RCT FILTER SYNTHESIS

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This is to certify that the

thesis entitled

RCΓ FILTER SYNTHESIS

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Ronald L. McNally

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ABSTRACT

RCΓ FILTER SYNTHESIS

by

Ronald L. McNally

A general basis for the synthesis of active RCΓ (resistor-capacitor-gyrator) filter is presented where all gyrators are grounded.

The approach requires the analysis of three terminal RC sections cascaded through active (unbalanced) grounded gyrators. Two important theorems are established as a consequence of this analysis:

1. If the RC sections satisfy the following conditions:
    i) Each RC section is connected;
    ii) For the input and intermediate RC sections the edges corresponding to the conductances form a connected graph when the input and output terminals (of the section) are grounded;
    iii) For the last or output RC section, the edges corresponding to the conductances form a connected graph when the input terminals are short circuited;
    iv) All RC sections, except the output section,

contain at least two terminals in addition to the ground terminal;

v) All RC sections contain one or more conductances;

then the RCΓ filter is stable and remains stable irrespective of RC component or gyrator parameter variation.

2. The minimum number of real poles of the voltage-ratio transfer function of a low-pass RCΓ filter can be determined from the gyrator placement.

It is demonstrated that the Calahan [CA] and Horowitz [TH] polynomial decompositions can be derived one from the other. Both polynomial decomposition methods are extended to include those polynomials which contain distinct negative real zeros.

It is also established that fourth degree low-pass (high-pass) voltage-ratio transfer functions of the form $T_v = k/P(s)$ ($T_v = ks^4/P(s)$), where P(s) is strictly Hurwitz, can always be realized with two-gyrator RCΓ networks.

Realization procedures using a computer program are established for realizing fourth or higher degree RCΓ

[CA] Calahan, D. A., "Restrictions on the Natural Frequencies of an RC-RL Network," Journal of the Franklin Institute, Vol. 272, pp. 112-133 (August 1961).

[TH] Thomas, R. E., "Polynomial Decomposition in Active Network Synthesis," IRE Transactions on Circuit Theory, CT-8, pp. 270-274 (September 1961).

filters from the voltage-ratio transfer functions. Practical examples are realized and displayed in the form of tables.

RCΓ FILTER SYNTHESIS

by

Ronald L. McNally

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LIST OF SYMBOLS

R	resistor - resistance
C	capacitor - capacitance
L	inductor - inductance
Γ	gyrator - either active or passive
NIC	negative impedance converter
$-R$	negative resistor or resistance
Δ	determinant of an admittance matrix
Δ_{ij}	determinant of admittance matrix when the ith row and jth column are removed
$\Delta(\frac{i}{j}, RC)$	determinant of an RC admittance matrix when the ith row and jth column are removed
$\Delta(\frac{i}{j}, RC\Gamma)$	determinant of an $RC\Gamma$ matrix when the ith row and jth column are removed

CHAPTER I

INTRODUCTION

1.1 Background

In the technical literature very little theoretical work has appeared concerning the synthesis of RCΓ filters. Calahan [CA1] has published some basic work on polynomial decomposition which is applicable to one-gyrator RCΓ networks. He also has shown that, whenever possible, a one-gyrator RCΓ filter realization of a transfer function is always less sensitive to parameter variations than an equivalent RC-NIC filter realization of the same transfer function [CA2]. To the best of the author's knowledge, no work has appeared in the literature on RCΓ filters containing more than one gyrator.

1.2 Purpose

The purpose of this thesis is to provide a theoretical basis for RCΓ filter synthesis using one or more gyrators and also to develop realization procedures for synthesizing low-pass voltage-ratio transfer functions with RC networks and one or more gyrators. This problem is of practical interest since the use of gyrators provides a means of realizing complex natural frequencies without inductors. The elimination of inductors is desirable for the following reasons:

i) the design of high quality inductors is difficult and costly at low frequencies

ii) inductors cannot be "grown" in integrated circuits

iii) inductors are difficult to miniaturize in micro-miniature circuit technology.

Two other methods for eliminating inductors in filter realizations are the use of NIC's (negative impedance converters) and controlled sources. Both of these methods introduce the possibility of instability into the network realization whereas ideal gyrators, being passive components cannot cause a network to become unstable. This property also holds for active gyrators in a restricted network topology as is shown in this thesis.

The use of one-gyrator filter synthesis is not possible, in most cases, due to rather stringent requirements on the transfer function [CA1]. By allowing more than one gyrator, these requirements can be relaxed or removed completely as is shown in Chapter IV. It is always possible to replace an inductor by a capacitor loaded gyrator or by using two grounded gyrators and one capacitor [HT]. The latter, although it is a practical method, is somewhat extravagant as the realization of a fourth degree voltage-ratio low-pass transfer function would require four grounded gyrators. It is shown in this thesis that such fourth degree voltage-ratio low-pass transfer functions can be realized with at most two grounded gyrators.

1.3 Summary of Chapters

In Chapter II, the basic form of the denominator polynomial of voltage-ratio transfer functions T_V is established for various RCΓ network configurations. Separating the denominator polynomial of a given T_V into these basic forms constitutes the first step in any RCΓ network synthesis procedure. A theorem is proved which establishes the stability of RCΓ filters consisting of a broad class of three terminal RC sections cascaded through active gyrators. Another theoren is proved which establishes the minimum number of real poles for a transfer function T_V corresponding to a RCΓ network consisting of low-pass RC ladders cascaded through gyrators.

In Chapter III, the derivation of the Calahan [CA1] and the Horowitz [HO] polynomial decomposition (separation into forms suitable for network realization) methods from each other is established. Extensions of these decomposition methods to polynomials containing distinct negative real zeros, in addition to complex zeros, are also established. The use of these decomposition methods is demonstrated with sample filter realizations.

In Chapter IV, a theorem is proved which establishes that any low-pass voltage-ratio transfer function T_V which has a strictly Hurwitz fourth degree denominator, can always be realized with a two-gyrator RCΓ filter. Analytic and computer realizations are given to sample problems. In addition, sample computer realizations are given for some fifth and sixth degree low-pass voltage-ratio transfer functions. An extended version of Calahan's angle condition is conjectured for cases where the number of gyrators is greater than one. Finally, a theorem is proved which establishes a two-gyrator RCΓ network realization for high-pass voltage-ratio transfer functions of the complex variable s, $T_V(s)$, when $T_V(\frac{1}{s})$ has a two-gyrator RCΓ realization.

1.4 Preliminary Definitions

Definition 1.4.1 An RC admittance function is a real rational function in the complex variable s of the form $N(s)/D(s)$ where the zeros of $N(s)$ and $D(s)$ alternate along the negative real axis and the largest zero of $N(s)$ is less than or equal to zero and greater than the largest zero of $D(s)$.

Definition 1.4.2 An RC impedance function has the properties of the reciprocal of an RC admittance function.

Definition 1.4.3 An RL admittance function has the properties of the reciprocal of an RC admittance function.

Definition 1.4.4 An RL impedance function has the properties of an RC admittance function.

Definition 1.4.5 A passive, ideal, (or balanced) gyrator is a 3 or 4 terminal network component which is represented by Fig. 1.4.1 and has the admittance matrix

$$\begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix} \tag{1.4.1}$$

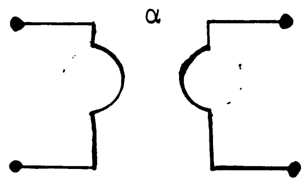
where $\alpha^2 > 0$.

Definition 1.4.6 An active or unbalanced gyrator is a 3 or 4 terminal network component which is represented by Fig. 1.4.2 and has the admittance matrix

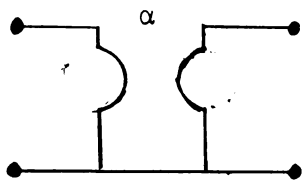
$$\begin{bmatrix} 0 & \tilde{\alpha} \\ -\alpha & 0 \end{bmatrix} \tag{1.4.2}$$

where $\alpha\,\tilde{\alpha} > 0$.

Definition 1.4.7 A polynomial in the complex variable s is called strictly Hurwitz if it has all of its zeros in the open left half of the s-plane.



Four terminal case



Three terminal case

Fig. 1.4.1 Gyrator representations.

CHAPTER II

THE DERIVATION OF SOME PROPERTIES OF RCΓ FILTERS

In this chapter, the properties of open-circuit voltage-ratio transfer functions, T_V, for grounded RCΓ (resister capacitor gyrator) networks are considered. The reason for developing these properties is to use them in synthesizing low pass filter configurations. The results, however, are applicable in the synthesis of band pass and high pass configurations, since the desired properties are developed in a more general context than the low-pass case. Initially an (n+1)-terminal RC network in which ideal gyrators are embedded is analyzed. Since the denominator polynomial of the transfer function T_V for such general networks does not give any clue to the realization of T_V, it is necessary to impose certain restrictions on this general network configuration. However, properties to be developed for this more general class of networks are applicable to the more restricted classes of RCΓ networks. From the practical point of view, the RCΓ network is restricted to RC sections cascaded through RC bridged and grounded gyrators. Further simplification is achieved by cascading grounded RC sections through grounded gyrators only.

A theorem is proved for a class of networks establishing the invariance of the denominator polynomial of T_V when active gyrators replace ideal gyrators. Another theoren is proved which establishes the stability of a class of RC-(active gyrator) networks. Finally, when the RCΓ network is restricted to low-pass RC ladder sections interconnected through active or passive gyrators, a theoren is established which gives the minimum number of real poles that the transfer function T_V can have.

2.1 The General Form of the Denominator Polynomial for RC Transfer Functions.

Consider an (n+1)-node connected RCΓ network in which each gyrator has three terminals. It is a known fact [KTK] that for the complete solvability of a network containing ideal gyrators, a formulation tree should exist such that both the edges corresponding to a gyrator are included in this tree or in its co-tree. If this topological condition is not satisfied, then the network cannot have a complete solution.

In order to insure that the branch equations for the RLΓ network can be written in a suitable form, the following assumptions are used throughout the thesis.

Assumptions:

i) The RC portion of the given RCΓ network is connected

ii) The Γ portion of the given RCΓ network contains no circuits.

Under the above assumptions both the edges corresponding to a gyrator can be included in a formulation tree. For such a formulation tree the branch equations for the RCΓ network can be written in the following form:

$$I = YV$$

Or, in detail,

$$\begin{bmatrix} I_1\\ 0\\ 0\\ \vdots\\ 0\\ 0\\ I_n \end{bmatrix} = \left\{ \begin{bmatrix} & & \\ & \text{RC admittance matrix} & \\ & & \end{bmatrix} + \begin{bmatrix} & \alpha_1 & & & & \\ -\alpha_1 & & & & & \\ & & & \alpha_2 & & \\ & & -\alpha_2 & & & \\ & & & & & \alpha_n \\ & & & & -\alpha_n & \end{bmatrix} \right\} \begin{bmatrix} V_1\\ V_2\\ \vdots\\ V_n \end{bmatrix}$$

$$I = (Y_{RC} + Y_{\Gamma})V \qquad (2.1.1)$$

By assumption, the RC portion of the RCΓ network is connected, hence the admittance matrix Y_{RC} in Eq. 2.1.1 is symmetric and non-singular; in fact, it is positive definite for real and poitive values of the complex variable "s". For this reason the quadratic form associated with the admittance matrix Y can be written as

$$X^TYX = X^TY_{RC}X + X^TY_{\Gamma}X = X^TY_{RC}X \qquad (2.1.2)$$

Since Y_{Γ} is skew symmetric, the quadratic form associated with Y_{Γ} vanishes identically and therefore Eq. w.1.2 implies that the admittance matrix Y (although nonsymmetric) is positive definite for all real and positive values of "s". This proves that Y is non-singular. The entries in the first and the last position in the first and the last position in the first column of Y^{-1} are Δ_{11}/Δ and Δ_{n1}/Δ, respectively, where $\Delta = |Y|$ and Δ_{11}, Δ_{n1} represent cofactors of order (n-1) of Y. In Eq. 2.1.1, if $I_n = 0$, then $V_1 = (\Delta_{11}/\Delta)I_1$, $V_n = (\Delta_{n1}/\Delta)I_,$, and the open circuit voltage-ratio transfer function is given by

$$T_V = \Delta_{n1}/\Delta_{11} \qquad (2.1.3)$$

The forms for Δ and Δ_{11} are identical, as can be seen from Eq. 2.1.1, and therefore, except for specific cases, only the properties of Δ will be discussed.

<u>Theorem 2.1.1</u>. If an (n+1)-terminal single gyrator RCΓ network satisfies assumptions i and ii, then

$$\Delta = \Delta(RC) + \alpha_1^2\Delta(\tfrac{i}{i},\tfrac{i+1}{i+1}, RC) \qquad (2.1.4)$$

where $\Delta(\tfrac{i}{j}, RC)$ denotes the determinant of the RC admittance matrix in Eq. 2.1.1 in which the i-th row and the j-th column are deleted. Proof: Consider the expression of the admittance matrix given in Eq. 2.1.1

$$Y = \begin{bmatrix} \text{RC} \\ \text{admittance} \\ \text{matric} \\ \\ \end{bmatrix} + \begin{array}{c} \begin{array}{cc} i & i+1 \end{array} \\ \left[\begin{array}{cc} \cdots\cdots & \alpha_1 \\ \cdots -\alpha_1 & \\ \end{array}\right] \end{array} \begin{array}{l} i \\ i+1 \end{array}$$

Forming the determinant of a sum of matrices, one obtains

$$\Delta = |Y| = \Delta(RC) + \alpha_1 \Delta({}^{\,i}_{i+1}, RC)(-1)^{2i+1}$$
$$-\alpha_1 \Delta({}^{i+1}_{\,i}, RC)\ (-1)^{2i+1} + \alpha_1^2 \Delta({}^{i}_{i}, {}^{i+1}_{i+1}, RC) \tag{2.1.5}$$

From the symmetry property of the RC admittance matrix, one has $\Delta({}^{\,i}_{i+1}, RC) = \Delta({}^{i+1}_{\,i}, RC)$. Therefore Eq. 2.1.5 takes the form

$$\Delta = \Delta(RC) + \alpha_1^2 \Delta({}^{i}_{i}, {}^{i+1}_{i+1}, RC) \tag{2.1.6}$$

This completes the proof.

Note: The expression for Δ_{11} corresponding to Eq. 2.1.6 is

$$\Delta_{11} = \Delta({}^{1}_{1}, RC) + \alpha_1^2 \Delta({}^{1}_{1}, {}^{i}_{i}, {}^{i+1}_{i+1}, RC) \tag{2.1.7}$$

If i = 1 the second term in equation 2.1.7 does not occur since the syrator constant α_1 (appearing in two places) is removed with the first row and the first column of Y. In this case

$$\Delta_{11} = \Delta(i, RC)$$

and the effect of the gyrator on the open-circuit voltage-ratio transfer function, $T_V = \Delta_{n1}/\Delta_{11}$, is lost.

It is desirable to use no more gyrators than are necessary, therefore the following assumption will be added to (i) and (ii).

Assumption:

iii) The RCΓ network will contain no gyrator branches connected directly across the imput terminal vertices.

Calahan has shown [CA1] that the zeros of the numerator of the sum of RC and RL immittance functions obey certain angle conditions. Since this Theoren is applicable to Eq. 2.1.6 and will be used in later developments, it is simply stated here. (For proof, see [CA1]).

<u>Theorem (After Calahan)</u>

Let the polynomial N(S) be of the form

$$N(S) = \prod_{i=1}^{n} (S + p_i) \prod_{i=1}^{m} (S + S_i)(S = \hat{S}_i),$$

then N(S) is the numerator of the sum of an RC and an RL immittance function if the only if

(a) for $n = 0$, $\sum_{i=1}^{n} \arg(S_i) \leq \frac{\pi}{2}$

(b) for $n > 0$, $p_i > 0$ and $\sum_{i=1}^{n} \arg(S_i) \leq \frac{\pi}{2}$

where the imaginary part of S_i is

$$\mathrm{Im}(S_i) > 0$$

<u>Theorem 2.1.2</u>. In Theoren 2.1.1, let

$$\Delta = [\prod_{i=1}^{n} (s+c_i)][\prod_{i=1}^{m} (S+a_i+jb_i)(s+a_i-jb_i)]$$

with c_i, a_i real and $b_i > 0$, then

$$\sum_{i-1} \tan^{-1} \frac{b_i}{a_i} = \sum_{i=1}^{m} \arg(a_i+jb_i) \leq \frac{\pi}{2}$$

and $c_i > 0$ for $n \neq 0$

Proof: From Thorem 2.1.1

$$\Delta = \Delta(RC) + \alpha_1^2 \Delta(\begin{smallmatrix} i & i+1 \\ i, & i+1 \end{smallmatrix}, RC)$$

Dividing both sides of the above equation by $\Delta(\begin{smallmatrix} i \\ i \end{smallmatrix}, RC)$ one can form

$$\frac{\Delta}{\Delta(\begin{smallmatrix} i \\ i \end{smallmatrix}, RC)} = \frac{\Delta(RC)}{\Delta(\begin{smallmatrix} i \\ i \end{smallmatrix}, RC)} + \alpha_1^2 \frac{\Delta(\begin{smallmatrix} i & i+1 \\ i, & i+1 \end{smallmatrix}, RC)}{\Delta(\begin{smallmatrix} i \\ i \end{smallmatrix}, RC)}$$

The right hand side of this equation can be recognized as the sum of an RC and an RL admittance function, i.e.

$$\frac{\Delta}{\Delta(\begin{smallmatrix} i \\ i \end{smallmatrix}, RC)} = y_{RC} + y_{RL}$$

with Calahan's Theorem this completes the proof.

<u>Theorem 2.1.3</u>. If an (n+1)-terminal RCΓ network satisfies assumptions (i), (ii), and (iii) and contains two gyrators Γ_1 and Γ_2 such that the edges corresponding to Γ_1 are the i-th and (i+1)-th edges of the formulation

tree while those corresponding to Γ_2 are the j-th and (j+1)-th edges of the formulation tree; then

$$\begin{aligned}
\Delta = {} & \Delta(RC) + \alpha_1^2\ \Delta\left({i \atop i},{i+1 \atop i+1},RC\right) \\
& + \alpha_2^2\Delta\ \left({j \atop j},{j+1 \atop j+1},RC\right) + \alpha_1^2\alpha_2^2\Delta\left({i \atop i},{i+1 \atop i+1},{j \atop j},{j+1 \atop j+1},RC\right) \\
& + 2\alpha_1\alpha_2\Delta\left({i \atop i+1},{j \atop j+1},RC\right) - 2\alpha_1\alpha_2\Delta\left({i \atop i+1},{j+1 \atop j},RC\right)
\end{aligned} \tag{2.1.8}$$

Proof: The form of the Y matrix is

$$Y = \begin{bmatrix} \text{RC} \\ \text{admittance} \\ \text{matrix} \end{bmatrix} + \begin{array}{c} \begin{array}{cccc} i & i+1 & j & j+1 \end{array} \\ \begin{bmatrix} -\alpha_1 & \alpha_1 & & \\ & & & \\ & & & \alpha_2 \\ & & -\alpha_2 & \end{bmatrix} \end{array} \begin{array}{l} i \\ i+1 \\ j \\ j+1 \end{array}$$

Forming the determinant of Y, one has

$$\begin{aligned}
\Delta = {} & \Delta(RC) + \alpha_1\ \Delta\left({i+1 \atop i},RC\right) \\
& - \alpha_1\Delta\left({i \atop i+1},RC\right) + \alpha_2\Delta\left({j+1 \atop j},RC\right) \\
& -\alpha_2\Delta\left({j \atop j+1},RC\right) + \alpha_1^2\ \Delta\left({i \atop i},{i+1 \atop i+1},RC\right) \\
& +\alpha_2^2\ \Delta\left({j \atop j},{j+1 \atop j+1},RC\right) + \alpha_1\alpha_2\ \Delta\left({j \atop j+k},{i \atop i+1},RC\right) \\
& +\alpha_1\alpha_2\ \Delta\left({j+1 \atop j},{i+1 \atop i},RC\right) - \alpha_1\alpha_2\Delta-\ \left({j \atop j+1},{i+1 \atop i},RC\right) \\
& -\alpha_1\alpha_2\ \Delta\left({j+1 \atop j},{i \atop i+1},RC\right) - \alpha_1^2\alpha_2\ \Delta\left({i \atop i},{i+1 \atop i+1},{j \atop j+1},RC\right) \\
& +\alpha_1^2\alpha_2\ \Delta\left({i \atop i},{i+1 \atop i+1},{j+1 \atop j},RC\right) - \alpha_2^2\alpha_1\ \Delta\left({i \atop i+1},{j \atop j},{j+1 \atop j+1},RC\right)
\end{aligned}$$

$$+ \alpha_2^2\alpha_1 \; \Delta\left(\begin{smallmatrix}i+1\\i\end{smallmatrix},\begin{smallmatrix}j\\j\end{smallmatrix},\begin{smallmatrix}j+1\\j+1\end{smallmatrix},RC\right)$$

$$+ \alpha_1^2\alpha_2^2 \; \Delta\left(\begin{smallmatrix}i\\i\end{smallmatrix},\begin{smallmatrix}i+1\\i+1\end{smallmatrix},\begin{smallmatrix}j\\j\end{smallmatrix},\begin{smallmatrix}j+1\\j+1\end{smallmatrix},RC\right) \qquad (2.1.9)$$

Due to the symmetry property of Y_{RC}, certain terms will be cancelled and Eq. 2.1.9. reduces to Eq. 2.1.8. Hence the proof.

Equation (2.1.8) is unsuitable for practical synthesis procedures due to the presence of terms corresponding to $\alpha_1\alpha_2$. In the following sections further developments will be considered for a restricted class of RCΓ network configurations to remove this difficulty.

2.2. RC Networks Cascaded Through Bridged Gyrators.

In this section a restricted class of RC networks is considered. As is shown in Fig. 2.2.1, 3-terminal RC networks are cascaded through bridged gyrators. For this class of networks the properties of the determinant Δ of the admittance matrix Y are given in the following theorem.

Theorem 2.2.1. If three grounded RC sections are cascaded through two grounded gyrators Γ_1 and Γ_2, where the gyrators are bridged with capacitors and/or resistors as shown in Fig. 2.2.1 and where the complete network satisfies assumptions i, ii, and iii, then

$$\Delta = \Delta(RC) + \alpha_1^2\,\Delta\left({i,\;i+1 \atop i,\;i+1},RC\right)$$
$$+\; \alpha_2^2\,\Delta\left({j,\;j+1 \atop j,\;j+1},RC\right)$$
$$+\; \alpha_1^2\alpha_2^2\,\Delta\left({i,\;i+1,\;j,\;j+1 \atop i,\;i+1,\;j,\;j+1},RC\right) \qquad (2.2.1)$$

Proof: Since both the RC sections and the gyrators are grounded, a Lagrangian formulation tree in the corresponding network graph can be chosen such that it contains both the edges corresponding to the gyrators. The Y matrix is of the form

$$Y = \begin{array}{c} \\ i \\ i+1 \\ \\ j \\ j+1 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} i & i+1 & j & j+1 \end{array} \\ \left[\begin{array}{c|c|c} Y_a & -Y_1(s) & \\ \hline -Y_1(s) & Y_b & -Y_2(s) \\ \hline & -Y_2(s) & Y_c \end{array}\right] \end{array} + \left[\begin{array}{c|c|c} & \alpha_1 & \\ \hline -\alpha_1 & & \alpha_2 \\ \hline & -\alpha_2 & \end{array}\right] \qquad (2.2.2)$$

where $Y_i(s) = c_i s + b_i$ and $c_i,\ b_i > 0 \quad i = 1,2$

Let

$$\overline{Y}_b = \left[\begin{array}{c|c} Y_B & \alpha_2 - Y_2(s) \\ \hline -\alpha_2 - Y_2(s) & Y_c \end{array}\right]$$

Then Eq. 2.2.2 can be written as

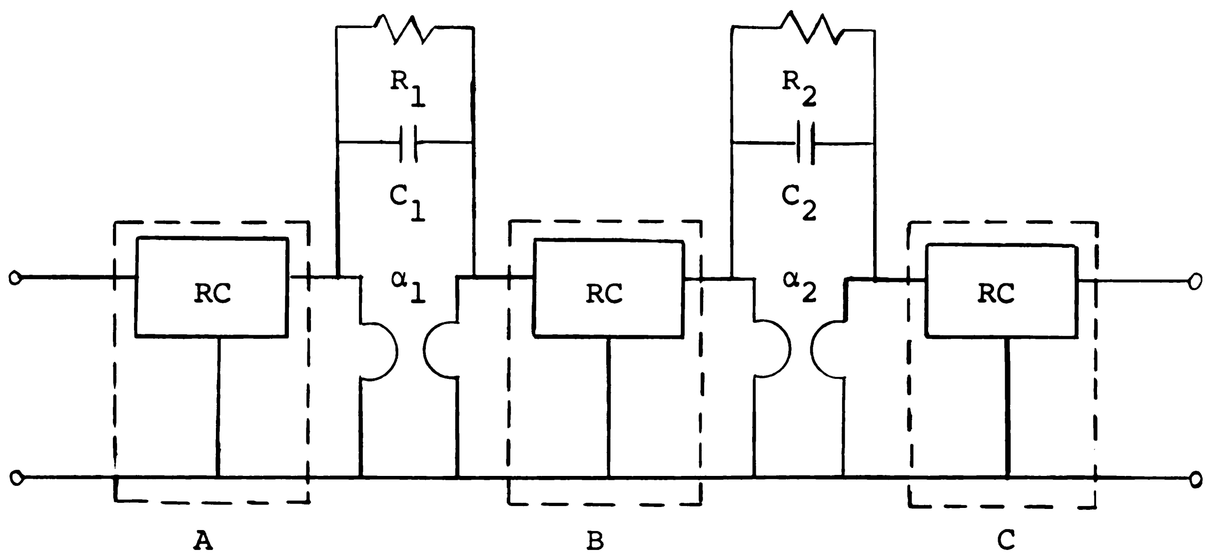


Fig. 2.2.1 Three terminal RC networks cascaded through RC-bridged-gyrators.

$$Y = \begin{array}{c} \\ i \\ i+1 \\ \\ \end{array} \left[\begin{array}{cc|cc} & i & i+1 & \\ Y_a & & & \\ \hline & -\alpha_1 - Y_1(s) & \alpha_1 - Y_1(s) & \\ & & & \\ & & \overline{Y}_B & \end{array}\right]$$

If the determinant of Y is expanded about the first i rows using Laplace's expansion, one has

$$\Delta = \Delta_a(RC)\ \overline{\Delta}_b(RC\Gamma_2)$$
$$- \ (Y_1^2(s) - \alpha_1^2)\ {}_a\left({i \atop i}, RC\right)\overline{\Delta}_b\left({i+1 \atop i+1}, RC\Gamma_2\right),$$

or

$$\Delta = \Delta_a(RC)\overline{\Delta}_b(RC\Gamma_2) - Y_1^2(s)\ \Delta_a\left({i \atop i}, RC\right)\overline{\Delta}_b\left({i+1 \atop i+1}, RC\Gamma_2\right)$$
$$+ \ \alpha_1^2\ \Delta_a\left({i \atop i}, RC\right)\overline{\Delta}_b\left({i+1 \atop i+1}, RC\Gamma_2\right).$$

This expression is equivalent to

$$\Delta = \Delta(RC\Gamma_2) + \alpha_1^2\ \Delta\left({i \atop i}, {i+1 \atop i+1}, RC\Gamma_2\right). \qquad (2.2.3)$$

Since $(RC\Gamma_2)$ and $\Delta\left({i \atop i}, {i+1 \atop i+1}, RC\Gamma_2\right)$ correspond to matrices of RC networks containing only one gyrator, Eq. 2.2.3 can be written as

$$\Delta = \Delta(RC) + \alpha_2^2\ \Delta\left({j \atop j}, {j+1 \atop j+1}, RC\right)$$
$$+ \ \alpha_1^2\left[\Delta\left({i \atop i}, {i+1 \atop i+1}, RC\right) + \alpha_2^2\ \Delta\left({i \atop i}, {i+1 \atop i+1}, {j \atop j}, {j+1 \atop j+1}, RC\right)\right]$$
$$(2.2.4)$$

This completes the proof.

The form of Eq. 2.2.4 is essentially suitable for synthesis procedures; however, if each term of the expansion could be identified with a specific RC section appearing in Fig. 2.2.1, then the hynthesis would be facilitated. For this purpose the bridged gyrators will be replaced by unbridged gyrators. This change also allows active or unbalanced gyrators to be used in place of passive gyrators.

2.3 RC Networks Cascaded with Gyrators

In this section 3-terminal RC networks cascaded through passive or active gyrators are considered. The complete network, as shown in Fig. 2.3.1, is assumed to satisfy assumptions i, ii, and iii. The properties of these networks are contained in the following theorems.

