\pi [g(x+t) - g(x-t)]P_r'(t)dt$ where

$P_r'(t) = \frac{-(1-r)^2 r \sin t}{(1-2r\cos t + r^2)^2}$ so that $|P_r'(t)| < \frac{2t}{(1-r)^3}$ and also

$|P_r'(t)| < \frac{2(1-r)t}{(4r\sin^2 t/2)^2}$. Hence

$$\left|\frac{\partial g(r,x)}{\partial x}\right| \le \frac{1}{(1-r)^3}\int_0^{1-r} O(t\eta(t))dt + (1-r)\int_{1-r}^{\pi} O\left(\frac{\eta(t)}{t^3}\right)dt$$

$$= O\left(\frac{1}{1-r}\cdot \eta(1-r)\right) + (1-r)\cdot O\left(\frac{\eta(t)}{t^2}\right)\Big|_{1-r}^{\pi}$$

which is, as in Lemma 1, on the order of $\frac{1}{1-r}\,\eta(1-r)$.

Now to prove the theorem,

let $g(t) = f(x+t) - f(x-t) \sim -\sum_1^\infty 2\sin nt[a_n \sin nx - b_n \cos nx]$

and $g(r,t) = -\sum_1^\infty 2\sin nt[a_n \sin nx - b_n \cos nx]r^n$.

For given x and r $(r < 1)$ the series

$$\frac{g(r,t)}{t^2}\,h(t) = -2\sum_1^\infty \frac{\sin nt}{t^2}\,h(t)[a_n \sin nx - b_n \cos nx]r^n$$

is uniformly convergent in t for $t > \epsilon > 0$. Hence we can integrate termwise over (ϵ,T). We note that

$$\left|\int_0^\epsilon \sin nt\,\frac{h(t)}{t^2}\,dt\right| \le n\int_0^\epsilon \frac{h(t)}{t}\,dt = O\left(n\epsilon\cdot\frac{h(\epsilon)}{\epsilon}\right) = O(n\,h(\epsilon))$$

and

$$\left|\int_T^\infty \sin nt \cdot \frac{h(t)}{t^2}\, dt\right| \le \frac{h(T)}{T} .$$

Thus we may conclude that

$$\int_o^\infty g(r,t)\, \frac{h(t)}{t^2}\, dt = -2 \sum_1^\infty (a_n \sin nx - b_n \cos nx) r^n \int_o^\infty \frac{\sin nt\ h(t)}{t^2}\, dt$$

where $\int_o^\infty \frac{\sin nt}{t^2}\, h(t)dt = \frac{n}{h(n)} \int_o^\infty \frac{\sin x}{x^2}\, h(x)dx = \frac{K\cdot n}{h(n)} \cong K\, \frac{1}{\Phi(n)}$

so that $K(h) \int_o^\infty \frac{g(r,t)}{t^2}\, h(t)dt = \sum_1^\infty (a_n \sin nx - b_n \cos nx) r^n \cdot \frac{1}{\Phi(n)}$

Therefore to complete the proof of the theorem it suffices to show that

$$D = \int_o^\infty \frac{g(r,t)}{t^2}\, h(t)dt - \int_{1-r}^\infty \frac{g(t)h(t)}{t^2}\, dt \quad \text{is} \quad o(1)$$

if $g \in \text{lip}\, \eta$, and $O(1)$ if $g \in \text{Lip}\, \eta$, as $r \to 1^-$, uniformly in x. The proofs being identical we let $g \in \text{Lip}\, \eta$.

$$D = D_1 + D_2 = \int_{1-r}^\infty \frac{g(r,t)-g(t)}{t^2}\, h(t)dt + \int_o^{1-r} \frac{g(r,t)h(t)}{t^2}\, dt .$$

The first integral is bounded since

$$|D_1| \le O(\eta(1-r)\cdot \frac{h(1-r)}{1-r}) = O(1) \quad \text{using our above assumptions}$$

on h. To establish the boundedness of D_2 we note that since $g(r,0) = 0$, we have $g(r,t) = t\cdot \frac{\partial g(r,t)}{\partial t}\Big|_{t=\theta}$, $0 < \theta < t$ which by Lemma 2 is on the order of $t\cdot \frac{\eta(1-r)}{1-r}$ so that

$$|D_2| = O(\frac{1}{1-r} \cdot \eta(1-r) \int_o^{1-r} \frac{h(t)}{t}\, dt) = O\ \frac{1}{1-r}\, \eta(1-r)\, \frac{h(1-r)}{1-r} \cdot (1-r)) = O(1)$$

This concludes the proof of the theorem. A nearly identical argument establishes the following:

<u>Theorem</u>. Let f be as in the previous theorem; then the difference

$$K_2(\eta)\int_{1-r}^{\infty} \frac{f(x+t)+f(x-t)-2f(x)}{t^2}h(t)dt - \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)r^n \cdot \frac{1}{\Phi(n)}$$

is $o(1)$ if $f \in \text{lip}\ \eta$ and is $O(1)$ if $f \in \text{Lip}\ \eta$.

2. Some results on Blaschke products: the class B_h

As is well known, any bounded holomorphic function in the unit disk D can be represented as a product of a non-vanishing function and a function containing all its zeros, this second function being a Blaschke product $B(z,a_\nu)$ where

$$B(z,a_\nu) = z^m \prod_{1}^{\infty} \frac{a_\nu - z}{1-z\bar{a}_\nu} \cdot \frac{\bar{a}_\nu}{|a_\nu|}, \ |z| < 1$$

and $\{a_\nu\}_1^\infty$, $0 < |a_\nu| < 1$, is a sequence of complex numbers such that $\sum\limits_{1}^{\infty} (1 - |a_\nu|) < \infty$. By this series condition we know that $|a_\nu| \to 1$ as $\nu \to \infty$, i.e. that the zeros a_ν cluster near the boundary as ν becomes large. By considering the subclass of the Blaschke products consisting of those whose associated sequence has the property that $\Sigma(1 - |a_n|)^{1-\alpha} < \infty$, $0 < \alpha < 1$. Frostman [7] in effect speeds up the rate of convergence, forcing the zeros to cluster sooner at the boundary. In doing so he obtains some results which were not possible in the general case. This subclass, B_α, is also discussed by Carleson [4] and he shows it to be a subset of the previously mentioned T_α. We are interested here in a local condition on the boundary in which Frostman shows that $B(z) \in B_\alpha$ tends to a radial limit at θ of modulus 1 provided $\sum\limits_{\nu=1}^{\infty} \frac{1-|a_\nu|}{|e^{i\theta}-a_\nu|} < \infty$. He then proceeds to show that this is true except possibly on a set whose $(1-\alpha)$-capacity is zero. Our discussion will be concerned with the class B_h which consists of all Blaschke products having the property that $\sum_1^\infty h(1 - |a_\nu|) < \infty$, where h is a measure function. Now let $\tau(x)$ be a function having the properties that τ is increasing, $\tau(o) = 0$, $\tau(x) \le x$ if $x \le 1$. The following generalization of Frostman's result, in the case $\tau(x) = x$, is due to Carleson:

<u>Theorem</u> 5. Let $\{z_\nu\}_{\nu=1}^\infty$ be a sequence of complex numbers in the unit circle and $\{A_\nu\}$ a sequence of real numbers, $0 < A_\nu < 1$. If $h \circ \tau$ is a measure function and

$$\sum_{\nu=1}^\infty \tau^{-1}(A_\nu)\int_{\tau^{-1}(A_\nu)}^3 \frac{h\circ\tau(r)}{r^2}dr < \infty \quad \text{then} \quad \sum_{\nu=1}^\infty \frac{A_\nu}{\tau(|z-z_\nu|)} < \infty \quad \text{except}$$

possibly on a set E with $h\circ\tau$-measure 0.

Proof.

We note first that $\sum h(A_\nu) < \infty$ and consequently, since $\tau(A_\nu) \le A_\nu$, $\sum h\circ\tau(A_\nu) < \infty$ as well. This follows since

$$\infty > \sum_N^\infty \tau^{-1}(A_\nu)\int_{\tau^{-1}(A_\nu)}^3 \frac{h\circ\tau(r)}{r^2}\,dr \ge \sum_N^\infty \tau^{-1}(A_\nu)h(A_\nu)\left(-\frac{1}{r}\right)\Big|_{\tau^{-1}(A_\nu)}^3$$

$$= \sum_N^\infty h(A_\nu)\left[1 - \frac{\tau^{-1}(A_\nu)}{2}\right] \ge \sum_N^\infty \frac{h(A_\nu)}{2} \quad \text{for } N \text{ sufficiently large.}$$

Also since $\sum_N^\infty \tau^{-1}(A_\nu)\int_{\tau^{-1}(A_\nu)}^3 \frac{h\circ\tau(r)}{r^2}\,dr \ge \sum_N^\infty \tau^{-1}(A_\nu)\int_1^3 \frac{h\circ\tau(r)}{r^2}\,dr$ for N sufficiently large, we have that $\sum \tau^{-1}(A_\nu)$ converges.

