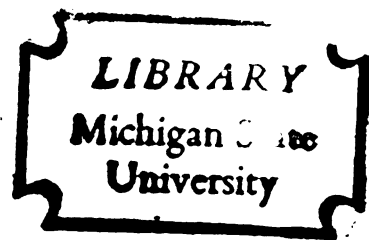


MODIFICATION OF
BACKSCATTERING OF A LOOP BY IMPEDANCE LOADING

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
JUANG-LU LIN
1967



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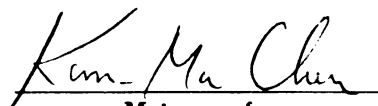
Modification of Backscattering
of a Loop by Impedance Loading

presented by

Juang-Lu Lin

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Major professor

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ABSTRACT

MODIFICATION OF BACKSCATTERING OF A LOOP BY IMPEDANCE LOADING

by Juang-Lu Lin

In this thesis the modification of the backscattering of a loop by the impedance loading method is investigated theoretically and experimentally.

First of all, the resonant phenomena of a loop when illuminated by an electromagnetic wave is studied. Next, the backscattering of a circular conducting loop is considered and the impedance loading method is applied to minimize its backscatter. Finally, the minimization of the backscattering of a rectangular loop is investigated.

In the theoretical analysis, a new method based on a differential equation for the loop current is developed for the case of a circular loop. For the case of a rectangular loop an existing method is applied with slight modification. Throughout the entire analysis, the principle of superposition is applied to simplify the problem of an illuminated loaded loop to the combination of an illuminated solid loop and a radiating loop. The major objectives of the theoretical analysis are to determine the induced current on a loaded loop, the scattered field from a loaded loop and the optimum impedance which, when loaded on the loop, makes the backscatter of the loop vanish.

Extensive experimental study was also conducted. The resonant phenomena of a solid loop was investigated carefully. The impedance loading technique was then applied to the circular and the rectangular

loops to minimize their backscatters. In the course of this experiment, the backscatter of a loop was successfully minimized to the noise level.

The major finding of this study is to show theoretically that the backscatter of a conducting loop can be eliminated by a properly chosen impedance loading, and to verify experimentally that indeed the backscatter of a loop can be minimized by a practical arrangement of the impedance loading technique. This investigation should prove significant in the understanding of the scattering phenomena of a conducting loop and in the practice of radar camouflage.

MODIFICATION OF BACKSCATTERING OF
A LOOP BY IMPEDANCE LOADING

By

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INTRODUCTION

In recent years much research has been conducted on reducing the radar cross sections of metallic objects. The conventional techniques include the use of radar absorbing material and the method of reshaping the body. Recently the impedance loading method has been found to be especially effective in reducing the radar cross sections of metallic objects with dimensions of the order of a wavelength. In this thesis, the minimization of the backscattering of a conducting loop, both circular and rectangular, by an impedance loading method is investigated.

Historically, Kouyoumjian¹ and Weston² are the first ones who investigated the backscattering cross sections of a thin solid circular loop with a result of fairly good agreement with the experiment. The methods employed by them are quite complicated and involved tedious computation. For our purpose, a mathematically simpler method has been developed. Our theoretical results also agree with experimental results quite satisfactorily. For the case of the solid rectangular loop, the method used by Chen and King³ was adopted in obtaining first-order solution for current distribution. Experimental results and those of theory appear to be in good agreement.

In solving the problem, the principle of superposition is employed. Based on the superposition principle, an illuminated loaded loop can be considered as the combination of an illuminated solid loop and a radiating loop. In this study, the two cases are solved separately and then combined together to yield the final solution for an illuminated loop.

For the minimization of the backscattering of a circular loop, a consideration is given to the perfectly conducting loop which is loaded symmetrically with two identical lumped impedances and is assumed to be illuminated by a plane wave at normal incidence. The induced current on the loaded loop is determined as a function of loop dimensions and loading impedance. The backscattered field produced by the loaded circular loop is calculated as a function of the loading impedance among other parameters. It is then possible to find an optimum impedance which makes the back scattered field equal to zero. An explicit expression for the optimum impedance for zero backscattering is obtained as a function of loop dimensions. Some numerical examples are included. The theory is later verified by an experiment.

A theory on the minimization of the backscatter of a conducting rectangular loop by the impedance loading method is developed in the last chapter. A rectangular loop loaded symmetrically with two identical lumped impedances at the centers of the long sides of the loop is assumed to be illuminated by a plane wave at normal incidence. The zeroth order and the first-order solutions of the induced current are evaluated as functions of the loop dimensions and the loading impedance. Based on the induced current, the backscattered field can be calculated. An optimum impedance which leads to zero backscattering from the rectangular loop is then found as a function of loop dimensions. Some numerical examples are included. An experiment was performed for the case of a square loop loaded with two identical reactive impedances and illuminated by a plane wave with vertical polarization at normal incidence. The experimental results indicate that it is possible to reduce the backscatter of a conducting, square loop to the noise level if the loading impedance is properly chosen.

CHAPTER I

BACKSCATTERING OF A SOLID LOOP

1.1. Definition of Backscattering Cross Section

An electromagnetic wave incident upon a metallic object will induce time-varying distributions of oscillating charges and currents in the object. The induced charges and currents will, in turn, maintain an electromagnetic field which is known as the scattered field. To characterize the reradiating properties of an object, total scattering cross section σ is defined as

$$\sigma = \frac{P^s}{S^i}$$

Where P^s is the total scattered power and S^i is the scalar magnitude of the real part of the incident complex Poynting vector at the location of the object. In radar, transmitter and receiver are installed at the same location so that only the power scattered by the object in the direction of the transmitter-receiver is observed. In this sense, backscattering cross section of an object is defined as a measure to characterize the quantity of the power scattered back toward the source by the object. In symbol, the backscattering cross section of an object is defined as

$$\sigma_B = \lim_{R \rightarrow \infty} 4\pi R^2 \frac{|E^s|^2}{|E^i|^2}$$

where E^s is the magnitude of the backscattered field and E^i is that of the incident electric field.

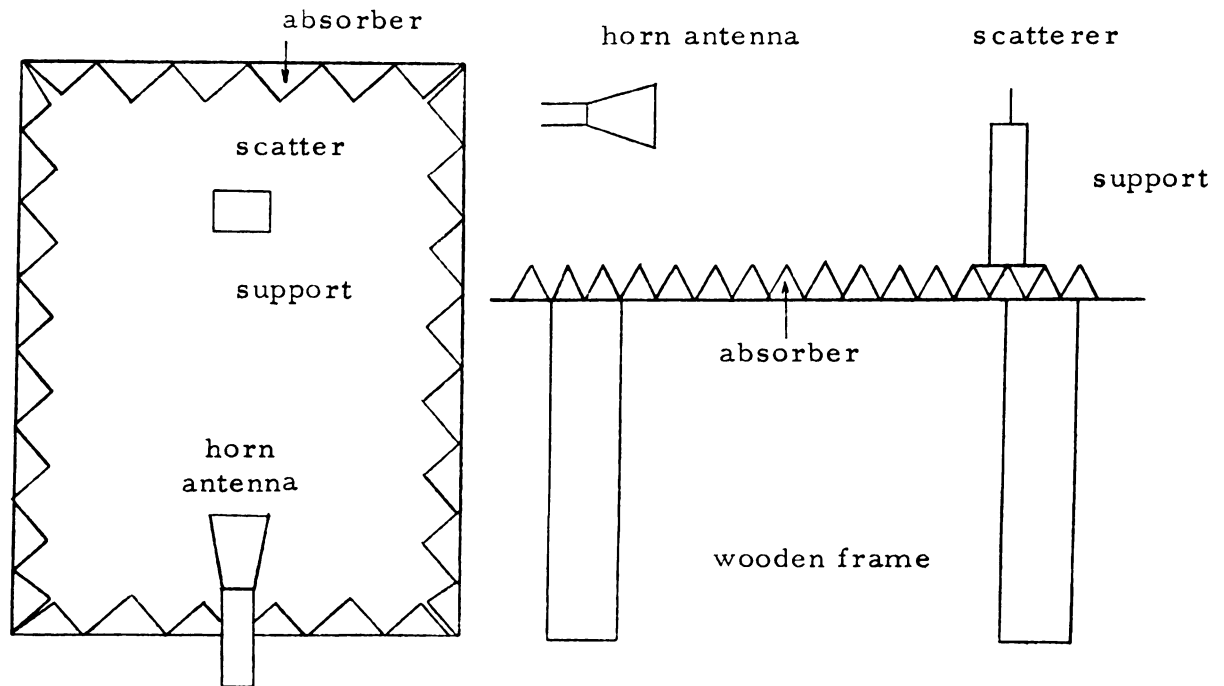
1.2. Experiment

A number of methods are available for measuring the back-scattering cross section of an object. In this research the cancellation method was used. The principle of this method is to cancel the signal received by the receiving antenna when the scatterer is absent. After this the scatterer is put in place. The signal received by the receiving antenna with the scatterer present will then represent the scattered field by the scatterer.

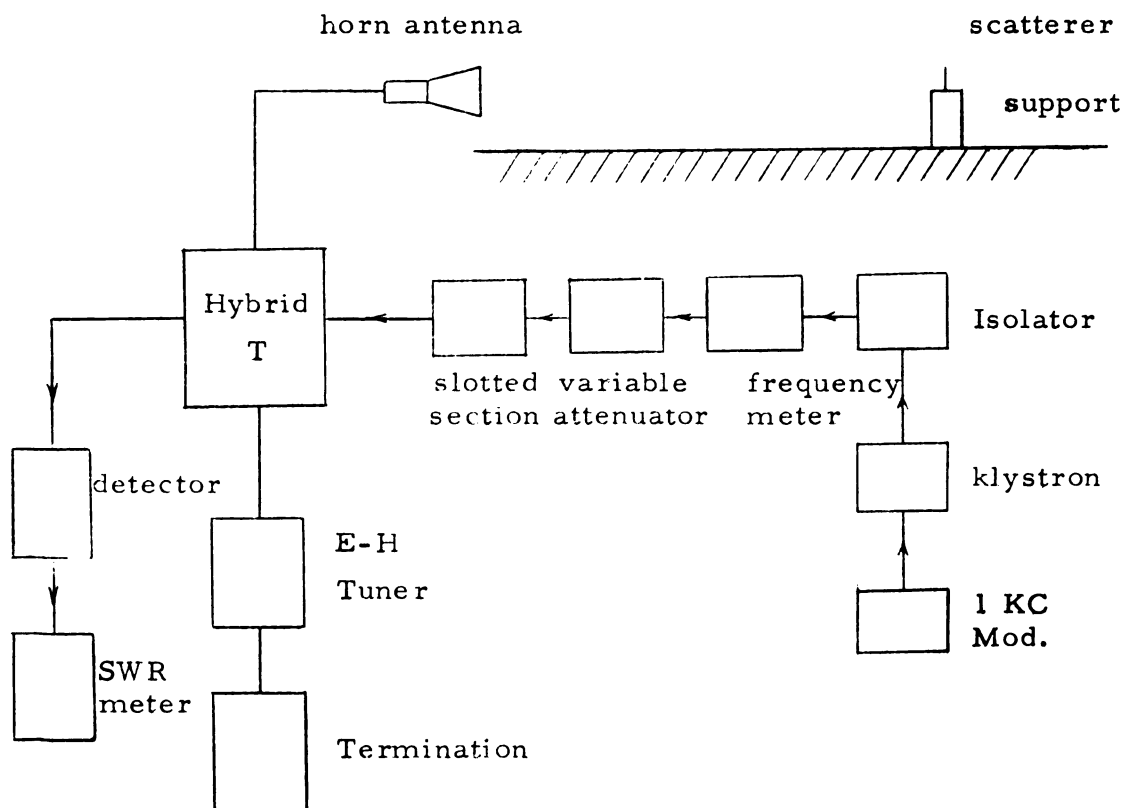
In this section experimental arrangement and the measuring technique of the backscattering cross section of rectangular and circular loops are discussed. The loops are illuminated by a plane wave for the cases of normal and horizontal incidence. Resonant phenomena of loops are also investigated.

1.2.1. Description of the Experimental Arrangement and the Measuring Technique

The experimental setup is shown in Figure 1.1. The experiment was conducted inside of an anechoic chamber (0.8 m x 1.4 m x 0.7 m in size) which is firmly mounted on a wooden frame. A horn antenna (HP X890A) is projected into the anechoic chamber through one of the narrow sides of the chamber. A cellular plastic column made of styrofoam is used as scatterer support. The support is placed in the far zone of the horn antenna and its height is so chosen that the scatterer on the support is on the axis of the antenna. The circuit diagram of the experiment is also shown in Figure 1.1. The klystron generator (F X R type X760A) modulated by a 1 KC square wave generator (HP 715A) is used as the R.F. source. The isolator (Polytechnic Res. and Dev. Co. type 1203) is used to protect the possible



Experimental Setup



Circuit Diagram

Figure 1.1. Experimental setup and circuit diagram.

backward energy from damaging the R.F. generator. Frequency is measured by a frequency meter (HP X532A). A variable attenuator (HP X375A) and slotted section (HP X810B) are used to control and probe the wave from the source. A hybrid T (HP X845A) is used to separate the incident wave to the antenna and the reflected wave back from the antenna. Two other terminals of the Hybrid T are connected to a matched load through an E-H tuner (HP X880A) and to a detector. The output of the detector is then measured by a SWR meter (HP 415B). The horn antenna in this experiment serves both as the radiating element and the receiving probe. In the experiment, a series of measurements have been made with various sizes of loops.

By the arrangement mentioned above, a plane wave with vertical polarization can be made to illuminate the loop either normally or horizontally. When the scatterer is absent, the reading of the SWR meter can be set to zero by adjusting the E-H tuner. After this balancing process is completed, the scatterer is introduced. The reading of the SWR meter will then indicate the back scattered field by the scatterer.

1.2.2. Measurement of the Backscattering Cross Section of Circular Loops

To measure the backscattering cross section of metallic circular loops, a total of 35 circular loops with radius ranging from 0.3 cm to 3.5 cm were constructed as experimental models. The diameter of the wire is 0.1 cm. The frequency of 9.61 GC was used. The loops are mounted upright on the support which is located 26 cm ($\approx 8\lambda$) away from the horn antenna. The loops are, therefore, considered to be in the far zone of the radiating element. Theoretically,

the distance between the antenna and the scatterer should be made as great as possible. However, to obtain a detectable scattered field from the scatterer one has to compromise for a finite distance. The radiated wave from the horn antenna at the location of the scatterer is then approximately a plane wave with a vertical E field.

In the experiment the plane of the loop is first oriented parallel to the horn aperture so that the loop is illuminated by a plane wave with vertical polarization at normal incidence. For the case of horizontal incidence, the loop is turned 90° from the vertical incidence position making its plane perpendicular to the horn aperture. The geometries of these two cases are shown in Figure 1.2. In Figure 1.3 and 1.4, relative backscattering cross section of the circular loops σ are plotted as functions of loop dimension $\beta_0 b$ where β_0 is the wave number and b is the radius of the loop. Figure 1.3 shows the backscattering cross section of the circular loop illuminated by a plane wave at normal incidence and Figure 1.4 shows that of horizontal incidence case. It is observed from these two figures that the first resonances for both cases occur when the half of the loop periphery is near $\frac{1}{2}$ wavelength (i.e., when $\beta_0 b = 1$). The first resonant peak in the case of normal incidence appears to be higher than that of the horizontal incidence case.

1.2.3. Measurement of the Backscattering Cross Section of Square Loops

The same technique used for the circular loops was adopted to measure the backscattering cross sections of the square loops. A number of square loops ranging from 0.5 cm x 0.5 cm to 3.5 cm x 3.5 cm in size were constructed as experimental models. The diameter of the metallic wire is 0.06 cm. The experiment was

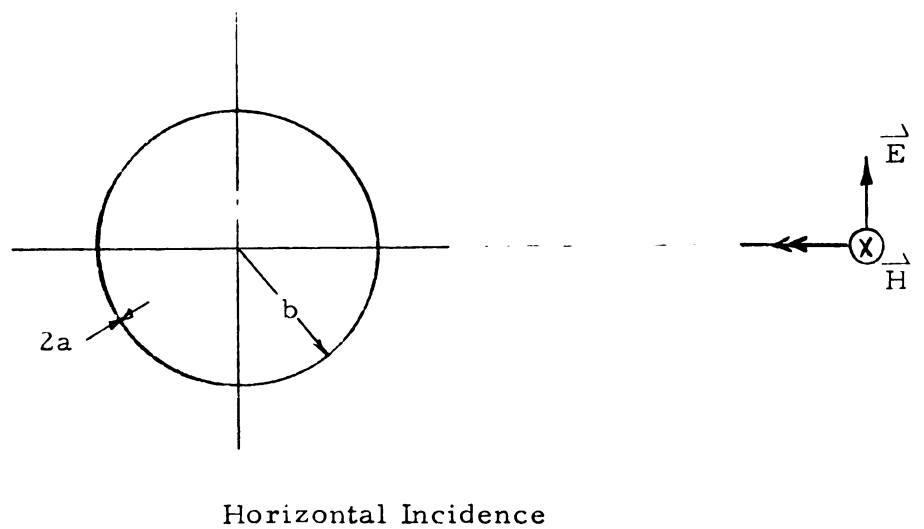
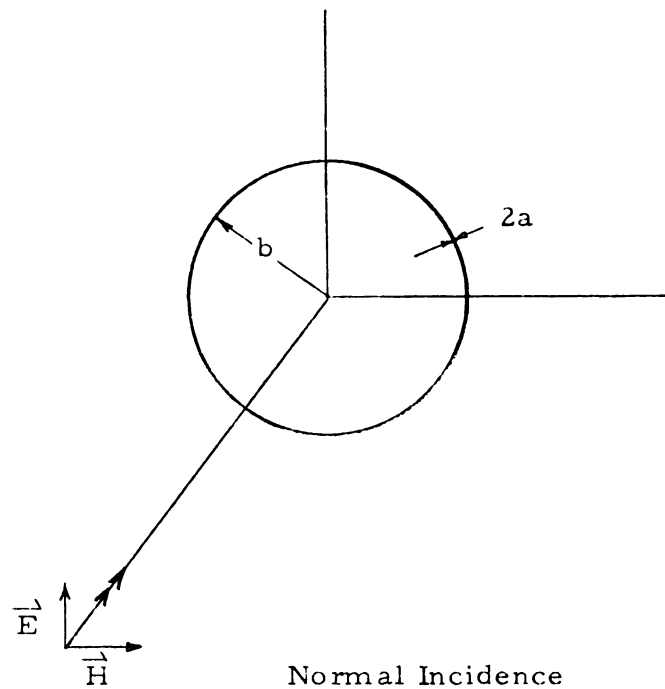


Figure 1.2. A circular loop is illuminated by a plane wave at either normal or horizontal incidence.

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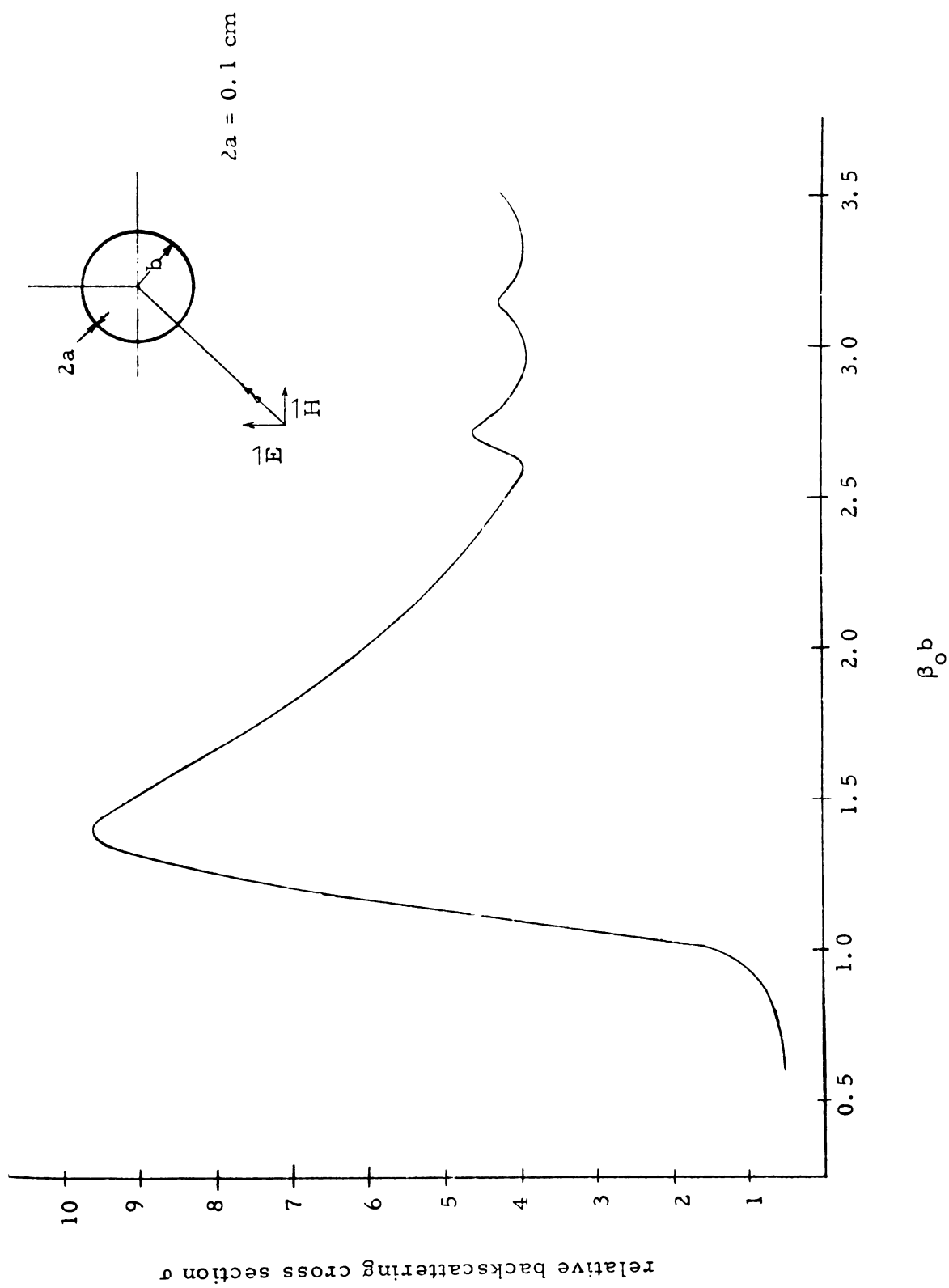


Figure 1.3. Backscattering cross section of circular loops illuminated by a plane wave at normal incidence.

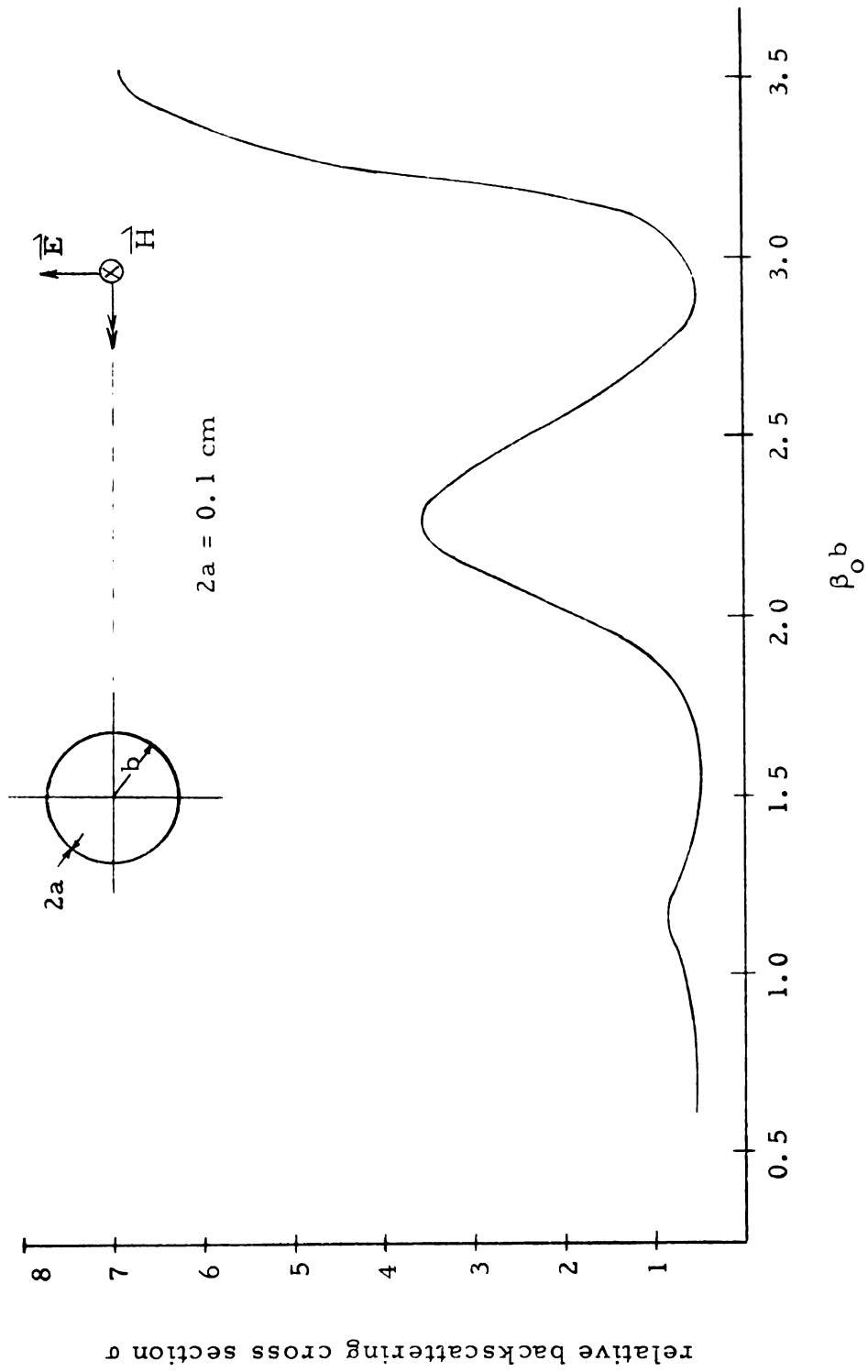


Figure 1.4. Backscattering cross section of circular loops illuminated by a plane wave at horizontal incidence.

conducted with the same setup described in previous sections and at the same frequency.

The backscattering cross sections of square loops are plotted as functions of loop dimensions $\beta_0 h$ (where $2h$ = one side of the square loop) in Figures 1.5 and 1.6. Figure 1.5 shows the backscattering cross section of square loops illuminated by the plane wave at normal incidence, and Figure 1.6 shows the backscattering cross section for the case of horizontal incidence.

Once again it is observed that the first resonances of both normal and horizontal cases occur when half of the loop periphery is near $\frac{1}{2}$ wavelength (i. e., when $\beta_0 h = 0.8$).

1.2.4. Measurement of the Resonances of Loops

The backscattering cross sections of a circular and a square loop described in the last two sections have a common nature that the first resonance tends to occur when the half of the loop periphery is near $\frac{1}{2}$ wavelength and the second resonance at $\frac{3}{2}$ wavelength and so on. At these resonant conditions the induced currents on the loops are maximum and likewise the scattered fields. However, it is anticipated that the loops with the same periphery but different geometrical shapes may have different resonant sizes. To testify these conjectures five kinds of rectangular loops were constructed. The five kinds of rectangular loops have the ratio of h_1/h_2 equal to 1, 1.5, 2.0, 2.5, and 3.0 where h_1 and h_2 are the short and the long side of the loop. The loops are illuminated by a plane wave with vertical polarization at normal incidence. The experiment was conducted inside of an anechoic chamber at the frequency of 9.61 GC with the same experimental setup mentioned previously.

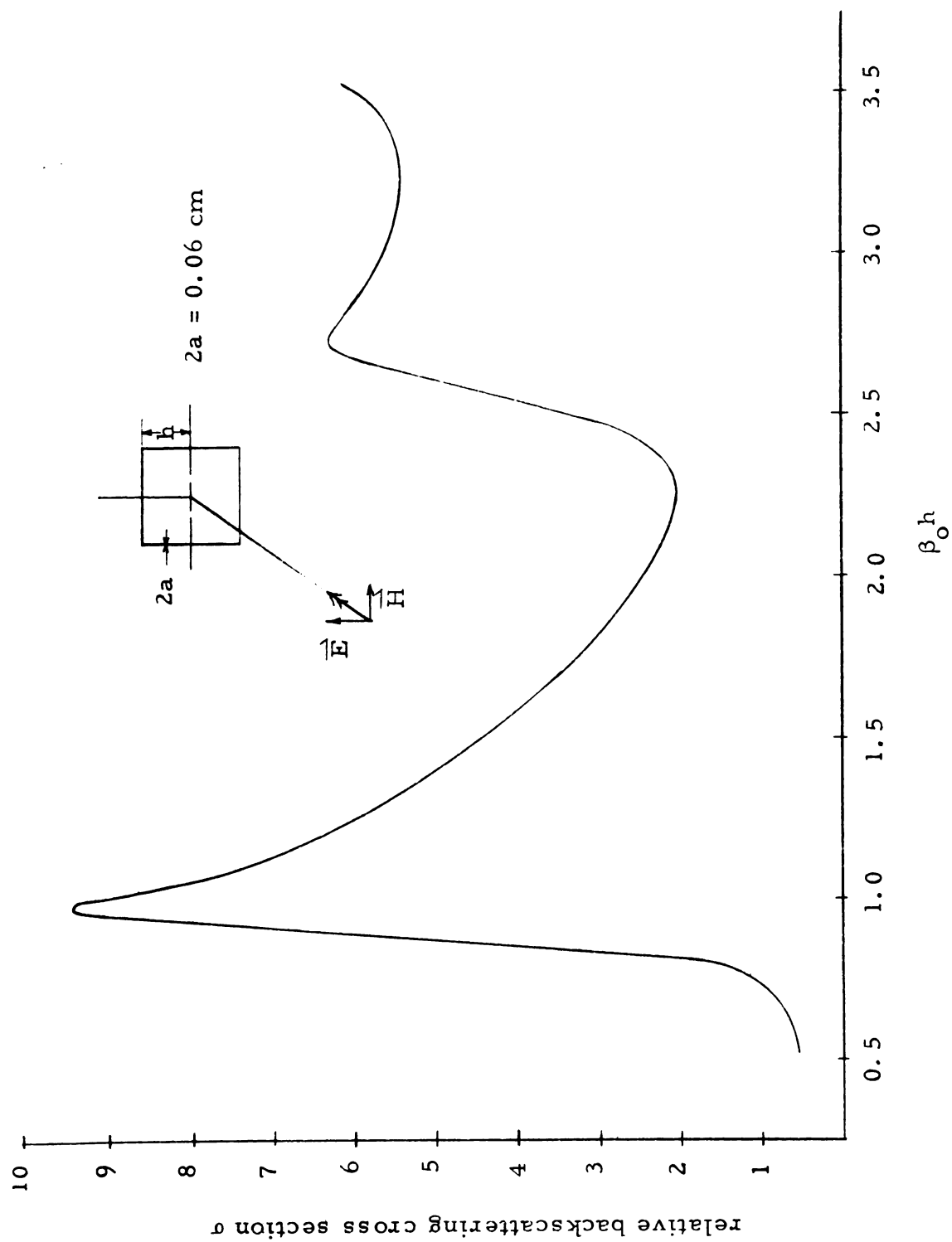


Figure 1.5. Backscattering cross section of square loops illuminated by a plane wave at normal incidence.

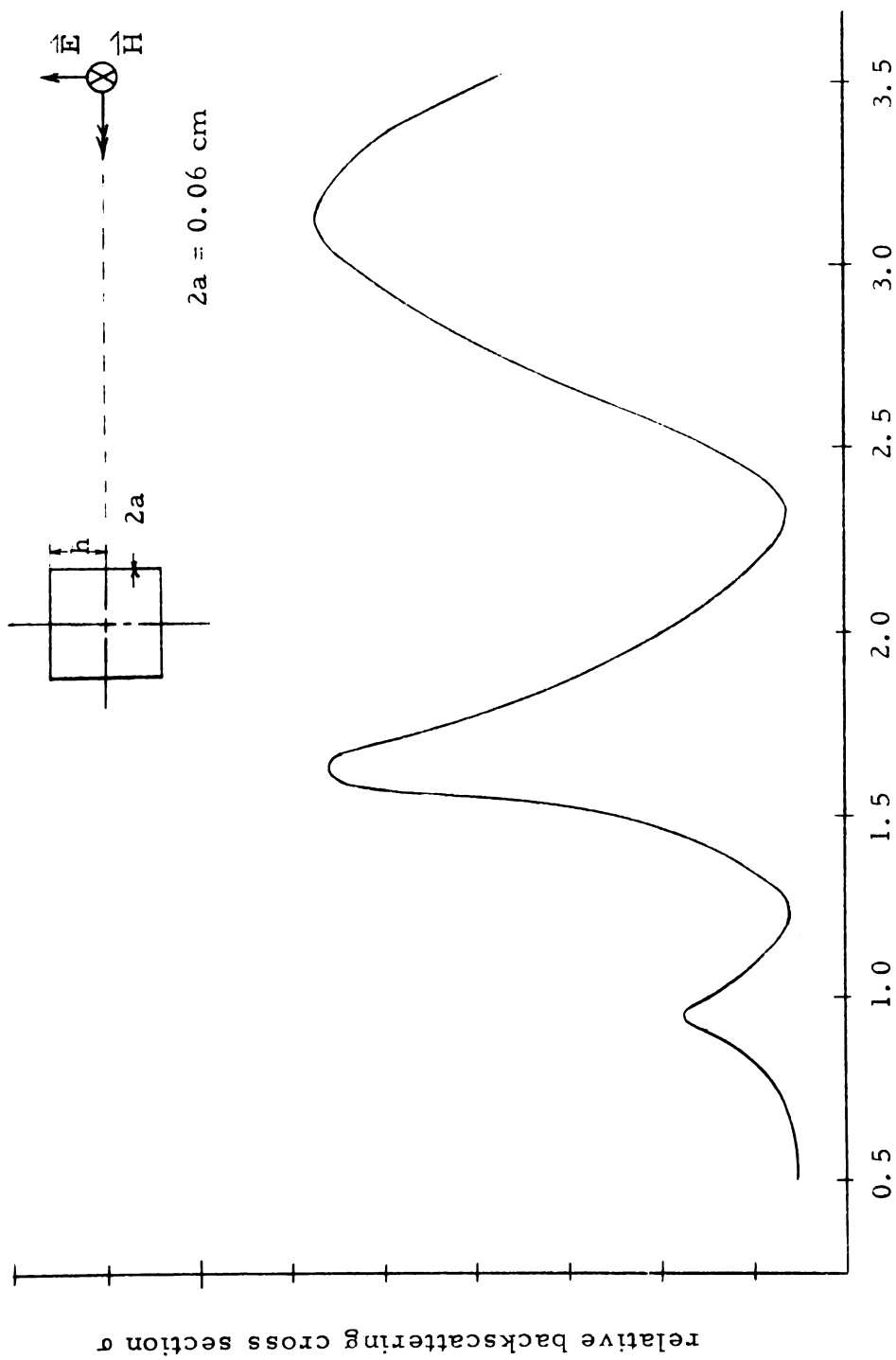


Figure 1.6. Backscattering cross section of square loops illuminated by a plane wave at horizontal incidence.

The experimental results are shown in Figures 1.7 and 1.8.

Figure 1.7 shows the first resonance curves of the loops. The resonant curve of a straight wire which corresponds to a loop with $h_1/h_2 = 0$ is also included in the figure. The scattering cross sections of the loops are plotted as functions of the loop sizes. From the peak of the curve the resonant size of the loop is determined.

It is observed that the first resonances of the loops with various shapes indeed occur when the half peripheries of loops are near $\frac{1}{2}$ wavelength. It is also seen that the first resonant size of a loop is dependent of its geometrical shape. These phenomena are summarized in Table 1.1. It is true that the first resonant size of a loop tends to increase as the loop becomes broader or the ratio of h_1/h_2 becomes larger.

Table 1.1.

Loop Geometry	first resonant size* ($\frac{1}{2}$ periphery)
straight wire ($h_1/h_2 = 0$)	0.433λ
rectangular loop ($h_1/h_2 = \frac{1}{3}$)	0.513λ
rectangular loop ($h_1/h_2 = \frac{1}{1.25}$)	0.516λ
rectangular loop ($h_1/h_2 = \frac{1}{2}$)	0.53λ
rectangular loop ($h_1/h_2 = \frac{1}{1.5}$)	0.561λ
rectangular loop ($h_1/h_2 = \frac{1}{1}$)	0.61λ
circular loop	0.705λ

* For the wire $\frac{1}{2}$ periphery corresponds to the total length

For rectangular loop $\frac{1}{2}$ periphery = $2(h_1+h_2)$

For circular loop $\frac{1}{2}$ periphery = πb where b = radius

Radius of wire of which the loops are made of = $a = 0.01\lambda$

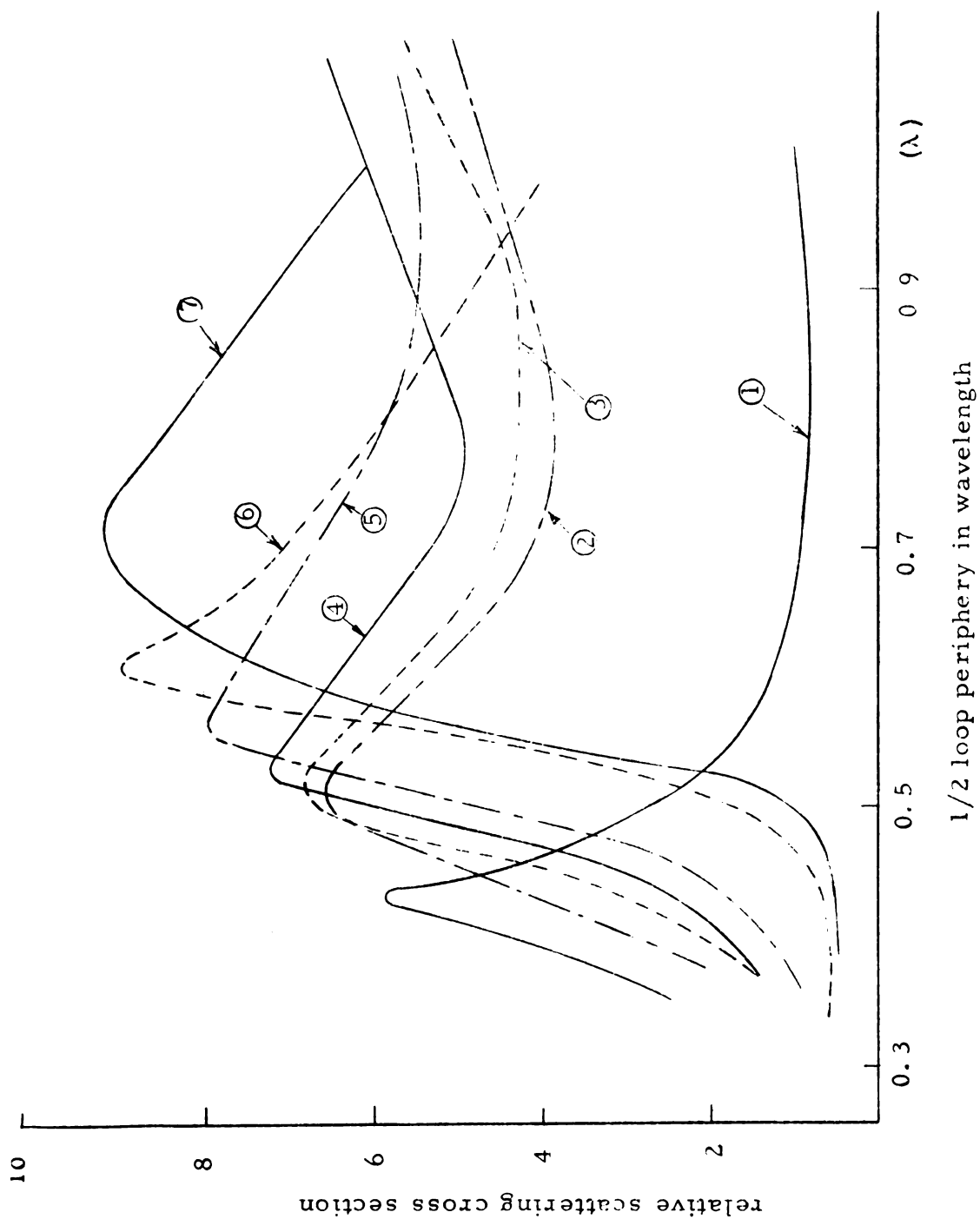


Figure 1.7. Scattering cross sections of loops as functions of loop sizes (first resonance curves).

The second resonance curves of the loops are shown in Figure 1.8. It is observed that the second resonance peaks are not clearly observable in the loops with large ratio of h_1/h_2 . The second resonances of the loops occur when the half peripheries of loops are in the neighborhood of $\frac{3}{2}$ wavelength. The second resonant sizes of the loops are summarized in Table 1.2.

Table 1.2.

Loop Geometry	Second resonant size ($\frac{1}{2}$ periphery)
straight wire ($h_1/h_2 = 0$)	1.44λ
rectangular loop ($h_1/h_2 = \frac{1}{3}$)	1.67λ
rectangular loop ($h_1/h_2 = \frac{1}{1.25}$)	1.68λ
rectangular loop ($h_1/h_2 = \frac{1}{2}$)	1.58λ
rectangular loop ($h_1/h_2 = \frac{1}{1.5}$)	not clear
rectangular loop ($h_1/h_2 = \frac{1}{1}$)	1.73λ
circular loop	not clear

1.3. Theory

A theory is presented here to calculate the backscattering cross section of a solid circular loop and a solid rectangular loop when they are illuminated by a plane wave with vertical polarization at normal incidence.

Some numerical results are obtained and are compared with the experimental results in Section 1.2.

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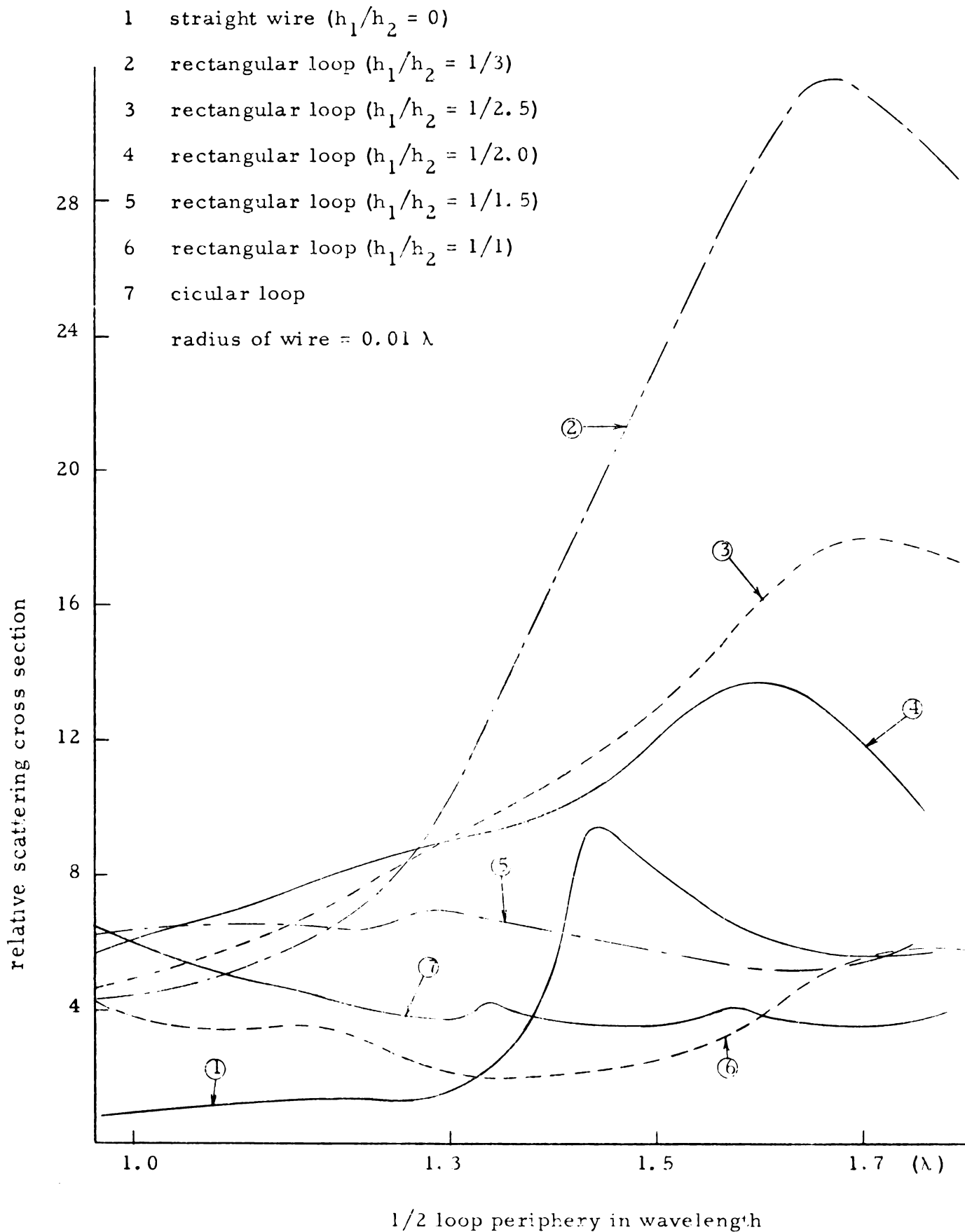


Figure 1.8. Scattering cross sections of loops as functions of loop sizes (second resonance curves)

1.3.1. Backscattering from a Solid Circular Loop

The backscattering cross section of a circular metallic loop was first explored by Kouyoumjian¹ who used a variational method to determine approximate formulas for the backscattering cross sections of thin wire loops. Numerically he was able to calculate the backscattering cross sections of a circular loop illuminated by a plane wave at either normal or horizontal incidence. A good agreement between the theory and the experiment was obtained in this study. The weakness of this method is the tedious computation. Later Weston² investigated the same case by solving the wave equation in toroidal coordinates to obtain general expressions for the electromagnetic field. Numerical values for an arbitrary incident angle were determined and his analysis is deemed as a generalization of earlier work by Kouyoumjian.

These two methods are too complicated to serve our purpose of solving the case of a loaded circular loop developed in the next chapter. A mathematically simpler method has been developed in this research.

(A) Geometry of Problem

The geometry of the problem is as shown in Figure 1.9. A loop of radius b is assumed to be made of perfectly conducting metallic wire of radius a . A plane electromagnetic wave with the E field parallel to the plane of the loop is incident normally upon the loop. The dimensions of interest are

$$a^2 \ll b^2, \quad \beta_0^2 a^2 \ll 1$$

where β_0 is the wave number. We assume that the wire is thin enough so that only the θ component of current is induced.

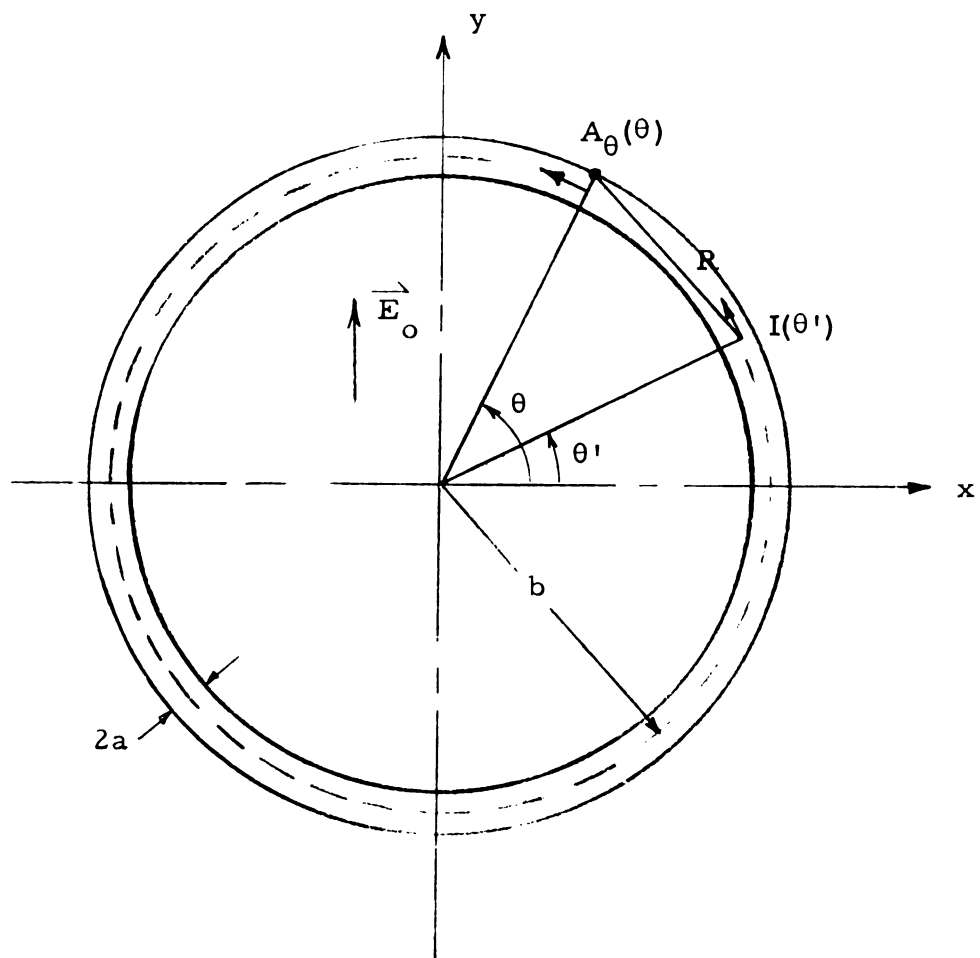


Figure 1.9. Geometry of solid circular loop.