<u>Theorem 2.3.1</u>. If grounded RC networks a, b, c,... are cascaded through grounded gyrators only, as shown in Fig. 2.3.1, then the following hold:

Case 1. For one gyrator and two RC networks

$$\Delta = \Delta_a(RC)\Delta_b(RC) + \alpha_1^2 \Delta_a(\tfrac{i}{i},RC)\Delta_b(\tfrac{i+1}{i+1},RC) \tag{2.3.1}$$

Case 2. For two gyrators and three RC networks

$$\Delta = \Delta_a(RC)\Delta_b(RC)\Delta_c(RC) \tag{2.3.2}$$

$$+ \alpha_1^2 \Delta_a(\tfrac{i}{i},RC)\Delta_b(\tfrac{i+1}{i+1},RC)\Delta_c(RC)$$

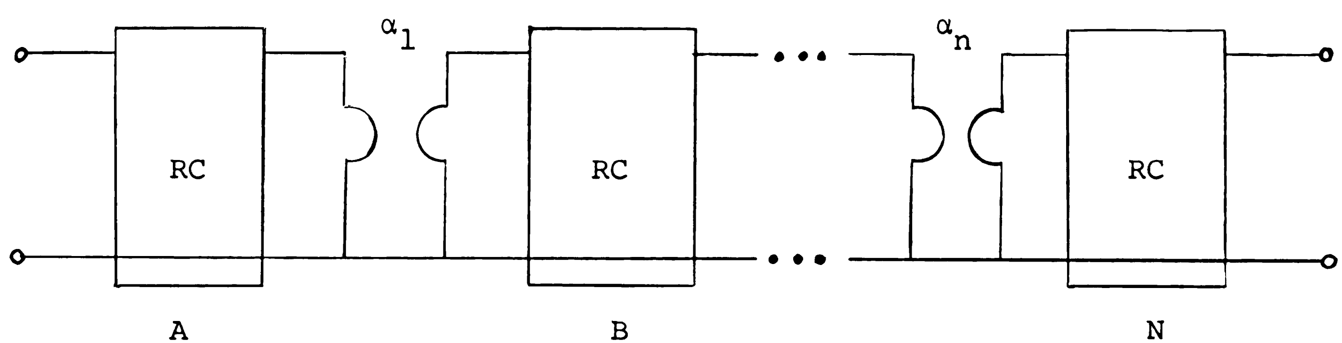


Fig. 2.3.1 Three terminal RC networks cascaded through gyrators.

$$+ \alpha_1^2 \; \Delta_a(RC)\Delta_b(\genfrac{}{}{0pt}{}{j}{j},RC)\Delta_c(\genfrac{}{}{0pt}{}{j+1}{j+1},RC)$$

$$+ \alpha_1^2\alpha_2^2 \; \Delta_a(\genfrac{}{}{0pt}{}{i}{i},RC)\Delta_b(\genfrac{}{}{0pt}{}{i+1}{i+1},\genfrac{}{}{0pt}{}{j}{j},RC)\Delta_c(\genfrac{}{}{0pt}{}{j+1}{j+1},RC)$$

$$j \neq i+1$$

Case 3. For three gyrators and four RC networks

$$\begin{aligned}
\Delta = {} & \Delta_a(RC)\Delta_b(RC)\Delta_c(RC)\Delta_d(RC) \qquad (2.3.3)\\
& + \alpha_1^2 \; \Delta_a(\genfrac{}{}{0pt}{}{i}{i},RC)\Delta_b(\genfrac{}{}{0pt}{}{i+1}{i+1},RC)\Delta_c(RC)\Delta_d(RC)\\
& + \alpha_2^2 \; \Delta_a(RC)\Delta_b(\genfrac{}{}{0pt}{}{j}{j},RC)\Delta_c(\genfrac{}{}{0pt}{}{j+1}{j+1},RC)\Delta_d(RC)\\
& + \alpha_3^2 \; \Delta_a(RC)\Delta_b(RC)\Delta_c(\genfrac{}{}{0pt}{}{k}{k},RC)\Delta_d(\genfrac{}{}{0pt}{}{k+1}{k+1},RC)\\
& + \alpha_1^2\alpha_2^2 \; \Delta_a(\genfrac{}{}{0pt}{}{i}{i},RC) \; \Delta_b(\genfrac{}{}{0pt}{}{i+1}{k+1},\genfrac{}{}{0pt}{}{j}{j},RC)\Delta_c(\genfrac{}{}{0pt}{}{j+1}{j+1},RC)\Delta_d(RC)\\
& + \alpha_1^2\alpha_3^2 \; \Delta_a(\genfrac{}{}{0pt}{}{i}{i},RC)\Delta_b(\genfrac{}{}{0pt}{}{i+1}{i+1},RC)\Delta_c(\genfrac{}{}{0pt}{}{k}{k},RC)\Delta_d(\genfrac{}{}{0pt}{}{k+1}{k+1},RC)\\
& + \alpha_2^2\alpha_3^2 \; \Delta_a(RC)\Delta_b(\genfrac{}{}{0pt}{}{j}{j},RC)\Delta_c(\genfrac{}{}{0pt}{}{j+1}{j+1},\genfrac{}{}{0pt}{}{k}{k},RC)\Delta_d(\genfrac{}{}{0pt}{}{k+1}{k+1},RC)\\
& + \alpha_1^2\alpha_2^2\alpha_3^2 \; \Delta_a(\genfrac{}{}{0pt}{}{i}{i},RC)\Delta_b(\genfrac{}{}{0pt}{}{i+1}{i+1},\genfrac{}{}{0pt}{}{j}{j},RC)\Delta_c(\genfrac{}{}{0pt}{}{j+1}{j+1},\genfrac{}{}{0pt}{}{k}{k},RC)\Delta_d(\genfrac{}{}{0pt}{}{k+1}{k+1},RC)
\end{aligned}$$

Proof:

Case 1. In this case the admittance matrix is

$$Y = \begin{array}{cc} & \begin{array}{cc} i & i+1 \end{array} \\ \begin{array}{c} i \\ i+1 \end{array} & \left[\begin{array}{c|c} Y_a & \alpha_1 \\ \hline -\alpha_1 & Y_b \end{array}\right] \end{array}$$

and

$$\Delta = \Delta_a(RC)\Delta_b(RC) + \alpha_1^2 \; \Delta_a(\genfrac{}{}{0pt}{}{i}{i},RC)\Delta_b(\genfrac{}{}{0pt}{}{i+1}{i+1},RC) \qquad (2.3.4)$$

Case 2. Replace Y_b in case 1 by

$$\overline{Y}_b = \begin{array}{c|c} & \begin{array}{ccc} i+1 & j & j+1 \end{array} \\ \begin{array}{c} i+1 \\ \\ j \\ j+1 \\ \\ \end{array} & \left[\begin{array}{c|c} Y_b & \alpha_2 \\ \hline -\alpha_2 & Y_c \end{array}\right] \end{array}$$

then from Case 1

$$\overline{\Delta}_b(RC\Gamma) = \Delta_b(RC)\Delta_c(RC) + \alpha_2^2\ \Delta_b\left(\begin{smallmatrix} j \\ j \end{smallmatrix}, RC\right)\Delta_c\left(\begin{smallmatrix} j+1 \\ j+1 \end{smallmatrix}, RC\right)$$

and

$$\begin{aligned} \overline{\Delta}_b\left(\begin{smallmatrix} i+1 \\ i+1 \end{smallmatrix}, RC\Gamma\right) &= \Delta_b\left(\begin{smallmatrix} i+1 \\ i+1 \end{smallmatrix}, RC\right)\Delta_c(RC) \\ &\quad + \alpha_2^2\ \Delta_b\left(\begin{smallmatrix} i+1 \\ i+1 \end{smallmatrix}, \begin{smallmatrix} j \\ j \end{smallmatrix}, RC\right)\Delta_c\left(\begin{smallmatrix} j+1 \\ j+1 \end{smallmatrix}, RC\right) \end{aligned}$$

If now $\Delta_b(RC)$ and $\Delta_b\left(\begin{smallmatrix} i+1 \\ i+1 \end{smallmatrix}, RC\right)$ are replaced respectively by $\overline{\Delta}_b(RC\Gamma)$ and $\overline{\Delta}_b\left(\begin{smallmatrix} i+1 \\ i+1 \end{smallmatrix}, RC\Gamma\right)$, then Eq. 2.3.4 becomes Eq. 2.3.2.

Case 3. Replace Y_c in case 2 by

$$\overline{Y}_c = \begin{array}{c|c} & \begin{array}{cc} k & k+1 \end{array} \\ \begin{array}{c} k \\ k+1 \end{array} & \left[\begin{array}{c|c} Y_c & \alpha_3 \\ \hline -\alpha_3 & Y_d \end{array}\right] \end{array}$$

then from Case 1

$$\overline{\Delta_c}(RC\Gamma) = \Delta_c(RC)\Delta_d(RC) + \alpha_3^2\ \Delta_c\left(\begin{smallmatrix} k \\ k \end{smallmatrix}, RC\right)\Delta_d\left(\begin{smallmatrix} k+1 \\ k+1 \end{smallmatrix}, RC\right)$$

and

$$\begin{aligned} \overline{\Delta_c}\left(\begin{smallmatrix} j+1 \\ j+1 \end{smallmatrix}, RC\Gamma\right) &= \Delta_c\left(\begin{smallmatrix} j+1 \\ j+1 \end{smallmatrix}, RC\right)\Delta_d(RC) \\ &\quad + \alpha_3^2\Delta_c\left(\begin{smallmatrix} j+1 \\ j+1 \end{smallmatrix}, \begin{smallmatrix} k \\ k \end{smallmatrix}, RC\right)\Delta_d\left(\begin{smallmatrix} k+1 \\ k+1 \end{smallmatrix}, RC\right) \end{aligned}$$

If now $\Delta_c(RC)$ and $\Delta_c(\begin{smallmatrix}j+1\\j+1\end{smallmatrix},RC)$ are replaced by $\overline{\Delta_c}(RC\Gamma)$ and $\Delta_c(\begin{smallmatrix}j+1\\j+1\end{smallmatrix},RC\Gamma)$, respectively, then Eq. 2.3.2 becomes Eq. 2.3.3

This completes the proof.

Note: The above Theorem could be extended to the general case; however, the number of terms in the expansion for Δ is 2^n which becomes prohibitively large for the number of gyrators $n \geq 3$.

Theorem 2.3.2. For the two gyrator case of Theorem 2.3.1, with the terms of Eq. 2.3.2 in the array

$$\begin{array}{l}
\Delta_a(RC)\Delta_b(RC)\,\Delta_c(RC)\\
\alpha_1^2\,\Delta_a(\begin{smallmatrix}i\\i\end{smallmatrix},RC)\,\Delta_b(\begin{smallmatrix}i+1\\i+1\end{smallmatrix},RC)\,\Delta_c(RC)\\
\alpha_2^2\Delta_a(RC)\,\Delta_b(\begin{smallmatrix}j\\j\end{smallmatrix},RC)\,\Delta_c(\begin{smallmatrix}j+1\\j+1\end{smallmatrix},RC)\\
\alpha_1^2\alpha_2^2\;\Delta_a(\begin{smallmatrix}i\\i\end{smallmatrix},RC)\,\Delta_b(\begin{smallmatrix}i+1\\k+1\end{smallmatrix},\begin{smallmatrix}j\\j\end{smallmatrix},RC)\,\Delta_c(\begin{smallmatrix}j+1\\j+1\end{smallmatrix},RC)
\end{array} \qquad (2.3.5)$$

the sum of each indicated pair forms a polynomial

$$N(S) = \prod_{i=1}^{n} (s + c_i) \prod_{i=1}^{m} (s + a + jb)(s + a - jb)$$

with $c_i \geq 0$ and

$$\sum_{i=1}^{m} \tan^{-1} \frac{b}{a} \leq \frac{\pi}{2} .$$

Proof: Consider the pair

$$N(S) = \Delta_a(RC)\Delta_b(RC)\Delta_c(RC) + \alpha_1^2\;\Delta_a(\begin{smallmatrix}i\\i\end{smallmatrix},RC)\Delta_b(\begin{smallmatrix}i+1\\i+1\end{smallmatrix},RC)\Delta_c(RC)$$

which can be written

$$N(S) = \Delta_c(RC)[\Delta_b(RC)\Delta_c(RC) + \alpha_1^2 \ \Delta_a(\tfrac{i+1}{i+1},RC))] \tag{2.3.6}$$

From Theorem 2.1.2, the terms within brackets in Eq. 2.3.6 can be replaced by

$$P(S) = \prod_{i=1}^{\ell} (s + c_i) \prod_{i=1}^{m} (s + a + jb)(s + a - jb) \tag{2.3.7}$$

with $c_i > 0$ and

$$\sum_{i=1}^{m} \tan^{-1} \frac{b}{a} \leq \frac{\pi}{2}$$

$\Delta_c(RC)$ is the determinant of an RC admittance matrix and so has non positive real zeros. Therefore Eq. 2.3.6 can be written as

$$N(S) = \prod_{i=\ell+1}^{n} (s + c_i)[\prod_{i=1}^{\ell} (s + c_i) \prod_{i=1}^{n} (s + a + jb)(s + a - jb)]$$

with $c_i > 0$ and

$$\sum_{i=1}^{m} \tan^{-1} \frac{b}{a} \leq \frac{\pi}{2}$$

The proof of the remaining three pairs follows similar lines. Q.E.D.

<u>Theorem 2.3.3</u>. Consider the three gyrator case in Theorem 2.3.1 with the terms in Eq. 2.3.3 represented by the power of α_i in the following array

$$\begin{array}{l} 1 \\ \alpha_1^2 \\ \alpha_2^2 \\ \alpha_3^2 \\ \alpha_1^2\alpha_2^2 \\ \alpha_1^2\alpha_3^2 \\ \alpha_2^2\alpha_3^2 \\ \alpha_1^2\alpha_2^2\alpha_3^2 \end{array}$$

In this array the sum of each indicated pair (from Eq. 2.3.3) forms a polynomial

$$N(S) = \prod_{i=1}^{n} (s + c_i) \prod_{i=1}^{m} (s + a + jb)(s + a - jb)$$

with $\sum_{i=1}^{m} \tan^{-1} \frac{b}{a} \leq \frac{\pi}{2}$ and $c_i \geq 0$

Proof: The proof is similar to that used in Theorem 2.3.2 and will not be repeated here.

Note: Theorems 2.3.2 and 2.3.3 establish necessary conditions which must be satisfied by any polynomial decomposition technique used in the synthesis of two and three gyrator RCΓ networks of the form shown in Fig. 2.3.1.

<u>Theorem 2.3.4</u>. If in Theorems 2.3.1, 2.3.2, and 2.3.3 the passive gyrators are replaced by active gyrators, then the form of Δ remains invariant.

Proof: Consider the Y matrix:

$$Y = \begin{array}{c c} & \begin{array}{cc} \quad i & \quad i+1 \end{array} \\ \begin{array}{c} i \\ i+1 \end{array} & \left[\begin{array}{c|c} \overline{Y}_a & \\ \hline & \beta_i \\ -\tilde{\beta}_i & \\ & \overline{Y}_b \end{array}\right] \end{array} \qquad \begin{array}{l} \beta_i \tilde{\beta}_i > 0 \\ \beta_i, \tilde{\beta}_i \quad \text{real} \end{array}$$

where $\overline{Y}_a$ and $\overline{Y}_b$ have the same form as Y. If the determinant of Y is expanded about the first i rows, then

$$\Delta = \overline{\Delta}_a(RC\Gamma)\overline{\Delta}_b(RC\Gamma)$$
$$+ \tilde{\beta}_i \ \beta_i \ \overline{\Delta}_a(\tfrac{i}{i}, RC\Gamma)\overline{\Delta}_b(\tfrac{i+1}{k+1}, RC\Gamma).$$

However taking $\alpha_i = \sqrt{\tilde{\beta}_i \ \beta_i}$, one has

$$\Delta = \Delta_a(RC\Gamma)\Delta_b(RC\Gamma) + \alpha_i^2 \ \Delta_a(\tfrac{i}{i}, RC\Gamma)\Delta_b(\tfrac{i+1}{i+1}, RC\Gamma).$$

Since an arbitrary active gyrator can always be replaced by an equivalent passive gyrator, Δ has the same form for active as well as for passive gyrators. Q.E.D.

2.4 Stability of the Transfer Function.

In the preceeding section, networks such as that in Fig. 2.3.1 were shown to have equally stable transfer functions using either balanced or unbalanced gyraors. That is, Δ and Δ_{11} are invariant when passive gyrators are replaced by active ones.

Complete stability can be shown for grounded RC sections cascaded with grounded gyrators (passive or active) where each RC section has the following properties:

1. Each RC section is connected.
2. The graph corresponding to the conductances contains all the nodes of the section and is either connected or is connected when the two terminal nodes of the RC section are connected (shorted) to the ground node. For the last section it is connected when the input node is connected to the ground.
3. Any RC section, except the last, has at least two nodes in addition to the ground node.
4. Each section contains one or more conductances.

Three important classes of RC sections which satisfy the properties 1-4 are low and high-pass RC ladders and Twin-T or Bridged-T RC notch filters.

Theorem 2.4.1. Let an RC section, with admittance matrix Y_S, satisfy conditions 1-4:

Case 1. If the section is an intermediate or leading section, then

$$\Delta_s(\tfrac{i}{i},\tfrac{j}{j},RC) > 0 \text{ for } s = 0$$

Case 2. If the section is the last or output section, then

$$\Delta_s(\tfrac{i}{i},RC) > 0 \text{ for } s = 0$$

where i is the input terminal and j is the output terminal of the RC section.

Proof:

Case 1. The Y_s matrix is

$$Y_s = Cs + G$$

Note that $Y_s|_{s=0} = G$

From condition 2, when both the input and output terminal is connected to the ground, the conductance graph becomes connected. This corresponds to the ith row and column and the jth row and column being removed from the matrix G. In general a conductance matrix is positive semi-definite. However, for a connected graph it is positive definite, hence it is nonsingular. It follows, therefore, that $\Delta_s(\frac{i}{i},\frac{j}{j},RC) > 0$ for $s = 0$. If i and j are the only nodes in Y_s, then $\Delta_s(\frac{i}{i},\frac{j}{j},RC) = 1$.

Case 2. From condition 2, when the input terminal is connected to the ground, the conductance graph becomes connected. This corresponds to the ith row and column being removed from the matrix G. For a connected graph a conductance matrix is positive definite. Therefore

$$\Delta_s(\tfrac{i}{i},RC) > 0 \text{ for } s = 0.$$

If $i = j$, then $\Delta_s(\frac{i}{i},RC) = 1$ Q.E.D.

Theorem 2.4.2. If a network is made up of RC sections satisfying conditions 1-4 cascaded through grounded gyrators, then Δ_{11} is strictly Hurwitz.

Proof: The proof is established using an induction on the number of gyrators. Consider first the one gyrator case:

$$\Delta_{11} = \Delta_a(\tfrac{1}{1}, RC)\,\Delta_b(RC) + \beta_1\bar{\beta}_1\;\Delta_a(\tfrac{1}{1}, \tfrac{i}{i}, RC)\,\Delta_b(\tfrac{i+1}{i+1}, RC) \tag{2.4.1}$$

Since conductance matrices are positive semi-definite, one has

$$\Delta_{11}\big|_{s=0} \geq \beta_1\bar{\beta}_1\;\Delta_a(\tfrac{1}{1}, \tfrac{i}{i}, RC)\,\Delta_b(\tfrac{i+1}{i+1}, RC)\big|_{s=0}$$

However from Theorem 2.4.1

$$\Delta_a(\tfrac{1}{1}, \tfrac{i}{i}, RC)\big|_{s=0} > 0 \text{ and } \Delta_b(\tfrac{i+1}{k+1}, RC)\big|_{s=0} > 0 \tag{2.4.2}$$

Therefore

$$\Delta_{11}\big|_{s=0} > 0 \tag{2.4.3}$$

If Eq. 2.4.1 is rewritten as

$$\frac{\Delta_{11}}{\Delta_b(RC)\,\Delta_a(\tfrac{1}{1}, \tfrac{i}{i}, RC)} = \frac{\Delta_a(\tfrac{1}{1}, RC)}{\Delta_a(\tfrac{1}{1}, \tfrac{i}{i}, RC)} + \beta_1\tilde{\beta}_1\;\frac{\Delta_b(\tfrac{i+1}{i+1}, RC)}{\Delta_b(RC)} \tag{2.4.4}$$

Then the right hand side of Eq. 2.4.4 can be recognized as the sum of RC and RL admittance functions. Consider the real part of the first term in Eq. 2.4.4 when $s = j\omega$

$$\mathrm{Re}\left\{\left.\frac{\Delta_a(\tfrac{1}{1}, RC)}{\Delta_a(\tfrac{1}{1}, \tfrac{i}{i}, RC)}\right|_{s=j\omega}\right\} = \mathrm{Re}\left\{h_0 j\omega + \sum_{i=1}^{n}\frac{h_i(j\omega)}{j\omega + c_i} + h_\infty\right\}$$

$$h_0,\ h_i,\ h_\infty \geq 0$$

$$\mathrm{Re}\left\{\left.\frac{\Delta_a(\tfrac{1}{1}, RC)}{\Delta_a(\tfrac{1}{1}, \tfrac{i}{i}, RC)}\right|_{s=j\omega}\right\} = h_\infty + \sum_{i=1}^{n}\frac{h_i\,\omega^2}{c_i^2 + \omega^2} \qquad (2.4.5$$

From condition 4, at least one h_i or h_∞ must be nonzero. Therefore Eqs. 2.4.4 and 2.4.5 imply that

$$\mathrm{Re}\left\{\left.\frac{\Delta_{11}}{\Delta_b(RC)\,\Delta_a(\tfrac{1}{1},\tfrac{i}{i},RC)}\right|_{x=j\omega}\right\} \geq \mathrm{Re}\left\{\left.\frac{\Delta_a(\tfrac{1}{1},RC)}{\Delta_a(\tfrac{1}{1},\tfrac{i}{i},RC)}\right|_{s=j\omega}\right\} > 0 \tag{2.4.6}$$

where $\Delta_b(RC)$ and $\Delta_a(\frac{1}{1},\frac{i}{i},RC)$ are the determinants of RC admittance matrices and can have no zeros on the $j\omega$ axis except at the origin. Therefore Eq. 2.4.3 and 2.4.6 imply that Δ_{11} has no zeros on the $j\omega$ axis. Since Δ_{11} is the numerator of a sum of RC and RL admittance functions, it is the numerator of a positive real function and can therefore have no zeros in the right-half of the s-plane. Hence Δ_{11} is strictly Hurwitz. This completes the proof for the one gyrator case. Assume now that Δ_{11} is strictly Hurwitz for K gyrators. Let Y be the admittance matrix of (K+2) RC sections satisfying conditions 1-4 cascaded through (K+1) gyrators:

$$Y = \left[\begin{array}{c|c} Y_a & \beta_1 \\ \hline -\tilde{\beta}_1 & Y_b \end{array}\right]$$

where Y_b is the admittance matrix of the last (K+1) RC networks cascaded through K gyrators. From Y, Δ_{11} is:

$$\Delta_{11} = \Delta_a(\tfrac{1}{1},RC)\,\Delta_b(RC\Gamma) + \beta_1\tilde{\beta}_1\;\Delta_a(\tfrac{1}{1},\tfrac{i}{i},RC)\,\Delta_b(\tfrac{i+1}{i+1},RC\Gamma) \tag{2.4.7}$$

where, by assumption, $\Delta_b(\begin{smallmatrix}i+1\\i+1\end{smallmatrix},RC)$ is strictly Hurwitz.

Observe first that $\Delta_{11}|_{s=0} > 0$. This can be shown as follows: $\Delta_a(\begin{smallmatrix}1\\1\end{smallmatrix},RC)|_{s=0} \geq 0$, since a conductance matrix is positive semi-definite. Also $\Delta_a(\begin{smallmatrix}1\\1\end{smallmatrix},\begin{smallmatrix}i\\i\end{smallmatrix},RC)|_{s=0} > 0$ by Theorem 2.4.1 and $\Delta_b(RC\ )|_{s=0} \geq 0$, since a conductance matrix is positive semi-definite. Furthermore, $\Delta_b(\begin{smallmatrix}i+1\\k+1\end{smallmatrix})RC\Gamma)|_{s=0} > 0$ since $\Delta_b(\begin{smallmatrix}i+1\\i+1\end{smallmatrix},RC\Gamma)$ is strictly Hurwitz by assumption. From the above inequalities and Eq. 2.4.7 one obtains

$$\Delta_{11}|_{s=0} > \beta_1\tilde{\beta}_1\ \Delta_a(\begin{smallmatrix}1\\1\end{smallmatrix},\begin{smallmatrix}i\\i\end{smallmatrix},RC)\Delta_b(\begin{smallmatrix}i+1\\i+1\end{smallmatrix},RC\Gamma)|_{s=0} > 0 \tag{2.4.8}$$

The second observation is the positiveness of $\Delta_{11}|_{s=j\omega}$. This can be shown as follows: Consider the ratio

$$\frac{\Delta_{11}}{\Delta_a(\begin{smallmatrix}i\\i\end{smallmatrix},RC)\Delta_b(\begin{smallmatrix}i+1\\i+1\end{smallmatrix},RC)} = \frac{\Delta_a(\begin{smallmatrix}1\\1\end{smallmatrix},\begin{smallmatrix}i\\i\end{smallmatrix},RC)}{\Delta_a(\begin{smallmatrix}i\\i\end{smallmatrix},RC)} + \tilde{\beta}_{n+1}\beta_{n+1}\frac{\Delta_b(RC\Gamma)}{\Delta_b(\begin{smallmatrix}i+1\\i+1\end{smallmatrix},RC\Gamma)} \tag{2.4.9}$$

Now $\Delta_b(RC\Gamma)/\Delta_b(\begin{smallmatrix}i+1\\i+1\end{smallmatrix},RC\Gamma)$ is positive real if all the gyrators are passive. However by Theorem 2.3.4. the form of Δ and Δ_{11} is invariant when passive gyrators are replaced by active gyrators. Therefore $\Delta_b(RC\Gamma)/\Delta_b(\begin{smallmatrix}i+1\\i+1\end{smallmatrix},RC\Gamma)$ is positive real and the real part of $\Delta_b(RC\Gamma)/\Delta_b(\begin{smallmatrix}i+1\\i+1\end{smallmatrix},RC\Gamma)|_{s=j\omega}$ is greater than zero. The first expression on the right hand side of Eq. 2.4.9 can be recognized as an RL admittance function with the following real part when $s = j\omega$:

$$\mathrm{Re}\left\{\left.\frac{\Delta_a\left({}^{1,\,i}_{1,\,i}, RC\right)}{\Delta_a\left({}^{i}_{i}, RC\right)}\right|_{s=j\omega}\right\} = \mathrm{Re}\left\{h_0 + \sum_{i=1}^{n} \frac{h_i}{s + c_i} + \frac{h_\infty}{s}\right\} \tag{2.4.10}$$

where c_i, h_0, h_i, $h_\infty \geq 0$. From condition 4 at least one h_i or h_0 is nonzero. Therefore

$$\mathrm{Re}\left\{\left.\frac{\Delta_a\left({}^{1,\,i}_{1,\,i}, RC\right)}{\Delta_a\left({}^{i}_{i}, RC\right)}\right|_{s=j\omega}\right\} = h_0 + \sum_{i=1}^{n} \frac{h_i\, c_i}{\omega^2 + c_1^2} > 0 \tag{2.4.11}$$

From Eqs. 2.4.8, 2.4.9, and 2.4.11 it follows that

$$\mathrm{Re}\left\{\left.\frac{\Delta_{11}}{\Delta_a\left({}^{i}_{i}, RC\right)\Delta_b\left({}^{i+1}_{i+1}, RC\Gamma\right)}\right|_{s=j\omega}\right\} > 0 \tag{2.4.12}$$

$\Delta_b\left({}^{i+1}_{i+1}, RC\Gamma\right)$ is strictly Hurwitz by assumption, $\Delta_a\left({}^{i}_{i}, RC\right)$ is the determinant of an RC admittance matrix and so has negative real zeros except possibly at the origin. Therefore Δ_{11} is the numerator of a positive real function with no zeros on the $j\omega$ axis. Eq. 2.4.8 shows that Δ_{11} can have no zeros at the origin and hence Δ_{11} is strictly Hurwitz. This completes the induction and the theorem is now proved.

2.5 Minimum Number of Real Poles of T_V for low-Pass RC Ladders Cascaded with Gyrators.

Theorem 2.4.2 considered in the preceeding section establishes the absolute stability of the transfer function T_V for RCΓ networks satisfying conditions 1-4. Necessary conditions on the number of real zeros of Δ_{11}

with respect to the number of capacitors and the gyrator placement can be established if the RC sections are low-pass RC ladders. This is given in the following theorem.

<u>Theorem 2.4.1</u>. If:

i) A connected network is composed of (n+1) low pass RC ladder networks cascaded through n grounded gyrators.

ii) The edges corresponding to the capacitors and the input voltage driver form a Lagrangian tree in the complete network graph.

iii) The total number of capacitors in the first RC section, third RC section, and so on alternately is denoted as $\#C^{(1)}$. The total number of capacitors remaining is denoted as $\#C^{(2)}$

then Δ_{11} has at least

$$\left|\#C^{(1)} - \#C^{(2)}\right| \tag{2.5.1}$$

real zeros.

Proof: The admittance matrix is of the form

$$Y = Cs + G + \Gamma \tag{2.5.2}$$

Now the admittance matrix Y' corresponding to Δ_{11} can be obtained from Y by removing from Y the first row and column:

$$Y' = C's + G' + \Gamma' \tag{2.5.3}$$

Condition ii ensures that C is diagonal with positive diagonal entries. Conditions i and ii ensure that $G' + \Gamma'$ is tridiagonal. Let $C' + \Gamma' = H'$ then

$$Y' = C's + H'$$

since C' is diagonal with positive diagonal entires, $[C']^{-1/2}$ exists and is diagonal with positive diagonal entries, and the determinant of $[C']^{-1/2}\, Y'[C']^{-1/2}$ differs from Δ_{11} by a constant. Let $Y'' = [C']^{-1/2}\, Y'\, [C']^{-1/2}$ then

$$Y'' = U\, s + [C']^{-1/2}$$

$$Y'' = U\, s + H''$$

The zeros of Δ_{11} are therefore the eigenvalues of -H". The proof follows as an adaption of a method described by Frame [FR]. Since the diagonal matrix $[C']^{-1/2}$ has all positive diagonal entries, the sign matrix of $Y'' = [C']^{-1/2}\, Y'\, [C']^{-1/2}$ is the same as that for Y'.

The rest of the proof can best be facilitated by a specific case before the general proof is completed.