For any integer p, denote by O_p the open set where

$$\sum_p^\infty \frac{A_\nu}{\tau(|z-z_\nu|)} > 1$$

Let $\mu \in \Gamma_{h\tau}$ where Γ_h is defined as the class of all non-negative completely additive functions of a set such that

$$\mu(r,a) \le h(r) \quad \text{for all} \quad a,$$

where $\mu(r,a)$ is the value taken by μ for the circle $C = \{z: |z - a| \le r\}$. For a discussion of these measures and their existence see Carleson [4, p. 11-12] and references cited there. Also we may choose μ so that it vanishes outside O_p.

About each point z_ν, $\nu \ge p$, we put a circle of radius $(\tau^{-1})^2(A_\nu)$, so that within these circles we have $\tau(|z - z_\nu|) < \tau^{-1}(A_\nu)$. Let G_p denote the exterior of these circles, i.e. where $\tau(|z - z_\nu|) \ge \tau^{-1}(A_\nu)$. On the circles, μ distributes a mass which does not exceed $\sum_p^\infty h\tau(A_\nu) = \epsilon'_p$. Then for p sufficiently large,

$$\mu(G_p) = \int_{G_p} d\mu(z) \le \int_{G_p} \sum \frac{A_\nu}{\tau(|z-z_\nu|)}\, d\mu(z)$$

$$\le \sum A_\nu \int_{\tau^{-1}(A_\nu)}^{3} \frac{d\mu(r,z_\nu)}{r}$$

$$\le \sum A_\nu \left.\frac{h\tau(r)}{r}\right|_{\tau^{-1}(A_\nu)}^{3} + A_\nu \int_{\tau^{-1}(A_\nu)}^{3} \frac{h\tau(r)}{r^2}\, dr.$$

The first term above is less than $K\cdot\sum A_\nu$, the lower limit being the negative of the positive expression $\sum A_\nu \dfrac{h(A_\nu)}{\tau^{-1}(A_\nu)}$. Both of these converge by virtue of the fact $A_\nu \le \tau^{-1}(A_\nu)$ and $\sum h(A_\nu) < \infty$. Likewise, since $A_\nu \le \tau^{-1}(A_\nu)$ and by our hypothesis, the second integral also converges.

Hence $\mu(G_p) \le \epsilon''_p$, and so by a result of Carleson [4, pp. 10-12] the $h\circ\tau$ measure of O_p is less than $32\cdot 36\cdot \epsilon_p$ where $\epsilon_p = \epsilon'_p + \epsilon''_p$. Finally we note that except for the points z_ν, $\nu = 1,\ldots,p-1$, O_p contains E so that

$$M_h(E) \le \lim_{p\to\infty} M_h(O_p) = 0.$$

The converse to this theorem is true under the hypothesis that $\int \frac{h(t)}{t^2} \cong O(\frac{h(t)}{t})$ as we now show. We shall make use of these results in Chapter II.

Theorem 6. Given a bounded set E, a necessary and sufficient condition for the existence of a Borel series $\sum_{\nu=1}^{\infty} \frac{A_\nu}{\tau(|z-z_\nu|)}$ divergent on E, with $\Sigma\, \tau^{-1}(A_\nu) \int_{\tau^{-1}(A_\nu)}^{3} \frac{h\circ\tau(r)}{r^2}\, dr < \infty$, is that $M_{h\circ\tau}(E) = 0$.

Proof.

That the condition is necessary is precisely the result of the previous theorem. It remains to show sufficiency: Let $M_{h\tau}(E) = 0$. Cover E by a family of circles with radii $\tau^{-1}(r_{n\mu})$ such that $\Sigma\, h\tau(\tau^{-1}(r_{n\mu})) = \Sigma\, h(r_{n\mu}) \le 2^{-n}$. Denote by $z_{n\mu}$ the centers of these circles: $|z - z_{n\mu}| < \tau^{-1}(r_{n\mu})$. The Borel series $\sum_{n=1}^{\infty} \sum_{\mu=1}^{\infty} \frac{r_{n\mu}}{\tau(|z-z_{n\mu}|)} = \sum_{\nu=1}^{\infty} \frac{A_\nu}{\tau(|z-z_{n\mu}|)}$ diverges on E while $\Sigma\, \tau^{-1}(A_\nu) \int_{\tau^{-1}(A_\nu)}^{3} \frac{h\tau(r)}{r^2}\, dr$ converges if $\Sigma\, h(A_\nu)$ does as we shall show below.

$$\therefore \sum_{\nu=1}^{\infty} h(A_\nu) = \sum_{n=1}^{\infty} \sum_{\mu=1}^{\infty} h(r_{n\mu}) \le \Sigma\, 2^{-n} = 1.$$

It remains to show $\Sigma\, \tau^{-1}(A_\nu) \int_{\tau^{-1}(A_\nu)}^{3} \frac{h\tau(r)}{r^2}\, dr$ converges if $\Sigma\, h(A_\nu)$ converges.

$$\Sigma\, \tau^{-1}(A_\nu) \int_{\tau^{-1}(A_\nu)}^{3} \frac{h\tau(r)}{r^2}\, dr = O\left(\Sigma\, \frac{h\tau(r)}{r}\Big|_{\tau^{-1}(A_\nu)}^{3}\right) \cdot \tau^{-1}(A_\nu)$$

$$= O(\Sigma\, \tau^{-1}(A_\nu)) + O\left(\Sigma\, h(A_\nu)\right).$$

3. Finiteness of the generalized fractional integral.

N. duPlessis [5] has shown the following concerning the finiteness of the fractional integral $f_{\alpha/q}$ of order α/q:

Theorem. If $f \in L^q[0,2\pi]$ then:

(a) For $0 < \alpha < 1$, $2 < q < \infty$, $f_{\alpha/q}$ is finite everywhere except in a set which is of zero β-capacity for every $\beta > 1-\alpha$.

(b) For $0 < \alpha < 1$, $1 \le q \le 2$, $f_{\alpha/q}$ is finite everywhere except possibly in a set of zero $(1-\alpha)$-capacity.

We wish here to extend this result to a generalized fractional integral with an exceptional set of the general h-measure; as with duPlessis, our two functions, ϕ and h, will not be independent of each other.

Let $h(x)$ be a measure function having the property that $\phi(x) = h(\frac{1}{1-x})$ is a non-negative, non-decreasing function, integrable on $(0,1)$. Further assume that $h(ab) \le h(a)h(b)$. Let $\Phi(n) = \int_{1-\frac{1}{n}}^{1} \phi(x)dx$ and note that $\Phi(n) \cong \frac{1}{n} h(n)$ (see §1). Extend Φ from a function of a discrete variable to a continuous variable by letting $\Phi(x) = \frac{1}{x} h(x)$. Finally we assume that

$$\int_0^{2\pi} \frac{1}{|x-t|} \Phi^{\varepsilon/q}\left(\frac{1}{|x-t|}\right)d\mu(t) \quad \text{and} \quad \int_0^{2\pi} \frac{1}{|u|} \cdot \frac{1}{|u-1|} \Phi^{\frac{1}{2}}\left(\frac{1}{|u|}\right)\Phi^{\frac{1}{2}}\left(\frac{1}{|u-1|}\right)du$$

both exist finitely.

Let $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$, $c_0 = 0$.

Define the generalized fractional integral as the function

$$f_{\Phi}(q,x) = \sum_{-\infty}^{\infty} \Phi^{1/q}(n) c_n e^{inx}$$

Next choose $a < 1$ such that $a = r/q$ where for $q > 2$, $\varepsilon > 0$, $r = \frac{q+\varepsilon(q-1)}{1+\varepsilon(q-1)}$, so that $2 < r < q$ and $r' - \varepsilon = q'$. (Here p' denotes the conjugate of p in the sense that $1/p + 1/p' = 1$)

Finally let $h_a(x) = x\,\Phi^a(x)$. We now prove the following:

<u>Theorem</u> 7. Let $f \in L^q[0,2\pi]$.

