# AN EFFICIENT ALGORITHM FOR THE ELEMENT VALUES OF MID-SERIES AND MID-SHUNT LOW-PASS LC LADDER NETWORKS

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ABSTRACT

AN EFFICIENT ALGORITHM FOR THE ELEMENT VALUES OF MID-SERIES AND MID-SHUNT LOW-PASS LC LADDER NETWORKS

by Chin Cheng Lin

In this thesis, an algorithm for the computation of element values of mid-shunt or mid-series low-pass ladder networks is given. The algorithm is based on the network interpretation of the Darlington formulas described in a recent article by Amstutz. It is shown that Fujisawa's procedure can readily be included in the algorithm. Thus, starting with an arbitrary attenuation pole sequence, the computer can find a realization with positive element values and a rearrangement of the sequence into a proper one. Compared with conventional methods, this algorithm gives more accurate results for ladder networks with large numbers of sections and it is insensitive to the change of variable used in filter synthesis to overcome the accuracy problem.

As an application of the algorithm, the synthesis of inverse Chebyshev filters are considered. Some useful properties and a design procedure for the inverse Chebyshev filters are established. Besides the practical application of the design method, the established properties are quite general and may be used for any low-pass ladder filters. The group delay characteristics of the inverse Chebyshev filters are also studied thoroughly.

The last part of this thesis considers a new approach to the synthesis of mid-shunt low-pass LC ladder networks by using state equations. Some common properties of the coefficient matrices of the state equations are established. Interpretation of the relations

that exist between functions defined in insertion-loss theory and the A-matrix of a state model are given. The formulation of the problem is such that it suggests some possible synthesis procedures. Although no general method for such synthesis is given, the results obtained from the analysis, and two suggestions for the solution of the synthesis problem are useful for further study in this area.

# AN EFFICIENT ALGORITHM FOR THE ELEMENT VALUES OF MID-SERIES AND MID-SHUNT LOW-PASS LC LADDER NETWORKS

By

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# CHAPTER I

## INTRODUCTION

In the synthesis of filter networks, on the basis of insertion-loss theory, the final step is the realization of the driving-point immittance function as either an impedance function, $Z_d(s)$, or an admittance function, $Y_d(s)$, which is seen from the input port 1-1' as indicated in Fig. I-1. Because of practical considerations, it is always preferable to realize the filter network in a ladder form. For low-pass filters, the most important forms are the mid-shunt and the mid-series ladders shown in Fig. I-2 in which $\omega_{\infty k}$ is the resonant frequency of a mid-shunt branch or a mid-series branch. Since at each $\omega_{\infty k}$ the corresponding branch becomes open circuited for the mid-shunt ladder and short circuited for the mid-series ladder, no signal can be transmitted at these frequencies. For this reason the $\omega_{\infty k}$'s are called the transmission zeros or the attenuation pole frequencies.

The establishment of the explicit formulas for the element values of a low-pass LC ladder filter from a given driving-point immittance function and a set of transmission zeros, is one of the outstanding problems in filter synthesis. Even in the simple LC ladder, where all transmission zeros occur at infinity, i.e., when in Fig. I-2 $\Gamma_k$'s (or $\tilde{\Gamma}_k$'s) are zeros, such explicit formulas are available only for special kinds of Butterworth and Chebyshev filters [BEN], [NO], [TA], [BE], [OR], [GR], [WE1], [NA1]. For non-simple LC low-pass ladder filters, i.e., when the $\omega_k$'s are finite, Darlington [DA] has given explicit formulas for the element values in terms of three distinct determinants, based on the

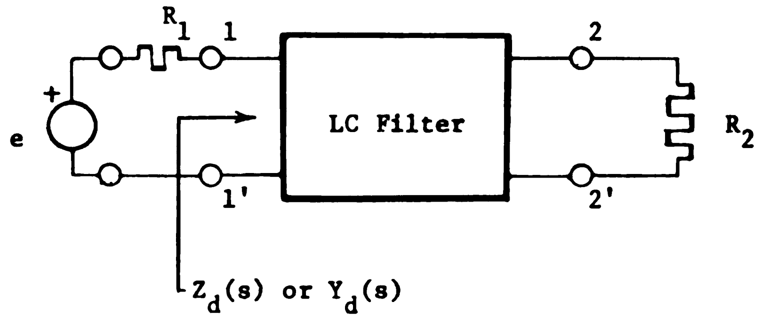
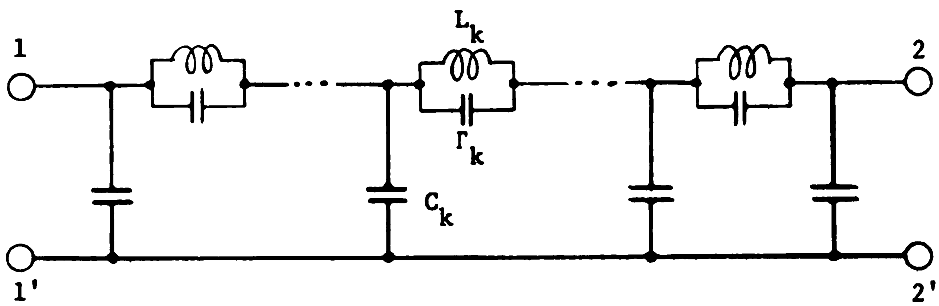
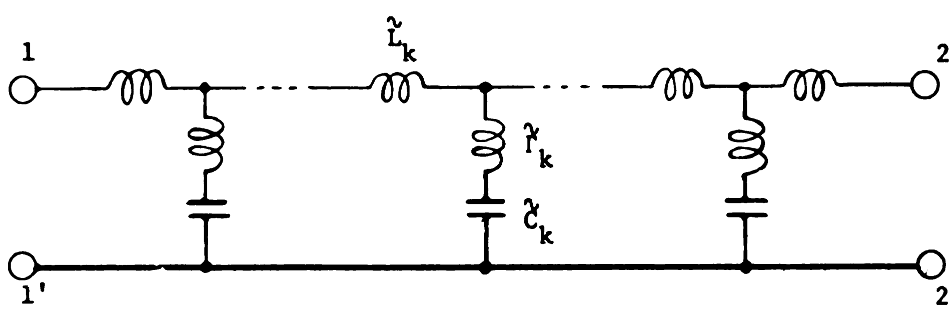



Fig. I-1




$$\omega_{\infty k} = 1/\sqrt{L_k \Gamma_k}$$

(a) Mid-shunt LC low-pass ladder network




$$\omega_{\infty k} = 1/\sqrt{\tilde{\Gamma}_k \tilde{C}_k}$$

(b) Mid-series LC low-pass ladder network

Fig. I-2

work by Norton [NO]. In practice, however, these formulas are useful only for a few sections [ZU], [SKZ], because as the number of sections increases, one has to cope with higher order determinants which have very small numerical values.

In this thesis, an attempt is made to develop new approaches to the solution of this problem which do not suffer from numerical inaccuracies. The first approach is complete and discussed in Chapter II and its application is given in Chapter III by considering a little known class of filters, namely, the Inverse Chebyshev filters. The second approach is based on the state model of low-pass ladder filters. This approach is developed in Chapter IV where the formulation of the problem is discussed thoroughly and two approaches to the synthesis problem are suggested. The contents of the various chapters may be briefly outlined as follows:

In Chapter II, an algorithm is given through which all the element values of the non-simple low-pass ladder networks (or their duals) in Fig. 2.3.1 can be determined iteratively from a knowledge of some initial parameters. Two of these parameters are, in fact, the element values in the first or last sections of the filter, the other parameters being used only for the intermediate computations. This algorithm automatically eliminates the difficulties encountered in the application of Darlington's formulas for complicated ladders. Such an algorithm is suggested and used, but not explicitly proved, in a recent article by Amstutz [AM]. The pair of general recursion formulas derived in this chapter are equivalent to those given for special cases in Table-II of Amstutz's paper. Because of the duality, only mid-shunt configurations are considered throughout this thesis. For the realization with positive element values, it is proved that Fujisawa's procedure [FU], [BA], [MB], [ME], [NE], [LE] can be included in the algorithm. A program was written for the computation of element values, based on the above mentioned algorithm in which one can specify the order of the attenuation poles so that the algorithm yields positive element values. If, however, the order of the attenuation poles is not specified, then the program starts with an arbitrary ordering of poles and rearranges the poles according to Fujisawa's criterion so that the algorithm produces positive element values. Examples given at the end of Chapter II demonstrate in detail the scheme of computations involved in the process for different cases.

In Chapter III, the amplitude and group delay characteristics of a low-pass ladder filter with inverse Chebyshev attenuation characteristics are studied. One reason the filters with inverse Chebyshev characteristics are studied is that in the work by Beletskiy [BL], [TR], where the linear phase characteristics are studied, the attenuation characteristics turned out to be very similar to those of the inverse Chebyshev attenuation characteristics. Another reason for considering inverse Chebyshev filters is the fact that, in the literature, no details of the properties or design formulas for this class of filters have been published.

In Chapter III, after a brief introduction of the insertion-loss theory, the relations that exist among filter functions for Butterworth, Chebyshev and inverse Chebyshev filters are studied. With the aid of these relations, formulas for the realization of inverse Chebyshev filters are derived which utilize the algorithm obtained in Chapter II. A design procedure, starting from the approximation of the specified attenuation characteristics, is presented with examples. In the last section of Chapter III, a general formula for the group delay function, $\tau(\omega)$, for low-pass filters is derived and some useful properties are established. By using this general formula, the group delay characteristics for inverse Chebyshev filters for different values of the design parameters D, $\omega_a$, and n, defined in Chapter III, are obtained.

The last part of this thesis considers a new method of realizing the non-simple low-pass LC ladder network with arbitrary attenuation characteristics by using state equations. In this approach, the natural frequencies and transmission zeros of the network are assumed to be given. An approach similar to this was first given by Marshall [MA1], [MA2] for simple low-pass ladders. The essence of this approach as included in Chapter IV is as follows:

Let

$$P\dot{\Psi}(t) = R\Psi(t) + Be(t)$$

be the state equations for the network shown in Fig. I-1, where $\Psi$ is the state vector, e(t) is the input voltage function, and P, R and B are coefficient matrices. Entries in P are linear combination of network elements while the entries in R and B are equal to 1's and the terminating resistances. The forms of P, R and B depend on the network configuration. Since the eigenvalues of $P^{-1}R$ are the specified natural frequencies and the forms of P and R are known, the network element values can be obtained from the given natural frequencies and terminating resistances. For simple ladders, P is diagonal and R is a special kind of tridiagonal matrix. Therefore, the element values can be obtained by transforming the companion matrix constructed from the given eigenvalues into the tridiagonal form of $P^{-1}R$. However, for non-simple ladders, P is tri-diagonal. Therefore, the form of $P^{-1}R$ is complicated and the problem cannot be solved by a simple transformation.

In Chapter IV, without loss of generality, only the mid-shunt configuration is used to demonstrate this approach for synthesis. Although no general solution for the synthesis problem is obtained, a careful formulation of the problem is presented which may be used in future research work. Two possible synthesis methods are suggested together with simple examples.

## CHAPTER II

## REALIZATION OF DRIVING-POINT IMMITTANCE INTO LOW-PASS LADDER NETWORKS

### 2.1. General

This chapter deals with the realization of driving-point impedance (admittance) in terms of the mid-series (mid-shunt) ladder network defined in Chapter I with positive element values, i.e., without mutual reactances. It is well known that only a certain class of driving-point immittances that satisfy the necessary and sufficient conditions found by Fujisawa can be realized in terms of mid-series or mid-shunt ladders without mutual reactances. In this chapter, without loss of generality, only the realization of driving-point immittances in terms of the mid-shunt ladder with non-negative elements is considered, since the mid-series network is the dual of the mid-shunt and their realizations are essentially the same. Two algorithms for the realization of a driving-point admittance are established under the assumption that all the attenuation pole frequencies of the corresponding filter are distinct.

### 2.2. Mid-shunt and Mid-series Ladder Network Synthesis

Given a driving-point immittance $Z_d$ or $Y_d$ which is to be realizable as a mid-series or mid-shunt ladder network, three conditions stated in the following realizability theorem must be satisfied. Let $Z_d$ or $Y_d$ be written in the form

$$Z_d(s) \quad \text{or} \quad Y_d(s) = \frac{m_1 + n_1}{m_2 + n_2}$$

where $m_1$, $n_1$ and $m_2$, $n_2$ are even and odd parts of the numerator and denominator polynomials of $Z_d$ or $Y_d$, respectively. Let $n_1 = sg_1$, $n_2 = sg_2$, where $g_1$ and $g_2$ are even polynomials, and

$$M(s) = m_1m_2 - n_1n_2$$

The zeros of M(s) are the zeros of transfer functions [VV] which are the transmission zeros of the filter shown in Fig. 1.1.1.

The necessary and sufficient conditions for realizability of a given $Z_d$ (or $Y_d$) were given by Fujisawa [FU], [MB], [BA], [WE2] and are stated in detail in the following theorem:

Realizability Theorem The Brune function $Z_d$ with $n_1 \not\equiv 0$, can be realized as the driving-point impedance of a resistance terminated mid-series low-pass ladder network if, and only if, the following three conditions are satisfied:

(1) $Z_d$ has a pole or zero at infinity.

(2) M(s) is a positive constant or
$M(s) = K[\ (s^2 + s_1^2)(s^2 + s_2^2)\ldots(s^2 + s_n^2)\ ]^2$
where n = degree of $Z_d$ and $0 < s_1^2 \leqslant s_2^2 \leqslant \ldots \leqslant s_n^2$

(3) Let m be the degree of $g_1$ in $s^2$ and
$g_1(s) = G(s^2 + \omega_1^2)(s^2 + \omega_2^2)\ldots(s^2 + \omega_m^2)$
where G > 0 and the $\omega_i$'s are distinct. Then $m \geqslant n$ and for each $s_i^2$ there exists at least i $\omega_j^2$'s which are not greater than $s_i^2$.

Note 1. In the special case when $n_1 \equiv 0$ the above three necessary and sufficient conditions reduce to the necessary and sufficient condition that

$$m_2 = R_2m_1$$

Note 2. The necessary and sufficient conditions for the realization of a given driving-point admittance function as a mid-shunt ladder network terminated in a resistor are the same except in condition (3) $g_1$ is replaced by $g_2$.

Note 3. If in the Realizability Theorem $Z_d$ is replaced by $Y_d$ then the term "mid-series" should be replaced by "mid-shunt".

The Realizability Theorem is important in low-pass ladder filter synthesis and can be interpreted as follows. Let $y_{11}$, $y_{22}$, $y_{12}$ and $z_{11}$, $z_{22}$, $z_{12}$ be the short-circuit and open-circuit parameters, respectively, for the 2-port LC network, then from Case A (low-pass ladder networks belong to this case) of Darlington's synthesis [BA] we have:

$$Z_d = \frac{m_1 + n_1}{m_2 + n_2} = \frac{m_1}{n_2} \frac{1 + (n_1/m_1)}{1 + (m_2/n_2)} = z_{11} \frac{1 + (1/y_{22})}{1 + \quad z_{22}} \qquad (2.2.1)$$

where the short-circuit admittance function $y_{22} = Y_{2s}$ corresponds to the reactive network looking from the output port of the filter with the input port short-circuited. The conditions of the Realizability Theorem simply say that for $Z_d$ to be realizable as a mid-series ladder network with all positive element values, at the i-th transmission zero of the mid-series ladder network, $Y_{2s}$ must have at least i poles on the imaginary axis with magnitudes not greater than that of the transmission zero.

For the mid-shunt case $z_{22}$ should replace $y_{22}$ in the above statement. A similar interpretation can be given when $Z_d$ is replaced by $Y_d$.

The necessity of the condition (3) interpreted above becomes evident if one looks at the zero-shifting procedure utilized in ladder network synthesis. There will not be a sufficient number of zeros to be shifted to generate the specified transmission zeros if condition (3) is not satisfied.

In the proof of sufficiency of the Realizability Theorem Fujisawa described a procedure based on a very simple criterion which is applied at each step of the conventional zero-shifting procedure used in the synthesis of a ladder network [VV]. This procedure by several authors is sometimes referred to as "Fujisawa's procedure". A driving-point immittance which satisfies the conditions of the Realizability Theorem, in general, may have more than one realization with all positive element values. This is due to the multiple choices of ordering the transmission zeros used in the realization procedure. Fujisawa's procedure, however, corresponds to one of these choices of ordering and will always give a realization whenever the given driving-point immittance satisfies the realizability conditions.

The algorithm to be shown in section 2.4. is based on Fujisawa's procedure which is given together with its network interpretation in the following.

Let $Y_d$ be the given driving-point admittance function satisfying the realizability conditions, then $Y_d$ can be realized as a mid-shunt ladder network without any negative element values by the following procedure:

<u>Step 1.</u> (A) If $Y_d(s)$ has a zero at $s=\infty$, remove a series inductance L from $Y_d$ as shown in Fig. 2.2.1-(a). For this case $1/Y_d$ has a pole at infinity with the residue L:

$$1/\tilde{Y}_d = 1/Y_d - Ls$$

(B) If $Y_d$ has a zero $s=j\omega$ which coincides with one of the transmission zeros, remove a parallel resonator as shown in

Fig. 2.2.1-(b). For this case $1/Y_d$ has a pole at $s=j\omega_k$ with the residue $L\omega_k^2/2$:

$$1/\tilde{Y}_d = 1/Y_d - \frac{L\omega_k^2 s}{s^2 + \omega_k^2}$$

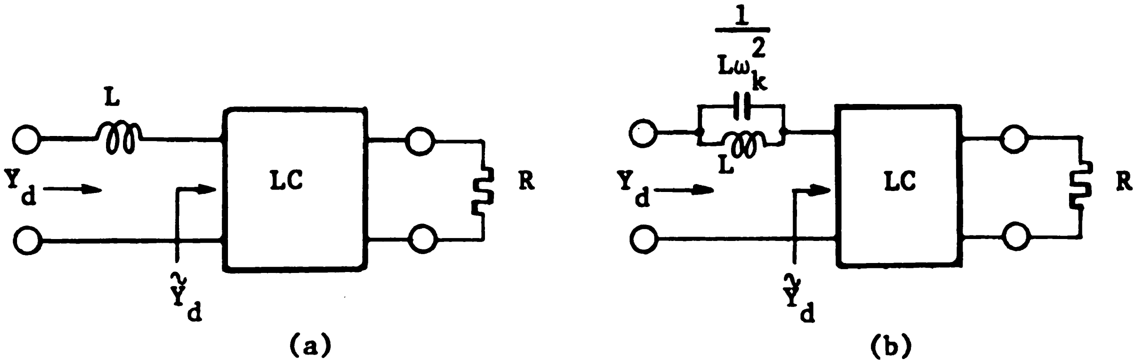



Fig. 2.2.1

Step 2. The next step is to carry out the zero-shifting procedure by removing a shunt capacitance $C_1$ determined by the following relation:

$$C_1 = \min.[Y_d(j\omega_1)/j\omega_1, \ldots, Y_d(j\omega_n)/j\omega_n, \lim_{s\to\infty}(Y_d/s)]$$

where $j\omega_1, \ldots, j\omega_n$ are the transmission zeros. In determining the minimum element of the set of numbers, all negative or infinite values are ignored.

If it is found that $C_1=Y_d(j\omega_k)/j\omega_k$, then the first section of the mid-shunt ladder to be realized has shunt capacitance $C_1$ and a mid-shunt arm $L_1$-$\Gamma_1$ (see Fig. 2.3.1 in the next section) with resonance frequency equal to $\omega_k$. Then the same procedure is applied to the new driving-point admittance obtained from the removal of the section ($C_1$, $L_1$, $\Gamma_1$) until all the transmission zeros are realized.

Darlington [DA] gave a pair of formulas for computing the values of the elements $C_1$ and $L_1$ as follows:

$$\left.\begin{aligned} C_1 &= F(z_k) \\ L_1 &= -1/[\frac{d}{dz} F(z)]_{z=z_k} \end{aligned}\right\} \quad (2.2\text{-}2)$$

where

$$F(z) = [Y_d(j\omega)/j\omega]_{z=1/\omega^2} \quad \text{and} \quad z_k = 1/\omega_k^2$$

In the above discussion realization of the 2-port LC network is accomplished by the consideration of $Y_d$. But this is not necessary. Indeed, one can always realize the same network by using short-circuit admittance functions $Y_{1s}$ and $Y_{2s}$ obtained from $Y_d$. This is preferred since $Y_{1s}$ and $Y_{2s}$ have much simpler forms as compared to $Y_d$. On the other hand, realizing the network from both $Y_{1s}$ and $Y_{2s}$ yields a check on the accuracy of the element values. In order to demonstrate the validity of the procedure for $Y_{1s}$ and $Y_{2s}$, let $y_{11}$, $y_{22}$, $y_{12}$ be the short-circuit admittance functions for the LC ladder network. Therefore

$$Y_d = y_{11} - y_{12}^2/(1 + y_{22})$$

Note that,

$$Y'_d = y'_{11} - 2y_{12}y'_{12}/(1 + y_{22}) + y_{12}^2 y'_{22}/(1 + y_{22})^2$$

Since the transmission zeros are the zeros of $y_{12}$, it follows from the last two equations that

$$Y_d(j\omega_k) = y_{11}(j\omega_k) = Y_{1s}(j\omega_k)$$

and

$$Y'_d(j\omega_k) = Y'_{1s}(j\omega_k)$$

Since in Darlington's formulas $C_1$ and $L_1$ are expressed in terms of $Y_d$ and $Y_d'$, this justifies the use of $Y_{1s}$ instead of $Y_d$ in the realization. A similar conclusion is valid for $Y_{2s}$.