(B) Differential Equation for Loop Current

Since the tangential electric field should vanish at the surface of the loop, the following equation is valid:

$$E_t^i + E_t^a = 0 \quad \text{for } -\pi \leq \theta \leq \pi \quad (1.1)$$

where E_t^a is the tangential electric field maintained by the current and the charge on the loop, E_t^i is the tangential electric field of the incident wave.

The symmetry of the geometry gives

$$I_\theta^s(\theta) = -I_\theta^s(\pi - \theta) \quad (1.2)$$

$$I_\theta^s\left(\pm \frac{\pi}{2}\right) = 0$$

where $I^s(\theta)$ is the induced current in the loop. The tangential component of the incident E field along the loop is

$$E_t^i = E_o \cos \theta \quad (1.3)$$

where E_o is a constant electric field.

The current and the charge on the loop maintain a tangential electric field at the surface which can be expressed as

$$E_t^a = -(\nabla \phi^s)_\theta - j\omega A_\theta^s \quad (1.4)$$

where ϕ^s is the scalar potential maintained by the charge on the loop and A_θ^s is the tangential component of the vector potential maintained by the current on the loop. ϕ^s can be expressed as

$$\phi^s(\theta) = \frac{1}{4\pi\epsilon_0} \int_{-\pi}^{\pi} q^s(\theta') \frac{e^{-j\beta_0 R}}{R} b d\theta' \quad (1.5)$$

where $q^s(\theta')$ is the density of charge induced on the loop at $\theta = \theta'$, and

$$R = b \sqrt{4 \sin^2 \frac{\theta - \theta'}{2} + \frac{a^2}{b^2}} = b \sqrt{2 - 2 \cos(\theta - \theta') + \frac{a^2}{b^2}} \quad (1.6)$$

Assume $\phi^s(\theta)$ can be approximated as

$$\phi^s(\theta) = \frac{\Phi_q^s}{4\pi\epsilon_0} q^s(\theta) \quad (1.7)$$

where Φ_q^s is the ratio between scalar potential to charge density and can be defined as

$$\Phi_q^s = 4\pi\epsilon_0 \frac{\phi^s(\theta)}{q^s(\theta)} \quad (1.8)$$

With (1.8)

$$(\nabla \phi^s)_\theta \Big|_{r=b} = \frac{\Phi_q^s}{4\pi\epsilon_0} \frac{1}{b} \frac{q^s(\theta)}{\partial \theta} \quad (1.9)$$

From the equation of continuity

$$q^s(\theta) = j \frac{1}{\omega} \nabla \cdot \vec{I}^s = j \frac{1}{\omega b} \frac{\partial I_{\theta}^s(\theta)}{\partial \theta} \quad (1.10)$$

it leads to

$$\left[(\nabla \phi^s)_{\theta} \right]_{r=b} = \frac{\Phi_q^s}{4\pi\epsilon_0} \frac{j}{\omega b^2} \frac{\partial^2 I_{\theta}^s(\theta)}{\partial \theta^2} \quad (1.11)$$

Writing A_{θ}^s in terms of the induced current $I_{\theta}^s(\theta)$ on the loop, we have

$$A_{\theta}^s(\theta) = \frac{\mu_0}{4\pi} \int_{-\pi}^{\pi} I_{\theta}^s(\theta') \frac{e^{-j\beta_0 R}}{R} \cos(\theta - \theta') b d\theta' \quad (1.12)$$

If $A_{\theta}^s(\theta)$ is assumed to be proportional to $I_{\theta}^s(\theta)$, (1.12) can be rewritten as

$$A_{\theta}^s(\theta) = \frac{\mu_0 \Phi_i^s}{4\pi} I_{\theta}^s(\theta) \quad (1.13)$$

where Φ_i^s is the ratio between the tangential component of vector potential to the current and is defined as

$$\Phi_i^s = \frac{4\pi}{\mu_0} \frac{A_{\theta}^s(\theta)}{I_{\theta}^s(\theta)} \quad (1.14)$$

With (1.11) and (1.13), (1.4) can be written as

$$-E_t^a = \frac{j\Phi_q^s}{4\pi\epsilon_o\omega b^2} \left[\frac{\partial^2 I_\theta^s(\theta)}{\partial \theta^2} + \beta_o^2 b^2 a^s I_\theta^s(\theta) \right] \quad (1.15)$$

where

$$a^s = \frac{\Phi_i^s}{\Phi_q^s} \quad (1.16)$$

The substitution of (1.3) and (1.15) in (1.1) gives

$$\left(\frac{\partial^2}{\partial \theta^2} + \beta_s^2 b^2 \right) I_\theta^s(\theta) = K^s E_o \cos \theta \quad (1.17)$$

for $-\pi \leq \theta \leq \pi$

where

$$\beta_s^2 = \beta_o^2 a^s = \beta_o^2 \frac{\Phi_i^s}{\Phi_q^s} \quad (1.18)$$

$$K^s = -j \frac{4\pi\epsilon_o\omega b^2}{\Phi_q^s} \quad (1.19)$$

(C) Solution for Loop Current

The solution for $I_\theta^s(\theta)$ can be expressed as

$$I_\theta^s(\theta) = C_1^s \cos \beta_s b \theta + C_2^s \sin \beta_s b \theta + P^s(\theta) \quad (1.20)$$

where C_1^s and C_2^s are arbitrary constants, and $P^s(\theta)$ is a particular integral. C_2^s is zero due to the symmetry and $P^s(\theta)$ can be found to be

$$P^s(\theta) = \frac{K^s E_o}{\beta_s^2 b^2 - 1} \cos \theta \quad (1.21)$$

Thus, (1.20) can be written as

$$I_\theta^s(\theta) = C_1^s \cos \beta_s b \theta + \frac{K^s E_o}{\beta_s^2 b^2 - 1} \cos \theta \quad (1.22)$$

If the boundary condition of $I_\theta^s(\frac{\pi}{2}) = 0$ is applied to (1.22), we obtain

$$C_1^s = 0 \quad (1.23)$$

Therefore, the final solution for $I_\theta^s(\theta)$ becomes

$$I_\theta^s(\theta) = \frac{K^s E_o}{\beta_s^2 b^2 - 1} \cos \theta \quad (1.24)$$

for $-\pi \leq \theta \leq \pi$

(D) Backscattering Cross Section

With the induced current found in (1.24), the vector potential in the direction of the incident wave and at the far zone of the loop can be calculated as

$$\begin{aligned}
A_y^s &= \frac{\mu_o}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2I_{\theta}^s(\theta') \cos \theta' \frac{e^{-j\beta_o R_1}}{R_1} b d\theta' \\
&= \frac{\mu_o}{4} \frac{e^{-j\beta_o R_1}}{R_1} b \frac{K^s E_o}{\beta_s^2 b^2 - 1}
\end{aligned} \tag{1.25}$$

where $R_1 = \sqrt{R_o^2 + b^2}$ and R_o is the distance between the center of the loop and an observation point on the axis of the loop as shown in Figure 1.10.

The back scattered field in the far zone is

$$\begin{aligned}
E_y^s &= -j\omega A_y^s \\
&= -j \frac{\zeta_o}{4} \beta_o b \frac{e^{-j\beta_o R_1}}{R_1} \frac{K^s E_o}{\beta_s^2 b^2 - 1}
\end{aligned} \tag{1.26}$$

where $\zeta_o = \sqrt{\frac{\mu_o}{\epsilon_o}} = 120\pi$ and β_s is to be determined in a following section.

The Poynting power density of the scattered field can be found to be

$$\begin{aligned}
P &= \frac{|E_y^s|^2}{2\zeta_o} \\
&= \frac{\zeta_o}{32} (\beta_o b)^2 \frac{E_o^2}{R_1^2} \left| \frac{K^s}{\beta_s^2 b^2 - 1} \right|^2
\end{aligned} \tag{1.27}$$

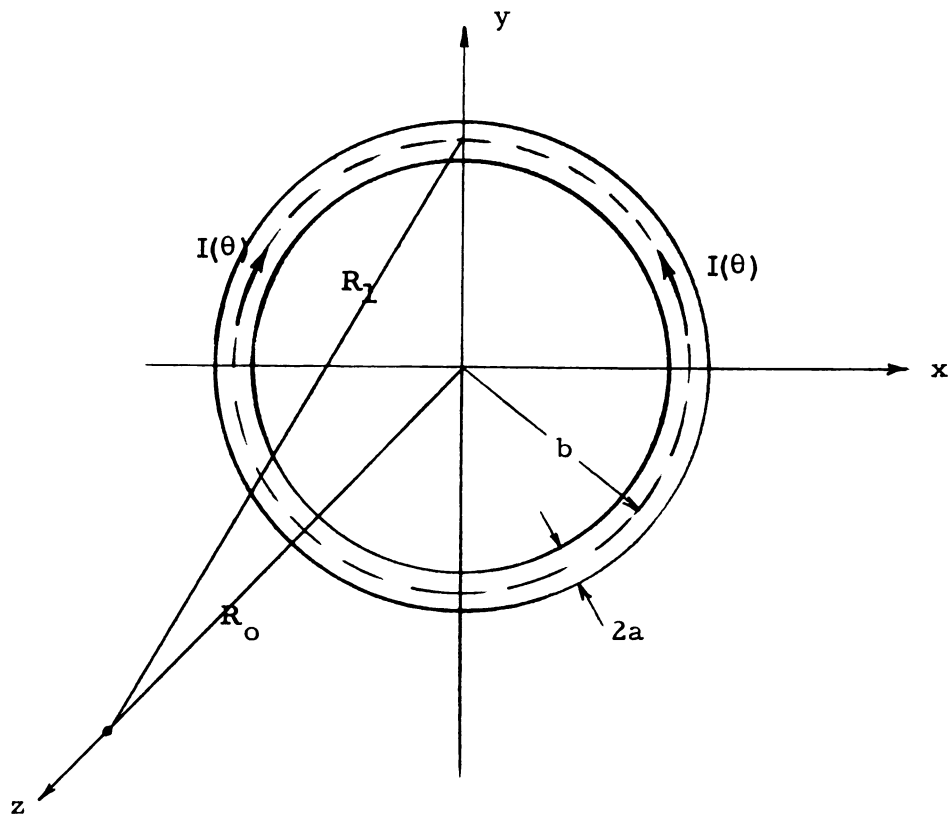


Figure 1.10. Geometry for the calculation of radiation field of a solid circular loop.

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The backscattering cross section is then found to be

$$\begin{aligned}\sigma_B &= \lim_{R_1 \rightarrow \infty} 4\pi R_1^2 \frac{|E_y^s|^2}{|E_o|^2} \\ &= \frac{\pi \zeta_o^2}{4} (\beta_o b)^2 \left| \frac{K^s}{\beta_s^2 b^2 - 1} \right|^2\end{aligned}\quad (1.28)$$

Substitution of (1.19) in (1.28) yields

$$\begin{aligned}\sigma_B &= 4\pi^3 \zeta_o^2 \epsilon_o^2 \omega^2 b^4 (\beta_o b)^2 \left| \frac{1}{\Phi_q^s (\beta_s^2 b^2 - 1)} \right|^2 \\ &= \lambda^2 \pi (\beta_o b)^6 \left| \frac{1}{\Phi_q^s (\beta_s^2 b^2 - 1)} \right|^2\end{aligned}$$

or

$$\frac{\sigma_B}{\lambda^2} = \pi (\beta_o b)^6 \left| \frac{1}{\Phi_q^s (\beta_s^2 b^2 - 1)} \right|^2 \quad (1.29)$$

where λ is the wavelength.

(E) Determination of Φ_q^s and Φ_i^s

(i) Using the equation of continuity (1.10) and (1.24), $q^s(\theta)$ can be found to be

$$\begin{aligned}
q^s(\theta) &= j \frac{1}{\omega b} \frac{\partial I_\theta^s(\theta)}{\partial \theta} \\
&= -j \frac{1}{\omega b} \frac{K^s E_o}{\beta_s^2 b^2 - 1} \sin \theta
\end{aligned} \tag{1.30}$$

$$\text{for } -\pi \leq \theta \leq \pi$$

With (1.5) and (1.30), Φ_q^s defined in (1.8) is found as

$$\begin{aligned}
\Phi_q^s &= 4\pi \epsilon_o \frac{\phi^s(\theta_o)}{q^s(\theta_o)} \\
&= \frac{\int_{-\pi}^{\pi} \sin \theta' \frac{e^{-j\beta_o R}}{R} b d\theta'}{\sin \theta_o}
\end{aligned} \tag{1.31}$$

where

$$R = b \sqrt{2 - 2 \cos(\theta_o - \theta') + \frac{a^2}{b^2}}$$

Choosing $\theta_o = \frac{\pi}{2}$, the point of maximum charge density, Φ_q^s becomes

$$\begin{aligned}
\Phi_q^s &= \int_{-\pi}^{\pi} \sin \theta' \frac{e^{-j\beta_s b \sqrt{2 - 2 \sin \theta' + \frac{a^2}{b^2}}}}{\sqrt{2 - 2 \sin \theta' + \frac{a^2}{b^2}}} d\theta' \\
&= \int_{-\pi}^{\pi} \sin \theta' K_1 \left(\theta', \frac{a^2}{b^2} \right) d\theta'
\end{aligned} \tag{1.32}$$

where

$$K_1\left(\theta', \frac{a^2}{b^2}\right) = \frac{e^{-j\beta_s b \sqrt{2 - 2 \sin \theta' + \frac{a^2}{b^2}}}}{\sqrt{2 - 2 \sin \theta' + \frac{a^2}{b^2}}} \quad (1.33)$$

In actual calculation of Φ_q^s , β_s will be replaced by β_o as an approximation.

(ii) Similarly, Φ_i^s is obtained by making use of (1.12) and (1.14) as

$$\begin{aligned} \Phi_i^s &= \frac{4\pi}{\mu_o} \frac{A_\theta^s(\theta_o)}{I_\theta^s(\theta_o)} \\ &= \frac{\int_{-\pi}^{\pi} \cos \theta' \cos(\theta_o - \theta') \frac{e^{-j\beta_s R}}{R} b d\theta'}{\cos \theta_o} \end{aligned} \quad (1.34)$$

where

$$R = b \sqrt{2 - 2 \cos(\theta_o - \theta') + \frac{a^2}{b^2}}$$

Choosing $\theta_o = 0$, the point of maximum current, Φ_i^s becomes

$$\begin{aligned}
\Phi_i^s &= \int_{-\pi}^{\pi} \cos^2 \theta' \frac{e^{-j\beta_s b \sqrt{2 - 2 \cos \theta' + \frac{a^2}{b^2}}}}{\sqrt{2 - 2 \cos \theta' + \frac{a^2}{b^2}}} d\theta' \\
&= 2 \int_0^{\pi} \cos^2 \theta' K_2\left(\theta', \frac{a^2}{b^2}\right) d\theta'
\end{aligned} \tag{1.35}$$

where

$$K_2\left(\theta', \frac{a^2}{b^2}\right) = \frac{e^{-j\beta_s b \sqrt{2 - \cos \theta' + \frac{a^2}{b^2}}}}{\sqrt{2 - 2 \cos \theta' + \frac{a^2}{b^2}}} \tag{1.36}$$

In actual calculation of Φ_i^s , β_s will be replaced by β_0 as an approximation.

(F) Numerical Results

To calculate the theoretical results of backscattering cross section, Φ_q^s and Φ_i^s are numerically calculated for the case of $\frac{a^2}{b^2} = 0.00179$ as functions of $\beta_0 b$. Numerical results of Φ_q^s and Φ_i^s are shown graphically in Figure 1.11 and Figure 1.12. It is observed that Φ_q^s and Φ_i^s vary only weakly over the range of interest. This agrees reasonably well with our original assumption of Φ_q^s and Φ_i^s being constant. The numerical result of backscattering cross section $\frac{\sigma}{\lambda^2}$ is plotted as functions of $\beta_0 b$ in Figure 1.13 in comparison with the experimental results.

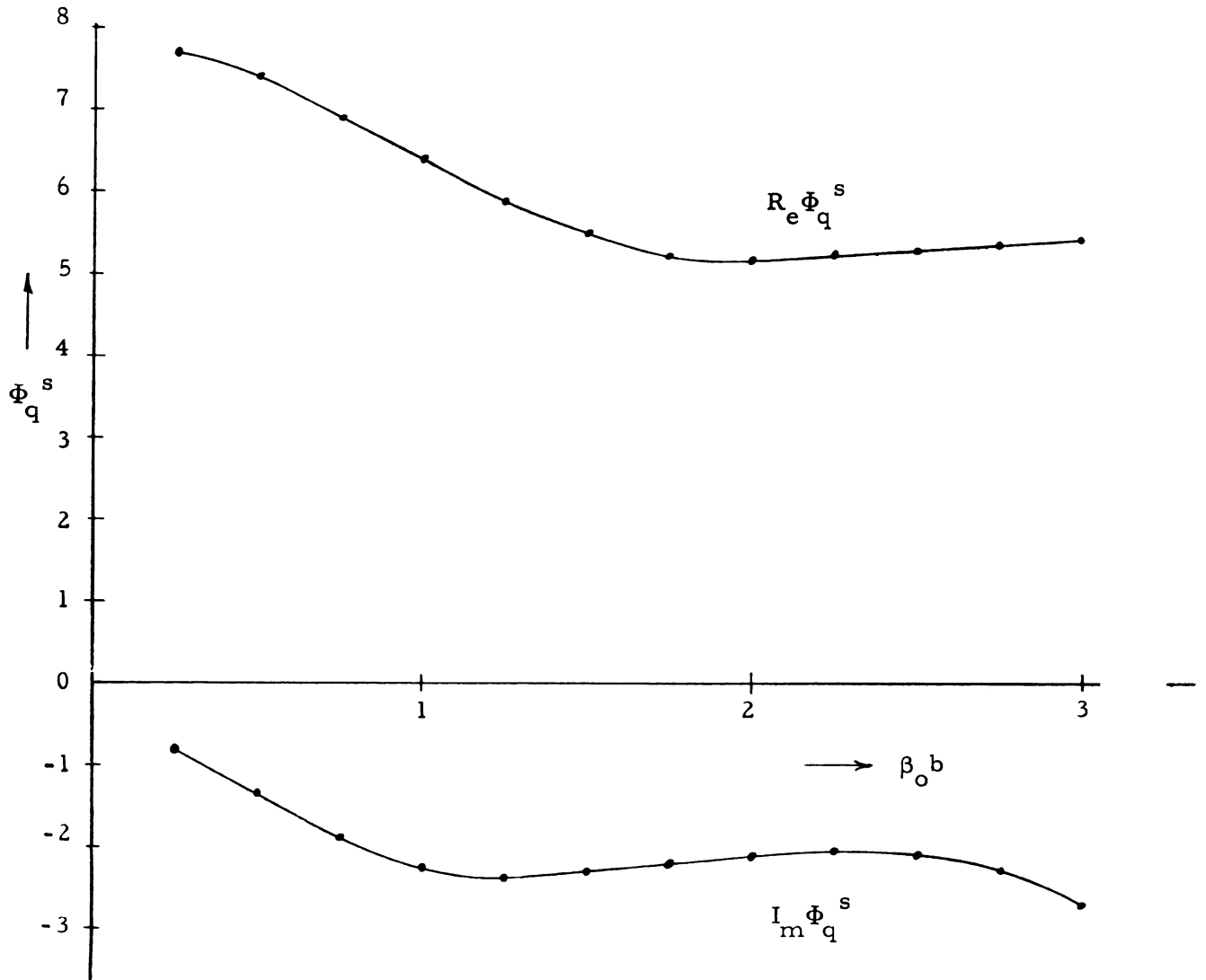


Figure 1.11. Φ_q^s as a function of $\beta_0 b$ ($\frac{a^2}{b^2} = 0.00179$).

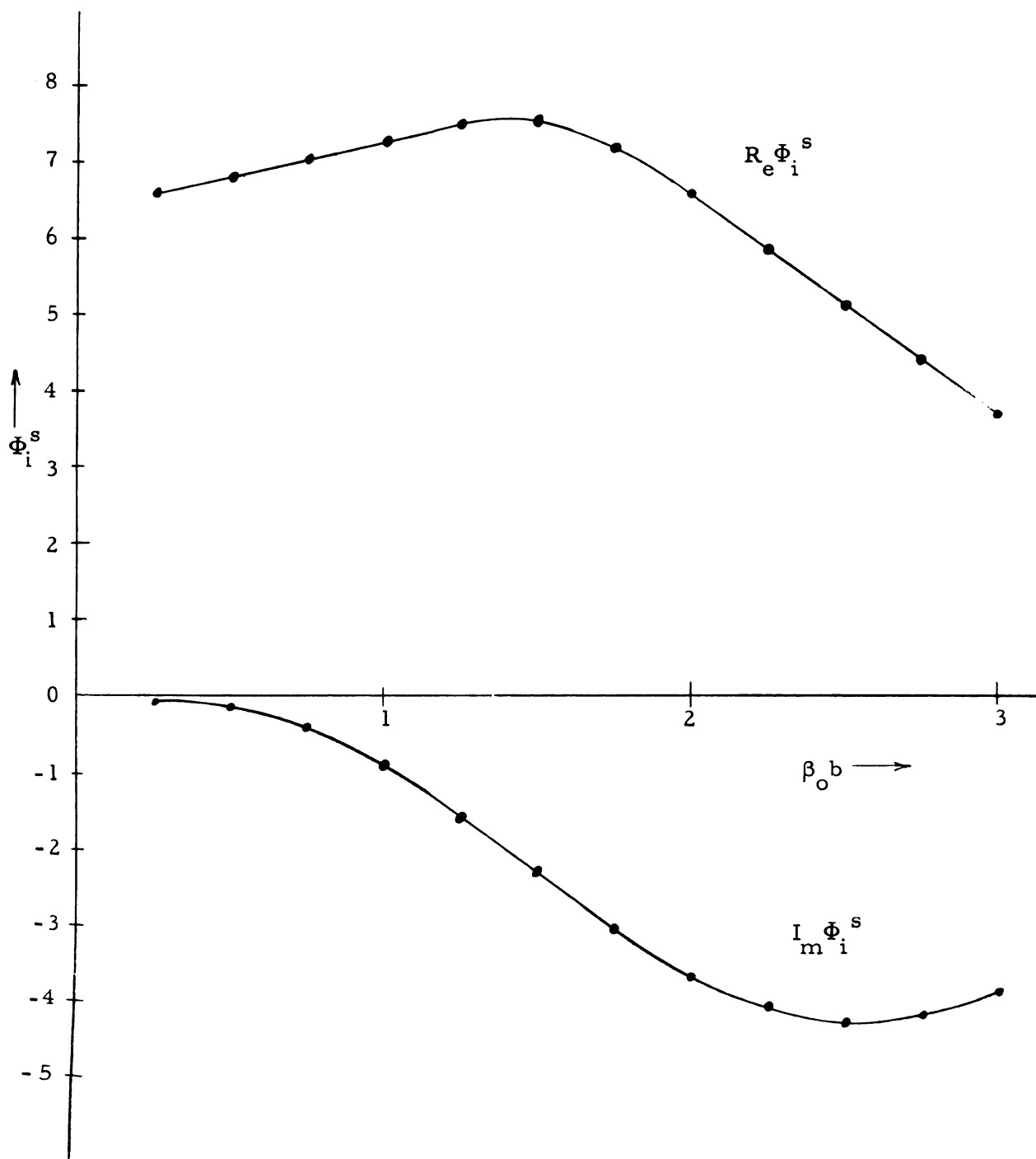


Figure 1.12. Φ_i^s as a function of $\beta_o b$ ($\frac{a^2}{b^2} = 0.00179$).

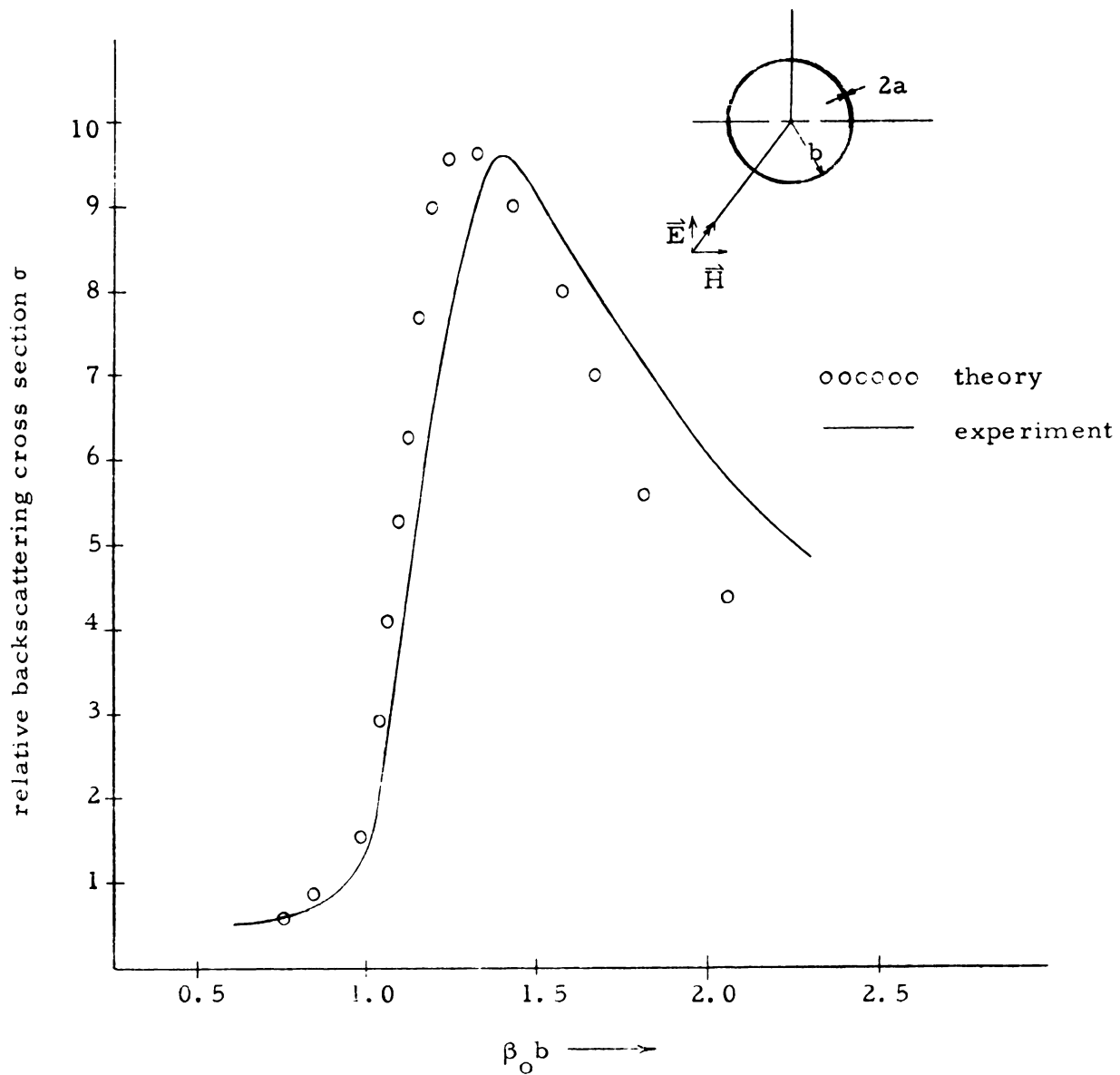


Figure 1.13. Comparison between theory and experiment for circular loops illuminated by plane wave at normal incidence.

The theory predicts fairly well for the loops with $\beta_0 b$ smaller than 1.5 which is the range of interest. For a larger loop the present theory fails to give accurate solution. A refined theory would give better results but we find the present theory adequate for further theoretical development.

1.3.2. Scattering from a Solid Rectangular Loop

To study the backscattering from a solid rectangular loop an approach which is different from that used for a circular loop will be used. The first order solution will be obtained by using the method developed by Chen and King³.

(A) Formulation of Problem

The geometry of the problem is as shown in Figure 1.14. A rectangular loop with short side $2h_1$ and long side $2h_2$ is assumed to be made of perfectly conducting metallic wire of radius a . A plane electromagnetic wave with the E field parallel to the plane of the loop is incident normally to the loop. The dimensions of interest are

$$a^2 \ll h_1^2 \text{ and } h_2^2, \quad \beta_0^2 a^2 \ll 1$$

where β_0 is the wave number. We assume that the wire is thin enough so that only the tangential component of current is induced.

(B) Integral Equations for Loop Currents

Since the tangential electric field should be continuous at the surface of the loop, we obtain the following equation:

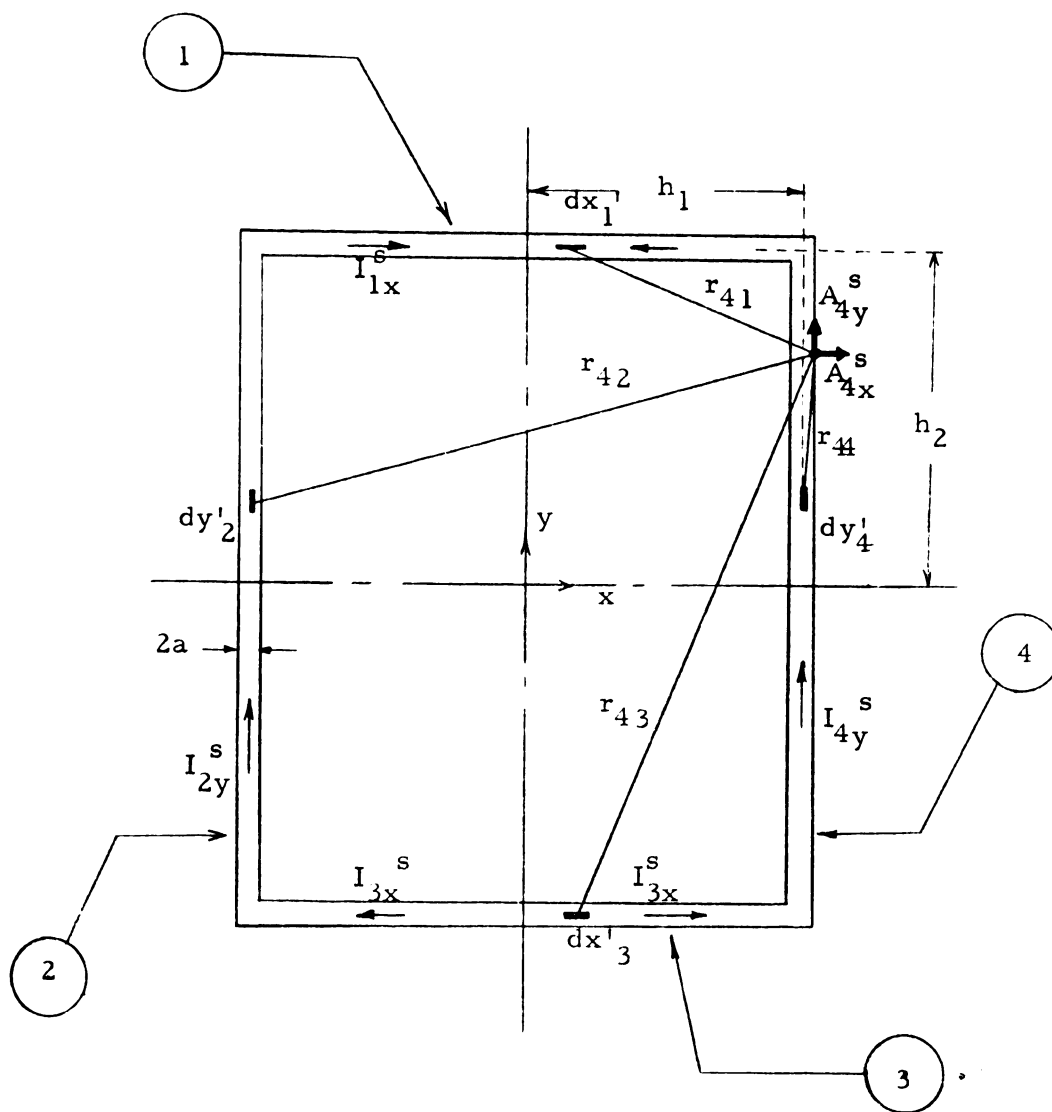


Figure 1.14. Geometry for solid rectangular loop.

For side 4

$$E_{4t}^i + E_{4t}^a = 0 \quad (1.37)$$

$$E_{4t}^i = E_0 \quad (1.38)$$

where E_{4t}^i is the tangential electric field of the incident wave and E_{4t}^a is the tangential E field at the surface of side 4 maintained by the induced current and charge on the loop.

Due to symmetrical configuration, it follows that

$$I_{2y}^s(y) = I_{4y}^s(y) \quad (1.39)$$

$$I_{1x}^s(x) = -I_{3x}^s(x) \quad (1.40)$$

$$q_2^s(y) = q_4^s(y) \quad (1.41)$$

$$q_1^s(x) = -q_3^s(x) \quad (1.42)$$

where $I_{4y}^s(y)$ is the induced current in side 4 and $I_{1x}^s(x)$ is that in side 1, etc., whereas q_i^s denotes the induced charge on side i , $i = 1, 2, 3$, and 4. E_{4t}^a can be expressed as

$$E_{4t}^a = -(\nabla \phi_4^s)_y - j\omega(\vec{A}_4^s)_y \quad (1.43)$$

where $\vec{A}_4^s$ is the vector potential at the surface of side 4 contributed by the induced current in the loop; ϕ_4^s is the scalar potential at the

surface of side 4 maintained by the induced charges on the loop. In symbols, $\vec{A}_4^s$ and ϕ_4^s can be expressed as

$$\vec{A}_4^s = A_{4x}^s \hat{x} + A_{4y}^s \hat{y} \quad (1.44)$$

$$\phi_4^s = \phi_{41}^s + \phi_{42}^s + \phi_{43}^s + \phi_{44}^s \quad (1.45)$$

where A_{4x}^s is the vector potential at the surface of side 4 contributed by the induced currents in sides 1 and 3. The induced currents in sides 2 and 4 do not contribute to A_{4x}^s because they do not have x component. A_{4y}^s carries the same meaning as A_{4x}^s except that the contribution is made by the induced currents in sides 2 and 4.

ϕ_{4i}^s is the scalar potential at the surface of side 4 maintained by the induced charges on side i, $i = 1, 2, 3$, and 4. A_{4y}^s can be expressed in terms of current by Helmholtz integrals as

$$\begin{aligned} A_{4y}^s(y) &= \frac{\mu_o}{4\pi} \int_{-h_2}^{h_2} I_{4y}^s(y') \frac{e^{-j\beta_o r_{44}}}{r_{44}} dy' \\ &+ \frac{\mu_o}{4\pi} \int_{-h_2}^{h_2} I_{2y}^s(y') \frac{e^{-j\beta_o r_{42}}}{r_{42}} dy' \\ &= \frac{\mu_o}{4\pi} \int_{-h_2}^{h_2} I_{4y}^s(y') K_{1A}(y, y') dy' \end{aligned} \quad (1.46)$$

where

$$K_{1A}(y, y') = \frac{e^{-j\beta_o r_{44}}}{r_{44}} + \frac{e^{-j\beta_o r_{42}}}{r_{42}} \quad (1.47)$$

$$r_{44} = \sqrt{(y' - y)^2 + a^2} \quad (1.48)$$

$$r_{42} = \sqrt{(y' - y)^2 + 4h_1^2} \quad (1.49)$$

With (1.44) and (1.45) , (1.43) can be rewritten as

$$E_{4t}^a = -\frac{\partial}{\partial y} (\phi_{42}^s + \phi_{44}^s) - j\omega A_{4y}^s - \frac{\partial}{\partial y} (\phi_{41}^s + \phi_{43}^s) \quad (1.50)$$

By the Lorentz condition, it follows that

$$\phi_{42}^s + \phi_{44}^s = \frac{j\omega}{\beta_o^2} \nabla \cdot (\vec{A}_{42}^s + \vec{A}_{44}^s) = \frac{j\omega}{\beta_o^2} \frac{\partial A_{4y}^s(y)}{\partial y} \quad (1.51)$$

After substituting (1.51) in (1.50) and after rearranging, (1.50) becomes

$$E_{4t}^a = \frac{-j\omega}{\beta_o^2} \left(\frac{\partial^2}{\partial y^2} + \beta_o^2 \right) A_{4y}^s(y) - \frac{\partial}{\partial y} (\phi_{41}^s + \phi_{43}^s) \quad (1.52)$$

Substitution of (1.38) and (1.52) in (1.37) gives

$$\left(\frac{\partial^2}{\partial y^2} + \beta_o^2 \right) A_{4y}^s(y) = -\frac{j\beta_o^2}{\omega} E_c + \frac{j\beta_o^2}{\omega} \frac{\partial}{\partial y} (\phi_{41}^s + \phi_{43}^s) \quad (1.53)$$

Similarly, the differential equation for the vector potential at side 1 can be obtained as follows:

For side 1

$$E_{1t}^i + E_{1t}^a = 0 \quad (1.54)$$

$$E_{1t}^i = 0 \quad (1.55)$$

where E_{1t}^i is the tangential E field of the incident wave and E_{1t}^a is the tangential E field at the surface of side 1 maintained by the induced current and charge on the loop.

E_{1t}^a can be expressed as

$$E_{1t}^a = -(\nabla \phi_1^s)_x - j\omega (\vec{A}_1^s)_x \quad (1.56)$$

where ϕ_1^s is the scalar potential at the surface of side 1 maintained by the induced charges on the loop and $\vec{A}_1^s$ is the vector potential at the surface of side 1 contributed by the induced currents in the loop. In symbols, $\vec{A}_1^s$ and ϕ_1^s can be expressed as

$$\vec{A}_1^s = A_{1x}^s \hat{x} + A_{1y}^s \hat{y} \quad (1.57)$$

$$\phi_1^s = \phi_{11}^s + \phi_{12}^s + \phi_{13}^s + \phi_{14}^s \quad (1.58)$$

where A_{1x}^s is the vector potential at the surface of side 1 contributed by the induced currents in sides 1 and 3; while A_{1y}^s is that contributed by the induced currents in sides 2 and 4. ϕ_{1i}^s is the scalar potential

at the surface of side 1 maintained by the induced charges on side i,
i = 1, 2, 3 and 4.

A_{1x}^s can be expressed in terms of current as

$$\begin{aligned}
 A_{1x}^s(x) &= \frac{\mu_o}{4\pi} \int_{-h_1}^{h_1} I_{1x}^s(x') \frac{e^{-j\beta_o r_{11}}}{r_{11}} dx' \\
 &+ \frac{\mu_o}{4\pi} \int_{-h_1}^{h_1} I_{3x}^s(x') \frac{e^{-j\beta_o r_{13}}}{r_{13}} dx' \\
 &= \frac{\mu_o}{4\pi} \int_{-h_1}^{h_1} I_{1x}^s(x') K_{2A}(x, x') dx' \tag{1.59}
 \end{aligned}$$

where

$$K_{2A}(x, x') = \frac{e^{-j\beta_o r_{11}}}{r_{11}} - \frac{e^{-j\beta_o r_{13}}}{r_{13}} \tag{1.60}$$

$$r_{11} = \sqrt{(x - x')^2 + a^2} \tag{1.61}$$

$$r_{13} = \sqrt{(x' - x)^2 + 4h_2^2} \tag{1.62}$$

With (1.57) and (1.58), (1.56) can be rewritten as

$$E_{1t}^a = -\frac{\partial}{\partial x} (\phi_{11}^s + \phi_{13}^s) - j\omega A_{1x}^s - \frac{\partial}{\partial x} (\phi_{12}^s + \phi_{14}^s) \tag{1.63}$$

By the Lorentz condition, we have

$$\phi_{11}^s + \phi_{13}^s = \frac{j\omega}{\beta_o^2} \nabla \cdot (\vec{A}_{11}^s + \vec{A}_{13}^s) = \frac{j\omega}{\beta_o^2} \frac{\partial A_{1x}^s}{\partial x} \quad (1.64)$$

Substituting (1.64) in (1.63), we can rearrange (1.63) as

$$E_{1t}^a = -\frac{j\omega}{\beta_o^2} \left(\frac{\partial^2}{\partial x^2} + \beta_o^2 \right) A_{1x}^s - \frac{\partial}{\partial x} (\phi_{12}^s + \phi_{14}^s) \quad (1.65)$$

With (1.55) and (1.65), (1.54) can be rewritten as

$$\left(\frac{\partial^2}{\partial x^2} + \beta_o^2 \right) A_{1x}^s(x) = \frac{j\beta_o^2}{\omega} \frac{\partial}{\partial x} (\phi_{12}^s + \phi_{14}^s) \quad (1.66)$$

Due to symmetry, we note that

$$A_{4y}^s(y) = A_{4y}^s(-y) \quad (1.67)$$

and

$$A_{1x}^s(x) = -A_{1x}^s(-x) \quad (1.68)$$

With (1.67), the general solution for $A_{4y}^s(y)$ in (1.53) is then

$$A_{4y}^s(y) = \frac{-j}{v_o} \left[C_4^s \cos \beta_o y + \theta_4^s(y) \right] \quad (1.69)$$

where C_4^s is an arbitrary constant, v_o is the velocity of light in free space and $\theta_4^s(y)$ is a particular integral which can be found as

$$\theta_4^s(y) = \frac{E_0}{\beta_o} - \int_0^y \frac{\partial}{\partial s} \left[\phi_{41}^s(s) + \phi_{43}^s(s) \right] \sin \beta_o(y - s) ds \quad (1.70)$$

Similarly, with (1.68) the general solution for $A_{1x}^s(x)$ in (1.66) can be expressed as

$$A_{1x}^s(x) = \frac{-j}{v_o} \left[C_1^s \sin \beta_o x + \theta_1^s(x) \right] \quad (1.71)$$

where C_1^s is an arbitrary constant and $\theta_1^s(x)$ is a particular integral which can be found as

$$\theta_1^s(x) = - \int_0^x \frac{\partial}{\partial s} \left[\phi_{12}^s(s) + \phi_{14}^s(s) \right] \sin \beta_o(x - s) ds \quad (1.72)$$

With (1.46) and (1.69), it follows that

$$\begin{aligned} \frac{4\pi}{\mu_o} A_{4y}^s(y) &= \int_{-h_2}^{h_2} I_{4y}^s(y') K_{1A}(y, y') dy' \\ &= - \frac{j4\pi}{\zeta_o} \left[C_4^s \cos \beta_o y + \theta_4^s(y) \right] \end{aligned} \quad (1.73)$$

Similarly, with (1.59) and (1.66), we obtain

$$\begin{aligned}
\frac{4\pi}{\mu_o} A_{1x}^s(x) &= \int_{-h_1}^{h_1} I_{1x}^s(x') K_{2A}(x, x') dx' \\
&= -\frac{j4\pi}{\zeta_o} \left[C_1^s \sin \beta_o x + \theta_1^s(x) \right]
\end{aligned} \tag{1.74}$$

(C) Zeroth Order Solutions for Loop Currents

For the first approximation, we assume that

$$\frac{A_{4y}^s(y)}{I_{4y}^s(y)} = \frac{\mu_o}{4\pi} \Phi_s(y) \tag{1.75}$$

and

$$\frac{A_{1x}^s(x)}{I_{1x}^s(x)} = \frac{\mu_o}{4\pi} \Phi_s(x) \tag{1.76}$$

where $\Phi_s(y)$ and $\Phi_s(x)$ are defined respectively as

$$\Phi_s(y) = \frac{4\pi}{\mu_o} \frac{A_{4y}^s(y)}{I_{4y}^s(y)} \tag{1.77}$$

and

$$\Phi_s(x) = \frac{4\pi}{\mu_o} \frac{A_{1x}^s(x)}{I_{1x}^s(x)} \tag{1.78}$$

Substituting (1.75) in (1.73), we obtain

$$I_{4y}^s(y) = \frac{-j4\pi}{\zeta_o \Phi_s(y)} \left[C_4^s \cos \beta_o y + \theta_4^s(y) \right] \tag{1.79}$$

Similarly, the substitution of (1.76) in (1.74) gives

$$I_{1x}^s(x) = \frac{-j4\pi}{\zeta_0 \Phi_s(x)} \left[C_1^s \sin \beta_0 x + \theta_1^s(x) \right] \quad (1.77)$$

By looking closely at (1.46) and (1.59), we note that the main contributions to $A_{4y}^s(y)$ and $A_{1x}^s(x)$ are due to the current elements located at $y' = y$ and $x' = x$ respectively. It is then reasonable to assume that $\Phi_s(y)$ and $\Phi_s(x)$ are constant over the entire conductor except at the corners. For simplicity, we assume that

$$\Phi_s(y) = \Phi_s(x) = \Phi_s \quad (1.81)$$

To determine C_4^s and C_1^s , the following boundary conditions will be used.

(a) Current at the corners being continuous.

$$\text{i.e.,} \quad I_{4y}^s(y = h_2) = -I_{1x}^s(x = h_1) \quad (1.82)$$

(b) Charge around the corners being continuous.

$$\text{i.e.,} \quad q_4^s(y = h_2) = q_1^s(x = h_1) \quad (1.83)$$

The substitution of (1.79), (1.80) and (1.81) in (1.82) yields

$$C_4^s \cos \beta_0 h_2 + \theta_4^s(h_2) = C_1^s \sin \beta_0 h_1 + \theta_1^s(h_1) \quad (1.84)$$

$q_4^s(y)$ and $q_1^s(x)$ can be found as follows by using the equation of continuity.

$$\begin{aligned}
q_4^s(y) &= \frac{j}{\omega} \frac{\partial}{\partial y} I_{4y}^s(y) \\
&= \frac{j}{\omega} \frac{-j4\pi}{\zeta_o \Phi_s} \left[-C_4^s \beta_o \sin \beta_o y + \frac{\partial \theta_4^s(y)}{\partial y} \right] \quad (1.85)
\end{aligned}$$

$$\begin{aligned}
q_1^s(x) &= \frac{j}{\omega} \frac{\partial}{\partial x} I_{1x}^s(x) \\
&= \frac{j}{\omega} \frac{-j4\pi}{\zeta_o \Phi_s} \left[C_1^s \beta_o \cos \beta_o x + \frac{\partial \theta_1^s(x)}{\partial x} \right] \quad (1.86)
\end{aligned}$$

Combining (1.85) and (1.86) with boundary condition (1.83), we have

$$-C_4^s \beta_o \sin \beta_o h_2 + \frac{\partial \theta_4^s(y)}{\partial y} \Big|_{y=h_2} = C_1^s \beta_o \cos \beta_o h_1 + \frac{\partial \theta_1^s(x)}{\partial x} \Big|_{x=h_1} \quad (1.87)$$

Solving (1.84) and (1.87) for C_4^s and C_1^s , we obtain

$$\begin{aligned}
C_4^s &= \frac{1}{\beta_o [\cot \beta_o h_1 \cos \beta_o h_2 + \sin \beta_o h_2]} \left[\frac{\partial \theta_1^s(x)}{\partial x} \Big|_{x=h_1} \right. \\
&\quad \left. - \frac{\partial \theta_4^s(y)}{\partial y} \Big|_{y=h_2} \right] + \frac{1}{\tan \beta_o h_1 \sin \beta_o h_2 - \cos \beta_o h_2} \times \\
&\quad \left[\theta_1^s(h_1) + \theta_4^s(h_2) \right] \quad (1.88)
\end{aligned}$$

$$C_1^s = \frac{1}{\beta_o [\sin \beta_o h_1 \tan \beta_o h_2 - \cos \beta_o h_1]} \times$$

$$\cdot \left[\left. \frac{\partial \theta_1^s(x)}{\partial x} \right|_{x=h_1} - \left. \frac{\partial \theta_4^s(y)}{\partial y} \right|_{y=h_2} \right]$$

$$- \beta_o \tan \beta_o h_2 \left[\theta_1^s(h_1) + \theta_4^s(h_2) \right] \quad (1.89)$$

For the zeroth order solutions, we assume that $\theta_4^s(y)$ in (1.70) and $\theta_1^s(x)$ in (1.72) can be approximated respectively as

$$\left[\theta_4^s(y) \right]_0 = \frac{E_o}{\beta_o} \quad (1.90)$$

$$\left[\theta_1^s(y) \right]_0 = 0 \quad (1.91)$$

With (1.90) and (1.91) substituted in (1.88) and (1.89), we obtain

$$\left[C_4^s \right]_0 = - \frac{E_o}{\beta_o} \frac{\cos \beta_o h_1}{\cos \beta_o (h_1 + h_2)} \quad (1.92)$$

$$\left[C_1^s \right]_0 = \frac{E_o}{\beta_o} \frac{\sin \beta_o h_2}{\cos \beta_o (h_1 + h_2)} \quad (1.93)$$

Consequently, the zeroth order solutions for the currents are obtained as

$$\left[I_{4y}^s(y) \right]_0 = \frac{-j4\pi}{\zeta_o \Phi_s} \left\{ - \frac{E_o}{\beta_o} \frac{\cos \beta_o h_1}{\cos \beta_o (h_1 + h_2)} \cos \beta_o y + \theta_4^s(y) \right\} \quad (1.94)$$

$$\left[I_{1x}^s(x) \right]_0 = \frac{-j4\pi}{\zeta_o \Phi_s} \left\{ \frac{E_o}{\beta_o} \frac{\sin \beta_o h_2}{\cos \beta_o (h_1 + h_2)} \sin \beta_o x + \theta_1^s(x) \right\} \quad (1.95)$$

If (1.90) and (1.91) are applied to (1.94) and (1.95), we have

$$\left[I_{4y}^s(y) \right]_{00} = \frac{j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o} \frac{\cos \beta_o h_1 \cos \beta_o y - \cos \beta_o (h_1 + h_2)}{\cos \beta_o (h_1 + h_2)} \quad (1.96)$$

$$\left[I_{1x}^s(y) \right]_{00} = \frac{-j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o} \frac{\sin \beta_o h_2}{\cos \beta_o (h_1 + h_2)} \sin \beta_o x \quad (1.97)$$

(D) First Order Solutions for Loop Currents

To obtain the first order solutions for the currents, the method used by Chen and King is adopted.