$f_\Phi(q,x)$ is finite except in a set of h-measure zero if $1 \le q \le 2$, in a set of h_a-measure zero if $q > 2$.

Letting $h(x) = x^{1-\alpha}$, $0 < \alpha < 1$, so that $\phi(x) = (1-x)^{\alpha-1}$ and $\Phi(n) = n^{-\alpha}$, we have $(1-\alpha)$-capacity and the ordinary fractional integral. The proof depends on the following lemma which is of independent interest. First we define

$$V_h = \sup_{x\in[0,2\pi]} \int_0^{2\pi} h\left(\frac{1}{|x-t|}\right) d\mu(t).$$

LEMMA:

(1) For every $\epsilon > 0$, $1 < q < 2$,

$$M_{q-\epsilon}\left[\int_0^{2\pi} \frac{1}{|x-t|}\,\Phi^{1/q'}\left(\frac{1}{|x-t|}\right) d\mu(t)\right] \le A(\Phi,\epsilon)V_h^{\frac{1}{(q-\epsilon)'}}$$

where A is a constant depending on Φ,ϵ only.

(2) For $2 \le q \le \infty$:

$$M_q\left[\int_0^{2\pi} \frac{1}{|x-t|}\,\Phi^{1/q'}\left(\frac{1}{|x-t|}\right) d\mu(t)\right] \le A(\Phi)V_h{}^{1/q'}$$

where A is a constant depending on Φ only.

Proof.

(1) $\displaystyle\int_0^{2\pi} \frac{1}{|x-t|}\,\Phi\left(\frac{1}{|x-t|}\right) d\mu(t) = \int_0^{2\pi} \Phi^{-1/q}\left(\frac{1}{|x-t|}\right) dv_x(t)$

where $\displaystyle v_x(t) = \int_0^t h\left(\frac{1}{|x-s|}\right) d\mu(s)$

$$\left(\int_0^{2\pi} \frac{1}{|x-t|}\,\Phi^{1/q'}\left(\frac{1}{|x-t|}\right) d\mu(t)\right)^{q-\epsilon} = \left(\int_0^{2\pi} \Phi^{-1/q}\left(\frac{1}{|x-t|}\right) dv_x^{1/q-\epsilon}(t)\,dv_x^{1/(q-\epsilon)'}\right)^{q-\epsilon}$$

$$\leq \left(\int_0^{2\pi} \Phi^{-\left(\frac{q-\varepsilon}{q}\right)} dv_x(t)\right) \cdot \left(\int_0^{2\pi} dv_x(t)\right)^{\frac{q-\varepsilon}{(q-\varepsilon)'}} \quad \text{by Hölder's inequality,}$$

$$(*) \leq \int_0^{2\pi} \frac{1}{|x-t|} \Phi^{\varepsilon/q}\left(\frac{1}{|x-t|}\right) d\mu(t) \cdot V_h^{\frac{q-\varepsilon}{(q-\varepsilon)'}}$$

The latter inequality holds since

$$\Phi^{-\left(\frac{q-\varepsilon}{q}\right)} \cdot h\left(\frac{1}{|x-t|}\right) = \Phi^{-1+\varepsilon/q} \cdot \frac{1}{|x-t|} \cdot \Phi = \frac{1}{|x-t|} \cdot \Phi^{\varepsilon/q}\left(\frac{1}{|x-t|}\right).$$

The inequality (*) and our assumption above gives (1).

Proof (2): The special case $q = 2$ is proved first and used to prove the remaining cases.

$$M_2^2\left[\int_0^{2\pi} \frac{1}{|x-t|} \Phi^{\frac{1}{2}}\left(\frac{1}{|x-t|}\right) d\mu(t)\right]$$

$$= \int_0^{2\pi} \int_0^{2\pi} \left[\int_0^{2\pi} \frac{1}{|x-t|} \Phi^{\frac{1}{2}}\left(\frac{1}{|x-t|}\right) \cdot \frac{1}{|x-s|} \Phi^{\frac{1}{2}}\left(\frac{1}{|x-s|}\right) dx\right] d\mu(t) d\mu(s)$$

Using the substitution $x-t = (s-t) \cdot u$ we find, on the innermost integral, that

$$\int_0^{2\pi} \frac{1}{|x-t|} \Phi^{\frac{1}{2}}\left(\frac{1}{|x-t|}\right) \frac{1}{|x-s|} \Phi^{\frac{1}{2}}\left(\frac{1}{|x-s|}\right) dx =$$

$$= \int_0^{2\pi} \frac{1}{|x-s|} \cdot \frac{1}{|u|} \Phi^{\frac{1}{2}}\left(\frac{1}{|u| \cdot |s-t|}\right) \cdot \frac{1}{|s-t|} \cdot \frac{1}{|1-u|} \cdot \Phi^{\frac{1}{2}}\left(\frac{1}{|1-u| \cdot |s-t|}\right) |s-t| \, du$$

$$\leq \frac{1}{|s-t|} \cdot \Phi\left(\frac{1}{|s-t|}\right) \int_0^{2\pi} \frac{1}{|u|} \cdot \frac{1}{|u-1|} \Phi^{\frac{1}{2}}\left(\frac{1}{|u|}\right) \cdot \Phi^{\frac{1}{2}}\left(\frac{1}{|u-1|}\right) du$$

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{1}{|x-t|} \Phi^{\frac{1}{2}}\left(\frac{1}{|x-t|}\right) \cdot \frac{1}{|x-s|} \Phi^{\frac{1}{2}}\left(\frac{1}{|x-s|}\right) dx \, d\mu(t) d\mu(s)$$

$$\leq B(\Phi) \int_0^{2\pi}\int_0^{2\pi} \frac{1}{|s-t|} \Phi\left(\frac{1}{|s-t|}\right) d\mu(t) d\mu(s)$$

$$\leq B(\Phi) \cdot V_h(\mu(2\pi) - \mu(o)) \quad \text{giving (2) for } q = 2.$$

Next for $q > 2$:

$$M_q^q\left[\int_0^{2\pi} \frac{1}{|x-t|}\, \Phi^{1/q'}\left(\frac{1}{|x-t|}\right) d\mu(t)\right]$$

$$= \int_0^{2\pi}\left\{\int_0^{2\pi} \frac{1}{|x-t|^{\frac{q-2}{q}}}\, \Phi^{\frac{q-2}{q}} \cdot \frac{1}{|x-t|^{2/q}}\, \Phi^{1/q} d\mu(t)\right\}^q dx$$

$$\leq \int_0^{2\pi}\left\{\left[\int_0^{2\pi} \frac{1}{|x-t|}\, \Phi\left(\frac{1}{|x-t|}\right) d\mu(t)\right]^{\frac{q-2}{q}} \cdot \left[\int_0^{2\pi} \frac{1}{|x-t|}\, \Phi^{\frac{1}{2}} d\mu(t)\right]^{2/q}\right\}^q dx$$

using Hölder's inequality and the fact that $\frac{q-2}{q} + \frac{2}{q} = 1$

$$\leq V_h^{q-2} \cdot M_2^2\left[\int_0^{2\pi} \frac{1}{|x-t|}\, \Phi^{\frac{1}{2}}\left(\frac{1}{|x-t|}\right) d\mu(t)\right]$$

$$\leq V_h^{q-2} \cdot \text{constant} \cdot V_h = \text{const. } V_h^{q-1}$$

This completes the proof of the lemma.

Proof of the theorem:

$$\text{Let } f \in L^q,\ f \sim \sum_{-\infty}^{\infty} c_k e^{ikx},\ f_\Phi(q,x) = \sum_{-\infty}^{\infty} \Phi^{1/q}(k) c_k e^{ikx}$$

Let $S_n = \sum_{-n}^{n} \Phi^{1/q}(k)\, c_k e^{ikx}$

To show: S_n is bounded outside a set of h-measure 0 if $1 \leq q \leq 2$, outside a set of h_a-measure 0 if $q > 2$.