Consider the network in Fig. 2.5.1 where the Y matrix is

$$Y = \left[\begin{array}{ccc|cc|cc}
G_1 & -G_1 & & & & & \\
-G_1 & C_1s+G_1+G_2 & -G2 & & & & \\
 & -G_2 & C_2s+G_2 & \beta_1 & & & \\
\hline
 & & -\tilde{\beta}_1 & C_3s+G_3 & G_3 & & \\
 & & & -G & C_4s+G_3 & \beta_2 & \\
\hline
 & & & & \tilde{\beta}_2 & C_5s+G_5 & -G_5 \\
 & & & & & -G_5 & C_6s+G_5+G_6
\end{array}\right]$$

For this case

$$
Y'' = \left[\begin{array}{cc|cc|cc}
s+a_1 & -b_1 & & & & \\
-b_1 & s+a_2 & c_1 & & & \\
\hline
 & -\tilde{c}_1 & s+a_3 & -b_3 & & \\
 & & -b_3 & s+a_4 & c_2 & \\
\hline
 & & & -\tilde{c}_2 & s+a_5 & -b_5 \\
 & & & & -b_5 & s+a_6
\end{array}\right]
$$

Multiplying the third and fourth column of Y" by -1, i.e. multiplying those columns corresponding to the second RC network the fourth RC network and so on alternately, one obtains

$$
Y''' = \left[\begin{array}{cc|cc|cc}
s+a_1 & -b_1 & & & & \\
-b_1 & s+a_2 & -c_1 & & & \\
\hline
 & -\tilde{c}_1 & -(s+a_3) & +b_3 & & \\
 & & +b_3 & -(s+a_4) & c_2 & \\
\hline
 & & & +\tilde{c}_2 & s+a_5 & -b_5 \\
 & & & & -b_5 & s+a_6
\end{array}\right]
$$

This results in a symmetric sign matrix and at worst changes the sign of the determinant. Now forming the determinants of the principal minors one obtains

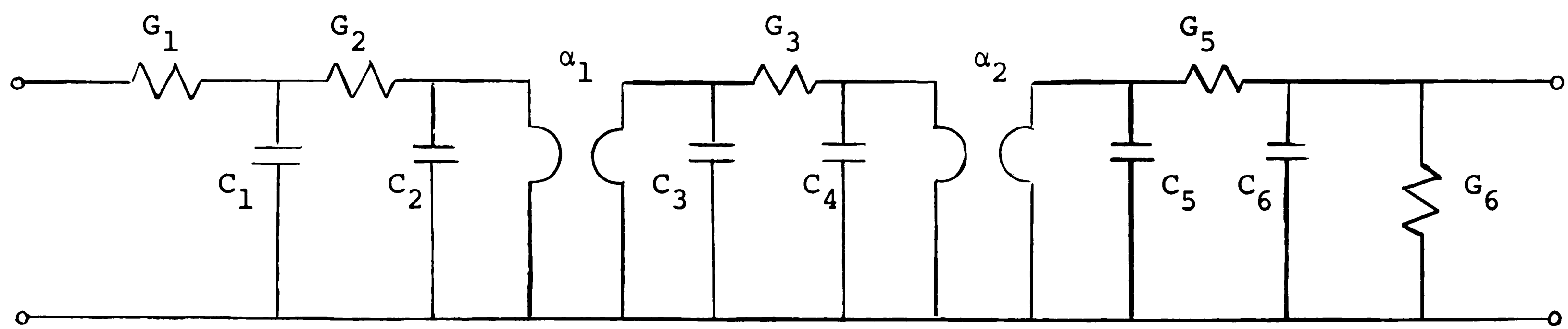


Fig. 2.5.1 RCΓ network for Theorem 2.5.1

$$P_0 \doteq 1$$
$$P_1 \equiv s + a_1$$
$$P_2 \equiv (s + a_2)P_1 - b_1^2 P_0$$
$$P_3 \equiv -(s + a_3)P_2 - c_1\tilde{c}_1 P_1$$
$$P_4 \equiv -(s + a_4)P_3 - b_3^2 P_2$$
$$P_5 \equiv (s + a_5)P_4 - c_2\tilde{c}_2 P_3$$
$$P_6 \equiv (s + a_6)P_5 - b_5^2 P_4 = \pm k\, \Delta_{11}$$

The polynomials P_6 through P_0 constitute a Sturm sequence. The Cauchy index [TO]

$$I_{-\infty}^{\infty} \frac{P_5}{P_6} = V(-\infty) - V(\infty)$$

gives a minimum number of real zeros of P_6, where $V(a)$ denotes the variation of sign in the Sturm sequence at $s = a$.

	$-\infty$	$+\infty$
P_0	+	+
P_1	-	+
P_2	+	+
P_3	+	-
P_4	+	+
P_5	-	+
P_6	+	+

$V(-\infty) = 4 \quad V(+\infty) = 2$

Therefore for the network under consideration Δ_{11} must have two or more real zeros. In the general case one would have

$$P_0(s) = 1$$

$$P_1(s) = (s + a_1)P_0$$

$$P_2(s) = (s + a_2)P_1(s) - b_1^2 P_0(s)$$

$$\vdots$$

$$P_K(s) = (s + a_K)P_{K-1}(s) - b_{K-1}^2 P_{K-2}(s)$$

K capacitors in the first RC section

$$P_{K+1}(s) = -(s + a_{K+1})P_K(s) - \beta_1 \tilde{\beta}_1 \; P_{K-1}(s)$$

$$P_{K+s}(s) = -(s + a_{K+2})P_{K+1}(s) - b_{K+1}^2 \; P_K(s)$$

$$\vdots$$

$$P_\ell(s) = -(s + a_\ell)P_{\ell-1}(s) - b_{\ell-1}^2 P_{\;-2}(s)$$

ℓ-k capacitors in next RC section

$$\vdots$$

may be more negative terms or may not be

$$P_n(s) = \pm\, k\, \Delta_{11} = \quad (s - a_K)P_{n-1}(s) - b_{n-1}^2 P_{n-2}(s).$$

Therefore,

	$V(-\infty)$		$V(\infty)$	
P_0	+		+	
P_1	−		+	
	+	always changes	+	never changes
	−		+	
	⋮		⋮	
			+	
P_k	$\pm$			
P_{k+1}	$\pm$		−	
	$\pm$			
P_{k+2}		does not change	+	always changes
	⋮		⋮	
P	$\pm$		$\pm$	
			$\pm$	
$P_{\;+1}$	$\mp$	always changes		does not change
			$\pm$	
	$\pm$		$\pm$	
	$\mp$		$\pm$	

From P_K to P_ℓ, $(\ell-k)$ sign changes were lost in $V(-\infty)$ and $(\ell-k)$ sign changes were gained in $V(+\infty)$. In the same way the number of sign changes lost and gained can be calculated for the 4th RC section, the 6th RC section and so on. Since $\ell-k$ is the number of capacitors in the second RC section, it follows that for n, the total number of capacitors,

$$V(-\infty) = n - \#C^{(2)} = \#C^{(1)}$$

$$V(+\infty) = \#C^{(2)}$$

$$V(-\infty) - V(+\infty) = \#C^{(1)} - \#C^{(1)}$$

From which it follows that

$$\mathop{I}_{-\infty}^{\infty} \frac{P_{n-1}}{P_n} = \#C^{(1)} - \#C^{(2)}$$

and so P_n has at least

$$|\#C^{(1)} - \#C^{(2)}| \text{ real zeros.} \qquad \text{Q.E.D.}$$

<u>Theorem 2.5.2</u>. Low-pass RC ladders cascaded through gyrators form a low-pass network.

Proof: The Y matrix is tridiagonal and the C matrix is diagonal. Therefore no "s" terms appear in the off diagonal entries of Y. Δ_{n1} is the determinant of the matrix obtained from Y when its first row and last column is removed. This matrix is upper triangular and if diagonal entries are the lower off diagonal entries of Y. Therefore Δ_{n1} is not a function of s. The proof follows from $T_V = \Delta_{n1}/\Delta_{11}$.

2.6 Conclusion.

In this chapter an analysis of RCΓ networks was carried out which developed certain necessary conditions on RCΓ filters. The one gyrator case was shown to have an angle condition [CA1]. The transfer function of RCΓ networks composed of "notch" filters or low-pass ladders cascaded through passive or active gyrators was shown to be stable. Finally a theorem was proved giving a minimum number of real poles in the transfer function for a certain class of low-pass RCΓ filters.

CHAPTER III

DEVELOPMENT OF REALIZATION FORMULAS FOR ONE-GYRATOR RCΓ FILTERS

In the second chapter of this thesis, the general form of the denominator polynomial for the open circuit voltage ratio transfer function of RCΓ networks containing one or more gyrators is established. Calahan [CA1] has given necessary and sufficient conditions under which a given set of complex frequencies may be realized as the natural frequencies of a network consisting of two interconnected 2-terminal networks, one being RC and the other RL. His network configuration, however, can be reduced to the form under consideration in this thesis, as shown in Fig. 3.0.1.

In this chapter, a new derivation of Calahan's polynomial decomposition [CA1] is given. This derivation also indicates the relationship between two seemingly different polynomial decomposition methods which are referred to, in the literature, as the Horowitz [HO] and as the Calahan [CA1] polynomial decompositions. That is, it is shown how these two decomposition methods can be developed from each other. Methods of extending the

Calahan and Horowitz decomposition methods to polynomials which have distinct negative real zeros are established. Examples are given to illustrate how these decomposition methods can be utilized to realize low pass voltage ratio transfer functions T_V. Furthermore it is shown that all polynomials with distinct negative real zeros (if any) can be realized as the natural frequencies of two RC networks connected through a gyrator, the latter loaded with a negative resistor as shown in Fig. 3.0.2. However, such a network will not be stable unless the denominator polynomial of T_V is Hurwitz.

3.1 Calahan's Decomposition

Calahan [CA1] has shown that any real polynomial N(S) with all complex zeros,

$$N(S) = \prod_{i=1}^{m} (s + s_1)(s + \hat{s}_i) \qquad (3.1.1)$$

where the imaginary part of s_i satisfies $I_m(s_i) > 0$, can be separated uniquely into

$$N(S) = a^2(s) + b^2(s) \qquad (3.1.2)$$

such that the degree of a(s) is one greater than that of b(s) and the polynomials a(s) and b(s) have alternating real zeros. Any polynomial decomposition of the form in Eq. 3.1.2 which satisfied the above conditions will be hereafter referred to as Calahan's decomposition.

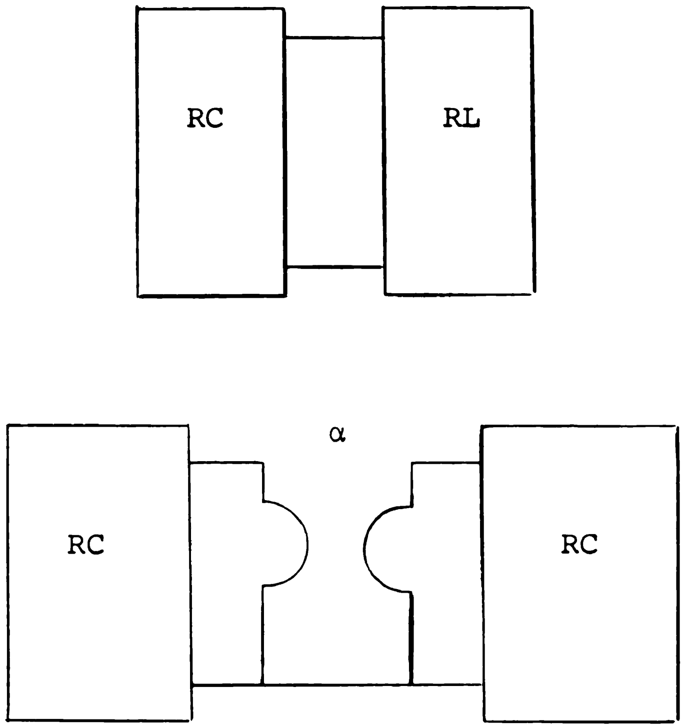


Fig. 3.0.1 RLC and RCΓ equivalent forms.

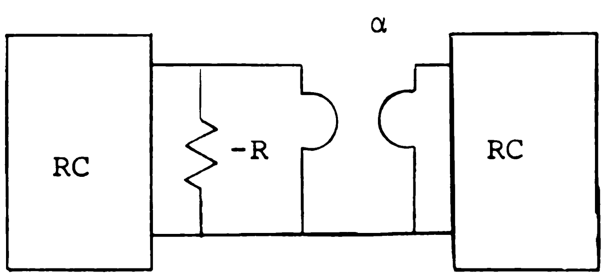


Fig. 3.0.2 RCΓ-R network representation.

Calahan has also shown that the rational function $a(s)/b(s)$ is an RC admittance function if, and only if

$$\sum_{i=1}^{m} \arg(s_i) \leq \pi/2 \qquad (3.1.3)$$

As will be shown, a development of the Calahan decomposition different from Calahan's [CA1] original lengthy derivation can be obtained from the Horowitz [HO] decomposition. Therefore, Horowitz's decomposition will be described briefly.

Horowitz [HO] has shown that if a real polynomial $P(S)$ has no non-positive real zeros, then $P(S)$ can be ce-composed uniquely as

$$P(S) = \pm[A^2(s) - sB^2(s)] \qquad (3.1.4)$$

such that $A(s)/B(s)$ and $sB(s)/A(s)$ are RC admittance functions. In order to establish the Horowitz decomposition, as indicated by Thomas [TH], consider the even polynomial $P(s^2)$ which can be written as

$$P(s^2) = \pm\, F(s)F(-s) \qquad (3.1.5)$$

where the plus (minus) sign holds if $P(s)$ is of even (odd) degree and the polynomial $F(s)$ is strictly Hurwitz. Indeed, by hypothesis, $P(s)$ has no zeros on the non-positive real axis, and hence the zeros of $P(s^2)$ have quadrantal symmetry and none can be on the imaginary axis. Thus, $F(s)$

can be formed by rejecting the right half-plane zeros of $P(s^2)$. If $F(s)$ is now separated into even and odd parts

$$F(s) = A(s^2) + sB(s^2) \tag{3.1.6}$$

one obtains

$$P(s^2) = \pm F(s)F(-s)$$
$$P(s^2) = \pm [A^2(s^2) - s^2B^2(s^2)] \tag{3.1.7}$$

and if s^2 is replaced by s, the desired form

$$P(s) = \pm [A^2(s) - sB^2(s)] \tag{3.1.8}$$

is obtained. The rational functions $A(s)/B(s)$ and $sB(s)/A(s)$ are RC admittance functions since $A(s)/sB(s^2)$ is a Foster function [TO].

Returning to the development of Calahan's decomposition, Horowitz's decomposition is used in establishing the following Theorems.

Theorem 3.1.1. Given a real polynomial $P(s)$ with all complex zeros, there exists a decomposition of the form

$$P(s) = [A'(s)]^2 + (s+\alpha)[B'(s)]^2$$

where the zeros of $A'(s)$ and $B'(s)$ alternate and $-\alpha <$ min{zeros of $A'(s)$}.

Proof: First "shift" the zeros of $P(s)$ into the right half plane with the transformation $s' = s + \alpha$ where α is a sufficiently large positive constant. This yields

$$P'(s') = P(s' - \alpha)$$

Then "folding" the zeros of $P'(s')$ into the left half plane by the transformation $\lambda = -s'$, one has

$$P''(\lambda) = P'(-\lambda) = P(-\lambda - \alpha)$$

Since $P(s)$ has no real zeros, $P''(\lambda) = P(-\lambda - \alpha)$ does not have. Therefore the Horowitz decomposition of $P''(\lambda)$ yields

$$P''(\lambda) = A^2(\lambda) - \lambda B^2(\lambda)$$

"Folding" and "shifting" $P''(\lambda)$ back to $P(s)$ with the transformation $s = -\lambda-\alpha$, the desired decomposition is obtained in the following form:

$$P(s) = A^2(-s-\alpha) + (s + \alpha)B^2(-s-\alpha)$$

Note that since $A(\lambda)$ and $B(\lambda)$ have negative real zeros, due to the properties of the Horowitz decomposition, $P(s)$ can be written as

$$P(s) = \prod_{i=1}^{n} (-s-\alpha + z_{a_i})^2 + b^2(s + \alpha) \prod_{i=1}^{n-1} (-s-\alpha + z_{b_i})^2$$

or

$$P(s) = \prod_{i=1}^{n} (s + \alpha - z_{a_i})^2 + b^2(s + \alpha) \prod_{i=1}^{n-1} (s + \alpha - z_{b_i})^2$$

From the properties of the Horowitz decomposition, one has

$$0 < z_{a_1} < z_{b_1} < z_{a_2} < z_{b_2} < \dots < z_{b_{n-1}} < z_{a_n}$$

or

$$\alpha - z_{a_n} < \alpha - z_{b_{n-1}} < \dots < \alpha - z_{b_1} < \alpha - z_{a_1} < \alpha$$

Therefore, if we let

$$A'(s) = \prod_{i=1}^{n} (s + \alpha - z_{a_i}) = \pm A(-s-\alpha) \qquad (3.1.9)$$

and

$$B'(s) = \prod_{i=1}^{n-1} (s + \alpha - z_{b_i}) = \pm B(-s-\alpha) \qquad (3.1.10)$$

it follows that the zeros of $A'(s)$ and $B'(s)$ do in fact alternate; and the minimum zero of $A'(s)$ is greater than $-\alpha$. Therefore $P(s) = [A'(s)]^2 + (s + \alpha)[B'(s)]^2$ and the Theorem is proved.

Note: If $\alpha - z_{a_n}$ is greater than zero, then the rational functions $A'(s)/B'(s)$ and $(s +\alpha)B'(s)/A'(s)$ represent RC and RL admittances respectively. However, since z_{a_n} is a function of α, $(\alpha - z_{a_n})$ may not be greater than zero.

Theorem 3.1.2. Given a real polynomial $P(s)$ with no real zeros, the decomposition

$$P(s) = [A'(s)]^2 + (s + \alpha)[B'(s)]^2$$

considered in Theorem 3.1.1 reduces to the Calahan decomposition

$$P(s) = \overline{A}^2(s) + \overline{B}^2(s)$$

as α approaches infinity.

Proof: The proof is established using an induction on n where $P_{2n}(s)$ denotes a real polynomial of degree $2n$ with complex zeros.

Part I. Let n = 1. The polynomial $P_2(s) = (s^2 + 2\sigma_1 s + \rho_1^2)$, where $\rho_1^2 > \sigma_1^2$, can be "shifted" and "folded" into $P_2''(\lambda)$ by the transformation $\lambda = -s-\alpha$:

$$P_2''(\lambda) = (\lambda^2 + 2(\alpha - \sigma_1)\lambda + \alpha^2 - 2\sigma_1\alpha + \rho_1^2).$$

To obtain the Horowitz decomposition of $P_2''(\lambda)$, consider

$$P_2''(\lambda^2) = (\lambda^4 + 2(\alpha - \sigma_1)\lambda + \alpha^2 - 2\sigma_1\alpha + \rho_1^2).$$

This polynomial can be written as

$$P_2''(\lambda^2) = F_1(\lambda)F_1(-\lambda)$$

where $F_1(\lambda) = (\lambda^2 + a_1\lambda + b_1)$ is strictly Hurwitz and

$$\begin{aligned} b_1 &= \sqrt{\alpha^2-2\sigma_1\alpha+\rho_1^2} > 0 \\ a_1 &= \sqrt{2b_1-2(\alpha-\sigma_1)} > 0 \end{aligned} \qquad (3.1.11)$$

If $F(\lambda)$ is now separated into even and odd parts, then $A_1(\lambda^2) = \lambda^2+b_1$ and $B_1(\lambda^2) = a_1$. Hence $A_1(\lambda) = \lambda + b_1$ and $B_1(\lambda) = a_1$. Performing the inverse transformation $\lambda = -s-\alpha$, one obtains $A_1(-s-\alpha) = -s-\alpha+b_1$ and $B_1(-s-\alpha) = a_1$. The polynomial $P_2(s)$ can now be written as

$$P_2(s) = (s + \alpha-b_1)^2 + (s + \alpha)a_1^2 \qquad (3.1.12)$$

A Lemma is necessary

<u>Lemma</u> 3.1.2. Let $b_i = \sqrt{\alpha^2-2\sigma_i\alpha+\rho_i^2}$ and $a_i = \sqrt{2b_i-2(\alpha-\sigma_i)}$,

then $\lim\limits_{\alpha\to\infty} \alpha - b_i = \sigma_i$, $\lim\limits_{\alpha\to\infty} a_i = 0$ and $\lim\limits_{\alpha\to\infty}\sqrt{\alpha}\, a_i = \omega_i$

Proof of Lemma: Consider the expression $(\alpha - \sigma_i)$. Since $b_i = \sqrt{\alpha^2 - 2\sigma_i\alpha + \rho_i^{\,2}}$,

$$\lim_{\alpha\to\infty} \alpha - \sqrt{\alpha^2 - 2\sigma_i\alpha + \rho_i^{\,2}} = \lim_{\alpha\to\infty} \frac{2\sigma_i + \rho_i^2\,\alpha}{1 + \sqrt{1 + \frac{2\sigma_i}{\alpha} + \frac{\rho_i^2}{\alpha^2}}} = \sigma_1.$$

or

$$\lim_{\alpha\to\infty} (\alpha - b_i) = \sigma_i$$

From the above result it follows that

$$\lim_{\alpha\to\infty} a_i = \lim_{\alpha\to\infty} \sqrt{2\sigma_i - 2(\alpha - \sigma_i)} = 0$$

However $\lim\limits_{\alpha\to\infty}\sqrt{\alpha}\, a_i$ does exist. Indeed

$$\lim_{\alpha\to\infty} \sqrt{\alpha}\, a_i = \lim_{\alpha\to\infty} \sqrt{\alpha}\,\sqrt{2\sigma_i - 2(\alpha-\sigma_i)} \qquad \text{or}$$

$$\lim_{\alpha\to\infty} \sqrt{\alpha}\, a_i = \lim_{\alpha\to\infty} \frac{(2\alpha)^{1/2}(\rho_i^2 - \sigma_i^{\,2})^{1/2}}{\left[(\alpha^2 + 2\sigma_i\alpha + \rho_i^2)^{1/2} + (\alpha - \sigma_i)\right]^{1/2}} = \omega_i$$

where

$\rho_i^{\,2} - \sigma_i^{\,2} = \omega_i^{\,2}$. This proves the lemma.

Continuing with the proof, Eq. 3.1.12 can be written as

$$P_2(s) = (s + \sigma_1)^2 + \omega_1^{\,2} \qquad (3.1.13)$$

This concludes the proof for part I.

Before continuing with Part II observe that $A_1'(s) = (s + \alpha - \sigma_1)$ and $B_1'(s) = a_1$. Also observe that the limits:

$$\lim_{\alpha\to\infty} A_i'(s) = (s + \sigma_1) = \bar{A}_1(s)$$

$$\lim_{\alpha\to\infty} B_1'(s) = \lim_{\alpha\to\infty} a_1 = 0$$

$$\lim_{\alpha\to\infty} \sqrt{\alpha}B_1'(s) = \omega_1 = \bar{B}_1(s)$$

hold for n = 1.

Part II. Assume now that the theorem is true for n = k-1, which is the same as assuming the limits

$$\lim_{\alpha\to\infty} A_n'(s) = \bar{A}_n(s) \qquad n = k-1$$

$$\lim_{\alpha\to\infty} B_n'(s) = 0 \qquad n = k-1$$

$$\lim_{\alpha\to\infty} \sqrt{\alpha}B_n'(s) = \bar{B}_n(s) \qquad n \equiv k-1$$

hold. Consider the expression of $P_{2n}(s)$ for n = k, where

$$P_{2k}(s) = \prod_{i=1}^{k} (s^2 + 2\sigma_i s + \rho_i^2) \text{ and } \rho_1^2 > \sigma_i^2.$$

Performing the transformation $\lambda^2 = -s - \alpha$ one has

$$P_{2k}''(\lambda^2) = \prod_{i=1}^{k} (\lambda^4 + 2(\alpha - \sigma_i)\lambda^2 + \alpha^2 - 2\sigma_i\alpha + \rho_i^2)$$

This polynomial can be written as

$$P''_{2k}(\lambda^2) = (\lambda^4 + 2(\alpha - \sigma_k)^2 + \alpha^2 - 2\sigma_k + \rho_k^2)F_{k-1}(\lambda)F_{k-1}(-\lambda)$$

where $F_{k-1}(\lambda)$ is formed for the case $n = k-1$. Collecting the terms corresponding to the left half plane zeros one has

$$F_k(\lambda) = (\lambda^2 + a_k\lambda + b_k)F_{k-1}(\lambda)$$

where
$$b_k = \sqrt{\alpha^2 - 2\sigma_k\alpha + \rho_k^2}$$

$$a_k = \sqrt{2b_k - 2(\alpha - \sigma_k)}.$$

Let the even and odd parts of $F_{k-1}(\lambda)$ be denoted as $A_{k-1}(\lambda^2)$ and $B_{k-1}(\lambda^2)$. Therefore the even and odd parts of $F_k(\lambda)$ can be written respectively as

$$A_k(\lambda^2) = (\lambda^2 + b_k)A_{k-1}(\lambda^2) + \lambda^2 a_k B_{k-1}(\lambda^2)$$

and

$$B_k(\lambda^2) = (\lambda^2 + b_k)B_{k-1}(\lambda^2) + a_k A_{k-1}(\lambda^2)$$

Transforming $P''_{2k}(\lambda^2)$ back to $P_{2k}(s)$ with the transformation $\lambda^2 = -s - \alpha$ one obtains

$$P_{2k}(s) = A_k^2(-s - \alpha) + (s + \alpha)\, B_k^2(-s - \alpha)$$

where

$$A_k(-s - \alpha) = (-s-\alpha+b_k)A_{k-1}(-s-\alpha) + (-s-\alpha)a_k B_{k-1}(-s-\alpha)$$

and

$$B_k(-s-\alpha) = (-s-\alpha+b_k)B_{k-1}(-s-\alpha) + a_k A_{k-1}(-s-\alpha)$$

Now since $A_k'(s) = \pm A_k(-s-\alpha)$, $B_k'(s) = \mp B_k(-s-\alpha)$ and the degree of $A_k(\lambda)$ is one greater than the degree of $B_k(\lambda)$, the polynomial $P_{2k}(s)$ can be written as

$$P_{2k}(s) = [A_k'(s)]^2 + (s+\alpha)\ B_k'(s)]^2$$

where

$$A_k'(s) = [(s+\alpha-b_k)A_{k-1}'(s) - (s+\alpha)a_k B_{k-1}'(s)] \tag{3.1.14}$$

and

$$B_k'(s) = [(s+\alpha-b_k)B_{k-1}'(s) + a_k A_{k-1}'(s)] \tag{3.1.15}$$

In order to find the expression for $P_{2k}(s)$ as α approaches infinity one can form the limits

$$\lim_{\alpha\to\infty} A_k'(s) = \lim_{\alpha\to\infty} [(s+\alpha-b_k)A_{k-1}'(s)-(s+\alpha)a_k B_{k-1}'(s)]$$

$$\lim_{\alpha\to\infty} B_k'(s) = \lim_{\alpha\to\infty} [(s+\alpha-b_k)B_{k-1}'(s)+a_k A_{k-1}'(s)]$$

and

$$\lim_{\alpha\to\infty} \sqrt{\alpha}B_k'(s) = \lim_{\alpha\to\infty} [(s+\alpha-b_k)\sqrt{\alpha}B_{k-1}'(s)+\sqrt{\alpha}a_k A_{k-1}'(s)]$$

The above limits reduce to

$$\lim_{\alpha\to\infty} A_k'(s) = (s+\alpha_k)A_{k-1}(s) - \omega_k B_{k-1}(s) = A_k(s) \tag{3.1.16}$$

$$\lim_{\alpha\to\infty} B_k'(s) = 0$$

$$\lim_{\alpha\to\infty} \sqrt{\alpha}B_k'(s) = (s+\sigma_k)B_{k-1}(s) + \omega_k A_{k-1}(s) = B_k(s) \tag{3.1.17}$$

using the assumption that the theorem holds for $n = k-1$ and Lemma 3.1.1. The expression for $P_{2k}(s)$ in the limit is

$$P_{2k}(s) = \bar{A}_k^2(s) + \bar{B}_k^2(s) \tag{3.1.18}$$

This completes the induction.

Now $A'(s)$ and $\sqrt{\alpha}B'(s)$ have alternating real zeros for all finite real values at α. Since the zeros of a polynomial are continuous functions of the coefficients and the coefficients are continuous functions of α, the zeros of $A'(s)$ and $\sqrt{\alpha}B'(s)$ must remain real in the limit. Suppose the zeros at $A'(s)$ and $\sqrt{\alpha}B'(s)$ do not alternate in the limit, then either of the following cases occur:

i) $A'(s)$ and $\sqrt{\alpha}B'(s)$ have some coincident zeros in the limit

ii) $A'(s)$ and $\sqrt{\alpha}B'(s)$ have no coincident zeros but the zeros no longer alternate in the limit.

Case i cannot happen since $P(s)$ has all complex zeros. Case ii cannot happen as it implies that $A'(s)$ and $\sqrt{\alpha}B'(s)$ have coincident zeros for some finite α. This completes the proof.

It is interesting to note that Eqs. 3.1.14, 3.1.15, 3.1.16 and 3.1.17 establish recursion relations which can be used to calculate $A_n'(s)$, $B_n'(s)$, $\bar{A}_n(s)$, and $\bar{B}_n(s)$. Collected and rewritten here for convenience they are:

$$\left.\begin{aligned}
&A_o'(s) \doteq 1 \qquad\qquad B_o'(s) \doteq 1 \text{ (by definition)}\\
&A_n'(s) \equiv (s+\alpha-b_n)A_{n-1}'(s) - (s+\alpha)a_n B_{n-1}'(s)\\
&B_n'(s) \equiv (s+\alpha-b_n)B_{n-1}'(s) + a_n a_{n-1}'(s)\\
&\text{where } \; b_k = \sqrt{\alpha^2 - 2\sigma_k\alpha + \rho_k^{\;2}}\\
&\qquad\qquad a_k = \sqrt{2b_k - 2(\alpha-\sigma_k)}
\end{aligned}\right\} \quad (3.1.19)$$

and

$$\left.\begin{aligned}
&\bar{A}_o(s) \doteq 1 \qquad\qquad \bar{B}_o(s) \doteq 1 \text{ (by definition)}\\
&\bar{A}_n(s) \equiv (s+\sigma_n)\bar{A}_{n-1}(s) - \omega_n\bar{B}_{n-1}(s)\\
&\bar{B}_n(s) \equiv (s+\sigma_n)\bar{B}_{n-1}(s) + \omega_n\bar{A}_{n-1}(s)\\
&\text{where } \omega_k = \rho_k^2 - \sigma_k^2
\end{aligned}\right\} \quad (3.1.20)$$

<u>Theorem 3.1.3</u>. Given $P_{2n}(s) \; \prod_{i=1}^{n} (s^2 + 2\sigma_i s + \rho_i^{\;2}), \rho_i^{\;2} > \sigma_i^{\;2}$ then the zeros of the polynomials $A_n'(s)$ and $A_{n-1}'(s)$ appearing in Eq. 3.1.19 alternate along the real axis.

Proof: From the recursive relation $A_n'(s) = (s+\alpha-b_n)A_{n-1}'(s) - (s+\alpha)a_n B_{n-1}(s)$ the following list of properties can be stated, from which the proof is established.

1. $A_n'(s)$, $A_{n-1}'(s)$, and $B_{n-1}'(s)$ have positive coefficients for the highest powers of s as can be seen from Eq. 3.1.13.
2. The zeros of $A_{n-1}'(s)$ are larger than $-\alpha$. This is established in Theorem 3.1.1.