The following equations are needed:

From (1.69) and (1.71)

$$A_{4y}^s(y) = \frac{-j}{v_o} \left[C_4^s \cos \beta_o y + \theta_4^s(y) \right]$$

$$A_{1x}^s(x) = \frac{-j}{v_o} \left[C_1^s \sin \beta_o x + \theta_1^s(x) \right]$$

From (1.75) and (1.76), we have

$$\left[I_{4y}(y) \right]_0 = \frac{-j4\pi}{\zeta_o \Phi_s} \left[C_4^s \cos \beta_o y + \theta_4^s(y) \right]$$

$$\left[I_{1x}^s(x) \right]_0 = \frac{-j4\pi}{\zeta_o \Phi_s} \left[C_1^s \sin \beta_o x + \theta_1^s(x) \right]$$

From (1.85) and (1.86) we have

$$\left[q_4^s(y) \right]_0 = \frac{4\pi}{\omega \zeta_o \Phi_s} \left[-C_4^s \beta_o \sin \beta_o y + \frac{\partial \theta_4^s(y)}{\partial y} \right]$$

$$\left[q_1^s(x) \right]_0 = \frac{4\pi}{\omega \zeta_o \Phi_s} \left[C_1^s \beta_o \cos \beta_o x + \frac{\partial \theta_1^s(x)}{\partial x} \right]$$

First order currents are formulated as follows:

$$\begin{aligned} \left[I_{4y}^s(y) \right]_1 &= \left[I_{4y}^s(y) \right]_0 + \frac{4\pi}{\mu_o \Phi_s} \left\{ A_{4y}^s - \frac{\mu_o}{4\pi} \right. \\ &\quad \cdot \left. \int_{-h_2}^{h_2} \left[I_{4y}^s(y') \right]_0 K_{1A}(y, y') dy' \right\} \end{aligned} \quad (1.98)$$

$$\begin{aligned} \left[I_{1x}^s(x) \right]_1 &= \left[I_{1x}^s(x) \right]_0 + \frac{4\pi}{\mu_o \Phi_s} \left\{ A_{1x}^s - \frac{\mu_o}{4\pi} \right. \\ &\quad \cdot \left. \int_{-h_1}^{h_1} \left[I_{1x}^s(x') \right]_0 K_{2A}(x, x') dx' \right\} \end{aligned} \quad (1.99)$$

In effect, (1.98) and (1.99) can be rewritten as

$$\left[I_{4y}^s(y) \right]_1 \cong 2 \left[I_{4y}^s(y) \right]_0 - \frac{1}{\Phi_s} \int_{-h_2}^{h_2} \left[I_{4y}^s(y') \right]_0 K_{1A}(y, y') dy' \quad (1.100)$$

$$\left[I_{1x}^s(x) \right]_1 \cong 2 \left[I_{1x}^s(x) \right]_0 - \frac{1}{\Phi_s} \int_{-h_1}^{h_1} \left[I_{1x}^s(x') \right]_0 K_{2A}(x, x') dx' \quad (1.101)$$

The substitution of the zeroth order solutions in (1.100) and (1.101) yields

$$\begin{aligned} \left[I_{4y}^s(y) \right]_1 &= \frac{-j4\pi}{\zeta_o \Phi_s} \left\{ 2 \left[C_4^s \cos \beta_o y + \theta_4^s(y) \right] \right. \\ &\quad - \frac{C_4^s}{\Phi_s} \int_{-h_2}^{h_2} \cos \beta_o y' K_{1A}(y, y') dy' \\ &\quad \left. - \frac{1}{\Phi_s} \int_{-h_2}^{h_2} \theta_4^s(y') K_{1A}(y, y') dy' \right\} \\ &= \frac{-j4\pi}{\zeta_o \Phi_s} \left\{ C_4^s \left[2 \cos \beta_o y - \frac{T_{ca}(y)}{\Phi_s} \right] + 2 \theta_4^s(y) - \frac{n_1^s(y)}{\Phi_s} \right\} \\ &= \frac{-j4\pi}{\zeta_o \Phi_s} \left\{ C_4^s L_1^s(y) + 2 \theta_4^s(y) - N_1^s(y) \right\} \end{aligned} \quad (1.102)$$

where

$$L_1^s(y) = 2 \cos \beta_o y - \frac{T_{ca}(y)}{\Phi_s} \quad (1.103)$$

$$N_1^s(y) = \frac{n_1^s(y)}{\Phi_s} \quad (1.104)$$

$$T_{ca}(y) = \int_{-h_2}^{h_2} \cos \beta_o y' K_{1A}(y, y') dy' \quad (1.105)$$

$$n_1^s(y) = \int_{-h_2}^{h_2} \theta_4^s(y') K_{1A}(y, y') dy' \quad (1.106)$$

$$\begin{aligned} \left[I_{1x}^s(x) \right]_1 &= \frac{-j4\pi}{\zeta_o \Phi_s} \left\{ 2 \left[C_1^s \sin \beta_o x + \theta_1^s(x) \right] - \frac{C_1^s}{\Phi_s} \right. \\ &\quad \cdot \left. \int_{-h_1}^{h_1} \sin \beta_o x' K_{2A}(x, x') dx' - \frac{1}{\Phi_s} \int_{-h_1}^{h_1} \theta_1^s(x') K_{2A}(x, x') dx' \right\} \\ &= \frac{-j4\pi}{\zeta_o \Phi_s} \left\{ C_1^s \left[2 \sin \beta_o x - \frac{T_{s*d}(x)}{\Phi_s} \right] + 2\theta_1^s(x) - \frac{d_1^s(x)}{\Phi_s} \right\} \\ &= \frac{-j4\pi}{\zeta_o \Phi_s} \left\{ C_1^s M_1^s(x) + 2\theta_1^s(x) - D_1^s(x) \right\} \end{aligned} \quad (1.107)$$

where

$$M_1^s(x) = 2 \sin \beta_o x - \frac{T_{s*d}(x)}{\Phi_s} \quad (1.108)$$

$$D_1^s(x) = \frac{d_1^s(x)}{\Phi_s} \quad (1.109)$$

$$T_{s*d}(x) = \int_{-h_1}^{h_1} \sin \beta_o x' K_{2A}(x, x') dx' \quad (1.110)$$

$$d_1^s(x) = \int_{-h_1}^{h_1} \theta_1^s(x') K_{2A}(x, x') dx' \quad (1.111)$$

The scalar potentials ϕ_{41}^s and ϕ_{43}^s in (1.45) can be expressed in terms of the charges as

$$\begin{aligned} \phi_{41}^s + \phi_{43}^s &= \frac{1}{4\pi\epsilon_o} \int_{-h_1}^{h_1} q_1^s(x') \frac{e^{-j\beta_o r_{41}}}{r_{41}} dx' \\ &+ \frac{1}{4\pi\epsilon_o} \int_{-h_1}^{h_1} q_3^s(x') \frac{e^{-j\beta_o r_{43}}}{r_{43}} dx' \\ &= \frac{1}{4\pi\epsilon_o} \int_{-h_1}^{h_1} q_1^s(x') K_{1B}(y, x') dx' \end{aligned} \quad (1.112)$$

where

$$K_{1B}(y, x') = \frac{e^{-j\beta_o r_{41}}}{r_{41}} - \frac{e^{-j\beta_o r_{43}}}{r_{43}} \quad (1.113)$$

$$r_{41} = \sqrt{(h_2 - y)^2 + (h_1 - x')^2} \quad (1.114)$$

$$r_{43} = \sqrt{(h_2 + y)^2 + (h_1 - x')^2} \quad (1.115)$$

Similarly,

$$\begin{aligned} \phi_{12}^s + \phi_{14}^s &= \frac{1}{4\pi\epsilon_o} \int_{-h_2}^{h_2} q_2^s(x') \frac{e^{-j\beta_o r_{12}}}{r_{12}} dy' \\ &+ \frac{1}{4\pi\epsilon_o} \int_{-h_2}^{h_2} q_4^s(x') \frac{e^{-j\beta_o r_{14}}}{r_{14}} dy' \\ &= \frac{1}{4\pi\epsilon_o} \int_{-h_2}^{h_2} q_4^s(x') K_{2B}(x, y') dy' \end{aligned} \quad (1.116)$$

where

$$K_{2B}(x, y') = \frac{e^{-j\beta_o r_{12}}}{r_{12}} + \frac{e^{-j\beta_o r_{14}}}{r_{14}} \quad (1.117)$$

$$r_{12} = \sqrt{(h_2 - y')^2 + (h_1 + x)^2} \quad (1.118)$$

$$r_{14} = \sqrt{(h_2 - y')^2 + (h_1 - x)^2} \quad (1.119)$$

With (1.51) and (1.112), (1.45) can be rewritten as

$$\begin{aligned} \phi_4^s &= \phi_{44}^s + \phi_{42}^s + \phi_{41}^s + \phi_{43}^s \\ &= \frac{j\omega}{\beta_o} \frac{\partial A_{4y}^s(y)}{\partial y} + \frac{1}{4\pi\epsilon_o} \int_{-h_1}^{h_1} q_1^s(x') K_{1B}(y, x') dx' \end{aligned} \quad (1.120)$$

Similarly, with (1.64) and (1.116), (1.58) becomes

$$\begin{aligned} \phi_1^s &= \phi_{11}^s + \phi_{13}^s + \phi_{12}^s + \phi_{14}^s \\ &= \frac{j\omega}{\beta_o} \frac{\partial A_{1x}^s}{\partial x} + \frac{1}{4\pi\epsilon_o} \int_{-h_2}^{h_2} q_4^s(y') K_{2B}(x, y') dy' \end{aligned} \quad (1.121)$$

If we substitute the zeroth order solutions in (1.120) and (1.121) we obtain

$$\begin{aligned}
\phi_4^s(y) &= \frac{j\omega}{\beta_o^2} \frac{-j}{v_o} \left[-C_4^s \sin \beta_o y + \frac{\partial \theta_4^s(y)}{\partial y} \right] \\
&\quad + \frac{1}{4\pi\epsilon_o} \int_{-h_1}^{h_1} \left[q_1^s(x') \right]_0 K_{1B}(y, x') dx' \\
&= -C_4^s \sin \beta_o y + \frac{1}{\beta_o} \frac{\partial \theta_4^s(y)}{\partial y} + \frac{C_1^s}{\Phi_s} \int_{-h_1}^{h_1} \cos \beta_o x' K_{1B}(y, x') dx' \\
&\quad + \frac{1}{\Phi_s \beta_o} \int_{-h_1}^{h_1} \frac{\partial \theta_1^s(x')}{\partial x'} K_{1B}(y, x') dx' \\
&= -C_4^s \sin \beta_o y + \frac{1}{\beta_o} \frac{\partial \theta_4^s(y)}{\partial y} + C_1^s \frac{T_{cd}(y)}{\Phi_s} + \frac{f_1^s(y)}{\Phi_s} \\
&= -C_4^s (\sin \beta_o y + \frac{1}{\beta_o} \frac{\partial \theta_4^s(y)}{\partial y}) + C_1^s L_2^s(y) + F_1^s(y)
\end{aligned} \tag{1.122}$$

where

$$L_2^s(y) = \frac{T_{cd}(y)}{\Phi_s} \tag{1.123}$$

$$F_1^s(y) = \frac{f_1^s(y)}{\Phi_s} \tag{1.124}$$

$$T_{cd}(y) = \int_{-h_1}^{h_1} \cos \beta_o x' K_{1B}(y, x') dx' \tag{1.125}$$

$$f_1^s(y) = \frac{1}{\beta_o} \int_{-h_1}^{h_1} \frac{\partial \theta_1^s(x')}{\partial x'} K_{1B}(y, x') dx' \quad (1.126)$$

$$\begin{aligned} \phi_1^s(x) &= \phi_{11}^s + \phi_{13}^s + \phi_{12}^s + \phi_{14}^s \\ &= \frac{j\omega}{\beta_o} \frac{\partial A_{1x}^s}{\partial x} + \frac{1}{4\pi\epsilon_o} \int_{-h_2}^{h_2} q_4^s(y') K_{2B}(x, y') dy' \\ &= \frac{j\omega}{\beta_o} \frac{-j}{v_o} \left\{ C_1^s \beta_o \cos \beta_o x + \frac{\partial \theta_1^s(x)}{\partial x} \right\} \\ &\quad + \frac{1}{4\pi\epsilon_o} \int_{-h_2}^{h_2} [q_4^s(y')]_0 K_{2B}(x, y') dy' \\ &= C_1^s \cos \beta_o x + \frac{1}{\beta_o} \frac{\partial \theta_1^s(x)}{\partial x} - \frac{C_4^s}{\Phi_s} T_{s*a}(x) + \frac{f_2^s(x)}{\Phi_s} \\ &= C_1^s \cos \beta_o x + \frac{1}{\beta_o} \frac{\partial \theta_1^s(x)}{\partial x} - C_4^s M_2^s(x) + F_2^s(x) \end{aligned} \quad (1.127)$$

where

$$M_2^s(x) = \frac{T_{s*a}(x)}{\Phi_s} \quad (1.128)$$

$$F_2^s(x) = \frac{f_2^s(x)}{\Phi_s} \quad (1.129)$$

$$T_{s^*a}(x) = \int_{-h_2}^{h_2} \sin \beta_0 y' K_{2B}(x, y') dy' \quad (1.130)$$

$$f_2^s(x) = \frac{1}{\beta_0} \int_{-h_2}^{h_1} \frac{\partial \theta_4^s(y')}{\partial y'} K_{2B}(x, y') dy' \quad (1.131)$$

The boundary conditions to be used for the first order solutions are:

(a) Current at the corners being continuous

i. e.,

$$I_{4y}^s(y = h_2) = -I_{1x}^s(x = h_1) \quad (1.132)$$

(b) Scalar potential at the corners being continuous

i. e.,

$$\phi_4^s(y = h_2) = \phi_1^s(x = h_1) \quad (1.133)$$

Substitution of (1.102) and (1.107) in (1.132) gives

$$\begin{aligned} & C_4^s L_1^s(h_2) + 2\theta_4^s(h_2) - N_1^s(h_2) \\ &= - \left[C_1^s M_1^s(h_1) + 2\theta_1^s(h_1) - D_1^s(h_1) \right] \end{aligned} \quad (1.134)$$

With (1.122) and (1.127), (1.133) becomes

$$\begin{aligned}
& - C_4^s \sin \beta_o h_2 + \frac{1}{\beta_o} \left. \frac{\partial \theta_4^s(y)}{\partial y} \right|_{y=h_2} + C_1^s L_2^s(h_2) + F_1^s(h_2) \\
& = C_1^s \cos \beta_o h_1 + \frac{1}{\beta_o} \left. \frac{\partial \theta_1^s(x)}{\partial x} \right|_{x=h_1} - C_4^s M_2^s(h_1) + F_2^s(h_1)
\end{aligned} \tag{1.135}$$

Solving (1.134) and (1.135) for C_4^s and C_1^s , we obtain

$$C_4^s = \frac{\left[\begin{aligned} & \left[2\theta_4^s(h_2) + 2\theta_1^s(h_1) - N_1^s(h_2) - D_1^s(h_1) \right] \left[\cos \beta_o h_1 - L_2^s(h_2) \right] \\ & + M_1^s(h_1) \left[\frac{1}{\beta_o} \left. \frac{\partial \theta_4^s(y)}{\partial y} \right|_{y=h_2} - \frac{1}{\beta_o} \left. \frac{\partial \theta_1^s(x)}{\partial x} \right|_{x=h_1} + F_1^s(h_2) - F_2^s(h_1) \right] \end{aligned} \right]}{M_1^s(h_1) \left[\sin \beta_o h_2 - M_2^s(h_1) \right] - L_1^s(h_2) \left[\cos \beta_o h_1 - L_2^s(h_2) \right]} \tag{1.136}$$

$$C_1^s = \frac{\left[\begin{aligned} & \left[2\theta_4^s(h_2) + 2\theta_1^s(h_1) - N_1^s(h_2) - D_1^s(h_1) \right] \left[\sin \beta_o h_2 - M_2^s(h_1) \right] \\ & + L_1^s(h_2) \left[\frac{1}{\beta_o} \left. \frac{\partial \theta_4^s(y)}{\partial y} \right|_{y=h_2} - \frac{1}{\beta_o} \left. \frac{\partial \theta_1^s(x)}{\partial x} \right|_{x=h_1} + F_1^s(h_2) - F_2^s(h_1) \right] \end{aligned} \right]}{M_1^s(h_1) \left[\sin \beta_o h_2 - M_2^s(h_1) \right] - L_1^s(h_2) \left[\cos \beta_o h_1 - L_2^s(h_2) \right]} \tag{1.137}$$

For the first approximation, we assume that

$$\theta_4^s(y) \approx \left[\theta_4^s(y) \right]_0 = \frac{E_o}{\beta_o} \quad (1.138)$$

$$\theta_1^s(x) \approx \left[\theta_1^s(x) \right]_0 = 0 \quad (1.139)$$

Then, (1.106), (1.109), (1.124) and (1.129) can be rewritten as

$$N_1^s(y) = \frac{\frac{E_o}{\beta_o}}{\Phi_s} \int_{-h_2}^{h_2} K_{1A}(y, y') dy' = \frac{E_o}{\Phi_s \beta_o} T_{ea}(y) \quad (1.140)$$

where

$$T_{ea}(y) = \int_{-h_2}^{h_2} K_{1A}(y, y') dy' \quad (1.141)$$

$$D_1^s(x) = 0 \quad (1.142)$$

$$F_1^s(y) = 0 \quad (1.143)$$

$$F_2^s(x) = 0 \quad (1.144)$$

With (1.138) and (1.139), (1.136) and (1.137) can be simplified respectively as follows if only the terms with the order of $\frac{1}{\Phi_s}$ are retained in both numerator and denominator:

$$\left[C_4^s \right]_1 = - \frac{E_o}{\beta_o} \frac{\cos \beta_o h_1 + \frac{1}{\Phi_s} B^s}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \quad (1.145)$$

$$\left[C_1^s \right]_1 = \frac{E_o}{\beta_o} \frac{\sin \beta_o h_2 + \frac{1}{\Phi_s} H^s}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \quad (1.146)$$

where

$$B^s = - \frac{1}{2} \left[2 T_{cd}(h_2) + T_{ea}(h_2) \cos \beta_o h_1 \right] \quad (1.147)$$

$$G = \frac{1}{2} \left[T_{s*d}(h_1) \sin \beta_o h_2 + 2 T_{s*a}(h_1) \sin \beta_o h_1 \right. \\ \left. - T_{ca}(h_2) \cos \beta_o h_1 - 2 T_{cd}(h_2) \cos \beta_o h_2 \right] \quad (1.148)$$

$$H^s = - \frac{1}{2} \left[T_{ea}(h_2) \sin \beta_o h_2 + 2 T_{s*a}(h_1) \right] \quad (1.149)$$

Substituting (1.81), (1.145) and (1.146) in (1.79) and (1.80), we obtain

$$\left[I_{4y}^s(y) \right]_1 = \frac{-j4\pi}{\zeta_o \Phi_s} \left\{ - \frac{E_o}{\beta_o} \frac{\cos \beta_o h_1 + \frac{1}{\Phi_s} B^s}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \cos \beta_o y + \theta_4^s(y) \right\} \quad (1.150)$$

$$\left[I_{1x}^s(x) \right]_1 = \frac{-j4\pi}{\zeta_o \Phi_s} \left\{ \frac{E_o}{\beta_o} \frac{\sin \beta_o h_2 + \frac{1}{\Phi_s} H^s}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \sin \beta_o x + \theta_1^s(x) \right\} \quad (1.151)$$

(E) Evaluation of Particular Integrals $\theta_4^s(y)$ and $\theta_1^s(x)$.

Integration by parts enables us to rewrite $\theta_4^s(y)$ and $\theta_1^s(x)$ in (1.70) and (1.72) respectively as

$$\begin{aligned}
\theta_4^s(y) &= \frac{E_o}{\beta_o} - \int_0^y \frac{\partial}{\partial s} \left[\phi_{41}^s(s) + \phi_{43}^s(s) \right] \sin \beta_o(y - s) ds \\
&= \frac{E_o}{\beta_o} + \left[\phi_{41}^s(0) + \phi_{43}^s(0) \right] \sin \beta_o y \\
&\quad + \beta_o \int_0^y \left[\phi_{41}^s(s) + \phi_{43}^s(s) \right] \cos \beta_o(y - s) ds \\
&= \frac{E_o}{\beta_o} + \beta_o \int_0^y \left[\phi_{41}^s(s) + \phi_{43}^s(s) \right] \cos \beta_o(y - s) ds
\end{aligned} \tag{1.152}$$

Equation (1.152) is arrived due to the condition of

$$\phi_{41}^s(0) = -\phi_{43}^s(0) \tag{1.153}$$

Similarly

$$\begin{aligned}
\theta_1^s(x) &= - \int_0^x \frac{\partial}{\partial s} \left[\phi_{12}^s(s) + \phi_{14}^s(s) \right] \sin \beta_o(x - s) ds \\
&= \left[\phi_{12}^s(0) + \phi_{14}^s(0) \right] \sin \beta_o x + \beta_o \int_0^x \left[\phi_{12}^s(0) + \phi_{14}^s(0) \right] \cos \beta_o(x - s) ds \\
&= 2\phi_{12}^s(0) \sin \beta_o x + \beta_o \int_0^x \left[\phi_{12}^s(s) + \phi_{14}^s(s) \right] \cos \beta_o(x - s) ds
\end{aligned} \tag{1.154}$$

Equation (1.154) is obtained because

$$\phi_{12}^s(0) = \phi_{14}^s(0) \quad (1.155)$$

For the first approximation, we substitute (1.138) and (1.139) in (1.85) and (1.86).

$$q_4^s(y) = -\frac{4\pi\epsilon_o}{\Phi_s} C_4^s \sin \beta_o y \quad (1.156)$$

$$q_1^s(x) = \frac{4\pi\epsilon_o}{\Phi_s} C_1^s \cos \beta_o x \quad (1.157)$$

The substitution of (1.157) in (1.112) yields

$$\begin{aligned} \phi_{41}^s(y) + \phi_{43}^s(y) &= \frac{C_1^s}{\Phi_s} \int_{-h_1}^{h_1} \cos \beta_o x' K_{1B}(y, x') dx' \\ &= \frac{C_1^s}{\Phi_s} T_{cd}(y) \end{aligned} \quad (1.158)$$

Similarly, with (1.156), (1.116) can be rewritten as

$$\begin{aligned} \phi_{12}^s(x) + \phi_{14}^s(x) &= \frac{-C_4^s}{\Phi_s} \int_{-h_2}^{h_2} \sin \beta_o y' K_{2B}(x, y') dy' \\ &= \frac{-C_4^s}{\Phi_s} T_{s^*a}(x) \end{aligned} \quad (1.159)$$

With (1.155), (1.159) yields

$$2\phi_{12}(0) = \frac{-C_4^s}{\Phi_s} T_{s^*a}(0) \quad (1.160)$$

Substituting (1.158) in (1.152), we obtain

$$\theta_4^s(y) = \frac{E_o}{\beta_o} + \frac{C_1^s \beta_o}{\Phi_s} \int_0^y T_{cd}(s) \cos \beta_o(y-s) ds \quad (1.161)$$

Similarly, with (1.158) and (1.159), (1.154) becomes

$$\theta_1^s(x) = -\frac{C_4^s}{\Phi_s} T_{s^*a}(0) \sin \beta_o x - \frac{C_4^s \beta_o}{\Phi_s} \int_0^x T_{s^*a}(s) \cos \beta_o(x-s) ds \quad (1.162)$$

With (1.146), (1.161) becomes

$$\begin{aligned} \left[\theta_4^s(y) \right]_1 &= \frac{E_o}{\beta_o} + \frac{\sin \beta_o h_2 + \frac{1}{\Phi_s} H^s}{\cos \beta_o(h_1 + h_2) + \frac{1}{\Phi_s} G} \frac{\beta_o}{\Phi_s} \frac{E_o}{\beta_o} \\ &\quad \cdot \int_0^y T_{cd}(s) \cos \beta_o(y-s) ds \\ &= \frac{E_o}{\beta_o} \left[1 + \frac{\beta_o}{\Phi_s} \frac{\sin \beta_o h_2 + \frac{1}{\Phi_s} H^s}{\cos \beta_o(h_1 + h_2) + \frac{1}{\Phi_s} G} t_{cd}(y) \right] \end{aligned} \quad (1.163)$$

where

$$t_{cd}(y) = \int_0^y T_{cd}(s) \cos \beta_o(y-s) ds \quad (1.164)$$

Similarly, the substitution of (1.145) in (1.162) gives

$$\begin{aligned}
 \left[\theta_1^s(x) \right]_1 &= \frac{E_o}{\beta_o \Phi_s} \frac{\cos \beta_o h_1 + \frac{1}{\Phi_s} B^s}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \\
 &\cdot \left[T_{s*a}(0) \sin \beta_o x + \beta_o \int_0^x T_{s*a}(s) \cos \beta_o (x - s) ds \right] \\
 &= \frac{E_o}{\beta_o} \frac{\cos \beta_o h_1 + \frac{1}{\Phi_s} B^s}{\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G \right]} \\
 &\cdot \left[T_{s*a}(0) \sin \beta_o x + \beta_o t_{s*a}(x) \right]
 \end{aligned} \tag{1.165}$$

where

$$t_{s*a}(x) = \int_0^x T_{s*a}(s) \cos \beta_o (x - s) ds \tag{1.166}$$

With (1.163), (1.150) becomes

$$\left[I_{4y}^s(y) \right]_{11} = \frac{j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o} x$$

$$\left[\cos \beta_o h_1 \cos \beta_o y - \cos \beta_o (h_1 + h_2) \right] + \frac{1}{\Phi_s} \left[B^s \cos \beta_o y - \beta_o \sin \beta_o h_2 t_{cd}(y) - G \right] \\ - \frac{1}{\Phi_s^2} \beta_o H^s t_{cd}(y)$$

$$\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G$$

$$= \frac{j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o}$$

$$\cdot \left[\frac{\cos \beta_o h_1 \cos \beta_o y - \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} P_{41}^s(y) + \frac{1}{\Phi_s^2} P_{42}^s(y)}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \right]$$

(1.167)

where

$$P_{41}^s(y) = B^s \cos \beta_o y - \beta_o \sin \beta_o h_2 t_{cd}(y) - G \quad (1.168)$$

$$P_{42}^s(y) = -\beta_o H^s t_{cd}(y) \quad (1.169)$$

Similarly, with (1.65), (1.51) becomes

$$\left[I_{1x}^s(x) \right]_{11} = \frac{-j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o} x$$

$$\begin{aligned} & \left[\sin \beta_o h_2 \sin \beta_o x + \frac{1}{\Phi_s} \left\{ H^s \sin \beta_o x + \left[T_{s*a}(0) \sin \beta_o x + \beta_o t_{s*a}(x) \right] \cos \beta_o h_1 \right\} \right. \\ & \left. + \frac{1}{\Phi_s^2} B^s \left[T_{s*a}(0) \sin \beta_o x + \beta_o t_{s*a}(x) \right] \right] \\ & \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G \\ & = \frac{-j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o} \\ & \cdot \frac{\sin \beta_o h_2 \sin \beta_o x + \frac{1}{\Phi_s} P_{11}^s(x) + \frac{1}{\Phi_s^2} P_{12}^s(x)}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \end{aligned} \quad (1.170)$$

where

$$P_{11}^s(x) = H^s \sin \beta_o x + \left[T_{s*a}(0) \sin \beta_o x + \beta_o t_{s*a}(x) \right] \cos \beta_o h_1 \quad (1.171)$$

$$P_{12}^s(x) = B^s \left[T_{s*a}(0) \sin \beta_o x + \beta_o t_{s*a}(x) \right] \quad (1.172)$$

$\left[I_{4y}^s(y) \right]_{11}$ and $\left[I_{1x}^s(x) \right]_{11}$ involve the double integrals which complicate

the problem.

In order to avoid the double integrals, the following method is devised at the expense of decreasing the accuracy. First of all,

substituting (1.138) to (1.144)

in (1.102), we obtain

$$\begin{aligned}
 \left[I_{4y}^s(y) \right]_{10} &= \frac{-j4\pi}{\zeta_o s} \left\{ -\frac{E_o}{\beta_o} \frac{\cos \beta_o h_1 + \frac{1}{\Phi_s} B^s}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \times \right. \\
 &\quad \left. \left[2 \cos \beta_o y - \frac{T_{ca}(y)}{\Phi_s} \right] + 2 \frac{E_o}{\beta_o} - \frac{E_o}{\beta_o \Phi_s} T_{ea}(y) \right\} \\
 &= \frac{j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o} \times \\
 &\quad \left[2 \left[\cos \beta_o h_1 \cos \beta_o y - \cos \beta_o (h_1 + h_2) \right] + \frac{1}{\Phi_s} \left[2 B^s \cos \beta_o y - 2G \right. \right. \\
 &\quad \left. \left. - \cos \beta_o h_1 T_{ca}(y) + \cos \beta_o (h_1 + h_2) T_{ea}(y) \right] \right. \\
 &\quad \left. + \frac{1}{\Phi_s^2} \left[G T_{ea}(y) - B^s T_{ca}(y) \right] \right] \\
 &\quad \cos \beta_o (h_1 + h_2) \frac{1}{\Phi_s} G \\
 &= \frac{j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o} \times \\
 &\quad \frac{2 \left[\cos \beta_o h_1 \cos \beta_o y - \cos \beta_o (h_1 + h_2) \right] + \frac{1}{\Phi_s} U_{41}^s(y) + \frac{1}{\Phi_s^2} U_{42}^s(y)}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \quad (1.173)
 \end{aligned}$$

where

$$U_{41}^s(y) = 2B^s \cos \beta_o y - 2G - \cos \beta_o h_1 T_{ca}(y) + \cos \beta_o (h_1 + h_2) T_{ea}(y) \quad (1.174)$$

$$U_{42}^s(y) = G T_{ea}(y) - B^s T_{ca}(y) \quad (1.175)$$

Similarly, the substitution of (1.138) to (1.144) in

(1.107) gives

$$\begin{aligned} \left[I_{1x}^s(x) \right]_{10} &= \frac{-j4\pi}{\zeta_o \Phi_s} \left\{ \frac{E_o}{\beta_o} \frac{\sin \beta_o h_2 + \frac{1}{\Phi_s} H^s}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \left[2 \sin \beta_o x - \frac{T_{s*d}(x)}{\Phi_s} \right] \right\} \\ &= \frac{-j4\pi}{\zeta_o s} \frac{E_o}{\beta_o} x \\ &\quad \frac{2 \sin \beta_o h_2 \sin \beta_o x + \frac{1}{\Phi_s} \left[2H^s \sin \beta_o x - \sin \beta_o h_2 T_{s*d}(x) \right] - \frac{H^s}{\Phi_s^2} T_{s*d}(x)}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \\ &\dots \\ &= \frac{-j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o} \frac{2 \sin \beta_o h_2 \sin \beta_o x + \frac{1}{\Phi_s} g_{11}^s(x) + \frac{1}{\Phi_s^2} g_{12}^s(x)}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \end{aligned} \quad (1.176)$$

where

$$g_{11}^s(x) = 2H^s \sin \beta_o x - \sin \beta_o h_2 T_{s*d}(x) \quad (1.177)$$

$$g_{12}^s(x) = -H^s T_{s*d}(x) \quad (1.178)$$

(F) Backscattering Cross Section of a Solid Rectangular Loop

The vector potential, maintained by the induced current in the solid rectangular loop, at an arbitrary point on the axis of the loop and in the far zone of the loop (Fig. 1.15) can be expressed as

$$\begin{aligned} A^s = A_y^s &= \frac{\mu_o}{4\pi} \int_{-h_2}^{h_2} 2 I_{4y}^s(y') \frac{e^{-j\beta_o R}}{R} dy' \\ &\cong \frac{\mu_o}{2\pi} \frac{e^{-j\beta_o R_o}}{R_o} \int_{-h_2}^{h_2} \left[I_{4y}^s(y') \right]_{11} dy' \end{aligned} \quad (1.179)$$

where

$$R = \sqrt{R_o^2 + y'^2 + h_1^2} \cong R_o \quad (1.180)$$

Substituting (1.167) in (1.179), we have

$$\left[A_y^s \right]_{11} = j \frac{2\mu_o}{\zeta_o \beta_o} E_o \frac{e^{-j\beta_o R_o}}{R_o} \frac{J_1}{\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right]} \quad (1.181)$$

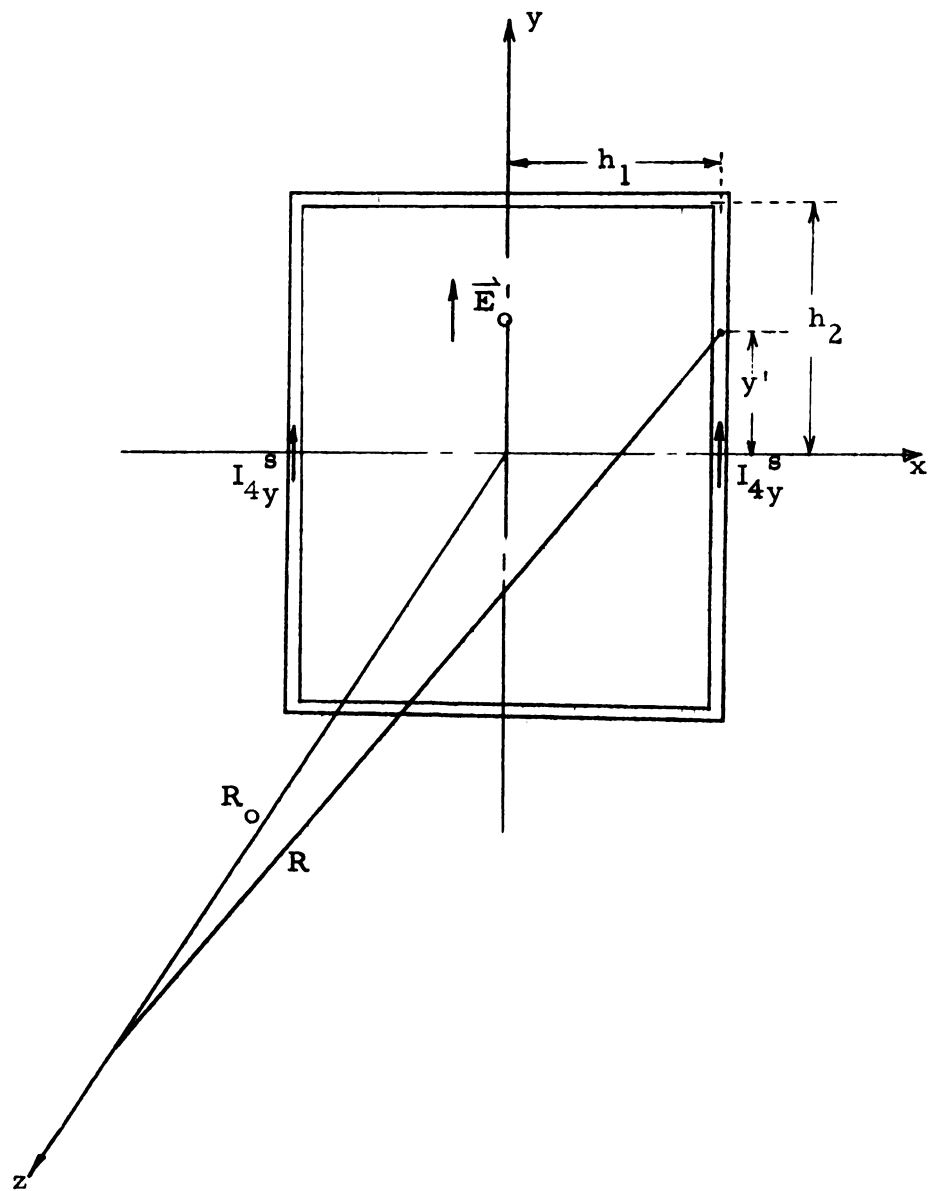


Figure 1.15. Geometry for the calculation of radiation field of a solid rectangular loop.

where

$$J_1 = \int_{-h_2}^{h_2} \left[\cos \beta_o h_1 \cos y' - \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} P_{41}^s(y') + \frac{1}{\Phi_s} P_{42}^s(y') \right] dy' \quad (1.182)$$

J_1 will be evaluated later. Consequently the backscattered electric field due to the solid rectangular loop is

$$\begin{aligned} \left[E^s \right]_{11} &= \left[E_y^s \right]_{11} = -j\omega \left[A_y^s \right]_{11} \\ &= 2E_o \frac{e^{-j\beta_o R_o}}{R_o} \frac{J_1}{\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right]} \end{aligned} \quad (1.183)$$

The backscattering cross section is found to be

$$\begin{aligned} \sigma_B &= \lim_{R_o \rightarrow \infty} 4\pi R_o^2 \frac{|E^s|^2}{|E_o|^2} \\ &= 16\pi \left| \frac{J_1}{\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right]} \right|^2 \end{aligned} \quad (1.184)$$

σ_B is plotted as the function of $\beta_o h$ in Figure 1.16 to compare with the experimental results for the case of the square loop where

$$h_1 = h_2 = h = \frac{5.15}{2} \text{ cm} \quad \text{and} \quad a = 0.03 \text{ cm} .$$

The agreement between the theory and the experiment is fairly good over the range where $\beta_o h$ is small.

A simpler method in obtaining the backscattering cross section is also presented here. The vector potential at an arbitrary point on the axis of the loop and in the far zone of the loop is calculated on the basis of $\left[I_{4y}^s(y) \right]_{10}$.

$$A^s = A_y^s = \frac{\mu_o}{4\pi} \int_{-h_2}^{h_2} 2 I_{4y}^s(y') \frac{e^{-j\beta_o R}}{R} dy'$$

$$\approx \frac{\mu_o}{2\pi} \frac{e^{-j\beta_o R_o}}{R_o} \int_{-h_2}^{h_2} \left[I_{4y}^s(y') \right]_{10} dy'$$

(1.185)

Substituting (1.173) in (1.185) , we have

$$\left[A_y^s \right]_{10} = j \frac{2\mu_o}{\epsilon_o \beta_o} E_o \frac{e^{-j\beta_o R_o}}{R_o} \frac{J_2}{\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right]}$$

(1.186)

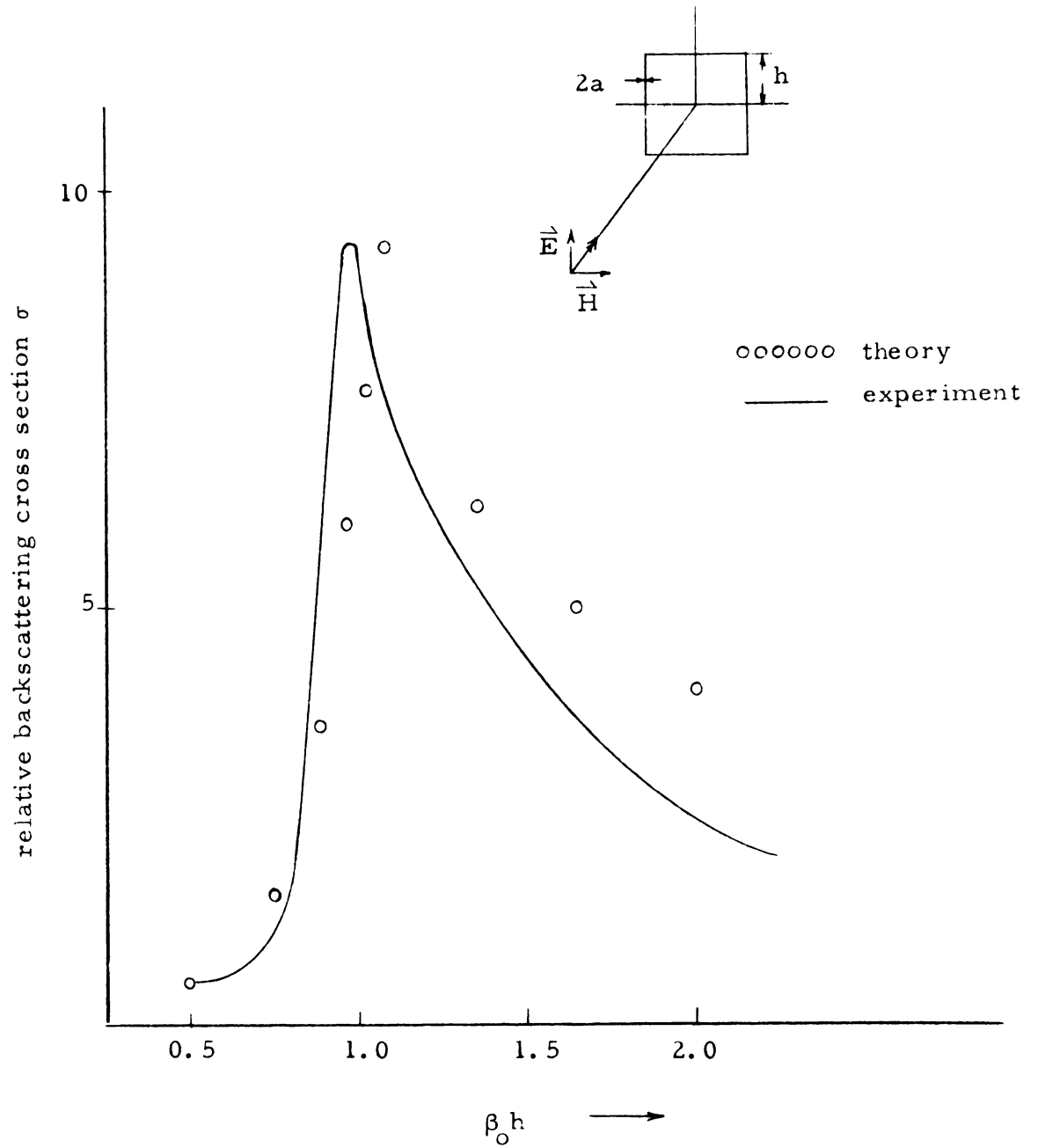


Figure 1.16. Comparison between theory and experiment for a square loop illuminated by a plane wave at normal incidence.

where

$$J_2 = \int_{-h_2}^{h_2} \left[2 \cos \beta_o h_1 \cos \beta_o y' - 2 \cos \beta_o (h_1 + h_2) + \frac{U_{41}^s(y')}{\Phi_s} + \frac{U_{42}^s(y')}{\Phi_s^2} \right] dy' \quad (1.187)$$

The evaluation of J_2 will be presented later.

The backscattered electric field is, therefore, calculated as

$$\begin{aligned} \left[E^s \right]_{10} &= \left[E_y^s \right]_{10} = -j\omega \left[A_y^s \right]_{10} \\ &= 2E_o \frac{e^{-j\beta_o R_o}}{R_o} \frac{J_2}{\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right]} \end{aligned} \quad (1.188)$$

Based on (1.188), the backscattering cross section of a solid rectangular loop is found to be

$$\begin{aligned} \sigma_B &= \lim_{R_o \rightarrow \infty} 4\pi R_o^2 \frac{|E^s|^2}{|E_o|^2} \\ &= 16\pi \left| \frac{J_2}{\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right]} \right|^2 \end{aligned} \quad (1.189)$$

(G) Expansion Parameter Φ_s

From (1.77) we have

$$\Phi_s(y) = \frac{4\pi}{\mu_o} \frac{A_{4y}^s(y)}{I_{4y}^s(y)}$$

Considering the zeroth order current

$$\left[I_{4y}^s(y) \right]_{00} = \frac{j4\pi}{\zeta_{os}} \frac{E_o}{\beta_o} \frac{\cos \beta_o h_1 \cos \beta_o y - \cos \beta_o (h_1 + h_2)}{\cos \beta_o (h_1 + h_2)}$$

we obtain

$$\Phi_s(y) = \frac{\int_{-h_2}^{h_2} [\cos \beta_o h_1 \cos \beta_o y' - \cos \beta_o (h_1 + h_2)] K_{1A}(y, y') dy'}{\cos \beta_o h_1 \cos \beta_o y - \cos \beta_o (h_1 + h_2)}$$

$$= \frac{\cos \beta_o h_1 T_{ca}(y) - \cos \beta_o (h_1 + h_2) T_{ea}(y)}{\cos \beta_o h_1 \cos \beta_o y - \cos \beta_o (h_1 + h_2)}$$

For a loop with $\beta_o (h_1 + h_2) \leq \frac{\pi}{2}$, the point of maximum current is at $y = 0$. We can then set

$$\Phi_s = \left[\Phi_s(y) \right]_{y=0} = \frac{\cos \beta_o h_1 T_{ca}(0) - \cos \beta_o (h_1 + h_2) T_{ea}(0)}{\cos \beta_o h_1 - \cos \beta_o (h_1 + h_2)} \quad (1.190)$$

For a loop with $\beta_o(h_1 + h_2) \geq \frac{\pi}{2}$, the point of maximum current is at $y = (h_1 + h_2) - \frac{\lambda}{4}$, so we set

$$\begin{aligned} \Phi_s &= \left[\Phi_s(y) \right]_{y=h_1 + h_2 - \frac{\lambda}{4}} \\ &= \frac{\cos \beta_o h_1 T_{ca}(h_1 + h_2 - \frac{\lambda}{4}) - \cos \beta_o(h_1 + h_2) T_{ea}(h_1 + h_2 - \frac{\lambda}{4})}{\cos \beta_o h_1 \sin \beta_o(h_1 + h_2) - \cos \beta_o(h_1 + h_2)} \end{aligned} \quad (1.191)$$

(H) Evaluation of J_1 and J_2

Since

$$\begin{aligned} J_1 &= \int_{-h_2}^{h_2} \left[\cos \beta_o h_1 \cos \beta_o y' - \cos \beta_o(h_1 + h_2) + \frac{1}{\Phi_s} P_{41}^s(y') \right. \\ &\quad \left. + \frac{1}{\Phi_s} P_{42}^s(y') \right] dy' \end{aligned}$$

where

$$P_{41}^s(y) = B^s \cos \beta_o y - \beta_o \sin \beta_o h_2 t_{cd}(y) - G$$

$$P_{42}^s(y) = -\beta_o H^s t_{cd}(y)$$

$$\begin{aligned}
J_1 = & \frac{2}{\beta_o} \cos \beta_o h_1 \sin \beta_o h_2 - 2h_2 \cos \beta_o (h_1 + h_2) \\
& + \frac{1}{\Phi_s} \left[\frac{2B^s}{\beta_o} \sin \beta_o h_2 - \beta_o \sin \beta_o h_2 b_{cd} - 2h_2 G \right] \\
& - \frac{1}{\Phi_s^2} \beta_o H^s b_{cd}
\end{aligned} \tag{1.192}$$

where

$$\begin{aligned}
b_{cd} &= \int_{-h_2}^{h_2} t_{cd}(y') dy' \\
&= \int_{-h_2}^{h_2} \left[\int_0^{y'} T_{cd}(s) \cos \beta_o (y' - s) ds \right] dy'
\end{aligned} \tag{1.193}$$

J_1 is difficult to evaluate because of the triple integral involved.