Suppose S_n is unbounded in a set E of positive H-measure where H is either h or h_a accordingly. Then there exists a distribution $\mu(x)$ concentrated on E such that $\int_0^{2\pi} H\left(\frac{1}{|x-t|}\right) d\mu(t)$ is bounded for all x. As noted earlier (§1), it can be shown [14, p. 253] that there exists a function $n(x) \leq n$ taking integer values such that $\int_0^{2\pi} S_{n(x)}(x) d\mu(x)$ exists and is unbounded as $n \to \infty$. We show this to be impossible.

$$\int_0^{2\pi} S_{n(x)}(x) d\mu(x) = \int_0^{2\pi} \sum_{-n(x)}^{n(x)} \Phi^{1/q}(k) c_k e^{ikx} d\mu(x)$$

$$= \int_0^{2\pi}\int_0^{2\pi} f(t) \sum_{-n(x)}^{n(x)} \Phi^{1/q}(k) e^{ik(x-t)} dt\, d\mu(x)$$

We note here that, as has been shown, (§1)

$$\sum_{K=1}^{n} \Phi^{1/q}(k) e^{ik(x-t)} = O\left(\frac{1}{|x-t|} \cdot \Phi^{1/q}\left(\frac{1}{|x-t|}\right)\right).$$

Therefore $\left|\int_0^{2\pi} S_{n(x)}(x)\, d\mu(x)\right| \le C \cdot \int_0^{2\pi} |f(t)| \cdot \left\{\int_0^{2\pi} \frac{1}{|x-t|} \Phi^{1/q}\left(\frac{1}{|x-t|}\right) d\mu(x)\right\} dt$

$$\le C \cdot M_q(f) \cdot M_{q'}\left[\int_0^{2\pi} \frac{1}{|x-t|} \Phi^{1/q}\left(\frac{1}{|x-t|}\right) d\mu(x)\right]$$

For $1 \le q \le 2$: Use (2) of the lemma, interchanging the roles of q, q'. The resulting contradiction proves the theorem for this case.

For $q > 2$: define a,r, as given earlier so that $\frac{1}{q} = \frac{a}{r}$, $2 < r < q$, and $q' = r' - \varepsilon$.

$M_{q'}\left[\int_0^{2\pi} \frac{1}{|x-t|} \Phi^{1/q}\left(\frac{1}{|x-t|}\right) d\mu(t)\right]$ becomes $M_{r'-\varepsilon}\left[\int_0^{2\pi} \frac{1}{|x-t|} \left(\Phi^{a}\left(\frac{1}{|x-t|}\right)\right)^{1/r} d\mu(t)\right]$

which is finite by (1) of the lemma with r' for q, r for q'. This completes the proof of the theorem. We note that the limiting case of h_a-measure, i.e. as $\varepsilon \to 0$ in the definition of r, gives precisely h-measure. This corresponds to the capacity case where $(1-\alpha+\varepsilon)$-capacity, i.e. every $\beta > 1-\alpha$, has $(1-\alpha)$-capacity as the limiting case.

CHAPTER II

On Generalized Tangential limits for Certain Function Classes

1. τ-tangential limits of functions in B_h

Let $B(z,\{a_n\}) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1-\bar{a}_n z}$, $0 < |a_n| < 1$, $n = 1,2,\ldots$

and $\sum_{n=1}^{\infty} (1-|a_n|) < \infty$, so that $B(z,\{a_n\})$ is a Blaschke product. It is known that $B(z,\{a_n\})$ has radial limits at almost every point of the unit circle C and further that this implies an angular limit at each such point (See Cargo [3] for a brief discussion and references).

Let $R[\tau,\theta] = \{z\colon 1 - |z| \geq \tau(|\arg z - \theta|)\}$ where $|\arg z - \theta|$ is the length of the shorter segment of C which joins $e^{i\theta}$ and $z/|z|$, and where τ satisfies the following properties:

$\tau(0) = 0$,

τ is increasing

$\tau'(x)$ exists finitely at 0

$\tau(\alpha x) \leq C_\alpha \tau(x)$ where α is constant and C_α is a constant depending on α and τ alone

$\frac{\tau'(x-\alpha\tau(x))}{\tau'(x)} \not\to 0$ as $x \to 0$

$$\tau\left(\frac{1}{|e^{i\theta}-a_k|}\right) \geq \frac{1}{\tau(|e^{i\theta}-a_k|)}$$

Next let h denote a measure function such that $h(ab) \le h(a)h(b)$. Let B_h denote the class of all Blaschke products having the property that $\Sigma\, h(1 - |a_n|) < \infty$.

In the case $\tau(r) = r^\gamma$, $\gamma \ge 1$, $R[\tau,\theta]$ is the path which meets the circle with order of tangency γ-1. It has been shown (see Cargo [3] for references cited there) that corresponding to each such path, there exists a Blaschke product having no tangential limit whatever on C. Cargo however has shown that for $1 \le \gamma < 1/\alpha$, tangential limits of order γ-1 exists for Blaschke products in the class B_h, with $h(r) = r^\alpha$, except for sets on the unit circle whose $\alpha\gamma$-capacity is zero. This includes, in the case $\gamma = 1$, the results of Frostman [7] noted in Chapter I concerning radial limits and α-capacity.

Following the lead of Cargo of composing the measure function with the tangential function to form a measure function, we show here that this result can be extended to show that every Blaschke product belonging to B_h has a τ-tangential limit at every point of C except possibly for a set of $h\circ\tau$-measure zero.

a. Local condition

<u>Theorem</u> 1. Let $\{a_n\}$ be a Blaschke sequence such that

$$(1) \qquad \sum_{n=1}^{\infty} \frac{1 - |a_n|}{\tau(|e^{i\theta}-a_n|)} < \infty .$$

Then $B(z,\{a_n\})$ has a τ-tangential limit of modulus 1 at $e^{i\theta}$.

Proof.

Without loss of generality we may assume $\theta = 0$. For if

$$\Sigma \frac{1 - |a_n|}{\tau(|1-a_n|)} < \infty \Rightarrow \lim_{z\to 1} B(z,\{a_n\}) \text{ exists then the same proof}$$

shows that

$$\Sigma \frac{1-|a_n e^{i\theta}|}{\tau(|1-a_n e^{-i\theta}|)} < \infty \Rightarrow \lim_{z\to 1} B(z,\{a_n e^{-i\theta}\}) \text{ exists i.e.}$$

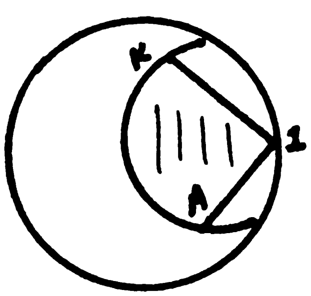
$$\Sigma \frac{1-|a_n|}{\tau(|e^{i\theta}-a_n|)} < \infty \Rightarrow \lim_{z\to 1} B(ze^{i\theta},\{a_n\}) \text{ exists}$$

but $z \to 1 \Rightarrow ze^{i\theta} \to e^{i\theta}$ and by change of notation (z for $ze^{i\theta}$) we have $\lim_{z\to e^{i\theta}} B(z,\{a_n\})$ exists.

By virtue of (1) and our definition of τ, we note since $|a_n| \to 1$ that at most finitely many of the a_n lie on the radius to 1; hence we may assume without loss of generality that no a_n lie on this radius.

We show first that $\sum_{n=1}^{\infty} \frac{1-|a_n|}{\tau(|\arg a_n|)} < \infty$. For suppose not, then we will show that $\Sigma \frac{1-|a_n|}{\tau(|1-a_n|)} = \infty$.

Let $K = D(1,1/\sqrt{2})$, let A be the right angle to 1 bisected by the radius to 1.

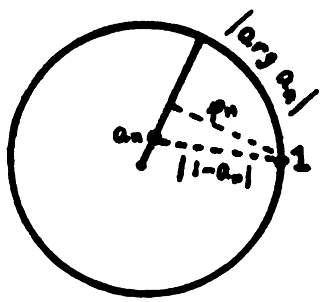


Since $\Sigma(1-|a_n|) < \infty$ and $\tau(|\arg a_n|)$ has a positive minimum value outside K we have

$$\sum_{a_n \in K} \frac{1-|a_n|}{\tau(|\arg a_n|)} = \infty .$$

If infinitely many $a_n \in K$ lie in A then $1-|a_n| > C\cdot\tau(|1-a_n|)$ where $C > 0$ and hence $\Sigma \frac{1-|a_n|}{\tau(|1-a_n|)}$ diverges.

If only finitely many $a_n \in K$ lie in A, consider only those $a_n \in K-A$. Let p_n be the perpendicular distance from 1 to the radius thru a_n.



Then $p_n < |\arg a_n|$, $\dfrac{p_n}{|1-a_n|} > \cos \pi/4$

$$|1-a_n| < p_n 2^{\frac{1}{2}} < |\arg a_n| \cdot 2^{\frac{1}{2}}$$

$$\tau(|1-a_n|) \le C\, \tau(|\arg a_n|),\ C > 0$$

$\dfrac{1}{\tau(|1-a_n|)} \ge \dfrac{1}{C} \dfrac{1}{\tau(|\arg a_n|)}$ so that the corresponding series must diverge.