Before giving a network interpretation to the Fujisawa's procedure, it is necessary to show first that $L_1=-1/F'(z_k)$ is always positive for any $z_k$. This property can be proved as follows: Without the transformation of the variable $z=1/\omega^2$, Darlington's formula for $L_1$ is

$$L_1 = -2/[(Y_d/s)'s^3]_{s=j\omega_k}$$

or

$$\frac{2}{L_1} = [-s^3(Y/s)']_{s=j\omega_k}$$

$$= -j\omega_k[j\omega_k Y_d'(j\omega_k) - Y_d(j\omega_k)]$$

From Foster's reactance theorem $Y_{1s}'(j\omega)$ is positive real. Furthermore, $Y_{1s}'(j\omega) > Y_{1s}(j\omega)/j\omega$ , which implies $2/L_1 > 0$, and the property, $L_1 > 0$. In fact this, in turn, also justifies Fujisawa's procedure: Let the reactance curve of $Y_{1s}$ be as shown in Fig. 2.2-2 in which $\omega_i'$s, i=1,2,3, are transmission zeros and $\theta_i=\tan^{-1}[Y_{1s}(j\omega_i)/j\omega_i]$ Fujisawa's procedure is to find $C_1$ from the condition

$$C_1 = \min.(\tan\theta_1, \tan\theta_3, \tan\theta_\infty)$$

From Fig. 2.2.2 , one can see that $C_1=\tan\theta_3$. Note that $\tan\theta_2 < 0$ must be ignored because that corresponds to negative capacitance. If $\tan\theta_1$, rather than $\tan\theta_3$ is used for $C_1$, then the difference of the $Y_{1s}$ and the line with slope $\tan\theta_1$ is not a reactance curve anymore. Therefore, the first transmission zero to be realized is $\omega_3$.

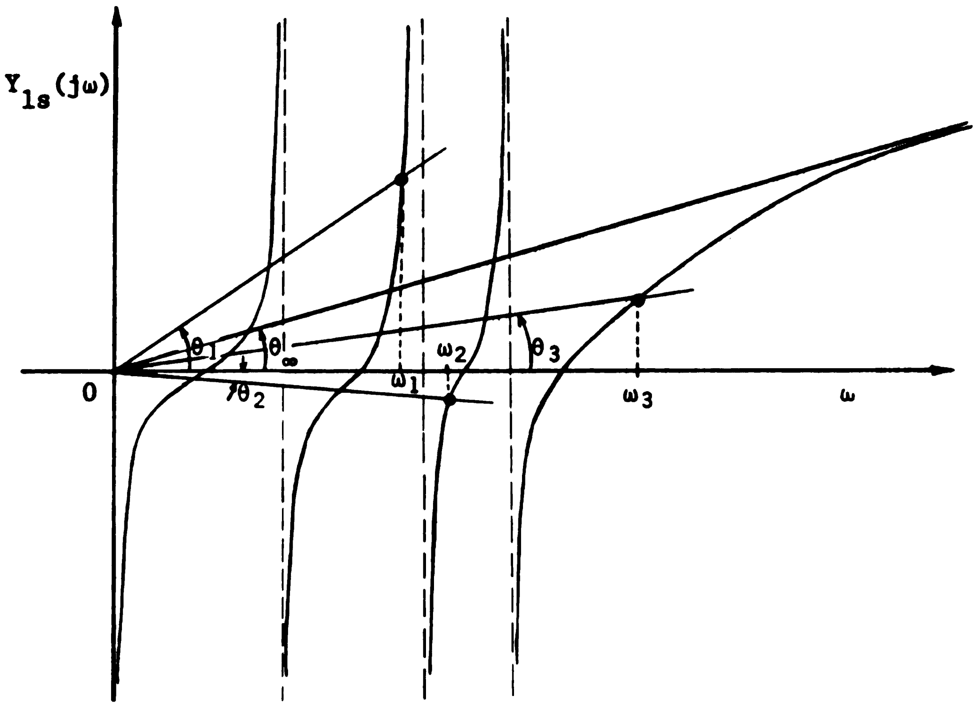



Fig. 2.2.2

A rather brief discussion about the mid-shunt or mid-series ladder network realization given above constitutes the background for the rest of this chapter.

Fujisawa's procedure given in this section is for the most general case. In practical applications, however, the ladder networks considered are always compact [BA]; that is, Step 1. given in the Fujisawa procedure will not occur. This is especially the case in filter applications. Therefore, in the next section only the type of network shown in Fig.2.3.1 will be considered.

## 2.3. Realization of $Y_d$ with a Prescribed Attenuation Pole Sequence

In this section an algorithm is derived for the computation of element values of the mid-shunt ladder network of Fig. 2.3.1. For the reason discussed in the preceding section, instead of realizing the given $Y_d$ we consider the realization of two short-circuit admittance functions for the LC ladder, $Y_{1s}$ and $Y_{2s}$, which are obtained from $Y_d$. The assumptions are that $Y_d$ satisfies the realizability conditions and is realizable as a compact [BA] mid-shunt network of Fig. 2.3.1 with all the attenuation poles distinct. The function $Y_{1s}(s)/s$ can be expanded into the following continued fraction:

$$\frac{Y_{1s}(s)}{s} = C_1 + \cfrac{1}{\cfrac{L_1 s^2}{L_1\Gamma_1 s^2+1} + \cfrac{1}{C_2 + \cfrac{1}{\cfrac{L_2 s^2}{L_2\Gamma_2 s^2+1} + \ddots \cfrac{1}{\cfrac{L_n s^2}{L_n\Gamma_n s^2+1} + \cfrac{1}{C_{n+1}}}}}} \qquad (2.3\text{-}1)$$

Following the same notations used by Darlington [DA], [CA], we have

$$F(\zeta) = [Y_{1s}(s)/s]_{\zeta=-1/s^2} = C_1 + \cfrac{1}{\cfrac{L_1}{\zeta_1-\zeta} + \cfrac{1}{C_2 + \cfrac{1}{\cfrac{L_2}{\zeta_2-\zeta} + \ddots \cfrac{1}{\cfrac{L_n}{\zeta_n-\zeta} + \cfrac{1}{C_{n+1}}}}}}$$

(2.3-2)

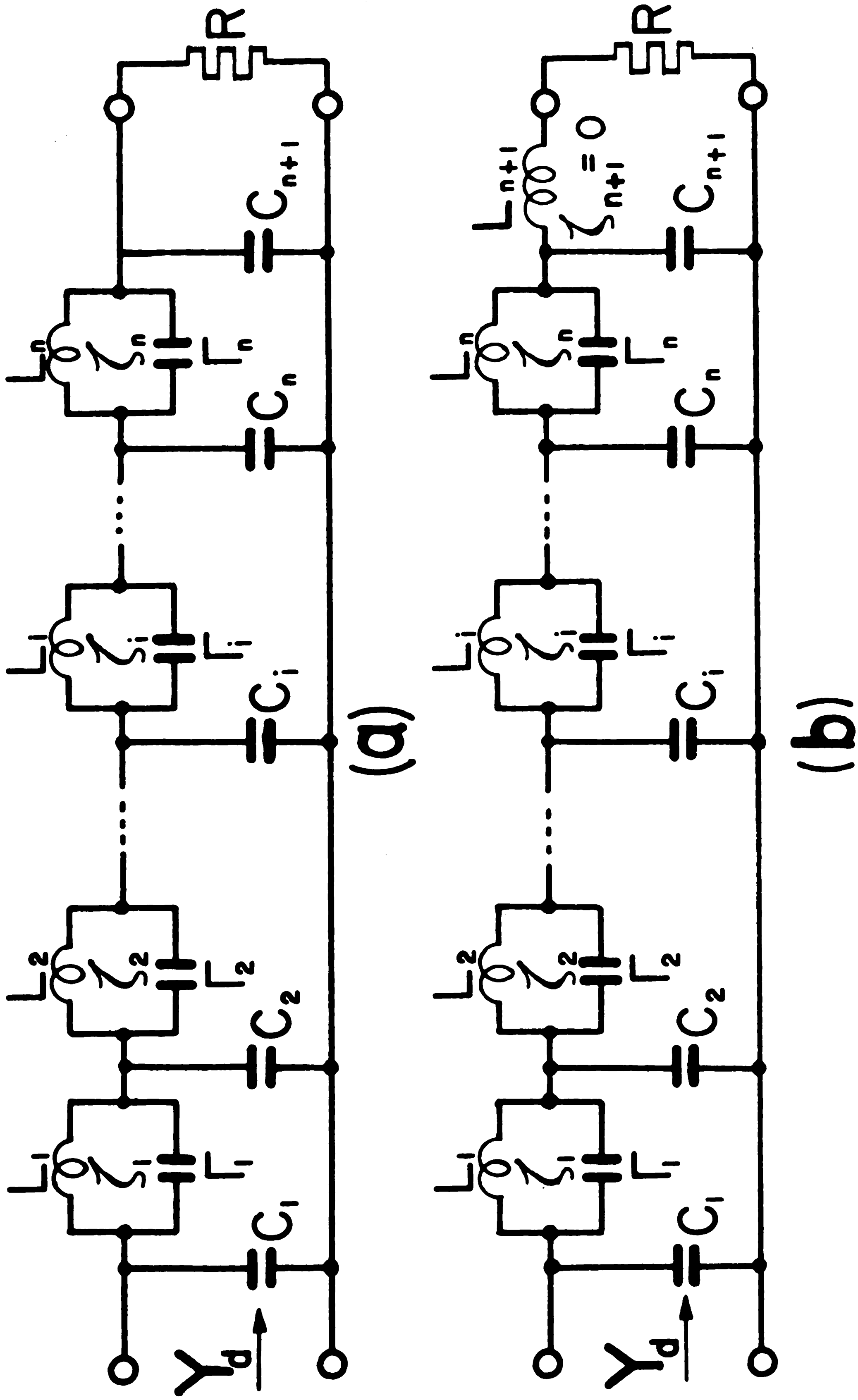



Fig. 2.3.1

From Eq.(2.3-2) it can be shown that

$$\left.\begin{aligned} C_1 &= F(\zeta_1) \\ L_1 &= -1/F'(\zeta_1) \end{aligned}\right\} \qquad (2.3\text{-}3)$$

where $\zeta_1=L_1\Gamma_1$ and $F'(\zeta_1)=[\frac{d}{d\zeta}F(\zeta)]_{\zeta=\zeta_1}$. If we let

$$\left.\begin{aligned} \zeta_k &= L_k\Gamma_k \geqslant 0 \\ C_{k1} &= F(\zeta_k) \\ L_{k1} &= -1/F'(\zeta_k) \end{aligned}\right\} \quad \text{for } k=1,2,\ldots,n \qquad (2.3\text{-}4)$$

then it will be proved that $C_i$ and $L_i$ ($i=1,2,\ldots,n$) can be generated by the following algorithm which is written in the form used by Wendroff [WD] where the abbreviations

$$P_{ji} = (\zeta_j - \zeta_{i-1})$$

$$B_{ji} = C_{j,i-1} - C_{i-1,i-1}$$

$$D_{ji} = P_{ji} + B_{ji}L_{i-1,i-1}$$

are used:

For $i=1$ (the first column)

For $k=1,2,\ldots,n$

$$C_{k1} = F(\zeta_k), \quad L_{k1} = -1/F'(\zeta_k)$$

For $i=2,3,\ldots,n$

For $j=i, i+1,\ldots,n$

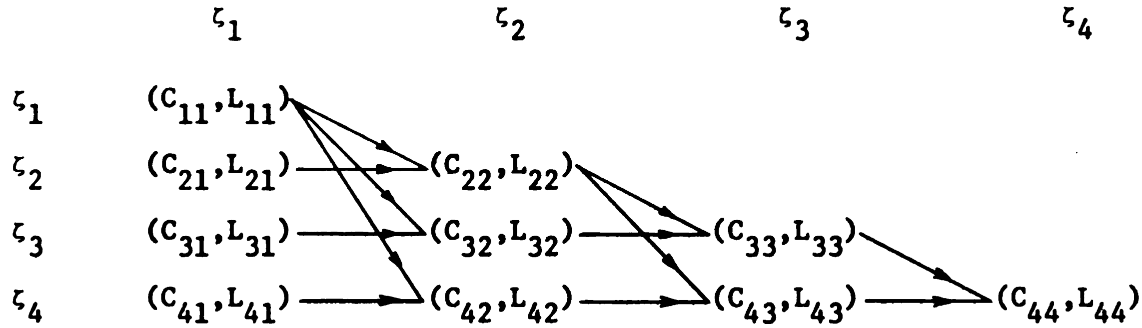
$$C_{ji} = P_{ji}B_{ji}/D_{ji} \qquad (2.3\text{-}5)$$

$$L_{ji} = D_{ji}^2/[(P_{ji}^2/L_{j,i-1}) - B_{ji}^2L_{i-1,i-1}] \qquad (2.3\text{-}6)$$

From this the element values are obtained as follows:

$$C_i = C_{ii}, \quad L_i = L_{ii} \quad \text{and} \quad \Gamma_i = \zeta_i/L_i \quad \text{for } i=1,2,\ldots, n$$

For example, for a 4-section ladder network with the sequence $<\zeta_1, \zeta_2, \zeta_3, \zeta_4>$ corresponding to the specified sequence of attenuation pole frequency, the steps involved in the scheme of computation can be put into a clearer form as follows:




In the above array arrows indicate how each pair of elements is obtained from two other pairs of already computed elements by the use of iterative formulas (2.3-5) and (2.3-6). For each pair $(C_{ii}, L_{ii})$ four parameters computed in the previous step are required, two of which called $C_{i-1,i-1}$ and $L_{i-1,i-1}$, are the actual element values and the other two, $C_{i,i-1}$ and $L_{i,i-1}$, can be considered as the element values if in the pole sequence the poles $\zeta_{i-1}$ and $\zeta_i$ are interchanged. Therefore, for an n-section ladder the total number of parameters that should be computed in the first step, by using Darlington's formulas in (2.3-2) is 2n, and the total number of computations needed is n(n+1), among which 2n appear in the leading positions of the array which give the actual element values and the remaining n(n-1) are parameters used for intermediate computations.

In order to prove the algorithm the following property is needed whose proof is given in Appendix I at the end of this thesis.

Let the one-port networks $N_1$ and $N_2$ in Fig.2.3-2 be obtained from two non-simple LC ladder network terminated by identical resistances. With the identical admittance functions, let $N_1$ and $N_2$ have identical parts indicated by N. Then $Y_1(s)=Y_2(s)$, if and only if $Y_{r1}(s)\equiv Y_{r2}(s)$.

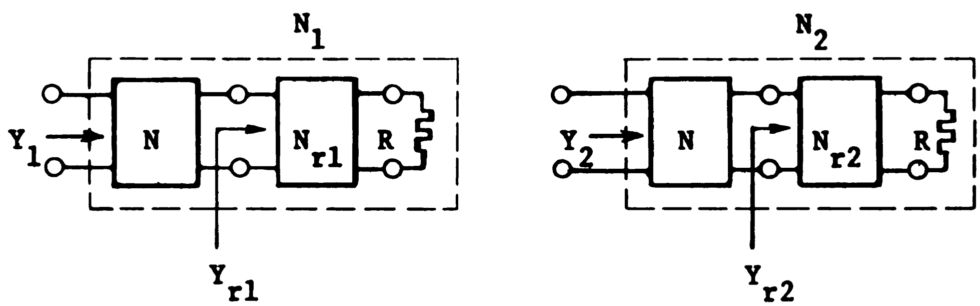



Fig. 2.3.2

For the proof of the algorithm, let us assume that $Y_{1s}$ is realized for two distinct attenuation pole sequences as shown in Fig. 2.3-3. Note that both sequences are identical up to the (k-1)-th element, but differ in that the k-th element and the (k+1)-th element are interchanged, whereas the rest of the elements in both sequences may have arbitrary orders. If we let

$$F_{r1}(\zeta) = C_k + \cfrac{1}{\cfrac{L_k}{\zeta_k-\zeta} + \cfrac{1}{C_{k+1} + \cfrac{1}{\cfrac{L_{k+1}}{\zeta_{k+1}-\zeta} + \ddots}}} \qquad (2.3\text{-}7)$$

and

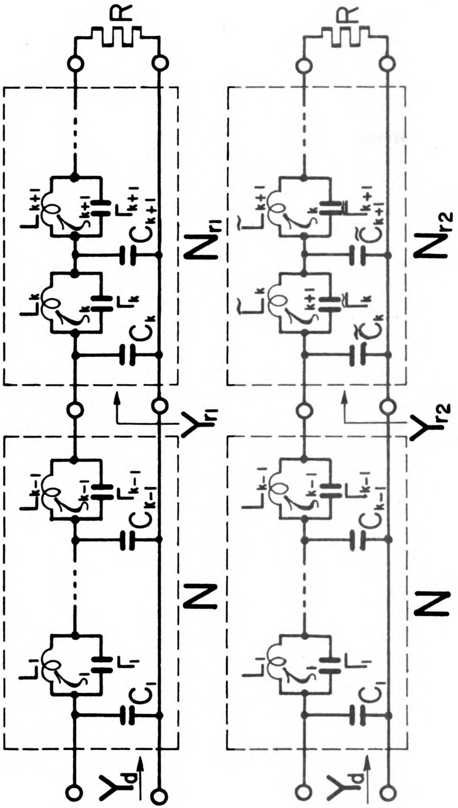



Fig. 2.3.3

$$F_{r2}(\zeta) = \tilde{C}_k + \cfrac{1}{\cfrac{\tilde{L}_k}{\zeta_{k+1} - \zeta} + \cfrac{1}{\tilde{C}_{k+1} + \ddots}} \qquad (2.3\text{-}8)$$

then according to the above theorem $F_{r1}(\zeta) \equiv F_{r2}(\zeta)$. Therefore, for $\zeta = \zeta_{k+1}$ we have

$$C_k + \cfrac{1}{\cfrac{L_k}{\zeta_k - \zeta_{k+1}} + \cfrac{1}{C_{k+1}}} = \tilde{C}_k$$

or

$$C_{k+1} = \frac{(\zeta_{k+1} - \zeta_k)(\tilde{C}_k - C_k)}{(\zeta_{k+1} - \zeta_k) + (\tilde{C}_k - C_k)L_k} \qquad (2.3\text{-}9)$$

This is Eq.(2.3-5) with different index notations.

To prove the iterative formula (2.3-6) the relation $F'_{r1}(\zeta) \equiv F'_{r2}(\zeta)$ is to be used. Applying Darlington's formulas, given in (2.3-3), to Eq. (2.3-8), we have

$$F'_{r2}(\zeta_{k+1}) = -1/\tilde{L}_k$$

From the derivative of $F_{r1}$ in Eq.(2.3-7), we have

$$F'_{r1}(\zeta_{k+1}) = \lim_{\zeta \to \zeta_{k+1}} F'_{r1}(\zeta)$$

$$= -\left[\frac{L_k}{(\zeta_k - \zeta_{k+1})^2} + \frac{1}{C_{k+1}^2}\right] / \left[\frac{L_k}{\zeta_k - \zeta_{k+1}} + \frac{1}{C_{k+1}}\right]^2$$

Then from the relation $F'_{r1}(\zeta_{k+1}) = F'_{r2}(\zeta_{k+1})$, we have

$$\frac{1}{L_{k+1}} = \frac{1}{\tilde{L}_k}\left[1 - \frac{C_{k+1}L_k}{\zeta_{k+1} - \zeta_k}\right] - L_k C_{k+1}^2/(\zeta_{k+1} - \zeta_k)^2$$

Substituting $C_{k+1}$ of equation (2.3-9) into the above equation gives

$$\frac{1}{L_{k+1}} = [\frac{(\zeta_{k+1} - \zeta_k)^2}{\tilde{L}_k} - L_k(\tilde{C}_k - C_k)^2]/[(\zeta_{k+1} - \zeta_k) + (\tilde{C}_k - C_k)L_k]^2$$

or finally

$$L_{k+1} = \frac{[(\zeta_{k+1} - \zeta_k) + (\tilde{C}_k - C_k)L_k]^2}{(\zeta_{k+1} - \zeta_k)^2/\tilde{L}_k - (\tilde{C}_k - C_k)^2 L_k}$$

which is identical to that in Eq.(2.3-6) with different index notations. This proves the algorithm.

Needless to say, if instead of $Y_{1s}(s)$, $Y_{2s}(s)$ is used for the computation of element values, the algorithm proved above is also valid. When $Y_{2s}$ is used that means we are realizing the LC ladder network from the output end, therefore, in order to obtain the realization identical to that of $Y_{1s}$ the pole sequence in reverse order should be used.

The algorithm as given, when applied to compute the element values in the network of Fig. 2.3.1-a can not give the last element in the opposite end, i.e., it misses $C_{n+1}$ when $Y_{1s}$ is used and $C_1$ can not be obtained when $Y_{2s}$ is used. But since $Y_d$ is given it is obvious that $C_{n+1}$ and $C_1$ can always be obtained separately. In actual ladder network synthesis, in order to check the accuracy, it is usual practice to realize both $Y_{1s}$ and $Y_{2s}$ and compare the element values obtained in both computations. The reason is that, because of the subsequent algebraic sum operations in iterative computations the errors will accumulate as the computation proceeds and cause a loss of significant digits. If both $Y_{1s}$ and $Y_{2s}$ are realized then all the element values can always be obtained.

For the network of Fig. 2.3.1-b which exists when the degree of $Y_d$ is even (see Table 3.3-2), the same algorithm is valid without any change. This network can be considered as the same network in Fig. 2.3.1-a with $\zeta_{n+1}=0$ and $C_{n+2}=0$, and for this case, since $C_{n+2}=0$, realization of $Y_{1s}$ yields all the element values.

## 2.4. Realization of $Y_d$ Starting with an Arbitrary Attenuation Pole Sequence

As mentioned in section 2.2, not all attenuation pole sequences $<\omega_j>_{j=1}^n$'s can yield positive element values in the realization of associated driving-point admittance functions $Y_d$, even though $Y_d$ satisfies the realizability conditions. In this section, it is shown that Fujisawa's procedure can readily be included in the algorithm described in the preceding section. Thus, starting with a sequence $<\zeta_j>_{j=1}^n$, $\zeta_j=1/\omega_j^2$, corresponding to a pole sequence $<\omega_j>_{j=1}^n$ selected arbitrarily from the set $\{\omega_i\}_{i=1}^n$, the computer can find a realization with positive element values by rearranging the given sequence into a proper one in the following manner: At any step of the algorithm, $C_{ki}$, $k=i, i+1,\ldots, n$, is computed for the remaining poles, $\zeta_i, \zeta_{i+1},\ldots,\zeta_n$, then the non-negative minimum element, $C_{mi}=\min.\{C_{ki}>0, k=i, i+1,\ldots, n\}$ is determined. Interchanging $(C_{mi}, L_{mi})$ with $(C_{ii}, L_{ii})$, and $\zeta_m$ with $\zeta_i$, also interchanging those elements in the same rows computed previously, i.e., interchanging $(C_{m\ell}, L_{m\ell})$ with $(C_{i\ell}, L_{i\ell})$ for $\ell=1,2,\ldots, i-1$ a rearranged array is obtained. $C_{mi}$'s always exist whenever $Y_d$ satisfies the realizability conditions given in section 2.2.