Equation (1.187) shows that

$$\begin{aligned}
J_2 = & \int_{-h_2}^{h_2} \left[2 \cos \beta_o h_1 \cos \beta_o y' - 2 \cos \beta_o (h_1 + h_2) \right. \\
& \left. + \frac{U_{41}^s(y')}{\Phi_s} + \frac{U_{42}^s(y')}{\Phi_s^2} \right] dy'
\end{aligned}$$

$$U_{41}^s(y) = 2B^s \cos \beta_o y - 2G - \cos \beta_o h_1 T_{ca}(y) + \cos \beta_o (h_1 + h_2) T_{ea}(y)$$

$$U_{42}^s(y) = G T_{ea}(y) - B^s T_{ca}(y)$$

or

$$\begin{aligned} J_2 = & \frac{4}{\beta_o} \cos \beta_o h_1 \sin \beta_o h_2 - 4h_2 \cos \beta_o (h_1 + h_2) \\ & + \frac{1}{\Phi_s} \left[\frac{4B^s}{\beta_o} \sin \beta_o h_2 - 4h_2 G - \cos \beta_o h_1 t_{ca} \right. \\ & \left. + \cos \beta_o (h_1 + h_2) t_{ea} \right] + \frac{1}{\Phi_s} \left[G t_{ea} - B^s t_{ca} \right] \end{aligned} \quad (1.194)$$

where

$$t_{ea} = \int_{-h_2}^{h_2} T_{ca}(y') dy' \quad (1.195)$$

$$t_{ca} = \int_{-h_2}^{h_2} T_{ea}(y') dy' \quad (1.196)$$

CHAPTER II

MINIMIZATION OF BACKSCATTERING OF A CIRCULAR LOOP

2.1. Introduction

The backscattering cross section of a metallic circular loop was studied in the previous chapter. It was found that when the loop is of resonant size, the induced current on the loop is maximum and likewise the backscattered field. In radar camouflage it is desirable to minimize the backscattering cross section of a loop, in particular, that of a resonant loop. Many investigations have been made in the recent years on the technique of minimizing the radar cross section of a metallic object. Two conventionally used techniques are to utilize radar absorbing material and to reshape the body to change the reflection patterns. Recently, a new method called the impedance loading method has been developed. This method was found to be especially effective in reducing the backscattering cross sections of metallic objects with dimensions of the order of a wavelength. Using this method Chen⁴⁻⁶ and others^{7, 8} have investigated the minimization of the backscattering from a cylindrical object. Liepa and Senior⁹ have applied the same technique to reduce the radar cross section of a conducting sphere.

The basic principle of the impedance loading method is to control the amplitude and phase of the induced current on the metallic object by inseting appropriate impedances at appropriate points on the object in such a way that the backscatter maintained by the induced current is minimized.

In this chapter, the minimization of the backscattering of a conducting, circular loop by an impedance loading method is investigated.

A perfectly conducting, circular loop which is loaded symmetrically with two identical lumped impedances, is assumed to be illuminated by a plane wave at normal incidence.

The induced current on the loaded loop is determined as a function of loop dimensions and loading impedance. The backscattered field maintained by the induced current in the loaded loop is calculated as a function of the loading impedance. It is then possible to find an optimum impedance which makes the back scattered field equal to zero. An explicit expression for the optimum impedance for zero backscattering is obtained as a function of loop dimensions. Some numerical examples are included.

Analytical study on the radiation of a loop is rare in the literature. Storer¹⁰ studied a loop antenna using a method of Fourier series expansion based on a Hallan's integral equation. This method is too complicated for our problem. A new method which is simple enough for our purpose is developed in this research. This new method is based on a differential rather than on an integral equation. The theory is later verified by an experiment.

2.2. Theory

Based on the principle of superposition, a loaded circular loop illuminated by a plane wave at normal incidence can be considered as the combination of (A) a solid loop illuminated by a normally incident plane wave, and (B) a loop driven by two identical voltages at $\theta = 0$ and $\theta = \pi$. The situation is shown graphically in Figure 2.1.1. Essentially, case (A) is the scattering of a solid circular loop and case (B) is the radiation of a circular loop antenna. The case (A) has already been solved in Chapter 1 and its results are rewritten here for further development. The problem of case (B) will be solved in this chapter.

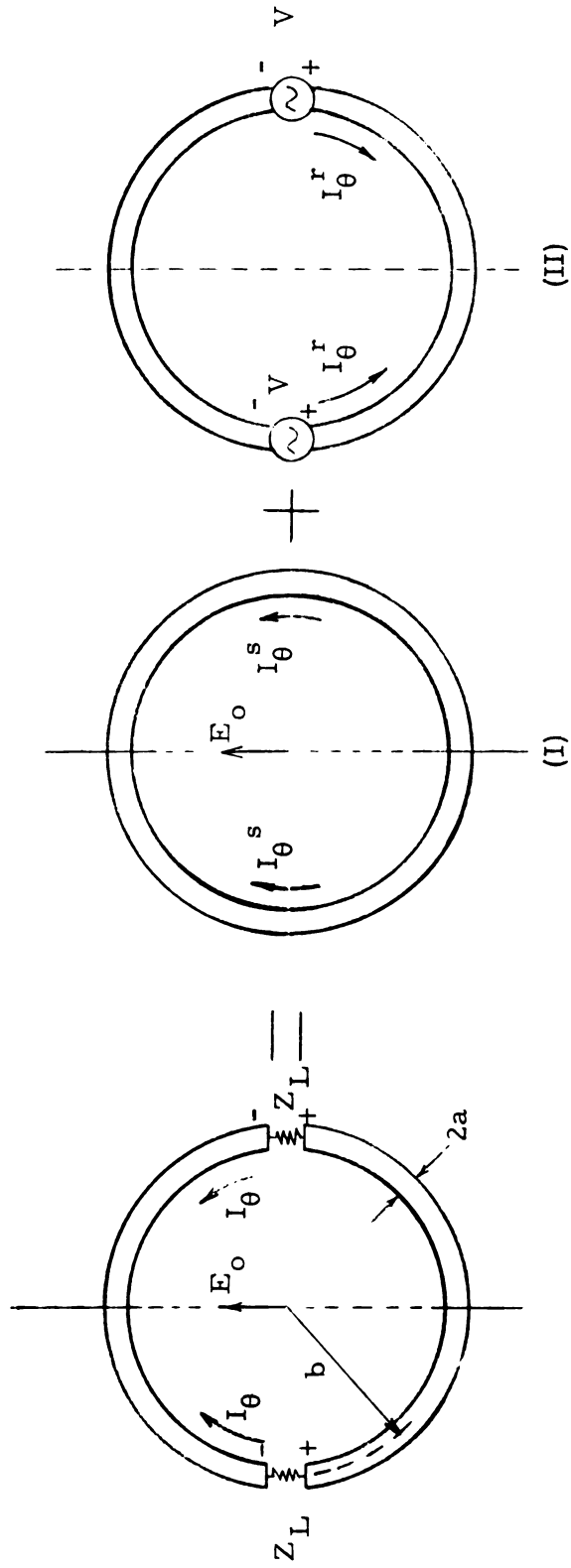


Figure 2.1.1.1. A loaded loop is the sum of a scattering loop and a radiating loop.

After this the results of case (A) will be combined with that of case (B) to produce the final solution for the problem of the scattering from a loaded loop.

2.2.1. Scattering from a Solid Circular Loop

The following equations from Chapter 1 are needed

$$I_{\theta}^s(\theta) = \frac{K^s E_o}{\beta_s^2 b^2 - 1} \cos \theta \quad (1.24)$$

$$\text{for } -\pi \leq \theta \leq \pi$$

$$E_y^s = -j \frac{\zeta_o}{4} \beta_o b \frac{e^{-\beta_o R_1}}{R_1} \frac{K^s E_o}{\beta_s^2 b^2 - 1} \quad (1.26)$$

where $I_{\theta}^s(\theta)$ is the induced current on the solid circular loop and E_y^s is the backscattered field on the axis and in the far zone of the loop maintained by the induced current.

2.2.2. Radiation from a Circular Loop

The geometry of the problem is as shown in Figure

2.1.2. The loop is driven by two identical voltages V at $\theta = 0$ and $\theta = \pi$. The dimensions of interest are

$$a^2 \ll b^2 \quad . \quad \beta_o^2 a^2 \ll 1$$

where β_o is the wave number. We assume that the wire is thin enough so that only the θ component of current is induced.

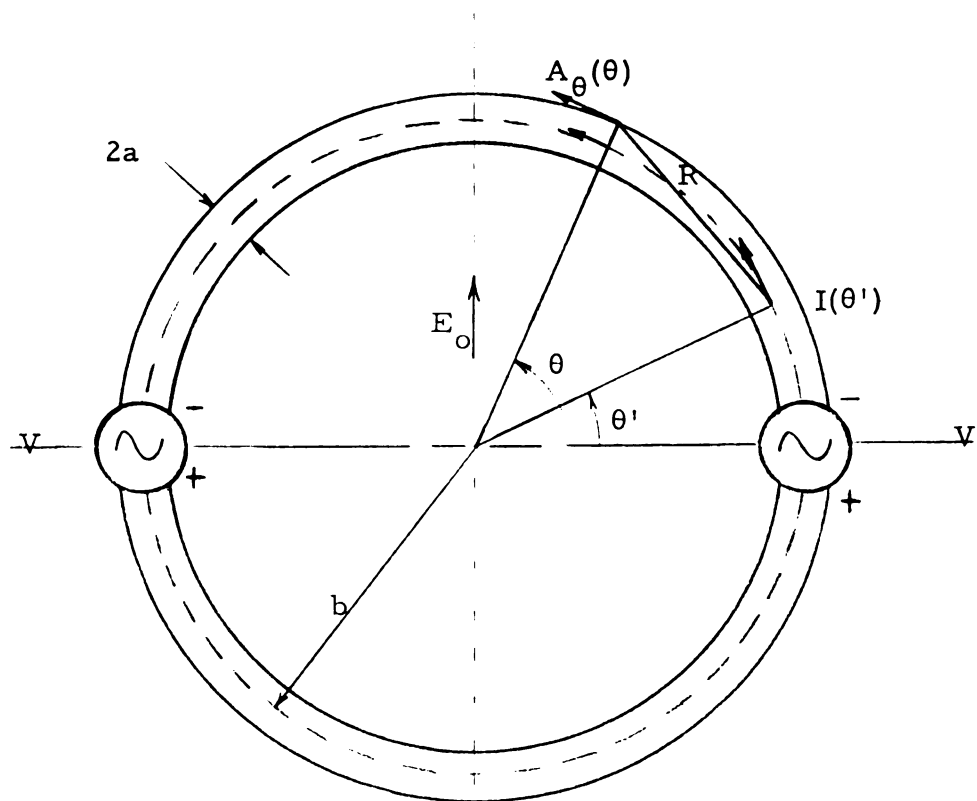


Figure 2.1.2. Geometry of a radiating loop.

(A) Differential Equation for Loop Current

The total tangential electric field should vanish at the surface of the loop except at the gaps at $\theta = 0$ and $\theta = \pi$ where voltages of V are maintained. Assuming that the gaps are very small, we obtain the following equation:

$$E_t^a = \frac{V}{b} \delta(\theta) - \frac{V}{b} \delta(\pi - \theta) \quad (2.1)$$

$$\text{for } -\pi \leq \theta \leq \pi$$

where E_t^a is the tangential electric field at the surface of the loop and $\delta(\theta)$ is a Dirac delta function. Due to the symmetrical configuration, the following conditions exist:

$$I_\theta^r(\theta) = -I_\theta^r(\pi - \theta) \quad (2.2)$$

$$I_\theta^r(\pm \frac{\pi}{2}) = 0 \quad (2.3)$$

where $I_\theta^r(\theta)$ is the current on the loop. This symmetry condition simplifies the problem and we only need to consider a half of the loop. In the right-half of the loop, (2.1) reduces to

$$E_t^a = \frac{V}{b} \delta(\theta) \quad (2.4)$$

$$\text{for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

The tangential electric field maintained by the current and the charge on the loop can be expressed as

$$E_t^a = -(\nabla \phi^r)_\theta - j\omega A_\theta^r \quad (2.5)$$

where ϕ^r is the scalar potential maintained by the charge on the loop and A_θ^r is the tangential component of the vector potential maintained by the current on the loop. $\phi^r(\theta)$ can be expressed as

$$\phi^r(\theta) = \frac{1}{4\pi\epsilon_0} \int_{-\pi}^{\pi} q^r(\theta') \frac{e^{-j\beta_0 R}}{R} b d\theta' \quad (2.6)$$

where $q^r(\theta')$ is the charge density induced on the loop at θ' , and

$$R = b \sqrt{4 \sin^2 \frac{\theta - \theta'}{2} + \frac{a^2}{b^2}} \quad \text{as mentioned in Chapter 1.}$$

Assume that $\phi^r(\theta)$ can be approximated as

$$\phi^r(\theta) = \frac{\Phi_q^r}{4\pi\epsilon_0} q^r(\theta) \quad (2.7)$$

where Φ_q^r is defined by

$$\Phi_q^r = 4\pi\epsilon_0 \frac{\phi^r(\theta)}{q^r(\theta)} \quad (2.8)$$

Φ_q^r is considered to be independent of θ because $\phi^r(\theta)$ in (2.6) is mainly contributed by the induced charge $q^r(\theta')$ in the vicinity of $\theta' = \theta$.

With (2.7), it leads to

$$\left[(\nabla \phi^r)_\theta \right]_{r=b} = \frac{\Phi_q^r}{4\pi\epsilon_0} \frac{1}{b} \frac{\partial q^r(\theta)}{\partial \theta} \quad (2.9)$$

By the equation of continuity,

$$q^r(\theta) = j \frac{1}{\omega} \nabla \cdot \vec{I}^r = j \frac{1}{\omega b} \frac{\partial I_{\theta}^r(\theta)}{\partial \theta} \quad (2.10)$$

(2.9) can then be rewritten as

$$\left[(\nabla \phi^r)_{\theta} \right]_{r=b} = \frac{\Phi_q^r}{4\pi\epsilon_o} \frac{j}{\omega b^2} \frac{\partial^2 I_{\theta}^r(\theta)}{\partial \theta^2} \quad (2.11)$$

Relating A_{θ}^r to the induced current $I_{\theta}^r(\theta)$ on the loop, we have

$$A_{\theta}^r(\theta) = \frac{\mu_o}{4\pi} \int_{-\pi}^{\pi} I_{\theta}^r(\theta') \frac{e^{-j\beta_o R}}{R} \cos(\theta - \theta') b d\theta' \quad (2.12)$$

$$\text{where } R = b \sqrt{4 \sin^2 \left(\frac{\theta - \theta'}{2} \right) + \frac{a^2}{b^2}}$$

Assume $A_{\theta}^r(\theta)$ can be approximated as

$$A_{\theta}^r(\theta) = \frac{\mu_o}{4\pi} \Phi_i^r I_{\theta}^r(\theta) \quad (2.13)$$

where Φ_i^r is defined by

$$\Phi_i^r = \frac{4\pi}{\mu_o} \frac{A_{\theta}^r(\theta)}{I_{\theta}^r(\theta)} \quad (2.14)$$

Φ_i^r here is presumed to be independent of θ since (2.12) shows that main contribution to the vector potential at θ is due to the current element located in the vicinity of θ .

With (2.11) and (2.13), (2.5) can be rewritten as

$$E_t^a = \frac{-j\Phi_q^r}{4\pi\epsilon_o\omega b^2} \left[\frac{\partial^2}{\partial\theta^2} + \beta_o^2 b^2 a^r \right] I_\theta^r(\theta) \quad (2.15)$$

where

$$a^r = \frac{\Phi_i^r}{\Phi_q^r} \quad (2.16)$$

The substitution of (2.15) in (2.4) gives

$$\left(\frac{\partial^2}{\partial\theta^2} + \beta_r^2 b^2 \right) I_\theta^r(\theta) = K^r \frac{V}{b} \delta(\theta) \quad (2.17)$$

$$\text{for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

where

$$\beta_r^2 = \beta_o^2 a^r = \beta_o^2 \frac{\Phi_i^r}{\Phi_q^r} \quad (2.18)$$

$$K^r = j \frac{4\pi\epsilon_o\omega b^2}{\Phi_q^r} \quad (2.19)$$

(B) Solution for Loop Current

The solution for $I_\theta^r(\theta)$ can be expressed as

$$I_\theta^r(\theta) = C_1^r \cos \beta_r b\theta + C_2^r \sin \beta_r b\theta + P^r(\theta) \quad (2.20)$$

where C_1^r and C_2^r are arbitrary constants, and $P^r(\theta)$ is a particular integral. C_2^r is zero due to the symmetry and $P^r(\theta)$ can be found to be

$$P^r(\theta) = \frac{K^r V}{2\beta_r b^2} \sin \beta_r b |\theta| \quad (2.21)$$

The application of the boundary condition of $I_\theta^r\left(\frac{\pi}{2}\right) = 0$ gives

$$C_1^r = \frac{-K^r V}{2\beta_r b^2} \tan \beta_r b \frac{\pi}{2} \quad (2.22)$$

Finally, we have

$$I_\theta^r(\theta) = -\frac{K^r V}{2\beta_r b^2} \sec \beta_r b \frac{\pi}{2} \sin \beta_r b \left(\frac{\pi}{2} - |\theta|\right) \quad (2.23)$$

$$\text{for } -\pi \leq \theta \leq \pi$$

The above expression for $I_\theta^r(\theta)$ is extended to cover the whole range of $-\pi \leq \theta \leq \pi$ by making use of the condition $I_\theta^r(\theta) = -I_\theta^r(\pi - \theta)$.

With the current found in (2.23), the vector potential on the axis of the loop and in the far zone of the loop can be calculated as

$$\begin{aligned} A_y^r &= \frac{\mu_o}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2I_\theta^r(\theta') \cos \theta' \frac{e^{-j\beta_o R_1}}{R_1} b d\theta' \\ &= \frac{\mu_o}{4\pi} \frac{e^{-j\beta_o R_1}}{R_1} \frac{K^r V}{\beta_r^2 b^2 - 1} \end{aligned} \quad (2.24)$$

where $R_1 = \sqrt{R_o^2 + b^2}$ and R_o is the distance between the center of the loop and an observation point on the axis of the loop.

The radiated electric field in the far zone of the loop is

$$E_y^r = -j\omega A_y^r = -j\omega \frac{\mu_o}{2\pi} \frac{e^{-j\beta_o R_1}}{R_1} \frac{K^r V}{\beta_r^2 b^2 - 1} \quad (2.25)$$

K^r and β_r are determined in a following section.

2.2.3. Total Current and Total Back Scattered Field

Total current can be obtained by superposing $I_\theta^s(\theta)$ upon $I_\theta^r(\theta)$ as follows:

$$\begin{aligned} I_\theta(\theta) &= I_\theta^s(\theta) + I_\theta^r(\theta) \\ &= \frac{K^s E_o}{\beta_s^2 b^2 - 1} \cos \theta \\ &\quad - \frac{K^r V}{2\beta_r b^2} \sec \beta_r b \frac{\pi}{2} \sin \beta_r b \left(\frac{\pi}{2} - |\theta| \right) \end{aligned} \quad (2.26)$$

$$\text{for } -\pi \leq \theta \leq \pi$$

Based on the principle of superposition, V is found to be

$$V = \left[I_\theta^s(0) + I_\theta^r(0) \right] Z_L \quad (2.27)$$

With (1.24) and (2.23), V is determined as

$$V = \frac{2 \beta_r b^2}{2 \beta_r b^2 + Z_L K^r \tan \beta_r b \frac{\pi}{2}} \frac{Z_L K^s E_o}{\beta_s^2 b^2 - 1} \quad (2.28)$$

The substitution of (2.28) in (2.26) yields a final expression for $I_\theta(\theta)$ as

$$I_\theta(\theta) = \frac{K^s E_o}{\beta_s^2 b^2 - 1} \left[\cos \theta - \frac{Z_L K^r \sec \beta_r b \frac{\pi}{2}}{2 \beta_r b^2 + Z_L K^r \tan \beta_r b \frac{\pi}{2}} \sin \beta_r b \left(\frac{\pi}{2} - |\theta| \right) \right] \quad (2.29)$$

for $-\pi \leq \theta \leq \pi$

Total E field in the far zone of the loop can be obtained as

$$\begin{aligned} E_y &= E_y^s + E_y^r \\ &= -j\omega \frac{\mu_o}{4\pi} \frac{e^{-j\beta_o R_1}}{R_1} \frac{K^s E_o b}{\beta_s^2 b^2 - 1} \times \\ &\quad \left[\pi + \frac{4 Z_L K^r \beta_r b}{(\beta_r^2 b^2 - 1) (2 \beta_r b^2 + Z_L K^r \tan \beta_r b \frac{\pi}{2})} \right] \end{aligned} \quad (2.30)$$

2.2.4. Optimum Impedance for Zero Back Scattering

To minimize the backscattering to zero, total field in (2.30) is set equal to zero. This leads to

$$\pi + \frac{4Z_L K^r \beta_r b}{(\beta_r b^2 - 1)(2\beta_r b^2 + Z_L K^r \tan \beta_r b \frac{\pi}{2})} = 0 \quad (2.31)$$

Solving (2.31) for Z_L , we obtain the optimum impedance for zero back scattering as

$$\left[Z_L \right]_0 = \frac{-2\pi\beta_r b^2 (\beta_r^2 b^2 - 1)}{K^r \left[4\beta_r b + \pi (\beta_r b^2 - 1) \tan \beta_r b \frac{\pi}{2} \right]} \quad (2.32)$$

With (2.19), $\left[Z_L \right]_0$ can be expressed as

$$\left[Z_L \right]_0 = \frac{j \zeta_o \sqrt{a^r} (\beta_r^2 b^2 - 1) \Phi_q^r}{2 \left[4\beta_r b + \pi (\beta_r b^2 - 1) \tan \beta_r b \frac{\pi}{2} \right]} \quad (2.33)$$

2.2.5. Determination of Φ_q^r and Φ_i^r

(A) By making use of (2.23) and the equation of continuity, we have

$$\begin{aligned} q^r(\theta) &= j \frac{1}{\omega b} \frac{\partial I_\theta^r(\theta)}{\partial \theta} \\ &= \pm j \frac{1}{\omega b} \frac{K^r V}{2b} \sec \beta_r b \frac{\pi}{2} \cos \beta_r b \left(\frac{\pi}{2} - |\theta| \right) \end{aligned} \quad (2.35)$$

— for $-\pi \leq \theta \leq 0$

+ for $0 \leq \theta \leq \pi$

Φ_q^r is then defined as

$$\Phi_q^r = \frac{- \int_{-\pi}^0 \cos \beta_r b \left(\frac{\pi}{2} + \theta' \right) \frac{e^{-j\beta_o R}}{R} b d\theta' + \int_0^{\pi} \cos \beta_r b \left(\frac{\pi}{2} - \theta' \right) \frac{e^{-j\beta_o R}}{R} b d\theta'}{\cos \beta_r b \left(\frac{\pi}{2} - \theta_o \right)} \quad (2.35)$$

where

$$R = b \sqrt{4 \sin^2 \left(\frac{\theta_o - \theta'}{2} \right) + \frac{a^2}{b^2}}$$

Choosing $\theta_o = \frac{\pi}{2}$, the point of maximum charge, Φ_q^r becomes

$$\Phi_q^r = \int_0^{\pi} \cos \beta_r b \left(\frac{\pi}{2} - \theta' \right) K_3 \left(\theta', \frac{a^2}{b^2} \right) d\theta' \quad (2.36)$$

where

$$K_3 \left(\theta', \frac{a^2}{b^2} \right) = \frac{e^{-j\beta_o b \sqrt{2 - 2 \sin \theta' + \frac{a^2}{b^2}}}}{\sqrt{2 - 2 \sin \theta' + \frac{a^2}{b^2}}} - \frac{e^{-j\beta_o b \sqrt{2 + 2 \sin \theta' + \frac{a^2}{b^2}}}}{\sqrt{2 + 2 \sin \theta' + \frac{a^2}{b^2}}} \quad (2.37)$$

In the actual calculation of Φ_q^r , β_r will be replaced by β_o as an approximation.

(B) Similarly, Φ_i^r can be obtained by making use of (2.23) as

$$\Phi_i^r = \frac{4\pi}{\mu_o} \frac{A_\theta^r(\theta)}{I_\theta^r(\theta)}$$

$$= \frac{\int_{-\pi}^{\pi} \sin \beta_r b \left(\frac{\pi}{2} - |\theta'| \right) \cos (\theta_o - \theta') \frac{e^{-j\beta_o R}}{R} b d\theta'}{\sin \beta_r b \left(\frac{\pi}{2} - |\theta_o| \right)}$$
(2.38)

where

$$R = b \sqrt{4 \sin^2 \left(\frac{\theta - \theta'}{2} \right) + \frac{a^2}{b^2}}$$

Choosing

$$\theta_o = \begin{cases} 0 & \text{for } \beta_r b \leq 1 \\ \frac{\pi}{2} - \frac{\lambda}{4b} & \text{for } \beta_r b \geq 1 \end{cases}$$

where θ_o is the point of maximum current. Φ_i^r then becomes

$$\Phi_i^r = \begin{cases} \frac{1}{\sin \beta_r b \frac{\pi}{2}} \int_{-\pi}^{\pi} \sin \beta_r b \left(\frac{\pi}{2} - |\theta'| \right) \cos \theta' K_4 \left(\theta', \frac{a^2}{b^2} \right) d\theta' & \text{for } \beta_r b \leq 1 \\ \int_{-\pi}^{\pi} \sin \beta_r b \left(\frac{\pi}{2} - |\theta'| \right) \sin \left(\frac{\lambda}{4b} + \theta' \right) K_5 \left(\theta', \frac{a^2}{b^2} \right) d\theta' & \text{for } \beta_r b \geq 1 \end{cases}$$
(2.39)

where

$$K_4\left(\theta', \frac{a^2}{b^2}\right) = \frac{e^{-j\beta_0 b \sqrt{2 - 2 \cos \theta' + \frac{a^2}{b^2}}}}{\sqrt{2 - 2 \cos \theta' + \frac{a^2}{b^2}}} \quad (2.40)$$

$$K_5\left(\theta', \frac{a^2}{b^2}\right) = \frac{e^{-j\beta_0 b \sqrt{2 - 2 \sin\left(\frac{\lambda}{4b} + \theta'\right) + \frac{a^2}{b^2}}}}{\sqrt{2 - 2 \sin\left(\frac{\lambda}{4b} + \theta'\right) + \frac{a^2}{b^2}}} \quad (2.41)$$

In the actual calculation of Φ_i^r , β_r will be replaced by β_0 as an approximation.

2.2.6. Numerical Examples

To show the theoretical results graphically, Φ_q^r and Φ_i^r are numerically calculated for the case of $a^2/b^2 = 0.00179$ as function of $\beta_0 b$. The optimum impedance, $[Z_L]_o$, is then calculated as a function of $\beta_0 b$ for the case of $a^2/b^2 = 0.00179$. Numerical results of Φ_q^r and Φ_i^r are shown graphically in Figure 2.2 and 2.3 respectively. The numerical result of $[Z_L]_o$ is shown in Figure 2.4. In Figure 2.2 and 2.3, Φ_q^r and Φ_i^r vary only weakly over the range of interest. This agrees reasonably well with our original assumption of Φ_q^r and Φ_i^r being constant. In Figure 2.4 the following facts are observed:

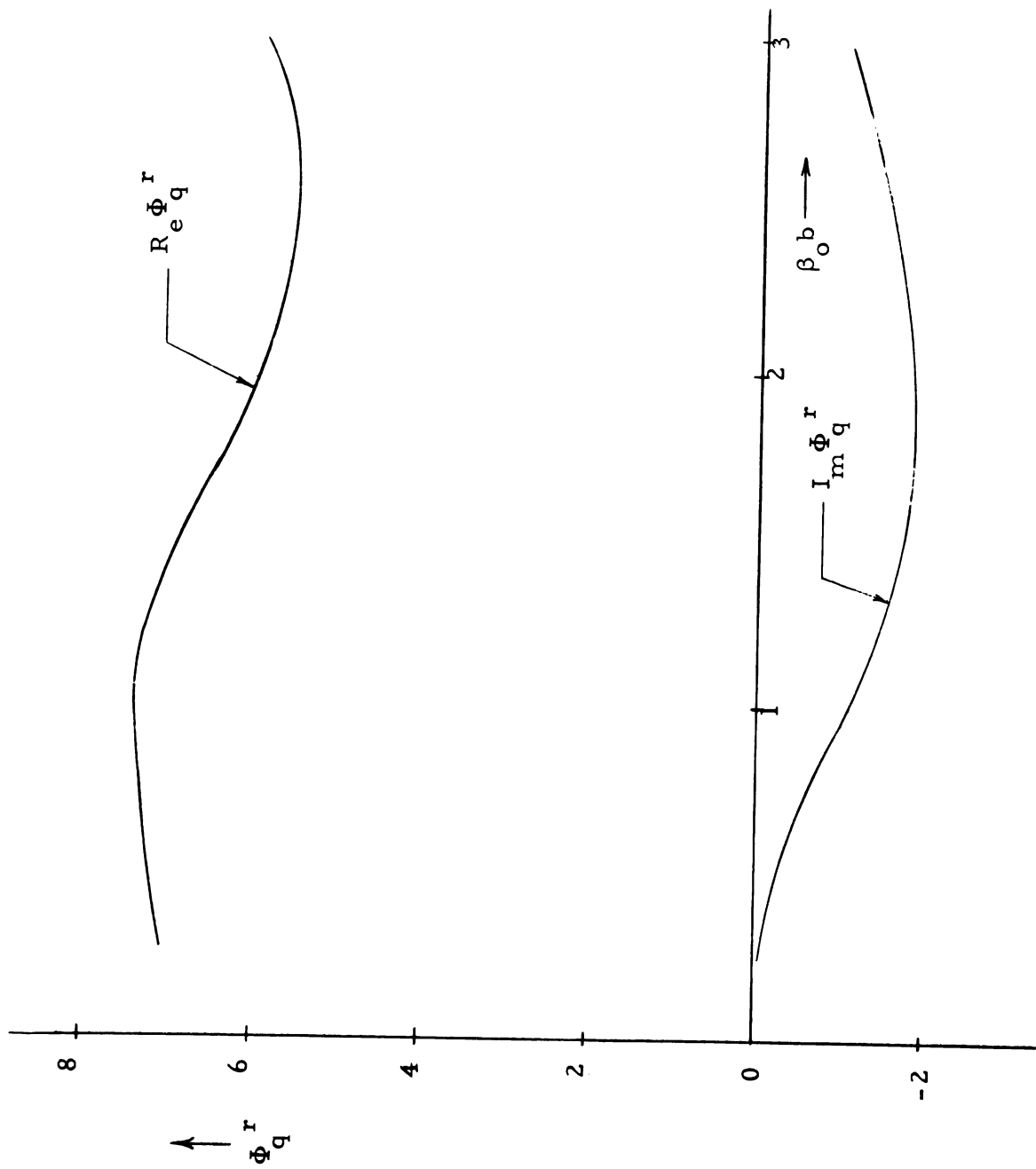


Figure 2.2. Φ_q^r as a function of $\beta_0 b$ ($a^2/b^2 = 0.00179$).

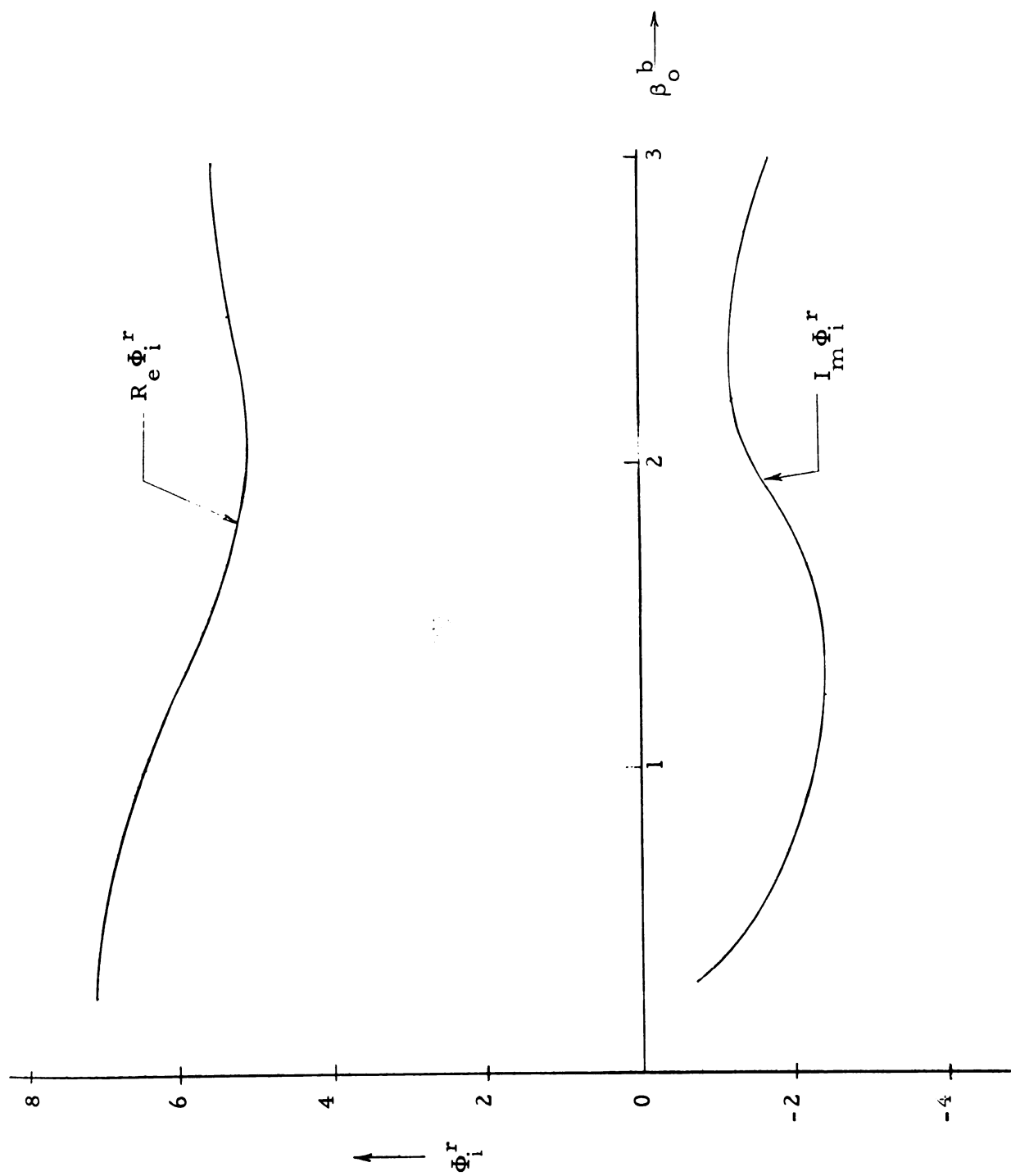


Figure 2.3. Φ_i^r as a function of β_o^b ($a^2/b^2 = 0.00179$).

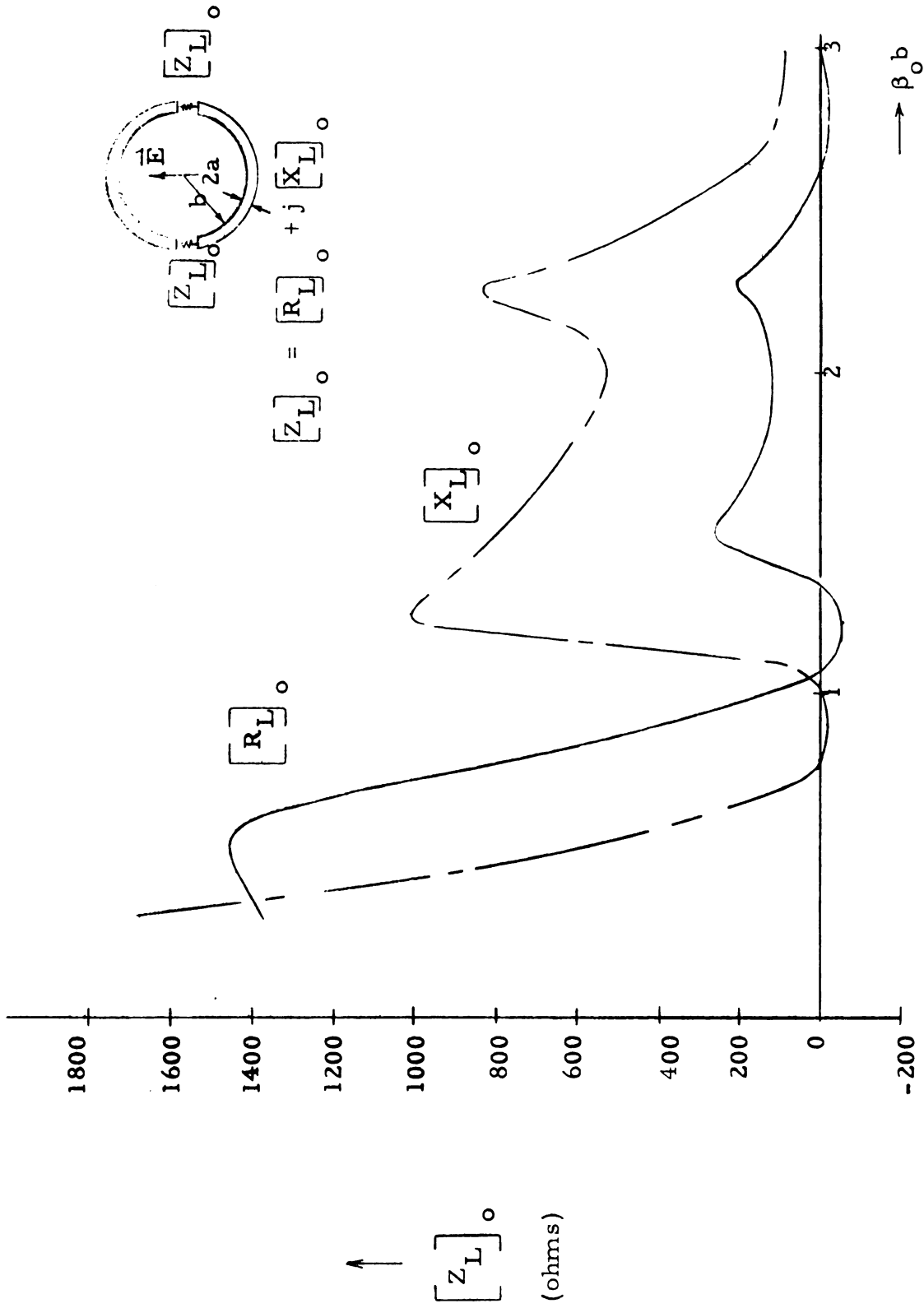


Figure 2.4. Optimum impedance for zero backscattering
 $(a^2/b^2 = 0.00179)$.

- (1) In general, the optimum impedance for zero back scattering needs both a resistive and a reactive component.
- (2) It requires a negative resistance around $\beta_o b = 1.25$ and a reactance with negative slope around $\beta_o b = 1$.

2.3. Experiment

As mentioned in previous section, the optimum impedance for zero backscattering, in general, requires a negative resistance. Practically, it is difficult to implement. To simplify the problem, an experiment was conducted for the case of a circular loop loaded with two purely reactive impedances.

In this section, experimental arrangement and experimental results are discussed. The comparison between the theory and the experimental results is also given.

2.3.1. Experimental Setup and Measuring Technique

The experimental setup is shown in Figure 2.5. The experiment was conducted inside of an anechoic chamber which is constructed on the top of an aluminum ground plane (10' x 12', 0.125" in thickness). In the experiment, the R.F. signal is radiated from a horn antenna (Scientific Atlanta model 12-1.7) located at one end of the ground plane and the circular loop is placed at the other end. With this arrangement, a plane wave is incident normally upon the loop. A slot is cut at the central part of the ground plane. A thin wire probe (Central Res. Lab. MX-1019/u) protruding out of the ground plane is movable along

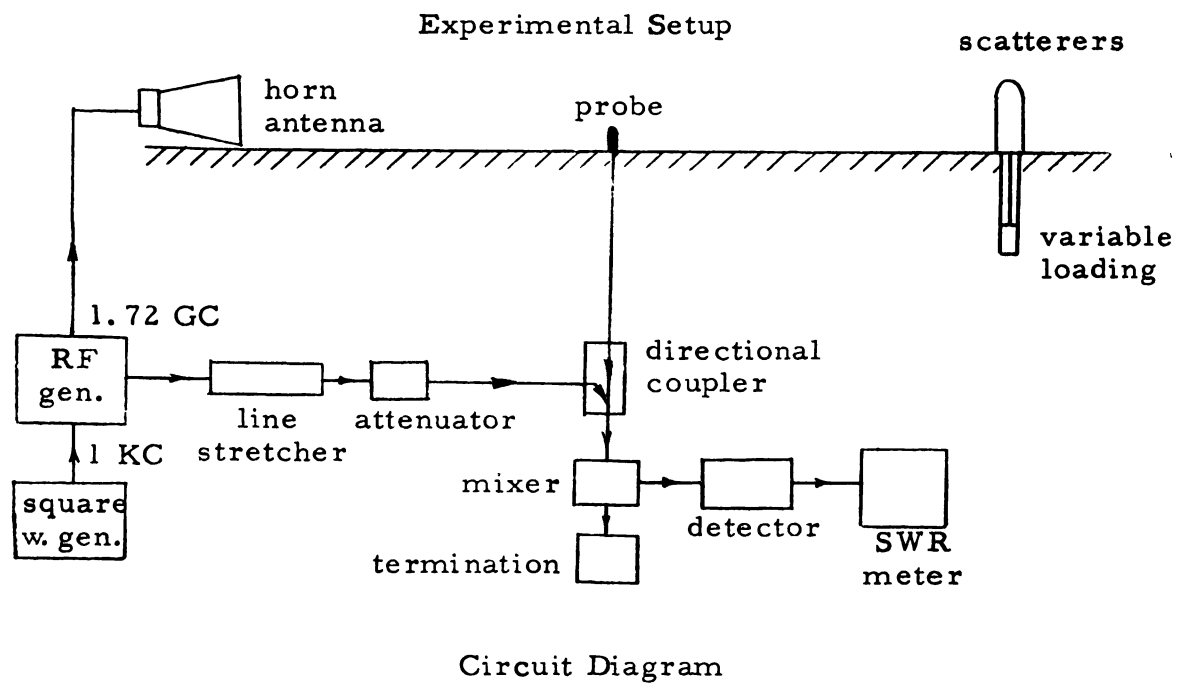
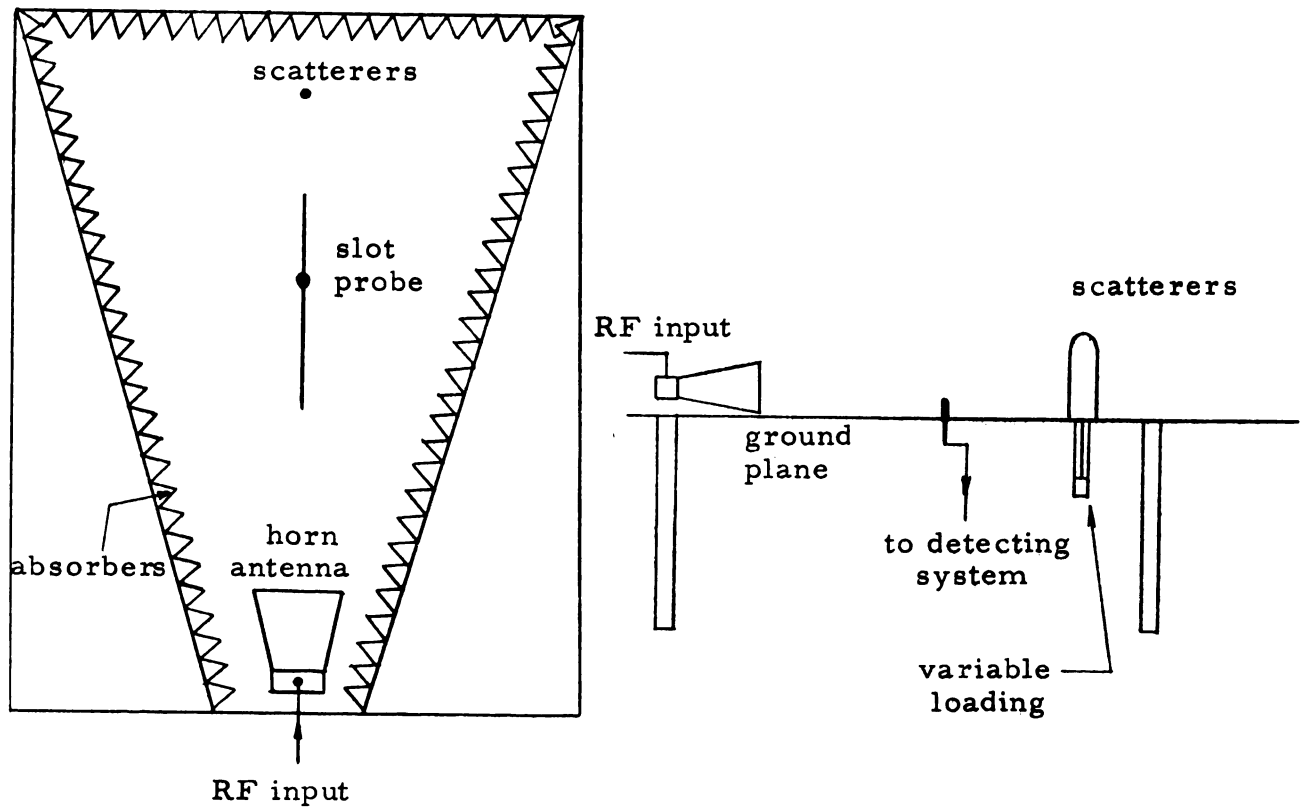


Figure 2.5. Experimental setup and circuit diagram.

the slot. The loop is loaded with a pair of identical coaxial lines (characteristic impedance $Z_c = 50 \Omega$) underneath the ground plane. This simple device provides a purely reactive loading to the loop. The approximate impedance of the coaxial cavity can be calculated from the well known expression $Z_L = jZ_c \tan \beta_0 l$ where Z_c is the characteristic impedance of the coaxial cavity, β_0 is the wave number and l is the length of the coaxial cavity.

The circuit diagram is also shown in Figure 2.5. The method of cancellation is employed in the experiment. The R.F. signal is generated from a microwave oscillator (GR 1360), modulated by a 1 KC square wave generator. The output of the R.F. oscillator is connected to the horn antenna. When the scatterer is absent, the signal received by the probe is cancelled by a reference signal from the R.F. oscillator through a line stretcher (GR874-LK20L) and an attenuator (ARRA 2414-20). The cancellation of the probe signal by the reference signal is accomplished by the combination of these two signals in a mixer after going through a directional coupler. After the cancellation process is completed, the loop is introduced. The output of the mixer or the reading of SWR meter will then represent the backscattered field by the loop.

2.3.2. Experimental Results

A circular loop (diameter = 5.15 cm) was constructed as an experimental model using cylindrical wire of 0.1 cm radius. The experiment was performed at various frequencies (1.6 GC - 2.5 GC). Experimental results are shown in Figures 2.6 - 2.20 in which the backscattering cross sections of the loop wire plotted as functions of loading impedances at each specified frequency.

The backscattering cross section of the loaded loop is represented by a solid curve and that of the solid loop by a solid straight line. It is observed that if the loading impedance (or the length of the coaxial line) is properly adjusted, the backscatters of the loop can be minimized to the noise level. At each frequency about 15 db reduction in the back scattering cross section was obtained. It was also observed that the scattering in the off broadside direction was reduced when the backscatter in the broadside direction was minimized.

2.4. Comparison Between Theory and Experiment

To check the accuracy of the theory, the optimum reactive impedance for the minimum backscattering was numerically calculated and then compared with the experimental results.