This establishes $\Sigma \dfrac{1 - |a_n|}{\tau(|\arg a_n|)} < \infty$.

Next, using a well-known device, choose a sequence $\{\omega_n\}$, $0 < \omega_n \le 1$, $\omega_n \to 0$ such that

$$(2) \qquad \sum_{n=1}^{\infty} \frac{1 - |a_n|}{\omega_n \tau(|\arg a_n|)} < \infty$$

Let $S_n = \{z: |z - a_n| < \omega_n \tau(|\arg a_n|)\}$.

We show that for any fixed positive integer k that $B(z,\{a_n\})$ converges uniformly on $D - \bigcup_{j=k}^{\infty} S_j$.

Let $b(z,\{a_n\}) = \frac{|a_n|}{a_n} \frac{a_n - z}{1-\bar{a}_n z}$ and $C(z,\{a_n\}) = b(z,\{a_n\}) - 1$

Then $|C(z,\{a_n\})| \le \frac{(1+|z|)(1-|a_n|)}{|1 - \bar{a}_n z|}$

$$< 2 \cdot \frac{1-|a_n|}{|a_n - z|} \cdot \frac{|a_n - z|}{|1-\bar{a}_n z|}$$

$$< 2 \frac{1-|a_n|}{|a_n - z|} \le \frac{2(1-|a_n|)}{\omega_n \tau(|\arg a_n|)} \quad \text{if} \quad z \in D - \cup S_j$$

This inequality, the convergence (2) and the fact that $|C(z,\{a_n\})| < 2$ implies that $\pi[1 + C(z,\{a_n\})]$ converges uniformly on $D - \bigcup_{j=k}^{\infty} S_j$.

It remains to show that $R[\tau,0]$ meets at most finitely many of the disks S_n; it suffices to show that for j sufficiently large, $z_o \in S_j \cap D$ implies $1 - |z_o| < \tau(|\arg z_o|)$.

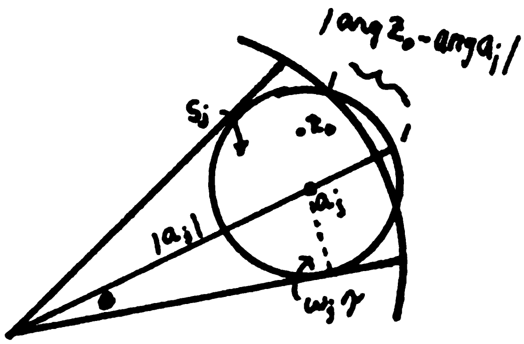
Since $z_o \in S_j$, $|z_o - a_j| < \omega_j \tau(|\arg z_o|)$ so that

(3) $\quad 1 - |z_o| < 1 - |a_j| + \omega_j \tau(|\arg z_o|)$

Also for j sufficiently large

$$|\arg z_o - \arg a_j| < \arcsin\left(\frac{\omega_j \tau(|\arg a_j|)}{|a_j|}\right) < \pi\omega_j \tau(|\arg a_j|)$$

as we see from the following diagram, where $\theta = \arcsin \frac{\omega_j \tau(|\arg a_j|)}{|a_j|}$



The latter inequality holds because $\pi\omega_j\tau(|\arg a_j|)$ is half the circumference of S_j.

Hence we can say that

$$(4) \qquad \tau(|\arg z_o|) \geq \tau(|\arg a_j| - |\arg z_o - \arg a_j|)$$

$$> \tau(|\arg a_j| - \pi\omega_j\tau(|\arg a_j|)).$$

If we can show that

$$\tau(|\arg a_j| - \pi\omega_j\tau(|\arg a_j|)) \geq 1 - |a_j| + \omega_j\tau(|\arg a_j|)$$

for j sufficiently large, then it follows from (3) and (4) that $\tau(|\arg z_o|) > 1 - |z_o|$ as was to be shown.

It suffices to show that

$$\frac{\tau(|\arg a_j| - \pi\omega_j\tau(|\arg a_j|))}{\tau(|\arg a_j|)} \geq \frac{1 - |a_j|}{\tau(|\arg a_j|)} + \omega_j \quad \text{for } j \text{ sufficiently}$$

large.

That the right hand side goes to 0 is immediate since $\omega_j \to 0$ and $\Sigma \dfrac{1 - |a_j|}{\tau(|\arg a_j|)} < \infty$.

That the left hand side does not go to 0 is a consequence of the facts that $|\arg a_j|$ is bounded, $\omega_j \to 0$, $|\arg a_j| \to 0$ as $j \to \infty$ and

$$\lim_{x\to 0} \frac{\tau(x - K\tau(x))}{\tau(x)} = \lim_{x\to 0} \frac{\tau'(x - K\tau(x))}{\tau'(x)} (1 - K\tau'(x))$$

where we cite our assumptions on τ and the fact that $K = \pi\omega_j \to 0$. Thus τ-tangential limits exist on $R[\tau,0] \subset \bigcup_{n=j}^{\infty} S_n$, for j sufficiently large.

Since $R[\tau,0]$ meets at most finitely many disks S_n, $B(z,\{a_n\}) = \prod_{n=1}^{\infty} b(z,a_n)$ converges uniformly on $R[\tau,0]$. For any fixed positive integer N, $\prod_{n=1}^{N} b(z,a_n)$ is a rational function with only finitely many poles, all of which lie outside of $D \cup C$. Therefore $\prod_{n=1}^{N} b(z,a_n) \to \prod_{n=1}^{N} b(1,a_n)$ as $z \to 1$, z within $R[\tau,0]$. Since the convergence is uniform

$$B(z,\{a_n\}) = \lim_{N\to\infty} \prod_{n=1}^{N} b(z,a_n) \to \lim_{N\to\infty} \prod_{n=1}^{N} b(1,a_n) = B(1,\{a_n\})$$

as $z \to 1$ on $R[\tau,0]$. Since $|b(1,a_n)| = 1$ for all n, we conclude that the limit is of modulus 1.

b. Global condition

<u>Theorem</u> 2. Let $\{a_n\}$ be a Blaschke sequence with $\Sigma\, h(1 - |a_n|) < \infty$. Let τ be defined so that $h \circ \tau$ is a measure function. Let $E_\tau = \{e^{i\theta}\colon \sum_{n=1}^{\infty} \frac{1 - |a_n|}{\tau(|e^{i\theta}-a_n|)} = \infty\}$.
Then E_τ has zero $h \circ \tau$-measure.

Proof.

For each positive integer n, let O_n be an open arc on C with center $\frac{a_n}{|a_n|}$ and length $\tau^{-1}(1 - |a_n|)$.

Let $G_n = \bigcup_{k=n}^{\infty} O_k$, $F_n = C - G_n$.

Then $\bigcup_{n=1}^{\infty} F_n$ and $\bigcap_{n=1}^{\infty} G_n$ are disjoint sets whose union is C.

Let $f_n = F_n \cap E$, $G = E \cap (\bigcap G_n)$.

We observe that for each N, $\bigcup_{k=N}^{\infty} O_k$ is a cover for $\bigcap G_n$.

Then $\lim_{N\to\infty} \sum_{k=N}^{\infty} h \circ \tau(\tau^{-1}(1 - |a_n|) = \lim_{N\to\infty} \sum h(1 - |a_n|) = 0$

Hence $h \circ \tau(G) = 0$.

Next let n be fixed, let $e^{i\theta} \in F_n$.