By referring to section 2.2 and from the proof of the algorithm it is easily seen that the above procedure is equivalent to applying

Fujisawa's procedure in the conventional zero-shifting synthesis method. As is shown in section 2.2, when Fujisawa's procedure is applied, at any step i, the $L_{ki}$, for k=i, i+1,..., n, are always positive. Negative element values, if such exist, will be among the $C_{ki}$, k=i, i+1,..., n, and the procedure is equivalent to the operations stated in the preceding paragraph.

The algorithm, including Fujisawa's procedure, is as follows:

For column 1 (i=1)

- For k=1,2,..., n
  - $C_{k1} = F(\zeta_k)$, $L_{k1} = -1/F'(\zeta_k)$
- Find the minimum non-negative element, $C_{m1}$; interchange $C_{m1}$ and $C_{11}$, $L_{m1}$ and $L_{11}$, also $\zeta_m$ and $\zeta_1$.

For i=2,3,..., n

- For j=i, i+1,..., n
  - $C_{ji} = P_{ji}B_{ji}/D_{ji}$
  - $L_{ji} = D_{ji}^2/[(P_{ji}^2/L_{j,i-1}) - B_{ji}^2 L_{i-1,i-1}]$
- Find the minimum non-negative element $C_{mi}$.
- For k=1,..., i
  - Interchange $C_{mk}$ and $C_{ik}$; $L_{mk}$ and $L_{ik}$
- Interchange $\zeta_m$ and $\zeta_i$.

From this procedure the element values and the attenuation pole sequence $<\zeta_i>_{i=1}^{n}$ are obtained as follows:

$$C_i = C_{ii}, \quad L_i = L_{ii}, \quad \Gamma_i = \zeta_i/L_{ii}, \quad i=1,2,\ldots,n$$

and

$$<\zeta_i>_{i=1}^{n} = <L_i \Gamma_i>_{i=1}^{n}$$

It should be emphasized, therefore, in both algorithms, that with or without Fujisawa's procedure, the expression for $Y_{1s}$ or $Y_{2s}$ is needed only for the computation of 2n parameters, $C_{k1}$ and $L_{k1}$, k=1,2,..., n, by the use of Darlington's formulas in (2.3-3). All other element values can be generated by the algorithm using these 2n parameters.

## 2.5. Numerical Examples

Both examples given in this section are constructed from tables obtained by Saal and Ulbrich [SU]. The degree of the driving-point admittance, $Y_d(s)$, for the first example is even which corresponds to the ladder of the type shown in Fig. 2.3.1-b. The filter is realized according to the pole sequence given in the table. For the second example, the degree of $Y_d$ is odd and $Y_d$ is realized into a ladder of the type shown in Fig. 2.3.1-a both with and without application of the Fujisawa criterion.

### Example 2.5.1

For the element values of the filter with specification C-08-20c, $\theta=85°$, the coefficients of the numerator and denominator polynomials of $Y_d(s)$ are found as follows:

| $(s^k)$ k | Numerator | Denominator |
|---|---|---|
| 8 | 2.6362218751+000 | 0.0000000000+000 |
| 7 | 2.3857211540+000 | 2.3863732874+000 |
| 6 | 9.1942771683+000 | 2.1596138348+000 |
| 5 | 6.9928022992+000 | 6.9946429436+000 |
| 4 | 1.1488488046+001 | 5.2927668720+000 |
| 3 | 6.7947365261+000 | 6.7963234209+000 |
| 2 | 5.9301643998+000 | 4.1329735998+000 |
| 1 | 2.1875999999+000 | 2.1880000000+000 |
| 0 | 1.0000000000+000 | 1.0000000000+000 |

From these calculations $Y_{1s}$ and $Y_{2s}$ can be obtained. Applying the algorithm to $Y_{1s}$ the following array is obtained:

| | | | | |
|---|---|---|---|---|
| $\zeta$ | 7.070395-001 | 9.454424-001 | 9.858527-001 | 0.000000+000 |

| | | | | |
|---|---|---|---|---|
| | 5.0010000000-001 | | | |
| | -8.6405977862-001 | 5.8560000000-001 | | |
| $C_{ij}$ | -1.5500865651-001 | 7.5304195949-001 | 3.5250000000-001 | |
| | 1.1046980324+000 | 1.4460739189+000 | 9.6746222531-001 | 7.4940000000-001 |

| | | | | |
|---|---|---|---|---|
| | 6.8050000000-001 | | | |
| | 1.6914529924-002 | 1.2670000000-001 | | |
| $L_{ij}$ | 1.3506378822-001 | 3.3406518193-001 | 2.7580000000-001 | |
| | 1.6263605667+000 | 1.4905283416+000 | 1.3772334767+000 | 1.1050000000+000 |

| | | | | |
|---|---|---|---|---|
| | 1.0390000000+000 | | | |
| | 5.8284368790+001 | 7.7810000000+000 | | |
| $\Gamma_{ij}$ | 6.9999695143+000 | 2.8301135561+000 | 3.4280000000+000 | |
| | 0.0000000000+000 | 0.0000000000+000 | 0.0000000000+000 | 0.0000000000+000 |

From the above array, element values with 4-digit accuracy, except $C_5$ can be written as follows:

$C_1$=5.001 $C_2$=0.5856 $C_3$=0.3525 $C_4$=0.7494

$L_1$=0.6805 $L_2$=0.1267 $L_3$=0.2758 $L_4$=1.105

$\Gamma_1$=1.039 $\Gamma_2$=7.781 $\Gamma_3$=3.428 $\Gamma_4$=0.0000

Applying the same algorithm to $Y_{2s}$, using the pole sequence in reverse order we have:

| | | | | |
|---|---|---|---|---|
| $\zeta$ | 0.000000+000 | 9.454424-001 | 9.858527-001 | 7.070395-001 |

| | | | | |
|---|---|---|---|---|
| | 0.0000000000+000 | | | |
| | 6.0373407151+000 | 7.4940000000-001 | | |
| $C_{ij}$ | 1.1305337200+000 | 4.9865486101-001 | 3.5250000000-001 | |
| | -1.4137388487+000 | 1.1688927606+000 | 8.1501834776-001 | 5.8560000000-001 |

| | | | | |
|---|---|---|---|---|
| | 1.1050000000+000 | | | |
| | 3.5665232712-003 | 2.7580000000-001 | | |
| $L_{ij}$ | 1.3061351187-002 | 6.8434818073-002 | 1.2670000000-001 | |
| | 1.6240692289-001 | 8.4093963802-001 | 7.9028547008-001 | 6.8050000000-001 |

| | | | | |
|---|---|---|---|---|
| | 0.0000000000+000 | | | |
| | 2.6508796610+002 | 3.4280000000+000 | | |
| $\Gamma_{ij}$ | 7.5478615182+001 | 1.4405718138+001 | 7.7810000000+000 | |
| | 4.3535059184+000 | 8.4077318756-001 | 8.9466341818-001 | 1.0390000000+000 |

Here $C_{ii}$, $L_{ii}$, $\Gamma_{ii}$, i=1,2,3,4 are the element values when labeled from the output end except $C_1$. Both realizations give identical element values.

Example 2.5.2

In this example, the filter is with specifications C-09-20, θ=85°. The coefficients of $Y_d(s)$ are:

| ($s^k$) k | Numerator | Denominator |
|---|---|---|
| 9 | 3.6604045718+000 | 0.0000000000+000 |
| 8 | 2.6837214038+000 | 2.6836561893+000 |
| 7 | 1.4131690981+001 | 1.9239848286+000 |
| 6 | 9.0626489556+000 | 9.0623464387+000 |
| 5 | 2.0345308683+001 | 5.3996189587+000 |
| 4 | 1.1069134057+001 | 1.1068719167+001 |
| 3 | 1.2934901904+001 | 5.0303220756+000 |
| 2 | 5.6902015400+000 | 5.6900239700+000 |
| 1 | 3.0609000000+000 | 1.5547000000+000 |
| 0 | 1.0000000000+000 | 1.0000000000+000 |

Applying the algorithm to $Y_{1s}$ we obtain:

| | | | | |
|---|---|---|---|---|
| ζ | 4.756319-001 | 9.722537-001 | 9.912000-001 | 8.716365-001 |

| | | | | |
|---|---|---|---|---|
| $C_{ij}$ | 9.6880000000-001 | | | |
| | -2.5387082592-001 | 9.2840000000-001 | | |
| | -8.5808331031-001 | 7.8248187472-001 | 3.1030000000-001 | |
| | 3.2801247137-001 | 1.2256282291+000 | 6.8160375866-001 | 5.2530000000-001 |
| $L_{ij}$ | 9.4110000000-001 | | | |
| | 6.7614671644-002 | 1.9090000000-001 | | |
| | 1.1586329922-002 | 7.3174915806-002 | 9.4400000000-002 | |
| | 3.2830884751-001 | 4.6983947323-001 | 4.1114083662-001 | 3.2830000000-001 |
| $\Gamma_{ii}$ | 5.0540000000-001 | 5.0930000000+000 | 1.0500000000+001 | 2.6550000000+000 |

From $Y_{2s}$ we have:

| | | | | |
|---|---|---|---|---|
| ζ | 8.716365-001 | 9.912000-001 | 9.722537-001 | 4.756319-001 |

| | | | | |
|---|---|---|---|---|
| $C_{ij}$ | 3.2810000000-001 | | | |
| | -8.5934091174-001 | 5.2530000000-001 | | |
| | -2.5407984338-001 | 6.4717406874-001 | 3.1030000000-001 | |
| | 9.6881200799-001 | 1.3666191216+000 | 9.9451958949-001 | 9.2840000000-001 |

| | | | | |
|---|---|---|---|---|
| | 3.2830000000-001 | | | |
| $L_{ij}$ | 1.1559163379-002 | 9.4400000000-002 | | |
| | 6.7540355836-002 | 2.1212508419-001 | 1.9090000000-001 | |
| | 9.4123135087-001 | 1.0825726775+000 | 1.0643853284+000 | 9.4110000000-001 |
| $\Gamma_{ii}$ | 2.6550000000+000 | 1.0500000000+001 | 5.0930000000+000 | 5.0540000000-001 |

As in Example 1 the realized element values can be written from the diagonal elements of the above arrays as follows:

$C_1$=0.9688 $C_2$=0.9284 $C_3$=0.3103 $C_4$=0.5253 $C_5$=0.3281

$L_1$=0.9411 $L_2$=0.1909 $L_3$=0.0944 $L_4$=0.3283

$\Gamma_1$=0.5054 $\Gamma_2$=5.093 $\Gamma_3$=10.50 $\Gamma_4$=2.655

For the same filter, applying the algorithm with Fujisawa's procedure to $Y_{1s}$ gives:

| | | | | |
|---|---|---|---|---|
| $\zeta$ | 8.716365-001 | 9.912000-001 | 9.722537-001 | 4.756319-001 |
| | 3.2801247137-001 | | | |
| $C_{ij}$ | -8.5808331031-001 | 5.2554323391-001 | | |
| | -2.5387082592-001 | 6.4750404909-001 | 3.1048753397-001 | |
| | 9.6880000000-001 | 1.3670045022+000 | 9.9457681501-001 | 9.2804475276-001 |
| | 3.2830884751-001 | | | |
| $L_{ij}$ | 1.1586329922-002 | 9.4326307134-002 | | |
| | 6.7614671644-002 | 2.1199179497-001 | 1.9083349449-001 | |
| | 9.4110000000-001 | 1.0826306257+000 | 1.0645244895+000 | 9.4123135087-001 |
| $\Gamma_{ii}$ | 2.6549284511+000 | 1.0508203174+001 | 5.0947749116+000 | 5.0532947034-001 |

For $Y_{2s}$ we have:

| | | | | |
|---|---|---|---|---|
| $\zeta$ | 4.756319-001 | 9.722537-001 | 9.912000-001 | 8.716365-001 |
| | 9.6881200799-001 | | | |
| $C_{ij}$ | -2.5407984338-001 | 9.2804475276-001 | | |
| | -8.5934091174-001 | 7.8209332515-001 | 3.1048753397-001 | |
| | 3.2810000000-001 | 1.2254063320+000 | 6.8199785079-001 | 5.2554323391-001 |

$L_{ij}$
9.4123135087-001
6.7540355836-002 1.9083349449-001
1.1559163379-002 7.3168006653-002 9.4326307134-002
3.2830000000-001 4.6964132660-001 4.1104882472-001 3.2830884751-001

$\Gamma_{ii}$ 5.0532947034-001 5.0947749116+000 1.0508203174+001 2.6549284511+000

From the above arrays, element values with 4-digit accuracy, are obtained as follows:

$C_1$=0.3280 $C_2$=0.5255 $C_3$=0.3105 $C_4$=0.9280 $C_5$=0.9688

$L_1$=0.3283 $L_2$=0.0943 $L_3$=0.1908 $L_4$=0.9412

$\Gamma_1$=2.655 $\Gamma_2$=10.51 $\Gamma_3$=5.095 $\Gamma_4$=0.5053

Note that in terms of $\zeta_i$'s in the order $\zeta_1 < \zeta_2 < \zeta_3 < \zeta_4$, the original attenuation pole sequence $\langle \zeta_1, \zeta_3, \zeta_4, \zeta_2 \rangle$ is now changed into $\langle \zeta_2, \zeta_4, \zeta_3, \zeta_1 \rangle$ in accordance with Fujisawa's procedure. Incidentally, in this particular example the pole sequence determined by Fujisawa's procedure is the reverse of the original one and the element values come out to be close to those found when the filter is reversed. What is observed above of course does not happen in general.

## 2.6. Some Properties of the Algorithms

Although a precise error analysis is not considered, the simplicity of the algorithm requires fewer algebraic sum operations than with the conventional zero-shifting procedure. Since the algebraic sum operations are the main cause of accumulation error it is expected that by using the algorithms more accurate results can be obtained. In the examples given in this chapter driving-point admittances are constructed from the 4-digit table given by Saal and Ulbrich [SU]. Thus errors are introduced during the computation of coefficients of polynomials for the driving-point admittance which involves many algebraic operations. Nevertheless, the experience gained through several computations showed that the 11-digit single precision computation is sufficient for 4-digit accuracy element values even for the most complicated 5-section filter given in the table.

One interesting property of the recurrence formulas used in the algorithms is that they remain identical for the case where a transformation of variable is applied in improving the accuracy problem [SZ], [BI]. Only the first step of the algorithms should be modified by using the transformed admittance function and different values of transmission zeros according to the transformations. Indeed, the transformations $\phi^2=s^2/(1+s^2)$ and $z^2=1+(1/s)^2$ followed by other transformations $\zeta=-1/\phi^2$ and $\zeta=z^2$, respectively, yield the same continued fraction expansion for $F(\zeta)$ as shown in equation (2.3-2) except that the term $L_i/(\zeta_i-\zeta)$ is now changed into $-L_i/[(1-\zeta_i)-\zeta]$. However, the relation $\zeta_i-\zeta_j=(1-\zeta_j)-(1-\zeta_i)$ shows clearly that the recurrence formulas have not been changed.

Very useful piece of information that is obtainable through the computations relates to the negative signs appearing in the array of $C_{ij}$. Negative $C_{i1}$ implies that when a pole sequence with $\zeta_i$ as its first element is used, it will yield a negative element value. Similarly the presence of any negative element in the other columns of the $C_{ij}$ array, say $C_{ik}< 0$, $k< i$, indicates that when a pole sequence which contains $\zeta_1$, $\ldots,\zeta_{k-1}$ as the one indicated by the array, except that $\zeta_k$ is replaced by $\zeta_i$, is used, a negative $C_k$ will be produced. Thus by inspecting the negative signs that appear in the $C_{ij}$ array some of the pole sequences that will produce negative element values can be detected.

CHAPTER III

# FILTERS WITH INVERSE CHEBYSHEV ATTENUATION CHARACTERISTICS

## 3.1. Introduction

In this chapter a design method is considered for filters, which uses the algorithm developed in Chapter II as the final step and with inverse Chebyshev attenuation charateristics. These filters are called the inverse Chebyshev filters and their group delay characteristics are also studied.

The inverse Chebyshev filters, which may be considered as the limiting cases of the more general elliptic filters [MB], are relatively simpler to study than the elliptic filters, since they are somewhat related to the Chebyshev filters and therefore one can obtain the explicit formulas for the zeros of the polynomials required in the course of synthesis. Nevertheless, the filter network, which is of the type of Fig.2.3.1-a, has the same network configuration as that of the elliptic filter and accordingly involves the same problem of ladder network realization.

## 3.2. Insertion-Loss Theory

In this section a brief introduction to Darlington's [DA], [TO] insertion-loss theory is given.

With reference to the quantities indicated in Fig. 3.2.1, the following definitions and notations are introduced:

Insertion voltage ratio $=V_{20}(s)/V_2(s)$

Reflection function $\Sigma(s)=[Z_d(s)-R_1]/[Z_d(s)+R_1]$ (3.2-1)

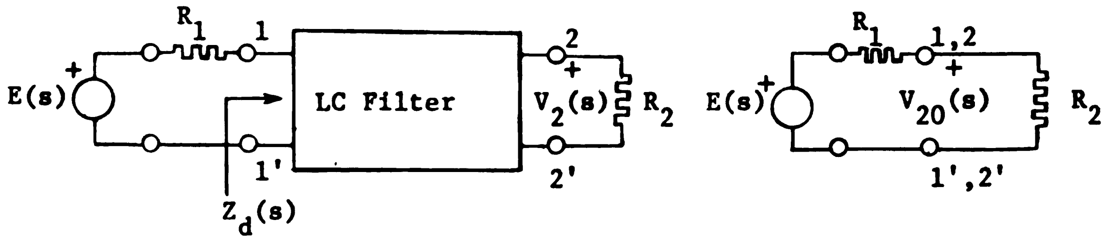



Fig. 3.2.1

$$\alpha^2 = 4R_1R_2/(R_1+R_2)^2 \qquad \text{or} \qquad \alpha^2 = 4r/(1+r)^2$$

where $r = R_1/R_2$. It is evident that $0 \leqslant \alpha^2 \leqslant 1$.

Transmission function $T(s) = \alpha[V_2(s)/V_{20}(s)]$ (3.2-2)

or

$T(s) = \alpha/(\text{Insertion voltage ratio})$

$$e^{2A(\omega)} = \left(\frac{V_{20}(s)}{V_2(s)}\right)\left(\frac{V_{20}(-s)}{V_2(-s)}\right)\Bigg|_{s=j\omega} = \left(\frac{V_{20}}{V_2}\right)\left(\frac{V_{20}}{V_2}\right)_{*}\Bigg|_{s=j\omega} \qquad (3.2\text{-}3)$$

where $A(\omega)$ is defined as the Insertion-loss function.

Among the quantities defined above, the following important relation can be derived either from the power relation or simply from the network equations [TO]:

$$\Sigma(s)\Sigma(-s) + T(s)T(-s) \equiv 1$$

Using the sub-star notation as in Eq. (3.2-3) this becomes

$$\Sigma\Sigma_{*} + TT_{*} \equiv 1 \qquad (3.2\text{-}4)$$

Eq. (3.2-4) is the basic relation in the theory of insertion-loss used in the design of filter networks with the specified attenuation characteristics and the specified ratio of terminating resistances.

In the synthesis of low-pass filters one approach is to consider the function $TT_{*}$ in the following form:

$$TT_* = \frac{\alpha^2}{1 + f^2(\omega)} \quad . \tag{3.2-5}$$

where $f(\omega)$ is called the filter function. $f(\omega)$ in general is a real rational function. There are other forms of $TT_*$, each of which is convenient for a certain case; however, none of them has significant advantages over the others as far as the synthesis is concerned. In Eq. (3.2-5), $f(\omega)$ is obtained from an approximation so that the function $TT_*$ will yield an attenuation function $A(\omega)$ which meets the specified attenuation constraints. From Eq's. (3.2-2), (3.2-3) and (3.2-5) $A(\omega)$ can be put into an explicit form:

$$A(\omega) = \frac{1}{2}\ln[1 + f^2(\omega)] \tag{3.2-6}$$

The rational character of $f(\omega)$ follows from two factors: (1) The filter to be realized is a lumped LC filter with a finite number of elements, hence the driving-point impedance $Z_d$ is a rational Brune function. (2) From the theory of approximation of a function by using rational functions, it is known that a rational function can always be found to meet the requirement.

Since $f(\omega)$ is a rational function, one can write

$$TT_* = \left(\frac{F}{Q}\right)\left(\frac{F}{Q}\right)_* \tag{3.2-7}$$

and

$$\Sigma\Sigma_* = 1 - TT_* = \left(\frac{H}{Q}\right)\left(\frac{H}{Q}\right)_* \tag{3.2-8}$$

where F, Q, H are real polynomials satisfying the relation (3.2-4), i.e.,

$$FF_* + HH_* \equiv QQ_* \tag{3.2-9}$$

Since the zeros of F, the transmission zeros, are also the zeros of $z_{12}$

of the LC filter network, F must be either even or odd. Therefore $FF_*=\mp F^2$, and the polynomial Q must be a strictly Hurwitz polynomial. For low-pass filters, since no transmission zero is allowed at zero frequency, F must be an even polynomial. This is a condition which must be satisfied when $f(\omega)$ is determined for a desired low-pass filter. In order that the filter be realizable as a mid-shunt or mid-series ladder network, additional restrictions for F and Q are required. These are discussed later.

With the notations introduced in the above discussion, a procedure for designing a filter is as follows:

Step 1 Find a filter function $f(\omega)$ that can yield a desired attenuation function $A(\omega)$ which meets specified attenuation characteristic. $f(\omega)$ also has to satisfy the conditions required for F and Q discussed above.

Step 2 (a) Find $FF_*=\mp F^2$ and $QQ_*$ from $f(\omega)$ determined in Step 1. (b) Find the zeros of F which are the transmission zeros. (c) Find the zeros of $QQ_*$ which are distributed with quadrantal symmetry on the s-plane. Using the zeros of $QQ_*$ in the left half plane, form the strictly Hurwitz polynomial Q. (d) From Eq. (3.2-9): $HH_*=QQ_*-FF_*$, find the zeros of $HH_*$ and choose half of them to form a real H. For the choice of H the only restriction is that H must have real coefficients. Each different choice of H will give a filter that yields identical attenuation and group delay characteristics but the element values are, in general, different.

It can be shown that if the degree of $HH_*$ is 2n, then the number of choices which exist for H are $2^{(n/2-1)}$ for n=even and $2^{(n-1)/2}$ for n=odd.