The theoretical value for the optimum reactive impedance is calculated as follows:

(1) In the expression for the total back-scattered field, equation (2.30), the loading impedance Z_L is replaced by jX_L ,

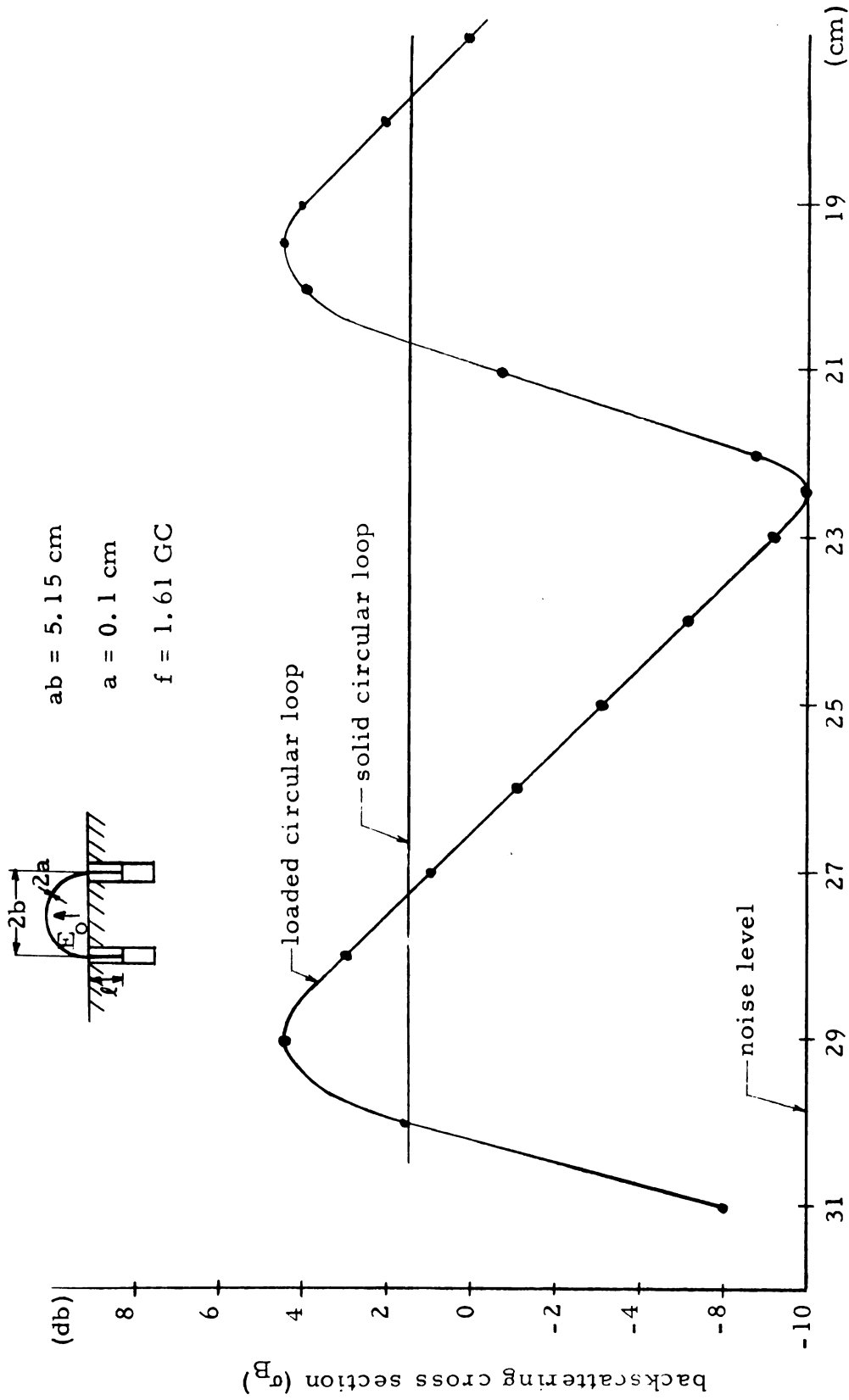
(2) An expression for the total backscattered field is numerically obtained for a particular frequency as the function of X_L ,

(3) A computer program is then set to calculate the particular value of X_L which gives the minimum value of the total backscattered field,

(4) Thus the optimum reactive impedance, $\left[X_L\right]_0$, is obtained as a function of $\beta_0 b$.

The experimental value for the optimum reactive impedance was obtained by taking into account of the stray capacitance which exists at the end of the coaxial line.

The experimental and theoretical results of the optimum reactive impedance for the minimum backscattering are shown graphically in Figure 2.21. The dotted curve represents the experimental results and the solid curve represents that of the theory. The agreement between theory and experiment is reasonably good.



(ℓ) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_o \ell$)

Figure 2.6. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 1.61$ GC).

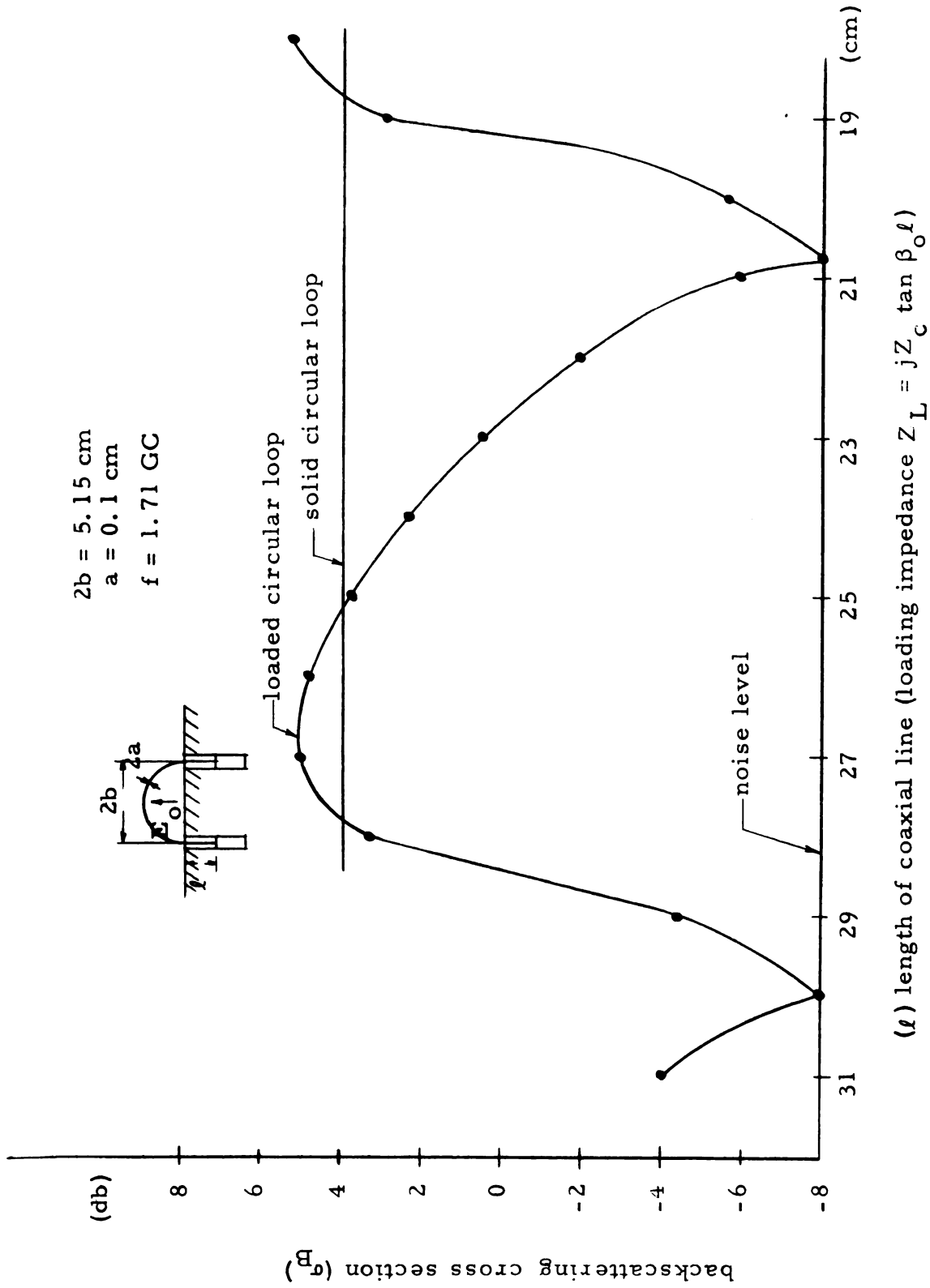
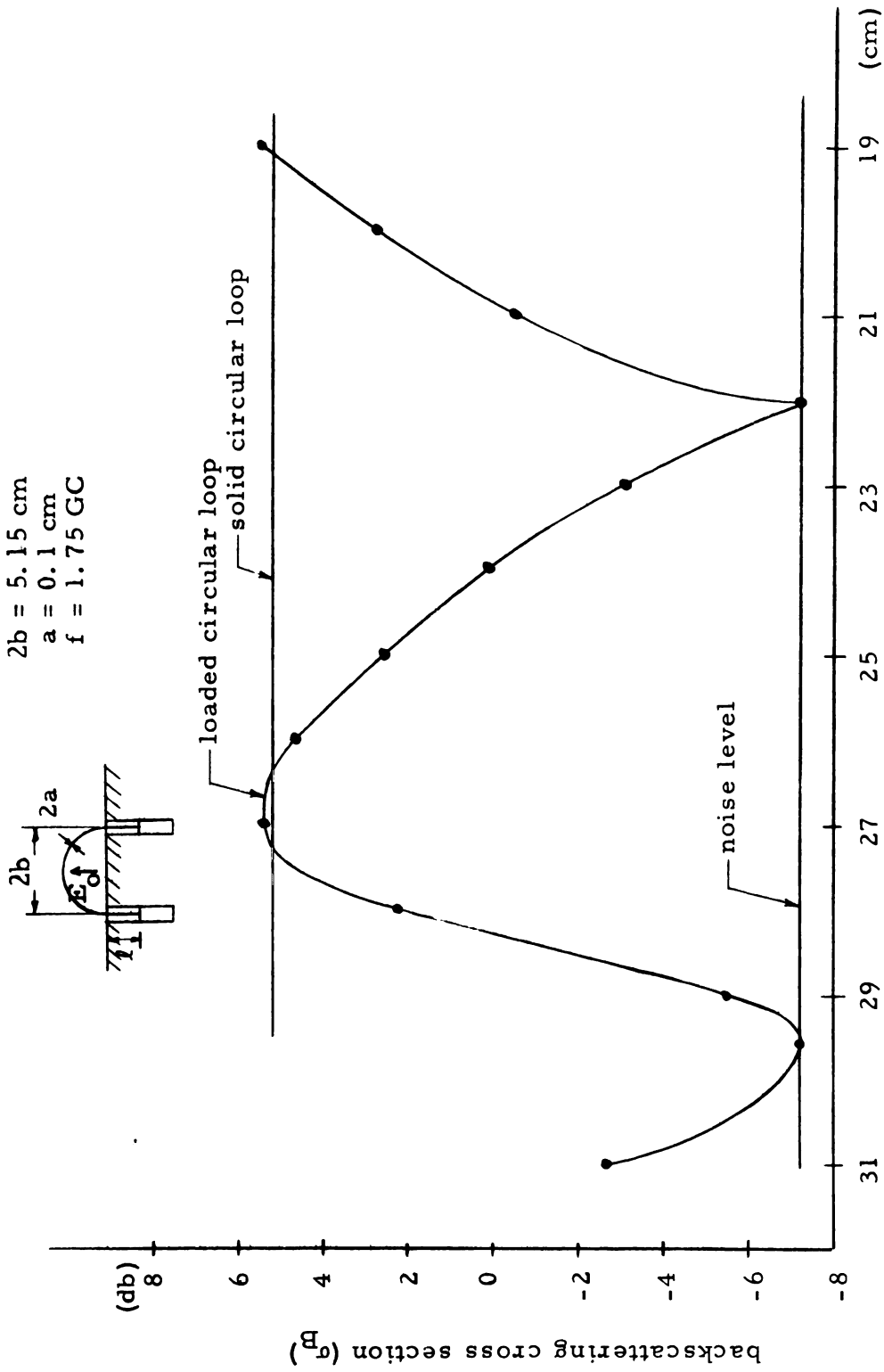


Figure 2.7. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 1.71 \text{ GC}$).



(ℓ) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_0 \ell$)

Figure 2.8. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 1.75$ GC)

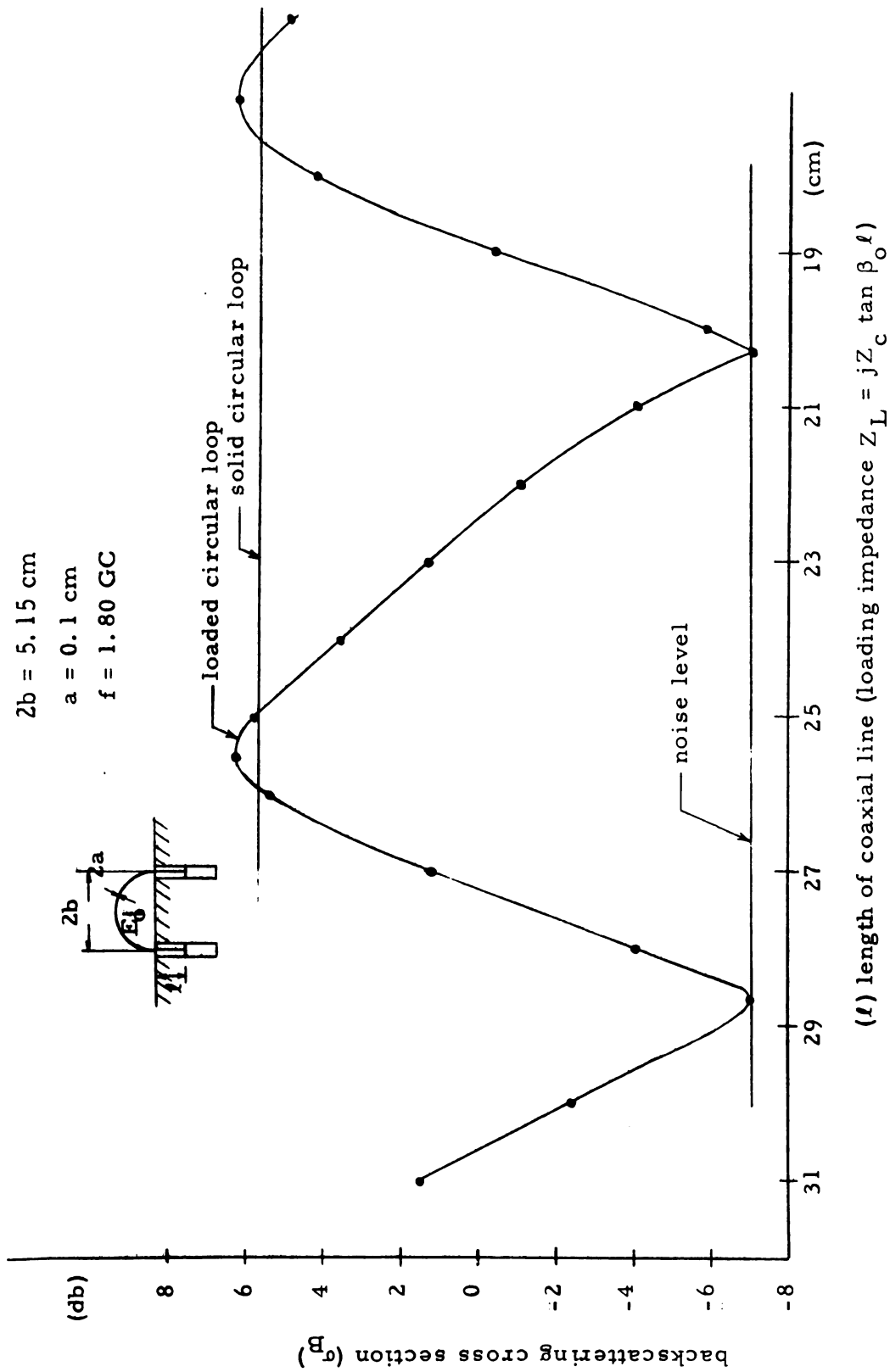


Figure 2.9. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 1.80 \text{ GC}$).

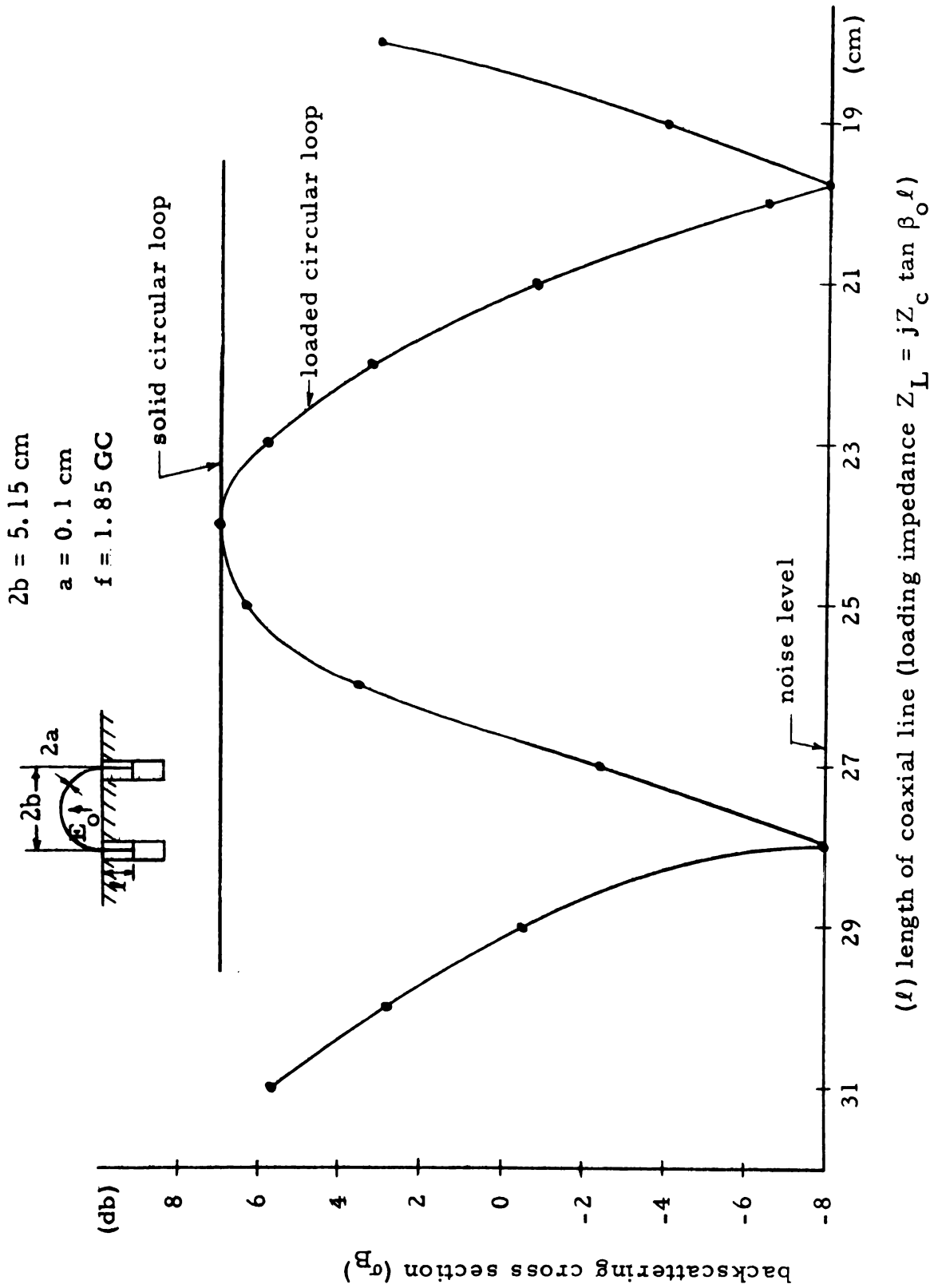
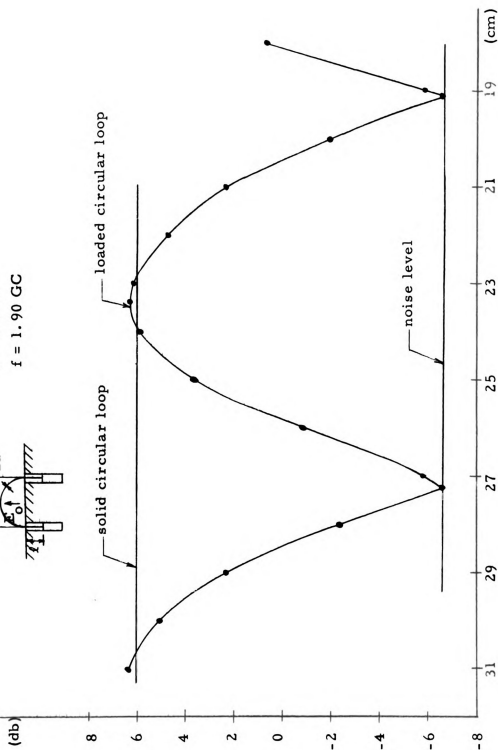
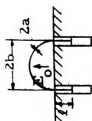


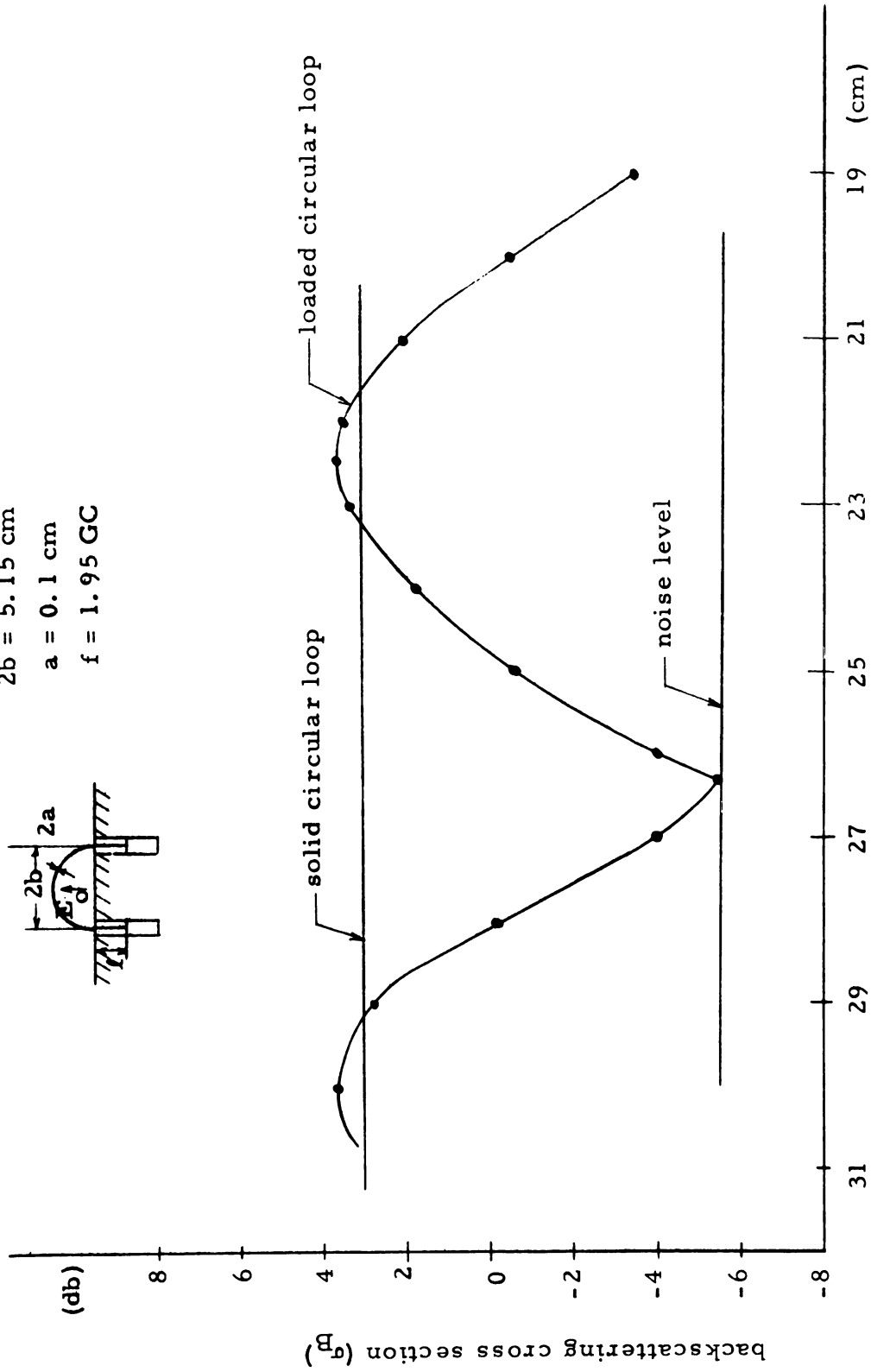
Figure 2.10. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 1.85 \text{ GC}$)

$2b = 5.15 \text{ cm}$
 $a = 0.1 \text{ cm}$
 $f = 1.90 \text{ GC}$



(l) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_o l$)

Figure 2.11. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 1.90 \text{ GC}$)



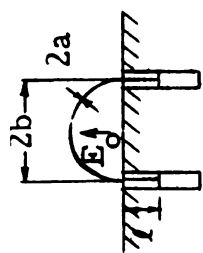
(ℓ) length of coaxial line (loading impedance $Z_L = jZ_C \tan \beta_O \ell$)

Figure 2.12. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 1.95 \text{ GC}$)

2.000000

2.000000

2.000000



$2b = 5, 15 \text{ cm}$
 $a = 0.1 \text{ cm}$
 $f = 2.00 \text{ GC}$

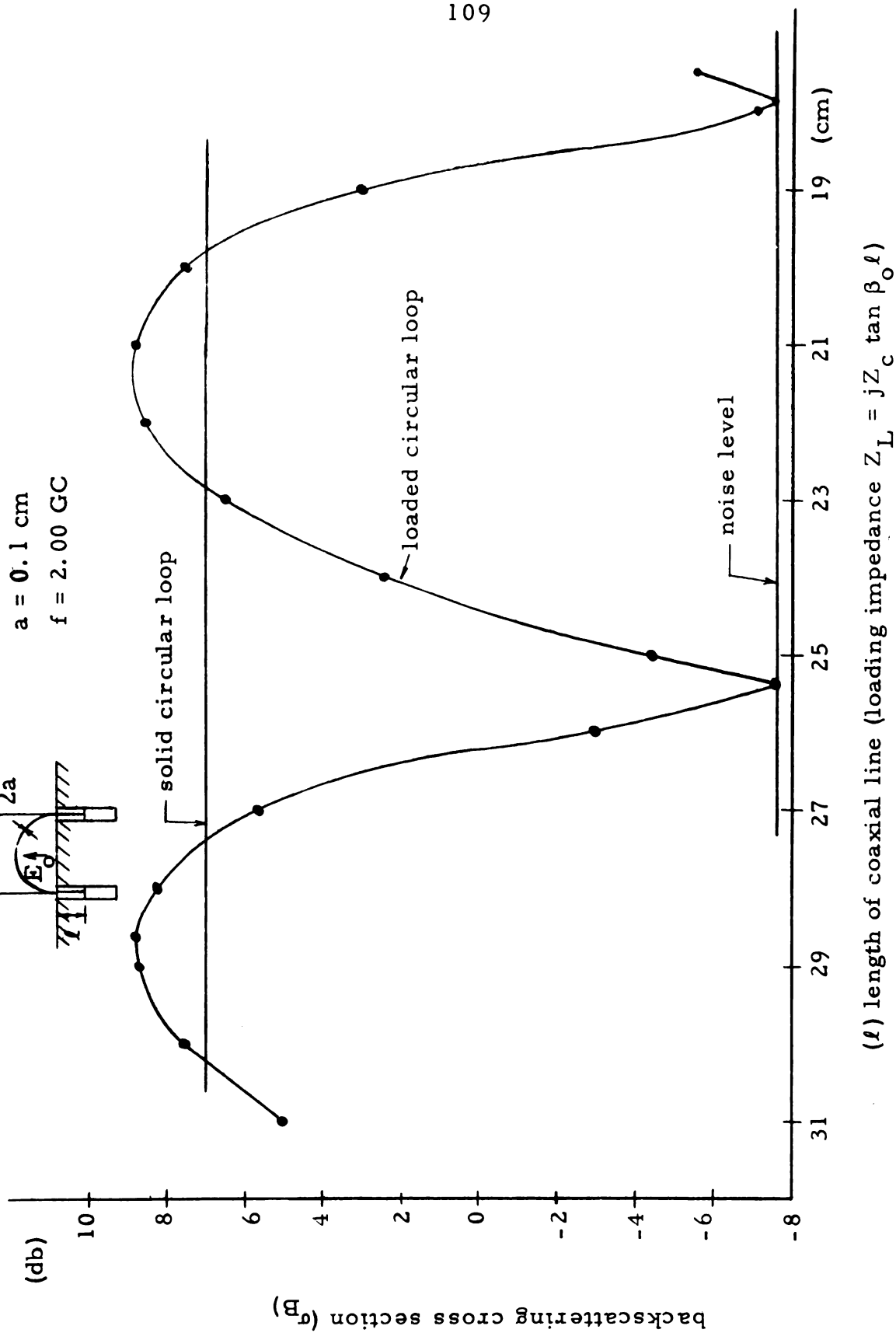


Figure 2.13. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 2.00 \text{ GC}$)

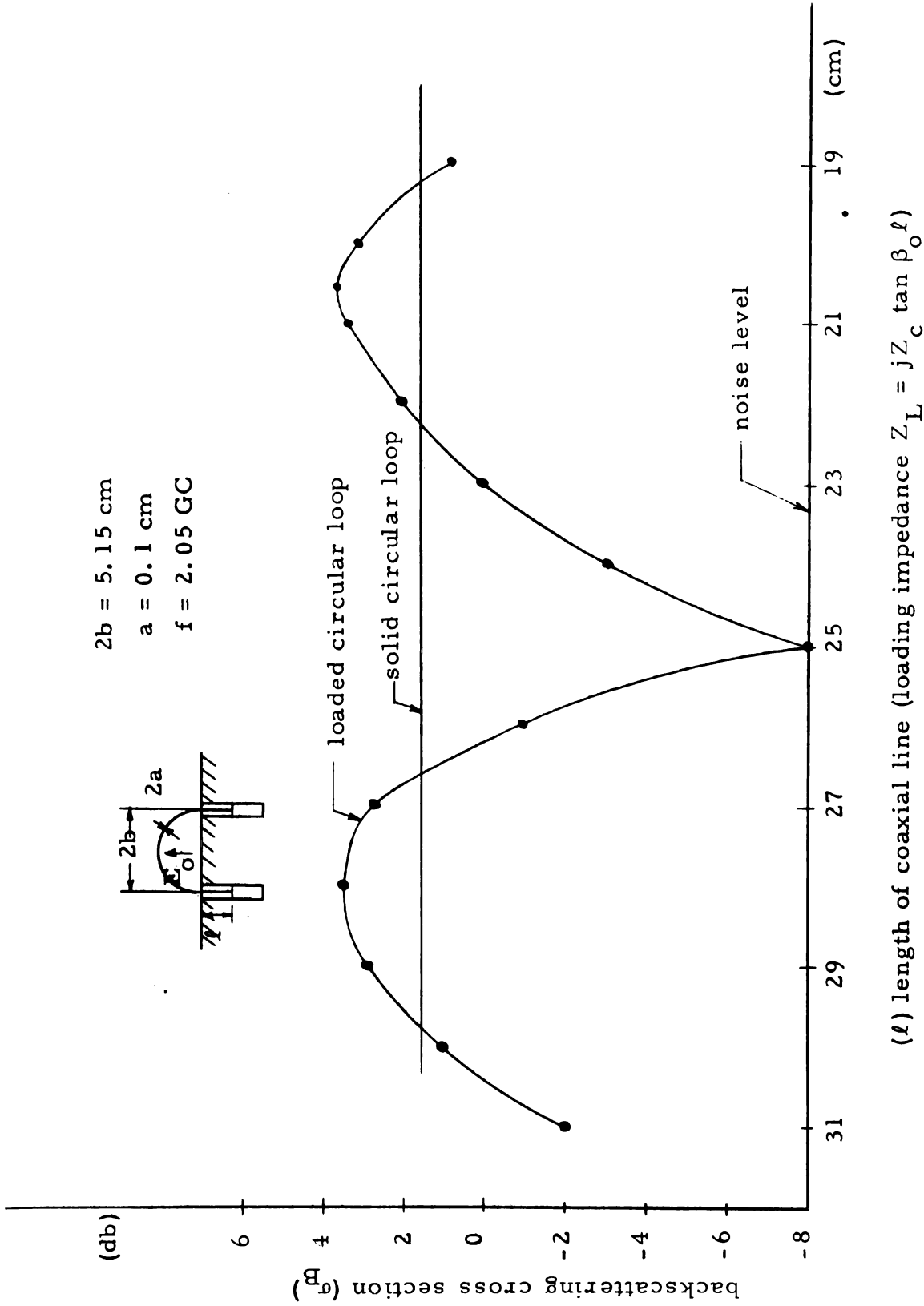
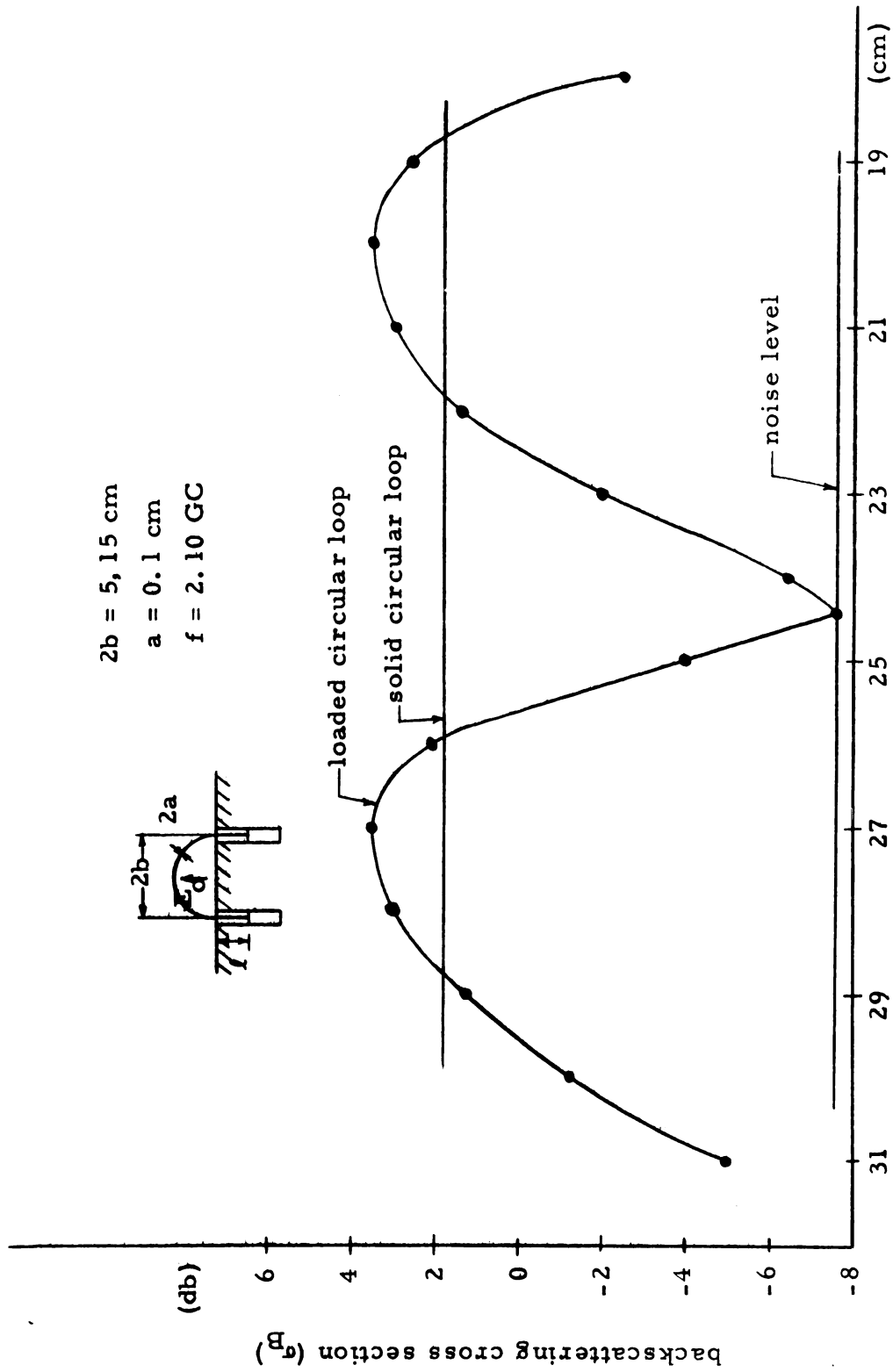


Figure 2.14. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 2.05 \text{ GC}$)



(l) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta l$)

Figure 2.15. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 2.10$ GC)

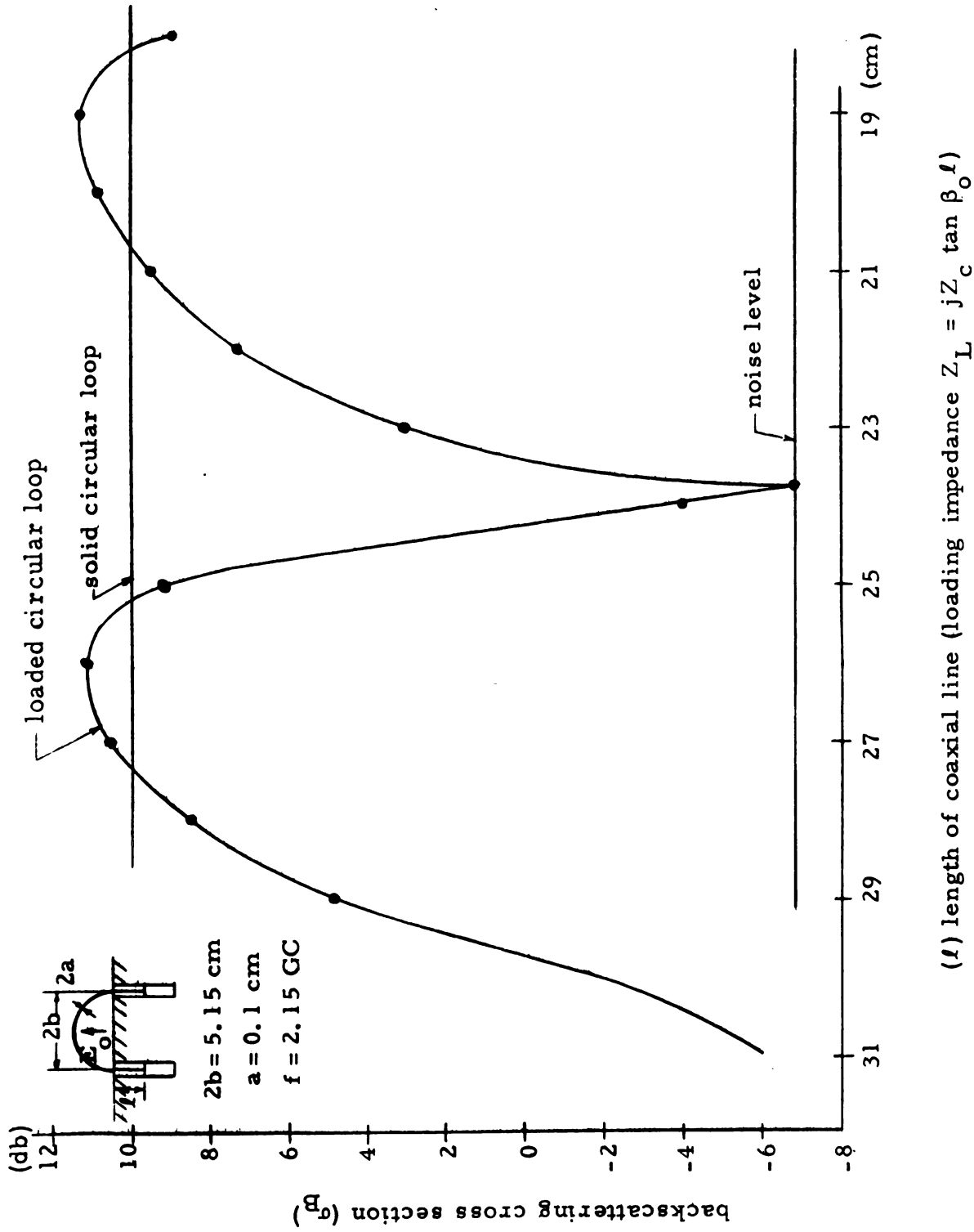
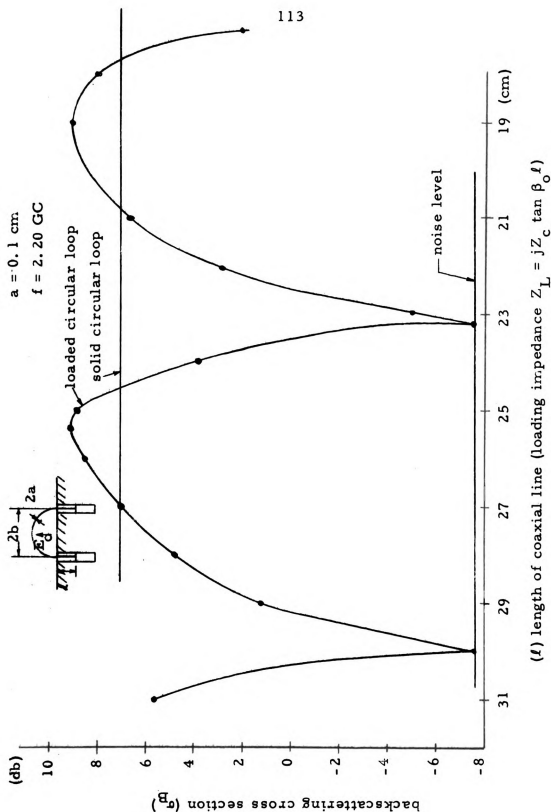


Figure 2.16. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 2.15 \text{ GC}$)

$$2b = 5.15 \text{ cm}$$

$$a = 0.1 \text{ cm}$$

$$f = 2.20 \text{ GC}$$



(l) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_0 l$)

Figure 2.17. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 2.20 \text{ GC}$)

1

2

3

4

5

6

7

8

9

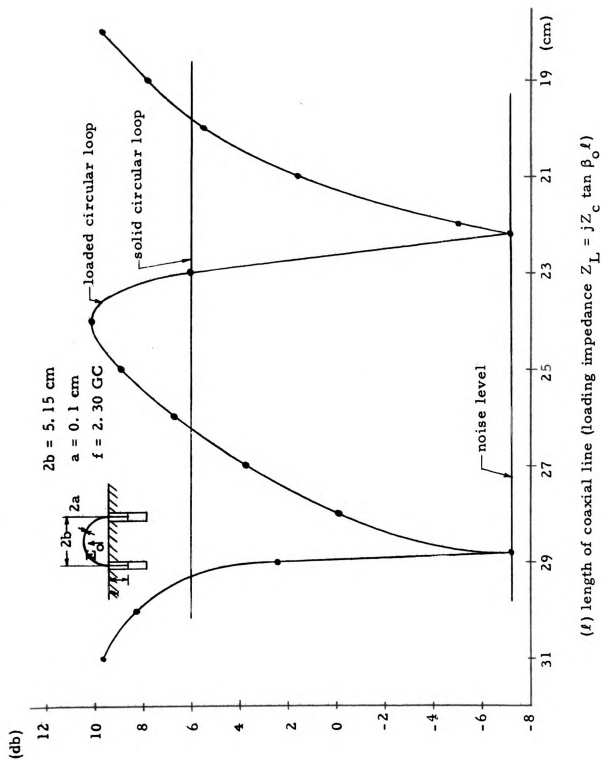
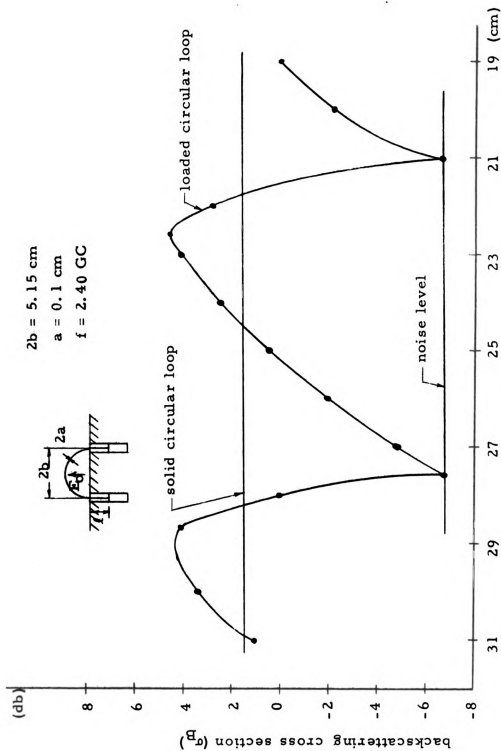
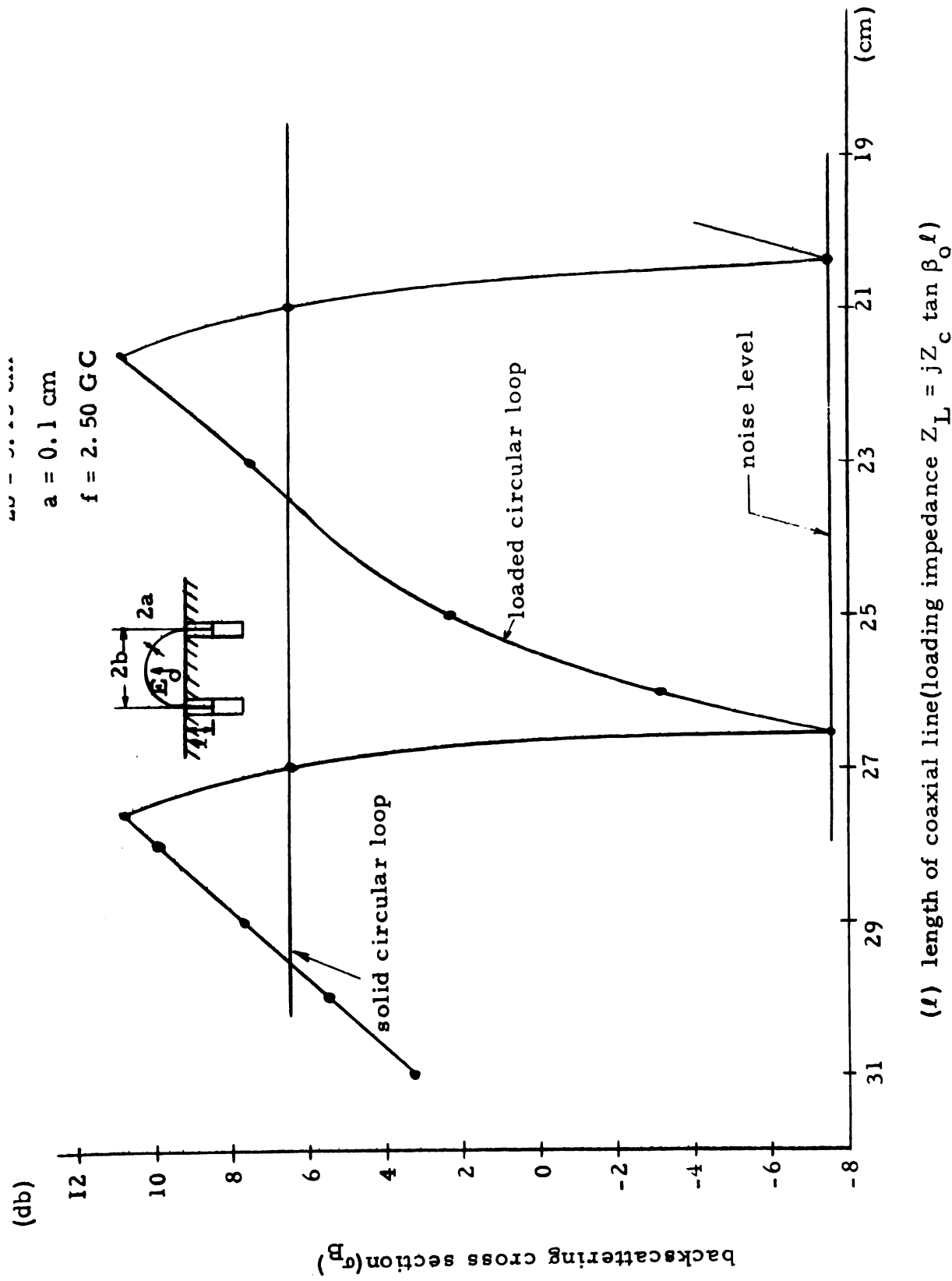


Figure 2.18. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 2.30 \text{ GC}$)



(#) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_o l$)

Figure 2.19. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 2.40$ GC)



(1) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_0 l$)

Figure 2.20. Backscattering cross section of a loaded loop as a function of loading impedance ($f = 2.50 \text{ GC}$)

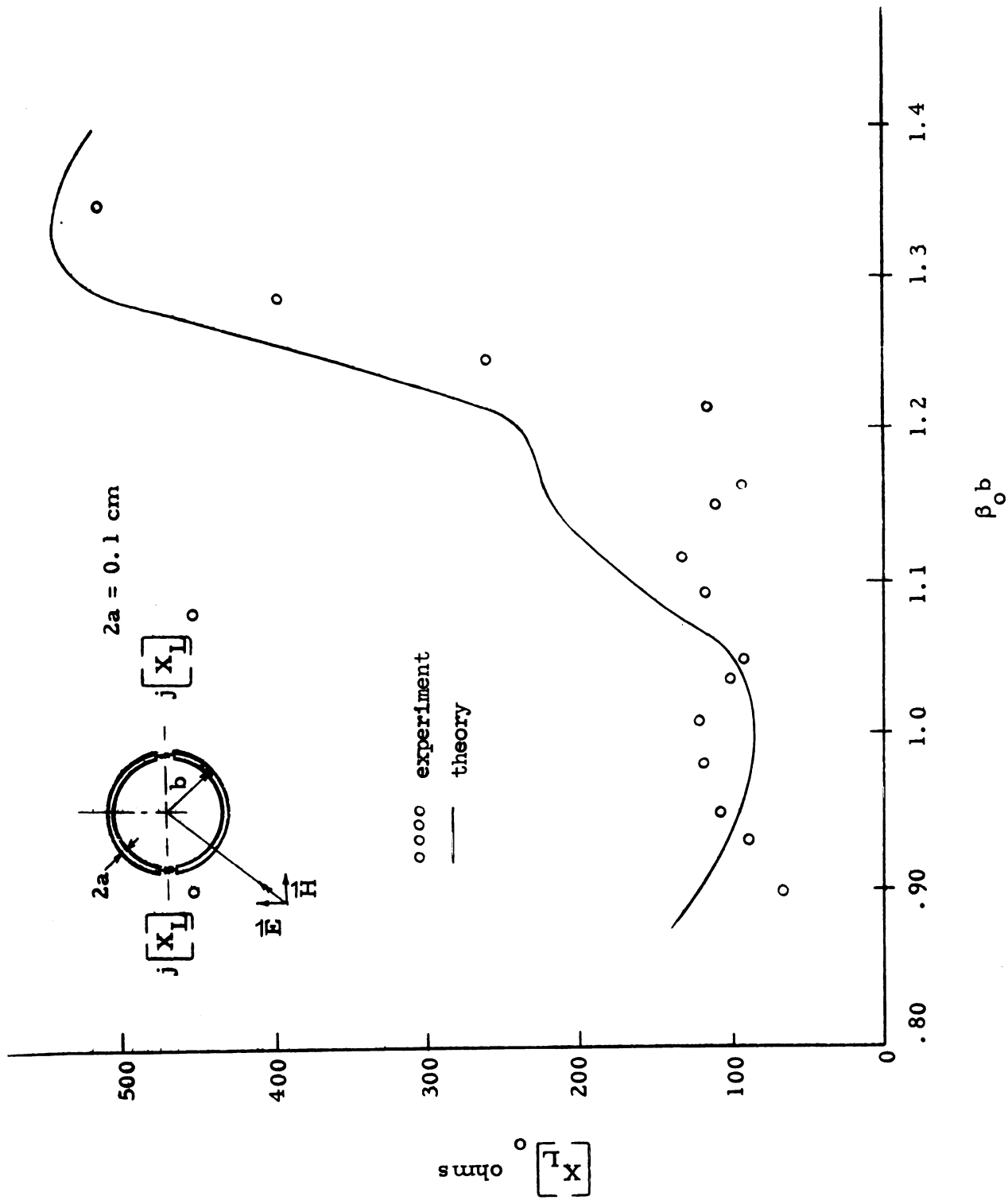


Figure 2.21. Optimum impedance for minimum backscattering from a circular loop illuminated by a plane wave at normal incidence.