3. $A'_{n-1}(s)$ and $B'_{n-1}(s)$ have alternate real zeros. This is established in Theorem 3.1.1.
4. $A'_n(s)$ has an odd number of zeros larger than the largest zero of $A'_{n-1}(s)$. This follows since 1, 2, and 3 imply that $A'_n(s)$ is negative at the largest zero of $A'_{n-1}(s)$. However 1 implies $A'_n(s)$ is positive for real and sufficiently large values of s.
5. $A'_n(s)$ has an odd number at zeros between each zero of $A'_{n-1}(s)$. This follows since the sign of $A'_n(s)$ is the negative of the sign of $B'_{n-1}(s)$ at each zero of $A'_{n-1}(s)$, and $A'_{n-1}(s)$ and $B'_{n-1}(s)$ have alternate zeros.
6. The degree of $A'_n(s)$ is one greater than $A'_{n-1}(s)$.

Returning to the proof of the theorem, from properties 4, 5, and 6 it follows that $A'_n(s)$ has one zero which is larger than the largest zero of $A'_{n-1}(s)$ and one zero between each zero $A'_{n-1}(s)$. The last zero must therefore be smaller than the smallest zero of $A'_{n-1}(s)$. This ends the proof.

<u>Theorem 3.1.4</u>. Consider the polynomials B'_n, $\overline{A}_n$ and $\overline{B}_n$ in Eq. 3.1.19 and Eq. 3.1.20, then

i) the zeros of $B'_n(s)$ and $B'_{n-1}(s)$ alternate

ii) the zeros of $\overline{A}_n(s)$ and $\overline{A}_{n-1}(s)$ alternate

iii) the zeros of $\overline{B}_n(s)$ and $\overline{B}_{n-1}(s)$ alternate

Proof: The proof follows an identical line to the proof for Theorem 3.1.3 and will not be repeated here.

A procedure for the derivation of Calahan's decomposition based on the Horowitz decomposition is now established through the use of the foregoing theorems. These theorems provide recursive formulas for obtaining the Calahan decomposition and they also establish some of the properties of the polynomials appearing in these recursive relations. The conditions under which $A_n'(s)/\sqrt{\alpha}B_n'(s)$ and $\overline{A}_n(s)/\overline{B}_n(s)$ are RC admittance functions have yet to be derived. It is possible to develop the conditions for which $\overline{A}_n(s)/\overline{B}_n(s)$ is an RC admittance function without regarding the conditions for which $A_n'(s)/\sqrt{\alpha}B_n'(s)$ is an RC admittance function. The next three theorems establish consitions for which $A_n'(s)/\sqrt{\alpha}B_n'(s)$ is an RC admittance function and also establish the fact that if $\overline{A}_n(s)/\overline{B}_n(s)$ is not an RC admittance function then $A_n'(s)/\sqrt{\alpha}B_n'(s)$ cannot be an RC admittance function for any value of α.

<u>Theorem 3.1.5</u>. Let $P_{2n}(s) = \prod_{i=1}^{n} (s^2+2\sigma_i s + \rho_i^2)$ with $\rho_i^2 > \sigma_i^2$ and let the polynomials $A_n'(s)$ and $B_n'(s)$ be derived from Eqs. 3.1.19. The rational function $A_n'(s)/\sqrt{\alpha}B_n(s)$ is an RC admittance function if, and only if, $A_n'(0) \geq 0$ and $A_k'(0) > 0$ for $k = 1,2,\ldots,n-1$.

Proof: Consider the if part of the theorem. Let $A_n'(0) \geq 0$ and $A_k'(0) > 0$, for $k = 1,2,\ldots n-1$, and assume that

$A_n'(s)/\sqrt{\alpha}B_n'(s)$ is not an RC admittance function. Since the zeros of $A_n'(s)$ and $B_n'(s)$ alternate along the real axis, the above assumption implies that $A_n'(s)$ or both $A_n'(s)$ and $B_n'(s)$ have some positive real zeros. The relation $A_n'(0) \geq 0$ implies $A_n'(s)$ must have an even number of zeros in the right half plane or possibly one at the origin and at least one in the right half-plane. Theorem 3.1.3 established that the zeros of $A_n'(s)$ and $A_{n-1}'(s)$ alternate along the real axis, therefore $A_{n-1}'(s)$ must have at least one positive real zero. However, the relation $A_{k-1}'(0) \geq 0$ implies that $A_{k-1}'(s)$ has an even number of positive real zeros, and therefore from Theorem 3.1.3 $A_{k-2}'(s)$ has some positive real zeros. Continuing this reasoning, one can conclude that $A_1'(s)$ must have some positive real zeros. However sine $A_1'(s)$ is of degree one, it follows that $A_1'(0) < 0$. Hence the contradiction.

Consider now the only if part of the theorem. If $A_n'(s)/\sqrt{\alpha}B_n'(s)$ is an RC admittance function, then $A_n'(s)$ has all negative real zeros or has at most one zero at the origin. Therefore $A_n'(0) \geq 0$. Since, from Theorem 3.1.3, the largest zero of $A_k'(s)$ is greater than the largest zero of $A_{k-1}'(s)$ $k_1 = 2,\ldots,n$; The zeros of $A_k'(s)$ $k = 1,\ldots,n-1$ are all in the left half plane, and $A_k'(0) > 0$ for $k = 1,2,\ldots,n-1$.

This completes the proof.

In order to establish the conditions under which both $A_n'(s)/\sqrt{\alpha}B_n'(s)$ and $\bar{A}_n(s)/\bar{B}_n(s)$ are RC admittance functions it is also necessary to consider the definition of $\Phi = \tan^{-1}\frac{B}{A}$ as a single valued function.

Definition 3.1.1. Let $\Phi = \tan^{-1}\frac{B}{A}$ where

$0 \leq \Phi \leq \frac{\ }{2}$	when $B \geq 0$ and $A \geq 0$
$\frac{\pi}{2} < \Phi \leq \pi$	when $B \geq 0$ and $A < 0$
$\pi < \Phi \leq \frac{3\pi}{2}$	when $B < 0$ and $A \leq 0$
$\frac{3\pi}{2} < \Phi_k < 2\pi$	when $B < 0$ and $A > 0$

Theorem 3.1.6. Let $A_k'(s)$ and $B_k'(s)$ be the polynomials defined in Eq. 3.1.13; then $A_n'(s)/\sqrt{\alpha}B_n'(s)$ is an RC admittance if, and only if,

$$0 < \Phi_n \leq \frac{\pi}{2}$$

$$0 < \Phi_k < \frac{\pi}{2} \qquad k = 1,2,\ldots n-1$$

where $\Phi_k = \tan^{-1}\dfrac{\sqrt{\alpha}B_k'(0)}{A_k'(0)}$ as defined in Definition 3.1.1.

Proof: Consider first the if part of the theorem. From Definition 3.1.1

$0 < \Phi_n \leq \frac{\pi}{2}$ implies $A_n'(0) \geq 0$ and

$0 < \Phi_k < \frac{\pi}{2}$ implies $A_k'(0) > 0$ for $k = 1,\ldots,n-1$.

Therefore it follows from Theorem 3.1.5 that $A_n'(s)/\sqrt{\alpha}B_n'(s)$ is an RC admittance function. Consider now the only if part of the theorem. If $A_n'(s)/\sqrt{\alpha}B_n'(s)$ is an RC admittance function then $A_n'(s)$ has non positive real zeros and $B_n'(s)$ has negative real zeros. From Theorems 3.1.3 and 3.1.4 it is known that the largest zero at $A_n'(s)$ is greater than the zeros of $A_k'(s)$ for $k = 1,\ldots,n-1$ and the largest zero of $B_n'(s)$ is greater than the zeros of $B_k'(s)$ for $k = 2,\ldots,n-1$. Note that Eq. 3.1.19 yields $B_1(s) = a_1$, where a_1 is positive as it follows from the properties of the Horowitz decomposition. Therefore one can conclude that $A_n'(0) \geq 0$, $B_n'(0) > 0$, and $A_k'(0) > 0$ $B_k'(0) > 0$ for $k = 1,\ldots,n-1$. Consequently from Definitions 3.1.1 it follows that $0 < \Phi_n \leq \frac{\pi}{2}$ and $0 < \Phi_k < \frac{\pi}{2}$ for $k = 1,\ldots,n-1$. This completes the proof.

Theorem 3.1.7. Let $A_n'(s)$ and $B_n'(s)$ be the polynomials defined by Eq. 3.19. If $A_n'(s)/\sqrt{\alpha}B_n'(s)$ is an RC admittance function for some $\alpha = \alpha'$, then $A_n'(s)/\sqrt{\alpha}B_n'(s)$ is an RC admittance function for all $\alpha > \alpha'$. Conversely, if $A_n'(s)/\sqrt{\alpha}B_n'(s)$ is not an RC admittance function for some $\hat{\alpha}$ it cannot be an RC admittance function for $\alpha \leq \hat{\alpha}$.

Proof: The following Lemma is needed.

Lemma 3.1.7. $A_n'(s)/\sqrt{\alpha}B_n'(s)$ is an RC admittance function only if $\alpha > b_i$ $i = 1,2,\ldots,n$

Proof of Lemma: If $\alpha < b_1$, then

$$\Phi_1 = \tan^{-1} \frac{\sqrt{\alpha}\ a_1}{\alpha - b_1} > \frac{\pi}{2}$$

and by Theorem 3.1.6, the rational function $A_n'(s)/\sqrt{\alpha}B_n'(s)$ cannot be a RC admittance function. On the other hand, since the Horowitz decomposition is unique, $A_n'(s)$ depends only on α and not on the ordering of the complex zeros of $P_{2n}(s)$, i.e.,

$$P_{2n}(s) = \prod_{i=1}^{n} (s^2 + 2\sigma_1 s + \rho_i^2), \quad \rho_i^2 > \sigma_i^2$$

where

$$b_i = \sqrt{\alpha^2 + 2\sigma_i \alpha + \rho_i^2}$$

Therefore, α must be greater than b_i, $i = 1,\ldots,n$. This proves the Lemma.

Returning to the proof of the theorem one observes from Lemma 3.1.7 that, only the case $\alpha > \max_i \{b_i\}$ need be considered.

Consider now the ratio

$$\frac{\sqrt{\alpha}B_n'(0)}{A_n'(0)} = \frac{(\alpha-b_n)\sqrt{\alpha}B_{n-1}'(0) + \sqrt{\alpha}a_n A_{n-1}'(0)}{(\alpha-b_n)A_{n-1}'(0) - \sqrt{\alpha}a_n \sqrt{\alpha}B_{n-1}'(0)}$$

which is formed from Eq. 3.1.19. This ratio can be re-written as

$$\frac{\sqrt{\alpha}B_n'(0)}{A_n'(0)} = \frac{\dfrac{\sqrt{\alpha}B_{n-1}'(0)}{A_{n-1}'(0)} + \dfrac{\sqrt{\alpha}a_n}{a-b_n}}{1 - \dfrac{\sqrt{\alpha}a_n}{(\alpha-b_n)} \dfrac{\sqrt{\alpha}B_{n-1}'(0)}{A_{n-1}'(0)}}$$

Introducing

$$\Phi_n = \tan^{-1}(\sqrt{\alpha}B_n(0)/A_n'(0)) \text{ and } \theta_n = \tan^{-1}(\sqrt{\alpha}a_n/(\alpha-b_n)),$$

defined in Definition 3.1.1, the above expression can be written as

$$\tan\Phi_n = \frac{\tan\Phi_{n-1} + \tan\theta_n}{1-\tan\Phi_{n-1}\tan\theta_n}$$

From this result it follows that $\Phi_n = \Phi_{n-1} + \theta_n$ and hence

$$\Phi_n = \sum_{i=1}^{n} \theta_i = \sum_{i=1}^{n} \tan^{-1} \frac{\sqrt{\alpha}\ a_i}{\alpha - b_i}.$$

Therefore to complete the proof it is sufficient to show that $\sqrt{\alpha}\ a_i$ is a decreasing function of α, and $\alpha-b_i$ is an increasing function of α. Indeed, considering the derivatives

$$\frac{\partial(\alpha-b_i)}{\partial\alpha} = \frac{\partial\left[\alpha-\sqrt{\alpha^2-2\sigma_i\alpha+\rho_i^2}\right]}{\partial\alpha} = \frac{a_1^{\ 2}}{2b_i}$$

and

$$\frac{\partial\sqrt{\alpha}\ a_i}{\partial\alpha} = \frac{\partial\left[\sqrt{\alpha}\ \sqrt{2b_i-2(\alpha-\sigma_i)}\right]}{\partial\alpha} = \frac{a_i(b_i-\alpha)}{\sqrt{\alpha}\ b_i}$$

the following observations can be made:

The terms a_i and b_i appearing in the above derivatives are positive as they follow from the properties of the Horowitz decomposition. (For proof see Theorem 3.1.1 and Eq. 3.1.11). Therefore $(\alpha-b_i)$ is an increasing function of α, while $\sqrt{\alpha}\ a_i$ is a decreasing function of α since $\alpha > \max\ \{b_i\}$. Therefore $\sqrt{\alpha}\ a_i/(\alpha-b_i)$ and $\Phi_n = \sum_{i=1}^{k} \tan^{-1}(\sqrt{\alpha}\ a_i/(\alpha-b_i))$ is a decreasing function of α, $\alpha > \max\ \{b_i\}$.

The theorem follows since $0 < \Phi_n \leq \frac{\pi}{2}$ and $0 < \Phi_i < \frac{\pi}{2}$ is necessary and sufficient for $A'(s)/\sqrt{\alpha}B'(s)$ to be an RC admittance function. This completes the proof.

Corollary 3.1.7

$A_n'(s)/\sqrt{\alpha}B_n'(s)$ is an RC admittance function only if $\bar{A}_n(s)\ \bar{B}_n(s)$ is an RC admittance function.

Proof: The proof follows immediately from the limits

$$\lim_{\alpha\to\infty} A_n'(s) = \bar{A}_n(s), \qquad \lim_{\alpha\to\infty} \sqrt{\alpha}\ B_n'(s) = \bar{B}_n(s) \quad \text{and}$$

Theorem 3.1.7.

The conditions for which $\bar{A}_n(s)/\bar{B}_n(s)$ represent an RC admittance can be developed directly from Eq. 3.1.20 and Theorem 3.1.4. A Theorem which establsihes these conditions follows.

Theorem 3.1.8. Let $\bar{A}_n(s)$ and $\bar{B}_n(s)$ be the polynomials defined by Eq. 3.1.20. Then $\bar{A}_n(s)/\bar{B}_n(s)$ represents an RC admittance, if and only if $\sum_{i=1}^{n} \tan^{-1} \frac{\omega_i}{\sigma_i} \leq \frac{\pi}{2}$ where $\omega_i^2 = \rho_i^2 - \sigma_i^2$ and $\sigma_i > 0$.

Proof: From Eq. 3.1.20 one can form the ratio

$$\frac{B_n(0)}{A_n(0)} = \frac{A_{n-1}(0)\omega_n + B_{n-1}(0)\sigma_n}{A_{n-1}(0)\sigma_n - \omega_n B_{n-1}(0)}$$

or

$$\frac{B_n(0)}{A_n(0)} = \frac{\frac{\omega_n}{\sigma_n} + \frac{B_{n-1}(0)}{A_{n-1}(0)}}{1 - \frac{\omega_n}{\sigma_n} \frac{B_{n-1}(0)}{A_{n-1}(0)}}$$

Introducing as before

$$\Phi_k = \tan^{-1} \frac{B_k(0)}{A_k(0)} \text{ and } \Theta_k = \tan^{-1} \frac{\omega_k}{\sigma_k}$$

where $\tan^{-1} \frac{B}{A}$ is defined in Definition 3.1.1 one has

$$\tan^{-1} \frac{B_n(0)}{A_n(0)} = \sum_{i=1}^{n} \Theta_i = \sum_{k=1}^{n} \tan^{-1} \frac{\omega_k}{\sigma_k}$$

Consider first the "only if" part of the theorem. Assume $\sum_{k=1}^{n} \tan^{-1} \frac{\omega_k}{\sigma_k} > \frac{\pi}{2}$, then $\tan^{-1} \frac{B_n(0)}{A_n(0)} > \frac{\pi}{2}$ and from Definition 3.1.1, $\bar{A}_n(0) < 0$ and/or $\bar{B}_n(0) \leq 0$. If $\bar{A}_n(0) < 0$ then $\bar{A}_n(s)$ has some zeros in the right half plane and $\bar{A}_n(s)/\bar{B}_n(s)$ is not an RC admittance function. If $\bar{B}_n(0) \leq 0$ then $\bar{B}_n(s)$ has a zero at the origin or in the right half-plane. Since the zeros of $\bar{A}_n(s)$ and $\bar{B}_n(s)$ alternate and the degree of $\bar{A}_n(s)$ is one greater than the degree of $\bar{B}_n(s)$, $\bar{A}_n(s)$ must have some zeros in the right half-plane and $\bar{A}_n(s)/\bar{B}_n(s)$ is not an RC admittance function. Consider now the "if" part of the theorem. Assume

$$\tan^{-1}\frac{\bar{B}_n(0)}{\bar{A}_n(0)} = \sum_{i=1}^{n}\tan^{-1}\frac{\omega_i}{\sigma_i} \leq \frac{\pi}{2}$$

Then $\bar{B}_n(0) > 0$ and $\bar{A}_n(0) \geq 0$. Note also that $\sigma_i > 0$ $i = 1,\ldots,n$, otherwise $\sum_{i=1}^{n}\tan^{-1}\frac{\omega_i}{\sigma_i}$ would be greater than $\frac{\pi}{2}$ as can be seen from Definition 3.1.1. Therefore

$$\tan^{-1}\frac{\bar{B}_k(0)}{\bar{A}_k(0)} = \sum_{i=1}^{k}\tan^{-1}\frac{\omega_i}{\sigma_i} < \frac{\pi}{2}$$

and $\bar{B}_k(0) > 0$, $A_k(0) > 0$ $k = 1,2,\ldots,n-1$. Suppose $\bar{A}_n(0) \geq 0$ and $\bar{A}_k(0) > 0$ $k = 1,\ldots,n-1$, but $\bar{A}_n(s)/\bar{B}_n(s)$ is not an RC admittance function.

$A_n(s)/B_n(s)$ not an RC admittance function implies $A_n(s)$ or $A_n(s)$ and $B_n(s)$ have some zeros in the right half-plane. Since $A_n(0) \geq 0$, $A_n(s)$ must have an even number of zeros in the right half-plane or possibly one at the origin and at least one in the right half-plane. Since by Theorem 3.1.4 $A_k(s)$ and $A_{k-1}(s)$ have alternate real zeros, $A_{n-1}(s)$ must have some zeros in the right half plane. By this process one concludes that $A_1(s)$ must have some zeros in the right half-plane. However $A_1(s)$ is of degree one and therefore $A_1(0) < 0$, hence the contradiction. This completes the proof.

The preceeding theorems show clearly that the Calahan decompositions can be established from the Horowitz decomposition. Conversely, by using the Calahan decomposition the Horowitz decomposition can be established. This is proved in the following theorem.

Theorem 3.1.9. Let P(s) be a real polynomial of degree n with all the real zeros (if any) positive, then the Horowitz decomposition

$$P(s) = \pm A^2(s) - s\,B^2(s)$$

can always be obtained from the Calahan decomposition

Proof: Using the transformation $s = -\lambda^2$ the polynomial P(s) takes on the form

$$P'_{2n}(\lambda) = P(-\lambda^2)$$

Now $P'_{2n}(\lambda)$ has no real zeros since the zeros z_{is} of P(s) are not negative real or zero by hypothesis and the zeros of $P'_{2n}(\lambda)$ are $\lambda_i = \pm \sqrt{-z_{is}}$ $i = 1,2,\ldots,n$. Therefore the Calahan decomposition can be formed to yield

$$P'_{2n}(\lambda) = [\,A_n(\lambda)]^2 + [B_n(\lambda)]^2$$

From Eq. 3.1.14 the following ratio can be formed

$$\frac{B_n(\lambda)}{A_n(\lambda)} = \frac{\dfrac{\omega_n}{\sigma_n+\lambda} + \dfrac{B_{n-1}(\lambda)}{A_{n-1}(\lambda)}}{1 - \dfrac{\omega_n}{\sigma_n+\lambda}\,\dfrac{B_{n-1}(\lambda)}{A_{n-1}(\lambda)}}$$

from which

$$\tan^{-1}\frac{B_n(\lambda)}{A_n(\lambda)} = \sum_{i=1}^{n}\tan^{-1}\frac{\omega_i}{\sigma_i+\lambda}, \lambda \text{ real, is obtained.}$$

Now since $P'_{2n}(\lambda) = P(-\lambda^2)$, $P'_{2n}(\lambda)$ is symmetric about the imaginary axis. It follows that

$$\tan^{-1}\frac{B_n(\lambda)}{A_n(\lambda)} = \sum_{i=1}^{k}\left(\tan^{-1}\frac{\omega_i}{\sigma_i+\lambda} + \tan^{-1}\frac{\omega_i}{-\sigma_i+\lambda}\right)$$

$$+ \sum_{i=1}^{\ell}\tan^{-1}\frac{\omega_i}{\lambda}$$

where 2k complex aeros are located symmetrically about the imaginary axis and ℓ zeros are on the imaginary axis. Since

$$\tan^{-1}\frac{\omega_i}{\sigma_i+\lambda} + \tan^{-1}\frac{\omega_i}{-\sigma_i+\lambda} = \tan^{-1}\frac{2\omega_i}{\lambda^2-\sigma_i^2-\omega_i^2}$$

the above equation can be written as

$$\tan^{-1}\frac{B_n(\lambda)}{A_n(\lambda)} = \sum_{i=1}^{k}\tan^{-1}\frac{2\omega_i\lambda}{\lambda^2-\sigma_i^2-\omega_i^2} + \sum_{i=1}^{\ell}\tan^{-1}\frac{\omega_i}{\lambda}$$

From which it follows that

$$\Phi_n(\lambda) = \tan^{-1}\frac{B_n(\lambda)}{A_n(\lambda)} = -\tan^{-1}\frac{B_n(-\lambda)}{A_n(-\lambda)}$$

Now $A_n(\lambda)$ equals zero for those values of λ where $\Phi_n(\lambda)$ equals an odd multiple of $\frac{\pi}{2}$. Similarly, $B_n(\lambda)$ equals zero where $\Phi_n(\lambda)$ equals an even multiple of $\frac{\pi}{2}$. Since $\Phi_n(\lambda) = -\Phi_n(-\lambda)$, the polynomials $A_n(\lambda)$ and $B_n(\lambda)$ must have zeros placed symmetrically about the origin, with one of them having a zero at the origin. From the equation

$$\Phi_n(0) = \sum_{j=1}^{\ell} \tan^{-1}(\infty) = \ell \frac{\pi}{2}$$

$B_n(\lambda)$ has a zero at the origin if ℓ is even and $A_n(\lambda)$ has a zero at the origin if ℓ is odd. From the above discussion, $P'_{2n}(\lambda)$ can be written as

$$P'_{2n}(\lambda) = \prod_{i=1}^{n/2} (-s-a_i^2)^2 - s \prod_{i=1}^{n/2\,-\,1} (-s-b_i^2)^2$$

or

$$P(s) = -\,s \prod_{i=1}^{\frac{n-1}{2}} (-s-a_i^2)^2 + \prod_{i=1}^{\frac{n-1}{2}} (-s-b_i^2)^2$$

Now the zeros of $A_n(\lambda)$ and $B_n(\lambda)$ alternate on the real axis as implied by the Calahan decomposition. Therefore one can write for each case

$$-a_n^2 < -b_n^2 < \ldots < -a_.^2 < 0 = b_o^2$$

or

$$-b_2^2 < -a_1^2 < -b_1^2 < 0 = a_o^2$$

Therefore P(s) can be written as

$$P(s) = \pm\,[A^2(s) - s\,B^2(s)] \text{ where}$$

A(s)/B(s) and s B(s)/A(s) are RC admittance functions. This completes the proof.

3.2 Decompositions for Polynomials with Real Zeros

In the preceeding section the Calahan decomposition was shown to exist for polynomials with no real zeros.

The Horowitz decomposition is known to exist for polynomials with no nonpositive real zeros. In this section a method is developed for systematically decomposing those polynomials with negative real zeros. To the best of the author's knowledge this type of decomposition has not been considered in the literature.

<u>Theorem 3.2.1</u>. Let $P_{2j}(s)$ and $P_{2k}(s)$ be two polynomials with the following Calahan decompositions

$$P_{2j}(s) = [\overline{A}_j(s)]^2 + [\overline{B}_j(s)]^2$$

$$P_{2k}(s) = [\overline{A}_k(s)]^2 + [\overline{B}_k(s)]^2$$

if

$$P_{2n}(s) = P_{2k}(s)P_{2j}(s)$$

then

$$P_{2n}(s) = [\overline{A}_n(s)]^2 + [\overline{B}_n(s)]^2$$

where

$$\overline{A}_n(s) = \overline{A}_j(s)\overline{A}_k(s) - \overline{B}_j(s)\overline{B}_k(s) \qquad (3.2.1)$$

and

$$\overline{B}_n(s) = \overline{A}_j(s)\overline{B}_k(s) + \overline{B}_j(s)\overline{A}_k(s) \qquad (3.2.2)$$

Proof: Consider Eq. 3.1.19. It is repeated here for convenience:

$$\overline{A}_n(s) = (s + \sigma_n)\overline{A}_{n-1}(s) - \omega_n\overline{B}_{n-1}(s)$$

$$\overline{B}_n(s) = (s + \sigma_n)\overline{B}_{n-1}(s) + \omega_n\overline{A}_{n-1}(s)$$

where

$$\bar{A}_o(s) \equiv 1 \quad \text{and } \bar{B}_o(s) \equiv 0$$

It can be seen that $A_n(s)$ and $B_n(s)$ can be obtained directly from

$$P_{2n}(s) = \prod_{i=1}^{n} (s^2 + 2\sigma_i s + \rho_i{}^2) = \prod_{i=1}^{n} (s+\sigma_i+j\omega_i)(s+\sigma_i-j\omega_i)$$

Indeed, since

$$P_{2n}(s) = [\bar{A}_n(s) + j\bar{B}_n(s)][\bar{A}_n(s) - j\bar{B}_n(s)]$$

one has

$$\bar{A}_n(s) + j\bar{B}_n(s) = \prod_{i=1}^{n} (s + \sigma_i + j\omega_i)$$

The recursion relation (Eq. 3.1.14) holds as can be seen from the following expressions

$$[\bar{A}_n(s) + j\bar{B}_n(s)] = [s + \sigma_n + j\omega_n][\bar{A}_{n-1}(s) + j\bar{B}_{n-1}(s)]$$

$$\bar{A}_n(s) = (s + \sigma_n)\bar{A}_{n-1}(s) - \omega_n\bar{B}_{n-1}(s)$$

$$\bar{B}_n(s) = (s + \sigma_n)\bar{B}_{n-1}(s) + \omega_n\bar{A}_{n-1}(s)$$

From the associative and accumulative laws of complex numbers the proof follows:

$$[\bar{A}_j(s) + j\bar{B}_j(s)][\bar{A}_k(s) + j\bar{B}_k(s)]$$

$$= \bar{A}_j(s)\bar{A}_k(s) - \bar{B}_j(s)\bar{B}_k(s)$$

$$+j[\bar{A}_j(s)\bar{B}_k(s) + \bar{B}_j(s)\bar{A}_k(s)]$$

Theorem 3.2.2. Let $P_n(s) = \prod_{i=1}^{\ell}(s+a_i)\prod_{i=1}^{k}((s+\sigma_i)^2+\omega_i^2)$ where $a_i > 0$, $a_i \neq a_j$ $i \neq j$, and $\sum_{i=1}^{k}\tan^{-1}(\omega_i/\sigma_i) < \frac{\pi}{2}$, then there exists a decomposition such that

$$P_n(s) = \overline{A}_{k+\ell}(s)\overline{A}_k(s) + \overline{B}_{k+\ell}(s)\overline{B}_k(s)$$

where $\overline{A}_{k+\ell}(s)/\overline{B}_{k+\ell}(s)$ and $\overline{A}_k(s)/\overline{B}_k(s)$ are RC admittance functions.

Proof: Consider $\overline{A}_{k+\ell}(s)$ and $\overline{B}_{k+\ell}(s)$ from Eq. 3.1.1 and Eq. 3.1.2

$$\begin{aligned}\overline{A}_{k+\ell}(s) &= \overline{A}_k(s)\overline{A}_\ell(s) - \overline{B}_k(s)\overline{B}_\ell(s)\\ \overline{B}_{k+\ell}(s) &= \overline{A}_k(s)\overline{B}_\ell(s) + \overline{B}_k(s)\ \overline{A}_\ell(s)\end{aligned} \qquad (3.2.3)$$

From Eq. 3.2.3 one can form the ratios

$$\frac{\overline{A}_{n+\ell}(s)}{\overline{A}_\ell(s)B_k(s)} = \frac{\overline{A}_k(s)}{\overline{B}_k(s)} - \frac{\overline{B}_\ell(s)}{\overline{A}_\ell(s)}$$

and

$$\frac{\overline{B}_{k+\ell}(s)}{\overline{A}_\ell(s)\overline{A}_k(s)} + \frac{\overline{B}_k(s)}{\overline{A}_k(s)} + \frac{\overline{B}_\ell(s)}{\overline{A}_\ell(s)} .$$

Adding the rational functions and clearing the denominators one obtains

$$\overline{A}_{k+\ell}(s)\overline{A}_k(s) + \overline{B}_{k+\ell}(s)\overline{B}_k(s) = \overline{A}_\ell(s)[\overline{A}_k^{\,2}(s) + \overline{B}_k^{\,2}(s)] \qquad (3.2.4)$$

Now it can be observed that the left hand side of Eq. 3.2.4 is of the desired form. To obtain a decomposition of P(s) in this form, one may calculate the Calahan decomposition of

$$\prod_{i=1}^{k}[s+\sigma_i)^2+\rho_i^{\,2}] = \overline{A}_k^{\,2}(s) + \overline{B}_k^{2}(s)$$

and set $\overline{A}_\ell(s) = \prod_{i=1}^{\ell}(s+a_i)$. One can now select $\overline{B}_\ell(S)$ such that $A_\ell(s)/B_\ell(s)$ is an RC admittance function, the degree of $\overline{B}_\ell(s)$ is one less than the degree of $\overline{A}_\ell(s)$ and

$$\tan^{-1}\frac{\overline{B}_\ell(0)}{\overline{A}_\ell(0)} + \tan^{-1}\frac{\overline{B}_k(0)}{\overline{A}_k(0)} < \frac{\pi}{2}\,.$$

This can always be done since the zeros of $\overline{A}_\ell(s)$ are distinct by hypothesis and $\overline{B}_\ell(s)$ can be modified, if necessary, by the multiplication of an arbitrary constant so that the angle condition is satisfied. Now using Eq. 3.2.3, $\overline{A}_{k+\ell}(s)$ and $\overline{B}_{k+\ell}(s)$ can be calculated. $\overline{A}_k(s)/\overline{B}_k(s)$ is an RC admittance function since

$$\tan^{-1}\frac{\overline{B}_k(0)}{\overline{A}_k(0)} = \sum_{i=1}^{k}\tan^{-1}\frac{\omega_i}{\sigma_i} < \frac{\pi}{2}$$

by hypothesis. $A_{k+\ell}\,/B_{k+\ell}(s)$ is an RC admittance function since

$$\tan^{-1}\frac{\overline{B}_{k+\ell}(0)}{\overline{A}_{k+\ell}(0)} = \tan^{-1}\frac{\overline{B}_\ell(0)}{\overline{A}_\ell(0)} + \tan^{-1}\frac{\overline{B}_k(0)}{\overline{A}_k(0)} < \frac{\pi}{2}$$

as can be seen from the ratio

$$\frac{\bar{B}_{k+\ell}(0)}{\bar{A}_{k+\ell}(0)} = \frac{\dfrac{\bar{B}_\ell(0)}{\bar{A}_\ell(0)} + \dfrac{\bar{B}_k(0)}{\bar{A}_k(0)}}{1-\dfrac{\bar{B}_\ell(0)}{\bar{A}_\ell(0)}\dfrac{\bar{B}_k(0)}{\bar{A}_k(0)}}$$

and since $\bar{A}_{k+\ell}(s)$ and $\bar{B}_{k+\ell}(s)$ form the Calahan decomposition of the polynomial

$$[\bar{A}_{k+\ell}(s)]^2+[\bar{B}_{k+\ell}(s)]^2 = \{[\bar{A}_\ell(s)]^2+[\bar{B}_\ell(s)]^2\}\{[\bar{A}_k(s)]^2 + [\bar{B}_k(s)]^2\}$$

This completes the proof.