For $k \geq n$,

$$|e^{i\theta} - a_k| > \frac{1}{\pi}\tau^{-1}(1 - |a_k|)$$

or

$$\frac{1 - |a_k|}{\tau(|e^{i\theta}-a_k|)} < \tau(\pi)$$

Suppose that $h \circ \tau(f_n) > 0$, then there exists a positive distribution μ concentrated on f_n such that for all z

$$\int_{f_n} h \circ \tau\left(\frac{1}{|e^{i\theta}-z|}\right) d\mu(\theta) < M < \infty$$

For $k \geq n$, using our assumptions on τ and h,

$$\int_{f_n} \frac{1 - |a_n|}{\tau(|e^{i\theta}-a_k|)} d\mu(\theta) = \int_{f_n} h(1-|a_k|) \cdot \frac{1 - |a_k|}{h(1-|a_k|)} \cdot \frac{1}{\tau(|e^{i\theta}-a_k|)}$$

$$\cdot \frac{1}{h\circ\tau\left(\frac{1}{|e^{i\theta}-a_k|}\right)} \, h\circ\tau \, d\mu$$

$$\leq \int_{f_n} h(1-|a_k|) \cdot h\circ\tau\left(\frac{1}{|e^{i\theta}-a_k|}\right) \cdot \frac{1 - |a_k|}{\tau(|e^{i\theta}-a_k|)} \cdot \frac{1}{h\left(\frac{1 - |a_k|}{\tau(|e^{i\theta}-a_k|)}\right)} d\mu(\theta)$$

$$< K \int_{f_n} h(1-|a_k|) \cdot h\circ\tau\left(\frac{1}{|e^{i\theta}-a_k|}\right) d\mu(\theta) < M \cdot K \cdot h(1 - |a_k|).$$

Then

$$\int_{f_n} \sum_{k=n}^{\infty} \frac{1-|a_k|}{\tau(|e^{i\theta}-a_k|)}\, d\mu(\theta) = \sum_{k=n}^{\infty} \int_{f_n} \frac{1-|a_k|}{\tau(|e^{i\theta}-a_k|)}\, d\mu(\theta)$$

$$< M\cdot K \sum h(1-|a_k|) < \infty$$

which contradicts the assumption that

$\sum \frac{1-|a_k|}{\tau(|e^{i\theta}-a_k|)} = \infty$ on $f_n \subset E_\tau$. Hence $h\circ\tau(f_n) = 0$.

That E_τ has zero $h\circ\tau$-measure follows from the above and the sub-additivity of the measure function.

c. An alternate proof using an analogue of a theorem of Carleson.

We assume here that $\int \frac{h(t)dt}{t^2} \cong O(\frac{h(t)}{t})$. Then, as noted in the proof of theorem 6, §2, chapter I, we have that

$\sum \tau^{-1}(A_\nu) \int_{\tau^{-1}(A_\nu)}^{3} \frac{h\tau(r)}{r^2}\, dr$ converges if $\sum h(A_\nu)$ converges.

In theorem 5, §2 of chapter I, we now let $A_\nu = 1 - |a_\nu|$. Therefore we have that the local condition $\sum \frac{1-|a_\nu|}{\tau(|e^{i\theta}-a_\nu|)} < \infty$ holds except possibly on a set E with $h\circ\tau$-measure 0. This result, together with theorem 1 of the present chapter, shows that the τ-tangential limits exist on the unit circle on the complement of E. These results may then be used to replace theorem 2 of this chapter.

In particular we note that the measure function $h(r) = r^\alpha$, $0 < \alpha < 1$, satisfies all the above properties. Thus the Carleson result serves as a proof of the Cargo result mentioned above.

2. τ-Tangential Limits of Functions in S_Φ and of Their Generalized Fractional Integrals.

Kinney [10] has established the following concerning functions f in the class

$$S_\alpha = \{f(z) = \sum_{n=0}^{\infty} c_n z^n : \sum n^\alpha |c_n|^2 < \infty\} :$$

(1) Given $0 < \gamma < \alpha$, $f \in S_\alpha$ has tangential limit with order of tangency $\tau-1$ for every $\tau < \frac{1-\gamma}{1-\alpha}$ except possibly for a set with $(1-\gamma)$-capacity 0.

(2) If $0 < \gamma < \alpha-2r$, then the fractional derivative of order r of f has tangential limit for every $\tau < \frac{1-\gamma}{1-\alpha+2r}$ except possibly for a set with $(1-\gamma)$-capacity 0.

(3) If $0 < \gamma < \alpha + 2q < 1$, then the fractional integral of order q of f has tangential limit for every $\tau < \frac{1-\gamma}{1-\alpha-2q}$ except possibly for a set of $(1-\gamma)$-capacity 0.

Let $h(x)$ be a measure function, let ϕ and ψ be functions such that $\phi(x)$ and $\frac{\psi^{\frac{1}{2}}(1-x)}{1-x}$ are non-negative, non-decreasing and integrable in $(0,1)$. Also assume that $\psi(\alpha x) \le C_\alpha \psi(x)$ where C_α is a constant depending on α and ψ alone. Define

$$\Phi(n) = \int_{1-\frac{1}{n}}^{1} \phi(x)dx \quad \text{and} \quad \Psi(n) = \int_{1-\frac{1}{n}}^{1} \frac{\psi^{\frac{1}{2}}(1-x)}{1-x}dx \quad \text{and let}$$

$S_\Phi = \{f(z) = \sum_{n=0}^{\infty} c_n Z^n : \sum \frac{|c_n|^2}{\Phi^2(n)} < \infty\}$. Define $S(\theta)$ to be the function whose Fourier Series is defined by $S(\theta) \sim \sum \frac{c_n e^{in\theta}}{\Phi(n)}$.

Extend ϕ and ψ to complex values by writing

$$\phi(z) \cong \sum \Phi(n)z^n \quad \text{and} \quad \frac{\psi^{\frac{1}{2}}(1-z)}{1-z} \cong \sum \Psi(n)z^n \quad .$$

Let $f_{\Psi/\Phi} = \Sigma \frac{\Psi(n)}{\Phi(n)} c_n e^{in\theta}$ be the Ψ/Φ-fractional integral of f.

Finally let $R[\tau,\nu] = \{z: 1 - |z| \geq \tau(|z - e^{i\nu}|)\}$ where τ is an increasing function such that $\tau(0) = 0$. We now show:

<u>Theorem</u> 3. If $\beta(1-x) = \frac{(1-x)^{1+\epsilon}}{\psi(1-x)}$ and $\tau(x) = \beta^{-1} o(\frac{1}{h})(\frac{1}{x})$ and $f \in S_\Phi$, then there exists a τ - $\lim_{z \to e^{i\nu}} f_{\Psi/\Phi}(z)$ except possibly for $e^{i\nu} \in E$ where E has h-measure 0.

Before proving the theorem we note that by taking $h(x) = x^{1-\gamma}$, $\gamma < 1$, $\phi(x) = (1-x)^{\alpha/2-1}$ and hence $\Phi(n) = n^{-\alpha/2}$ then we obtain Kinney's results given above by choosing in turn

(1) $\psi(x) = x^\alpha$, $\Psi(n) = n^{-\alpha/2}$

(2) $\psi(x) = x^{\alpha-2r}$, $\Psi(n) = n^{-\alpha/2+r}$

(3) $\psi(x) = x^{\alpha+2q}$, $\Psi(n) = n^{-\alpha/2-q}$

For (1) we choose $\beta(1-x) = (1-x)^{1-\alpha+\epsilon}$, $\tau(x) = x^{\frac{1-\gamma}{1-\alpha+\epsilon}}$ and similarly for the remaining cases.

Proof: Note that

$$\int_{-\pi}^{\pi} |S(\theta)|^2 d\theta = \int_{-\pi}^{\pi} \Sigma \frac{c_n e^{in\theta}}{\Phi(n)} \cdot \Sigma \frac{\bar{c}_n e^{-in\theta}}{\Phi(n)} d\theta = \text{const.} \Sigma \frac{|c_n|^2}{\Phi^2(n)} < \infty$$

so that $S \in L^2$.

We write

$$f_{\Psi/\Phi}(z) = k \int \frac{\psi^{\frac{1}{2}}(1-ze^{-\theta})}{1-ze^{-i\theta}} S(\theta) d\theta$$

In I, §3 we have shown that if $\delta(x) = h(\frac{1}{1-x})$ and $\Delta(n) = \int_{1-\frac{1}{n}}^{1} \delta(x)dx$, $q = 1$, then $g_\Delta(x) = \Sigma \Delta(k) a_k e^{ikx} = k \int \delta(e^{i(x-k)}) g(t) dt$ (where $g(x) = \Sigma a_k x^k$) is finite if $g \in L$ except on a set E of h-measure 0.

Let $g = |S^2|$, $\delta(e^{i(x-t)}) = h(\frac{1}{|1-e^{i(x-t)}|}) = h(\frac{1}{|e^{ix}-e^{it}|})$. Then

$|S_\Delta^2(\theta)| = \int_{-\pi}^{\pi} h(\frac{1}{|e^{i\nu}-e^{i\theta}|})|S(\theta)|^2 d\theta$ is finite off E, where

$E = \{e^{i\nu}: \int_{-\pi}^{\pi} h(\frac{1}{|e^{i\nu}-e^{i\theta}|})|S(\theta)|^2 d\theta = \infty\}$ and h-measure $(E) = 0$.