Step 3 From H and Q obtained in Step 2 obtain:

$$\Sigma = \pm \frac{H}{Q}$$

Substituting this relation into Eq. (3.2-1) and solving for $Z_d$, we have

$$Z_d = R_1 \frac{1 + \Sigma}{1 - \Sigma} \qquad (3.2\text{-}10)$$

or

$$Z_d = R_1 \frac{1 - \Sigma}{1 + \Sigma} \qquad (3.2\text{-}11)$$

corresponding to the choice of positive or negative sign for $\Sigma$, respectively. The filter networks for these $Z_d$'s are dual to each other.

Step 4 Realize the filter network from the driving-point impedance $Z_d$, or driving-point admittance $Y_d = 1/Z_d$, obtained in Step 3 together with the transmission zeros, i.e., the attenuation poles computed in Step 2.

The subject studied in Chapter II is a special case of the final step(Step 4) where the mid-shunt or mid-series filter network configurations are considered.

The filter design based on the insertion-loss theory was established by Darlington [DA] and Piloty [PI] in 1939. However it did not become practical until the late 1950's when the large scale electronic computers became available. The reason for this delay is mainly because the computations involved in Step 2 and Step 4 are too complicated and require too many significant digits even for a filter with only 3 sections. Since the late 1950's several papers have appeared which give both formulas and tables for element values for polynomial filters, i.e., $f(\omega)$ is a polynomial, and for elliptic filter where $f(\omega)$ is a special kind of rational function. For two special polynomial filters, namely the Butterworth and Chebyshev filters, explicit formulas for the element values were obtained [BE], [TA], and tables of element values for normalized filters now exist [WE2]. The network configuration for polynomial filters is a simple-ladder and the computation of element values can be obtained easily by using Routh arrays [TO]. For elliptic filters, formulas for less than 4-section filters are available [DA], [TT], [GRO], [SK], and tables of element values for normalized filters were prepared by several authors [SU], [SK]. In the rest of this chapter the inverse Chebyshev filter is considered. Thus far in the literature no detail of the properties or design formulas for the inverse Chebyshev filter has been published.

## 3.3. Synthesis of Inverse Chebyshev Filters

In this section the relations that exist among Butterworth, Chebyshev and Inverse Chebyshev filter characteristics are studied. Then by utilizing these relations, formulas for the realization of inverse Chebyshev filters are derived by making use of the algorithm obtained in Chapter II.

The filter functions for n-th order Butterworth, Chebyshev, and inverse Chebyshev filters are defined as follows:

| | |
|---|---|
| Butterworth: | $\omega^n$ |
| Chebyshev: | $C_n(\omega)$ |
| Inverse Chebyshev: | $1/C_n(1/\omega)$ |

where $C_n(\omega)$ is the Chebyshev polynomial whose properties are listed in

Table 3.3-1. The above filter functions have unit value at $\omega=1$. The specification of a required attenuation characteristic is usually given as shown in Fig. 3.3.1.

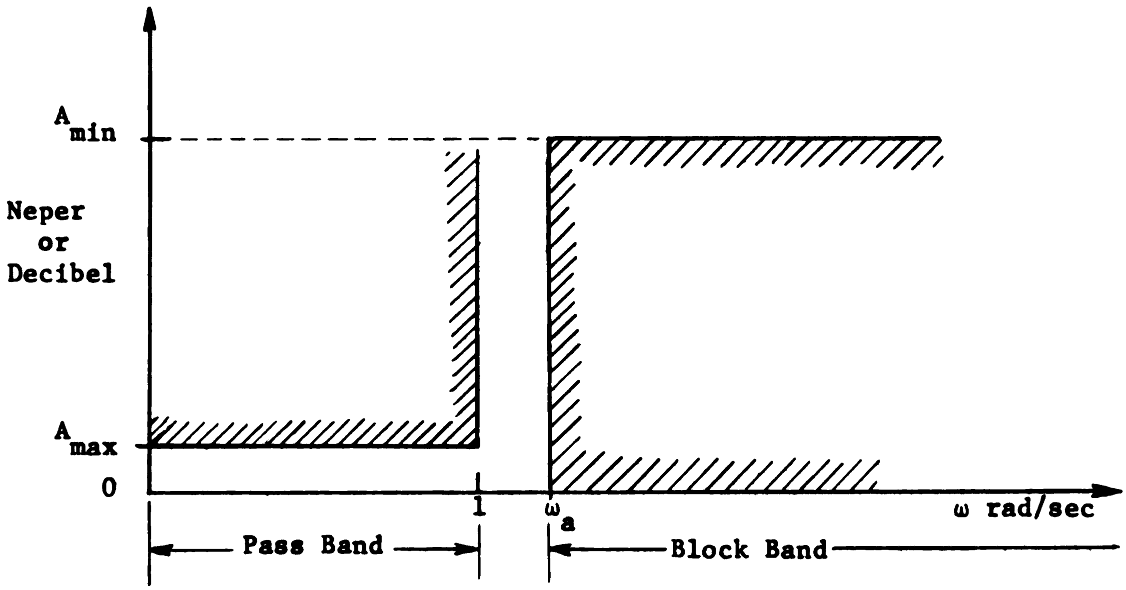



Fig. 3.3.1

It is convenient to introduce a positive constant $\varepsilon$ such that the attenuation at $\omega=1$ will equal $\frac{1}{2}\ln(1+\varepsilon^2)$ nepers or $10\log(1+\varepsilon^2)$ db which is equal to or less than $A_{max}$, the maximum allowable attenuation in the pass-band. Therefore at $\omega=\omega_a$ the attenuation will be $\frac{1}{2}\ln[1+\varepsilon^2 f^2(\omega_a)]$ nepers which is equal to or greater than $A_{min}$, the minimum allowable attenuation over the stop-band. The filter functions after the necessary modification to incorporate the above properties are as follows:

| | | |
|---|---|---|
| Butterworth: | $\varepsilon\omega^n$ | (3.3-1) |
| Chebyshev: | $\varepsilon C_n(\omega)$ | (3.3-2) |
| Inverse Chebyshev: | $\varepsilon C_n(\omega_a)/C_n(\omega_a/\omega)$ | (3.3-3) |

For these filter functions, the function $QQ_*=1+f^2$ defined in

Eq. (3.2-5) for the Butterworth filter, is $1+\varepsilon^2 s^{2n}$, $s=j\omega$. The zeros of $1+\varepsilon^2 s^{2n}$ are distributed on a circle of radius $\rho=(1/\varepsilon)^{1/n}$ and are separated by equal distances. Fig. 3.3.2-(a) shows the zeros of the 3rd order

TABLE 3.3-1

SOME USEFUL PROPERTIES OF THE CHEBYSHEV POLYNOMIALS

$$C_n(x) = \begin{cases} \cos(n \cdot \cos^{-1} x) & \text{for } |x| < 1 \\ \cosh(n \cdot \cosh^{-1} x) & \text{for } |x| \geqslant 1 \end{cases} \tag{3.3-1}$$

or

$$C_n(x) = \frac{1}{2}\left[(x - \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^{-n}\right]$$

$$= \frac{1}{2}\left[(x + \sqrt{x^2-1})^n + (x + \sqrt{x^2-1})^{-n}\right] \tag{3.3-2}$$

Note that $x + \sqrt{x^2 - 1} = 1/(x - \sqrt{x^2 - 1})$.

$$C_o(x) = 1$$

$$C_{n+1}(x) = 2xC_n(x) - C_{n-1}(x) \tag{3.3-3}$$

$$C_{2n}(x) = 2C_n^2(x) - 1 \tag{3.3-4}$$

$C_{n=even}$ = even polynomial with constant term = $(-1)^{n-1}$

$C_{n=odd}$ = odd polynomial

Coefficient of leading term = $2^{n-1}$

Coefficient of $x^{n-2k}$, $k = 0, 1, 2, \ldots$ is given by

$$(-1)^k 2^{n-2k-1}\left[\, 2\binom{n-k}{k} - \binom{n-k-1}{k} \,\right]$$

Butterworth filter with $\varepsilon=0.5$. For the Chebyshev filter, the zeros of $QQ_*=1+\varepsilon^2 C_n^2(s)$ lie on an ellipse as shown in Fig. 3.3.2-(b) with semiaxes a and b given as follows [GU]:

$$2a = K^{1/n} - K^{-1/n} \tag{3.3-4}$$

$$2b = K^{1/n} + K^{-1/n} \tag{3.3-5}$$

where

$$K = \sqrt{1/\varepsilon^2+1} + 1/\varepsilon \tag{3.3-6}$$

The zeros of $QQ_*$ are

$$s_k = a \cos\theta_k + jb \sin\theta_k \tag{3.3-7}$$

where

$$\left.\begin{array}{ll} \theta_k = \frac{2k-1}{2n}\pi & \text{when n is even} \\ \theta_k = \frac{2k-2}{2n}\pi & \text{when n is odd} \end{array}\right\} k=1,2,3,\ldots,2n$$

The $\theta_k$'s are the same for the corresponding Butterworth case. In fact for both cases, with the same $\varepsilon$, the polynomial $QQ_*$ for the Chebyshev filter can be transformed from the $QQ_*$ of the Butterworth filter:

$$1 + \varepsilon^2 C_n^2(s) = T[1 + \varepsilon^2 \omega^{2n}]$$

where T represents the transformation $s=\frac{1}{2}(w+1/w)$ whose mapping is conformal [AL]. As indicated in Fig. 3.3.2 the circles ($\rho$ = constant) and radial lines ($\theta$ = constant) in w-plane are mapped by T onto the s-plane as ellipses and hyperbolas, respectively. All have the same foci $\mp j$ and the zeros of $1 + \varepsilon^2 C_n^2(s)$ are located at the intersections of these curves. By using this transformation, formulas for the zeros of $QQ_*$ for the Chebyshev filter can be obtained from the formulas for the zeros of $QQ_*$ for the corresponding Butterworth filter. Note that the inverse of T is $w = s \pm\sqrt{s^2+1}$ and therefore for the filters to have the same $\varepsilon$, it is necessary that the constant $1/\varepsilon$ for the Butterworth filter be replaced by the constant $1/\varepsilon \pm\sqrt{1/\varepsilon^2+1}$. Depending upon the positive or negative sign, the constant has two different values which are reciprocal to each other. For both values the same a and b given by Eq. (3.3-4) and Eq. (3.3-5) are obtained.

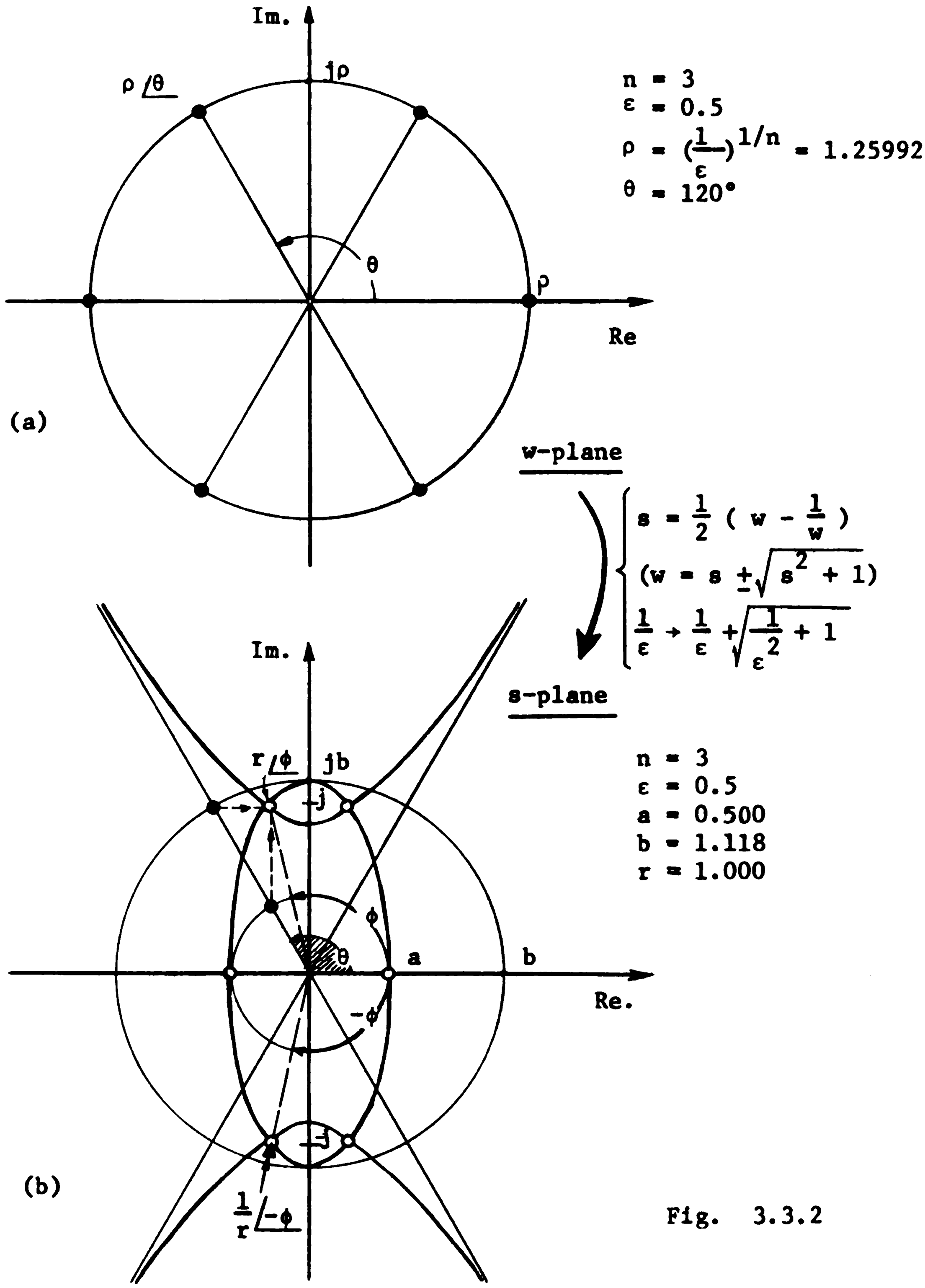



Fig. 3.3.2

The relation between the inverse Chebyshev filter and the Chebyshev filter is as shown in Eq. (3.3-2) and Eq. (3.3-3), from which it is evident that the zeros of $QQ_*$ for the Chebyshev filter are $\omega_a/s_k$, where $s_k$ are the corresponding zeros for the Chebyshev filter. The equation of the ellipse, on which the zeros of $QQ_*$ for the Chebyshev filter are located, can be written in polar coordinate form as follows:

$$\frac{(r\cos\phi)^2}{a^2} + \frac{(r\sin\phi)^2}{b^2} = 1 \qquad (3.3\text{-}9)$$

where

$$b^2 - a^2 = 1 \qquad (3.3\text{-}10)$$

Therefore, for the inverse Chebyshev filter, the equation for the curve on which the zeros of $QQ_*$ are located can be found by replacing r in Eq. (3.3-9) by $\omega_a/r$ and $\phi$ by $-\phi$:

$$r^2 = \frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}$$

Now using the relation in Eq. (3.3-10), the above equation can be written as

$$r^2 = \frac{1}{a^2b^2}\,[\,a^2 + \cos^2\phi\,] \qquad (3.3\text{-}11)$$

The curves corresponding to Eq. (3.3-11) with $D = \epsilon^2 C_n^2(\omega_a) = 10^4$, $\omega_a = 1.3$ for n=7 and n=13, respectively, are given in Fig. 3.3.3 where the dots on the curves indicate the zeros of $QQ_*$.

In the above discussion it is shown that the Butterworth, the Chebyshev, and the inverse Chebyshev filter characteristics are related and hence the explicit formula for the zeros of $QQ_*$ for the inverse

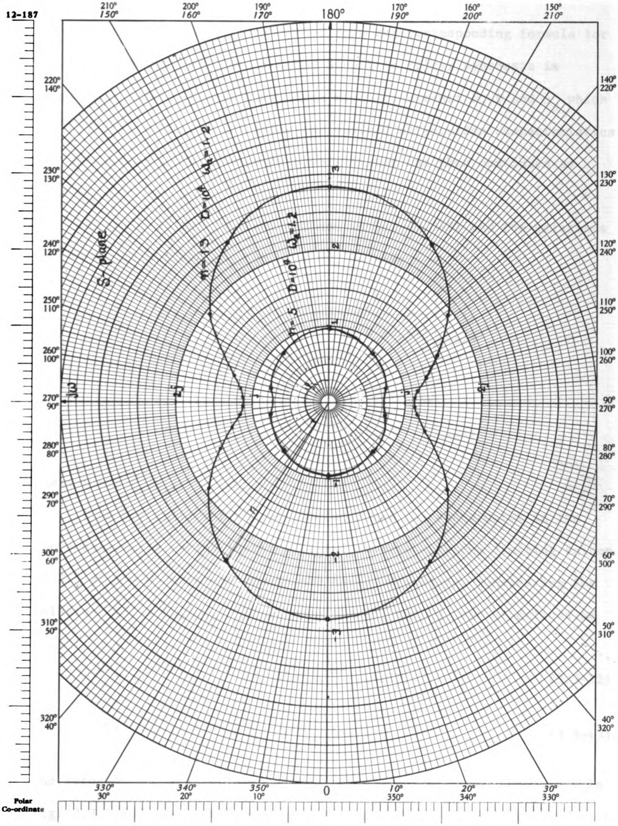

12-187
S-plane
m=15 D=10^4 ω_k=1.2
m=5 D=10^4 ω_k=1.2
jω
180°
0
Polar
Co-ordinate

Fig. 3.3.3

Chebyshev filters can be established from the corresponding formula for the Chebyshev filters. Assume that Step 1 in filter synthesis is completed; namely, the determination of the constants n, D and $\omega_a$ which will produce a filter satisfying the desired attenuation characteristics. The synthesis of the inverse Chebyshev filter can then continued as follows.

From the filter function defined in Eq. (3.3-3) and the relations among the polynomials $\Sigma\Sigma_*$, $HH_*$ and $QQ_*$ we have

$$\Sigma\Sigma_* = 1 - TT_* = 1 - \left.\frac{\alpha^2}{1 + D/C_n^2(\omega_a/\omega)}\right|_{\omega=s/j}$$

$$= \left.\frac{1 + \frac{1-\alpha^2}{D} C_n^2(\omega_a/\omega)}{1 + \frac{1}{D} C_n^2(\omega_a/\omega)}\right|_{\omega=s/j}$$

where $D = \varepsilon^2 C_n^2(\omega_a)$.

Upon multiplying both the numerator and the denominator in the above expression by the factor $s^{2n}$, and observe that the limiting values of $\Sigma\Sigma_*$ at $s = 0$ and $s = \infty$ remain the same. Therefore, considering the relation $\Sigma\Sigma_* = HH_*/QQ_*$, the polynomials $HH_*$ and $QQ_*$ can be defined as

$$HH_* = s^{2n} \left[ 1 + \frac{1-\alpha^2}{D} C_n^2(\omega_a/\omega) \right]_{\omega=s/j} \qquad (3.3\text{-}12)$$

$$QQ_* = s^{2n} \left[ 1 + \frac{1}{D} C_n^2(\omega_a/\omega) \right]_{\omega=s/j} \qquad (3.3\text{-}13)$$

The zeros of $HH_*$ and $QQ_*$ are located on curves similar to that shown in Fig. 3.3.3 and the curve on which the zeros of $HH_*$ are located is enclosed within the curve for $QQ_*$. In the special case for the filter with equal terminations ($\alpha = 1$), $HH_*$ reduces to $s^{2n}$.

The explicit formulas for the zeros of $QQ_*$ mentioned earlier in this section can be derived from equations (3.3-4) through (3.3-10) in the following manner. Replace $\varepsilon^2$ in Eq. (3.3-6) by 1/D. Then the $(\omega_a/s_k)$'s become the zeros of $QQ_*$ for the inverse Chebyshev filter. Let $s_i = r_i\underline{/\phi_i}$ be the zeros of $QQ_*$ for the inverse Chebyshev filter, then

$$\begin{aligned} 1/r_i^2 &= a^2\cos^2\theta_i + b^2\sin^2\theta_i \\ &= a^2\cos^2\theta_i + (a^2 + 1)\sin^2\theta_i \\ &= a^2 + \sin^2\theta_i \end{aligned}$$

i.e.,

$$r_i = 1/\sqrt{a^2 + \sin^2\theta_i} \tag{3.3-14}$$

and

$$\phi_i = \tan^{-1}\left[\ (a/b)\tan\theta_i\ \right] \tag{3.3-15}$$

where

$$\left.\begin{aligned} 2a &= K^{1/n} - K^{-1/n} \\ 2b &= K^{1/n} + K^{-1/n} \end{aligned}\right\} \tag{3.3-16}$$

$$K = \sqrt{D+1} + \sqrt{D} \tag{3.3-17}$$

$$\theta_i = \frac{i-1}{n}\pi$$

for $i = 1,2,3,\ldots,2n$, with n odd. It will be shown later that this filter is not realizable when n is even.

The formulas for the zeros of $HH_*$, when $\alpha \neq 1$, are the same as those for $QQ_*$ except that D is replaced in Eq. (3.3-17) by $D/(1 - \alpha^2)$. This is due to the fact that $HH_*$ and $QQ_*$, as shown in Eq's. (3.3-12) and (3.3-13), are identical in form.

From the zeros of $HH_*$ and $QQ_*$, Q and H are obtained by the method described in Step 2 of the synthesis procedure in the preceding section. As mentioned earlier, Q is strictly Hurwitz and is formed from the left half plane zeros of $QQ_*$. However, there are $2^{(n-1)/2}$ different choices for H to be determined from $HH_*$ for odd n. It is interesting to note that for n odd, if H is formed by selecting the alternating zeros of $HH_*$, then it can be shown that

$$H(s) = \left(\frac{s}{j}\right)^n [\ 1 + j\sqrt{(1 - \alpha^2)/D}\ C_n(j\omega_a/s)\ ] \tag{3.3-18}$$

Thus H(s) can be obtained without computing the zeros of $HH_*$. For this case, however, the attenuation pole frequencies $\omega_k$ are the zeros of $C_n(j\omega_a/s)$ and therefore $H(\omega_k) = \omega_k^n$.