<u>Theorem 3.2.3</u>. Given $P(s) = P^{(1)}(s)P^{(2)}(s)$ with no non-positive real zeros, where $P^{(1)}(s)$ and $P^{(2)}(s)$ have Horowitz decompositions of the form

$$P^{(1)}(s) = \pm\ [A^{(1)}(s)]^2 - s[B^{(1)}(s)]^2$$

and

$$P^{(2)}(s) = \pm\ [A^{(2)}(s)]^2 - s[B^{(2)}(s)]^2 \ .$$

Then $P(s)$ has Horowitz decomposition

$$P(s) = \pm\ [A^2(s) - s\ B^2(s)]$$

where

$$A(s) = A^{(1)}(s)A^{(2)}(s) + s\ B^{(1)}(s)B^{(2)}(s)$$

and (3.2.5)

$$B(s) = A^{(1)}(s)B^{(2)}(s) + B^{(1)}(s)A^{(2)}(s)$$

Proof: From the Thomas method of decomposition we have

$$P(\lambda^2) = P^{(1)}(\lambda^2)P^2(\lambda^2) = F^{(1)}(\lambda)F^{(1)}(-\lambda)F^{(2)}(\lambda)F^{(2)}(-\lambda)$$

with

$$F^{(1)}(\lambda)F^{(2)}(\lambda) = [A^{(1)}(\lambda^2)+ B^{(1)}(\lambda^2)][A^{(2)}(\lambda^2)+ B^{(2)}(\lambda^2)]$$

$$F(\lambda) = F^{(1)}(\lambda)F^{(2)}(\lambda) = [A^{(1)}(\lambda^2)A^{(2)}(\lambda^2) + \lambda^2B^{(1)}(\lambda^2)B^{(2)}(\lambda^2)] = \lambda[A^{(1)}(\lambda^2)B^{(2)}(\lambda^2) +B^{(1)}(\lambda^2)A^{(2)}(\lambda^2)]$$

From which the even and odd part of $F(\lambda)$ are

$$A(\lambda^2) = A^{(1)}(\lambda^2)A^{(2)}(\lambda^2) + \lambda^2B^{(1)}(\lambda^2)B^{(2)}(\lambda^2)$$

and

$$B(\lambda^2) = \lambda[A^{(1)}(\lambda^2)B^{(2)}(\lambda^2) + B^{(1)}(\lambda^2)A^{(2)}(\lambda^2)]$$

In these relations replacing λ^2 by s yields the desired result. This completes the proof.

Theorem 3.2.4. Given $P(s) = A^{(2)}(s)P^{(1)}(s)$ where $A^{(2)}(s)$ has distinct negative real zeros and $P^{(1)}(s)$ has no non-positive real zeros, then

$$P(s) = A^{(3)}(s)A^{(1)}(s) - s\,B^{(3)}(s)B^{(1)}(s)$$

where

$$A^{(i)}(s)/B^{(i)}(s) \text{ and } sB^{(i)}(s)/A^{(i)}(s)$$

are RC admittance functions for $i = 1,3$.

Proof: From equation (3.2.5) one has

$$\frac{A^{(3)}(s)}{A^{(2)}(s)B^{(1)}(s)} = \frac{A^{(1)}(s)}{B^{(1)}(s)} + \frac{s\ B^{(1)}(s)}{A^{(1)}(s)} \tag{3.2.6}$$

and

$$\frac{s\ B^{(3)}(s)}{A^{(2)}(s)A^{(2)}(s)} = \frac{s\ B^{(2)}(s)}{A^{(2)}(s)} + \frac{s\ B^{(1)}(s)}{A^{(1)}(s)} \tag{3.2.7}$$

Subtracting Eqs. 3.2.6 and 3.2.7 and clearing the denominators one can obtain

$$A^{(3)}(s)A^{(1)}(s) - s\ B^{(3)}(s)B^{(1)}(s) = A^{(2)}(s)\ \{[A^{(1)}(s)]^2 - s\ [B^{(1)}(s)]^2\}$$

which is of the desired form. To obtain a decomposition of P(s) in this form, one may calculate the Horowitz decomposition of $P^{(1)}(s) = \pm\ [A^{(1)}(s)\ ^2 - s[B^1(s)^2]$ and select $B^2(s)$ such that $A^{(2)}(s)/B^{(2)}(s)$ and $s\ B^{(2)}(s)/A^{(2)}(s)$ are RC admittance functions. Then one may calculate $A^{(3)}(s)$ and $B^{(3)}(s)$ using Eq. 3.2.5. $A^{(1)}(s)/B^{(1)}(s)$ and $sB^{(1)}(s)/A^{(1)}(s)$ are the desired RC admittance functions since they are derived from the Horowitz decomposition of $P^{(1)}(s)$. $A^{(3)}(s)/B^{(3)}(s)$ and $sB^{(3)}(s)/A^{(3)}(s)$ are also RC admittance functions since they are also derived by the Horowitz decomposition of a polynomial $P^{(3)}(s) = P^{(1)}(s)P^{(2)}(s)$ where $P^{(1)}(s)$ and $P^{(2)}(s)$ have the Horowitz decompositions

$$P^{(1)}(s) = \{[A^{(1)}(s)]^2 - s\ [B^{(1)}(s)]^2\}$$

$$P^{(2)}(s) = \pm\{[A^{(2)}(s)]^2 - s\,[B^{(2)}(s)]^2\}$$

This completes the proof.

3.3 Realization Techniques Using Calahan and Horowitz Type Decompositions.

In the preceeding sections, relationships between two of the existing decomposition techniques are established. Furthermore these decomposition techniques are extended to polynomials containing distinct negative real zeros as well as complex zeros. In this section these techniques are used to synthesize low-pass RC networks. The extended Horowitz decomposition can be used for RCNIC and RCΓ-R filter synthesis. Therefore the configurations corresponding to RCNIC and RCΓ-R networks are established as well as the configurations for RCΓ networks.

3.3.1 RCΓ Filter Realizations

Given the open circuit voltage-ratio transfer function $T_V = kN(s)\,D(s)$ where $D(s)$ has a Calahan type decomposition

$$D(s) = \bar{A}_k(s)\bar{A}_j(s) + \bar{B}_k(s)\bar{B}_j(s)$$

It will be shown that a RCΓ network of the form shown in Fig. 3.3.1 can be realized.

It is well known that the open-circuit voltage-ratio T_V for a network as shown in Fig. 3.3.1 is [MI]

$$T_V = \frac{-\alpha\, Y_{21\alpha} Z_{21b}}{Y_{22\alpha} + \alpha^2 Z_{11b}}$$

Therefore $T_V = kN(s)/D(s)$ is written in the form

$$T_V = \frac{\dfrac{-\alpha k_1 N_1(s)}{B_j(s)} \quad \dfrac{k_2 N_2(s)}{A_k(s)}}{\dfrac{A_j(s)}{B_j(s)} + \alpha^2 \dfrac{1}{\alpha^2} \dfrac{B_k(s)}{A_k(s)}}$$

where $k_1 k_2 N_1(s) N_2(s) = kN(s)$, and the functions Y_{22a}, Y_{21a}, Z_{21b}, and Z_{21b} can be identified as

$$Y_{22a} = \frac{A_j(s)}{B_j(s)}, \quad Y_{21a} = \frac{k_1 N_1(s)}{B_j(s)}, \quad Z_{21b} = \frac{k_2 N_2(s)}{A_k(s)}$$

and $Z_{11b} = \dfrac{1}{\alpha^2} \dfrac{B_k(s)}{A_k(s)}$. Note Y_{22a} and Z_{11b} are RC driving point functions, as required, from the properties of the Calahan decomposition. The realization is completed when the "a" network is synthesized from $-Y_{21a}$ and Y_{22a} and the "b" network from Z_{21b} and Z_{11b}. The realization of RC networks from $-Y_{21}$ and Y_{22} or Z_{21} and Z_{22} is well known and will not be explained here. A low pass filter example follows.

Example 3.3.1 Given the open circuit voltage ratio T_V

$$T_V = \frac{k}{(s+1)[s+2)^2+s^2]}$$

Realize a low pass RC gyrator filter of the form shown in Fig. 3.3.1.

Realization: First one must decompose the denominator polynomial into the form established in Theorem 3.2.2. From $P(s) = (s+1)[(s+2)^2 + 2^2]$ let

$$\bar{A}_1(s) = s+2, \quad \bar{B}_1(s) = 2, \text{ and } \tilde{A}\ (s) = s+1.$$

Selecting $\tilde{B}_1(s) = \frac{1}{2}$, i.e., such that $\tilde{A}_1(s)/\tilde{B}_1(s)$ is an RC admittance function and

$$\tan^{-1} \frac{\tilde{B}_1(0)}{\tilde{A}_1(0)} + \tan^{-1} \frac{\bar{B}_1(0)}{\bar{A}_1(0)} < \frac{\pi}{2}$$

or

$$\tan^{-1} \frac{2}{2} + \tan^{-1} \frac{1/2}{1} < \frac{1}{2}$$

one can now calculate $A_2(s)$ and $B_2(s)$ using Eq. 3.2.1 to obtain

$$\bar{A}_2(s) = s^2 + 3s + 1, \quad \bar{B}_2(s) = (2.5s + 3), \text{ and}$$

$$P(s) = (s^2+3s+1)\ (s + 2) + 2(2.5s + 3).$$

Arranging T_V into the form

$$_V = \frac{1/(2.5s + 3) \quad 2/(s+2)}{(s^2 + 3s + 2)/(2.5s + 3) + 2/(s+2}$$

one can detect that

$$Y_{22a} = (s^2 + 3s+2)/(2.5s+3),$$

$$-Y_{12a} = 1/(2.5s+3),$$

$$Z_{21} = Z_{22} = 2/(s+2), \text{ and } \alpha = 1.$$

The network realization is shown in Fig. 3.3.2.

3.3.2 RCVNIC Realizations

Given the open circuit voltage ratio $T_V = \frac{kN(s)}{D(s)}$ where $D(s)$ has a Horowitz type decomposition

$$D(s) = A^{(3)}(s)A^{(1)}(s) + s\ B^{(3)}(s)B^{(1)}(s)$$

It will be shown that a RCΓ network of the form shown in Fig. 3.3.2 can be realized. It is known that the voltage-ratio transfer function T_V for a network as shown in Fig. 3.3.3 is [CA3],

$$T_V = \frac{\pm\ k_V(-Y_{21a})(Z_{21b})}{k_V - (Y_{22a})(Z_{11b})}$$

and so T_V is arranged as

$$T_V = \frac{\pm \dfrac{k_1N_1(s)}{A^{(1)}(s)}\ \dfrac{k_2N_2(s)}{A^{(3)}(s)}}{1 - \dfrac{s\ B^{(1)}(s)}{A^{(1)}(s)}\ \dfrac{B^{(3)}(s)}{A^{(3)}(s)}\ k_V\ \dfrac{1}{k_V}}$$

from which one can identify

$$-Y_{21a} = k_1N_1(s)/A^{(1)}(s), \quad Y_{22a} = SB^1(s)/A^{(1)}(s),$$

$$Z_{21b}(s) = k_2N_2(s)/A^{(3)}(s), \text{ and } Z_{11b} = (B^{(3)}(s)/A^{(3)}(s))k_V.$$

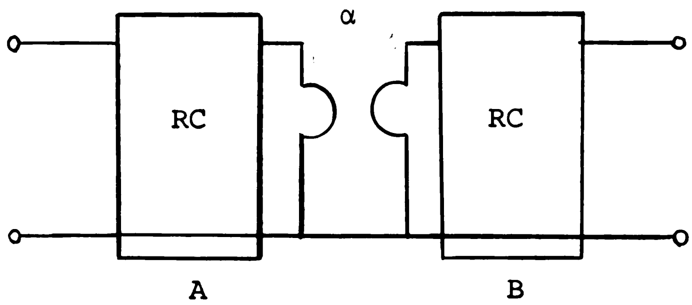
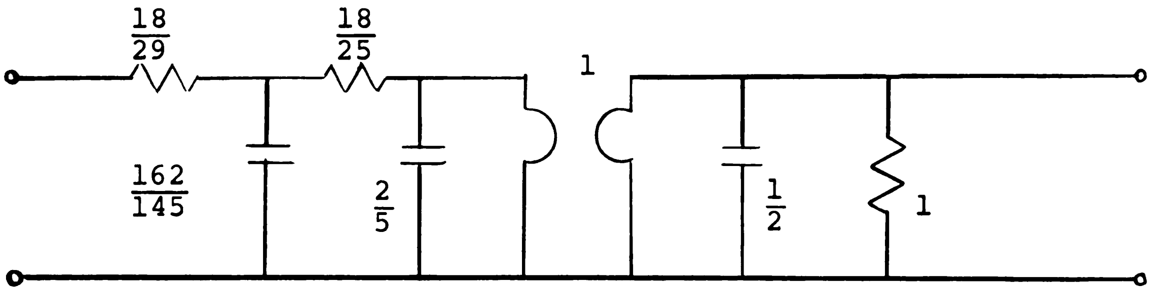


Fig. 3.3.1 General one-gyrator RCΓ filter.



All values are in mhos and farads.

$$T_v = \frac{2}{(s+1)\,(s+2)^2+2^2)}$$

Fig. 3.3.2 Realization of RCΓ filter for Example 3.3.1.

Y_{22a} and Z_{11b} are RC driving point functions from the properties of the Horowitz decomposition. Therefore using standard RC network realization techniques [GU], a network of the form shown in Fig. 3.3.3 can be realized.

Note: One could also divide numerator and denominator of T_V by $sB^{(1)}(s)/B^{(3)}(s)$ and obtain similar results. A low pass filter example is given in the following.

Example 3.3.2 Given the open circuit voltage ratio transfer function T_V

$$T_V = \frac{k}{(s+1)(s^2+4s+9)}$$

realize an RCVNIC low pass network as shown in Fig. 3.3.3.

Realization: First one must decompose the denominator polynomial into the form shown in Theorem 3.2.4:

$$P(s) = (s+1)(s^2+2s+9) = (s+1)[(s+3)^2 - s(s^2)]$$

where $A^{(1)}(s) = (s+3)$, $B^{(1)}(s) = 2$, and $A^{(2)}(s) = s+1$. One can select $B^{(2)}(s) = 2$ so that $(s+1)/2$ and $2s/s+1$ are RC admittance functions. Now one can calculate $A^{(3)}(s)$ and $B^{(3)}(s)$ using Eq. 3.2.2, to obtain

$$A^{(3)}(s) = s^2+ 8s+3, \quad B^{(3)}(s) = 2s+8$$

and

$$P(s) = (s^2+ 8s+3)(s+3) - s(2s+8)(2)$$

At this stage rearranging T_V into the form

$$T_V = \frac{-\frac{3}{2s+8} \quad \frac{3}{2s}}{1 - \frac{s^2+8s+3}{2s+8} \quad \frac{s+3}{2s}}$$

The parameters $K_V=1$, $Y_{22a} = (s^2+8s+3)/(2s+8)$, $-Y_{21a} = 3/(2s+8)$, $Z_{21} = 3/2s$, and $Z_{11} = (s+3)/2s$ can be identified. The network given in Fig. 3.3.4 is the realization of T_V.

3.3.3 RCΓ(-R) Filter Realizations

Given an open circuit voltage-ratio transfer function $T_V = k\,\frac{N(s)}{D(s)}$ where D(s) has a Horowitz type decomposition $D(s) = A^{(3)}(s)A^{(1)}(s) - sB^{(3)}(s)sB^{(1)}(s)$, realize an RCΓ(-R) filter as is shown in Fig. 3.3.5.

Realization: The open circuit voltage-ratio transfer function for a network as shown in Fig. 3.3.1 is

$$T_V = \frac{-\alpha Y_{12a}Z_{21b}}{Y_{22a} + \alpha^2 Z_{11b}}$$

However $T_V = kN(s)/D(s)$ can be rearranged to obtain

$$T_V = \frac{\dfrac{K_1N_1(s)}{B^{(1)}(s)} \qquad \dfrac{K_2N_2(s)}{A^{(3)}(s)}}{\dfrac{A^{(1)}(s)}{B^{(1)}(s)} - \dfrac{sB^3(s)}{A^{(3)}(s)}}$$

Now every negative RL impedance can be realized as a positive RC impedance and a negative R, as can be shown by the general expression

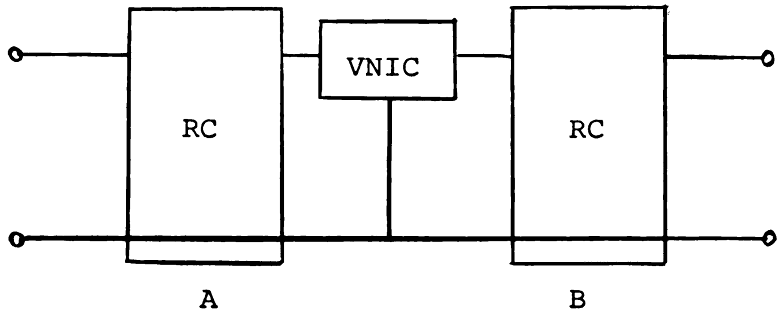
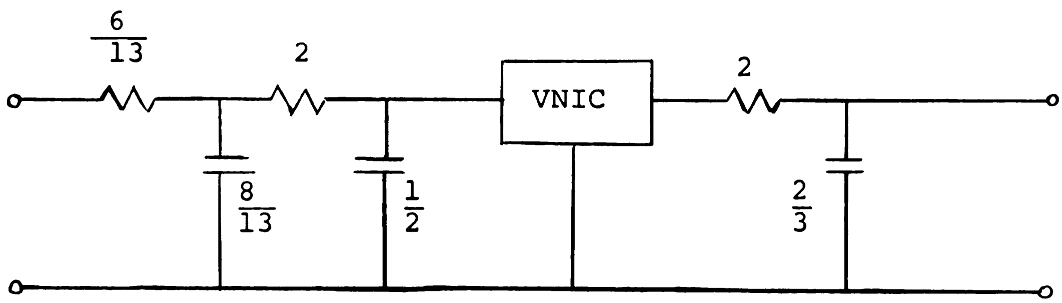


Fig. 3.3.3 General RCNIC filter



All numbers are in mhos or farads.

$$T_V = \frac{-9}{(s+1)(s^2+4s+9)}$$

Fig. 3.3.4 RCNIC realization for Example 3.3.2.

$$-Z_{RL}(s) = -\left[R_o + \sum_{i=1}^{n} \frac{a_i s}{s+b_i} \right]$$

$$-Z_{RL}(s) + R_o + \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \frac{a_i b_i}{s+b_i} = Z_{RC}(s)$$

or

$$-Z_{RL}(s) = Z_{RC}-R$$

that is

$$T_V = \frac{K_1 \dfrac{N_1(s)}{B^1(s)} K_2 \dfrac{N_2(s)}{B^2(s)}}{\dfrac{A^1(s)}{B^1(s)} - R + Z_{RC}}$$

where R is selected sufficiently large so that $-\dfrac{sB^3(s)}{A^3(s)} + R = Z_{RC}$ is a positive RC admittance function. At this point the following identifications can be made:

$$Y_{12a} = K_1N_1(s)/B^{(1)}(s), \quad Y_{22a} = (A^{(1)}(s)/B^{(1)}(s) - R),$$

$$Z_{21b} = K_2N_2(s)/B^{(2)}(s), \text{ and } Z_{11b} = Z_{RC}.$$

The network whose configuration is as shown in Fig. 3.35 can be realized with standard RC transfer function synthesis techniques.

Example 3.3.3

Given the same open circuit voltage-ratio transfer function T_V as in example 3.3.2 realize a RC gyrator negative R filter as shown in Fig. 3.3.5.

Realization: From example 3.3.2

$$T_V = \frac{k}{(s^2 + 8s + 3)(s+3) - s(2s+8)(2)}$$

or

$$T_V = \frac{\dfrac{k}{(s+3)(2s+8)}}{\dfrac{s^2+8s+3}{2s+8} - \dfrac{2s}{s+3}}$$

But every negative RL impedance can be realized as a positive RC impedance and a negative R.

That is

$$-\frac{2s}{s+3} + 2 - 2 = \frac{6}{s+3} - 2$$

Therefore

$$T_V = \frac{\dfrac{k}{(s+3)(2s+8)}}{\dfrac{s^2+8s+3}{2s+8} + \dfrac{6}{s+3} - 2}$$

From $T_V = \dfrac{-(Y_{21a})(Z_{11b})}{Y_{22a} + \alpha^2 Z_{11b}}$

one obtains

$$-Y_{21a} = \frac{3}{2s+8}, \quad Y_{22a} = \frac{s^2+8s+3}{2s+8} - 2,$$

$$Z_{11b} = Z_{21b} = \frac{6}{s+3} \quad \text{and } \alpha = 1$$

The realization is shown in Fig. 3.3.6.

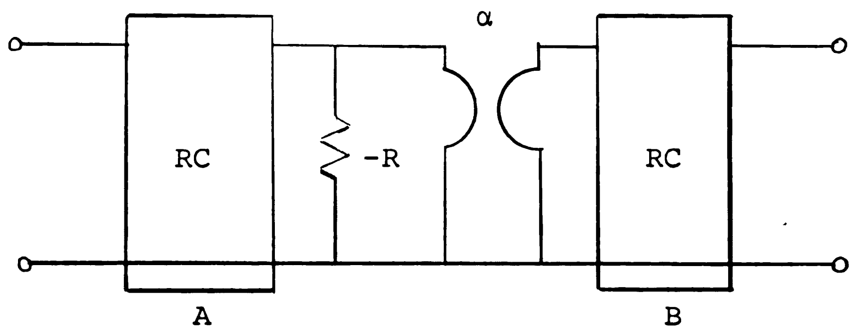
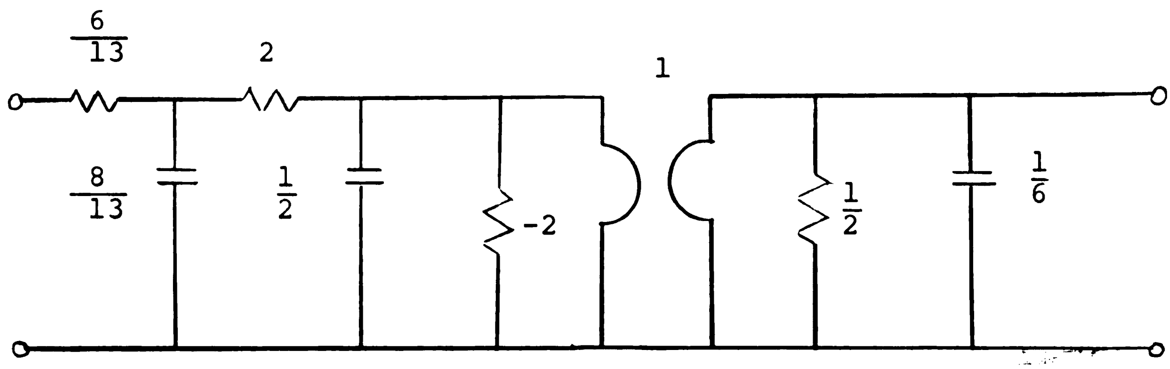


Fig. 3.3.5 General RCΓ-R filter.



All values are in mhos or farads.

$$T_v = \frac{18}{(s+1)(s^2+4s+9)}$$

Fig. 3.3.6 RCΓ-R realization for Example 3.3.3.

3.4. Conclusions

In this chapter, Calahan's decomposition is developed starting from the Horowitz decomposition. The development of the Horowitz decomposition from the Calahan decomposition is also given. Polynomial decompositions of the Calahan and Horowitz-type are developed for polynomials which contain some negative real distinct zeros. No systematic methods have been given in the literature for such decompositions. Sample low-pass transfer functions were synthesized by using Calahan and Horowitz-type decompositions. Finally, a low-pass RCΓ-R filter was synthesized using a Horowitz-type decomposition. This particular filter configuration, which has not appeared in the literature, can be utilized to synthesize any pole configuration (zeros of denominator) as long as the real poles (if any exist) are negative and distinct.

CHAPTER IV

LOW-PASS RCΓ FILTER REALIZATIONS

In Chapter II, it is established that the natural frequencies of a one-gyrator RCΓ network obey a rather restrictive angle condition. In Chapter III, it is demonstrated that this angle condition (satisfied with an inequality) is, in fact, sufficient to permit realization of single-gyrator RCΓ low-pass filters. However, the poles of most low-pass voltage-ratio transfer functions, T_V, do not obey this angle condition beyond the third degree case. This has necessitated, in practice, factoring the denominator of T_V into polynomials each of which do satisfy the angle condition, and then realizing each polynomial with an RCΓ section. These sections must then be connected through isolation amplifiers not only to realize T_V, but also to prevent loading effects. It is, of course, always possible to realize such filters with active devices such as NIC's, -R's, or controlled sources along with RC networks. Such realization techniques are well represented in the literature [LI] [SK] [YO] [HA]. However, in all these techniques the possibility of instability is introduced into the network realization. As is shown in Chapter II, RC networks satisfying the conditions of Theorem 2.4.2 cannot become unstable.

In the first two sections of this chapter, it will be established that forth degree low-pass voltage-ratio transfer functions, whose denominator polynomial being strictly Hurwitz, can always be realized with a two-gyrator RCΓ network satisfying Theorem 2.4.2. An example realized by the proposed method is compared with a computer realization. Also computer realizations are given for three practical low-pass filter networks.

In section 4.3, some necessary conditions, in the form of an extension of Calahan's angle condition, are conjectured and feasibility arguments are used to support them.

In section 4.4, computer realizations of two-gyrator RCΓ networks are given for some fifth and sixth degree low-pass voltage-ratio transfer functions which are of practical interest. In addition, a sample three-gyrator RCΓ computer realization is given to illustrate extensions of the method.

In section 4.5, the established methods of low-pass RCΓ filter realizations are extended to both the band-pass and the high-pass RCΓ filter.

4.1 Two-Gyrator RCΓ Realizations for Fourth Degree Low-Pass Transfer Functions

In Chapter II the expression

$$\Delta_{11} = \Delta_a(\substack{1\\1},RC)\Delta_b(RC)\Delta_c(RC)$$
$$+\alpha_1\tilde{\alpha}_1 \ \Delta_a(\substack{1\\1},\substack{i\\i},RC)\Delta_b(\substack{i+1\\i+1},RC)\Delta_c(RC)$$
$$+\alpha_2\tilde{\alpha}_2 \ \Delta_a(\substack{1\\1},RC)\Delta_b(\substack{j\\j},RC)\Delta_c(\substack{j+1\\j+1},RC)$$
$$+\alpha_1\tilde{\alpha}_1\alpha_2\tilde{\alpha}_2 \ \Delta_a(\substack{1\\1},\substack{i\\i},RC)\Delta_b(\substack{i+1\\k+1},\substack{j\\J},RC)\Delta_c(\substack{j+1\\j+1},RC)$$

is established for the denominator polynomial of the voltage-ratio transfer function of the network given in Fig. 4.1.1.

Consider the network in Fig. 4.1.2. This network satisfied the conditions for Theorem 2.4.2, and it has the following admittance matrix

$$Y = \begin{bmatrix} G_1 & -G_1 & & & \\ -G_1 & C_1S+G_1 & \tilde{\alpha}_1 & & \\ & -\alpha_1 & C_2S+G_2 & -G_2 & \\ & & -G_2 & C_3S+G_3 & \tilde{\alpha}_3 \\ & & & -\alpha_3 & C_4S+G_4 \end{bmatrix} \tag{4.1.1}$$

Let Y' denote the resultant matrix when the first row and column are deleted from Y, one has

$$\Delta_{11} = \det(Y') \tag{4.1.2}$$

and

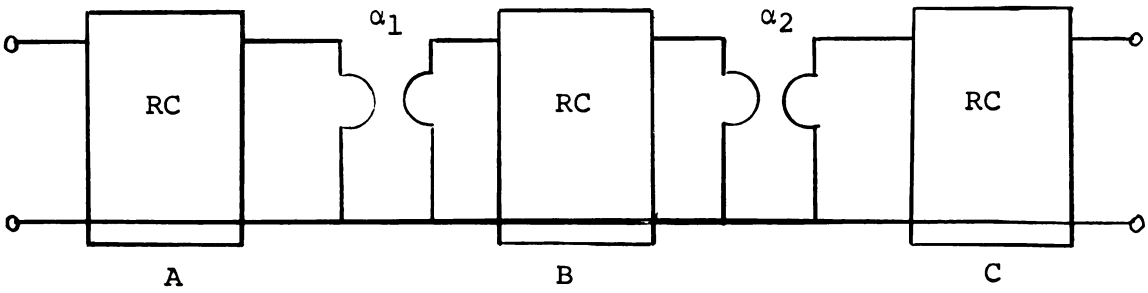


Fig. 4.1.1 General two-gyrator RCΓ filter.