CHAPTER III

MINIMIZATION OF BACKSCATTERING OF A LOADED RECTANGULAR LOOP

3.1. Introduction

A theory of the minimization of the backscatter of a metallic rectangular loop by impedance loading method is developed in this chapter. A rectangular loop loaded symmetrically with two identical lumped impedances at the centers of the long sides of the loop is assumed to be illuminated by a plane wave at normal incidence. The induced current on the loaded loop is determined as a function of the loop dimensions and the loading impedance. Based on the induced current, the backscattered field can be calculated. It is then possible to find an optimum impedance which leads to zero backscattering from the loop as a function of loop dimensions. Some numerical examples are included.

3.2. Theory

Based on the principle of superposition, the problem of the scattering from a loaded loop illuminated by a plane wave at normal incidence can be considered as the combination of the scattering of a solid loop and the radiation of a driven loop as described in Chapter II. Thus the following two cases will be considered separately and the results will then be combined to yield the final solution for the scattering from loaded loop:

(A) A solid rectangular loop illuminated by a normally incident plane wave.

- (B) A rectangular loop driven by two identical voltages at the centers of the long sides.

The situation is illustrated in Figure 3.1. In fact, case (A) is the scattering of a solid rectangular loop and case (B) is the radiation of a rectangular loop antenna. Case (A) has already been solved in Chapter 1 and its results are quoted here for further theoretical development. Case (B) will be considered in this section. The results of Case (B) will be superposed upon those of case (A) to obtain the solution for an illuminated, loaded rectangular loop.

3.2.1. Scattering from a Solid Rectangular Loop

The following expressions from Chapter 1 are needed for further development:

$$\left[I_{4y}^s(y) \right]_{11} = \frac{j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o} \chi$$

$$\left[\frac{\cos \beta_o h_1 \cos \beta_o y - \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} P_{41}^s(y) + \frac{1}{\Phi_s} P_{42}^s(y)}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \right]_{11} \quad (1.167)$$

$$\left[I_{1x}^s(x) \right]_{11} = \frac{-j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o} \chi$$

$$\left[\frac{\sin \beta_o h_2 \sin \beta_o x + \frac{1}{\Phi_s} P_{11}^s(x) + \frac{1}{\Phi_s} P_{12}^s(x)}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \right]_{11} \quad (1.170)$$

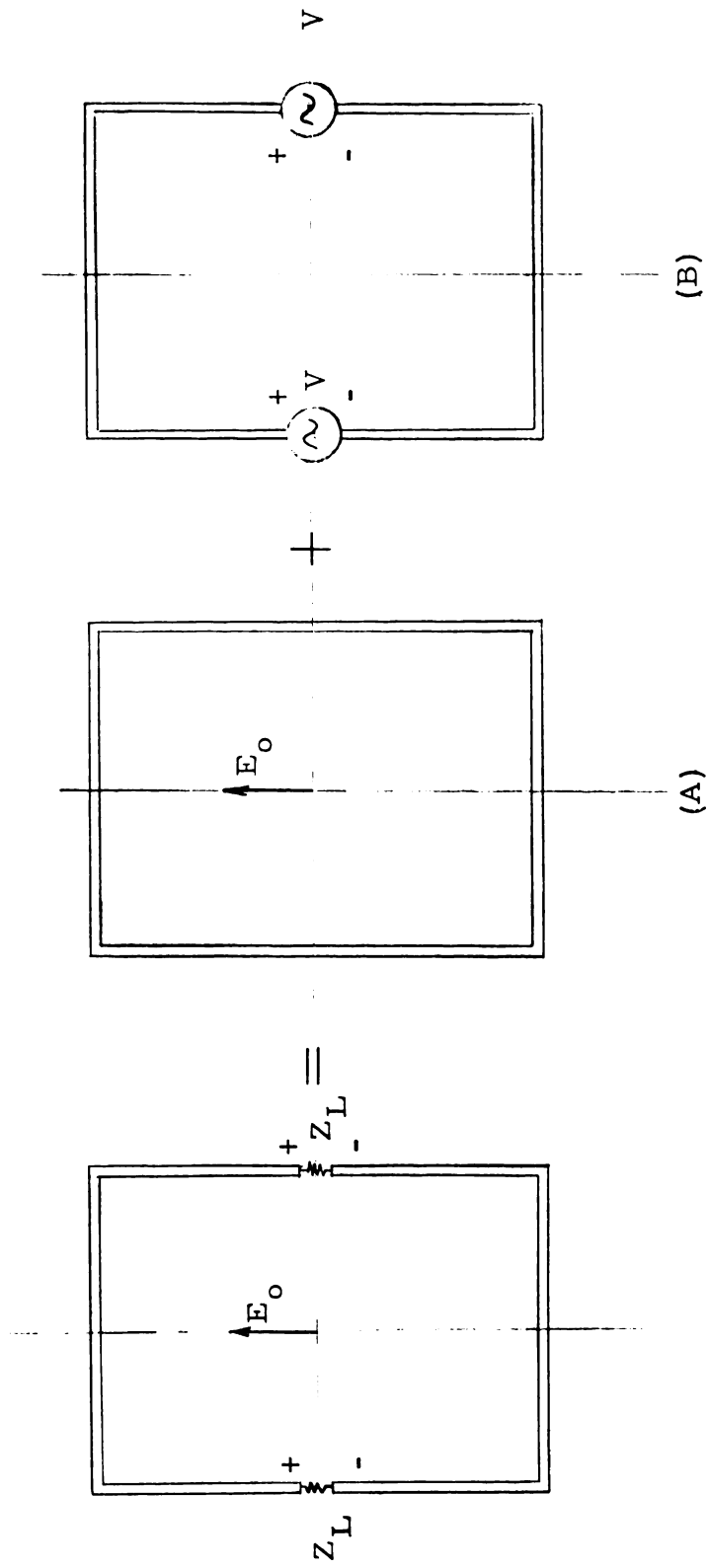


Figure 3.1. A loaded loop is the sum of a scattering loop and a radiating loop.

$$\left[I_{4y}^s(y) \right]_{10} = \frac{j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o} \times$$

$$\left[\frac{2 \left[\cos \beta_o h_1 \cos \beta_o y - \cos \beta_o (h_1 + h_2) \right] + \frac{1}{\Phi_s} U_{41}^s(y) + \frac{1}{\Phi_s} U_{42}^s(y)}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \right] \quad (1.173)$$

$$\left[I_{1x}^s(x) \right]_{10} = \frac{-j4\pi}{\zeta_o \Phi_s} \frac{E_o}{\beta_o} \times$$

$$\left[\frac{2 \sin \beta_o h_2 \sin \beta_o x + \frac{1}{\Phi_s} g_{11}^s(x) + \frac{1}{\Phi_s} g_{12}^s(x)}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G} \right] \quad (1.176)$$

$$\left[E^s \right]_{11} = 2 E_o \frac{e^{-j\beta_o R_o}}{R_o} \frac{J_1}{\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G \right]} \quad (1.183)$$

$$\left[E^s \right]_{10} = 2 E_o \frac{e^{-j\beta_o R_o}}{R_o} \frac{J_2}{\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G \right]} \quad (1.188)$$

where $I_{4y}^s(y)$ and $I_{1x}^s(x)$ are the induced current on side 4 and side 1 of the rectangular loop respectively. E^s is the backscattered field on the axis and in the far zone of the loop maintained by the induced

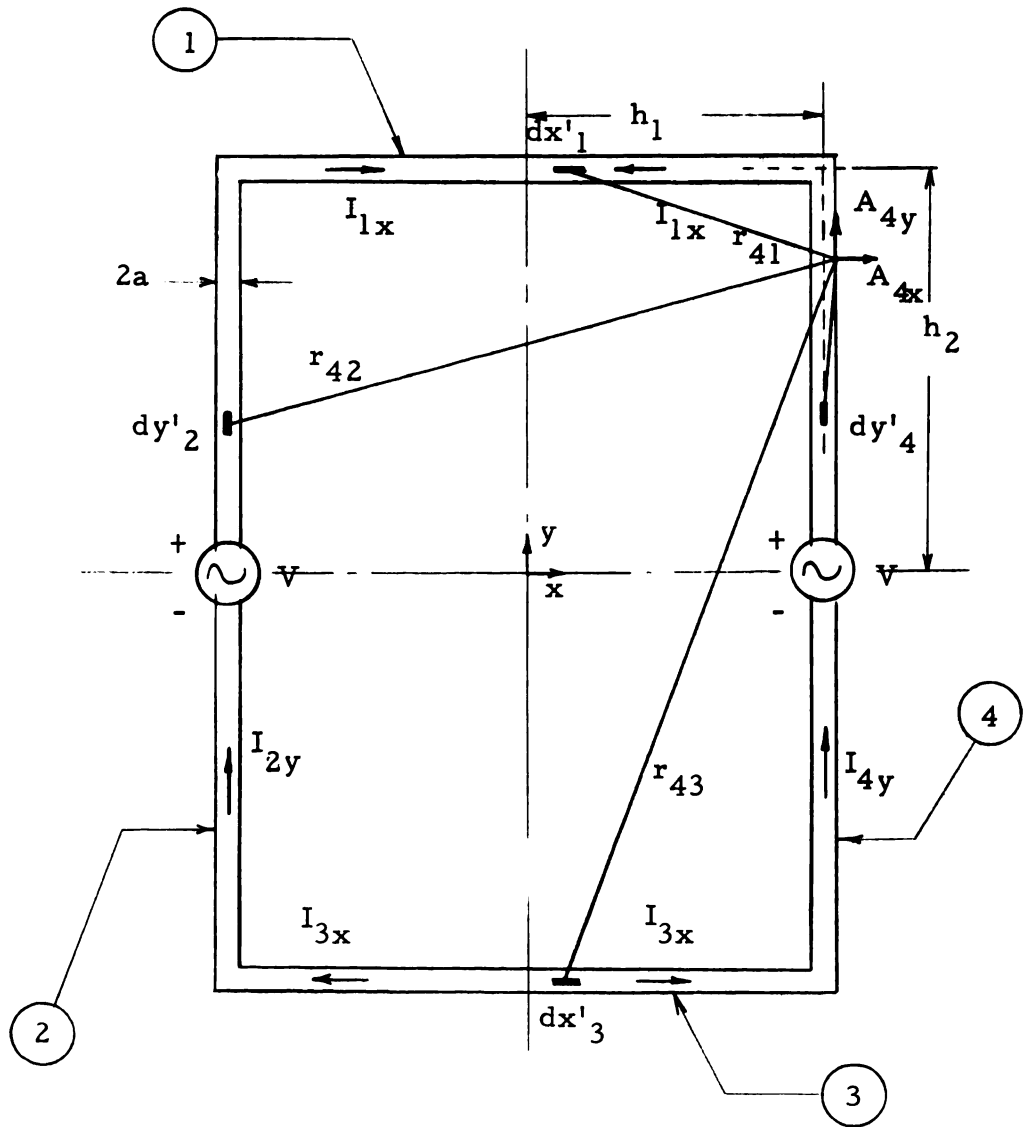


Figure 3.2. Geometry of a radiating loop.

currents. The subscripts outside the brackets in the above expressions are referred to the order of solutions.

3.2.2. Radiation from a Rectangular Loop

(A) Geometry of Problem

The geometry of the problem is as shown in Figure 3.2. A rectangular loop with short side $2h_1$ and large side $2h_2$ is assumed to be made of perfectly conducting wire of radius a . Two identical voltages V are connected across the gaps at the centers of the long sides of the loop. The dimensions of interest are

$$a^2 \ll h_1^2 \text{ and } h_2^2, \quad \beta_0^2 a^2 \ll 1$$

where β_0 is the wave number. We assume that the wire is thin enough so that only the tangential component of current is induced.

(B) Integral Equations for Loop Currents

The tangential electric field should vanish at the surface of the loop except at the gaps at the centers of the long sides where voltages of V are maintained. Assuming that the gaps are very small, we obtain the following equation:

For side 4

$$E_{4t}^a = -V \delta(y) \quad (3.1)$$

where E_{4t}^a is the tangential electric field at the surface of side 4 and $\delta(y)$ is a Dirac delta function. Due to symmetrical configuration, it follows that

$$I_{2y}^r(y) = I_{4y}^r(y) \quad (3.2)$$

$$I_{1x}^r(x) = -I_{3x}^r(x) \quad (3.3)$$

$$q_2^r(y) = q_4^r(y) \quad (3.4)$$

$$q_1^r(x) = -q_3^r(x) \quad (3.5)$$

where $I_{4y}^r(y)$ is the induced current on side 4 and $I_{1x}^r(x)$ is that on side 1, etc. q_i^r denotes the induced charge on side i , $i = 1, 2, 3$, and 4. E_{4t}^a can be expressed as

$$E_{4t}^a = -(\nabla \phi_4^r)_y - j\omega(\vec{A}_4^r)_y \quad (3.6)$$

where $\vec{A}_4^r$ is the vector potential at the surface of side 4 contributed by the currents in the loop; ϕ_4^r is the scalar potential at the surface of side 4 maintained by the induced charges on the loop.

In symbols, A_4^r and ϕ_4^r can be expressed as

$$\vec{A}_4^r = A_{4x}^r \hat{x} + A_{4y}^r \hat{y} \quad (3.7)$$

$$\phi_4^r = \phi_{41}^r + \phi_{42}^r + \phi_{43}^r + \phi_{44}^r \quad (3.8)$$

where A_{4x}^r is the vector potential at the surface of side 4 contributed by the currents in sides 1 and 3. The currents in sides 2 and 4 do not contribute to A_{4x}^r because they do not have x component.

Similarly, A_{4y}^r has the same meaning as A_{4x}^r except that the contribution is made by the currents in sides 2 and 4. ϕ_{4i}^r is the scalar potential at the surface of side 4 maintained by the charges on side i , $i = 1, 2, 3$, and 4.

A_{4y}^r can be expressed in terms of current as

$$\begin{aligned}
 A_{4y}^r(y) &= \frac{\mu_o}{4\pi} \int_{-h_2}^{h_2} I_{4y}^r(y') \frac{e^{-j\beta_o r_{44}}}{r_{44}} dy' \\
 &\quad + \frac{\mu_o}{4\pi} \int_{-h_2}^{h_2} I_{2y}^r(y') \frac{e^{-j\beta_o r_{42}}}{r_{42}} dy' \\
 &= \frac{\mu_o}{4\pi} \int_{-h_2}^{h_2} I_{4y}^r(y') K_{1A}(y, y') dy'
 \end{aligned} \tag{3.9}$$

where

$$K_{1A}(y, y') = \frac{e^{-j\beta_o r_{44}}}{r_{44}} + \frac{e^{-j\beta_o r_{42}}}{r_{42}} \tag{3.10}$$

$$r_{44} = \sqrt{(y' - y)^2 + a^2} \tag{3.11}$$

$$r_{42} = \sqrt{(y' - y)^2 + 4h_1^2} \tag{3.12}$$

With (3.7) and (3.8), (3.6) can be rewritten as

$$E_{4t}^r = -\frac{\partial}{\partial y} (\phi_{42}^r + \phi_{44}^r) - j\omega A_{4y}^r - \frac{\partial}{\partial y} (\phi_{41}^r + \phi_{43}^r) \quad (3.13)$$

By the Lorentz condition, it follows that

$$\phi_{42}^r + \phi_{44}^r = \frac{j\omega}{\beta_o^2} \nabla \cdot (\overline{A}_{42}^r + \overline{A}_{44}^r) = \frac{j\omega}{\beta_o^2} \frac{\partial A_{4y}^r(y)}{\partial y} \quad (3.14)$$

After substituting (3.14) in (3.13) and after rearranging, (3.13)

becomes

$$E_{4t}^a = -\frac{j\omega}{\beta_o^2} \left(\frac{\partial^2}{\partial y^2} + \beta_o^2 \right) A_{4y}^r(y) - \frac{\partial}{\partial y} (\phi_{41}^r + \phi_{43}^r) \quad (3.15)$$

The substitution of (3.15) in (3.1) gives

$$\left(\frac{\partial^2}{\partial y^2} + \beta_o^2 \right) A_{4y}^r(y) = -\frac{j\beta_o^2 V}{\omega} \delta(y) + \frac{j\beta_o^2}{\omega} \frac{\partial}{\partial y} (\phi_{41}^r + \phi_{43}^r) \quad (3.16)$$

where

$$\begin{aligned} \phi_{41}^r + \phi_{43}^r &= \frac{1}{4\pi\epsilon_o} \int_{-h_1}^{h_1} q_1^r(x') \frac{e^{-j\beta_o r_{41}}}{r_{41}} dx' \\ &\quad + \frac{1}{4\pi\epsilon_o} \int_{-h_1}^{h_1} q_3^r(x') \frac{e^{-j\beta_o r_{43}}}{r_{43}} dx' \\ &= \frac{1}{4\pi\epsilon_o} \int_{-h_1}^{h_1} q_1^r(x') K_{1B}(y, x') dx' \end{aligned} \quad (3.17)$$

where

$$K_{1B}(y, x') = \frac{e^{-j\beta_o r_{41}}}{r_{41}} - \frac{e^{-j\beta_o r_{43}}}{r_{42}} \quad (3.18)$$

$$r_{41} = \sqrt{(h_2 - y)^2 + (h_1 - x')^2} \quad (3.19)$$

$$r_{43} = \sqrt{(h_2 + y)^2 + (h_1 - x')^2} \quad (3.20)$$

Similarly, the differential equation for the vector potential at side 1 can be obtained as follows:

Since the tangential electric field should vanish at the surface of side 1, we have

$$E_{1t}^a = 0 \quad (3.21)$$

where E_{1t}^a is the tangential E field at the surface of side 1 maintained by the current and charge on the loop.

E_{1t}^a can be expressed as

$$E_{1t}^a = -(\nabla \phi_1^r)_x - j\omega (\vec{A}_1^r)_x \quad (3.22)$$

where ϕ_1^r is the scalar potential at the surface of side 1 maintained by the induced charges on the loop and $\vec{A}_1^r$ is the vector potential at the surface of side 1 contributed by the currents in the loop.

In symbols, $\vec{A}_1^r$ and ϕ_1^r can be expressed as

$$\vec{A}_1^r = A_{1x}^r \hat{x} + A_{1y}^r \hat{y} \quad (3.23)$$

$$\phi_1^r = \phi_{11}^r + \phi_{12}^r + \phi_{13}^r + \phi_{14}^r \quad (3.24)$$

where A_{1x}^r is the vector potential at the surface of side 1 contributed by the currents in sides 1 and 3; while A_{1y}^r is that contributed by the currents in sides 2 and 4. ϕ_{1i}^r is the scalar potential at the surface of side 1 maintained by the induced charges on side i , $i = 1, 2, 3$ and 4. A_{1x}^r can be expressed in terms of current as

$$\begin{aligned}
 A_{1x}^r &= \frac{\mu_o}{4\pi} \int_{-h_1}^{h_1} I_{1x}^r(x') \frac{e^{-j\beta_o r_{11}}}{r_{11}} dx' \\
 &+ \frac{\mu_o}{4\pi} \int_{-h_1}^{h_1} I_{3x}^r(x') \frac{e^{-j\beta_o r_{13}}}{r_{13}} dx' \\
 &= \frac{\mu_o}{4\pi} \int_{-h_1}^{h_1} I_{1x}^r(x') K_{2A}(x, x') dx'
 \end{aligned} \tag{3.25}$$

where

$$K_{2A}(x, x') = \frac{e^{-j\beta_o r_{11}}}{r_{11}} - \frac{e^{-j\beta_o r_{13}}}{r_{13}} \tag{3.26}$$

$$r_{11} = \sqrt{(x' - x)^2 + a^2} \tag{3.27}$$

$$r_{13} = \sqrt{(x' - x)^2 + 4h_2^2} \tag{3.28}$$

With (3.23) and (3.24), (3.22) can be rewritten as

$$E_{1t}^a = -\frac{\partial}{\partial x} (\phi_{11}^r + \phi_{13}^r) - j\omega A_{1x}^r - \frac{\partial}{\partial x} (\phi_{12}^r + \phi_{14}^r) \tag{3.29}$$

By the Lorentz condition, we have

$$\phi_{11}^r + \phi_{13}^r = \frac{j\omega}{\beta_o} \nabla \cdot \left(\vec{A}_{11}^r + \vec{A}_{13}^r \right) = \frac{j\omega}{\beta_o} \frac{\partial A_{1x}^r}{\partial x} \quad (3.30)$$

Substituting (3.30) in (3.29), we can rearrange (3.29) as

$$E_{1t}^a = - \frac{j\omega}{\beta_o} \left(\frac{\partial^2}{\partial x^2} + \beta_o^2 \right) A_{1x}^r - \frac{\partial}{\partial x} \left(\phi_{12}^r + \phi_{14}^r \right) \quad (3.31)$$

where ϕ_{12}^r and ϕ_{14}^r can be expressed in terms of charge as

$$\begin{aligned} \phi_{12}^r + \phi_{14}^r &= \frac{1}{4\pi\epsilon_o} \int_{-h_2}^{h_2} q_2^r(y') \frac{e^{-j\beta_o r_{12}}}{r_{12}} dy' \\ &+ \frac{1}{4\pi\epsilon_o} \int_{-h_2}^{h_2} q_4^r(y') \frac{e^{-j\beta_o r_{14}}}{r_{14}} dy' \\ &= \frac{1}{4\pi\epsilon_o} \int_{-h_2}^{h_2} q_4^r(y') K_{2B}(x, y') dy' \end{aligned} \quad (3.32)$$

where

$$K_{2B}(x, y') = \frac{e^{-j\beta_o r_{12}}}{r_{12}} + \frac{e^{-j\beta_o r_{14}}}{r_{14}} \quad (3.33)$$

$$r_{12} = \sqrt{(h_2 - y')^2 + (h_1 + x)^2} \quad (3.34)$$

$$r_{14} = \sqrt{(h_2 - y')^2 + (h_1 - x)^2} \quad (3.35)$$

With (3.31), (3.21) can be rewritten as

$$\left(\frac{\partial^2}{\partial x^2} + \beta_o^2\right) A_{1x}^r(x) = \frac{j\beta_o^2}{\omega} \frac{\partial}{\partial x} \left(\phi_{12}^r + \phi_{14}^r \right) \quad (3.36)$$

Due to symmetrical configuration, we note that

$$A_{4y}^r(y) = A_{4y}^r(-y) \quad (3.37)$$

$$A_{1x}^r(x) = -A_{1x}^r(-x) \quad (3.38)$$

(3.37) implies that $A_{4y}^r(y)$ is an even function of y and, therefore, the general solution for $A_{4y}^r(y)$ in (3.16) can be expressed as

$$A_{4y}^r(y) = \frac{-j}{v_o} \left[C_4^r \cos \beta_o y + \theta_4^r(y) \right] \quad (3.39)$$

where C_4^r is an arbitrary constant, and $\theta_4^r(y)$ is a particular integral which can be found to be

$$\theta_4^r(y) = \frac{V}{2} \sin \beta_o |y| - \int_0^y \frac{\partial}{\partial s} \left[\phi_{41}^r(s) + \phi_{43}^r(s) \right] \sin \beta_o (y - s) ds \quad (3.40)$$

Similarly, (3.38) implies that $A_{1x}^r(x)$ is an odd function of x and, therefore, the general solution for $A_{1x}^r(x)$ in (3.36) can be expressed as

$$A_{1x}^r(x) = \frac{-j}{v_o} \left[C_1^r \sin \beta_o x + \theta_1^r(x) \right] \quad (3.41)$$

where C_1^r is an arbitrary constant and $\theta_1^r(x)$ is a particular integral which is found as

$$\theta_1^r(x) = - \int_0^x \frac{\partial}{\partial s} \left[\phi_{12}^r(s) + \phi_{14}^r(s) \right] \sin \beta_o (x - s) ds \quad (3.42)$$

With (3.9) and (3.39), it follows that

$$\begin{aligned} \frac{4\pi}{\mu_o} A_{4y}^r(y) &= \int_{-h_2}^{h_2} I_{4y}^r(y') K_{1A}(y, y') dy' \\ &= \frac{-j4\pi}{\zeta_o} \left[C_4^r \cos \beta_o y + \theta_4^r(y) \right] \end{aligned} \quad (3.43)$$

Meanwhile, with (3.25) and (3.41) we obtain

$$\begin{aligned} \frac{4\pi}{\mu_o} A_{1x}^r(x) &= \int_{-h_1}^{h_1} I_{1x}^r(x') K_{2A}(x, x') dx' \\ &= \frac{-j4\pi}{\zeta_o} \left[C_1^r \sin \beta_o x + \theta_1^r(x) \right] \end{aligned} \quad (3.44)$$

(C) Zeroth Order Solutions for Loop Currents

At the first approximation, we assume that

$$\frac{A_{4y}^r(y)}{I_{4y}^r(y)} = \frac{\mu_o}{4\pi} \Phi_r(y) \quad (3.45)$$

$$\frac{A_{1x}^r(x)}{I_{1x}^r(x)} = \frac{\mu_o}{4\pi} \Phi_r(x) \quad (3.46)$$

where $\Phi_r(y)$ and $\Phi_r(x)$ are defined respectively as

$$\Phi_r(y) = \frac{4\pi}{\mu_o} \frac{A_{4y}^r(y)}{I_{4y}^r(y)} \quad (3.47)$$

$$\Phi_r(x) = \frac{4\pi}{\mu_o} \frac{A_{1x}^r(x)}{I_{1x}^r(x)} \quad (3.48)$$

Substituting (3.45) in (3.39), we obtain

$$I_{4y}(y) = \frac{-j4\pi}{\zeta_o \Phi_r(y)} \left[C_4^r \cos \beta_o y + \theta_4^r(y) \right] \quad (3.49)$$

Similarly, the substitution of (3.48) in (3.43) gives

$$I_{1x}^r(x) = \frac{-j4\pi}{\zeta_o \Phi_r(x)} \left[C_1^r \sin \beta_o x + \theta_1^r(x) \right] \quad (3.50)$$

As we can see from (3.9) and (3.25) respectively that $A_{4y}^r(y)$ is primarily contributed by the current elements in the neighborhood of y and that $A_{1x}^r(x)$ is mainly due to the current elements located

nearby x . It is, therefore, quite reasonable to assume that $\Phi_r(y)$ in (3.47) is independent of y and $\Phi_r(x)$ in (3.48) independent of x . For simplicity, we assume that

$$\Phi_r(y) = \Phi_r(x) = \Phi_r \quad (3.51)$$

The boundary conditions for determining C_4^r and C_1^r are

(a) Current at the corners being continuous.

$$\text{i.e.,} \quad I_{4y}^r(y = h_2) = -I_{1x}^r(x = h_1) \quad (3.52)$$

(b) Charge around the corners being continuous.

$$\text{i.e.,} \quad q_4^r(y = h_2) = q_1^r(x = h_1) \quad (3.53)$$

The substitution of (3.49), (3.50) and (3.51) in (3.52) yields

$$C_4^r \cos \beta_o h_2 + \theta_4^r(h_2) = -C_1 \sin \beta_o h_1 - \theta_1^r(h_1) \quad (3.54)$$

$q_4^r(y)$ and $q_1^r(x)$ can be found as follows by making use of the equation of continuity.

$$\begin{aligned} q_4^r(y) &= \frac{j}{\omega} \frac{\partial}{\partial y} I_{4y}^r(y) \\ &= \frac{j}{\omega} \frac{-j4\pi}{\zeta_o \Phi_r} \left[-C_4^r \beta_o \sin \beta_o y + \frac{\partial \theta_4^r(y)}{\partial y} \right] \end{aligned} \quad (3.55)$$

$$\begin{aligned}
q_1^r(x) &= \frac{j}{\omega} \frac{\partial}{\partial x} I_{1x}^r(x) \\
&= \frac{j}{\omega} \frac{-j4\pi}{\zeta_o \Phi_r} \left[C_1^r \beta_o \cos \beta_o x + \frac{\partial \theta_1^r(x)}{\partial x} \right] \quad (3.56)
\end{aligned}$$

Combining (3.55) and (3.56) with boundary condition (3.53), we have

$$\begin{aligned}
- C_4^r \beta_o \sin \beta_o h_2 + \left. \frac{\partial \theta_4^r(y)}{\partial y} \right|_{y=h_2} &= C_1^r \beta_o \cos \beta_o h_1 + \left. \frac{\partial \theta_1^r(x)}{\partial x} \right|_{x=h_1} \\
&\quad (3.57)
\end{aligned}$$

Solving (3.56) and (3.57) for C_4^r and C_1^r , we obtain

$$\begin{aligned}
C_4^r &= \frac{1}{\beta_o [\cot \beta_o h_1 \cos \beta_o h_2 - \sin \beta_o h_2]} \left[\left. \frac{\partial \theta_1^r(x)}{\partial x} \right|_{x=h_1} - \left. \frac{\partial \theta_4^r(y)}{\partial y} \right|_{y=h_2} \right] \\
&+ \frac{1}{\tan \beta_o h_1 \sin \beta_o h_2 - \cos \beta_o h_2} \left[\theta_1^r(h_1) + \theta_4^r(h_2) \right] \quad (3.58)
\end{aligned}$$

$$\begin{aligned}
C_1^r &= \frac{1}{\beta_o [\sin \beta_o h_1 \tan \beta_o h_2 - \cos \beta_o h_1]} \times \\
&\cdot \left\{ \left. \frac{\partial \theta_1^r(x)}{\partial x} \right|_{x=h_1} - \left. \frac{\partial \theta_4^r(y)}{\partial y} \right|_{y=h_2} \right. \\
&\quad \left. - \beta_o \tan \beta_o h_2 \left[\theta_1^r(h_1) + \theta_4^r(h_2) \right] \right\} \quad (3.59)
\end{aligned}$$

For zeroth order solution, we assume that $\theta_4^r(y)$ in (3.40) and $\theta_1^r(x)$ in (3.42) can be approximated respectively as

$$\left[\theta_4^r(y) \right]_0 = \frac{V}{2} \sin \beta_o |y| \quad (3.60)$$

$$\left[\theta_1^r(x) \right]_0 = 0 \quad (3.61)$$

With (3.60) and (3.61) substituted in (3.58) and (3.59), it follows that

$$\left[C_4^r \right]_0 = \frac{-V}{2} \tan \beta_o (h_1 + h_2) \quad (3.62)$$

$$\left[C_1^4 \right]_0 = \frac{V}{2} \frac{1}{\cos \beta_o (h_1 + h_2)} \quad (3.63)$$

Consequently, zeroth order solution of currents are obtained as

$$\left[I_{4y}^r(y) \right]_0 = \frac{-j4\pi}{\zeta_o \Phi_r} \left[-\frac{V}{2} \tan \beta_o (h_1 + h_2) \cos \beta_o y + \theta_4^y(y) \right] \quad (3.64)$$

$$\left[I_{1x}^r(x) \right]_0 = \frac{-j4\pi}{\zeta_o \Phi_r} \left[\frac{V}{2} \frac{\sin \beta_o x}{\cos \beta_o (h_1 + h_2)} + \theta_1^r(x) \right] \quad (3.65)$$

If (3.60) and (3.61) are applied to (3.64) and (3.65), we have

$$\left[I_{4y}^r(y) \right]_{00} = \frac{j2\pi V}{\zeta_o \Phi_r} \frac{\sin \beta_o (h_1 + h_2 - |y|)}{\cos \beta_o (h_1 + h_2)} \quad (3.66)$$

$$\left[I_{1x}^r(x) \right]_{00} = \frac{-j2\pi V}{\zeta_o \Phi_r} \frac{\sin \beta_o x}{\cos \beta_o (h_1 + h_2)} \quad (3.67)$$

(D) First Order Solutions for Loop Currents

To obtain first order solutions of currents, the following equations are needed:

From (3.39) and (3.41)

$$A_{4y}^r(y) = \frac{-j}{v_o} \left[C_4^r \cos \beta_o y + \theta_4^r(y) \right]$$

$$A_{1x}^r(x) = \frac{-j}{v_o} \left[C_1^r \sin \beta_o x + \theta_1^r(x) \right]$$

From (3.49), (3.50) and (3.51), it follows that

$$\left[I_{4y}^r(y) \right]_0 = \frac{-j4\pi}{\zeta_o \Phi_r} \left[C_4^r \cos \beta_o y + \theta_4^r(y) \right]$$

$$\left[I_{1x}^r(x) \right]_0 = \frac{-j4\pi}{\zeta_o \Phi_r} \left[C_1^r \sin \beta_o x + \theta_1^r(x) \right]$$

From (3.55) and (3.56)

$$\left[q_4^r(y) \right]_0 = \frac{4\pi}{\omega \zeta_o \Phi_r} \left[-C_4^r \beta_o \sin \beta_o y + \frac{\partial \theta_4^r(y)}{\partial y} \right]$$

$$\left[q_1^r(x) \right]_0 = \frac{4\pi}{\omega \zeta_o \Phi_r} \left[C_1^r \beta_o \cos \beta_o x + \frac{\partial \theta_1^r(x)}{\partial x} \right]$$

First order currents are formulated as follows:

$$\begin{aligned} \left[I_{4y}^r(y) \right]_1 &= \left[I_{4y}^r(y) \right]_0 + \frac{4\pi}{\mu_o \Phi_r} \left\{ A_{4y}^r - \frac{\mu_o}{4\pi} \right. \\ &\quad \cdot \left. \int_{-h_1}^{h_1} \left[I_{4y}^r(y') \right]_0 K_{1A}(y, y') dy' \right\} \end{aligned} \quad (3.68)$$

$$\begin{aligned} \left[I_{1x}^r(x) \right]_1 &= \left[I_{1x}^r(x) \right]_0 + \frac{4\pi}{\mu_o \Phi_r} \left\{ A_{1x} - \frac{\mu_o}{4\pi} \right. \\ &\quad \cdot \left. \int_{-h_1}^{h_1} \left[I_{1x}^r(x') \right]_0 K_{2A}(x, x') dx' \right\} \end{aligned} \quad (3.69)$$

In fact, (3.68) and (3.69) can be rewritten as

$$\left[I_{4y}^r(y) \right]_1 \cong 2 \left[I_{4y}^r(y) \right]_0 - \frac{1}{\Phi_r} \int_{-h_2}^{h_2} \left[I_{4y}^r(y') \right]_0 K_{1A}(y, y') dy' \quad (3.70)$$

$$\left[I_{1x}^r(x) \right]_1 \cong 2 \left[I_{1x}^r(x) \right]_0 - \frac{1}{\Phi_r} \int_{-h_1}^{h_1} \left[I_{1x}^r(x') \right]_0 K_{2A}(x, x') dx' \quad (3.71)$$

Substituting zeroth order solutions in (68) and (69) yields

$$\begin{aligned} \left[I_{4y}^r(y) \right]_1 &= \frac{-j4\pi}{\zeta_o \Phi_r} \left\{ 2 \left[C_4^r \cos \beta_o y + \theta_4^r(y) \right] \right. \\ &\quad - \frac{C_4^r}{\Phi_r} \int_{-h_2}^{h_2} \cos \beta_o y' K_{1A}(y, y') dy' \\ &\quad \left. - \frac{1}{\Phi_r} \int_{-h_2}^{h_2} \theta_4^r(y') K_{1A}(y, y') dy' \right\} \\ &= \frac{-j4\pi}{\zeta_o \Phi_r} \left\{ C_4^r \left[2 \cos \beta_o y - \frac{T_{ca}(y)}{\Phi_r} \right] + 2 \theta_4^r(y) - \frac{n_1^r(y)}{\Phi_r} \right\} \\ &= \frac{-j4\pi}{\zeta_o \Phi_r} \left[C_4^r L_1^r(y) + 2 \theta_4^r(y) - N_1^r(y) \right] \end{aligned} \quad (3.72)$$

where

$$L_1^r(y) = 2 \cos \beta_o y - \frac{T_{ca}(y)}{\Phi_r} \quad (3.73)$$

$$N_1^r(y) = \frac{n_1^r(y)}{\Phi_r} \quad (3.74)$$

$$T_{ca}(y) = \int_{-h_2}^{h_2} \cos \beta_o y' K_{1A}(y, y') dy' \quad (3.75)$$

$$n_1^r(y) = \int_{-h_2}^{h_2} \theta_4^r(y') K_{1A}(y, y') dy' \quad (3.76)$$

$$\begin{aligned} \left[I_{1x}^r(x) \right]_1 &= \frac{-j4\pi}{\zeta_o \Phi_r} \left\{ 2 \left[C_1^r \sin \beta_o + \theta_1^r(x) \right] \right. \\ &\quad - \frac{C_1^r}{\Phi_r} \int_{-h_1}^{h_1} \sin \beta_o x' K_{2A}(x, x') dx' \\ &\quad \left. - \frac{1}{\Phi_r} \int_{-h_1}^{h_1} \theta_1^r(x') K_{2A}(x, x') dx' \right\} \\ &= \frac{-j4\pi}{\zeta_o \Phi_r} \left\{ C_1^r \left[2 \sin \beta_o x - \frac{T_{s*d}(x)}{\Phi_r} \right] + 2\theta_1^r(x) - \frac{d_1^r(x)}{\Phi_r} \right\} \\ &= \frac{-j4\pi}{\zeta_o \Phi_r} \left[C_1^r M_1^r(x) + 2\theta_1^r(x) - D_1^r(x) \right] \end{aligned} \quad (3.77)$$

where

$$M_1^r(x) = 2 \sin \beta_o x - \frac{T_{s*d}(x)}{\Phi_r} \quad (3.77)$$

$$D_1^r(x) = \frac{d_1^r(x)}{\Phi_r} \quad (3.78)$$

$$T_{s*d}(x) = \int_{-h_1}^{h_1} \sin \beta_o x' K_{2A}(x, x') dx' \quad (3.79)$$

$$d_1^r(x) \int_{-h_1}^{h_1} \theta_1^r(x') K_{2A}(x, x') dx' \quad (3.80)$$

With (3.14) and (3.17), (3.8) can be rewritten as

$$\phi_4^r(y) = \frac{j\omega}{\beta_o^2} \frac{\partial A_{4y}^r(y)}{\partial y} + \frac{1}{4\pi\epsilon_o} \int_{-h_1}^{h_1} q_1^r(x') K_{1B}(y, x') dx' \quad (3.81)$$

Similarly, with (3.30) and (3.32), (3.24) can be rewritten as

$$\phi_1^r(x) = \frac{j\omega}{\beta_o^2} \frac{\partial A_{1x}^r(x)}{\partial x} + \frac{1}{4\pi\epsilon_o} \int_{-h_2}^{h_2} q_4^r(y') K_{2B}(x, y') dy' \quad (3.82)$$

If the zeroth order solutions are substituted in (3.81) and (3.82) respectively, we obtain

$$\begin{aligned}
\phi_4^r(y) &= \frac{j\omega}{\beta_o} \frac{-j}{2v_o} \left[-C_4^r \beta_o \sin \beta_o y + \frac{\partial \theta_4^r(y)}{\partial y} \right] \\
&+ \frac{1}{4\pi\epsilon_o} \int_{-h_1}^{h_1} \left[q_1^r(x') \right]_0 K_{1B}(y, x') dx' \\
&= -C_4^r \sin \beta_o y + \frac{1}{\beta_o} \frac{\partial \theta_4^r(y)}{\partial y} + \frac{C_1^r}{\Phi_r} \int_{-h_1}^{h_1} \cos \beta_o x' K_{1B}(y, x') dx' \\
&+ \frac{1}{\Phi_r \beta_o} \int_{-h_1}^{h_1} \frac{\partial \theta_1^r(x')}{\partial x'} K_{1B}(y, x') dx' \\
&= -C_4^r \sin \beta_o y + \frac{1}{\beta_o} \frac{\partial \theta_4^r(y)}{\partial y} + C_1^r \frac{T_{cd}(y)}{\Phi_r} + \frac{f_1^r(y)}{\Phi_r} \\
&= -C_4^r \sin \beta_o y + \frac{1}{\beta_o} \frac{\partial \theta_4^r(y)}{\partial y} + C_1^r L_2^r(y) + F_1^r(y)
\end{aligned} \tag{3.83}$$

where

$$L_2^r(y) = \frac{T_{cd}(y)}{\Phi_r} \tag{3.84}$$

$$F_1^r(y) = \frac{f_1^r(y)}{\Phi_r} \tag{3.85}$$

$$T_{cd}(y) = \int_{-h_1}^{h_1} \cos \beta_o x' K_{1B}(y, x') dx' \quad (3.86)$$

$$f_1^r(y) = \frac{1}{\beta_o} \int_{-h_1}^{h_1} \frac{\partial \theta_1^r(x')}{\partial x'} K_{1B}(y, x') dx' \quad (3.87)$$

$$\begin{aligned} \phi_1^r(x) &= \frac{j\omega}{\beta_o^2} \frac{-j}{v_o} \left[C_1^r \beta_o \cos \beta_o x + \frac{\partial \theta_1^r(x)}{\partial x} \right] \\ &+ \frac{1}{4\pi\epsilon_o} \int_{-h_2}^{h_2} \left[q_4^r(y') \right]_0 K_{2B}(x, y') dy' \\ &= C_1^r \cos \beta_o x + \frac{1}{\beta_o} \frac{\partial \theta_1^r(x)}{\partial x} - \frac{C_4^r}{\Phi_r} T_{s*a}(x) + \frac{f_2^r(x)}{\Phi_r} \\ &= C_1^r \cos \beta_o x + \frac{1}{\beta_o} \frac{\partial \theta_1^r(x)}{\partial x} - C_4^r M_2^r(x) + F_2^r(x) \end{aligned} \quad (3.88)$$

where

$$M_2^r(x) = \frac{T_{s*a}(x)}{\Phi_r} \quad (3.89)$$

$$F_2^r(x) = \frac{f_2^r(x)}{\Phi_r} \quad (3.90)$$

$$T_{s*a}(x) = \int_{-h_2}^{h_2} \sin \beta_o y' K_{2B}(x, y') dy' \quad (3.91)$$

$$f_2^r(x) = \frac{1}{\beta_o} \int_{-h_2}^{h_2} \frac{\partial \theta_4^r(y')}{\partial y'} K_{2B}(x, y') dy' \quad (3.92)$$

The boundary conditions to be used for the first order solutions are:

(a) Current at the corners is continuous

$$\text{i.e.,} \quad I_{4y}^r(y = h_2) = -I_{1x}^r(x = h_1) \quad (3.93)$$

(b) Scalar potential at the corners is continuous

$$\text{i.e.,} \quad \phi_4^r(y = h_2) = \phi_1^r(x = h_1) \quad (3.94)$$

The substitution of (3.72) and (3.77) in (3.93) gives

$$\begin{aligned} C_4^r L_1^r(h_2) + 2\theta_4^r(h_2) - N_1^r(h_2) \\ = - \left[C_1^r M_1^r(h_1) + 2\theta_1^r(h_1) - D_1^r(h_1) \right] \end{aligned} \quad (3.95)$$

With (3.83) and (3.88), (3.94) becomes

$$\begin{aligned}
 & - C_4^r \sin \beta_o h_2 + \frac{1}{\beta_o} \left. \frac{\partial \theta_4^r(y)}{\partial y} \right|_{y=h_2} + C_1^r L_2^r(h_2) + F_1^r(h_2) \\
 & = C_1^r \cos \beta_o h_1 + \frac{1}{\beta_o} \left. \frac{\partial \theta_1^r(x)}{\partial x} \right|_{x=h_1} - C_4^r M_2^r(h_2) + F_2^r(h_1)
 \end{aligned} \tag{3.96}$$

Solving (3.95) and (3.96) for C_4^r and C_1^r , we obtain

$$C_4^r = \frac{\left[\begin{aligned} & [2\theta_4^r(h_2) + 2\theta_1^r(h_1) - N_1^r(h_2) - D_1^r(h_1)] [\cos \beta_o h_1 - L_2^r(h_2)] \\ & + M_1^r(h_1) \left[\frac{1}{\beta_o} \left. \frac{\partial \theta_4^r(y)}{\partial y} \right|_{y=h_2} - \frac{1}{\beta_o} \left. \frac{\partial \theta_1^r(x)}{\partial x} \right|_{x=h_1} + F_1^r(h_2) - F_2^r(h_1) \right] \end{aligned} \right]}{M_1^r(h_1) [\sin \beta_o h_2 - M_2^r(h_1)] - L_1^r(h_2) [\cos \beta_o h_1 - L_2^r(h_2)]} \tag{3.97}$$

$$C_1^r = \frac{\left[\begin{aligned} & [2\theta_4^r(h_2) + 2\theta_1^r(h_1) - N_1^r(h_2) - D_1^r(h_1)] [\sin \beta_o h_2 - M_2^r(h_1)] \\ & + L_1^r(h_2) \left[\frac{1}{\beta_o} \left. \frac{\partial \theta_4^r(y)}{\partial y} \right|_{y=h_2} - \frac{1}{\beta_o} \left. \frac{\partial \theta_1^r(x)}{\partial x} \right|_{x=h_1} + F_1^r(h_2) - F_2^r(h_1) \right] \end{aligned} \right]}{M_1^r(h_1) [\sin \beta_o h_2 - M_2^r(h_1)] - L_1^r(h_2) [\cos \beta_o h_1 - L_2^r(h_2)]} \tag{3.98}$$