After Q and H have been obtained, the driving-point impedance can be formed from Eq. (3.2-10) or Eq. (3.2-11) :

$$Z_d(s) = R_1 \frac{Q + H}{Q - H} \tag{3.3-19a}$$

or

$$Z_d(s) = R_1 \frac{Q - H}{Q + H} \tag{3.3-20a}$$

or, alternatively, the driving-point admittance is

$$Y_d(s) = G_1 \frac{Q - H}{Q + H} \tag{3.3-19b}$$

or

$$Y_d(s) = G_1 \frac{Q + H}{Q - H} \tag{3.3-20b}$$

where $G_1 = 1/R_1$. The filter with desired attenuation characteristics can now be obtained by realizing the above driving-point function $Z_d$ or $Y_d$ with transmission zeros (attenuation poles), $\omega_k$, which are the zeros of $T = F/Q$. For the inverse Chebyshev filter, these zeros of F are the positive roots of the equation

$$C_n(\omega_a/\omega) = \cos[\ n \cos^{-1}(\omega_a/\omega)\ ]$$

These roots are

$$\omega_k = \frac{\omega_a}{\cos(\frac{2k-1}{2n}\pi)} \qquad (3.3\text{-}21)$$

for k = 1,2,...,(n-1)/2, with n odd. However, in order that the given $Z_d$ or $Y_d$ be realizable as a mid-series or mid-shunt ladder network, the degree of Q + H written as $\delta(Q + H)$ should be exactly one more than that of Q - H. i.e.,

$$\delta(\ Q + H\ ) = \delta(\ Q - H\ ) + 1 \qquad (3.3\text{-}22)$$

This necessary condition can also be obtained from the condition (1) of the Realizability Theorem given in Chapter II and from the fact that $Z_d$ and $Y_d$ are Brune which require that the difference of the degrees of their numerator and denominator polynomials is at most one. Eq. (3.3-22) implies that the leading coefficients of Q and H are identical, furthermore the identity $HH_* + FF_* \equiv QQ_*$ implies that

$$\delta(QQ_* - HH_*) = \delta(FF_*) = \delta F^2$$

and hence we have

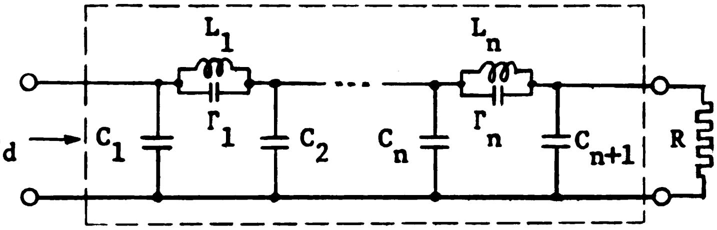
$$\delta F + 1 = \delta Q = \delta H \qquad (3.3\text{-}23)$$

Consider the inverse Chebyshev filter characteristics:

$$TT_* = \frac{FF_*}{QQ_*} = \frac{(s/j)^{2n}C_n^2(j\omega_a/s)\ \alpha^2/D}{(s/j)^{2n}[\ 1 + C_n^2(j\omega_a/s)/D\ ]}$$

If n is even, $C_n$ contains a constant term. Therefore, the filter is not

**TABLE 3.3-2**




$\delta N$ = Degree of numerator polynomial
$\delta D$ = Degree of denominator polynomial

$z_{11}, z_{22}, z_{12}$

$n \geqslant 2$

| Elements which have zero values | $\delta N/\delta D$ for $Z_d$ | $z_{11}$ or $Z_{10}$ | $z_{22}$ or $Z_{20}$ | $y_{11}$ or $Y_{1s}$ | $y_{22}$ or $Y_{2s}$ |
|---|---|---|---|---|---|
| All non-zero | | | | | |
| Some or all $\Gamma_i$'s | $\frac{2n}{2n+1}$ | $\frac{2n}{2n+1}$ | | $\frac{2n}{2n-1}$ | |
| $C_1$ or $C_{n+1}$ or both | | | | | |
| $C_1$, $\Gamma_1$ | $\frac{2n}{2n-1}$ | $\frac{2n}{2n-1}$ | $\frac{2n-2}{2n-1}$ | | $\frac{2n}{2n-1}$ |
| $C_{n+1}$, $\Gamma_n$ | $\frac{2n-1}{2n}$ | $\frac{2n-2}{2n-1}$ | $\frac{2n}{2n-1}$ | | $\frac{2n-2}{2n-1}$ |

**Notes:**

(1) $Z_d = \dfrac{R_2 z_{11} + |z|}{R_2 + z_{22}}, \quad Y_d = \dfrac{1}{Z_d} = \dfrac{y_{11} + R_2|y|}{1 + R_2 y_{22}}$

(2) $n \geqslant 2$ not necessary for all cases.

(3) The results are obtained by induction under the assumptions that $0 < R < \infty$ and $C_i$, $L_i$, $\Gamma_i$ are non-negative.

realizable in terms of a ladder network, since in this case $\delta F = \delta Q$, and this does not satisfy the condition given in Eq. (3.3-23). In Table 3.3-2 the degree relations of the numerator and the denominator polynomials of $Z_d$ for the mid-shunt ladder network are given. When n is odd, from Table 3.3-2 and from the fact that Eq. (3.3-21) yields all finite and non-zero $\omega_k$'s, it can be concluded that, if the filter is realizable, it is either a full mid-shunt ladder (all elements are non-zero) realized from Eq. (3.3-20b), or a full mid-series ladder realized from Eq. (3.3-19a).

Under the assumptions that Q and H yield a realizable $Y_d$ (or $Z_d$), two methods of realization of $Y_d$ by using the algorithm established in Chapter II can be given. These are discussed in the following:

Method 1

In this method, the initial 2n parameters, $C_{k1}$ and $L_{k1}$ (k=1,2,...,n) needed as required by the algorithm for the computations of remaining parameters are obtained directly from polynomials Q and H.

The initial 2n parameters are computed by Darlington's formulas as

$$C_{k1} = Y_d(s)/s \Big|_{s=j\omega_k} \tag{3.3-24}$$

and

$$L_{k1} = \frac{-2}{s^3[\ Y_d(s)/s\ ]'}\Bigg|_{s=j\omega_k} \tag{3.3-25}$$

for k = 1,2,...,(n-1)/2.

Since the normalized driving-point admittance (with respect to $G_1$) is

$$Y_d = (Q + H)/(Q - H) \tag{3.3-26}$$

we have

$$\frac{d}{ds}\left[\frac{Y_d}{s}\right] = \frac{1}{s}\,\frac{2(H'Q - HQ')}{(Q - H)^2} - \frac{1}{s^2}\,\frac{Q + H}{Q - H} \tag{3.3-27}$$

Both functions are expressed in terms of H, H', Q and Q'. Therefore, the parameters $C_{k1}$ and $L_{k1}$ can be computed in terms of H, H', Q and Q', for $s=j\omega_k$. However, another way to compute the parameters is as follows. Let

$$H = \prod_{i=1}^{n} (s - s_i') \quad \text{and} \quad Q = \prod_{i=1}^{n} (s - s_i)$$

Then by using the relations

$$\frac{H'}{H} = \sum_{i=1}^{n} \frac{1}{s - s_i'} \quad \text{and} \quad \frac{Q'}{Q} = \sum_{i=1}^{n} \frac{1}{s - s_i}$$

Eq. (3.3-27) can be written as

$$\left(\frac{Y_d}{s}\right)' = \frac{2QH}{(Q - H)^2 s} \sum_{i=1}^{n} \left[ \frac{1}{s - s_i'} - \frac{1}{s - s_i} \right] - Y_d/s^2 \qquad (3.3\text{-}29)$$

Therefore, $C_{k1}$ and $L_{k1}$ can be computed from H, Q and the summation that appears in the right hand side of Eq. (3.3-29).

In this method the computation of the coefficients in the numerator and the denominator polynomials are avoided and the computation of $Y_d(s_k)$ and $[Y_d/s]'_{s=s_k}$ involves only the factors $(s_k - s_i')$ or $(s_k - s_i)$. The parameters $C_{k1}$, $L_{k1}$ obtained by this method will therefore be much more accurate.

In the design of filters, especially for filters with large number of sections, it is convenient to compute the element values from both the input and output ends as an accuracy check. This method, with slight modification, can also yield two sets of $C_{k1}$ corresponding to those generated from the values of $Y_{1s}$ and $Y_{2s}$ at $s=j\omega_k$, respectively. Indeed, if we let

$$Q(\omega_k) + H(\omega_k) = E_{1k} + jO_{1k}$$

and

$$Q(\omega_k) - H(\omega_k) = E_{2k} + jO_{2k}$$

then

$$(E_{1k}/O_{2k})/\omega_k = Y_{1s}(j\omega_k)/j\omega_k$$

and (3.3-30)

$$(E_{2k}/O_{2k})/\omega_k = Y_{2s}(j\omega_k)/j\omega_k$$

are the $C_{kl}$'s obtained from $Y_{1s}$ and $Y_{2s}$ when $s=j\omega_k$. Note that for $L_{kl}$'s obtainable from $Y'_{1s}$ and $Y'_{2s}$, since there are no simple formulas for the values of $Y'_{1s}$ and $Y'_{2s}$ at $s=j\omega_k$, the $L_{kl}$'s cannot be computed as easily as in the case of the parameters $C_{kl}$. However, since the $L_{kl}$'s are the same from both computations, except for the ordering, we may use the same $L_{kl}$'s obtained from Eq. (3.3-25) for the out-put side and they are now ordered consistently with the pole sequence seen from the output end.

Method 2

In this method the coefficients of the numerator polynomial (Q + H) and those of the denominator polynomial (Q - H) of $Y_d(s)$ are obtained by using Newton's formulas [CH]. The algorithm established in Chapter II then is utilized to realize the $Y_{1s}$ and $Y_{2s}$ as discussed in Chapter II.

Let

$$Q(s) = \prod_{i=1}^{n}(s - s_i) = (s - s_o)\prod_{j=1}^{(n-1)/2}(s - s_j)(s - \bar{s}_j)$$

$$= s^n + p_1 s^{n-1} + p_2 s^{n-2} + \dots + p_{n-1}s + p_n \qquad (3.3\text{-}31)$$

$$H(s) = \prod_{i=1}^{n}(s - s'_i) = (s - s'_o)\prod_{j=1}^{(n-1)/2}(s - s'_j)(s - \bar{s}'_j)$$

$$= s^n + p'_1 s^{n-1} + p'_2 s^{n-2} + \dots + p'_{n-1}s + p'_n \qquad (3.3\text{-}32)$$

then

$$\left.\begin{matrix} Q + H = 2s^n \\ Q - H = 0 \end{matrix}\right\} + (p_1 \pm p'_1)s^{n-1} + \dots + (p_{n-1} \pm p'_{n-1})s + (p_n \pm p'_n)$$

(3.3-33)

Let

$$\delta_k = \sum_{i=1}^{n} s_i^k \qquad \text{and} \qquad \delta'_k = \sum_{i=1}^{n} s_i'^k \tag{3.3-34}$$

then from Newton's formulas we have

$$p_k \pm p'_k = -\frac{1}{k}[\,(\delta_k + \delta_{k-1}p_1 + \dots + \delta_2 p_{k-2} + \delta_1 p_{k-1})$$

$$\pm (\delta'_k + \delta'_{k-1}p'_1 + \dots + \delta'_2 p'_{k-2} + \delta'_1 p'_{k-1})\,] \tag{3.3-35}$$

for k=1,2,...,n. If $s_i = r_i \underline{/\phi_i}$, $s'_i = r'_i \underline{/\phi'_i}$, then by combining the complex conjugate pairs, Eq's. (3.3-34) become

$$\delta_k = s_o^k + 2 \sum_{j=1}^{(n-1)/2} r_j^k \cos(k\phi_j) \tag{3.3-36}$$

$$\delta'_k = s_o'^k + 2 \sum_{j=1}^{(n-1)/2} r_j'^k \cos(k\phi'_k) \tag{3.3-37}$$

where $r_j$ and $\phi_j$ are computed by formulas (3.3-14) and (3.3-15) respectively, and $s_o$ and $s'_o$ are real. Note that $s_o = -a$ and $s'_o = \pm a'$ are obtained by Eq. (3.3-16) for Q and H respectively and the sign of $s'_o$ depends on the choice of H.

The coefficients in the numerator (Q + H) and the denominator (Q - H) of $Y_d$ are found first by computing $\delta_k$ and $\delta'_k$ (k=1,2,...,n) and then carrying out the iterative computations given in Eq. (3.3-35). The advantage of the second method lies in the fact that the coefficients of $Y_{1s}$ and $Y_{2s}$ are also determined once the coefficients for $Y_d$ are computed. Therefore the realization from both the input and the output ends of the filter can now be accomplished by applying the algorithm to $Y_{1s}$ and $Y_{2s}$.

## 3.4. Design Procedure and Numerical Examples

In this section a design procedure for the inverse Chebyshev filter is outlined. The design procedure also includes the approximation step. This step enables one to select the design parameters D, $\varepsilon$ and n so that when the filter is designed based on these parameters, it will meet the required specifications i.e., the attenuation characteristics.

As indicated in the preceding section, when the filter function is $\varepsilon C_n(\omega_a)/C_n(\omega_a/\omega)$ then for a given set of parameters, $A_{min}$, $A_{max}$ and $\omega_a$(see Fig. 3.3.1), the design parameters D, $\varepsilon$ and n are determined according to the following relations:

$$\frac{1}{2}\ell n(1 + D) \geqslant A_{min} \quad \text{nepers} \tag{3.4-1}$$

and

$$\frac{1}{2}\ell n(1 + \varepsilon^2) \leqslant A_{max} \quad \text{nepers} \tag{3.4-2}$$

or if db(decibel) unit is used, then

$$10 \ \ell og(1 + D) \geqslant A_{min} \quad \text{db} \tag{3.4-1a}$$

$$10 \ \ell og(1 + \varepsilon^2) \leqslant A_{max} \quad \text{db} \tag{3.4-2a}$$

$$D = \varepsilon^2 C_n^2(\omega_a) \tag{3.4-3}$$

First the minimum value of D and the maximum value of $\varepsilon^2$ are determined from Eq's. (3.4-1) and (3.4-2), respectively, and then they are substituted into Eq. (3.4-3). Since $\omega_a$ is known, this relation yields n. In general the parameter n so determined is not an integer. Therefore for the value of n, the minimum integer which is greater than the computed n must be taken. For the determination of n, the formula in Eq.(3.3-2) of Table 3.3-1 may be used:

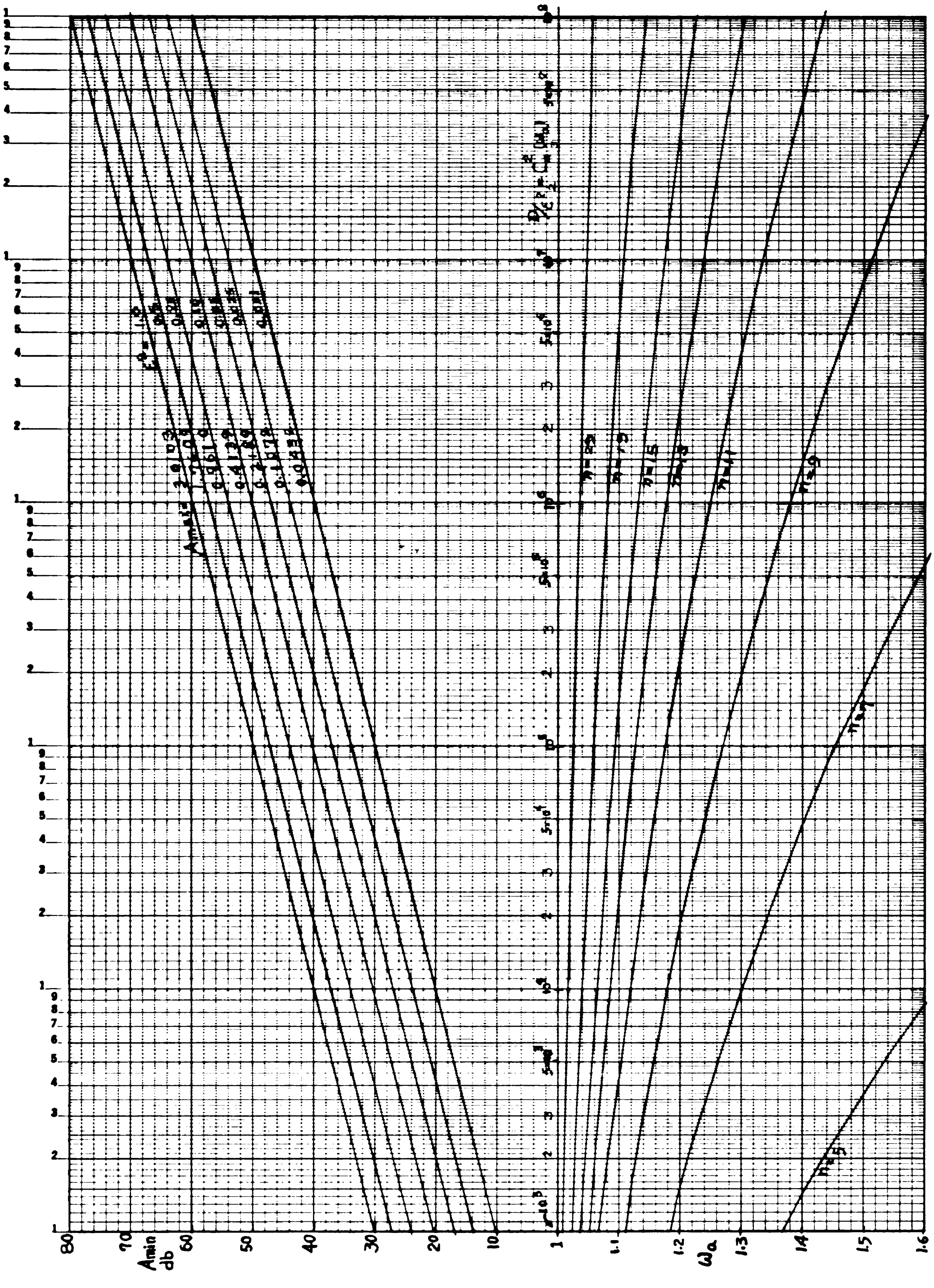


Fig. 3.4.1

$$C_n(\omega) = \frac{1}{2}[\ (\omega + \sqrt{\omega^2 - 1})^n + (\omega + \sqrt{\omega^2 - 1})^{-n}\ ] \qquad (3.4\text{-}4)$$

When n is large, the second term in the right hand side of the above equation is small compared with the first term and it may be ignored to simplify the computation. Note that this omission will increase the value of n slightly.

The computations needed in determining the values of the parameters D, $\varepsilon^2$ and n may be done graphically. One always has the freedom of choosing D and $\varepsilon^2$ for a fixed n as long as the inequalities (3.4-1) and (3.4-2) are satisfied. Therefore a graphical method is convenient to the determination of a most suitable set of parameters without the repeated computations. Two families of curves plotted in Fig. 3.4.1 can be used for finding these parameters. The first set of curves (in the upper part of the graph) are computed from Eq's. (3.4-1a) and (3.4-3) in which each curve corresponds to a different $\varepsilon^2$ which in turn implies $A_{max}$ through Eq. (3.4-2a). The second set of curves (in the lower part of the graph) are the plots of the relation, $C_n^2(\omega_a) = D/\varepsilon^2$, in which each curve corresponds to a different integer n. The following example illustrates how the graph is used in finding the desired parameters.