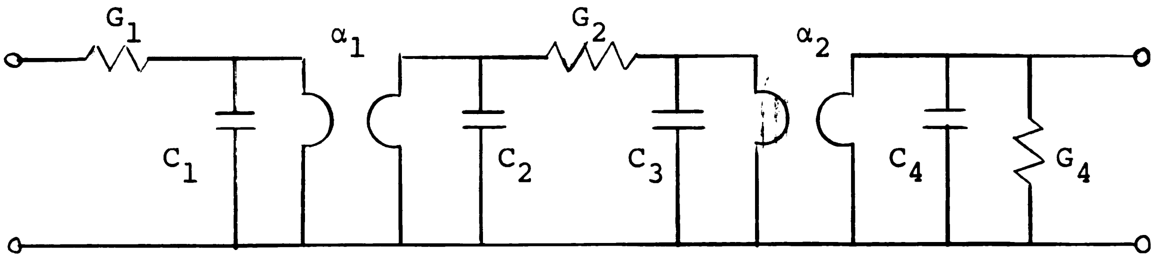


Fig. 4.1.2 Fourth degree low-pass RCΓ filter

$$Y' = \begin{bmatrix} C_1S+G_1 & \tilde{\alpha}_1 & & \\ -\alpha_1 & C_2S+G_2 & -G_2 & \\ & -G_2 & C_3S+G_2 & \tilde{\alpha}_2 \\ & & -\alpha_2 & C_4S+G_4 \end{bmatrix} \qquad (4.1.3)$$

From the relations

$$Y' = CS + H$$

and

$$Y'' = C^{-1/2}Y'C^{-1/2} \qquad (4.1.4)$$

one can obtain

$$Y'' = \begin{bmatrix} S + X_1 & \sqrt{X_5\tilde{\alpha}_1/\alpha_1} & & \\ -\sqrt{X_5\alpha_1/\tilde{\alpha}_1} & S + X_2 & -\sqrt{X_2X_3} & \\ & -\sqrt{X_2X_3} & S + X_3 & \sqrt{X_6\tilde{\alpha}_2/\alpha_2} \\ & & -\sqrt{X_6\alpha_2/\tilde{\alpha}_2} & S + X_4 \end{bmatrix} \qquad (4.1.5)$$

with

$$\begin{aligned} X_1 &= G_1/C_1 & X_4 &= G_4/C_4 \\ X_2 &= G_2/C_2 & X &= \alpha_1\bar{\alpha}_1/C_1C_2 \\ X_3 &= G_3/C_3 & X &= \alpha_2\bar{\alpha}_2/C_3C_4 \end{aligned} \qquad (4.1.6)$$

From Eq. 4.1.6 it follows that

$$X_i > 0 \quad i = 1,2,\ldots,6.$$

Conversely, if all the X_i are positive, one can calculate all the element values in terms of G_1, $\alpha_1\bar{\alpha}_1$, and $\alpha_2\bar{\alpha}_2$. This results in the set of equations

$$\begin{aligned} C_1 &= G_1/X_1 & C_3 &= G_2/X_3 \\ C_2 &= \alpha_1\bar{\alpha}_1/X_5C_1 & C_4 &= \alpha_2\tilde{\alpha}_2/X_6C_3 \\ G_2 &= C_2X_2 & G_4 &= C_3/X_4 \end{aligned} \tag{4.1.7}$$

Forming the determinant of Y", one obtains from Eq. 4.1.4

$$\det(Y'') = \det(C^{-1})\Delta_{11} \tag{4.1.8}$$

and from Eq. 4.1.3

$$\begin{aligned} \det\,(Y'') = {} & S(S + X_1)(S + X_2 + X_3)(S + X_4) \\ & + X_5(S + X_4)(S + X_3) + X_6(S + X_1)(S + X_2) + X_5X_6 \end{aligned} \tag{4.1.9}$$

Therefore the synthesis problem is now reduced to finding a set of positive X_i $(i = 1,2,\ldots,6)$ for a monic polynomial $P(S) = \det\,(Y'')$ where the low-pass voltage-ratio transfer function is

$$T_V = k/P(S) = \Delta_{n1}/\Delta_{11} = \alpha_1\alpha_2G_1G_2/\Delta_{11} \tag{4.1.10}$$

with $\Delta_{11} = \det(C)\det(Y'')$.

Note 1: Every network realization of $T_V = k/P(S)$ represents two possible network realizations since

$$\Delta_{11} = \det(Y'') = \det\,(U'Y'U')$$

where U' has the form

$$U' = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}$$

In terms of the network shown in Fig. 4.1.2 a second realization can be obtained by interchanging the following parameter values:

$$G_1 \text{ and } G_4$$

$$C_1 \text{ and } C_4$$

$$C_2 \text{ and } C_3$$

$$\alpha_1\tilde{\alpha}_1 \text{ and } \alpha_2\tilde{\alpha}_2$$

Note 2: The zero frequency gain for the network in Fig. 2.1.2 is given by

$$T_V\Big|_{S=0} = \frac{\Delta_{41}}{\Delta_{11}}\Big|_{S=0} = \frac{G_1G_2\alpha_1\alpha_2}{G_1G_2\alpha_1\tilde{\alpha}_2 + G_2G_4\alpha_1\tilde{\alpha}_1 + \alpha_1\tilde{\alpha}_1\alpha_2\tilde{\alpha}_2}$$

which can be calculated from the Y matrix in Eq. 4.1.1. Since the denominator polynomial is dependent on the products $\alpha_1\tilde{\alpha}_1$ and $\alpha_2\tilde{\alpha}_2$, it is apparent that unbalanced or active gyrators can be used to improve the voltage gain. For example, $\alpha_1 = 10$ and $\tilde{\alpha}_1 = \frac{1}{10}$ would give ten times the voltage gain that could be achieved for $\alpha_1 = \tilde{\alpha}_1 = 1$.

<u>Theorem 4.1.1</u>. Let P(S) be a real strictly Hurwitz polynomial of degree four, then P(S) can be put in the form

$$P(S) = [(S + a_1)(S + A_2) + K_1][(S + a_2)(S + a_3) + K_2] \quad (4.1.11)$$

where the parameters a_1, a_2, a_3, K_1, and K_2 are all positive and $0 < a_2 < a_1 + a_3$.

Proof: Consider first the polynomial

$$P(S) = [(S + \sigma_1)^2 + \omega_1^2][(S + \sigma_2)^2 + \omega_2^2] \quad (4.1.12)$$

which has all complex roots. If $\sigma_1 = \sigma_2$ Eq. 4.1.12 is already in the desired form. Therefore assume, without loss of generality, that $\sigma_1 < \sigma_2$. Hence,

$$P(S) = [(S + \sigma_1)^2 + \omega_1^2][(S + \sigma_1)(S + 2\sigma_2 - \sigma_1) + \omega_2^2 + (\sigma_2 - \sigma_1)^2]$$

where $2\sigma_2 - \sigma_1$ and $\omega_2^2 + (\sigma_2 - \sigma_1)^2$ are positive. Selecting $a_1 = \sigma_1$, $a_2 = \sigma_1$. $a_3 = 2\sigma_2 - \sigma_1$, $K_1 = \omega_1^2$ and $K_2 = \omega_2^2 + (\sigma_2 - \sigma_1)^2$ the relations $0 < a_1 \leq a_2 \leq a_3$, $0 < K_1$, and $0 < K_2$ hold. This proves the theorem for the case where $P(S)$ has all complex roots.

Consider now the real polynomial

$$P(S) = (S + z_1)(S + z_2)(S + z_3)(S + z_4).$$

Since $P(S)$ is strictly Hurwitz, one can select a_2 so that

$$0 < a_2 < \underset{i=1,\ldots,4}{\text{minimum}} \{\text{Re}(z_i)\} \quad (r.1.13)$$

Now if z_1 and z_2 are real, one has

$$(S = z_1)(S + z_2) = (S + a_2)(S + z_1 + z_2 - a_2) + (z_1 - a_2)(z_2 - a_2)$$

and if z_1 and z_2 are complex conjugates, letting $Z_1 = \hat{z}_2 = \sigma_1 = j\ \omega_1$, one has

$$(S + z_1)(S + z_2) = (S + a_2)(S + 2\sigma_1 - a_2) + \omega_1^2 + (\sigma_1 - a_2)^2.$$

Note that all the constant terms in both expansions are positive. Indeed, this is evident from Eq. 4.1.13. The same type of representation holds for $(S + z_3)(S + z_4)$ and therefore

$$P(S) = [(S + a_2)(S + a_1) + K_1][(S + a_2)(S + a_3) + K_2]$$

where a_1, a_2, a_3, K_1, and K_2 are positive as required. The condition $0 < a_2 < a_1 + a_3$ is satisfied since $a_2 < \underset{i=1,\ldots,4}{\text{minimum}} \{Re(z_i)\}$ implies $a_2 < Re(z_1) + Re(z_2) - a_2 = a_1$. This completes the proof.

<u>Theorem 4.1.2.</u> Given the polynomial

$$P(S) = [(S + a_1)(S + a_2) + K_1][(S + a_2)(S + a_3) + K_2]$$

where all the parameters are positive and $0 < a_2 < a_1 + a_3$, then there exists $X_i > 0$ $(i = 1,\ldots,6)$ such that

$$P(S) = S(S + X_1)(S + X_2 + X_3)(S + X_4) + X_5(S + X_3)(S + X_4) + X_6(S + X_1)(S + X_2) + X_5X_6$$

Proof: Let P(S) be written in the form

$$P(S) = (S + a_2)^2 S(S + a_1 + a_3) + K_1(S + a_2)(S + a_3) + K_2(S + a_1)(S + a_2) + a_1a_3(S + a_2)^2 + K_1K_2 \quad (4.1.14)$$

In order to identify the X_i parameters it is desirable to put P(S) in the following form

$$P(S) = (S + a_2)^2 S(S + a_1 + a_3) + h_1(S + a_1 + a_3 - \alpha)(S + a_2) + h_2(S + \alpha)(S + a_2) + h_1h_2 \quad (4.1.15)$$

with h_1, h_2, $a_1 + a_3 - \alpha$, and α positive. Director comparison of Eq. 4.1.14 and 4.1.15 leads to the equations

$$\begin{aligned} h_1 + h_2 &= K_1 + K_2 + a_1a_3 \\ h_1h_2 &+ K_1K_2 \\ h_1(a_1 + a_3 - \alpha) + h_2(\alpha) &= K_1a_3 + K_2a_1 + a_1a_3a_2 \end{aligned} \quad (4.1.16)$$

That the relation $(h_1 + h_2)^2 > 4\, h_1h_2$ holds follows immediately from Eq. 4.1.16. Therefore h_1 and h_2 are real and positive. In order to continue with the proof the following lemma is needed.

Lemma 4.1.2. Let, in Eq. 4.1.16, K_1, K_2, and a_1a_3 be positive, then

$$\max\{h_1,h_2\} > \max\{K_1,K_2\} + a_1a_3$$

and

$$\min\{h_1,h_2\} < \min\{K_1,K_2\}$$

Proof of Lemma: Let $h' = \min\{K_1,K_2\}$ and $h_2' =$ $\max\{K_1,K_2\} + a_1a_3$, then from Eq. 4.1.16 one obtains

$$\begin{aligned} h_1 + h_2 &= h_1' + h_2' \\ h_1h_2 &< h_1'h_2' \end{aligned} \qquad (4.1.17)$$

From which it follows that

$$h_1(h_1' + h_2' - h_1) < h_1'h_2' \qquad (4.1.18)$$

or

$$(h_1 - h_1')(h_1 - h_2') > 0$$

Without loss of generality, let h_1 be less than h_2. Then from Eq. 4.1.19 and Eq. 4.1.17 it follows that

$$h_1 < h_1' \quad \text{and} \quad h_2 > h_2' \; .$$

This proves the lemma.

Returning to the proof of the main theorem, from Eq. 4.1.16 one can write

$$(h_2 - h_1)\alpha = K_1a_3 + K_2a_1 + a_1a_3a_2 - h_1(a_1 + a_3) \qquad (4.1.20)$$

Now the condition $0 < \alpha < a_1 + a_3$ is necessary if all the X_i are to be positive. Let h_2 be greater than h_1, then the condition $0 < (h_2 - h_1)\alpha < (h_2 - h_1)(a_1 + a_3)$ is necessary if all the X_i are to be positive.

First consider the condition $(h_2 - h_1)\alpha > 0$. From Eq. 4.1.20 it follows that

$$(h_2 - h_1)\alpha = (K_1 - h_1)a_3 + (K_2 -h_1)a_1 + a_1a_2a_3 > 0$$

since $h_1 < \min\{K_1,K_2\}$.

Consider now the condition $(h_2 - h_1)\alpha < (h_2 - h_1)$ $(a_3 + a_1)$. From Eq. 4.1.20 this is equivalent to

$$K_1a_3 + K_2a_1 + a_1a_3a_2 < h_2(a_3 - a_1).$$

On the other hand, sinc $a_2 < a_1 + a_3$, it follows that

$$K_1a_3 + K_2a_1 + a_1a_3a_2 < (K_1 + a_1a_3)a_1 + (K_2 + a_1a_3)a_3$$

or

$$K_1a_3 + K_2a_1 + a_1a_3a_2 < \max\{K_1,K_2\} + a_1a_3](a_1 + a_3).$$

From lemma 4.1.2 $h_2 > \max\{K_1,K_2\} + a_1a_3$ and therefore $K_1a_3 + K_2a_1 + a_1a_2a_4 < h_2(a_1 + a_3)$ or

$$(h_2 - h_1)\alpha < (h_2 - h_1)(a_1 + a_3)$$

Eq. 4.1.15 is now established with

$$P(S) = S(S + a_2)^2(S + a_1 + a_3) + h_1(S + a_1 + a_3 - \alpha)(S + a_2)$$

$$+ h_2(S + \alpha)(S + a_2) + h_1h_2 \qquad (4.1.21)$$

where all parameters including $a_1 + a_3 - \alpha$ are positive. From Eq. 4.1.21, one choice of X_i is

$$\begin{array}{ll} X_i = a_2 & X_4 = a_2 \\ X_2 = \alpha & X_5 = h_1 \\ X_3 = a_1 + a_3 - \alpha & X_6 = h_2 \end{array} \qquad (4.1.22)$$

This completes the proof.

Example 4.1.4. Let, in the voltage-ratio transfer function $T_V = k/P(S)$, the polynomial $P(S)$ be given as

$$P(S) = ((S + 1)^2 + 2^2)((S + 2)^2 + 4^2)$$

Notice that

$$\tan^{-1}(2/1) + \tan^{-1}(4/2) > \pi/2$$

and so no one- gyrator realization is possible. T_V is to be realized as a low-pass filter with the configuration shown in Fig. 4.1.2.

Realization: Using the method established in Theorems 4.1.1 and 4.1.2, $P(S)$ can be put in the form

$$P(S) = [(S + 1)^2 + 2^2][(S + 1)(S + 3) + 17]$$

or

$$P(S) = S(S + 1)^2(S + 4) + 20(S + 1)^2 + 4(S + 1)(S + 3)$$
$$+ (4)(17)$$

From Eq. 4.1.20, it can be seen that

$$h_1 + h_2 = 24$$
$$h_1 h_2 = (4)(17)$$
$$h_2(\alpha) + h_1(4 - \alpha) = 32$$

or

$$h_1 = 3.28$$
$$h_2 = 20.72$$
$$\alpha = 1.082$$

P(S) can now be put in the following form

$$P(S) = S(S + 1)^2(S + 4) + 20.74\ (S + 1)\ (S + 1.082)$$
$$+ 3.28(S + 1)(S + 2.918) + (3.28)(20.72)$$

One choice of X_i, for P(S), is

$$X_1 = 1 \qquad X_4 = 1$$
$$X_2 = 1.082 \qquad X_5 = 3.28$$
$$X_3 = 2.018 \qquad X_6 = 20.72$$

with

$G_1 = 1$ $\quad$ $C_1 = 1$

$C_2 = .3048$ $\quad$ $G_2 = .33$

$C_3 = .1132$ $\quad$ $C_4 = .427$

$G_4 = .427$ $\quad$ $\alpha_1\tilde{\alpha}_1 = 1$

$\alpha_2\tilde{\alpha}_2 = 1$

where all component values are either in mhos or farads. The corresponding network configuration is in Fig. 4.1.2.

4.2 Computer Synthesis of Low-Pass Fourth Degree Filters

In the preceeding section an analytical method of realizing fourth degree low-pass voltage-ratio transfer functions with the network configuration shown in Fig. 4.1.2 is established. It is also possible to realize low-pass filters by solving the set of nonlinear equations in X_i defined by the coefficients of $P(S) = S^4 + \sum_{i=1}^{3} P_i S^i$, Indeed, since

$$
\begin{aligned}
P(S) = {} & S(S + X_1)(S + X_2 + X_3)(S + X_4) + X_3(S + X_3)(S + X_4) \\
& + X_6(S + X_1)(S + X_2) \qquad (4.2.1) \\
& + X_5X_6
\end{aligned}
$$

Equating the coefficients of $P(S)$ in these two expressions, one can obtain a set of nonlinear equations of the following form:

$$\left.\begin{aligned}
P_0 &= P_0(X_i) \equiv X_5X_6 + X_6X_1X_2 + X_5X_3X_4 \\
P_1 &= P_1(X_i) \equiv X_6(X_1 + X_2) + X_5(X_3 + X_4) \\
&\qquad + X_1X_4(X_2 + X_3) \\
P_2 &= P_x(X_i) \equiv X_6 + X_5 + X_1(X_2 + X_3 + X_4) \\
&\qquad + X_4(X_2 + X_3) \\
P_3 &= P_3(X_i) \equiv X_1 + X_2 + X_3 + X_4
\end{aligned}\right\} \qquad (4.2.2)$$

Since there are four equations and six unknowns, there is no unique solution. A computor program (described in Appendix A) has been written to solve such nonlinear equations. An error criterion E is used to test the validity of the solutions. The error criterion used is taken in the form

$$E = \sum_{i=0}^{3} (\max\{P_0, P_1, P_2, P_3\}) \left|1 - \frac{P_i(X_i)}{P_i}\right|.$$

A solution is accepted when E is sufficiently small, so that each coefficient is accurate to at least 8 places.

<u>Example 4.2.1</u>. Let T_V be the same as in Example 4.1.1.

Realization: Using the computer program, with $X_1 = X_4 = 1$ and $\alpha_1\tilde{\alpha}_1 = \alpha_2\tilde{\alpha}_2 = 1$ preset, the following results were obtained:

$X_1 = 1.$	$G_1 = 1.$
$X_2 = 1.0823371$	$C_1 = 1.$
$X_3 = 2.9176629$	$G_2 = .32975942$
$X_4 = 1.$	$C_2 = .30467350$
$X_5 = 3.2822021$	$C_3 = .11302177$
$X_6 = 20.717798$	$G_4 = .42706533$
	$C_4 = .42706533$

where the component values are either in mhos or farads. These results agree up to three significant figures with those found in Example 4.1.1, however, they are much more accurate, as Example 4.1.1 was calculated with a slide rule. For comparison purposes it is interesting to note that the error coefficient E is

$$E = .065 \text{ for Example 4.1.1}$$

$$E < 10^{-8} \text{ for Example 4.2.1}$$

Considerable freedom is available in the choice of X_i to solve Eqs. 4.2.5. One can, to some extent, take advantage of this freedom by specifying additional constraints. For example a useful constraint is $G_1 = k\, G_4$, and it can be written in terms of the parameters X_i as

$$X_1X_2X_6 = k'X_3X_4X_5 \tag{4.2.3}$$

or

$$\frac{G_1G_2\alpha_2\tilde{\alpha}_2}{C_1C_2C_3C_4} = k'\,\frac{G_2G_4\alpha_1\tilde{\alpha}_1}{C_3C_4C_1C_2} \tag{4.2.4}$$

where $K' = k\alpha_2\tilde{\alpha}_2/\alpha_1\tilde{\alpha}_1$. It is evident from 4.2.4 that different ratios of G_1/G_4 are also possible by modifying the ratio $\alpha_2\overline{\alpha}_2/\alpha_1\overline{\alpha}_1$ for a given set of X_i.

Computer solutions with various G_1/G_4 ratios are offered in Tables 4.2.1, 4.2.2, and 4.2.3 to the Linear Delay, the Butterworth, and the Chebyshev fourth degree low-pass voltage-ratio transfer functions. The polynomial coefficients for the Linear Delay filter are from Van Valkenburg [VA2, the polynomial coefficients for the Butterworth filter are from formulas in Van Valkenburg generated to the necessary eight place accuracy with a computer program, and the polynomial coefficients for the Chebyshev filter are from formulas in Guillemin [GU] for the case $\varepsilon^2 = 1/5$. Both the Butterworth and Chebyshev filter coefficients are for filters with a cut-off frequency of $\sqrt{10}$ radians. This cut-off frequency was selected so that the coefficients of P(S) would roughly compare to those for the Linear Delay filter.

The solutions presented are by no means exhaustive, but they are representative of the several hundred solutions obtained while developing the computer algorithm.

In order to test the practicality of the solutions, a frequency response curve for a Butterworth realization (generated with a computer program) is shown in Fig. 4.2.2. This curve is for the case $G_1 = G_4 = 1.$ in table 4.2.2 with all the component values rounded off to three significant figures. As can be seen from the graphs (Fig. 4.2.1 and Fig. 4.2.2), the difference between the theoretical and actual frequency response is that ± .02 d b from 0 to 100 radians/sec for both cases.

4.3 Conjectured Necessary Conditions

In the second chapter, Calahan's angle condition [CA1] is seen to be a necessary condition for a one-gyrator RCΓ network. In the third chapter this angle condition, satisfied with the inequality, is also seen to be sufficient for a one-gyrator RCΓ low-pass filter realization. An angle condition extended to the case of n gyrators is conjectured to be as follows:

Let $T_V = k/P(S)$ be the voltage-ratio transfer function of an RCΓ network consisting of low-pass RCΓ ladders cascaded through n grounded gyrators, then

$$\prod_{i=1}^{r} \tan^{-1} \frac{\omega_i}{\sigma_i} < n \frac{\pi}{2} \qquad (4.3.1)$$

where

$$P(S) = \prod_{i=1}^{\ell} (S + a_i) \prod_{i=1}^{r} [(S + \sigma_i)^2 + \omega_i]$$

Arguments: The case n = 1 is already proved [CA1]. For the case n = 2 with P(S) of degree 4 or 5, Eq. 4.3.1 is always satisfied since $P(S) = \det(C^{-1})\Delta_{11}$ and Δ_{11} is strictly Hurwitz. (For proof see Theorem 2.4.2).

G_1	C_1	G_2	C_2	C_3	G_4	C_4
1.00000000+0	3.11827711+0	4.31231210-1	1.12415224-1	1.41914292-1	1.00000000+0	3.56559694-1
2.00000000+0	3.54996125+0	4.32723971-1	1.17018184-1	1.43677812-1	1.00000000+0	3.66713199-1
3.00000000+0	3.96356699+0	4.32094583-1	1.17513616-1	1.54555090-1	1.00000000+0	3.60953762-1
4.00000000+0	4.36746541+0	4.32182852-1	1.20544985-1	1.55242992-1	1.00000000+0	3.68326352-1
5.00000000+0	4.80062795+0	4.18225666-1	1.12281028-1	1.82146158-1	1.00000000+0	3.40418720-1
6.00000000+0	5.15979640+0	4.30306908-1	1.29225544-1	1.42245206-1	1.00000000+0	4.02873878-1
7.00000000+0	5.58080284+0	4.13237743-1	1.13525925-1	1.88807535-1	1.00000000+0	3.42818134-1
8.00000000+0	5.90821005+0	4.28309890-1	1.32080198-1	1.44432987-1	1.00000000+0	4.10223964-1
9.00000000+0	6.27803932+0	4.22693418-1	1.23086238-1	1.75224600-1	1.00000000+0	3.67645390-1
1.00000000+1	6.62952297+0	4.26535564-1	1.33803998-1	1.48002286-1	1.00000000+0	4.12902485-1

$$T_V = k/(s^4 + 10S^3 + 45S^2 + 105S + 105)$$

$$\alpha_1\bar{\alpha}_1 = \alpha_2\bar{\alpha}_2 = 1.$$

Table 4.2.1.--Fourth Degree Linear Delay Low Pass Filter Realizations. The Network Configuration is in Fig. 4.1.2.

G_1	C_1	G_2	C_2	C_3	G_4	C_4
1.00000000+0	2.99224558+0	3.00302663-1	8.50495827-2	1.22702713-1	1.00000000+0	5.12578838-1
2.00000000+0	3.41907118+0	2.84560669-1	8.90159473-2	1.00482887-1	1.00000000+0	6.06131939-1
3.00000000+0	3.73960301+0	2.81517346-1	8.50641494-2	1.17913113-1	1.00000000+0	5.66817519-1
4.00000000+0	4.10599394+0	2.73217255-1	8.38878311-2	1.18174031-1	1.00000000+0	5.81287866-1
5.00000000+0	4.56315747+0	2.53211066-1	8.67717445-2	8.86526567-2	1.00000000+0	7.17692742-1
6.00000000+0	4.84807952+0	2.54533736-1	8.51563537-2	9.93189485-2	1.00000000+0	6.78417480-1
7.00000000+0	5.18972088+0	2.49131166-1	8.41073266-2	9.97232337-2	1.00000000+0	6.87606055-1
8.00000000+0	5.66897725+0	2.28649693-1	8.34073404-2	7.92372124-2	1.00000000+0	8.16164329-1
9.00000000+0	5.85168609+0	2.40597260-1	8.19951190-2	1.02513057-1	1.00000000+0	6.92457212-1
1.00000000+1	6.24015425+0	2.36006176-1	7.46752664-2	1.29549006-1	1.00000000+0	5.95691901-1

$T_v = k/P(S)$ $P_0=100.$ $P_1=82.6342975$ $P_2=34.1421356$ $P_3=8.26342975$ $P_4=1.$

$\alpha_1\tilde{\alpha}_1 = \alpha_2\tilde{\alpha}_2 = 1.$

Table 4.2.2.--Fourth Degree Butterworth Low Pass Filter Realizations. The Network Configuration is in Fig. 4.1.2.

G_1	C_1	G_2	C_2	C_3	G_4	C_4
1.0000000+0	4.3797661-1	2.2827471-2	7.7133301-1	5.4393266-2	1.0000000+0	1.8585096+0
2.0000000+0	8.8750650-1	2.3123005-2	3.8991054-1	4.9814378-2	1.0000000+0	2.0260528+0
3.0000000+0	1.3462708+0	2.2204772-2	2.6211292-1	5.7721763-2	1.0000000+0	1.7458610+0
4.0000000+0	1.8040620+0	1.8951141-2	1.9714356-1	6.8359305-2	1.0000000+0	1.4706181+0
5.0000000+0	2.3021295+0	2.2607625-2	1.6032137-1	5.4711413-2	1.0000000+0	1.8367855+0
6.0000000+0	2.7988785+0	2.2778106-2	1.3484719-1	5.2949273-2	1.0000000+0	1.8948692+0
7.0000000+0	3.3042857+0	2.2607173-2	1.1646602-1	5.4222110-2	1.0000000+0	1.8482410+0
8.0000000+0	3.8298678+0	2.2869055-2	1.0300245-1	4.4481413-2	1.0000000+0	2.2443340+0
9.0000000+0	4.3163808+0	2.0824885-2	9.1323775-2	6.2483238-2	1.0000000+0	1.6021517+0
1.0000000+1	4.9116608+0	2.2922255-2	8.3624743-2	4.5806500-2	1.0000000+0	2.1735952+0

$T_v = k/P(S)$ $P_0=30.618620$ $P_1=26.265697$ $P_2=15.348287$ $P_3=3.2705619$ $P_4=1.$

$\alpha_1\tilde{\alpha}_1 = \alpha_2\tilde{\alpha}_2 = 1.$

Table 4.2.3.--Fourth Degree Chebyshev ($\varepsilon^2=1/5$) Low Pass Filter Realizations. The Network Configuration is in Fig. 4.1.2.

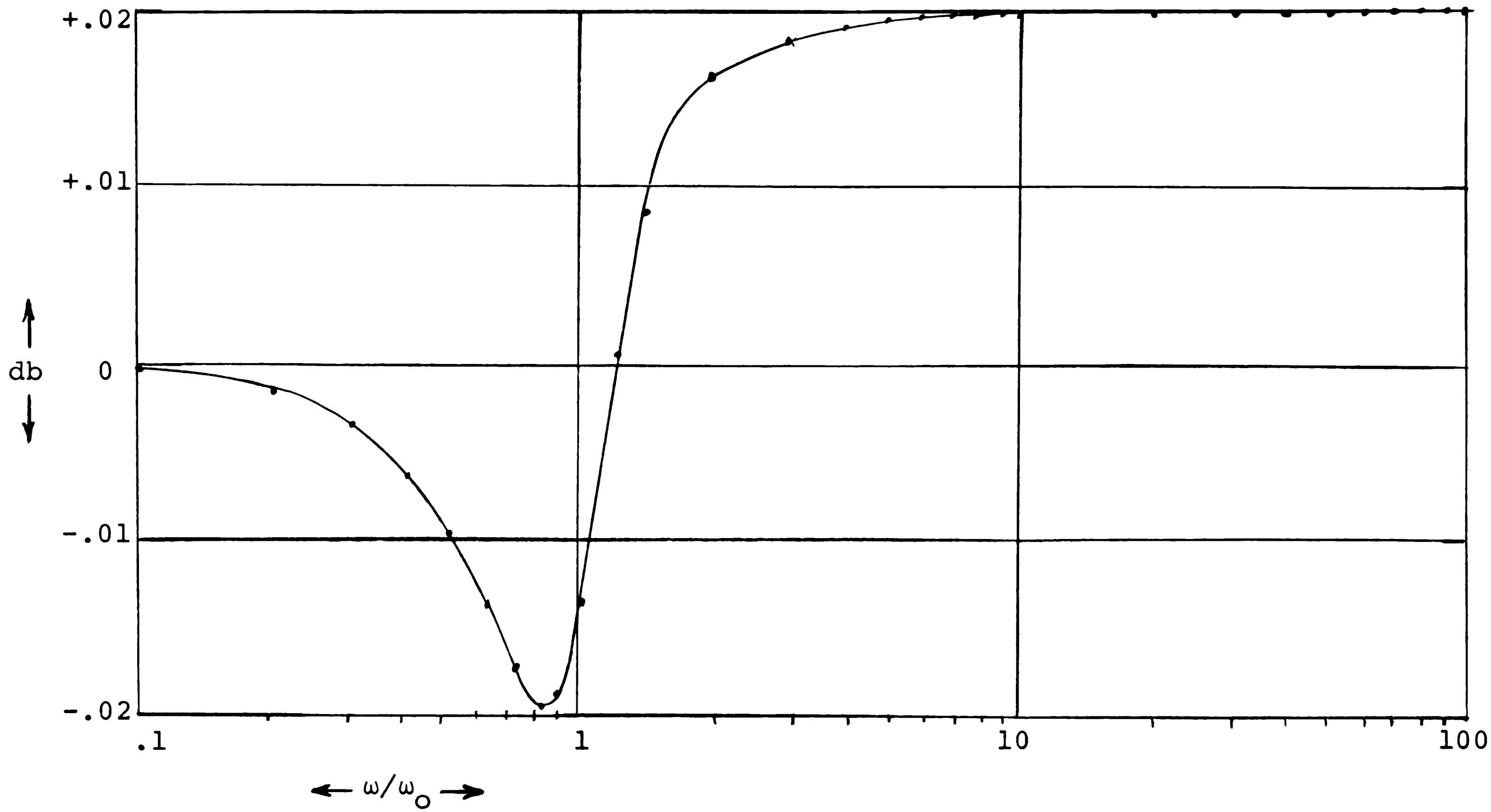


Fig. 4.2.1 Frequency Response Butterworth Filter. Theoretical minus actual response in "db" versus normalized frequency.