For our first approximation, we assume that

$$\theta_4^r(y) \cong \left[\theta_4^r(y) \right]_0 = \frac{V}{2} \sin \beta_o |y| \quad (3.99)$$

$$\theta_1^r(x) \cong \left[\theta_1^r(x) \right]_0 = 0 \quad (3.100)$$

Then (3.74), (3.78), (3.85) and (3.90) can be rewritten as

$$N_1^r(y) = \frac{V}{\Phi_r} \int_{-h_2}^{h_2} \sin \beta_o |y'| K_{1A}(y, y') dy' = \frac{V}{\Phi_r} T_{sa}(y) \quad (3.101)$$

where

$$T_{sa}(y) = \int_{-h_2}^{h_2} \sin \beta_o |y'| K_{1A}(y, y') dy' \quad (3.102)$$

$$D_1^r(x) = 0 \quad (3.103)$$

$$F_1^r(y) = 0 \quad (3.104)$$

$$F_2^r(x) = \frac{V}{\Phi_r} \int_{-h_2}^{h_2} \pm \cos \beta_o y' K_{2B}(x, y') dy' \quad \begin{array}{l} + \text{ for } y' > 0 \\ - \text{ for } y' < 0 \end{array}$$

$$= \frac{V}{\Phi_r} T_{ba}(x) \quad (3.105)$$

where

$$T_{ba}(x) = \int_{-h_2}^{h_2} \pm \cos \beta_o y' K_{2B}(x, y') dy' \quad \begin{array}{l} + \text{ for } y' > 0 \\ - \text{ for } y' < 0 \end{array} \quad (3.106)$$

Using (3.99) to (3.106), C_4^r in (3.97) and C_1^r in (3.98) can be obtained as follows if only the terms with the order of $\frac{1}{\Phi_r}$ are retained in both numerator and denominator:

$$\left[C_4^r \right]_1 = - \frac{\frac{V}{2} \left[\sin \beta_o (h_1 + h_2) + \frac{B^r}{\Phi_r} \right]}{\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r}} \quad (3.107)$$

$$\left[C_1^r \right]_1 = \frac{\frac{V}{2} \left[1 + \frac{H^r}{\Phi_r} \right]}{\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r}} \quad (3.108)$$

where

$$B^r = - \frac{1}{2} \left[T_{sa}(h_2) \cos \beta_o h_1 + 2 T_{cd}(h_2) \sin \beta_o h_2 \right. \\ \left. + T_{s*d}(h_1) \cos \beta_o h_2 + 2 T_{ba}(h_1) \sin \beta_o h_1 \right] \quad (3.109)$$

$$\begin{aligned}
G &= \frac{1}{2} \left[T_{s*d}(h_1) \sin \beta_o h_2 + 2T_{s*a}(h_1) \sin \beta_o h_1 \right. \\
&\quad \left. - T_{ca}(h_2) \cos \beta_o h_1 - 2T_{cd}(h_2) \cos \beta_o h_2 \right]
\end{aligned} \tag{3.110}$$

$$\begin{aligned}
H^r &= -\frac{1}{2} \left\{ \sin \beta_o h_2 \left[T_{sa}(h_2) + 2T_{s*a}(h_1) \right] \right. \\
&\quad \left. + \cos \beta_o h_2 \left[T_{ca}(h_2) + 2T_{ba}(h_1) \right] \right\}
\end{aligned} \tag{3.111}$$

With (3.51), (3.107) and (3.108), (3.49) and (3.50) can be rewritten as

$$\left[I_{4y}^r(y) \right]_1 = \frac{-j4\pi}{\zeta_o \Phi_r} \left\{ -\frac{\frac{V}{2} \left[\sin \beta_o (h_1 + h_2) + \frac{B_r}{\Phi_r} \right]}{\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r}} \cos \beta_o y + \theta_4^r(y) \right\} \tag{3.112}$$

$$\left[I_{1x}^r(x) \right]_1 = \frac{-j4\pi}{\zeta_o \Phi_r} \left\{ \frac{\frac{V}{2} \left[1 + \frac{H_r}{\Phi_r} \right]}{\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r}} \sin \beta_o x + \theta_1^r(x) \right\} \tag{3.113}$$

(E) Evaluation of Particular Integrals $\theta_4^r(y)$ and $\theta_1^r(x)$:

Integrating by parts, we can rewrite $\theta_4^r(y)$ and $\theta_1^r(x)$ in (3.40) and (3.42) as

$$\begin{aligned}
\theta_4^r(y) &= \frac{V}{2} \sin \beta_o |y| - \int_0^y \frac{\partial}{\partial s} \left[\phi_{41}^r(x) + \phi_{43}^r(s) \right] \sin \beta_o (y-s) ds \\
&= \frac{V}{2} \sin \beta_o |y| + \left[\phi_{41}^r(o) + \phi_{43}^r(o) \right] \sin \beta_o y \\
&\quad + \beta_o \int_0^y \left[\phi_{41}^r(s) + \phi_{43}^r(s) \right] \cos \beta_o (y-s) ds \\
&= \frac{V}{2} \sin \beta_o |y| + \beta_o \int_0^y \left[\phi_{41}^r(s) + \phi_{43}^r(s) \right] \cos \beta_o (y-s) ds
\end{aligned} \tag{3.114}$$

(3.114) is derived based on the relation of

$$\phi_{41}^r(o) = -\phi_{43}^r(o) \tag{3.115}$$

Similarly,

$$\begin{aligned}
\theta_1^r(x) &= - \int_0^x \frac{\partial}{\partial s} \left[\phi_{12}^r(s) + \phi_{14}^r(s) \right] \sin \beta_o (x-s) ds \\
&= \left[\phi_{12}^r(o) + \phi_{14}^r(o) \right] \sin \beta_o x + \beta_o \int_0^x \left[\phi_{12}^r(s) + \phi_{14}^r(s) \right] \\
&\quad \cdot \cos \beta_o (x-s) ds \\
&= 2 \phi_{12}^r(o) \sin \beta_o x + \beta_o \int_0^x \left[\phi_{12}^r(s) + \phi_{14}^r(s) \right] \cos \beta_o (x-s) ds
\end{aligned} \tag{3.116}$$

based on the relation of

$$\phi_{12}^r(o) = \phi_{14}^r(o) \quad (3.117)$$

For the first approximation, we substitute (3.99) and (3.100) in (3.55) and (3.56). This leads to

$$q_4^r(y) = + \frac{4\pi\epsilon_o}{\Phi_r} \left[-C_4^r \sin \beta_o y \pm \frac{V}{2} \cos \beta_o y \right] \begin{array}{l} + \text{ for } y > 0 \\ - \text{ for } y < 0 \end{array} \quad (3.118)$$

$$q_1^r(x) = + \frac{4\pi\epsilon_o}{\Phi_r} C_1^r \cos \beta_o x \quad (3.119)$$

The substitution of (3.119) in (3.17) gives

$$\begin{aligned} \phi_{41}^r(y) + \phi_{43}^r(y) &= + \frac{C_1^r}{\Phi_r} \int_{-h_1}^{h_1} \cos \beta_o x' K_{1B}(y, x') dx' \\ &= \frac{C_1^r}{\Phi_r} T_{cd}(y) \end{aligned} \quad (3.120)$$

Similarly, with (3.118), (3.32) can be rewritten as

$$\begin{aligned}
\phi_{12}^r(x) + \phi_{14}^r(x) &= -\frac{C_4^r}{\Phi_r} \int_{-h_2}^{h_2} \sin \beta_o y' K_{2B}(x, y') dy' \\
&+ \frac{\frac{V}{2}}{\Phi_r} \int_{-h_2}^{h_2} \pm \cos \beta_o y' K_{2B}(x, y') dy' \\
&\quad + \text{for } y' > 0 \\
&\quad - \text{for } y' < 0 \\
&= -\frac{C_4^r}{\Phi_r} T_{s*a}(x) + \frac{\frac{V}{2}}{\Phi_r} T_{ba}(x)
\end{aligned} \tag{3.121}$$

With (3.117), (3.121) yields

$$2\phi_{12}^r(o) = -\frac{C_4^r}{\Phi_r} T_{s*a}(o) + \frac{\frac{V}{2}}{\Phi_r} T_{ba}(o) \tag{3.122}$$

Substituting (3.120) in (3.114), we obtain

$$\theta_4^r(y) = \frac{V}{2} \sin \beta_o |y| + \frac{C_1^r \beta_o}{\Phi_r} \int_0^y T_{cd}(s) \cos \beta_o (y-s) ds \tag{3.123}$$

Similarly, with (3.121) and (3.122), (3.116) becomes

$$\begin{aligned}
\theta_1^r(x) = & \frac{1}{\Phi_r} \left\{ \left[-C_4^r T_{s*a}(o) + \frac{V}{2} T_{ba}(o) \right] \sin \beta_o x \right. \\
& - C_4^r \int_0^x T_{s*a}(s) \cos \beta_o(x-s) ds + \frac{V}{2} \\
& \cdot \left. \int_0^x T_{ba}(s) \cos \beta_o(x-s) ds \right\}
\end{aligned}
\tag{3.124}$$

With (3.108), (3.123) becomes

$$\begin{aligned}
\left[\theta_4^r(y) \right]_1 &= \frac{V}{2} \sin \beta_o |y| \\
&+ \frac{-\frac{V}{2} \left[\sin \beta_o(h_1 + h_2) + \frac{B^r}{\Phi_r} \right]}{\cos \beta_o(h_1 + h_2) + \frac{G}{\Phi_r}} \frac{\beta_o}{\Phi_r} \int_0^y T_{cd}(s) \cos \beta_o(y-s) ds \\
&= \frac{V}{2} \frac{\sin \beta_o |y| \left[\cos \beta_o(h_1 + h_2) + \frac{G}{\Phi_r} \right] - \frac{\beta_o}{\Phi_r} \left[\sin \beta_o(h_1 + h_2) + \frac{B^r}{\Phi_r} \right] t_{cd}(y)}{\cos \beta_o(h_1 + h_2) + \frac{G}{\Phi_r}}
\end{aligned}
\tag{3.125}$$

where

$$t_{cd}(y) = \int_0^y T_{cd}(s) \cos \beta_o(y-s) ds \tag{3.126}$$

Similarly, the substitution of (3.107) in (3.124) gives

$$\begin{aligned} \left[\theta_1^r(x) \right]_1 &= \frac{1}{\Phi_r} \frac{V}{2} \left\{ \frac{\sin \beta_o(h_1 + h_2) + \frac{B^r}{\Phi_r}}{\cos \beta_o(h_1 + h_2) + \frac{G}{\Phi_r}} T_{s*a}(o) + T_{ba}(o) \right\} \sin \beta_o x \\ &\quad + \left\{ \frac{\sin \beta_o(h_1 + h_2) + \frac{B^r}{\Phi_r}}{\cos \beta_o(h_1 + h_2) + \frac{G}{\Phi_r}} t_{s*a}(x) + t_{ba}(x) \right\} \end{aligned} \quad (3.127)$$

where

$$t_{s*a}(x) = \int_0^x T_{s*a}(s) \cos \beta_o(x - s) ds \quad (3.128)$$

$$t_{ba}(x) = \int_0^x T_{ba}(s) \cos \beta_o(x - s) ds \quad (3.129)$$

With (3.125), (3.112) becomes

$$\begin{aligned}
\left[I_{4y}^r(y) \right]_{11} &= \frac{j2\pi V}{\zeta_o \Phi_r} \times \\
&\frac{\left[\sin \beta_o (h_1 + h_2 - |y|) + \frac{1}{\Phi_r} \left[B^r \cos \beta_o y - G \sin \beta_o |y| \right. \right. \\
&\left. \left. + \beta_o \sin \beta_o (h_1 + h_2) t_{cd}(y) \right] + \frac{1}{\Phi_r^2} \beta_o B^r t_{cd}(y) \right]}{\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r}} \\
&= \frac{j2\pi V}{\zeta_o \Phi_r} \frac{\sin \beta_o (h_1 + h_2 - |y|) + \frac{1}{\Phi_r} P_{41}^r(y) + \frac{1}{\Phi_r^2} P_{42}^r(y)}{\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} G}
\end{aligned} \tag{3.130}$$

where

$$P_{41}^r(y) = B^r \cos \beta_o y - G \sin \beta_o |y| + \beta_o \sin \beta_o (h_1 + h_2) t_{cd}(y) \tag{3.131}$$

$$P_{42}^r(y) = \beta_o B^r t_{cd}(y) \tag{3.132}$$

Similarly, with (3.127), (3.113) becomes

$$\left[I_{\mathbf{L}\mathbf{x}}^{\mathbf{r}}(\mathbf{x}) \right]_{11} = \frac{-j2\pi V}{\zeta_o \Phi_r} \mathbf{x}$$

$$\begin{aligned} & \left[\sin \beta_o \mathbf{x} + \frac{1}{\Phi_r} \left\{ H^{\mathbf{r}} \sin \beta_o \mathbf{x} + \left[\sin \beta_o (h_1 + h_2) T_{s*a}(o) \right. \right. \right. \\ & \quad \left. \left. + \cos \beta_o (h_1 + h_2) T_{ba}(o) \right] \sin \beta_o \mathbf{x} + \sin \beta_o (h_1 + h_2) t_{s*a}(\mathbf{x}) \right. \\ & \quad \left. \left. + \cos \beta_o (h_1 + h_2) t_{ba}(\mathbf{x}) \right\} \right. \\ & \quad \left. + \frac{1}{\Phi_r} \left\{ \left[B^{\mathbf{r}} T_{s*a}(o) + G T_{ba}(o) \right] \sin \beta_o \mathbf{x} + B^{\mathbf{r}} t_{s*a}(\mathbf{x}) + G t_{ba}(\mathbf{x}) \right\} \right] \\ & \quad \cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r} \end{aligned}$$

$$= \frac{-j2\pi V}{\zeta_o \Phi_r} \frac{\sin \beta_o \mathbf{x} + \frac{1}{\Phi_r} P_{11}^{\mathbf{r}}(\mathbf{x}) + \frac{1}{\Phi_r} P_{12}^{\mathbf{r}}(\mathbf{x})}{\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r}}$$

(3.133)

where

$$\begin{aligned} P_{11}^{\mathbf{r}}(\mathbf{x}) &= H^{\mathbf{r}} \sin \beta_o \mathbf{x} + \left[\sin \beta_o (h_1 + h_2) T_{s*a}(o) + \cos \beta_o (h_1 + h_2) T_{ba}(o) \right] \\ & \quad \cdot \sin \beta_o \mathbf{x} + \sin \beta_o (h_1 + h_2) t_{s*a}(\mathbf{x}) + \cos \beta_o (h_1 + h_2) t_{ba}(\mathbf{x}) \end{aligned}$$

(3.134)

$$p_{12}^r(x) = \left[B^r T_{s*a}(o) + G T_{ba}(o) \right] \sin \beta_o x$$

$$+ B^r t_{s*a}(x) + G t_{ba}(x) \quad (3.135)$$

$$\left[I_{4y}^r(y) \right]_{11} \quad \text{and} \quad \left[I_{1x}^r(x) \right]_{11} \quad \text{involve the double integrals which in}$$

turn complicate the problem. In order to avoid double integral, the following method is presented at the expense of sacrificing the accuracy of the theory:

Substituting (3.99) to (3.105) in (3.72), we obtain

$$\left[I_{4y}^r(y) \right]_{10} = \frac{-j4\pi}{\zeta_o \Phi_r}$$

$$\left\{ - \frac{\frac{V}{2} \left[\sin \beta_o (h_1 + h_2) + \frac{B^r}{\Phi_r} \right]}{\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r}} \left[2 \cos \beta_o y - \frac{T_{ca}(y)}{\Phi_r} \right] \right.$$

$$\left. + V \sin \beta_o |y| - \frac{\frac{V}{2} T_{sa}(y)}{\Phi_r} \right\}$$

$$= \frac{j2\pi V}{\zeta_o \Phi_r} \times \frac{2 \sin \beta_o ((h_1 + h_2) - |y|) + \frac{1}{\Phi_r} U_{41}^r(y) + \frac{1}{\Phi_r} U_{42}^r(y)}{\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r}}$$

(3.136)

where

$$\begin{aligned}
 U_{41}^r(y) = & 2B^r \cos \beta_o y - \sin \beta_o (h_1 + h_2) T_{ca}(y) - G \sin \beta_o |y| \\
 & + \cos \beta_o (h_1 + h_2) T_{sa}(y)
 \end{aligned} \tag{3.137}$$

$$U_{42}^r(y) = -B^r T_{ca}(y) + G T_{sa}(y) \tag{3.138}$$

Similarly, the substitution of (3.99) to (3.105) in (3.77) yields

$$\begin{aligned}
 \left[I_{1x}^r(x) \right]_{10} &= \frac{-j4\pi}{\zeta_o \Phi_r} \left\{ \frac{\frac{V}{2} \left[1 + \frac{H^r}{\Phi_r} \right]}{\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r}} \left(2 \sin \beta_o x - \frac{T_{s*d}(x)}{\Phi_r} \right) \right\} \\
 &= \frac{-j2\pi V}{\zeta_o \Phi_r} \frac{2 \sin \beta_o x + \frac{1}{\Phi_r} g_{11}^r(x) + \frac{1}{\Phi_r} g_{12}^r(x)}{\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r}}
 \end{aligned} \tag{3.139}$$

where

$$g_{11}^r(x) = 2H^r \sin \beta_o x - T_{s*d}(x) \tag{3.140}$$

$$g_{12}^r(x) = -H^r T_{s*d}(x) \tag{3.141}$$

(F) Expansion Parameter, Φ_r

With zeroth order current in (3.66), $\Phi_r(y)$ in (3.45) can be rewritten as

$$\Phi_r(y) = \frac{\int_{-h_2}^{h_2} \sin \beta_o(h_1 + h_2 - |y'|) K_{1A}(y, y') dy'}{\sin \beta_o(h_1 + h_2 - |y|)}$$

$$= \frac{\sin \beta_o(h_1 + h_2) T_{ca}(y) - \cos \beta_o(h_1 + h_2) T_{sa}(y)}{\sin \beta_o(h_1 + h_2 - |y|)}$$

For a loop with $\beta_o(h_1 + h_2) \leq \frac{\pi}{2}$, the point of maximum current is at $y = 0$, so

$$\Phi_r = \left[\Phi_r(y) \right]_{y=0} = T_{ca}(0) - \cot \beta_o(h_1 + h_2) T_{sa}(0) \quad (3.142)$$

For a loop with $\beta_o(h_1 + h_2) \geq \frac{\pi}{2}$, the point of maximum current is at $y = (h_1 + h_2) - \frac{\lambda}{4}$, so

$$\Phi_r = \left[\Phi_r(y) \right]_{y=(h_1 + h_2) - \frac{\lambda}{4}}$$

$$= \sin \beta_o(h_1 + h_2) T_{ca}(h_1 + h_2 - \frac{\lambda}{4}) - \cos \beta_o(h_1 + h_2) T_{sa}(h_1 + h_2 - \frac{\lambda}{4})$$

(3.143)

3.3. Total Loop Currents

By the principle of superposition we note that

$$I_y(y) = I_{4y}^r(y) + I_{4y}^s(y) \quad (3.144)$$

$$I_x(x) = I_{1x}^r(x) + I_{1x}^s(x) \quad (3.145)$$

$$V = -I_y(o)Z_L = -\left[I_{4y}^r(o) + I_{4y}^s(o)\right] Z_L \quad (3.146)$$

(A) Considering the first order solutions of currents

$$\left[I_{4y}^r(y)\right]_{11} \text{ in (3.130), } \left[I_{4y}^s(y)\right]_{11} \text{ in (1.167), } \left[I_{1x}^r(x)\right]_{11} \text{ in}$$

$$(3.133) \text{ and } \left[I_{1x}^s(x)\right]_{11} \text{ in (1.170), } V \text{ in (3.146) can be solved as}$$

$$V = \frac{-j4\pi Z_L \frac{E_o}{\beta_o} \Phi_r \left[\cos \beta_o(h_1 + h_2) + \frac{G}{\Phi_r}\right]}{\Phi_s \left[\cos \beta_o(h_1 + h_2) + \frac{G}{\Phi_s}\right]} x$$

$$\frac{\left[\cos \beta_o h_1 - \cos \beta_o(h_1 + h_2) + \frac{1}{\Phi_s} P_{41}^s(o) + \frac{1}{\Phi_s 2} P_{42}^s(o)\right]}{\left[\zeta_o \Phi_r \left[\cos \beta_o(h_1 + h_2) + \frac{G}{\Phi_r}\right] + j2\pi Z_L \left[\sin \beta_o(h_1 + h_2) + \frac{1}{\Phi_r} P_{41}^r(o) + \frac{1}{\Phi_r 2} P_{42}^r(o)\right]\right]} \quad (3.147)$$

With (3.147), (3.130) can be rewritten as

$$\left[I_{4y}^r(y) \right]_{11} = \frac{8\pi^2 Z_L}{\zeta_o} \frac{E_o}{\beta_o} x$$

$$\frac{\left[\begin{aligned} & \left[\cos \beta_o h_1 - \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} P_{41}^s(o) + \frac{1}{\Phi_s^2} P_{42}^s(o) \right] x \\ & \left[\sin \beta_o (h_1 + h_2 - |y|) + \frac{1}{\Phi_r} P_{41}^r(y) + \frac{1}{\Phi_r^2} P_{42}^r(y) \right] \end{aligned} \right]}{\left[\begin{aligned} & \Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right] \left\{ \zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r} \right] \right. \\ & \left. + j2\pi Z_L \left[\sin \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} P_{41}^r(o) + \frac{1}{\Phi_r^2} P_{42}^r(o) \right] \right\} \end{aligned} \right]} \quad (3.148)$$

Similarly, (3.133) can also be rewritten as

$$\left[I_{1x}^r(x) \right]_{11} = - \frac{8\pi^2 Z_L}{\zeta_o} \frac{E_o}{\beta_o} x$$

$$\frac{\left[\begin{aligned} & \left[\cos \beta_o h_1 - \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} P_{41}^s(o) + \frac{1}{\Phi_s^2} P_{42}^s(o) \right] x \\ & \left[\sin \beta_o x + \frac{1}{\Phi_r} P_{11}^r(x) + \frac{1}{\Phi_r^2} P_{12}^r(x) \right] \end{aligned} \right]}{\left[\begin{aligned} & \Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right] \left\{ \zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r} \right] \right. \\ & \left. + j2\pi Z_L \left[\sin \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} P_{41}^r(o) + \frac{1}{\Phi_r^2} P_{42}^r(o) \right] \right\} \end{aligned} \right]} \quad (3.149)$$

Thus, the total loop currents in (3.144) and (3.145) can be expressed as the functions of the loop dimensions and the loading impedance Z_L .

(B) If consideration is given to the sub-first order solutions of currents

$$\left[I_{4y}^r(y) \right]_{10} \text{ in (3.136), } \left[I_{1x}^r(x) \right]_{10} \text{ in (3.139), } \left[I_{4y}^s(y) \right]_{10} \text{ in (1.173)}$$

and $\left[I_{1x}^s(x) \right]_{10}$ in (1.176), V in (3.146) is obtained as

$$V = \frac{-j4\pi Z_L \frac{E_o}{\beta_o} \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r} \right]}{\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right]} x$$

$$\frac{\left[2 \cos \beta_o h_1 - 2 \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} U_{41}^s(o) + \frac{1}{\Phi_s} U_{42}^s(o) \right]}{\left[\zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r} \right] + j2\pi Z_L \left[2 \sin \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} U_{41}^r(o) + \frac{1}{\Phi_r} U_{42}^r(o) \right] \right]} \quad (3.150)$$

With (3.150), (3.136) can be rewritten as

$$\left[I_{4y}^r(y) \right]_{10} = \frac{8\pi^2 Z_L}{\zeta_o} \frac{E_o}{\beta_o} x$$

$$\frac{\left[\begin{aligned} & \left[2 \cos \beta_o h_1 - 2 \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} U_{41}^s(o) + \frac{1}{\Phi_s 2} U_{42}^s(o) \right] x \\ & \left[2 \sin \beta_o (h_1 + h_2 - y) + \frac{1}{\Phi_r} U_{41}^r(y) + \frac{1}{\Phi_r 2} U_{42}^r(y) \right] \end{aligned} \right]}{\left[\begin{aligned} & \Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right] x \left\{ \zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r} \right] \right. \\ & \left. + j2\pi Z_L \left[2 \sin \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} U_{41}^r(o) + \frac{1}{\Phi_r 2} U_{42}^r(o) \right] \right\} \end{aligned} \right]} \quad (3.151)$$

Similarly, (3.139) can be rewritten as

$$\left[I_{1x}^r(x) \right]_{10} = - \frac{8\pi^2 Z_L}{\zeta_o} \frac{E_o}{\beta_o} x$$

$$\frac{\left[\begin{aligned} & \left[2 \cos \beta_o h_1 - 2 \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} U_{41}^s(o) + \frac{1}{\Phi_s 2} U_{42}^s(o) \right] x \\ & \left[2 \sin \beta_o x + \frac{1}{\Phi_r} U_{11}^r(x) + \frac{1}{\Phi_r 2} U_{12}^r(x) \right] \end{aligned} \right]}{\left[\begin{aligned} & \Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right] \left\{ \zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r} \right] \right. \\ & \left. + j2\pi Z_L \left[2 \sin \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} U_{41}^r(o) + \frac{1}{\Phi_r 2} U_{42}^r(o) \right] \right\} \end{aligned} \right]} \quad (3.152)$$

3.4. Total Backscattered Field

The vector potential, maintained by the induced current in the radiating loop, at an arbitrary point on the axis of the loop and in the far zone of the loop (Fig. 3.3) can be expressed as

$$\begin{aligned}
 A^r &= A_y^r = \frac{\mu_o}{4\pi} \int_{-h_2}^{h_2} 2I_{4y}^r(y') \frac{e^{-j\beta_o R}}{R} dy' \\
 &\approx \frac{\mu_o}{2\pi} \frac{e^{-j\beta_o R_o}}{R_o} \int_{-h_2}^{h_2} I_{4y}^r(y') dy'
 \end{aligned} \tag{3.153}$$

where

$$R = \sqrt{R_o^2 + y'^2 + h_1^2} \approx R_o \tag{3.154}$$

3.4.1. Total Backscattered Field Based on First Order Currents

With (3.148), (3.153) can be rewritten as

$$\begin{aligned}
 \left[A^r \right]_{11} &= \left[A_y^r \right]_{11} = \frac{4\pi\mu_o Z_L}{\zeta_o \beta_o} E_o \frac{e^{-j\beta_o R_o}}{R_o} \times \\
 &\frac{\left[\cos \beta_o h_1 - \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} P_{41}^s(o) + \frac{1}{\Phi_s^2} P_{42}^s(o) \right] K_1}{\left[\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right] \left\{ \zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r} \right] \right. \right.} \\
 &\left. \left. + j2\pi Z_L \left[\sin \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} P_{41}^r(o) + \frac{1}{\Phi_r^2} P_{42}^r(o) \right] \right\} \right]
 \end{aligned} \tag{3.155}$$

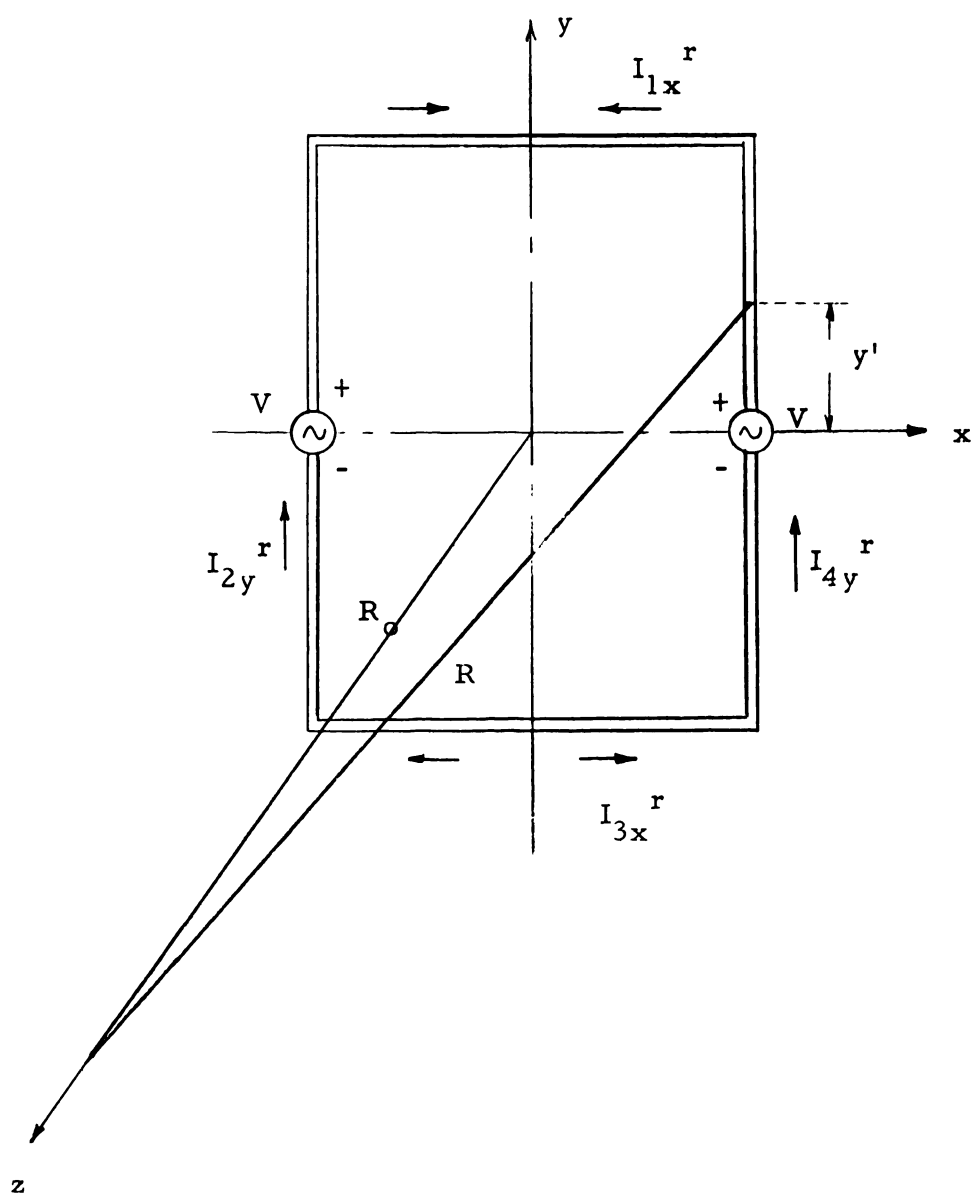


Figure 3.3 Geometry for the calculation of the radiation field of a radiating rectangular loop.

where

$$K_1 = \int_{-h_2}^{h_2} \left[\sin \beta_o (h_1 + h_2 - |y'|) + \frac{P_{41}^r(y')}{\Phi_r} + \frac{P_{42}^r(y')}{\Phi_r^2} \right] dy' \quad (3.156)$$

K_1 will be evaluated in a following section.

The backscattered electric field due to the radiating loop is

$$\begin{aligned} \left[E_y^r \right]_{11} &= -j\omega \left[A_y^r \right]_{11} = -j4\pi E_o Z_L \frac{e^{-j\beta_o R_o}}{R_o} x \\ &\frac{\left[\cos \beta_o h_1 - \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} P_{41}^s(o) + \frac{1}{\Phi_s^2} P_{42}^s(o) \right] K_1}{\left[\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right] \left\{ \zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r} \right] \right. \right.} \\ &\left. \left. + j2\pi Z_L \left[\sin \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} P_{41}^r(o) + \frac{1}{\Phi_r^2} P_{42}^r(o) \right] \right\} \right] \end{aligned} \quad (3.157)$$

The total backscattered field from a loaded rectangular loop is obtained with (1.183) and (3.157)

$$\begin{aligned}
\begin{bmatrix} E_y \end{bmatrix}_{11} &= \begin{bmatrix} E_y^s \end{bmatrix}_{11} + \begin{bmatrix} E_y^r \end{bmatrix}_{11} \\
&= 2E_o \frac{e^{-j\beta_o R_o}}{R_o} \frac{1}{\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right]} \times \left\{ J_1 - j_2 \pi Z_L \times \right. \\
&\quad \left. \frac{\left[\cos \beta_o h_1 - \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} P_{41}^s(o) + \frac{1}{\Phi_s 2} P_{42}^s(o) \right] K_1}{\left[\zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r} \right] + j_2 \pi Z_L \left[\sin \beta_o (h_1 + h_2) \right. \right.} \\
&\quad \left. \left. + \frac{1}{\Phi_r} P_{41}^r(o) + \frac{1}{\Phi_r 2} P_{42}^r(o) \right] \right\}
\end{aligned}
\tag{3.158}$$

3.4.2. Total Backscattered Field Based on Sub-first Order Currents

With (3.151), (3.153) can be rewritten as

$$\begin{aligned}
\begin{bmatrix} A^r \end{bmatrix}_{10} &= \begin{bmatrix} A_y^r \end{bmatrix}_{10} \approx \frac{\mu_o}{2\pi} \frac{e^{-j\beta_o R_o}}{R_o} \int_{-h_2}^{h_2} \begin{bmatrix} I_{4y}^r(y') \end{bmatrix}_{10} dy' \\
&= \frac{4\pi\mu_o Z_L}{\zeta_o \beta_o} E_o \frac{e^{-j\beta_o R_o}}{R_o} \\
&\quad \left[2 \cos \beta_o h_1 - 2 \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} U_{41}^s(o) + \frac{1}{\Phi_s 2} U_{42}^s(o) \right] K_2 \\
&\quad \frac{\left[\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right] \left\{ \zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} G \right] \right. \right.}{\left. \left. + j_2 \pi Z_L \left[2 \sin \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} U_{41}^r(o) + \frac{1}{\Phi_r 2} U_{42}^r(o) \right] \right\} \right]}
\end{aligned}
\tag{3.159}$$

where

$$K_2 = \int_{-h_2}^{h_2} \left[2 \sin \beta_o (h_1 + h_2 - |y'|) + \frac{1}{\Phi_r} U_{41}^r(y') + \frac{1}{\Phi_r} \frac{1}{2} U_{42}^r(y') \right] dy' \quad (3.160)$$

Then, the backscattered electric field due to the radiating loop is obtained as

$$\begin{aligned} \begin{bmatrix} E^r \end{bmatrix}_{10} &= \begin{bmatrix} E_y^r \end{bmatrix}_{10} = -j\omega \begin{bmatrix} A_y^r \end{bmatrix}_{10} = -j4\pi E_o Z_L \frac{e^{-j\beta_o R_o}}{R_o} \mathbf{x} \\ &\frac{\left[2 \cos \beta_o h_1 - 2 \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} U_{41}^s(o) + \frac{1}{\Phi_s} \frac{1}{2} U_{42}^s(o) \right] K_2}{\left[\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_s} \right] \left\{ \zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{G}{\Phi_r} \right] \right. \right.} \\ &\left. \left. + j2\pi Z_L \left[2 \sin \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} U_{41}^r(o) + \frac{1}{\Phi_r} \frac{1}{2} U_{42}^r(o) \right] \right\} \right]} \end{aligned} \quad (3.161)$$

The total backscattered electric field due to the loaded rectangular loop can be calculated by superposing $\begin{bmatrix} E^s \end{bmatrix}_{10}$ in (1.188) upon $\begin{bmatrix} E^r \end{bmatrix}_{10}$

in (3.160) as

$$\begin{bmatrix} E_y \end{bmatrix}_{10} = \begin{bmatrix} E_y^s \end{bmatrix}_{10} + \begin{bmatrix} E_y^r \end{bmatrix}_{10}$$

$$= 2E_o \frac{e^{-j\beta_o R_o}}{R_o} \frac{1}{\Phi_s \left[\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} G \right]} x$$

$$\left\{ J_2 - j2\pi Z_L X \right.$$

$$\left[2 \cos \beta_o h_1 - 2 \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} U_{41}^s(o) + \frac{1}{\Phi_s^2} U_{42}^s(o) \right] K_2$$

$$\left[\zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} G \right] + j2\pi Z_L \left[2 \sin \beta_o (h_1 + h_2) \right. \right. \\ \left. \left. + \frac{1}{\Phi_r} U_{41}^r(o) + \frac{1}{\Phi_r^2} U_{42}^r(o) \right] \right]$$

(3.162)

3.4.3. Evaluations of K_1 and K_2

From (3.156) we note that

$$K_1 = \int_{-h_2}^{h_2} \left[\sin \beta_o (h_1 + h_2 - |y'|) + \frac{1}{\Phi_r} P_{41}^r(y') + \frac{1}{\Phi_r^2} P_{42}^r(y') \right] dy'$$

where

$$P_{41}^r(y) = B^r \cos \beta_o y - G \sin \beta_o |y| + \sin \beta_o (h_1 + h_2) t_{cd}(y)$$

$$P_{42}^r(y) = \beta_o B^r t_{cd}(y)$$

then

$$\begin{aligned} K_1 = & \frac{2}{\beta_o} \left[\cos \beta_o h_1 - \cos \beta_o (h_1 + h_2) \right] \\ & + \frac{1}{\Phi_r} \left\{ 2 B^r \sin \beta_o h_2 - G [1 - \cos \beta_o h_2] + \beta_o \sin \beta_o (h_1 + h_2) b_{cd} \right\} \\ & + \frac{1}{\Phi_r^2} \beta_o B^r b_{cd} \end{aligned} \quad (3.163)$$

where

$$b_{cd} = \int_{-h_2}^{h_2} t_{cd}(y') dy' = \int_{-h_2}^{h_2} \left[\int_0^{y'} T_{cd}(s) \cos \beta_o (y' - s) ds \right] dy' \quad (3.164)$$

Similarly, from (3.160), we have

$$K_2 = \int_{-h_2}^{h_2} \left[2 \sin \beta_o (h_1 + h_2 - |y'|) + \frac{1}{\Phi_r} U_{41}^r(y') + \frac{1}{\Phi_r^2} U_{42}^r(y') \right] dy'$$

where

$$\begin{aligned} U_{41}^r(y) = & 2B^r \cos \beta_o y - \sin \beta_o (h_1 + h_2) T_{ca}(y) - G \sin \beta_o |y| \\ & + \cos \beta_o (h_1 + h_2) T_{sa}(y) \end{aligned}$$

$$U_{42}^r(y) = -B^r T_{ca}(y) + G T_{sa}(y)$$

Then

$$\begin{aligned}
 K_2 = & \frac{4}{\beta_o} \left[\cos \beta_o h_1 - \cos \beta_o (h_1 + h_2) \right] \\
 & + \frac{1}{\Phi_r} \left[\frac{4B^r}{\beta_o} \sin \beta_o h_2 - \sin \beta_o (h_1 + h_2) t_{ca} - \frac{2G[1 - \cos \beta_o h_2]}{\beta_o} \right. \\
 & \left. + \cos \beta_o (h_1 + h_2) t_{sa} \right] + \frac{1}{\Phi_r^2} \left[G t_{sa} - B^r t_{ca} \right]
 \end{aligned} \tag{3.165}$$

where

$$t_{ca} = \int_{-h_2}^{h_2} T_{ca}(y') dy' \tag{3.166}$$

$$t_{sa} = \int_{-h_2}^{h_2} T_{sa}(y') dy' \tag{3.167}$$

3.5. Optimum Impedance for Zero Backscattering

The optimum impedance for zero backscattering can be obtained by letting total backscattered field equal to zero.

3.5.1. First Order Optimum Impedance for Zero Backscattering

Letting $\left[E_y \right]_{11}$ in (3.158) equal to zero and solving for Z_L ,

we have

$$\begin{aligned} \left[Z_L \right]_0 &= \frac{-jJ_1 \zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} G \right]}{2\pi \left[\begin{aligned} &K_1 \left[\cos \beta_o h_1 - \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} P_{41}^s(o) + \frac{1}{\Phi_s^2} P_{42}^s(o) \right] \\ &- J_1 \left[\sin \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} P_{41}^r(o) + \frac{1}{\Phi_r^2} P_{42}^r(o) \right] \end{aligned} \right]} \end{aligned} \quad (3.168)$$

3.5.2. Sub-first Order Optimum Impedance for Zero Backscattering

Letting $\left[E_y \right]_{10}$ in (3.161) equal to zero and solving for Z_L , we

get

$$\begin{aligned} \left[Z_L \right]_0 &= \frac{-jJ_2 \zeta_o \Phi_r \left[\cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} G \right]}{2\pi \left[\begin{aligned} &K_2 \left[2 \cos \beta_o h_1 - 2 \cos \beta_o (h_1 + h_2) + \frac{1}{\Phi_s} U_{41}^s(o) + \frac{1}{\Phi_s^2} U_{42}^s(o) \right] \\ &- J_2 \left[2 \sin \beta_o (h_1 + h_2) + \frac{1}{\Phi_r} U_{41}^r(o) + \frac{1}{\Phi_r^2} U_{42}^r(o) \right] \end{aligned} \right]} \end{aligned} \quad (3.169)$$

3.6. Numerical Examples

The numerical results are obtained for a square loop with

$\frac{a}{h} = 0.0388$. Φ_r is numerically calculated as functions of $\beta_o h$

and is plotted in Figure 3.4. The optimum loading for zero back-

scattering $\left[Z_L \right]_0$ in (3.168) is then calculated as a function of

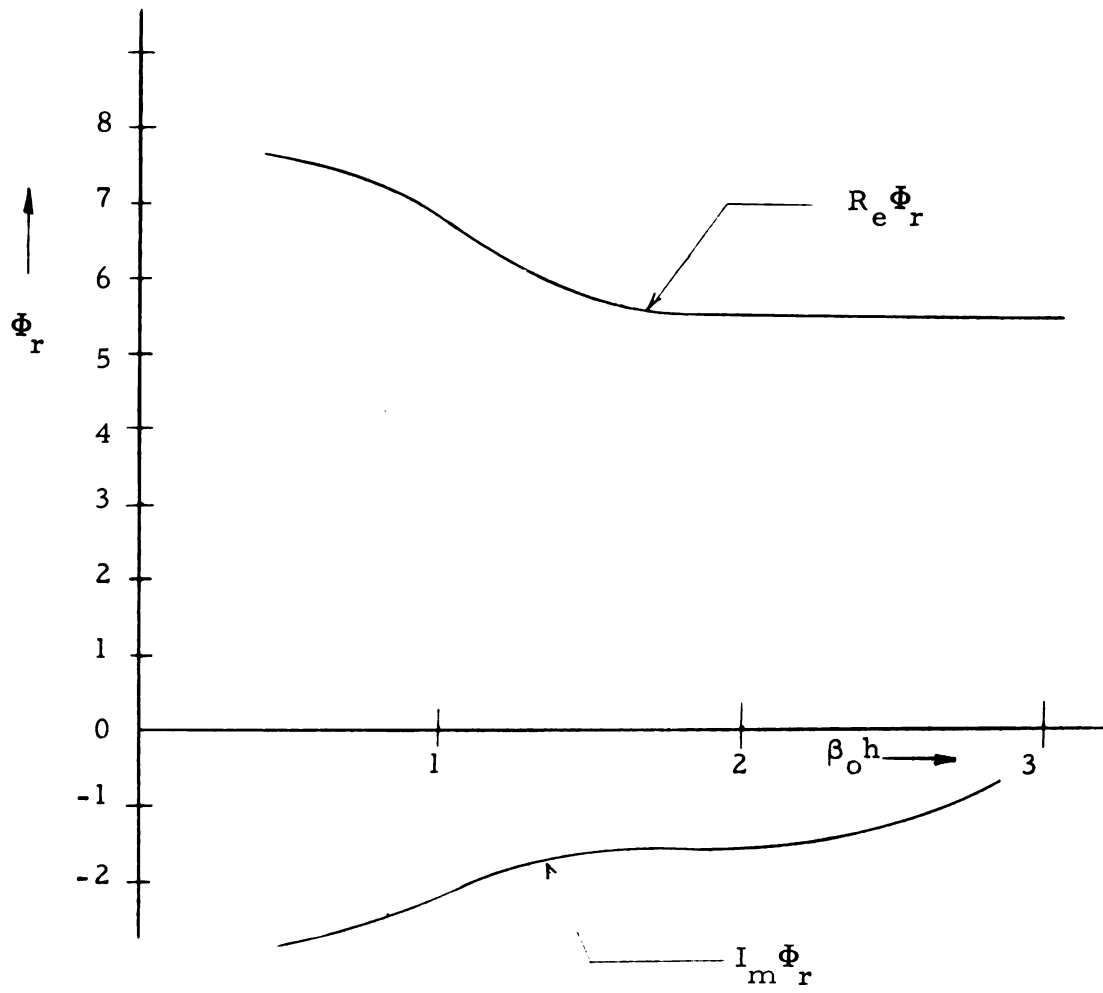


Figure 3.4. Φ_r as a function of $\beta_o h$ ($a/h = 0.0388$).

$\beta_0 h$ for the case of $\frac{a}{h} = 0.0388$. Numerical results of $\left[Z_L \right]_0$ are

shown in Figure 3.5.

3.7. Experiment

The optimum impedance for zero backscattering from a rectangular loop requires both the resistive part and the reactive part shown in Figure 3.5. To simplify the problem, an experiment was conducted for a conducting square loop loaded with two identical reactive impedances. Experimentally, this reactive loading method proved to be quite adequate in reducing the backscattering cross section of a square loop.

3.7.1. Experimental Arrangement and Measuring Technique

The experimental setup and measuring technique were identical to the case of the circular loop mentioned in Chapter 2 and no further description seems to be necessary.