<u>Example 3.4-1</u>

Find the design parameters D, $\varepsilon$ and n for an inverse Chebyshev filter satisfying the following specifications:

$$A_{max} = 1 \text{ db}$$

$$A_{min} = 55 \text{ db}$$

and

$$\omega_a \leqslant 1.5$$

With the given specifications, regions in Fig. 3.4.1 in which the specifications are not satisfied can be determined. In this case the graph will appear as shown in Fig. 3.4.2. From Fig. 3.4.2, a solution for the parameters will exist if $D/\varepsilon^2 > 1.2\ 10^6$ and $n \geqslant 9$. Of course the

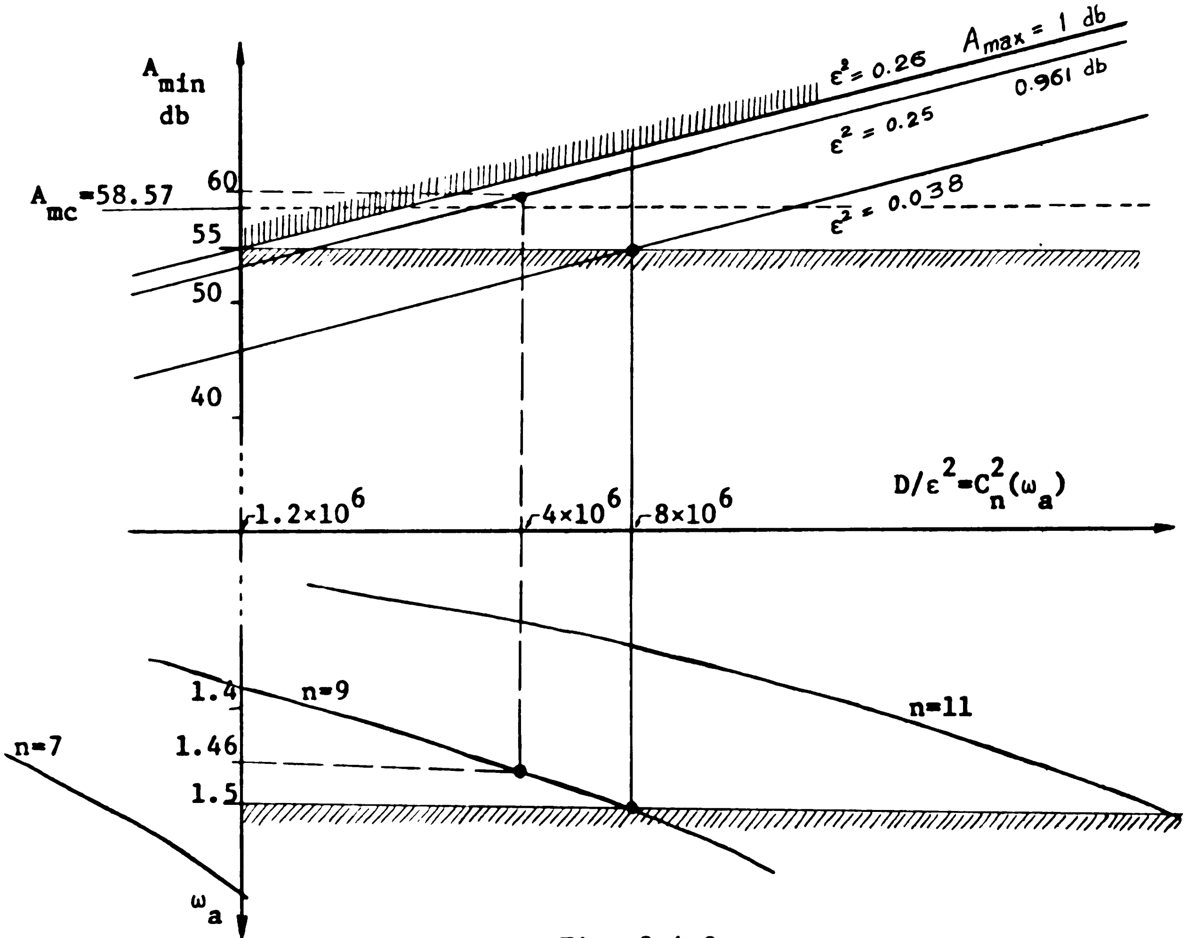



Fig. 3.4.2

simplest filter is obtained if n=9. The selection of n=9 imposes an upper bound on $D/\varepsilon^2$, $8\times10^6$, and a lower bound on $\varepsilon^2$, 0.038. Therefore,

$$1.2\times10^6 \leqslant \frac{D}{\varepsilon^2} \leqslant 8\times10^6$$

and

$$0.038 \leqslant \varepsilon^2 \leqslant 0.26$$

If we select $D/\varepsilon^2 = 4\times10^6$ and $\varepsilon^2 = 0.25$ then with n=9 we have,

$$\omega_a = 1.46$$

$A_{min} = 60$ db

$A_{max} = 0.961$ db (at $\omega=1$)

$D = \varepsilon^2 \times 4 \times 10^6 = 10^6$

It is evident from the above example that a set of design parameters always exist for any given set of specifications: $\omega_a \geq 1$ and $0 < A_{max} \leq A_{min}$. However, the driving-point admittance $Y_d(s)$ constructed on the basis of the approximation may or may not satisfy the conditions stated in the Realizability Theorem given in Chapter II. If Fujisawa's condition is violated then there will be negative shunt capacitors in the mid-shunt realization (or negative series inductance in the mid-series realization). Meinguet and Belevitch [MB] stated a relation, which is also applicable to inverse Chebyshev filters, and gave a table from which one can see clearly that the minimum attenuation $A_{min}$ in the block-band cannot be

TABLE 3.4-1

| n | $A_{mc}$ (nepers) | $A_{mc}$ (decibels) |
|---|---|---|
| 5 | 2.764276+000 | 2.401019+001 |
| 7 | 4.827812+000 | 4.193384+001 |
| 9 | 6.743005+000 | 5.856900+001 |
| 11 | 8.599218+000 | 7.469186+001 |
| 13 | 1.042546+001 | 9.055439+001 |
| 15 | 1.223426+001 | 1.062654+002 |
| 17 | 1.403198+001 | 1.218802+002 |
| 19 | 1.582221+001 | 1.374299+002 |
| 21 | 1.760712+001 | 1.529335+002 |

smaller than a critical value $A_{mc}$ otherwise a ladder realization is impossible. The derivation of this relation and a more accurate table of values for $A_{mc}$'s appeared later in a Doctoral Dissertation [ME]. This table is extended and given in Table 3.4-1. For example, from this table, for the inverse Chebyshev filter with n=9, we have $A_{mc}$=6.743005 nepers or 58.56900 decibels. Note that in Example 1 we have selected $A_{min}$=60 db which is larger than $A_{mc}$, hence Fujisawa's conditions will be satisfied for this selection. As we have seen, in the above example, the choice for $A_{min}$ was any value greater than 55 db. Therefore if $D/\varepsilon^2$ is selected, which is close to its lower bound $1.2\times10^6$, then, although the filter specifications are met, realizability of the filter in a ladder form would not be possible.

After obtaining the design parameters ε, D, and n either by direct computations using Eq's. (3.4-1), (3.4-2) and (3.4-4) or by the graphical method as in Example 1, the next step is to determine the zeros of the polynomials Q and H from the formulas (3.3-14) through (3.3-17). Therefore using either Method 1 or 2 given in the preceding section, the necessary data needed for the application of the algorithm established in Chapter II can be obtained. This constitutes the final step of the filter synthesis. In the following example, the design parameters obtained in Example 3.4-1 are utilized to realize the filter. In this example, Method 2 is used.

Example 3.4-2

Design parameters are:

$$\varepsilon^2 = 0.25, \quad D = 10^6, \quad \omega_a = 1.46$$

$$R_1 = R_2 = 1$$

Zeros, $r_i\angle\phi_i$, of $-Q_*$ are found by formulas (3.3-14) and (3.3-15):

| r | φ(radians) |
|---|---|
| 1.5391 | 0 |
| 1.4479 | ±0.48647 |
| 1.2742 | ±0.88387 |
| 1.1367 | ±1.1926 |
| 1.0678 | ±1.4500 |

Coefficients in the numerator polynomial of $Y_d$, i.e., Q + H, computed by formula (3.3-33), are:

| k | Coefficients for $s^k$ |
|---|---|
| 9 | 2.0000 |
| 8 | 6.8117 |
| 7 | 23.200 |
| 6 | 51.725 |
| 5 | 83.217 |
| 4 | 100.54 |
| 3 | 91.865 |
| 2 | 62.157 |
| 1 | 28.871 |
| 0 | 7.7164 |

Note that for $R_1=R_2$, $H(s)=s^n$ and, therefore, the coefficients in the denominator polynomial, Q - H, are identical to those of Q + H except that the term $s^9$ is missing. Hence, in this case ($R_1=R_2$) we have $Y_{1s}=Y_{2s}$ and the coefficients in the numerator and denominator polynomials of $Y_{1s}$ or $Y_{2s}$ are those given in the above table for k=8,6,4,2,0 and k=7,5,3,1, respectively. Attenuation poles, i.e., the zeros of F(s) computed from Eq. (3.3-21), are as follows:

| k | $\omega_k$(rad/sec) | $\zeta_k = 1/\omega_k^2$ |
|---|---|---|
| 1 | 1.4825 | 0.45499 |
| 2 | 1.6859 | 0.35185 |
| 3 | 2.2714 | 0.19383 |
| 4 | 4.2688 | 0.054878 |

Using the algorithm which includes the Fujisawa procedure for the functions $Y_{1s}$ and $Y_{2s}$ with the pole sequence in the order as computed above, $< \zeta_1, \zeta_2, \zeta_3, \zeta_4 >$, the computer arranged the pole sequence according to Fujisawa's procedure as $< \zeta_3, \zeta_1, \zeta_2, \zeta_4 >$ and gave the element values as listed below:

| | | | |
|---|---|---|---|
| $C_1 = 0.011771$ | $C_2 = 1.0420$ | $C_3 = 1.3817$ | $C_4 = 1.0804$ |
| $L_1 = 0.54464$ | $L_2 = 1.1726$ | $L_3 = 1.2585$ | $L_4 = 0.76581$ |
| $\Gamma_1 = 0.35589$ | $\Gamma_2 = 0.38801$ | $\Gamma_3 = 0.27958$ | $\Gamma_4 = 0.071660$ |
| $L_1\Gamma_1 = \zeta_3$ | $L_2\Gamma_2 = \zeta_1$ | $L_3\Gamma_3 = \zeta_2$ | $L_4\Gamma_4 = \zeta_4$ |
| $C_5 = 0.22564$ | | | |

Since the element values are all positive, the filter can be realized in a ladder form.

In the above example, although Newton's formula involves a large number of algebraic sum operations for n=9, in the 4-section filter computed with single precisions arithmetic (11 digits) both element values realized from $Y_{1s}$ and $Y_{2s}$ check up to at least 6 digits. The numerical values given in Example 3.4-2 are rounded to 5 digits.

## 3.5. The Group Delay Characteristics

In section 3.2 the following definitions are given:

$$\left(\frac{V_2}{V_{20}}\right)\left(\frac{V_2}{V_{20}}\right)_* = \frac{TT_*}{\alpha^2} = \frac{1}{\alpha^2}\left(\frac{F}{Q}\right)\left(\frac{F}{Q}\right)_* \tag{3.5-1}$$

where $V_{20}(s) = [E(s)R_2/(R_1 + R_2)]$, $\alpha^2 = 4R_1R_2/(R_1 + R_2)^2$, and $V_2(s)$ is the output voltage function and T(s) is the transmission function of the filter network as indicated in Fig. 3.2.1. For LC filters, F(s) is an even or odd polynomial and Q is a strictly Hurwitz polynomial. Since the filter network is a linear system, Q is the characteristic polynomial of the system. In other words, Q characterizes the transient response of the network. The zeros of Q are the natural frequencies which are complex valued and are located in the left half plane. A more explicit form for Q, in terms of the configuration and the element values of the filter have been established and will be given in the next chapter.

Two functions, the phase function B(ω) and the group delay function τ(ω), which are important in the study of transient response characteristic of transmission network, are defined as follows:

$$B(\omega) = \arg T(j\omega) = \arg \left.\frac{F}{Q}\right|_{s=j\omega} \tag{3.5-2}$$

$$\tau(\omega) = -\frac{dB(\omega)}{d\omega} \tag{3.5-3}$$

From Eq. (3.5-1) and Eq. (3.5-2) the phase function B(ω) can be expressed in terms of polynomial Q(jω). Indeed, since argT(jω) is an odd function of ω, we have,

$$B(\omega) = \arg T(j\omega) = -\arg T(-j\omega)$$

then for s=jω

$$2B(\omega) = \arg T - \arg T_*$$

$$2B(\omega) = \arg \frac{T}{T_*}$$

$$= \arg[ \left(\frac{F}{Q}\right) / \left(\frac{F}{Q}\right)_* ]$$

$$= \arg[ \frac{F}{F_*} \frac{Q_*}{Q} ] \qquad (3.5\text{-}4)$$

Since F is an even or odd polynomial, we have $F/F_* = \mp 1$ and

$$2B(\omega) = \arg \frac{Q_*}{Q}\bigg|_{s=j\omega} \qquad \text{for } F = \text{even}$$

$$2B(\omega) = -\arg \frac{Q_*}{Q}\bigg|_{s=j\omega} \qquad \text{for } F = \text{odd}$$

We note that for $s=j\omega$

$$\frac{T}{T_*} = \left|\frac{T}{T_*}\right| e^{j \arg(T/T_*)}$$

Substituting Eq. (3.5-4) into the last expression we have,

$$\frac{T}{T_*} = e^{j2B(\omega)}$$

or

$$j2B(\omega) = \ln \frac{T}{T_*} = \ln[ \left(\frac{F}{Q}\right) / \left(\frac{F}{Q}\right)_* ]$$

Therefore

$$B(\omega) = \pm \frac{1}{2j} \ln \frac{Q_*}{Q}\bigg|_{s=j\omega} \qquad (3.5\text{-}5)$$

where the positive or negative sign will be used according as F is even and odd, respectively.

From Eq. (3.5-5) $\tau(\omega)$ is obtained as follows:

$$\tau(\omega) = \mp \frac{1}{2j} \frac{dB(\omega)}{d\omega} = \mp \frac{1}{2j} \frac{d}{d\omega}[ \ln \frac{Q(-j\omega)}{Q(j\omega)} ]$$

$$= \mp \frac{1}{2} \frac{d}{dj\omega} [ \ln \frac{Q(-j\omega)}{Q(j\omega)} ]$$

$$= \mp \frac{1}{2} \frac{Q'Q_* - Q(Q_*)'}{QQ_*}$$

Since $(Q_*)' = -(Q')_*$, we further have

$$\tau(\omega) = \pm \frac{1}{2}\left[ \left(\frac{Q'}{Q}\right) + \left(\frac{Q'}{Q}\right)_* \right]_{s=j\omega} \tag{3.5-6}$$

or

$$\tau(\omega) = \text{Even part of } \left.\frac{Q'}{Q}\right|_{s=j\omega}$$

In what follows we shall consider only the low-pass filter where F is even and hence the upper sign of Eq. (3.5-6) is to be used. It is shown in section 3.3 that for inverse Chebyshev filters only the odd degree case is realizable in terms of low-pass ladders. This implies that the degree of Q is odd and, therefore, the characteristic polynomial Q has the following form

$$Q = (s - s_o) \prod_{j=1}^{n-1} (s - s_i) \tag{3.5-7}$$

where $s_o$ is on the negative real axis and the $s_i$'s are complex and occur in complex conjugate pairs. It can be shown that

$$\frac{Q'}{Q} = \sum_{i=0}^{n-1} \frac{1}{s - s_i} \quad \text{and} \quad \left(\frac{Q'}{Q}\right)_* = -\sum_{i=0}^{n-1} \frac{1}{s + s_i}$$

and hence

$$\tau(\omega) = \frac{s_o}{s^2 + s_o^2} + \sum_{i=1}^{n-1} \frac{s_i}{s^2 - s_i^2}$$

Since the $s_i$'s occur in complex conjugate pairs, letting $s_i = \sigma_i + j\omega_i$ and $\bar{s}_i = \sigma_i - j\omega_i$ we have

$$\left.\frac{s_i}{s^2 - s_i^2}\right|_{s=j\omega} = \frac{(\sigma_i + j\omega_i)(\omega_i^2 - \omega^2 - \sigma_i^2 + 2j\sigma_i\omega_i)}{(\omega_i^2 - \omega^2 - \sigma_i^2)^2 + 4\sigma_i^2\omega_i^2}$$

and

$$\left[ \frac{s_i}{s^2 - s_i^2} + \frac{\bar{s}_i}{s^2 - \bar{s}_i^2} \right]_{s=j\omega} = - 2\mathrm{Re} \frac{s_i}{\omega^2 + s_i^2}$$

$$= \frac{- 2\sigma_i(\omega_i^2 + \omega^2 + \sigma_i^2)}{(\omega_i^2 - \omega^2 - \sigma_i^2)^2 + 4\sigma_i^2\omega_i^2}$$

from which

$$\tau(\omega) = \frac{|s_o|}{\omega^2 + s_o^2} + 2 \sum_{i=1}^{(n-1)/2} \frac{|\sigma_i|(\omega^2 + \sigma_i^2 + \omega_i^2)}{(\omega^2 + \sigma_i^2 - \omega_i^2)^2 + 4\sigma_i^2\omega_i^2} \qquad (3.5\text{-}8)$$

Eq. (3.5-8) is a general formula, i.e., it is true for any low-pass filter with a characteristic polynomial Q as shown in Eq. (3.5-7). $\tau(\omega)$ is a positive function of $\omega$. The expression in Eq. (3.5-8) can also be obtained by applying the result proved by Gilbert [GI]. Indeed, if

$$T = A[ \prod^{M} (s - \tilde{s}_i)/ \prod^{N}(s - s_i) ]$$

where A is an arbitrary constant, $\tilde{s}_i = \tilde{\sigma}_i + j\tilde{\omega}_i$, and $s_i = \sigma_i + j\omega_i$ then, as Gilbert has shown, $\tau(\omega)$ can be written as

$$\tau(\omega) = \sum_{i=1}^{M} \frac{\tilde{\sigma}_i}{\tilde{\sigma}_i^2 + (\omega - \tilde{\omega}_i)^2} - \sum_{i=0}^{N} \frac{\sigma_i}{\sigma_i^2 + (\omega - \omega_i)^2} \qquad (3.5\text{-}9)$$

For filter networks, $\tilde{s}_i$ are the zeros of polynomial F all of which lie on the imaginary axis. This implies that the first summation on the right hand side of Eq. (3.5-9) is zero. The second summation, after combining the conjugate pairs, yields the expression in Eq. (3.5-8).

It is interesting to note that the m-th derivative of $\tau(\omega)$ is

$$\frac{d^m}{d\omega^m} \tau(\omega) \bigg|_{\omega=0} = \frac{m!}{2} \sum_{i=1}^{n} [ (-1)^m + 1 ]/s_i^{m+1}$$

i.e.,

$$\frac{d^m}{d\omega^m}\tau(\omega)\Bigg|_{\omega=0} = \begin{cases} 0 & \text{for } m = \text{odd} \\ m! \sum_{i=1}^{n} [\ 1/s_i^{m+1}\ ] & \text{for } m = \text{even} \end{cases}$$

This result can also be predicted from the fact that $B(\omega)=\arg T(j\omega)$ is an odd rational function. Thus the group delay curve is flat at $\omega=0$.

Another property of the group delay function is that it depends only on Q and Q' and is independent of the values of terminating resistances, $R_1$, $R_2$, of the filter.

For inverse Chebyshev filters, $\tau(0)$ can be obtained from Eq.(3.5-8) by letting $\omega=0$, $s_o = -\omega_a/a$, $\sigma_i^2 + \omega_i^2 = \omega_a^2/(a^2 + \sin^2\theta_i)$, and $\sigma_i = -\omega_a/a\cos\theta_i$ :

$$\tau(0) = \frac{1}{\omega_a}[\ a + 2 \sum_{i=1}^{(n-1)/2} \frac{a^2 + \sin^2\theta_i}{a\cos\theta_i}\ ] \qquad (3.5\text{-}10)$$

where $\omega_a$ and a are as defined in section 3.4 and $\theta_i=\pi/2n$.

Although for the inverse Chebyshev filter the explicit formulas for the zeros of Q exist, it is difficult to establish further analytical properties of $\tau(\omega)$. To sketch the $\tau(\omega)$ characteristics, $\tau(\omega_i)$'s with $\omega_{i+1} = \omega_i + \Delta\omega$ are computed for a particular set of parameters, D, $\omega_a$, and n. By fixing two of the parameters and varying the third parameter some properties of $\tau(\omega)$ are obtained which are listed below:

(A) $\tau(\omega)$ has a single peak in the vicinity of $\omega=1$ and is strictly increasing and decreasing before and after the peak respectively.

(B) For constant D and $\omega_a$, the peak increases and moves away from $\omega=0$ as the parameter n increases. The normalized delay curves (with respect to $\tau(0)$) sketched in Fig. 3.5.1, show clearly this property.

(C) When n and $\omega_a$ are kept constant and D is varying, changes similar to those described in (B) occur as D decreases, except that in this case the changes are smaller and the curves before the peaks do not change much. The normalized curves in Fig.3.5.2 show the property.

(D) When n and D are kept constant for different $\omega_a$'s, the peak values remain almost constant and the peak points move away from $\omega$=0 as $\omega_a$ increases.

It is also interesting to note that $\tau(0)$ does not change considerably over a wide ranges of n, D, or $\omega_a$, and the curves remain fairly flat up to the middle of the passband, i.e., $\omega$=0.5.