Let $\beta(1-x) = \frac{(1-x)^{1+\epsilon}}{\psi(1-x)}$, $\tau(x) = \beta^{-1} \circ (\frac{1}{h})(\frac{1}{x})$. Then $\beta(1-x)$ is decreasing, so β is an increasing function and

$\beta(\alpha(1-x)) \geq k_\alpha \beta(1-x)$ from the above assumptions on ψ.

Now let $z \in R[\tau,\nu]$, then for suitable $k > 0$,

$$|e^{i\theta}-z| \geq \min_{z \in R[\tau,\nu]} |e^{i\theta}-z| \geq k\cdot\beta^{-1} \circ \frac{1}{h}(\frac{1}{|e^{i\nu}-e^{i\theta}|})$$

$$\beta(|e^{i\theta}-z|) \geq k_\beta \cdot \frac{1}{h}(\frac{1}{|e^{i\nu}-e^{i\theta}|})$$

so that $$\frac{\psi(|1-ze^{-i\theta}|)}{|1-ze^{-i\theta}|^{1+\epsilon}} \leq k\cdot h(\frac{1}{|e^{i\nu}-e^{i\theta}|})$$

Choosing $a > 0$,

$$\left|\int_{\nu-a}^{\nu+a} \frac{\psi^{\frac{1}{2}}(1-ze^{-i\theta})}{1-ze^{-i\theta}} \cdot S(\theta)d\theta\right|^2 \leq$$

$$\left|\int_{\nu-a}^{\nu+a} \frac{|\psi(1-ze^{-i\theta})|}{|e^{i\theta}-z|^{1+\epsilon}} \cdot |S(\theta)|^2 d\theta\right| \cdot \left|\int_{\nu-a}^{\nu+a} |e^{i\theta}-z|^{\epsilon-1} d\theta\right|$$

by Schwarz inequality using $f = \frac{\psi^{\frac{1}{2}}\cdot S}{(1-ze^{-i\theta})^{\frac{1}{2}+\epsilon/2}}$, $g = (1-ze^{-i\theta})^{\frac{\epsilon-1}{2}}$

$$\leq k\left|\int_{-\pi}^{\pi} h(\frac{1}{|e^{i\nu}-e^{i\theta}|}) \cdot |S(\theta)|^2 d\theta\right| \cdot \left|\int_{\nu-a}^{\nu+a} |e^{i\theta}-z|^{\epsilon-1} d\theta\right|$$

Choose $e^{i\nu} \notin E$, $z \in R[\tau,\nu]$. Then the first factor is finite and the second factor goes uniformly to 0 with a. Since

$\int_{-\pi}^{\pi}\left(\frac{\psi^{\frac{1}{2}}(1-ze^{-i\theta})}{1-ze^{-i\theta}}\right)S(\theta)d\theta = \int_{-\pi}^{\nu-a} + \int_{\nu-a}^{\nu+a} + \int_{\nu+a}^{\pi}$ and the first and the last are analytic at $e^{i\nu}$, then the τ-limit exists.

3. τ-Tangential Limits of Functions in $A(\Psi,\phi)$

Let $h(r)$ be a measure function, D the unit disk. The following theorem, which follows from a lemma of Ahlfors, has been established by Rung [11]:

Theorem A. Let $U(z)$ be a real-valued, non-negative, measurable function defined in D such that

$$\int_D\int U(z)dxdy < \infty.$$

Then $\lim\limits_{r\to 0}\frac{1}{h(r)}\int\int_{D(e^{i\theta},r)} U(z)dxdy = 0.$

This theorem is used by Rung to obtain results concerning the class A^s, $s > -1$: we say that f holomorphic in D belongs to the class A^s if

$$\int_D\int |f'(z)|^2(1-|z|)^s dxdy < \infty .$$

He goes on to determine the "statistical" orders of certain functions of the Picard type; the order of a function is said to be "statistical" if this order is obtained by restricting the choice of z, as for example in Rung's case, to a Stolz domain.

Let ϕ be a function which satisfies either of the following properties:

(i) $\phi(x)$ is non-negative, non-decreasing and integrable in $(0,1)$

or (ii) there is an integer $p \geq 1$ such that

$\phi(x), \phi'(x), \ldots, \phi^{(p-1)}(x)$ are absolutely continuous in $[0,1]$ and vanish at 1. Also we suppose that $\phi^{(p)}(x)$ has constant sign and $|\phi^{(p)}(x)|$ is non-decreasing in the subset of $(0,1)$ where $\phi^{(p)}(x)$ exists.

In the first case define $\Phi(n) = \int_{1-\frac{1}{n}}^{1} \phi(x)dx$; in the second case define $\Phi(n) = n^{-p}|\phi^{(p-1)}(1 - 1/n)|$.

The function $\phi(x) = (1-x)^s$ serves as an example of (i) if $-1 < s \leq 0$ and as an example of (ii) if $s > 0$.

We shall discuss here the class $A(\Psi,\phi)$ where Ψ is a general function and ϕ satisfies either of the properties given above. Then f holomorphic in D belongs to $A(\Psi,\phi)$ if $\int_D\int |f_\Psi(z)|^2 \phi(|z|)dxdy < \infty$.

Here we shall write $f(z) = \Sigma\, a_n z^n$, $z = re^{i\theta}$

$$f_\Psi(z) = \Sigma\, \Psi(n) a_n z^n$$

By letting $\Psi(n) = n$, $\phi(x) = (1-x)^s$ we have the class A^s. Also the Stolz domain is the example $\tau(r) = r$ of the tangential domain $R[\tau,\theta] = \{z: 1 - |z| \geq \tau(|e^{i\theta}-z|)\}$ in terms of which our "statistical" results are given.

a. An equivalent series condition

We state here a result of P.B. Kennedy [9] (Thms. 1 & 2).

<u>Theorem</u> B. Let $f(x) = \Sigma\, c_n x^n$; $0 \leq x < 1$; $c_n \geq 0$ and let ϕ be as above. Then $f(x)\phi(x) \in L(0,1)$ if and only if $\Sigma\, \Phi(n) c_n$ is convergent.

The following theorem gives an equivalence between $A(\Psi,\phi)$ and a class of functions with a certain order on their Taylor coefficients.

Theorem 4. Let $f(z) = \Sigma a_n z^n$; $z = re^{i\theta}$. Then $f \in A(\Psi,\phi)$ if and only if $\Sigma \Psi^2(n)|a_n|^2\Phi(n) < \infty$.

Proof.

$$\int_D\int |f_\Psi(z)|^2\phi(|z|)dxdy =$$

$$\int_0^{2\pi}\int_0^1 \Sigma\, \Psi(n)a_n r^n e^{in\theta} \cdot \Sigma\, \Psi(n)\bar{a}_n r^n e^{-in\theta}\phi(r)rdrd\theta =$$

$$\int_0^1 \Sigma\, \Psi^2(n)|a_n|^2 r^{2n+1} \cdot \phi(r)dr$$

These equalities together with theorem B, above where we are using $c_n = [\Psi(n)|a_n|]^2$, gives the desired result.

To obtain the proof of the following theorem, we make the following assumptions on the functions ϕ, ψ, f and h:

(1) If $\alpha > 0$, $h(\alpha r) \le C_\alpha h(r)$ where C_α is a constant depending on α alone.

(2) $|f_\Psi(z)|^k$ is subharmonic in D for $k > 0$.

(3) If $|z - \xi| < (1 - |\xi|)t$, $0 < t < 1$ then $\phi(|z|) \ge C_{\phi,t}\phi(|\xi|)$.

Theorem 5. Let $f(z)$ be holomorphic in D; $k > 0$ and

$$\int_D\int |f_\Psi(z)|^k\phi(|z|)dxdy < \infty.$$

Then $\displaystyle\lim_{z\to e^{i\theta}} \frac{|f_\Psi(z)|^k\phi(|z|)(1 - |z|)^2}{h(|z-e^{i\theta}|)} = 0$ except possibly for a set of $e^{i\theta}$ with zero h-measure.