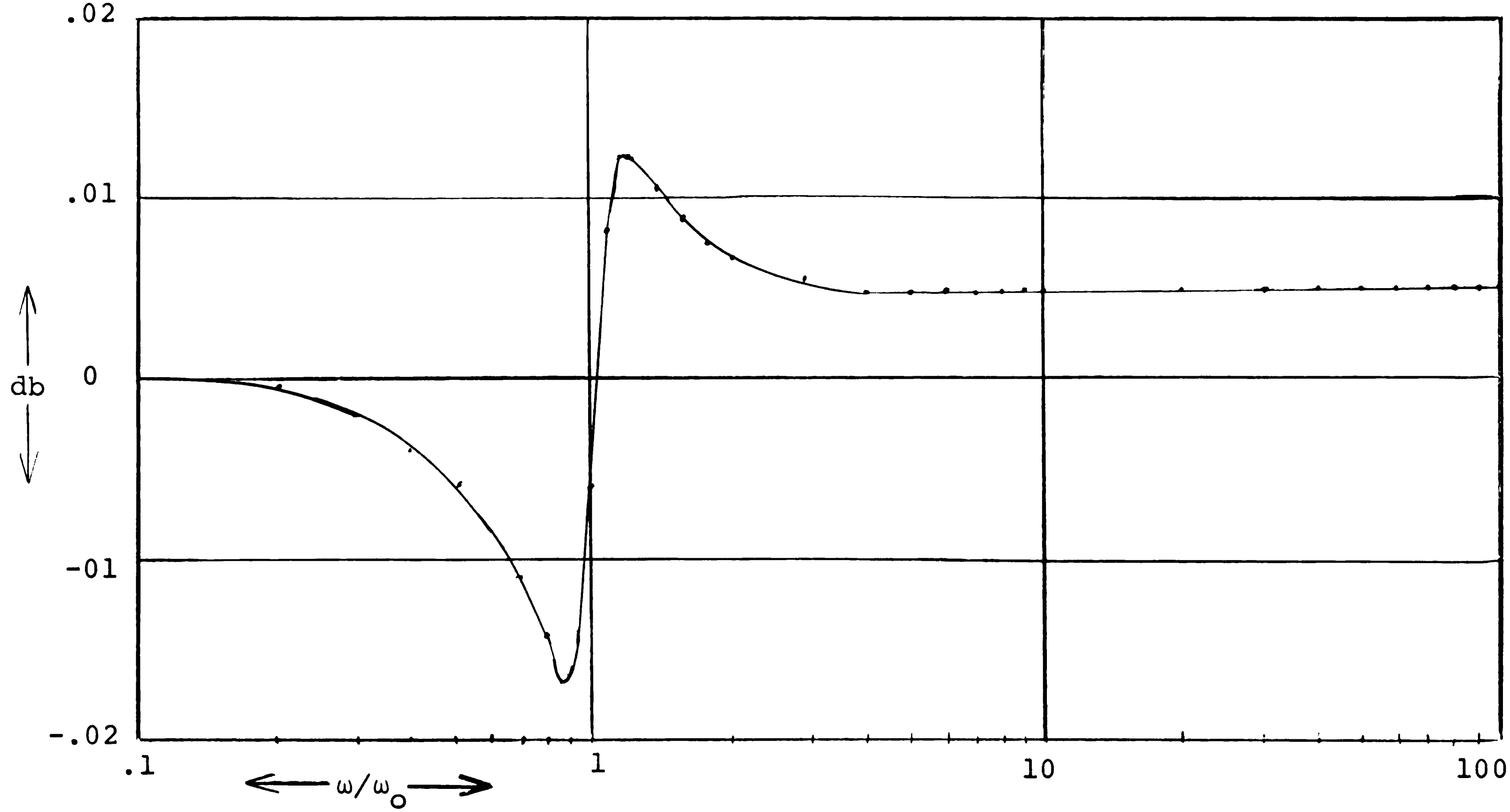
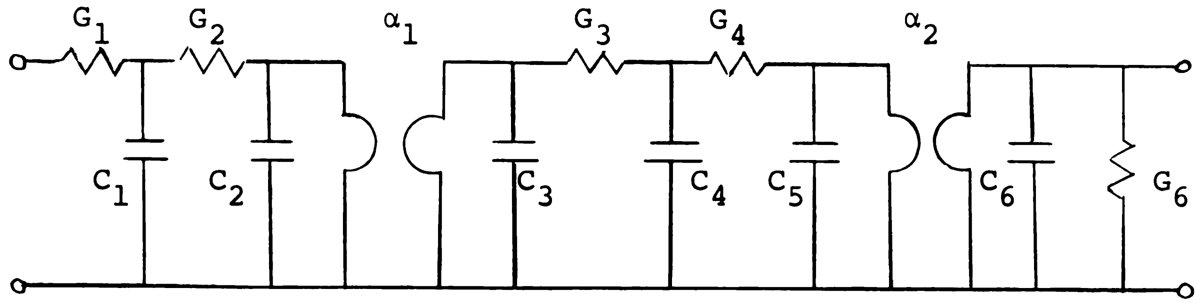


Fig. 4.2.2 Frequency Response Chebyshev Filter. Theoretical minus actual response in "db" versus normalized frequency.

For the cases where $n \geq 2$ and $P(S)$ is of degree greater than five, it appears impossible to supply an analytic proof. For this reason a random analysis program was carried out on the computer for the case where $P(S)$ was of sixth degree using the two networks shown in Fig. 4.3.1 and Fig. 4.3.2, respectively. It is not necessary to consider the other possible configurations which yield $P(S)$ of sixth degree, since by Theorems 2.4.2 and 2.5.1 such $P(S)$ must be strictly Hurwitz and have at lease two real zeros thereby satisfying the conjecture. For networks 1 and 2 the parameters X_i are defined as:

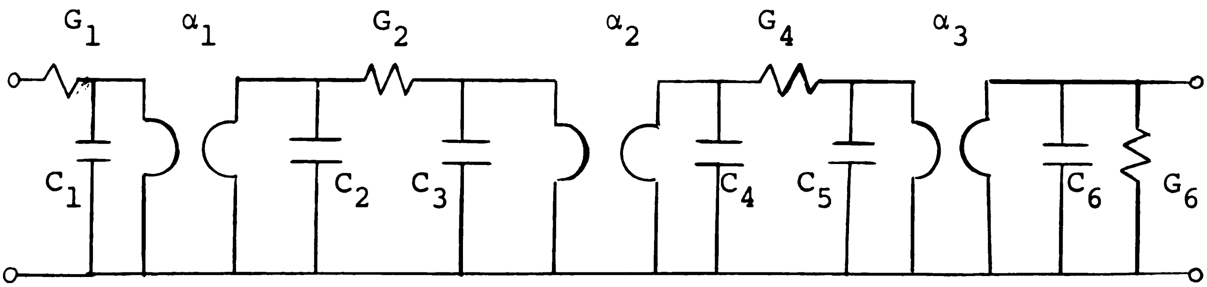
$$\left.\begin{array}{ll}
X_1 = G_1/C_1 & X_6 = G_4/C_4 \\
X_2 = G_2/C_1 & X_7 = G_4/C_5 \\
X_3 = G_2/C_2 & X_8 = G_6/C_6 \\
X_4 = G_3/C_3 & X_9 = \;1\;1/C_2C_3 \\
X_5 = G_3/C_4 & X_{10} = \alpha_2\tilde{\alpha}_2/C_5C_6
\end{array}\right\} \quad \begin{array}{c}\text{Network 1} \\ (4.3.2)\end{array}$$

$$\left.\begin{array}{ll}
X_1 = G_1/C_1 & X_6 = G_6/C_6 \\
X_2 = G_2/C_2 & X_7 = \alpha_1\tilde{\alpha}_1/C_1C_2 \\
X_3 = G_2/C_3 & X_8 = \alpha_2\tilde{\alpha}_2/C_3C_4 \\
X_4 = G_4/C_4 & X_9 = \alpha_2\tilde{\alpha}_3/C_5C_6 \\
X_5 = G_4/C_5 &
\end{array}\right\} \quad \begin{array}{c}\text{Network 2} \\ (4.3.3)\end{array}$$



Network 1

Fig. 4.3.1 Two-gyrator sixth degree RCΓ low-pass filter.



Network 2

Fig. 4.3.2 Three-gyrator sixth degree RCΓ low-pass filter.

From each set of X_i the corresponding set of element values can be calculated in terms of G_1 and the parameters $\alpha_i \tilde{\alpha}_i$.

A description of the method used in the random analysis program is as follows: Each X_i is selected randomly using a uniform distribution (computer library random number generator) and then the polynomial

$$P(S) = \prod_{i=1}^{\ell} (S + a_i) \prod_{i=1}^{r} [S + G_i)^2 + w_i^2]$$

is obtained from which the summation

$$\Phi = \sum_{i=1}^{r} \tan^{-1}(w_i/G_i) \qquad (4.3.4)$$

is formed by computing the zeros of the polynomial.

In using this program, the range for each X_i was adjusted after some preliminary runs, so as to maximize Φ in Eq. 4.3.4. It was observed that the conjecture was not violated in a sample of several hundred runs for each network.

Networks 1 and 2 were also analysed by a program which starts from a randomly selected set of X_i and adjusts the X_i sequentially so that Φ, in Eq. 4.3.4, is maximized within a bounded set of X_i. Due to the extreme time requirements (each iteration required the solution for the zeros of a sixth degree polynomial) only a few runs were made using this program. In no case, however, was the conjecture violated. In addition it was also observed, through several examples, that the proposed angle condition could not have been made stricter.

Finally attempts in realizing $T_V = k/P(S)$ with network configurations which do not satisfy the conjecture have not been successful, whereas when the network configuration satisfied the conjecture and Theorem 2.5.1, realizations have been found on the computer in a straightforward manner. (See Appendix A for the computer algorithm.) Some examples follow in the next section.

4.4 Computer Realizations for Fifth and Sixth Degree Low-Pass Transfer Functions Using RCΓ Configurations.

Although an analytic proof of the existence of network realizations satisfying $T_V = k/P(S)$, where P(S) is strictly Hurwitz and has degree greater than four, has not been established; realizations have been obtained, using the computer algorithm discussed in Appendix A to solve the nonlinear equations (similar to Eq. 4.2.2) derived from P(S) = det(Y"). In the following examples, computer realizations for three practical low-pass voltage-ratio transfer functions, are given.

Example 4.4.1. Let $T_V = k/P(S)S$ where

$$P(S) = S^5 + 15S^4 + 105S^3 + 420S^2 + 945S + 945 \tag{4.41.}$$

The coefficients for P(S) are for a Linear Delay (Thomson) filter and are taken from Van Valkenburg [VA]. Consider the network in Fig. 4.4.1 which has an admittance matrix

$$Y = \begin{bmatrix} G_1 & -G_1 & & & & \\ -G_1 & C_1S+G_1 & \alpha_1 & & & \\ & -\tilde{\alpha}_1 & C_2S+G_2 & -G_2 & & \\ & & -G_2 & C_3S+G_2 & \alpha_2 & \\ & & & -\tilde{\alpha}_2 & C_4S+G_4 & -G_4 \\ & & & & -G_4 & C_5S+G_4+G_5 \end{bmatrix}$$

The matrix obrained by deleting the first row and column of Y is

$$Y' = \begin{bmatrix} C_1S+G_1 & \alpha_1 & & & \\ -\tilde{\alpha}_1 & C_2S+G_2 & -G_2 & & \\ & -G_2 & C_3S+G_2 & \alpha_2 & \\ & & -\tilde{\alpha}_2 & C_3S+G_4 & -G_4 \\ & & & -G_4 & C_5S+G_4+G_5 \end{bmatrix}$$

The determinant Δ_{11} = det(Y') has a minimum of one real zero (for proof see Theorem 2.5.1). Other possible two-gyrator five-capacitor RCΓ low-pass filter configurations have minimums of two or more real zeros in Δ_{11} and they cannot be used since P(S) is known to have only one real zero [VA]. The matrix Y' can now be pre and post multiplied by $C^{-1/2}$ to obtain

$$Y'' = \begin{bmatrix} s + X_1 & \sqrt{X_7\alpha_1/\tilde{\alpha}_1} & & & \\ -\sqrt{X_7\tilde{\alpha}_1/\alpha_1} & s + X_2 & -\sqrt{X_2X_3} & & \\ & -\sqrt{X_2X_3} & s + X_3 & \sqrt{X_8\alpha_2/\tilde{\alpha}_2} & \\ & & -\sqrt{X_8\tilde{\alpha}_2/\alpha_2} & s + X_4 & -\sqrt{X_4X_5} \\ & & & -\sqrt{X_4X_5} & s + X_5 + X_6 \end{bmatrix} \tag{4.4.2}$$

where $\Delta_{11} = \det(C)\det(Y'')$ and

$$\begin{aligned} X_1 &= G_1/C_1 & X_5 &= G_4/C_5 \\ X_2 &= G_2/C_2 & X_6 &= G_5/C_5 \\ X_3 &= G_2/C_3 & X_7 &= \alpha_1\bar{\alpha}_1/C_1C_2 \\ X_4 &= G_4/C_4 & C_8 &= \alpha_2\bar{\alpha}_2/C_3C_4 \end{aligned} \tag{4.4.3}$$

The parameters X_i must be positive since the capacitors, conductances, and the parameters $\alpha_i\bar{\alpha}_i$ $i = 1,2$ are all positive. Conversely, if $X_i > 0$ and $\alpha_1\bar{\alpha}_1, \alpha_2\bar{\alpha}_2$ and G_1 are known then the network elements are

$$\begin{array}{ll} C_1 = G_1/X_1 & C_4 = \alpha_2\bar{\alpha}_2/C_3X_8 \\ C_2 = \alpha_1\bar{\alpha}_1/C_1X_7 & G_4 = C_4X_4 \\ G_2 = C_2X_2 & C_5 = G_4/X_5 \\ C_3 = G_2/X_3 & G_5 = C_5X_6 \end{array} \tag{4.4.4}$$

If a set of positive X_i can be found such that $\det(Y'') = P(S)$, then the transfer function $T_V = \Delta_{41}/\Delta_{11} = k/P(S)$ has been realized since $\Delta_{11} = \det[(C)(Y'')]$. The nonlinear equations in X_i, defined by $\det(Y'') = P(S)$, can be solved by the computer (see Appendix). Since there is considerable freedom in the choice of X_i ($\det(Y'') = P(S)$ yields five equations in eight unknowns) additional constraints can be added such as $G_1 = k\ G_5$. Expressing this constraint in terms of X_i, one has

$$X_1X_2X_5X_8 = K'X_3X_4X_6X_7$$

where $K' = k\ \alpha_2\tilde{\alpha}_2/\alpha_1\tilde{\alpha}_1$.

Computor realizations for three different values of K are

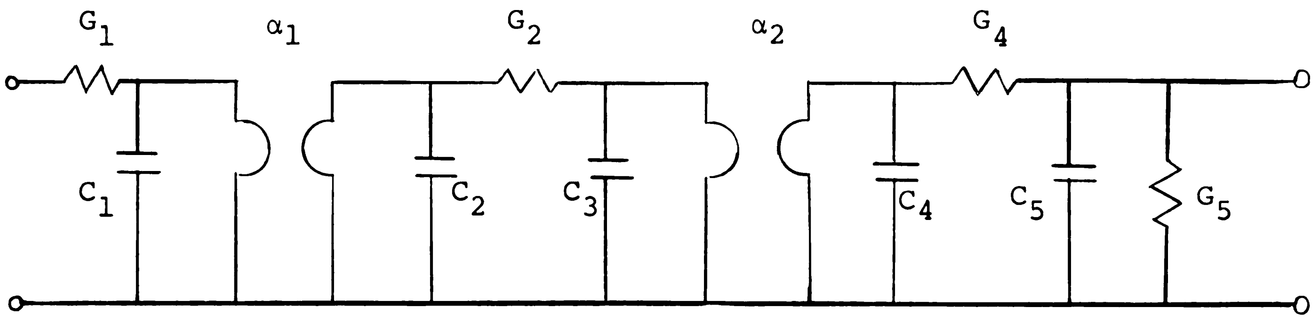


Fig. 4.4.1 Fifth degree low-pass RCΓ filter

$G_1 = 1.$	$G_1 = 1.$	$G_1 = 1.$
$C_1 = .19772492$	$C_1 = .16552402$	$C_1 = .16965992$
$G_2 = 2.1143563$	$G_2 = 1.2299518$	$G_2 = 2.8729672$
$C_2 = 1.1606858$	$C_2 = .97598661$	$C_2 = 1.3117841$
$C_3 = 1.0804736$	$C_3 = .36074834$	$C_3 = 1.4740298$
$G_4 = .13470185$	$G_4 = .17139153$	$G_4 = .075790874$
$C_4 = .049224786$	$C_4 = .11972908$	$C_4 = .02652635$
$G_5 = 1.$	$G_5 = .1$	$G_5 = .01$
$C_5 = .33105995$	$C_5 = .094977605$	$C_5 = .040669095$
$\alpha_1\bar{\alpha}_1 = \alpha_2\bar{\alpha}_2 = 1.$	$\alpha_1\bar{\alpha}_1 = \alpha_2\bar{\alpha}_2 = 1.$	$\alpha_1\bar{\alpha}_1 = \alpha_2\bar{\alpha}_2 = 1.$

All the element values given above are in mhos. or in farads. The realized network is given in Fig. 4.4.1.

Example 4.4.2. Let $T_V = k/P(S)$ where

$$P(S) = S^6 + 21S^5 + 210S^4 + 1260S^3 + 472S^2 + 10395S + 10395. \tag{4.4.5}$$

The coefficients for P(S) are for a sixth degree linear delay filter and are taken from Van Valkenburg [VA]. Consider the network in Fig. 4.3.1. This network satisfies the necessary conditions established in Theorem 2.5.1 and so is suitable to realize T_V. The admittance matrix for this network is

$$Y=\begin{bmatrix} G_1 & -G_1 & & & & & \\ -G_1 & C_1S+G_1+G_2 & -G_2 & & & & \\ & -G2 & C_2S+G_2 & \alpha_1 & & & \\ & & -\tilde{\alpha}_1 & C_3S+G_3 & -G_3 & & \\ & & & -G_3 & C_4S+G_3+G_4 & -G_4 & \\ & & & & -G_4 & C_5S+G_4 & \alpha_2 \\ & & & & & -\tilde{\alpha}_2 & C_6S+G_6 \end{bmatrix} \tag{4.4.6}$$

Forming Y" directly from this matrix, one obtains

$$Y''=\begin{bmatrix} S+X_1+X_2 & -\sqrt{X_2X_3} & & & & & \\ -\sqrt{X_2X_3} & S+X_3 & \sqrt{X_9\alpha_1/\tilde{\alpha}_1} & & & & \\ & -\sqrt{X_9\tilde{\alpha}_1/\alpha_1} & S+X_4 & -\sqrt{X_4X_5} & & & \\ & & -\sqrt{X_4X_5} & S+X_5+X_6 & -\sqrt{X_6X_7} & & \\ & & & -\sqrt{X_6X_7} & S+X_7 & \sqrt{X_{10}\alpha_2/\tilde{\alpha}_2} & \\ & & & & -\sqrt{X_{10}\tilde{\alpha}_2/\alpha_2} & S+X_8 & \end{bmatrix} \tag{4.4.7}$$

where $\det(C^{1/2}Y''C^{1/2}) = \Delta_{11}$ and

$$X_1 = G_1/C_1 \qquad X_6 = G_4/C_4$$
$$X_2 = G_2/C_1 \qquad X_7 = G_4/C_5$$
$$X_3 = G_2/C_2 \qquad X_8 = G_6/C_6 \tag{4.4.8}$$
$$X_4 = G_3/C_3 \qquad X_9 = \alpha_1\tilde{\alpha}_1/C_2C_3$$
$$X_5 = G_3/C_4 \qquad X_{10} = \alpha_2\tilde{\alpha}_2/C_5C_6$$

The relation $X_i > 0$, $(i = 1,\ldots,10)$ clearly holds since all the network parameters are positive. Conversely, if a set of positive X_i is found such that det(Y") = P(S), then the network parameters can be solved in terms of G_1, $\alpha_1\tilde{\alpha}_1$ and $\alpha_2\tilde{\alpha}_2$ giving the realization of T_V. The equations for the network parameters are:

$$C_1 = G_1/X_1 \qquad C_4 = G_3/X_5$$
$$G_2 = C_1X_2 \qquad G_4 = C_4X_6$$
$$C_2 = G_2/X_3 \qquad C_5 = G_4/X_7 \tag{4.4.9}$$
$$C_3 = \alpha_1\tilde{\alpha}_1/C_2X_9 \qquad C_6 = \alpha_2\tilde{\alpha}_2/C_5X_{10}$$
$$G_3 = C_3X_4 \qquad G_6 = C_6X_8$$

The nonlinear equations in X_i defined by

$$\det(Y'') = P(S) \tag{4.4.10}$$

can be solved by a computer program to realize $T_V = k/P(S)$. Since the relation det(Y") = P(S) yields six equations in ten unknowns, considerable freedom in the choice of

X_i is possible. A useful constraint which can be added is $G_1 = k\ G_6$. Expressing this constraint in terms of the X_i one has

$$X_1X_3X_4X_6X_{10} = K'X_2X_5X_7X_8X_9 \qquad (4.4.11)$$

where $K' = k\alpha_2\tilde{\alpha}_2/\alpha_1\tilde{\alpha}_1$. Computor solutions for three ratios of G_1/G_6 are:

G_1 = 1.	G_1 = 1.	G_1 = 1.
C_1 = .30561664	C_1 = .39875898	C_1 = .26994128
G_2 = .54369763	G_2 = .58810805	G_2 = .18377659
C_2 = .099232055	C_2 = .21466229	C_2 = .031468068
G_3 = 1.0379997	G_3 = .47470337	G_3 = 3.6152459
C_3 = .37053873	C_3 = .16927982	C_3 = 1.7360973
G_4 = .28599644	G_4 = 1.0119594	G_4 = 5.8383642
C_4 = .4818776	C_4 = .35106470	C_4 = 2.5482323
C_5 = .11172470	C_5 = 3.5776846	C_5 = 18.287234
G_6 = 1.	G_6 = .1	G_6 = .01
C_6 = .42353046	C_6 = .014376383	C_6 = .021445128
$\alpha_1\tilde{\alpha}_1 = \alpha_2\tilde{\alpha}_2 = 1.$	$\alpha_1\tilde{\alpha}_1 = \alpha_2\tilde{\alpha}_2 = 1.$	$\alpha_1\tilde{\alpha}_1 = \alpha_2\tilde{\alpha}_2 = 1.$

The above element values are in mhos or farads, and the network configuration is given in Fig. 4.3.1.

Example 4.4.3. Consider Example 4.2.2 when the polynomial $P(S) = \sum_{i=0}^{6} P_iS^i$ has the coefficients

$$P_0 = 15625.$$
$$P_1 = 12074.073$$
$$P_2 = 4665.0635$$
$$P_3 = 1142.7025 \qquad (4.4.12)$$
$$P_4 = 186.60254$$
$$P_5 = 19.318516$$
$$P_6 = 1.$$

The coefficients for P(S) were calculated on a computer with eight digit accuracy for a sixth degree low-pass Butterworth filter [VA]. This was frequency scaled for a cut-off at $\omega = 5$, so the coefficients would roughly match those in Example 4.4.2. Again the network configuration in Fig. 4.3.1 is used since P(S) has no real zeros. Computer solutions for three ratios of G_1/G_6 are:

$G_1 = 1.0$	$G_1 = 1.0$	$G_1 = 1.0$
$C_1 = .24379647$	$C_1 = .33142963$	$C_1 = 1.0010717$
$G_2 = .11840130$	$G_2 = .30864407$	$G_2 = 2.6430899$
$C_2 = .020970984$	$C_2 = .043334407$	$C_2 = .50873174$
$G_3 = 2.9354488$	$G_3 = .81051727$	$G_3 = .18560747$
$C_3 = 1.6751027$	$C_3 = .72286262$	$C_3 = .064120396$
$G_4 = .070774711$	$G_4 = .64364059$	$G_4 = 1.3184473$
$C_4 = .63416227$	$C_4 = .32733992$	$C_4 = .30267577$
$C_5 = .061371579$	$C_5 = .71763355$	$C_5 = 9.8780364$
$G_6 = 1.0$	$G_6 = .1$	$G_6 = .01$
$C_6 = .69490787$	$C_6 = .055954909$	$C_6 = .0040216537$
$\alpha_1\tilde{\alpha}_1 = \alpha_2\tilde{\alpha}_2 = 1.$	$\alpha_1\tilde{\alpha}_1 = \alpha_2\tilde{\alpha}_2 = 1.$	$\alpha_1\tilde{\alpha}_1 = \alpha_2\tilde{\alpha}_2 = 1.$

The above parameters are in mhos. and farads, and the network configuration is given in Fig. 4.3.1.

Example 4.4.4. Consider Example 4.4.2 when

$$P(S) = [(S + 1)^2 + 2^2]^3 \tag{4.4.13}$$

Note that Conjecture 4.3.1 yields

$$3 \tan^{-1} \frac{2}{1} < n \frac{\pi}{2} \tag{4.4.14}$$

which is satisfied only if $n \leq 3$. Therefore the configuration shown in Fig. 4.3.1 cannot be used to realize $T_V = k/P(S)$. Consider the network in Fig. 4.3.2 which contains 3 gyrators. By theorem 2.5.1, taking the maximum number of real poles of T_V as zero, one can realize $T_V = k/P(S)$. The admittance matrix is

$$Y = \begin{bmatrix} G_1 & -G_1 & & & & & \\ -G_1 & C_1S+G_1 & \alpha_1 & & & & \\ & -\tilde{\alpha}_1 & C_2S+G_2 & -G_2 & & & \\ & & -G_2 & C_3S+G_2 & \alpha_2 & & \\ & & & -\tilde{\alpha}_2 & C_4S+G_4 & -G_4 & \\ & & & & -G_4 & C_5S+G_4 & 3 \\ & & & & & -\tilde{\alpha}_3 & C_6S+G_6 \end{bmatrix}$$

Forming Y" directly from Y, one can obtain

$$Y'' = \begin{bmatrix} S+X_1 & \sqrt{X_7\alpha_1/\tilde{\alpha}_1} & & & & \\ -\sqrt{X_2\tilde{\alpha}_1/\alpha_1} & S+X_2 & -\sqrt{X_2X_3} & & & \\ & -\sqrt{X_2X_3} & S+X_3 & \sqrt{X_8\alpha_2/\tilde{\alpha}_2} & & \\ & & -\sqrt{X_8\tilde{\alpha}_2/\alpha_2} & S+X_4 & -\sqrt{X_4X_5} & \\ & & & -\sqrt{X_4X_5} & S+X_5 & \sqrt{X_9\alpha_3/\tilde{\alpha}_3} \\ & & & & \sqrt{X_9\tilde{\alpha}_3/\alpha_3} & S+X_6 \end{bmatrix}$$

where $\det(Y'') = \det(C^{-1})\Delta_{11}$ and

$$\begin{aligned} X_1 &= G_1/C_1 & X_6 &= G_6/C_6 \\ X_2 &= G_2/C_3 & X_7 &= \alpha_1\tilde{\alpha}_1/C_1C_2 \\ X_3 &= G_2/C_3 & X_8 &= \alpha_2\tilde{\alpha}_2/C_3C_4 \\ X_4 &= G_4/C_4 & X_9 &= \alpha_3\tilde{\alpha}_3/C_5C_6 \end{aligned} \qquad (4.4.15)$$

The X_i $i = 1,\ldots,9$ must be positive; conversely if they are positive Eq. 4.4.15 can be used to solve for the network components in terms of G_1, $\alpha_1\tilde{\alpha}_1$, $\alpha_2\tilde{\alpha}_2$, and $\alpha_3\tilde{\alpha}_3$. The solution of the six nonlinear equations in X_i derived from

$$P(S) = \det(Y'')$$

with positive X_i realizes $T_V = k/P(S)$. The computer solution is

$G_1 = 1.$ $\quad$ $G_4 = .33940651$

$C_1 = 1.3115185$ $\quad$ $C_4 = .54931147$

$G_2 = .43199501$ $\quad$ $C_5 = 3.0489273$

$C_2 = .41741739$ $\quad$ $G_6 = .098963398$

$C_3 = .25824894$ $\quad$ $C_6 = .054960727$

$$\alpha_1\tilde{\alpha}_1 = \alpha_2\tilde{\alpha}_2 = \alpha_3\tilde{\alpha}_3 = 1.$$

where all the elements are in mhos. or in farads. The realized network is given in Fig. 4.3.2.

4.5 Band-pass and High-pass Filter Realizations

Although the purpose of this thesis is to develop low-pass RCΓ realizations, it is possible, in some cases, to extend the low-pass realization into other filters. This is established by the following two theorems.