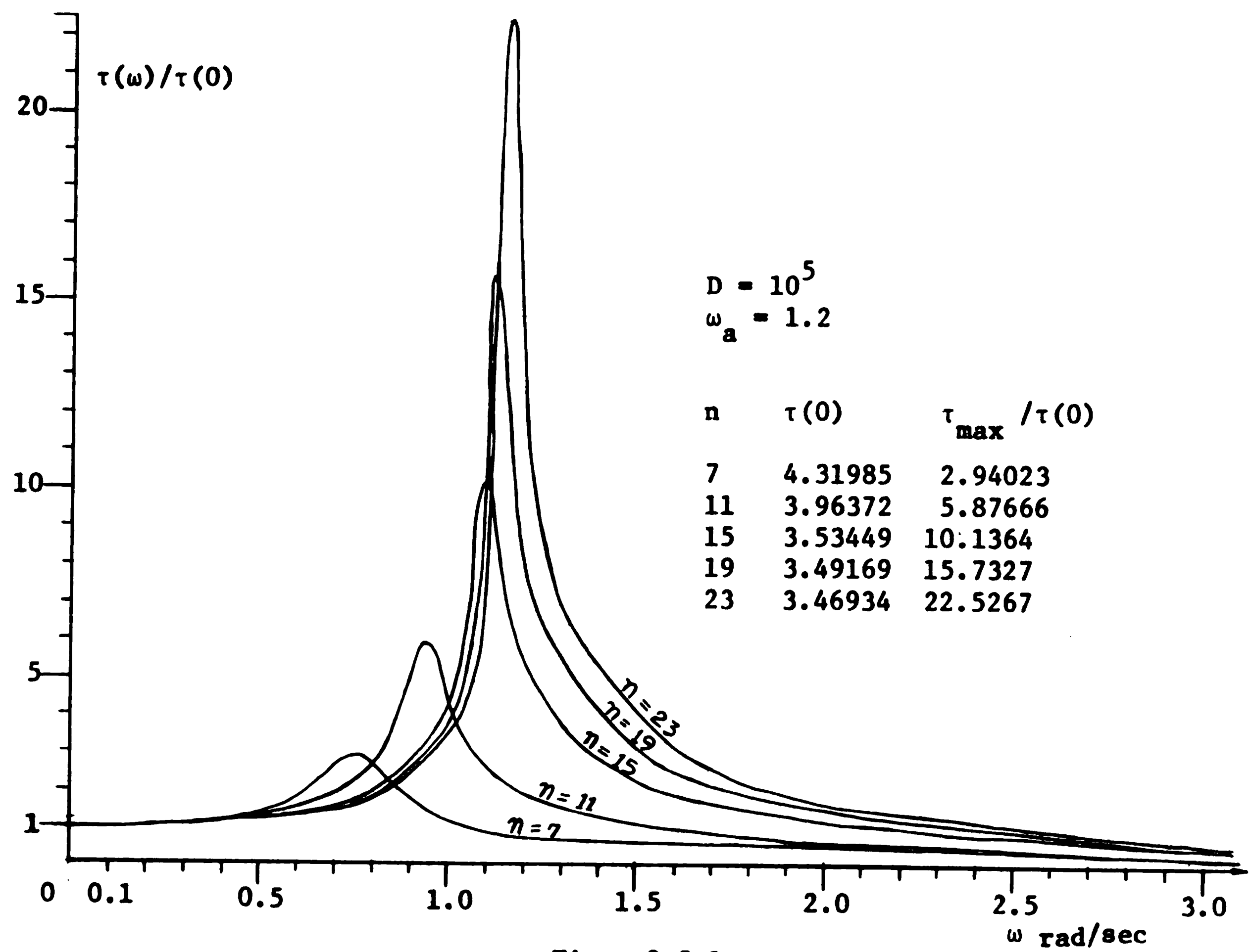



Fig. 3.5.1

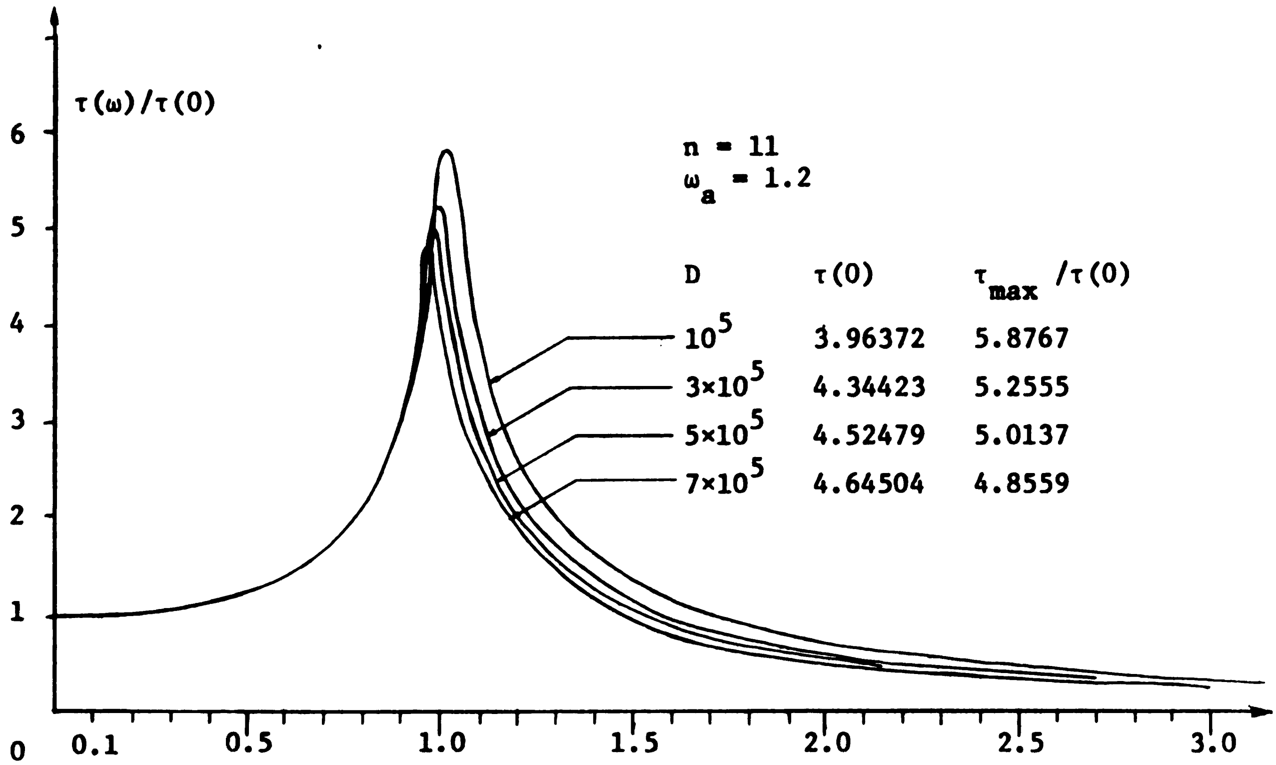



Fig. 3.5.2

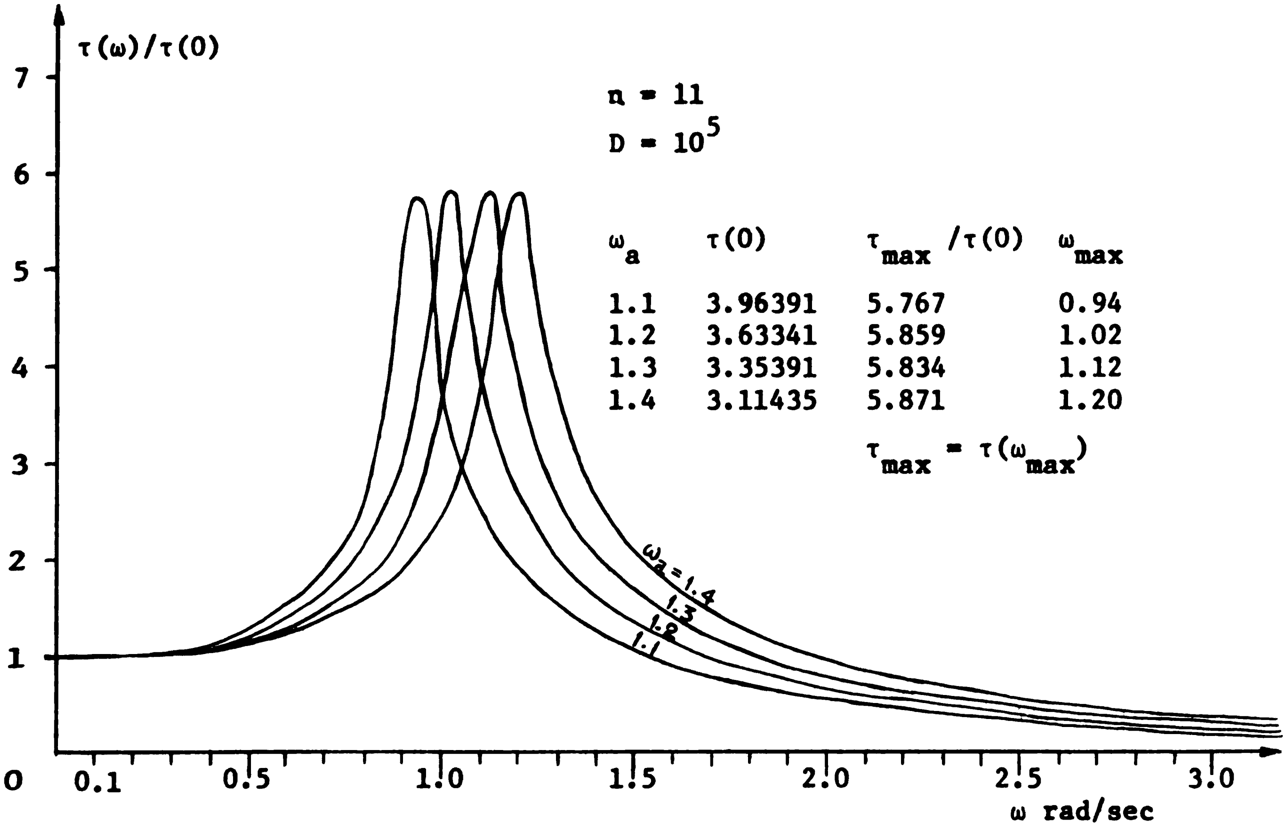



Fig. 3.5.3

# CHAPTER IV

# STATE EQUATIONS APPROACH TO THE SYNTHESIS OF LOW-PASS LC LADDER NETWORK

## 4.1. Introduction

In this chapter a new method for the realization of non-simple low-pass LC ladder networks, from the specified natural frequencies and the transmission zeros, is considered. Although only the mid-shunt configuration is used in this demonstration of the new approach, other cases may be treated in a similar manner.

The use of state equations in low-pass ladder network synthesis was first given by Marshall [MA1], [MA2]. The network considered by Marshall is the simple low-pass ladder shown in Fig.4.1.1 whose realization, by conventional methods, is straightforward. However the approach considered by Marshall suggests that it may also be used for non-simple ladder networks. This possibility is explored in the following sections. Here the essence of Marshall's method is briefly summarized. For the loss-less simple ladder, considering the system graph and the selected formulation tree (shown in heavy lines) in Fig.4.1.1, the state equations are as in Eq. (4.1-1) or in short hand notations as in Eq. (4.1-1a), where the eigenvalues of the matrix A' are the natural frequencies of the network and are assumed to be given. Applying a similarity transformation T to A', one can change it into the form of A in Eq. (4.1-2). For the synthesis, the companion matrix C as shown in Eq. (4.1-3) is written down directly and then by applying a similarity transformation, similar to Lanczos' algorithm [RA], C can be transformed into a similar matrix $\tilde{A}$ which has the same form as A. Of course, not any C can be

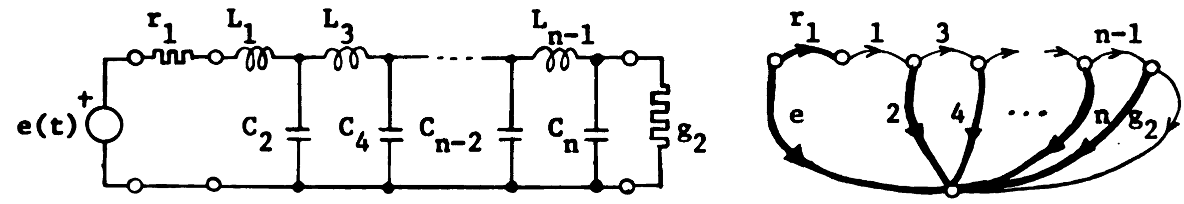



Fig. 4.1.1

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ v_2 \\ \vdots \\ i_{n-1} \\ v_n \end{bmatrix} = \begin{bmatrix} -r_1/L_1 & -1/L_1 & & & \\ 1/C_2 & 0 & -1/C_2 & & \\ & 1/L_3 & 0 & -1/L_3 & \\ & & \ddots & 0 & -1/L_{n-1} \\ & & & 1/C_n & -g_2/C_n \end{bmatrix}\begin{bmatrix} i_1 \\ v_2 \\ \vdots \\ i_{n-1} \\ v_n \end{bmatrix} + \begin{bmatrix} v_g/L_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

or (4.1-1)

$$\dot{\Psi} = A'\Psi + B \qquad (4.1\text{-}1a)$$

$$TA'T^{-1} = \begin{bmatrix} -r_1/L_1 & -1/L_1C_2 & & & \\ 1 & 0 & -1/C_2L_3 & & \\ & \ddots & \ddots & \ddots & \\ & & & 0 & -1/C_{n-2}L_{n-1} \\ & & & 1 & -g_2/C_n \end{bmatrix} = A \qquad (4.1\text{-}2)$$

where

$$T = \text{diag.}[L_1, L_1C_2, L_1C_2L_3, \ldots, L_1C_2\ldots C_n]$$

$$C = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_o \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & & \\ 1 & 0 & \ddots & \\ & \ddots & 0 & a_{n-1\,n} \\ & & 1 & a_{nn} \end{bmatrix} = \tilde{A}$$

(4.1-3)

transformed into the desired special form of $\tilde{A}$ in Eq. (4.1-3), since a certain set of conditions is not always satisfied. After obtaining $\tilde{A}$ one can obtain the element values by equating the corresponding upper diagonal elements of the matrices A and $\tilde{A}$. Note that $r_1$ and $g_2$ are assumed to be known.

For the non-simple ladders, considered in the following section, the state equations become much more complicated and the matrix corresponding to A for a non-simple ladder is no longer tridiagonal. Furthermore, in this general case, not only the natural frequencies but also the transmission zeros have to be realized. No general solution is obtained. However, a useful formulation of the problem is presented which may be useful for future research. Two possible approaches are suggested together with their applications to simple examples.

## 4.2. State Equations for Low-pass Mid-shunt Ladders

In this section the state equations for the full mid-shunt ladder network as shown in Case 1 of Table 4.2-1, are derived [KTK]. Then, from the state equations, the insertion voltage ratio function and hence the characteristic polynomial Q are obtained.

Based on the formulation tree and the orientations of the edges indicated on the system graph for the network in Table 4.2-1, the derivation can be carried out as follows. The component equations are:

$$\left[\begin{array}{ccc|ccc} L_2 & & & & & \\ & \ddots & & & & \\ & & L_{2n} & & & \\ \hline & & & C_1 & & \\ & & & & C_3 & \\ & & & & & \ddots \; C_{2n+1} \end{array}\right] \frac{d}{dt} \left[\begin{array}{c} i_2 \\ \vdots \\ i_{2n} \\ \hline v_1 \\ \vdots \\ v_{2n+1} \end{array}\right] = \left[\begin{array}{c} v_2 \\ \vdots \\ v_{2n} \\ \hline i_1 \\ \vdots \\ i_{2n+1} \end{array}\right] \qquad (4.2\text{-}1)$$

or with compact matrix notation

$$\left[\begin{array}{c|c} L & \\ \hline & C \end{array}\right] \dot{\Psi} = \overline{\Psi} \qquad (4.2\text{-}1a)$$

where L and C are diagonal submatrices indicated in Eq. (4.2-1) $\dot{\Psi}$ is the derivative of the state vector and $\overline{\Psi}$ represents the complementary vector of $\Psi$. In the following in order to maintain the simplicity, we shall not explicitly indicate the zero submatrices.

The fundamental circuit and cut-set equations, for the selected tree (indicated in heavy lines in the system graph)

$$\left[\begin{array}{c} v_2 \\ | \\ v_{2n} \\ \hline i_1 \\ | \\ i_{2n+1} \end{array}\right] = \left[\begin{array}{c|c} & \begin{matrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{matrix} \\ \hline \begin{matrix} -1 & & \\ 1 & \ddots & \\ & \ddots & -1 \\ & & 1 \end{matrix} & \end{array}\right] \left[\begin{array}{c} i_2 \\ | \\ i_{2n} \\ \hline v_1 \\ | \\ v_{2n+1} \end{array}\right] + \left[\begin{array}{c} \\ \hline \begin{matrix} -1 & & \\ 1 & \ddots & \\ & \ddots & -1 \\ & & 1 \end{matrix} \end{array}\right] \left[\begin{array}{c} i_{2'} \\ | \\ i_{2n'} \end{array}\right]$$

$$+ \left[\begin{array}{c} \\ \hline 1 \\ 0 \\ | \\ 0 \end{array}\right] i_{g_1} + \left[\begin{array}{c} \\ \hline 0 \\ | \\ 0 \\ -1 \end{array}\right] i_{g_2} \qquad (4.2\text{-}2)$$

By using the relations $i_{g_1} = (e - v_1)\times g_1$ and $i_{g_2} = v_{2n+2}g_2$ Eq. (4.2-2) can be written in the following form.

$$\overline{\Psi} = \left[\begin{array}{c|c} & -D^t \\ \hline D & \begin{matrix} -g_1 & & \\ & \ddots & \\ & & -g_2 \end{matrix} \end{array}\right] + \left[\begin{array}{c} \\ \hline D \end{array}\right] \left[\begin{array}{c} i_{2'} \\ | \\ i_{2n'} \end{array}\right] + g_1 \left[\begin{array}{c} \\ \hline 1 \\ 0 \\ | \\ 0 \end{array}\right] e \qquad (4.2\text{-}3)$$

On the other hand, from the component equations for $C_1$, $C_3$, ..., $C_{2n+1}$ and from Eq. (4.2-2) it follows that

$$\begin{bmatrix} i_{2'} \\ | \\ i_{2n'} \end{bmatrix} = \begin{bmatrix} C_2 & & \\ & \ddots & \\ & & C_{2n} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v_2 \\ | \\ v_{2n} \end{bmatrix}$$

$$= C' \frac{d}{dt} \begin{bmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ | \\ v_{2n+1} \end{bmatrix} \qquad (4.2\text{-}4)$$

or

$$\begin{bmatrix} i_{2'} \\ | \\ i_{2n'} \end{bmatrix} = -C'D^t \frac{d}{dt} \begin{bmatrix} v_1 \\ | \\ v_{2n+1} \end{bmatrix} \qquad (4.2\text{-}4a)$$

From Eq's. (4.2-1) to (4.2-4) we have

$$\left[\begin{array}{c|c} L & \\ \hline & C \end{array}\right] \dot{\Psi} = \left[\begin{array}{c|c} & -D^t \\ \hline D & \begin{matrix} -g_1 & \\ & -g_2 \end{matrix} \end{array}\right] \Psi + \left[\begin{array}{c} \\ \hline D \end{array}\right] C' \left[\begin{array}{c|c} & -D^t \end{array}\right] \Psi + \left[\begin{array}{c} \\ \hline g_1 \\ \\ \end{array}\right] e$$

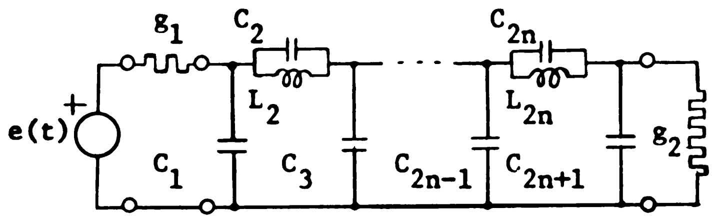
or finally the state equations are

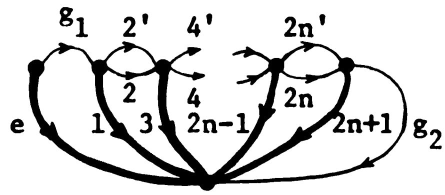
$$\left[\begin{array}{c|c} L & \\ \hline & C + DC'D^t \end{array}\right] \dot{\Psi}(t) = \left[\begin{array}{c|c} & -D^t \\ \hline D & \begin{matrix} -g_1 & \\ & -g_2 \end{matrix} \end{array}\right] \Psi(t) + \left[\begin{array}{c} \\ \hline g_1 \\ \\ \end{array}\right] e(t) \qquad (4.2\text{-}5)$$

or

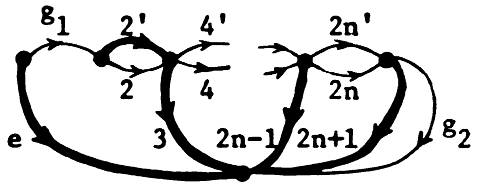
$$P\dot{\Psi}(t) = R\Psi(t) + Be(t) \qquad (4.2\text{-}5a)$$

TABLE 4.2-1



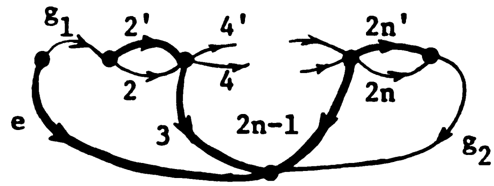




Case 1 $\quad \Psi^t = [i_2 \dots i_{2n} \mid v_1 \dots v_{2n+1}]$




Case 2 ($C_1 = 0$)

$\Psi^t = [i_2 \dots i_{2n} \mid v_2 v_3 v_5 \dots v_{2n+1}]$




Case 3 ($C_1 = C_{2n+1} = 0$)

$\Psi^t = [i_2 \dots i_{2n} \mid v_2 v_3 \dots v_{2n-1} v_{2n}]$

$$P\dot{\Psi}(t) = R\Psi(t) + Be(t), \quad P = \left[\begin{array}{c|c} L & \\ \hline & C + DC'D^t \end{array}\right] \quad R = \left[\begin{array}{c|c} & -D^t \\ \hline D & G \end{array}\right]$$

| | Case 1 | Case 2 | Case 3 |
|---|---|---|---|
| $L =$ | $\text{diag.}[L_2 \dots L_{2n}]$ | $\text{diag.}[L_2 \dots L_{2n}]$ | $\text{diag.}[L_2 \dots L_{2n}]$ |
| $C =$ | $\text{diag.}[C_1 \dots C_{2n+1}]$ | $\text{diag.}[C_2 C_3 C_5 \dots C_{2n+1}]$ | $\text{diag.}[C_2 C_3 \dots C_{2n-1} C_{2n}]$ |
| $C' =$ | $\text{diag.}[C_2 \dots C_{2n}]$ | $\text{diag.}[0, C_4 \dots C_{2n}]$ | $\text{diag.}[0, C_4 \dots C_{2n-2}, 0]$ |

$$D = \begin{bmatrix} -1 & & \\ 1 & \ddots & \\ & \ddots & -1 \\ & & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & & \\ 0 & -1 & & \\ & 1 & \ddots & \\ & & \ddots & -1 \\ & & & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & & & \\ 0 & -1 & & & \\ & 1 & \ddots & & \\ & & \ddots & -1 & 0 \\ & & & 1 & 0 \\ & & & 0 & -1 \end{bmatrix}$$

$$G = \begin{bmatrix} -g_1 & & \\ & & \\ & & -g_2 \end{bmatrix} \qquad \begin{bmatrix} -g_1 & -g_1 & & \\ -g_1 & -g_1 & & \\ & & & \\ & & & -g_2 \end{bmatrix} \qquad \begin{bmatrix} -g_1 & -g_1 & & \\ -g_1 & -g_1 & & \\ & & -g_2 & g_2 \\ & & g_2 & -g_2 \end{bmatrix}$$

$$B^t = [\ \ \vdots g_1 \quad ] \qquad [\ \ \vdots g_1, g_1 \quad ] \qquad [\ \ \vdots g_1, g_1 \quad ]$$

where $C + DC'D^t$ is tridiagonal, symmetric, strictly diagonally dominant and nonsingular. R is determined by the configuration of the network and is similar to the following tridiagonal matrix:

$$R \sim R_1 = \begin{bmatrix} -g_1 & -1 & & \\ 1 & 0 & \ddots & \\ & \ddots & 0 & -1 \\ & & 1 & -g_2 \end{bmatrix}$$

it can easily be shown that

$$\det.R = \det.R_1 = \begin{cases} g_1 g_2 + 1 & \text{for R of even order} \\ -(g_1 + g_2) & \text{for R of odd order} \end{cases}$$

Therefore, both P and R are nonsingular.

The submatrices in the state equations for the full mid-shunt network and two special cases (Case 2 and Case 3) are listed in Table 4.2-1. As can be seen from the table neither of the special cases are limiting cases of Case 1.

The Laplace transformation of Eq.(4.2-5a), with zero initial conditions, is

$$(Ps - R)\tilde{\Psi}(s) = BE(s) \tag{4.2-6}$$

Let $v_2(s) = v_{2n+1}(s)$ be the output voltage, and $J = [0,\dots,0,1]$ then,

$$v_2 = v_{2n+1} = J\tilde{\Psi}$$

or

$$v_2 = J(Ps - R)^{-1}BE$$

Hence the insertion voltage ratio function defined in section 3.2 takes on the form

$$\frac{v_{20}(s)}{v_2(s)} = \frac{r/(1 + r)}{J(Ps - R)^{-1}B} \tag{4.2-7}$$

where $r = g_1/g_2$

Note that the denominator of Eq.(4.2-7) can be considered as a matrix of order one and therefore,

$$J(Ps - R)^{-1}B = 1 + J(Ps - R)^{-1}B - 1$$

$$= \det.[\ 1 + J(Ps - R)^{-1}B\ ] - 1 \qquad (4.2\text{-}8)$$

Eq.(4.2-8) may be put into a more convenient form by using the following Lemma.