Proof. Let $\xi \in D$

Let $r = (1 - |\xi|)t$, $0 < t < 1$ so that $D(\xi,r) \subset D$. By (2) above,

$$\pi r^2 |f_\Psi(\xi)|^k \le \iint\limits_{D(\xi,r)} |f_\Psi(z)|^k dxdy$$

Let $z \in D(\xi,r)$. By (3),

$$(4) \quad \pi(1 - |\xi|)^2 t^2 |f_\Psi(\xi)|^k \cdot C_{\phi,t}\phi(|\xi|) \le \iint\limits_{D(\xi,r)} |f_\Psi(z)|^k \phi(|z|)dxdy$$

Here we apply theorem A of Rung [11] cited above with $U(z) = |f_\Psi(z)|^k \phi(|z|)$, $R = 2|\xi - e^{i\theta}|$ and using the assumption (1) so that $h(2|\xi - e^{i\theta}|) \le C\, h(|\xi - e^{i\theta}|)$. From this we find

$$\lim_{\xi \to e^{i\theta}} \frac{\iint\limits_{D(e^{i\theta},R)} |f_\Psi(z)|^k \phi(|z|)dxdy}{h(R)} = 0$$

except on a set of $e^{i\theta}$ with h-measure 0. Together with the inequality (4), this gives the desired result.

Next we let $\gamma(t)$ be a function such that $\phi \cdot \gamma$ is a function satisfying either properties (i) or (ii) as described above. We define $\chi(n)$ to be the function which is defined so that $\frac{\Phi(n)\Psi^2(n)}{\chi^2(n)}$ is the discrete function corresponding to the continuous function $\phi\gamma$. For example, if $\phi \cdot \gamma$ satisfies property (i) then

$$\frac{\Phi(n)\Psi^2(n)}{\chi^2(n)} = \int_{1-\frac{1}{n}}^{1} \phi\gamma(t)dt \; .$$

Let $f_\chi(z) = \Sigma\, \chi(n) a_n z^n$. We now use χ to extend the previous theorem.

Theorem 6. Let $f(z) = \Sigma\, a_n z^n$, $f \in A(\Psi,\phi)$. Then

$$\lim_{z \to e^{i\theta}} \frac{|f_\chi(z)|\{\phi\gamma(|z|)\}^{\frac{1}{2}}(1 - |z|)}{\sqrt{h(|z - e^{i\theta}|)}} = 0$$

except possibly on a set of $e^{i\theta}$ of h-measure 0.

Proof.

Let $g(z) = \Sigma \frac{\chi(n)}{\Psi(n)} a_n z^n$.

We note that $\Psi^2 \cdot \frac{\chi^2}{\Psi^2} \cdot |a_n|^2 \cdot (\frac{\Psi^2 \cdot \Phi}{\chi^2}) = \Psi^2 \cdot \Phi \cdot |a_n|^2$ so that by theorem 4, $g \in A(\Psi,\phi\gamma)$ since here $\phi\gamma$ is to $\frac{\Phi\Psi^2}{\chi^2}$ as ϕ is to Φ in theorem 4. Therefore we apply theorem 5 with $k = 2$ to obtain:

$$\lim_{z \to e^{i\theta}} \frac{|g_\Psi(z)|^2 \phi\gamma(|z|)(1 - |z|)^2}{h(|e^{i\theta} - z|)} = 0$$

except possibly on a set of $e^{i\theta}$ of h-measure 0.

The theorem follows upon noting that $g_\Psi(z) = f_\chi(z)$ and taking of square roots.

b. The generalized tangential limit

Let $G \in A(\Psi,\phi)$. We use the results of either of the two preceding theorems to determine an order for G.

Let τ be an increasing function with $\tau(0) = 0$ and such that $h \circ \tau$ is a measure function.

Let $z \in R[\tau,\theta]$ so that

$$1 - |z| \geq \tau(|e^{i\theta} - z|)$$

and $h(1 - |z|) \geq h\circ\tau(|e^{i\theta} - z|)$,

$$\frac{1}{h\circ\tau(|e^{i\theta}-z|)} \geq \frac{1}{h(1-|z|)}$$

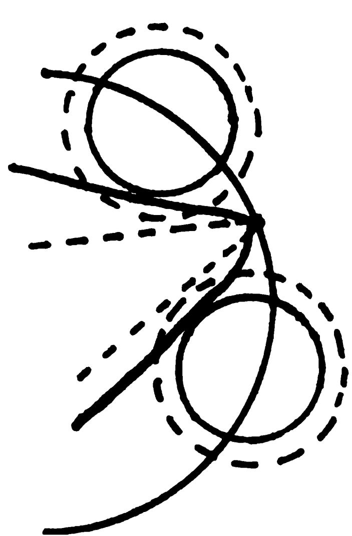
Then $\dfrac{|G_\Psi(z)|^2\phi(|z|)(1-|z|)^2}{h\circ\tau(|e^{i\theta}-z|)} \to 0$ as $z \to e^{i\theta}$, $z \in R[\tau,\theta]$

except possibly for a set of $e^{i\theta}$ whose $h\circ\tau$-measure is zero, which implies that for the same exceptional set

$$\frac{|G_\Psi(z)|^2\phi(|z|)(1-|z|)^2}{h(1-|z|)} \to 0 \quad \text{as} \quad z \to e^{i\theta},\ z \in R[\tau,\theta] \ .$$

As an example, let $h(r) = r^\alpha$, $0 < \alpha \leq 1$, and let $\tau(r) = r^\beta$, $1 \leq \beta \leq 1/\alpha$. Then the resulting order holds for z confined to a tangential approach of order of tangency $\beta-1$ for all $e^{i\theta}$ outside a set whose $\alpha\beta$-capacity is zero. In addition, the statistical result is more simply stated if we choose $\phi(r) = (1-r)^s$, $s > -1$.

In determining a tangential approach, as in [10], for instance we note that we are covering the circles bad points, i.e. points where limits fail to exist, by a system of circles with certain radii. The smaller the radii, the larger the permissible tangential approach, i.e. the approach missing these circles.



By describing our measure function $h \circ \tau$ in the previous example using the technique of Cargo (see §1) of composing the measure function h with the approach function τ, we are able to realize a wider approach with a still small exceptional set. Naturally this leads to a loss in the result which describes the order of the function.

Applying theorem 5 to the above example we find

$$|G_\Psi(z)|(1 - |z|)^{s/2+1-\alpha/2} \to 0, \; z \to e^{i\theta}, \; z \in R[\tau,\theta]$$

except possibly on a set of $e^{i\theta}$ of $\alpha\beta$-capacity 0. Where $\alpha > 0$ is small we are permitted to choose values of $\beta \geq 1$ which are large, maintaining the relation $\alpha\beta \leq 1$. This permits then a tangential approach of larger order of tangency, while at the same time, once β is fixed, gives rise to an extremely small exceptional set since we can choose α (and hence $\alpha\beta$) to be arbitrarily small. However the smaller α then diminishes the order of the overall result. Therefore the wider approach and smaller exceptional set are obtained at the expense of the statistical type order.

If, in theorem 6, we choose $\Psi(n) = n$, $\phi(t) = (1-t)^s$, $s > -1$ and so $\Phi(n) = n^{-s-1}$, and let $\gamma(t) = (1-t)^{2\beta-2}$ where $s + 2\beta > 1$, we then find $\chi(n) = n^\beta$ so that $f_\chi(z) = f^\beta(z)$, the fractional derivative of order β of f. We have then the following result obtained by Rung [11, p. 329]: Except on a set of h-measure 0,

$$\lim_{z \to e^{i\theta}} \frac{|f^\beta(z)|(1 - |z|)^{\frac{s+2\beta}{2}}}{\sqrt{h(|e^{i\theta}-z|)}} = 0 .$$

This result is applied to the Picard function $Q(z)$ which is holomorphic in D and omits there the values $\pm 2\pi n i$ and which belongs to the class $A^{2+\varepsilon}$ (see Rung [11, pp. 329-330] for details). Choosing $\beta = 0$ we find

$$\lim_{z \to e^{i\theta}} \frac{|Q(z)|(1 - |z|)^{1+\varepsilon}}{\sqrt{h \circ \tau(|e^{i\theta} - z|)}} = 0$$

except for $e^{i\theta}$ which belong to a set of $h \circ \tau$-measure 0.
Again letting $h(r) = r^{\alpha}$, $\tau(r) = r^{\delta}$, $0 < \alpha \le 1$ and $\delta \ge 1$, with $z \in R[\tau, \theta]$ we find

$$\frac{|Q(z)|(1 - |z|)^{1+\varepsilon}}{(1 - |z|)^{\alpha/2}} \to 0 \quad \text{as} \quad z \to e^{i\theta}$$

except at most on a set of $\alpha\delta$-capacity 0. The result of Rung that $\lim_{z \to e^{i\theta}} |Q(z)|(1 - |z|)^{\frac{1}{2}+\varepsilon} = 0$, z confined to a Stolz angle, for almost every θ, is obtained by choosing $\alpha = \delta = 1$.

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