<u>Theorem 4.5.1</u>. Let $T_V = k\ S^n/P(S)$, where P(S) is of degree n. Then there exists a high-pass RC filter realization of T_V if there exists an RCΓ low-pass filter realization for $T_V' = T_V(\frac{1}{S})$ satisfying Theorem 2.4.3 which uses exactly two gyrators.

Proof: Since there is a network realizing $T_V' = T_V(\frac{1}{S})$, there is a set of positive X_i such that $\det(Y'') = P(\frac{1}{S})S^n$ where $\det(Y'') = \det(C^{-1})\Delta_{11}$. From this set of X_i one can calculate a set of components so that

$\alpha_1 = \tilde{\alpha}_1 = \alpha_2 = \alpha_2 = 1$. The complex variable S can now be replaced by $\frac{1}{S}$ to obtain realization of a RLΓ network corresponding to T_V, as shown in Fig. 4.5.1. Making extensive use of the equivalent forms in Fig. 4.5.2, the network is transformed into the RCΓ network shown in Fig. 4.5.3. Finally recognizing that the current-ratio transfer function, without the terminating gyrators, is the same as the voltage-ratio transfer functions and also using Thevenin and Norton equivalents, the desired RCΓ network can be obtained as shown in Fig. 4.5.4. Q.E.D.

<u>Corollary 4.5.1</u>. Let P(S) be a strictly Hurwitz polynomial of degree four, then $T_V = k\ S^4/P(S)$ can always be realized.

Proof: Since P(S) is strictly Hurwitz, so is $S^4 P(\frac{1}{S})$. The transfer function $T_V' = k/P'(S)$, where $P'(S)$ is strictly Hurwitz and of degree four, is always realizable from Theorem 4.1.1. Q.E.D.

Note: Theorem 4.5.1 can be extended to the networks containing any even number of gyrators, however the number two is probably the practical limit.

Note: A high-pass filter as shown in Fig. 4.5.4 satisfies the conditions for Theorem 2.4.2. Therefore such a filter remains stable for R, C, or Γ parameter variation.

<u>Theorem 4.5.2</u>. Let $T_V = k\ N(S)/P(S)$ where $N(S) = (S^2 + a^2)$ and $T_V' = k/P(S)$ has a low pass network realization of the form shown in Fig. 4.5.5. Then, T_V can be realized with an RCΓ network.

Proof: Consider the chain matrix representation of the network shown in Fig. 4.5.5.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$$

From which one can obtain

$$A = A_1A_2 + B_1C_2$$

Since $A = 1/T_V$, A' can be formed as

$$A' = 1/T_V' = \frac{A_1A_2 + B_1C_2}{kN(S)}$$

Note that A_1/B_1 and $1/B_1$ define Y22 and -Y21 respectively for network 1. Similarly, A_2/C_2 and $1/C_2$ define Z_{11} and Z_{21} for network 2. Let $A_1' = A_1/K\ N(S)$ and $B_1' = B_1/K\ N(S)$; then $A'_1/B_1' = A_1/B_1$ defines $Y_{22}^{(1)}$ and $1/B_1' = k\ N(S)/B_1$ defines $-Y_{21}^{(1)}$ for the desired realization. Since the degree of $N(S)$ equals the degree of A_1, the synthesis of the network from $-Y_{21}^{(1)}$ and $Y_{22}^{(1)}$ is possible using standard synthesis techniques. The final realization is shown in Fig. 4.5.6. Q.E.D.

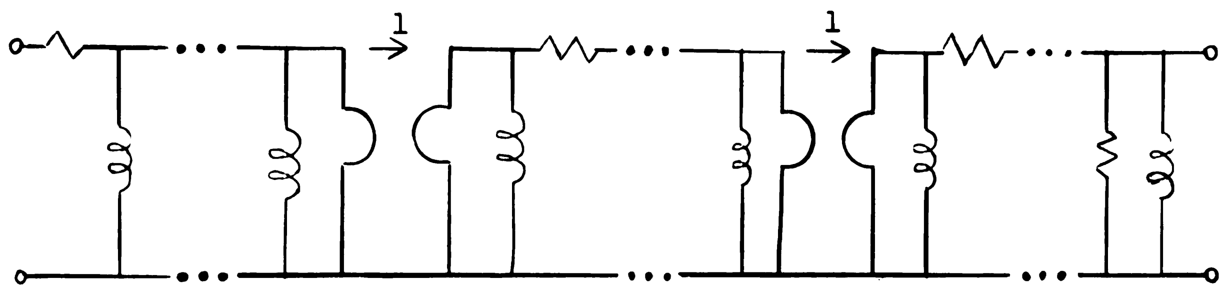


Fig. 4.5.1 RLΓ network.

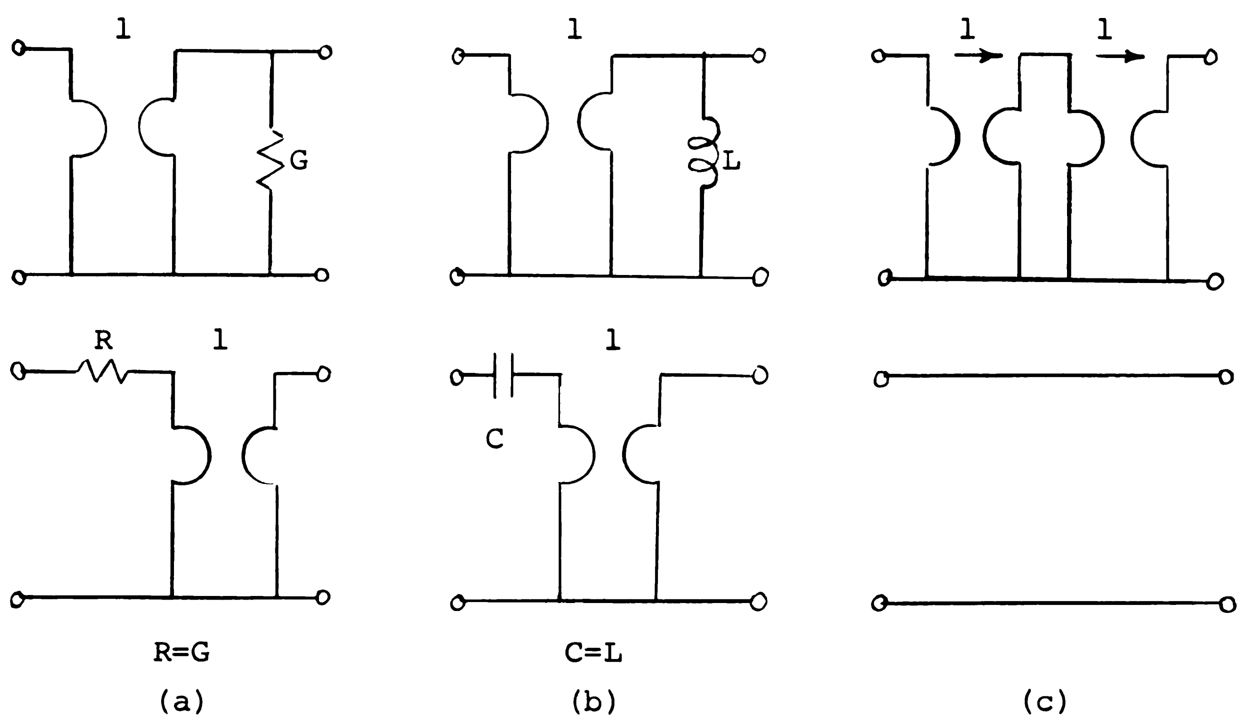


Fig. 4.5.2 Equivalent forms.

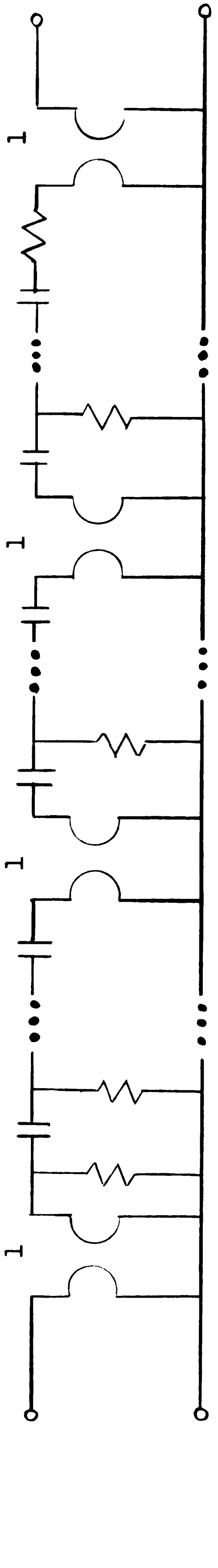


Fig. 4.5.3 RCΓ network.

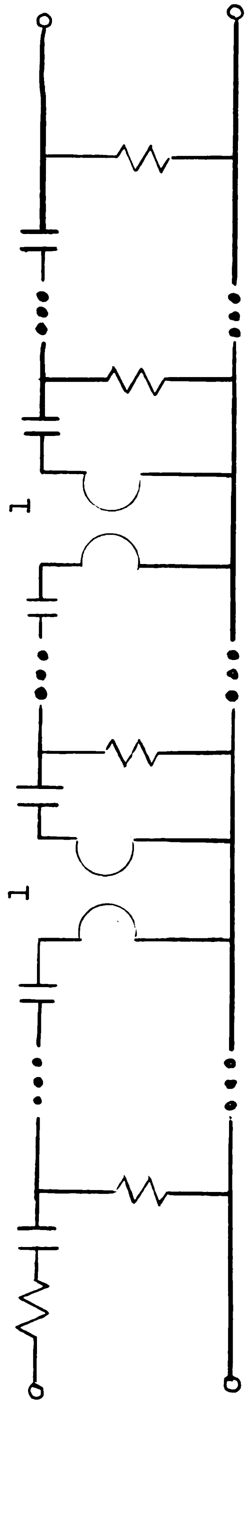


Fig. 4.5.4 RCΓ high-pass filter.

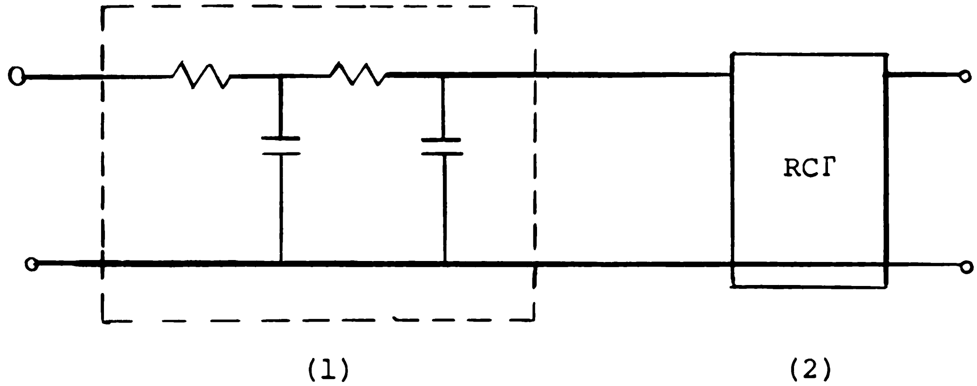
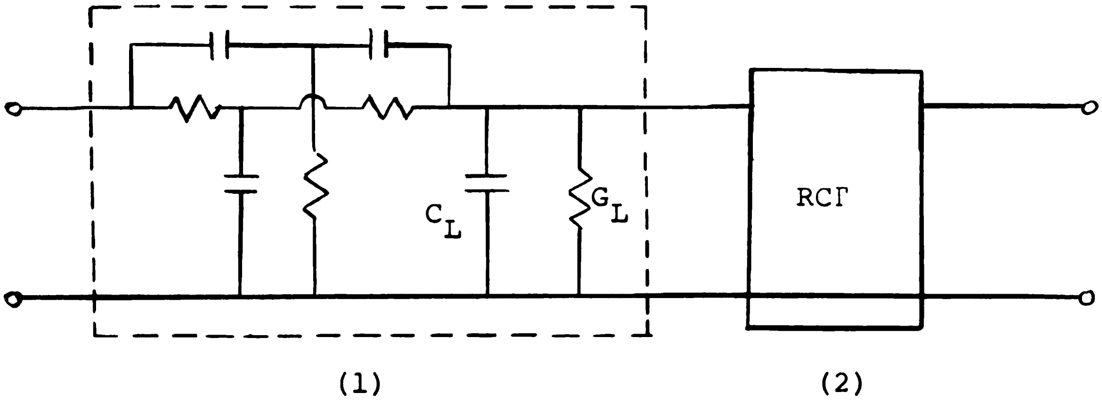


Fig. 4.5.5 Initial network for Theorem 4.5.2.



In general, either C_L or G_L or both are zero.

Fig. 4.5.6 Final network for Theorem 4.5.2.

Note: The network in Fig. 4.5.6 satisfies Theorem 2.4.2 and therefore remains stable for R, C, or Γ parameter variation.

Note: All the results of this chapter remain true if practical gyrators which have admittance matrices of the form

$$Y_\Gamma = \begin{bmatrix} G_{11} & G_{12} \\ -G_{21} & G_{22} \end{bmatrix}$$

where G_{11}, G_{12}, G_{21}, and G_{22} are positive parameters, are used rather than ideal active gyrators.

Indeed the form of Y_Γ suggests that a practical gyrator is equivalent to an ideal active gyrator loaded by conductances at its ports. Consider $T_V = k/P(S)$ where P(S) is strictly Hurwitz. Since one can find a positive constant, γ, such that

$$P'(S) = P(S - \gamma)$$

is strictly Hurwitz, the voltage-ratio transfer function

$$T_V' = k/P'(S)$$

can be realized as a low-pass RCΓ filter by methods established in this chapter. In this realization replacing S by S + γ, the RCΓ filter is now modified and it becomes the realization of

$$T_V = k/P(S)$$

In this modified network every capacitor C_i is connected in parallel with a conductance of value γC_i. Since each gyrator branch is always in parallel with a capacitor, each gyrator branch can now be considered as being in parallel with a conductance. This permits one to replace ideal active gyrators by practical ones.

For the high-pass case the process is similar except each gyrator branch is in series with a resistance. This results in a gyrator impedance matrix of the form

$$Z_\Gamma = \begin{bmatrix} R_{11} & R_{12} \\ -R_{21} & R_{22} \end{bmatrix}$$

which represents a practical gyrator.

4.6. Conclusions

In this chapter it is established that every fourth degree low-pass voltage-ratio transfer function whose denominator is a strictly Hurwitz polynomial, can be realized using a two-gyrator low-pass RCΓ filter. This also holds for the high-pass filter case by a simple extension established in Theorem 4.5.1. Computer

RCΓ realizations are given for some practical fourth, fifth, and sixth degree low-pass voltage-ratio transfer functions which utilize two gyrators. A proposed necessary condition is given and supported with feasability arguments. A computer realization using three gyrators is given to illustrate extensions of the method. Finally, it is demonstrated that, for certain cases, the method can be extended to high-pass and band-pass filter realizations.

CHAPTER V

CONCLUSIONS

The purpose for this thesis is to provide a basis for RCΓ filter synthesis and to specifically develop realization procedures for low-pass voltage-ratio transfer functions. A general basis for RCΓ filter synthesis is established in Chapters II and III, whereas synthesis procedures for low-pass voltage-ratio transfer functions are established in Chapter IV.

The main contribution of this thesis can be listed as follows:

i) Theorem 2.4.2 is established. As a consequence of this theorem, it is shown that a large class of active RCΓ filter realizations are stable and remain stable irrespective of the variation in the RC components of the gyrator parameters. All the RCΓ filter realizations considered in this thesis belong to this class defined in section 2.4.

ii) It is proved that the Calahan and Horowitz polynomial decomposition methods can be derived one from the other.

iii) The Calahan and the Horowitz polynomial decomposition methods are extended to polynomials which contain distinct negative real zeros.

iv) Low-pass RCΓ filter realizations are shown to exist for voltage-ratio transfer functions $T_v = k/P(s)$, where P(s) is strictly Hurwitz and of degree four. Through the use of a network transformation, it is shown that high-pass RCΓ filter realizations exist for the voltage-ratio transfer function $T_v = k\, s^4/P(s)$, where P(s) is a strictly Hurwitz polynomial of the fourth degree.

v) Computer realization procedures are established to realize $T_v = k/P(s)$, where P(s) is strictly Hurwitz.

Suggestions for Future Work:

i) To establish the necessary and sufficient conditions for network realizations when the degree of the polynomial P(s) appearing in $T_v = k/P(s)$ is greater than four.

ii) To establish parameter sensitivity comparisons between the general RCΓ filter realizations and The RCNIC filter realizations.

iii) To develop analytic techniques for the realization of $T_v = k/P(s)$ when P(s) is of degree greater than four.

APPENDIX A

Consider the solution of a set of nonlinear equations of the general form

$$
\begin{aligned}
p_1 &= P_1(x_1,x_2,\ldots,x_{nv}) \\
p_2 &= P_2(x_1,x_2,\ldots,x_{nv}) \qquad \text{(A.0.1)} \\
&\vdots \\
p_{nc} &= P_{nc}(x_1,x_2,\ldots,x_{nv})
\end{aligned}
$$

where the subscripts nv and nc are the number of variables and the number of equations respectively. Several computer programs have been written and applied to the problems of the above form considered in this thesis. One of these, a direct search scheme based on Hooke and Jeeves [WI] pattern search, proved to be successful when the number of variables nv is less than seven. This program is now available from the Michigan State University program library [MC1].

To solve the nonlinear equations in Eq. A.0.1 when the number of variables is greater than seven, an algorithm was adapted based on the Taylor least squares reduction scheme [CT]. This algorithm is discussed in the following section.

A.1 Computer Algorithm

Consider the error vector F where

$$F_1 = p_1 - P_1(x_1,x_2,\ldots,x_{nv})$$
$$F_2 = p_2 - P_2(x_1,x_2,\ldots,x_{nv})$$
$$\vdots$$
$$F_{nc} = p_{nc} - P_{nc}(x_1,x_2,\ldots,x_{nv})$$

Let x be the column vector with the variables $x_1,x_2,\ldots,$ x_{nv}, and let $\hat{x}$ be a solution for Eq. A.0.1. A Taylor series expansion of F about $\hat{x}$ is

$$0 = F(x) + \frac{\partial F}{\partial x}(\hat{x} - x) + \ldots \qquad \text{(A.1.2)}$$

It is interesting to note at this point that, for the network problems considered in this thesis, the functions $P_i(x_1,x_2,\ldots,x_{nv})$ are always of the form

$$P_i(x_1,x_2,\ldots,x_{nv}) = g_i(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{nv})x_j$$
$$+h_i(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{nv}) \qquad \text{(A.1.3)}$$

This property, in general, also holds for the nonlinear equations generated from passive netowrk functions. Equation A.1.3 allows the matrix $\partial F/\partial x$ to be evaluated very accurately and simply by a computer program.

In Eq. A.0.1, if nv equals nc, the matrix $\partial F/\partial x$ is the Jacobian of F and neglecting higher order terms of the Taylor expansion, $\hat{x}$ can be calculated iteratively using the relation

$$x^{(i+1)} - x^{(i)} = - [\partial F/\partial x]^{-1} F \qquad \text{(A.1.4)}$$

where $x^{(i)}$ is the ith iteration. Equation A.1.4 represents the familiar Newton-Raphson method.

If nv is less than nc no solutions exist in general. However, one can use

$$x^{(i+1)} - x^{(i)} = - [\frac{\partial F}{\partial x}^T \frac{\partial F}{\partial x}]^{-1} \frac{\partial F}{\partial x}^T F \qquad \text{(A.1.5)}$$

which gives a least squares estimate of $\hat{x}$ [CT].

Finally, consider the case where nv is greater than nc. Solutions exist for this case, however, the matrix $\partial F/\partial x$ is not square and the matrix $[\partial F/\partial x]^T[\partial F/\partial x]$ is singular. Therefore neither Eq. A.1.4 or Eq. A.1.5 can be used for this case. However, F can be augmented with a set of trivial functions such that they become zero as F becomes zero, and the relation in A.1.5 can be used for this case. Indeed let

$$F' = \begin{bmatrix} F \\ \sqrt{\lambda}(x-x^{(i)}) \end{bmatrix}$$

then Eq. A.1.5 takes on the form

$$x^{(i+1)} - x^{(i)} = - \left[\frac{\partial F^T}{\partial x} \frac{\partial F}{\partial x} + \lambda u \right]^{-1} \left[\frac{\partial F^T}{\partial x} \middle| \sqrt{\lambda} u \right] \begin{bmatrix} F \\ \hline 0 \end{bmatrix}$$

or

$$x^{(i+1)} - x^{(i)} = - \left[\frac{\partial F^T}{\partial x} \frac{\partial F}{\partial x} + \lambda u \right]^{-1} \frac{\partial F^T}{\partial x} F \qquad \text{(A.1.6)}$$

It is interesting to note that E . A.1.6 reduces to a Newton-Raphson reduction scheme if nv equals nc and λ equals zero. On the other hand if λ is large this method has the characteristics of a steepest descent method [CT]. In general the reduction scheme represented in Eq. A.1.6 simply minimizes the step size as well as the error vector F.

Let E be the error function defined by

$$E = \sum_{i=1}^{nc} \left| F_i \right| \qquad \text{(A.1.7)}$$

Since the Newton Raphson scheme converges rapidly in the vicinity of a solution and the steepest descent method converges more rapidly far from a solution, a value of λ based on E is practical. For the examples considered in the thesis the criterion

$$\lambda = \text{minimum} \; \{E, \sqrt{E}\} \qquad \text{(A.1.8)}$$

has worked very well.

In order to equalize the accuracy among the coefficients P_i, Eq. A.0.2 is modified so that

$$F_i = w_i[p_i - P_i(x_1,x_2,\ldots,x_{nv})] \qquad \text{(A.1.9)}$$

where

$$w_i = (\underset{j}{\text{maximum}}\ \{P_j\})/P_i.$$

The introduction of these weighing factors improves the performance of the program as well. Two possible heuristic explanations for this are: the addition of the w_i improves the conditioning of the matric $[\frac{\partial F}{\partial x}^T \frac{\partial F}{\partial x} + \lambda u]$, or the addition of the w_i promotes uniform reduction of the terms in the error vector F.

The problem of oscillation about a solution is prevented by introducing a damping factor α into the reduction scheme. This is represented by the following equation.

$$x^{(i+1)} - x^{(i)} = -\alpha \left[\frac{\partial F}{\partial x}^T \frac{\partial F}{\partial x} + u\right] \frac{\partial F}{\partial x}^T F \qquad \text{(A.1.10)}$$

Initially α is set equal to 1. If the condition

$$E(x^{(i+1)}) \geq E(x^{(i)})$$

holds, α is reduced by one half and $x^{(i+1)}$ recalculated. This is repeated up to seven times before the attempt is given up.

The algorithm described here converges rapidly from a good initial point x, however it does not necessarily converge from an arbitrary initial point. For this reason a Monte Carlo random search program [WI] was written to provide a good initial point. Experience gained using this program indicates that the Monte Carlo initial point search becomes increasingly important as the number of variables increases.

A flow diagram of the entire program is given in Fig. A.1.1.

A.2 Usage

Program documentation which describes how to use this program and the associated program source deck is available from the Michigan State University program library [MC2]. Unfortunately, a certain amount of practice is required to successfully use this program for solving a set of nonlinear equations. This is true since the boundaries for the Monte Carlo search must be set by the program user. If the boundaries are set too large the region searched becomes too large and conversely, if they are too small, the region may not contain any solutions. If no previous knowledge is available about a particular problem, it becomes necessary to try several exploratory runs on the computer before satisfactory search boundaries are established. Fortunately even very "rough" results obtained by hand calculations or knowledge about the roots, etc., provide sufficient information for the establishment of search boundaries.

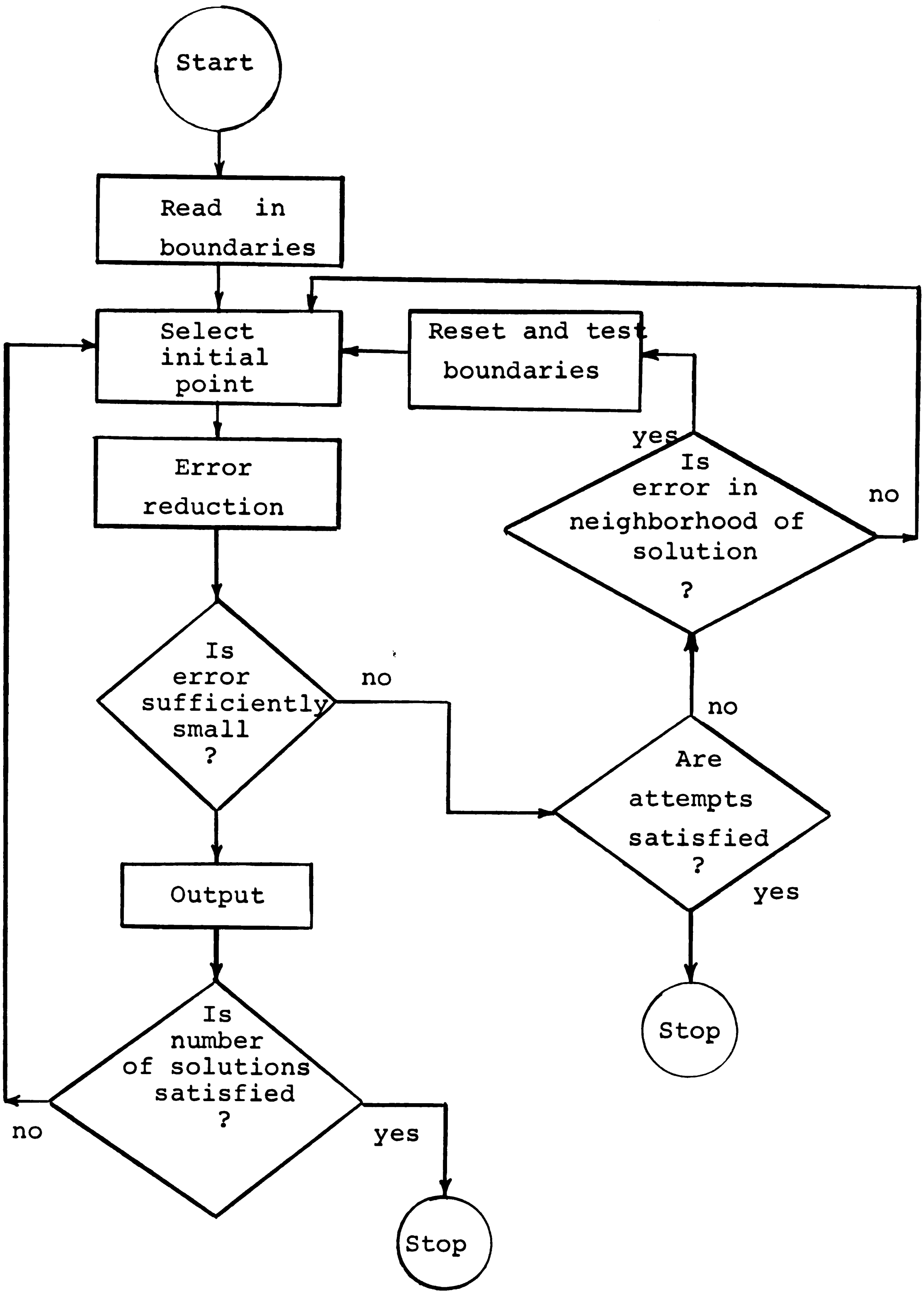


Fig. A.1.1 Computer algorithm flow chart.

BIBLIOGRAPHY

[CA1] Calahan, D. A.: "Restrictions on the Natural Frequencies of an RC-RL Network," Journal of the Franklin Institute, Vol. 272. pp. 112-133; August 1961.

[CA2] Calahan, D. A.: "Sensitivity Minimization in Active RC Synthesis," IRE Trans. on Circuit Theory, Vol. CT-9, pp. 38-42; March 1962.

[CA3] Calahan, D. A.: Modern Network Synthesis, Vol. II, Hayden, New York, 1964.

[CT] Calahan, D. A. and Temes, G. C.: "Computer Aided Network Optimization," Proc. of the IEEE, Vol. 55, p. 1839; November 1967.

[FR] Frame, J. S.: "Matrix Functions and Applications," IEEE Spectrum, Vol. 1, No. 7, pp. 103-109, July 1964.

[GU] Guillemin, E. A.: Synthesis of Passive Networks, John Wiley and Sons, Inc., New York, 1957.

[HA] Hakim, S. S.: "RC Active Filters Using an Amplifier as the Active Element," Proc. IEE (London), Vol. 112, pp. 901-912; May, 1955.

[HO] Horowitz, I. M.: "Optimization of Negative-Impedance Methods of Active RC Synthesis," IRE Trans. on Circuit Theory, Vol. CT-6, pp. 296-303; September, 1959.

[HT] Holt, A. G. and Taylor, J.: "Method of Replacing Ungrounded Inductors by Grounded Gyrators," Electronic Letters, Vol. 1, No. 4, pp. 105; June 1965.

[KTK] Koenig, H. E., Tokad, Y. and Kesavan, H. K.: Analysis of Discrete Physical Systems, McGraw-Hill, New York, 1967.

[LI] Linvill, J. G. "RC Active Filters," Proc. of the IRE, Vol. 42, pp. 555-564; March 1954.

[MC1] McNally, R. L.: "Optimization with Hooke and Jeeves Pattern Search," Michigan State University program library, No. 00000384.

[MC2] McNally, R. L.: "Network Synthesis with Optimization," Michigan State University program library, No. 00000406.

[MI] Mitra, S. K.: Analysis and Synthesis of Linear Active Networks, Wiley, New York, 1969.

[SK] Sallen, R. P. and Key, E. L.: "Practical Method of Designing RC Active Filters," IRE Trans. on Circuit Theory, Vol. CT-2, pp. 74-85; March 1955.

[TH] Thomas, R. E.: "Polynomial Decomposition in Active Network Synthesis," IRE Trans. on Circuit Theory, Vol. CT-8, pp. 270-274; September 1961.

[TO] Tokad, Y.: "Foundations of Passive Electrical Network Synthesis," part 1, (Bound Notes), Michigan State University, East Lansing, Michigan, 1966.

[VA] Van Valkenburg, M. E.: Introduction to Modern Network Synthesis, John Wiley and Sons, Inc., New York, 1960.

[WI] Wilde, D. J.: Optimum Seeking Methods, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[YO] Yonagisawa, T.: "RC Active Networks Using Current Inversion Type NIC," IRE Trans. on Circuit Theory, Vol. CT-4, pp. 140-144; September 1957.