Lemma: Let the matrices A and B be of orders n×m and m×n, respectively. Then

$$|U_n \pm AB| = |U_m \pm BA| \qquad [\ |\cdot| = \det.(\cdot)\ ]$$

Proof Consider

$$K = \begin{bmatrix} U_n & -A \\ B & U_m \end{bmatrix}$$

then

$$|K| = \left| \begin{bmatrix} U_n & A \\ 0 & U_m \end{bmatrix} \begin{bmatrix} U_n & -A \\ B & U_m \end{bmatrix} \right| = \begin{vmatrix} U_n+AB & 0 \\ B & U_m \end{vmatrix} = |U_n + AB|$$

also

$$|K| = \left| \begin{bmatrix} U_n & -A \\ B & U_m \end{bmatrix} \begin{bmatrix} U_n & A \\ 0 & U_m \end{bmatrix} \right| = \begin{vmatrix} U_n & 0 \\ B & U_m+BA \end{vmatrix} = |U_m + BA|$$

hence

$$|U_n + AB| = |U_m + BA|$$

By using the above Lemma for n → 1 and m → 2n+1, Eq.(4.2-8) can be written as

$$J(Ps - R)^{-1}B = |U_{2n+1} + (Ps - R)^{-1}BJ| - 1$$

$$= |(Ps - R)^{-1}|\,|(Ps - R) + BJ| - 1$$

or

$$J(Ps - R)^{-1}B = (|Ps - R + BJ| - |Ps - R|)/|Ps - R| \qquad (4.2\text{-}9)$$

Substituting Eq.(4.2-9) into Eq.(4.2-7) one obtains

$$\frac{V_{20}}{V_2} = \frac{[r/(1 + r)]|Ps - R|}{|Ps - R + BJ| - |Ps - R|} \qquad (4.2\text{-}9)_*$$

or

$$\frac{V_{20}}{V_2} = \frac{|Us - P^{-1}R|}{\frac{1 + r}{r|P|}(|Ps - R + BJ| - |Ps - R|)} = \frac{Q}{F} \qquad (4.2\text{-}10)$$

where $|Us - P^{-1}R| = Q$ is the monic strictly Hurwitz polynomial defined in section 3.2. Note that $|Us - P^{-1}R|$ is the characteristic polynomial of the RLC network and its zeros are the natural frequencies. The denominator of Eq.(4.2-10) is the even or odd polynomial F defined in section 3.2, whose zeros are the transmission zeros of the filter network. Considering the explicit expressions of various matrices, it can be shown that,

$$|Ps - R + BJ| - |Ps - R| = -g_1 \prod_{k=1}^{n} (1 + L_{2k}C_{2k}s^2) \qquad (4.2\text{-}11)$$

Indeed the matrices [Ps - R + BJ] and [Ps - R] are the same except that the former has an extra element $g_1$ contributed by BJ. Therefore the difference of the two determinants can be combined into one determinant as follows:

$$
\left|\begin{array}{ccccc|cccccc}
L_2 s & & & & & -1 & 1 & & & & \\
& L_4 s & & & & & -1 & 1 & & & \\
& & \ddots & & & & & \ddots & \ddots & & \\
& & & L_{2n-2}s & & & & & -1 & 1 & \\
& & & & L_{2n}s & & & & & -1 & 1 \\
\hline
0 & & & & & 0 & 0 & & & & g_1 \\
-1 & 1 & & & & -C_2 s & a_1 s & -C_4 s & & & \\
& -1 & 1 & & & & -C_4 s & a_2 s & -C_6 s & & \\
& & \ddots & \ddots & & & & \ddots & \ddots & \ddots & \\
& & & -1 & 1 & & & & -C_{2n-2}s & a_{n-1}s & -C_{2n}s \\
& & & & -1 & & & & & -C_{2n}s & g_2+a_{2n}s
\end{array}\right|
$$

where $a_i = C_{2i} + C_{2i+1} + C_{2i+2} (i=1,2,\ldots,n-1)$ and $a_{2n} = C_{2n} + C_{2n+1}$. Expanding the above determinant with respect to its (n+1)-st row, it reduces to a new determinant with special properties which can be expressed as the product of factors indicated in Eq.(4.2-11). Therefore, the polynomials Q and F used in the synthesis based on the insertion-loss theory are now expressed in terms of the circuit parameters $C_i$, $L_i$, $g_1$ and $g_2$ appearing in the coefficient matrices P, R and G. It can be shown that the polynomial H involved in the insertion-loss theory is also related to P and R. In fact

$$H(s) = \frac{-1}{|P|}|Ps - R|_{g_1 \to -g_1} \qquad (4.2\text{-}12)$$

This relation can be obtained as follows:

Let

$$\mathfrak{J} = [1, 0 —— 0 \,\vdots\, (C_1 + C_3)s, -C_2 s, 0 —— 0]$$

Then the input current I(s) is given by

$$I(s) = \mathfrak{J}(Ps - R)^{-1}BE(s)$$

Since the input voltage is $V(s) = E(s) - I(s)/g_1$, the driving-point impedance takes on the form

$$Z_d = \frac{V}{I} = \frac{E - I/g_1}{I} = \frac{1}{\tilde{J}(Ps - R)^{-1}B} - r_1$$

from which the reflexion function is obtained as follows:

$$\Sigma = \frac{Z_d - r_1}{Z_d + r_1} = 1 - 2r_1\tilde{J}(Ps - R)^{-1}B \qquad (4.2\text{-}13)$$

By using the Lemma stated above, Eq.(4.2-13) becomes

$$\Sigma = \frac{|Ps - R - \tilde{B}\tilde{J}|}{|Ps - R|} = \frac{H|P|}{Q|P|} = \frac{H}{Q} \qquad (4.2\text{-}13a)$$

where $\tilde{B} = 2r_1B$. As can be seen from the expression of $\tilde{J}$ and $\tilde{B}$, the subtraction of $\tilde{B}\tilde{J}$ from Ps − R changes only one row of Ps − R. After multiplying that row by (-1), |Ps − R − BJ| becomes identical to |Ps − R| except for the sign of $g_1$.

## 4.3. Synthesis Methods

In this section two synthesis methods and examples are given. The main idea here is to use the explicit forms of Q and H obtained from the state equations in section 4.2. The first method involves the solution of a set of nonlinear equations and is applicable to networks with a small number of sections. The second method uses an approach similar to that of Marshall [MA1] except that, instead of the A-matrix, the inverse of the A-matrix is used. Both methods assume the existence of a mid-shunt ladder network with the specified eigenvalues and the transmission zeros.

### Method 1

Let

$$Q(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_o \qquad (4.3\text{-}1)$$

be the characteristic polynomial constructed from the given eigenvalues, and let $\zeta_i$ ($i=1,\dots,n/2$ or $(n-1)/2$) be the specified transmission zeros. Let

$$|Ps - R| = b_n s^n + b_{n-1}s^{n-1} + \dots + b_1 s + b_o = |P|Q(s) \qquad (4.3\text{-}2)$$

Note that $b_o = |Ps - R|_{s=o} = |-R|$ is equal to $g_1 + g_2$ when n is odd, or $-(1 + g_1 g_2)$ when n is even. That is, $b_o$ is known when $g_1$ and $g_2$ are specified. From Table 3.3-2 it is known that the number of coefficients $a_i$ in Eq.(4.3-1) plus the number of parameters $\zeta_i$ equals to the number of elements in the network. Since $b_1,\dots,b_n$ are functions of the element values, by letting

$$(b_o/a_o)Q(s) \equiv |Ps - R|$$

and defining $d_i = (b_o/a_o)a_i$, the element values can be obtained by solving the following set of equations

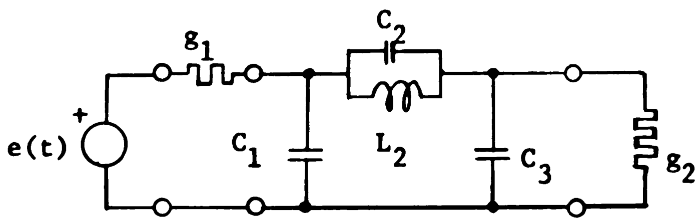
$$d_i = (b_o/a_o)a_i = b_i(C_1,C_2,\dots,C_{2n+1},L_2,L_4,\dots,L_{2n}) \qquad (i = 1,\dots,n) \qquad (4.3\text{-}3)$$

subject to the restriction that

$$L_{2i}C_{2i} = \zeta_i \qquad [i = 1,\dots,n/2 \text{ or } (n-1)/2] \qquad (4.3\text{-}4)$$

However, equations (4.3-3) and (4.3-4) are non-linear in nature and become more complicated as n increases. For a single section network when n=3, the matrices P and R are as shown in Fig. 4.3.1. The form of equations (4.3-3) and (4.3-4) for this case is:

$$\left.\begin{aligned} d_3 &= L_2[C_1C_3 + (C_1 + C_3)C_2] \\ d_2 &= L_2[C_1g_2 + C_3g_1 + (g_1 + g_2)C_2] \\ d_1 &= L_2g_1g_2 + C_1 + C_3 \\ \zeta &= L_2C_2 \end{aligned}\right\} \qquad (4.3\text{-}5)$$




$$P\Psi = R\Psi + Be$$

$$P = \left[\begin{array}{c|cc} L_2 & & \\ \hline & C_1+C_2 & -C_2 \\ & -C_2 & C_3+C_2 \end{array}\right] \qquad R = \left[\begin{array}{c|cc} & 1 & -1 \\ \hline -1 & -g_1 & \\ 1 & & -g_2 \end{array}\right]$$

Fig. 4.3.1

**In this case, letting $g_1 = g_2 = g$, the explicit formulas of the element value can be given as follows:**

$$L_2 = [d_1 \pm \sqrt{d_1^2 - 4g(d_2 - 2g\zeta)}]/(2g^2)$$

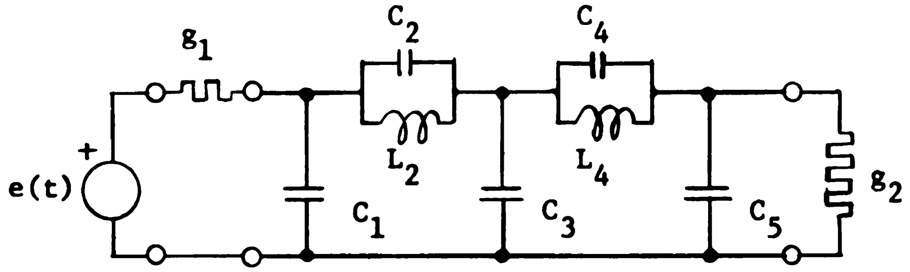
$$C_1 = [K \pm \sqrt{K^2 - 4(d_3 - K\zeta)/L_2}]/2$$

$$C_3 = K - C_1 \qquad (4.3\text{-}6)$$

**and**

$$C_2 = L_2/\zeta$$

**where** $K = d_1 - L_2g^2$




$$P = \left[\begin{array}{cc|ccc} L_2 & & & & \\ & L_4 & & & \\ \hline & & C_1+C_2 & -C_2 & \\ & & -C_2 & C_3+C_2+C_4 & -C_4 \\ & & & -C_4 & C_5+C_4 \end{array}\right] \qquad R = \left[\begin{array}{ccc|ccc} & & & 1 & -1 & \\ & & & & 1 & -1 \\ \hline -1 & & & -g_1 & & \\ 1 & -1 & & & & \\ & 1 & & & & -g_2 \end{array}\right]$$

Fig. 4.3.2

For the two-section network indicated in Fig. 4.3.2, equations (4.3-3) and (4.3-4) take on the following forms:

$$\left.\begin{aligned}
d_5 &= L_2L_4\{(C_1+C_2)[\,(C_3+C_2+C_4)(C_5+C_4) - C_4^2\,] - C_2^2(C_5+C_4)\} \\
d_4 &= L_2L_4\{(C_3+C_2+C_4)[\,g_1(C_5+C_4) + g_2(C_1+C_2)] - g_1C_4^2 - g_2C_2^2\} \\
d_3 &= L_2L_4g_1g_2(C_3+C_2+C_4) + L_2(C_1+C_2)(C_3+C_5+C_2) - \zeta_1C_2 \\
&\qquad + L_4C_5(C_1+C_3) + \zeta_2(C_1+C_3+C_5) \\
d_2 &= L_2g_2(C_1+C_2) + L_2g_1(C_3+C_5+C_2) + L_4(g_1C_5+g_2C_1+g_2C_3) + \zeta_2(g_1+g_2) \\
d_1 &= g_1g_2(L_2+L_4) + C_1 + C_3 + C_5 \\
\zeta_1 &= L_2C_2 \\
\zeta_2 &= L_4C_4
\end{aligned}\right\} \qquad (4.3\text{-}7)$$

As can be seen from the above set of equations, they are quite complicated even for two-section case. Although the analytical solution may be extremely difficult, the equations can be solved by some optimization methods.

Example 4.3-1

For a two-section inverse Chebyshev filter with $\omega_a = 1.36$, $D=5000$, and $g_1=g_2=1$, the coefficients of the characteristic polynomial Q(s) and the transmission zeros are given as follows:

$$a_5 = 1 \qquad \zeta_1 = 4.89029$$
$$a_4 = 3.11448 \qquad \zeta_2 = 1.86793$$
$$a_3 = 4.84535$$
$$a_2 = 4.72436$$
$$a_1 = 2.90675$$
$$a_o = 1.05276$$

Since n=5 is odd we have $b_o=g_1 + g_2=2$. Then from Eq.(4.3-3) the $d_i$ are obtained as follows:

$$d_5 = 2a_5/a_o = 1.89977$$
$$d_4 = 2a_4/a_o = 5.91678$$
$$d_3 = 2a_3/a_o = 9.20505$$
$$d_2 = 2a_2/a_o = 8.97518$$
$$d_1 = 2a_1/a_o = 5.52215$$

with these given specifications and the expressions in Eq.(4.3-7), one of the solutions obtained by optimization method [MC] yields the following element values:

$$C_1 = 0.308$$
$$C_2 = 0.400 \qquad L_2 = 1.22$$
$$C_3 = 1.93$$
$$C_4 = 0.121 \qquad L_4 = 1.53$$
$$C_5 = 0.527$$

which agree with the values realized by the method discussed in Chapter III.

Method 2

The idea involved in this method is based on the transformation from the companion matrix C, constructed from reciprocals of the given eigenvalues, to the inverse of the A-matrix. Since $A^{-1}$ has the general form of $A^{-1}=R^{-1}P$, and the form of P is known (see Table 4.2-1), the entries of $R^{-1}P$ are the linear combinations of the network elements. After transforming C into the desired form of $R^{-1}P$, the rest of the problem is to solve the linear system of equations. For the full mid-shunt network given as Case A in Table 4.2-1, let

$$R^{-1} = \begin{bmatrix} 0 & -D^t \\ D & G \end{bmatrix}^{-1} = \frac{-1}{g_1+g_2}\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

Then with a simple manipulation on matrices it can be shown that all the entries in $R_{11}$ are $g_1g_2$, all the entries in G are 1, and the entries in $R_{12} = -R_{21}^t$ are either $g_1$ or $-g_2$. For example if R is a 7×7 matrix, then

$$R^{-1} = \frac{-1}{g_1+g_2}\left[\begin{array}{ccc|cccc} g_{12} & g_{12} & g_{12} & -g_2 & g_1 & g_1 & g_1 \\ g_{12} & g_{12} & g_{12} & -g_2 & -g_2 & g_1 & g_1 \\ g_{12} & g_{12} & g_{12} & -g_2 & -g_2 & -g_2 & g_1 \\ \hline g_2 & g_2 & g_2 & 1 & 1 & 1 & 1 \\ -g_1 & g_2 & g_2 & 1 & 1 & 1 & 1 \\ -g_1 & -g_1 & g_2 & 1 & 1 & 1 & 1 \\ -g_1 & -g_1 & -g_1 & 1 & 1 & 1 & 1 \end{array}\right] \tag{4.3-8}$$

where $g_{12} = g_1g_2$.

From the general form of $R^{-1}$ given in Eq.(4.3-8) and the form of P for Case 1 in Table 4.2-1, it will be evient that the submatrices of $R^{-1}P$ have some useful properties. Indeed, if we let

$$R^{-1}P = \frac{-1}{g_1+g_2}\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} L & 0 \\ 0 & C+DC'D^t \end{bmatrix}$$

$$= \frac{-1}{g_1+g_2}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

then it follows that $A_{11}$ has identical rows each of which is equal to $(g_{12}L_2, g_{12}L_4, \ldots, g_{12}L_{2n})$. Similarily $A_{22}$ has identical rows each of which has the form $(C_1, C_3, \ldots, C_{2n+1})$. Since $A_{21}$ is equal to $R_{21}L$, it also has a very simple form. $C+DC'D^t$ is tridiagonal and each element of $A_{21}=R_{22}(C+DC'D^t)$ is a linear combination of $C_{2i}$, $C_{2i+1}$, and $C_{2i+2}$. For example, for the single section network given in Fig.4.3.1 we have

$$R^{-1}P = \frac{-1}{g_1+g_2}\left[\begin{array}{c|cc} g_1g_2 & g_2 & -g_1 \\ \hline -g_2 & 1 & 1 \\ g_1 & 1 & 1 \end{array}\right]\left[\begin{array}{c|cc} L_2 & 0 & 0 \\ \hline 0 & C_1+C_2 & -C_2 \\ 0 & -C_2 & C_3+C_2 \end{array}\right]$$

$$= \frac{-1}{g_1+g_2}\left[\begin{array}{c|cc} g_1g_2L_2 & g_2C_1+(g_1+g_2)C_2 & -g_1C_3-(g_1+g_2)C_2 \\ \hline -g_2L_2 & C_1 & C_3 \\ g_1L_2 & C_1 & C_3 \end{array}\right] \qquad (4.3\text{-}9)$$

where $g_1$, $g_2$ and $\zeta = L_2C_2$ are known.

Let

$$C = \begin{bmatrix} 0 & 0 & -d_o \\ 1 & 0 & -d_1 \\ 0 & 1 & -d_2 \end{bmatrix}$$

where the entries $d_o$, $d_1$ and $d_2$ are the coefficients of the characteristic polynomial and are known. The problem now is to find a similarity transformation that transforms C into a matrix which satisfies the constraints imposed by the forms of the submatrices $A_{11}$, $A_{21}$, and $A_{22}$.

Example 4.3-2

Let the companion matrix constructed from the given eigenvalues be

$$C = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

Further, let $\zeta = 1$ and $g_1 = g_2 = 1$. Since the order of C is 3, from Eq. (4.3-9) we have

$$R^{-1}P = -\frac{1}{2}\left[\begin{array}{c|cc} L_2 & C_1+2C_2 & -(C_3+2C_2) \\ \hline -L_2 & C_1 & C_3 \\ L_2 & C_1 & C_3 \end{array}\right]$$

The following transformation yields the desired form for $A^{-1}$:

$$TCT^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1/2 & -1/2 \\ 0 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ -3/2 & 0 & 1 \\ -1/2 & 0 & 1 \end{bmatrix}$$

$$= -\frac{1}{2}\left[\begin{array}{c|cc} 2 & 2 & -2 \\ \hline -2 & 1 & 1 \\ 2 & 1 & 1 \end{array}\right] = A^{-1}$$

From the relations, $A^{-1} = R^{-1}P$ and $\zeta = L_2C_2 = 1$, element values are obtained as

$$C_1 = 1, \quad C_2 = 1/2, \quad L_2 = 1, \quad C_3 = 1$$

Note that the above example is constructed from a network with assigned element values. The transformation then is obtained by going from $R^{-1}P$ to C by using Krylov's method [FR] with initial vector col.[0, 1, 0]. So far no simple way of determining the transformation matrix is found. This is left as an open question.

# CHAPTER V

## CONCLUSIONS

The main contribution of this thesis is the establishment of an efficient method for the computation of element values for mid-series or mid-shunt LC low-pass ladder networks. This is accomplished by the introduction of an algorithm in Chapter II, which is based on classical synthesis procedure.

The synthesis procedure given in Chapter III utilizes the algorithm established in Chapter II. This chapter can be considered as the application of the algorithm, as well as a general study of the inverse Chebyshev filters.

A different approach which utilizes the state equations for the realization of mid-shunt LC low-pass ladder network is discussed in Chapter IV. Some properties of the coefficient matrices in the state equations and their connections to the functions which appear in the insertion loss theory are established. Two possible synthesis methods are suggested. However, the general realizability conditions and the synthesis procedures are left for future work.

# APPENDIX I

<u>Proof of the Property:</u>

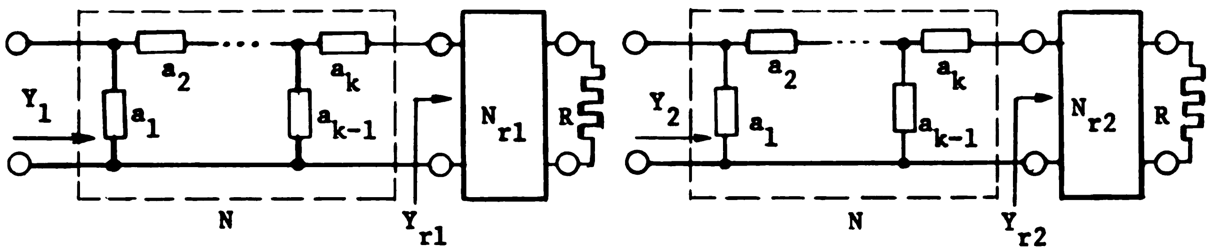



Fig. A-I

Consider the ladder networks in Fig. A-I. Then using the bracket-symbol or continuant notation of Stieltjes[ST], $Y_1$ and $Y_2$ can be written as follows:

$$Y_1 = \frac{[a_1,a_2,a_3,\ldots,a_k,Y_{r1}]}{[a_2,a_3,\ldots,a_k,Y_{r1}]}, \quad Y_2 = \frac{[a_1,a_2,a_3,\ldots,a_k,Y_{r2}]}{[a_2,a_3,\ldots,a_k,Y_{r2}]} \tag{A-1}$$

From these expressions the proof of sufficiency is evident. Proof of necessity follows from the following property of continuants

$$[a_1,\ldots,a_k][a_2,\ldots,a_{k-1}] - [a_1,\ldots,a_{k-1}][a_2,\ldots,a_k] \equiv (-1)^k \tag{A-2}$$

which can be found in Muir and Metzler [MU]. If the identity in Eq.(A-2) is abbreviated as $AD-BC \equiv (-1)^k$, then equating Eqs.(A-1) and expanding the continuants we have

$$\frac{AY_{r1} + B}{CY_{r1} + D} = \frac{AY_{r2} + B}{CY_{r2} + D}$$

This relation, together with the identity in Eq. (A-2) yields $Y_{r1} \equiv Y_{r2}$.

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