3.7.2. Experimental Results

A square loop (side length = 5.15 cm) were constructed as an experimental model using cylindrical wire of 0.1 cm radius. The experiments were conducted at various frequencies. The experimental results are shown in Figure 3.6 to Figure 3.16 in which the backscattering cross sections of the square loop were plotted as functions of loading impedances at each particular frequency. The solid curve represents the backscattering cross section of a loaded square loop

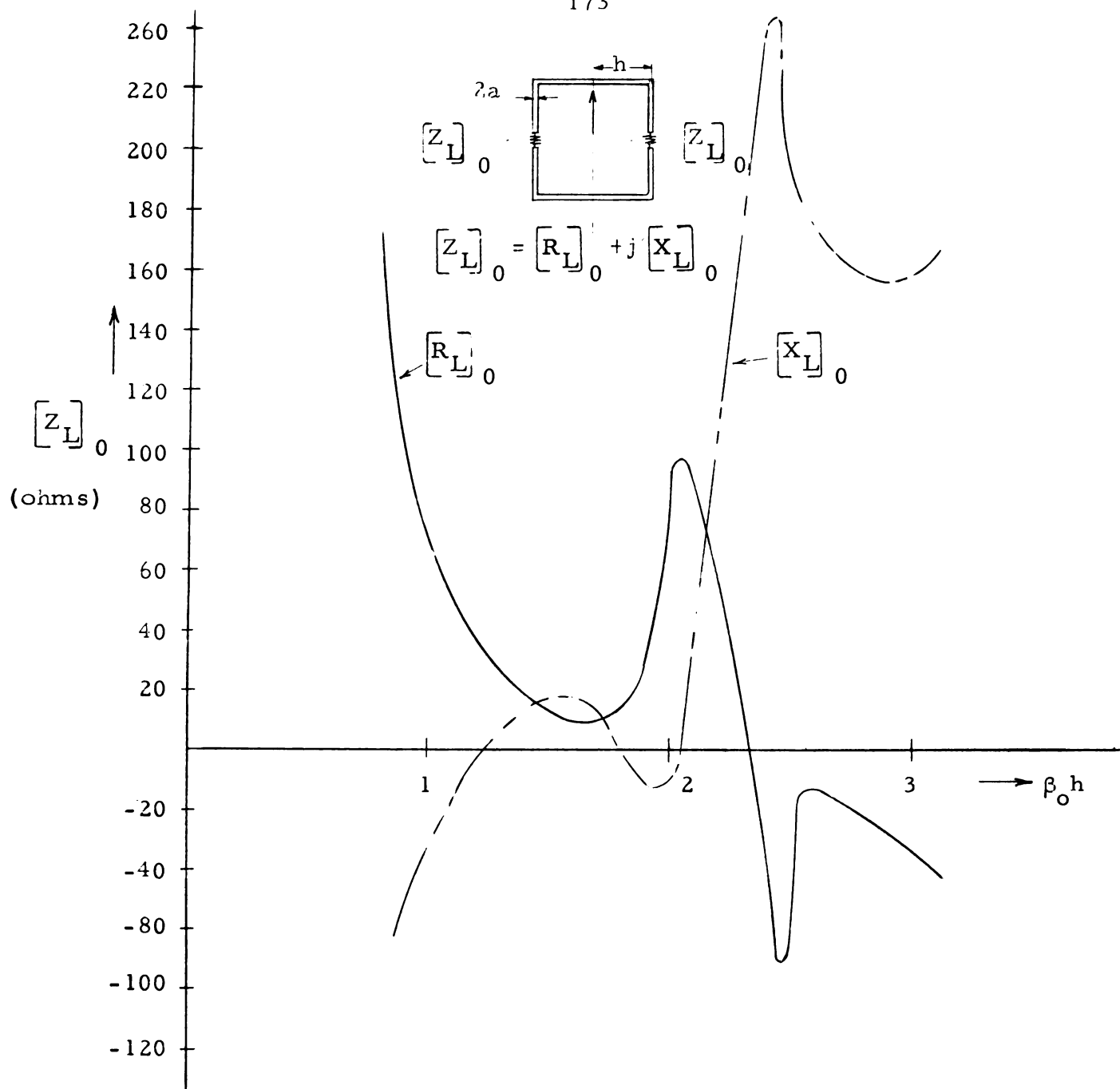
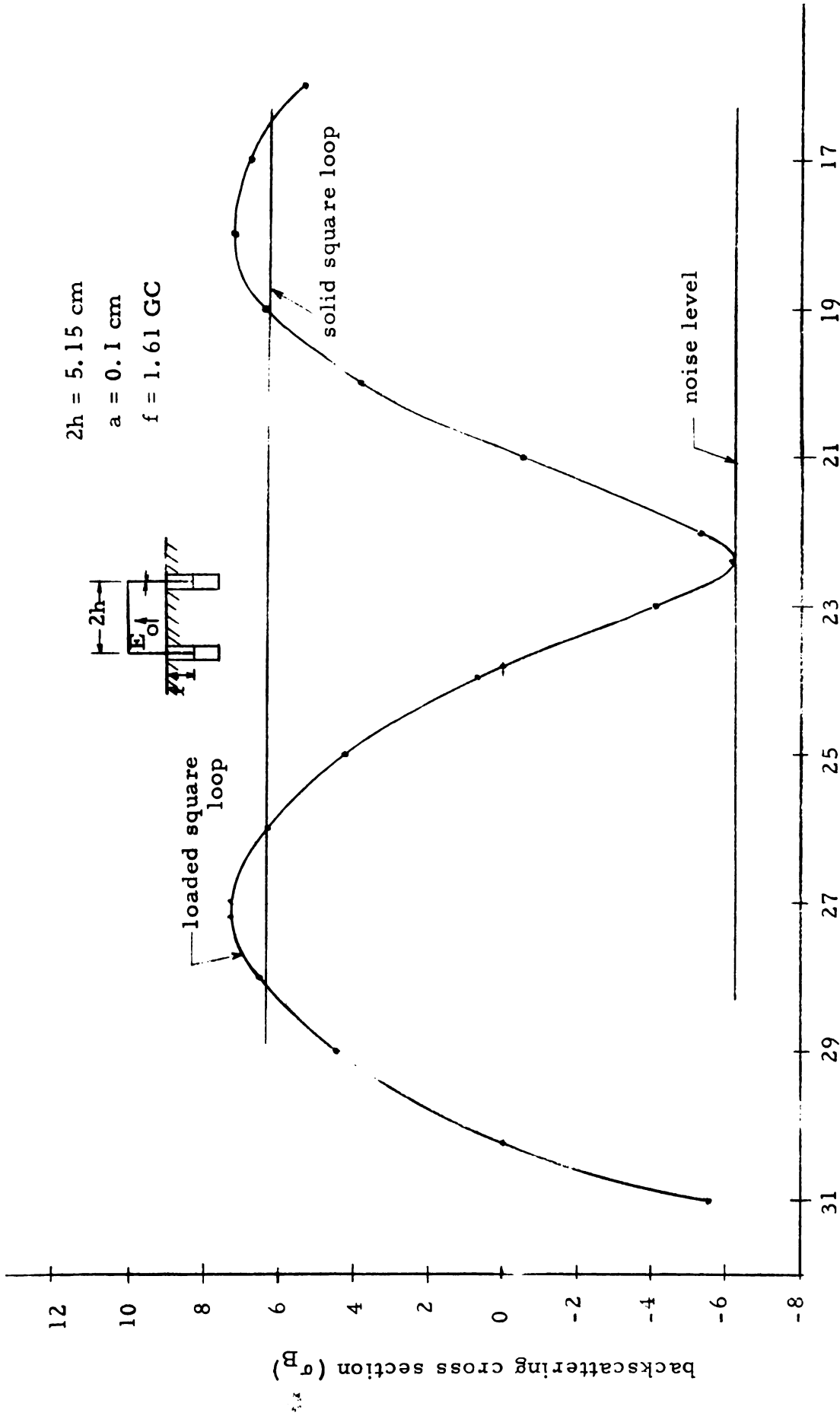


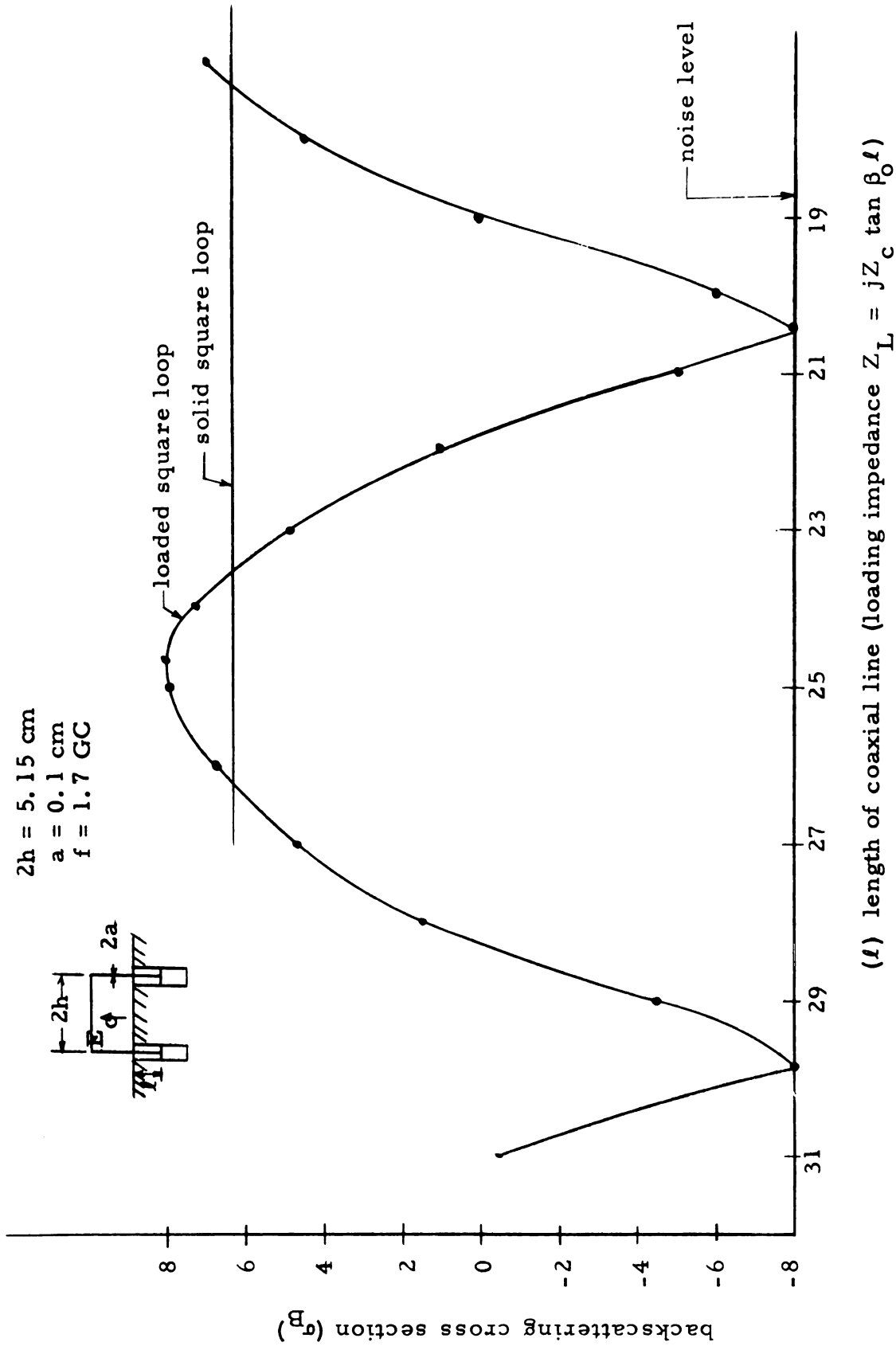
Figure 3.5. Optimum impedance for zero backscattering from a square loop ($a/h = 0.0388$).

and the solid straight line represents that of the solid square loop. It is observed that if the loading impedance (or the length of the coaxial line) is properly adjusted, the backscatters of the square loops can be minimized to the noise level. About 15 db reduction in the backscattering cross section was obtained in the experiments. In Figure 3.17, the optimum reactive impedances for minimum backscattering are plotted as a function of $\beta_0 h$.



(ℓ) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_0 \ell$)

Figure 3.6. Backscattering cross section of a loaded square loop as a function of loading impedance ($f = 1.61$ GC).



(1) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_0 l$)

Figure 3.7. Backscattering cross section of a loaded square loop as a function of loading impedance ($f = 1.7$ GC.)

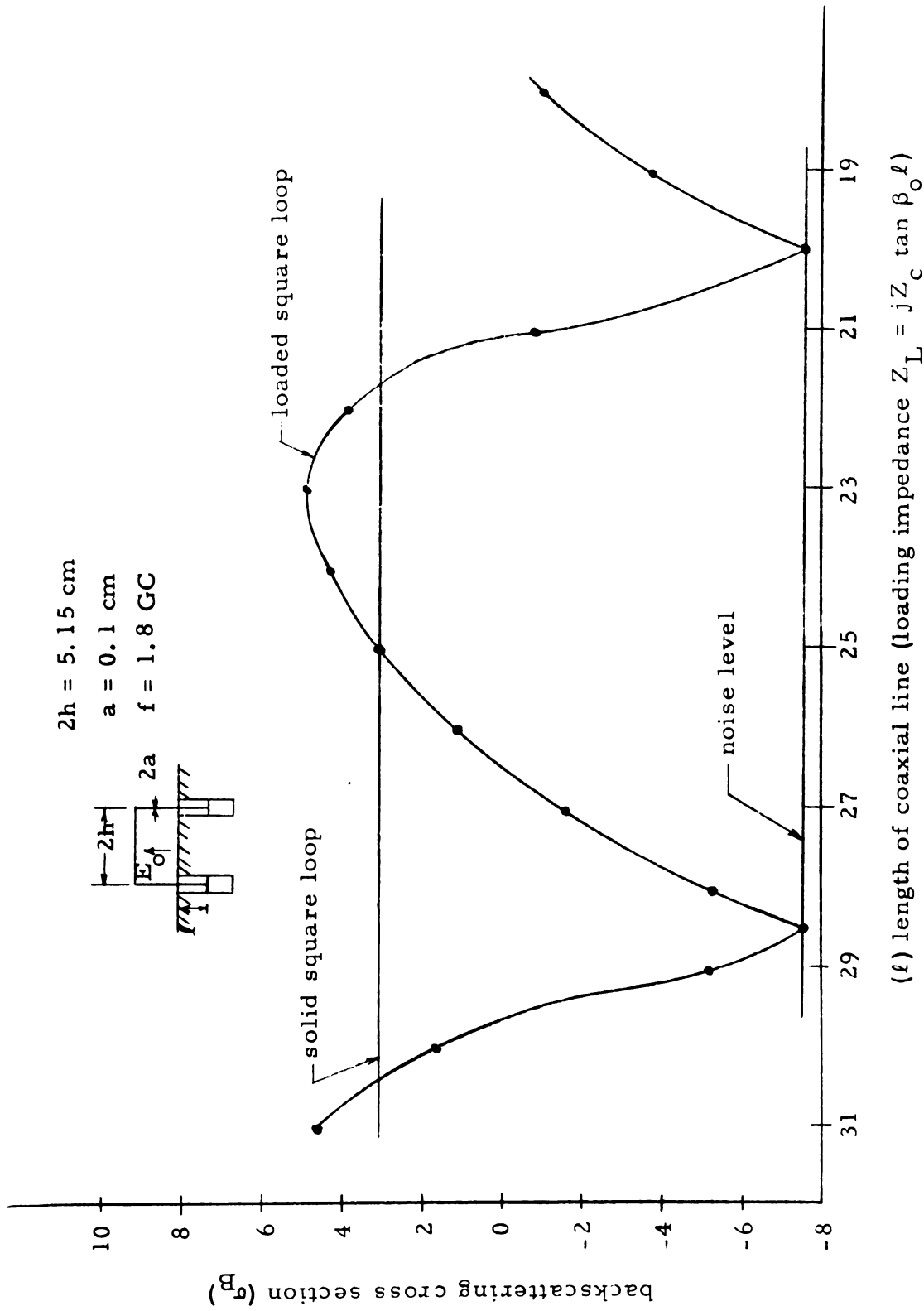
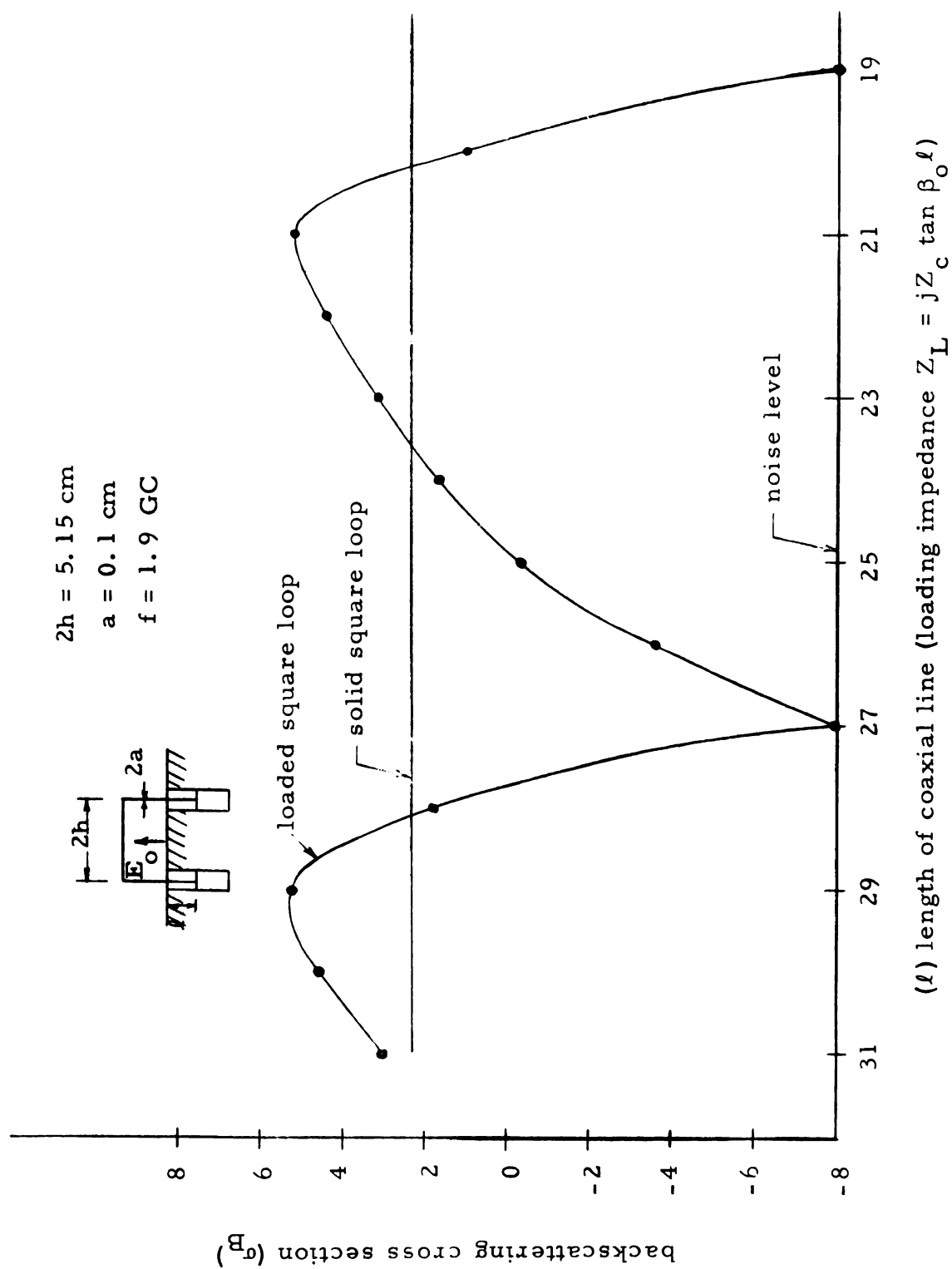
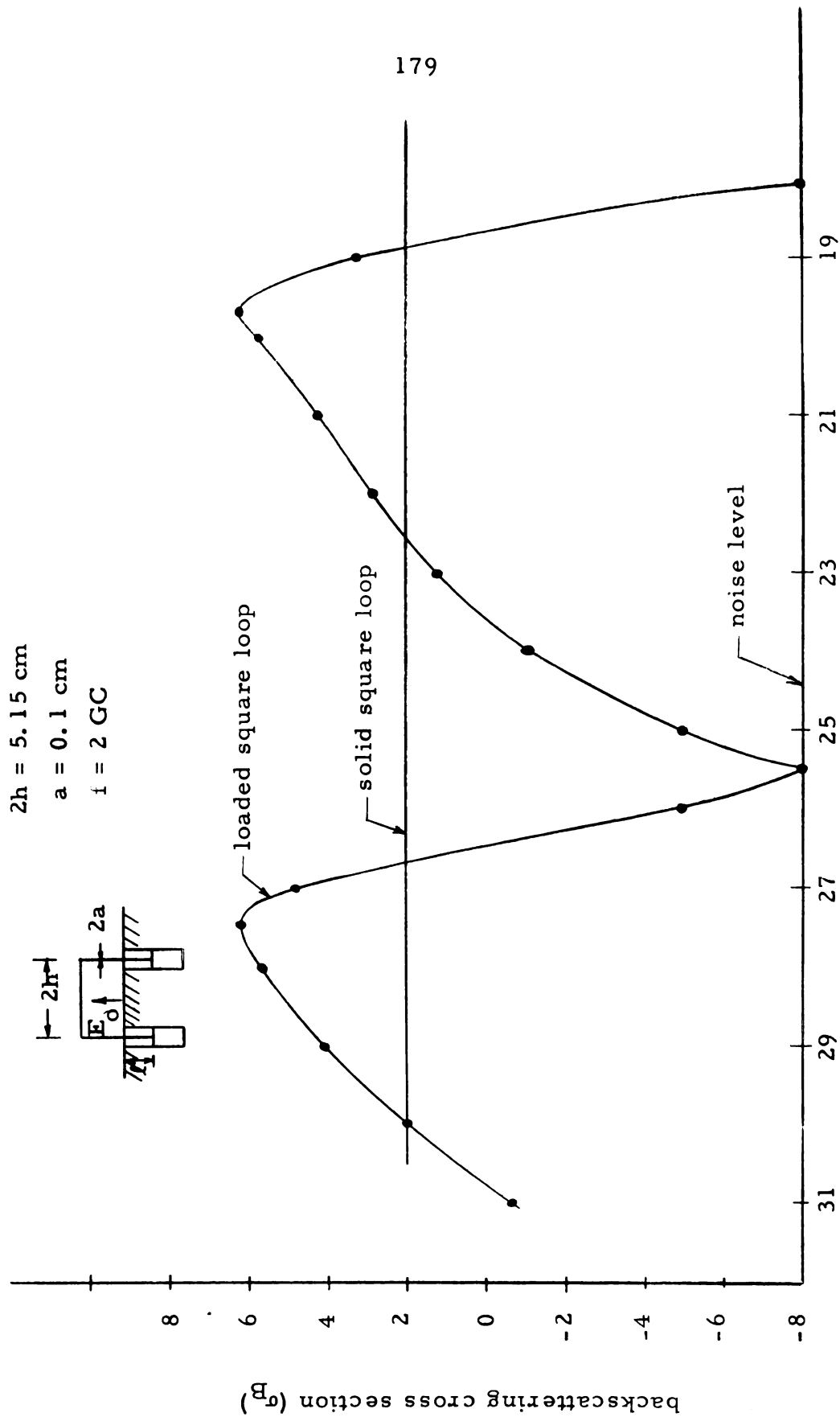


Figure 3.8. Backscattering cross section of a loaded square loop as a function of loading impedance ($f = 1.8$ GC).



(ℓ) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_o \ell$)

Figure 3.9. Backscattering cross section of a loaded square loop as a function of loading impedance ($f = 1.9$ GC).



(l) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_o l$)

Figure 3.10. Backscattering cross section of a loaded square loop as a function of loadin impedance ($f = 2 \text{ GC}$).

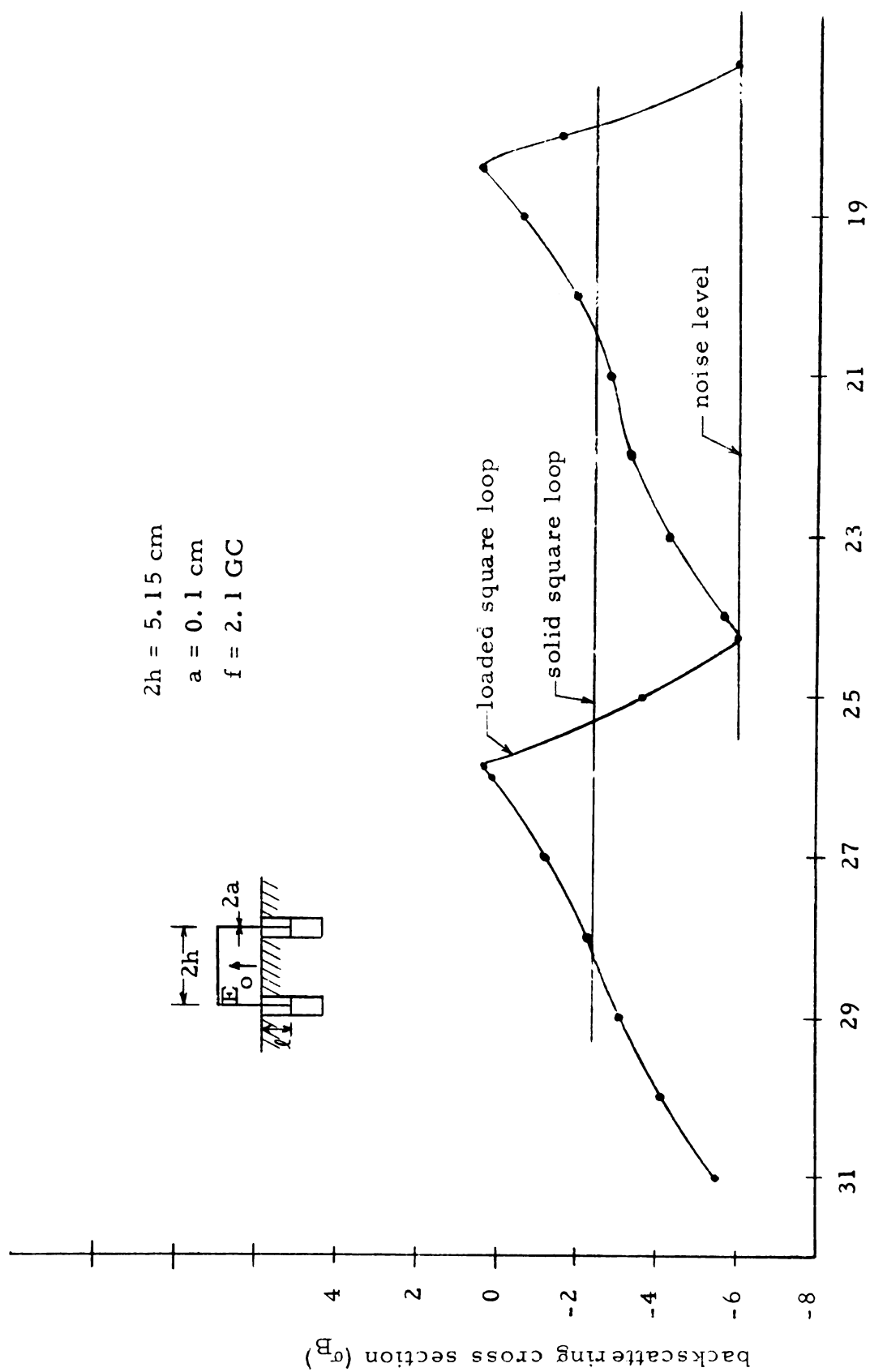
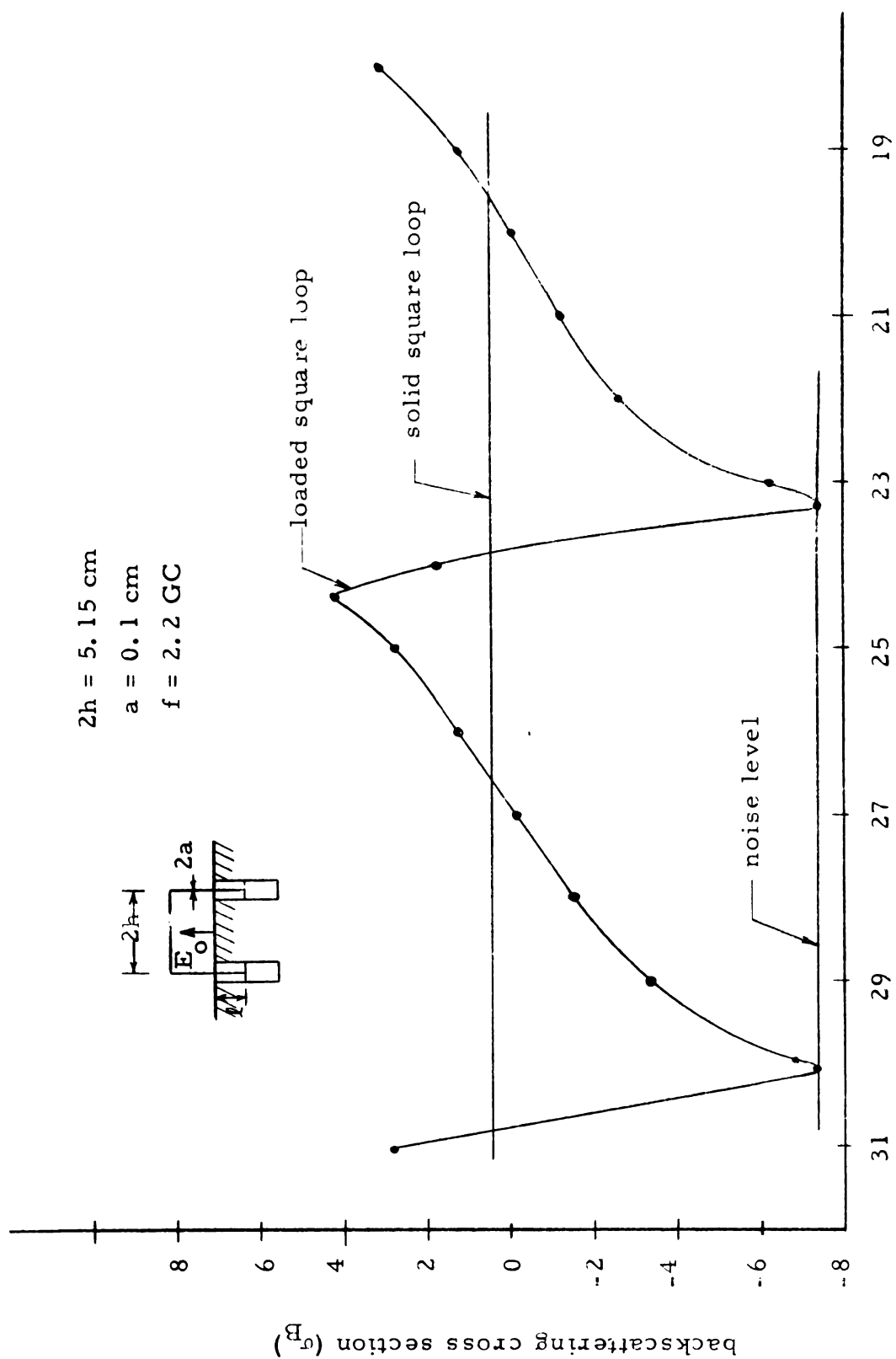


Figure 3.11. Backscattering cross section of a loaded square loop as a function of loading impedance $Z_L = jZ_c \tan \beta_o \ell$ (ℓ length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_o \ell$)).



(ℓ) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_0 \ell$)
 Figure 3.12. Backscattering cross section of a loaded square loop as a function of loading impedance ($f = 2.2$ GC).

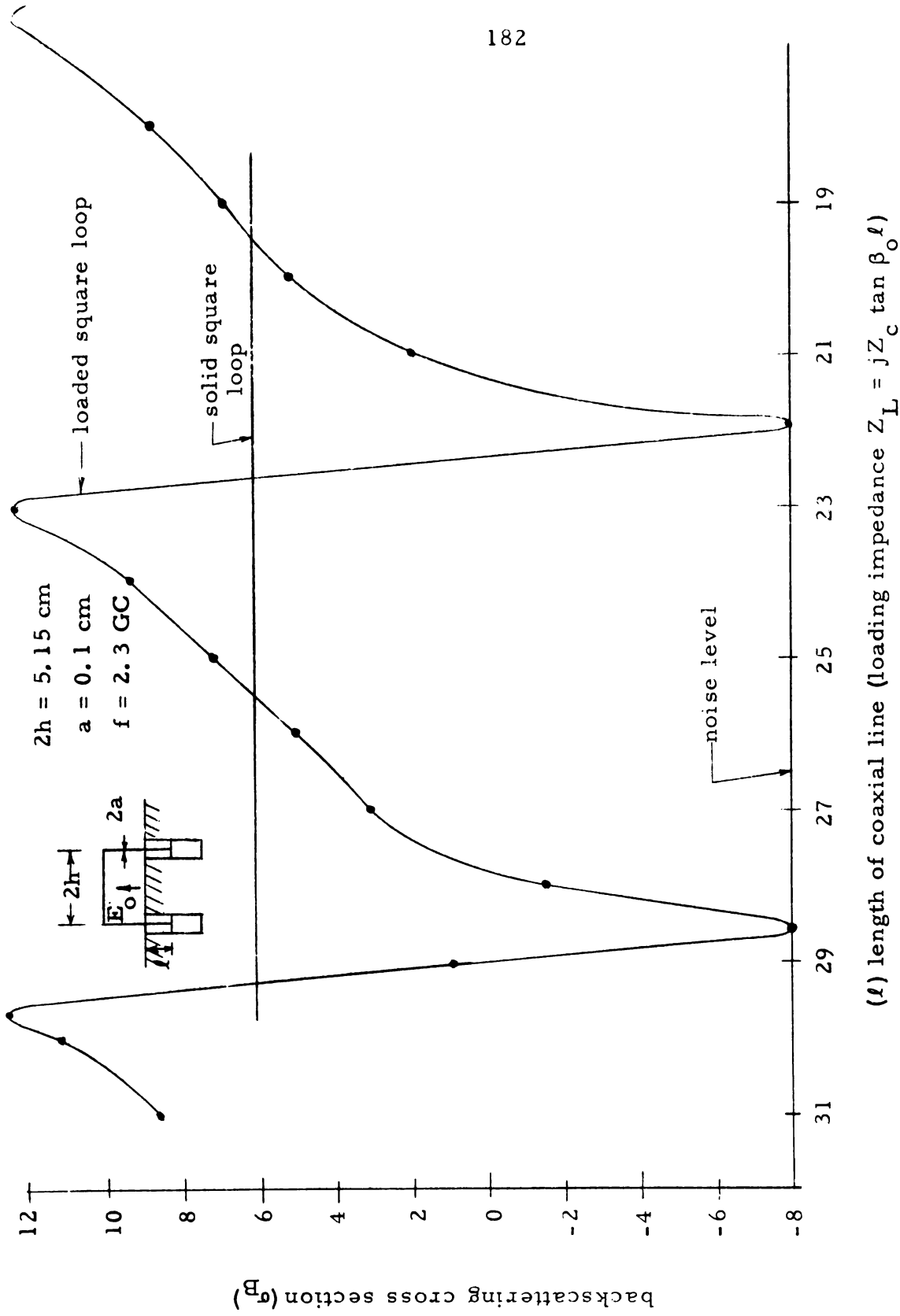


Figure 3.13. Backscattering cross section of a loaded square loop as a function of loading impedance ($Z_L = jZ_C \tan \beta_0 \ell$).

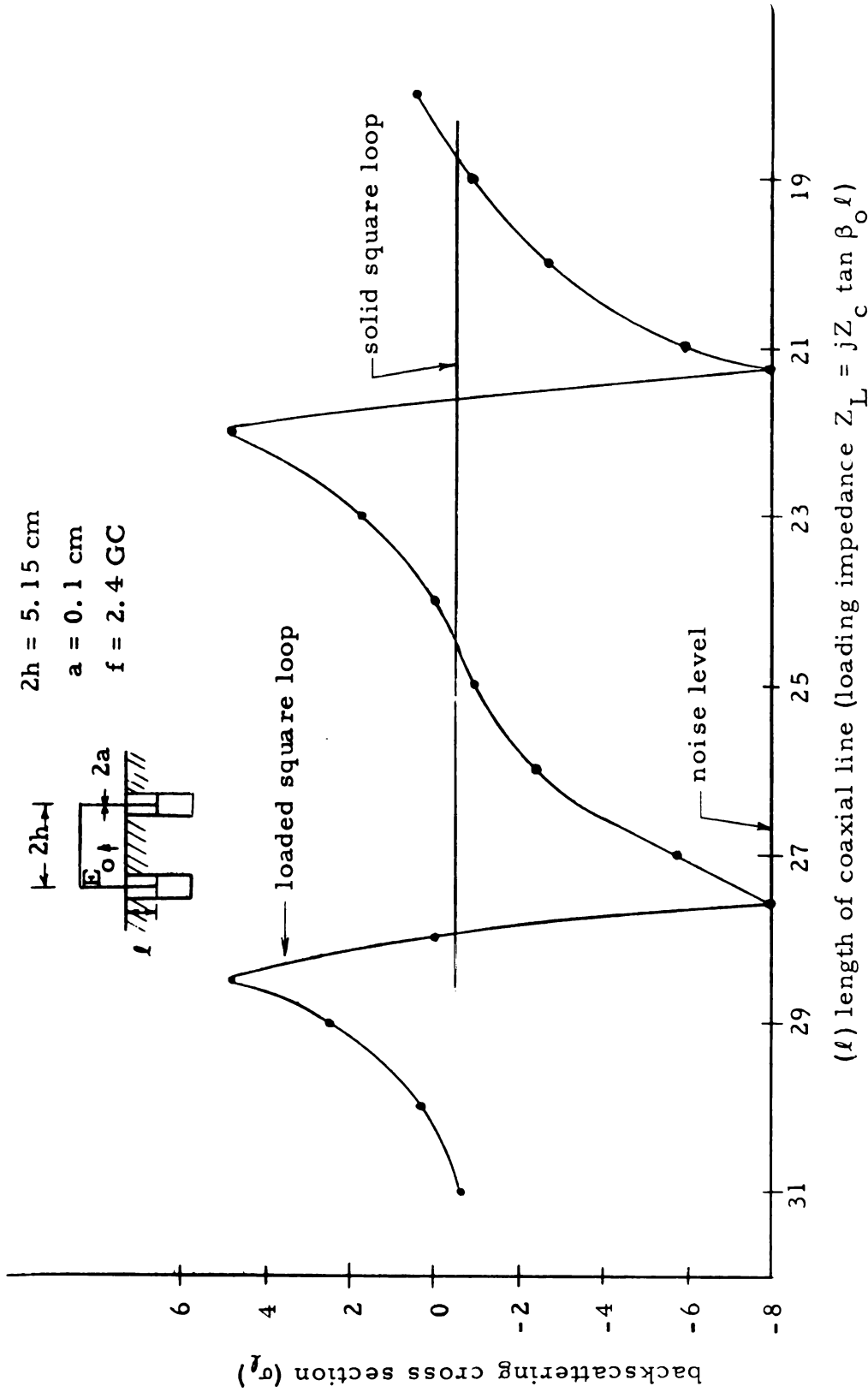
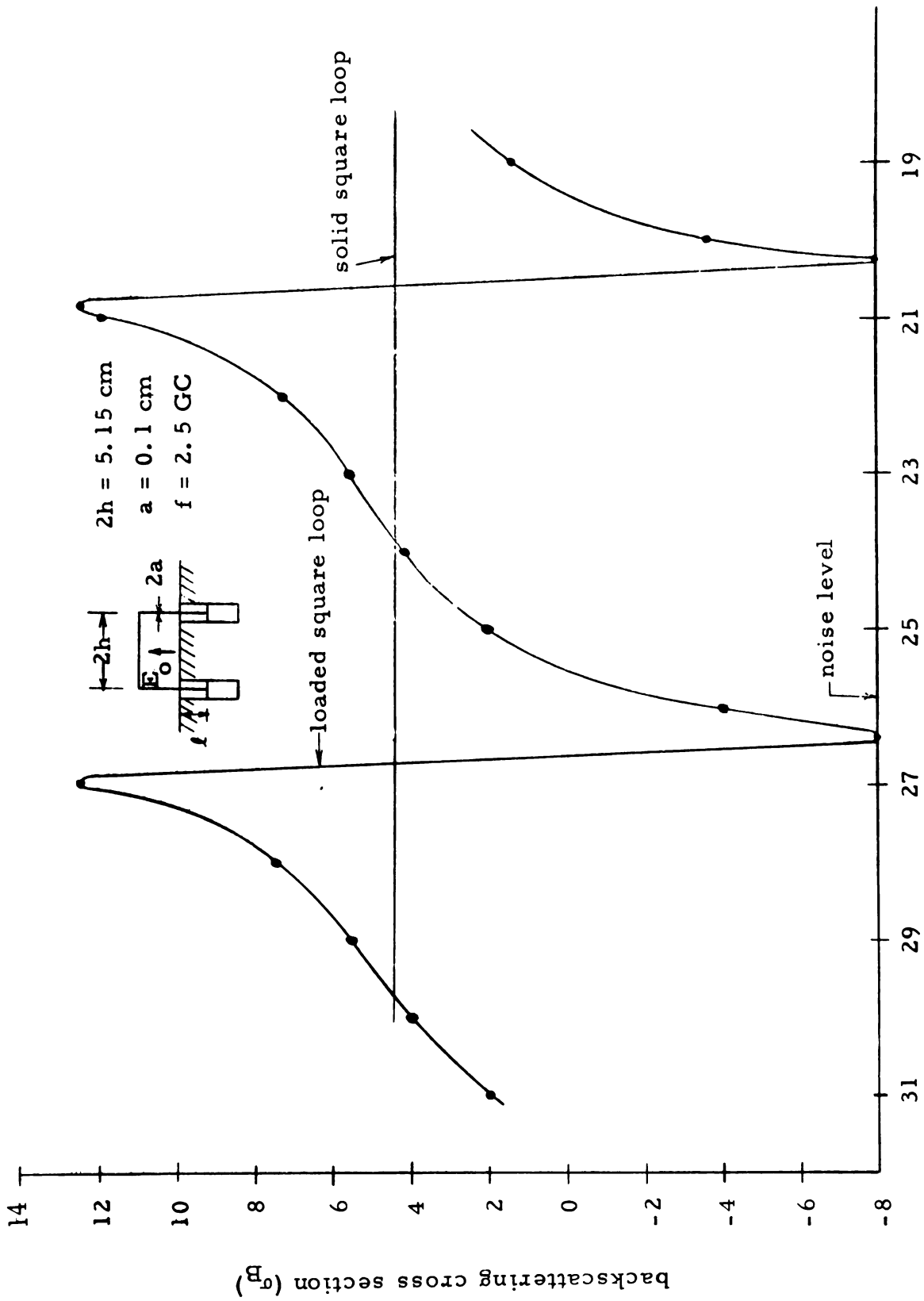
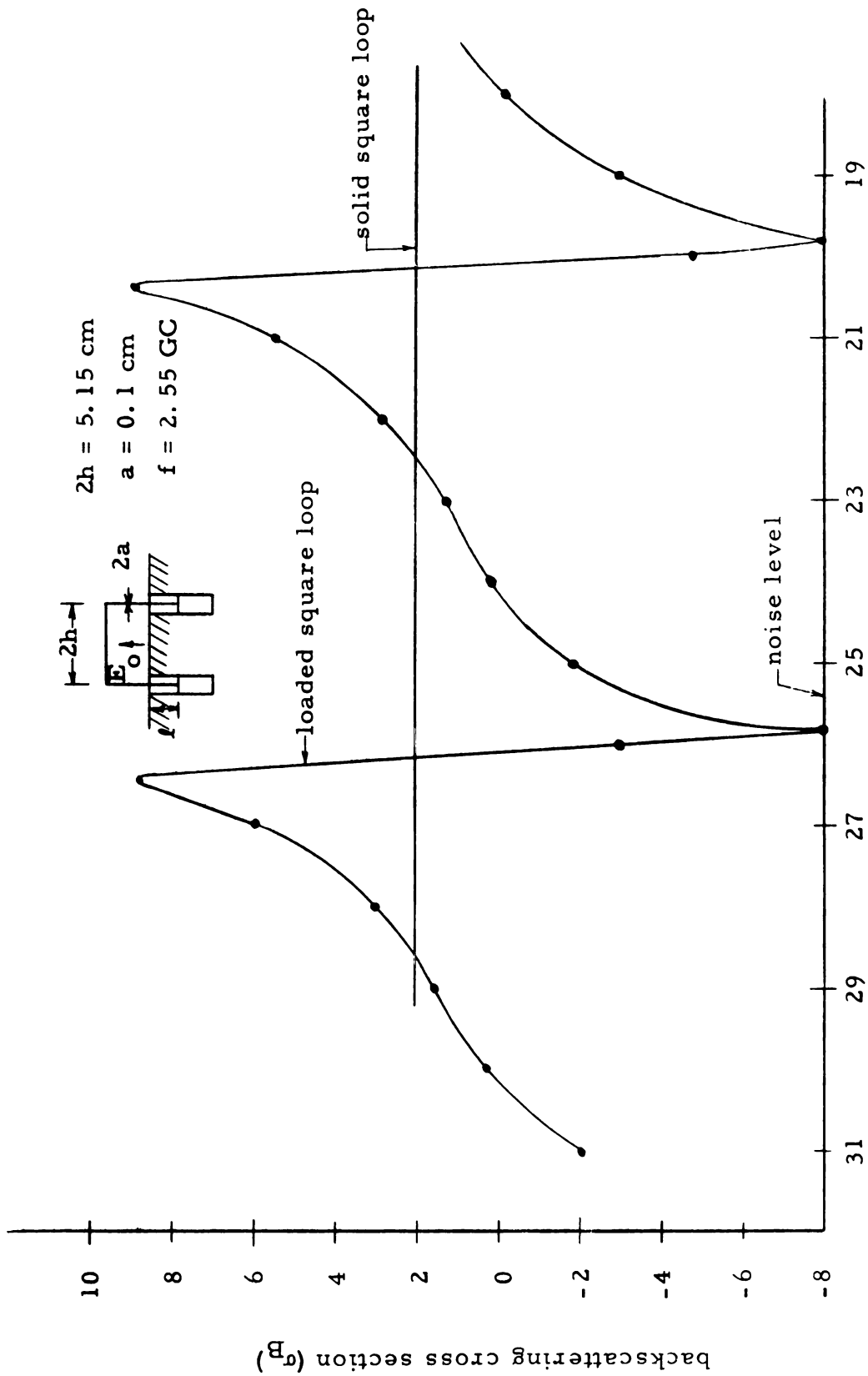


Figure 3.14. Backscattering cross section of a loaded square loop as a function of loading impedance ($f = 2.4 \text{ GC}$).



(1) length of coaxial line (loading impedance $Z_L = jZ_C \tan \beta_o l$)

Figure 3.15. Backscattering cross section of a loaded square loop as a function of loading impedance ($f = 2.5 \text{ GC}$).



(l) length of coaxial line (loading impedance $Z_L = jZ_c \tan \beta_o l$)

Figure 3.16. Backscattering cross section of a loaded square loop as a function of loading impedance ($f = 2.55$ GC).

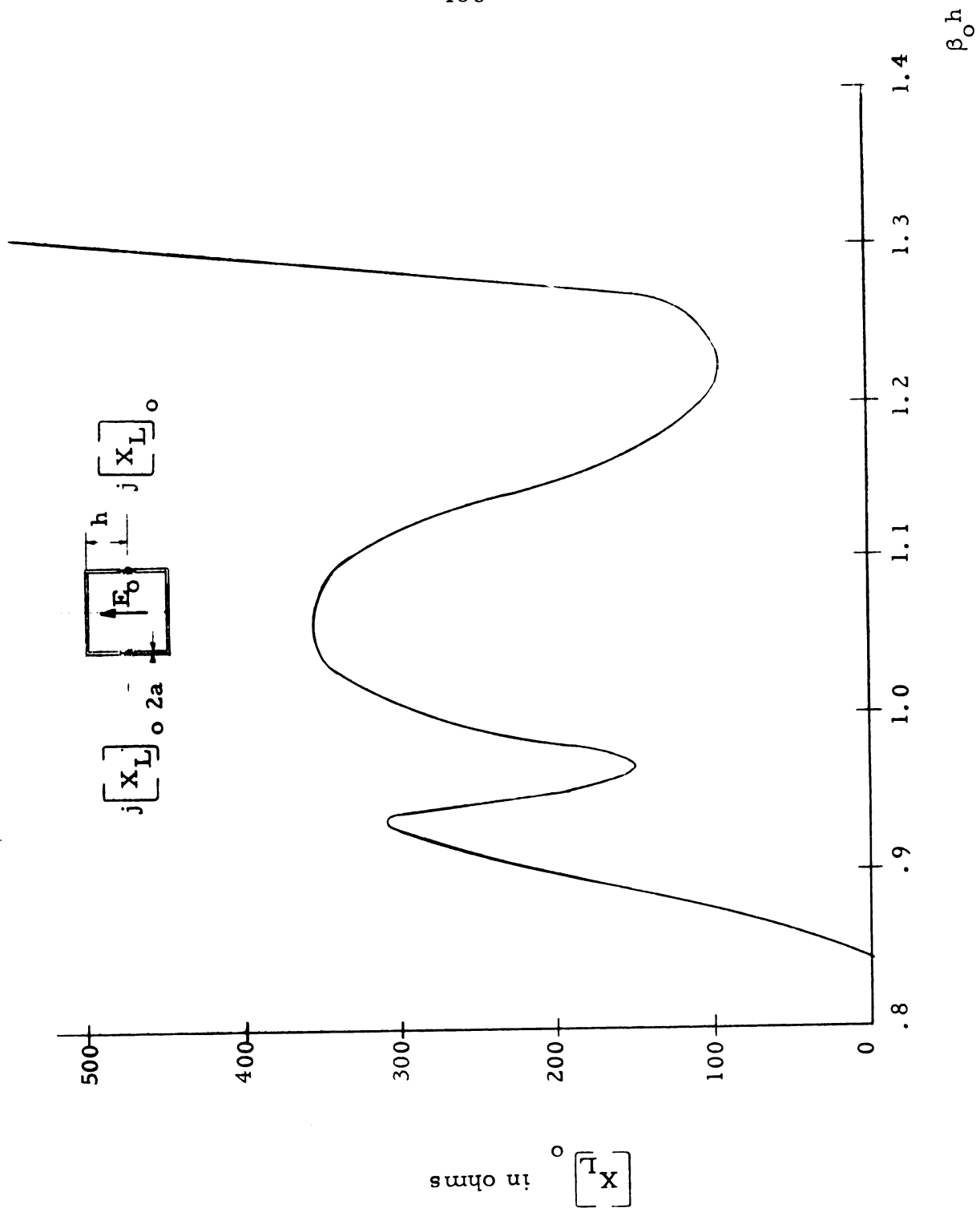


Figure 3.17. Optimum reactive impedance for minimum backscattering from a square loop.

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