SAMPLED-DATA CONTROL OF SYSTEMS WITH SLOW AND FAST MODES

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ABSTRACT

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The class of linear time-invariant singularly perturbed discrete-time systems is considered. Different sources and typical representations of this class of systems is surveyed. The infinite-time optimal regulator problem and the asymptotic behavior of the resulting algebraic Riccati equations, as the perturbation parameter tends to zero, are studied. It is shown that, analogous to the continuous-time case, a near-optiomal solution can be obtained by applying slow-fast decompositions. An iterative technique for solving the full algebraic Riccati equation which uses the solution of slow and fast modes is introduced. This technique has a high degree of convergence and alleviates the curse of dimensionality by eliminating the stiffness and reducing the order of the system.

Furthermore, feedback stabilization and control of this class of systems is considered. The two-time-scale nature of the system is exploited to decompose the design problem into two lower-order design problems. Moreover, we address the important issue of "multirate measurements" or "multirate sampling." Composite control strategies are developed for the case of single-rate measurements as well as for the case of multirate measurements. Stability results and closeness of trajectories are shown under the application of these composite controls.

Our findings are applied to the deterministic model of the longitudinal motions of an F-8 aircraft and simulation results supporting the theory are presented.

This Dissertation Is Dedicated
to
My Parents
Whom Through Their Love,
The Sun Always Shines For
Me.

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CHAPTER 1

INTRODUCTION

Simplification of mathematical models for many physical and engineering problems is a common practice of control engineers. In analysis and design of large scale control systems the need for such simplifications emerge quite naturally.

Methods of reduced-order modeling and control have received a great deal of attention in recent years. Of these methods, aggregation [Aoki, 1978], and singular perturbation [Kokotovic et al., 1976], seem to be the most well-known. A typical simplification is to neglect some small "parasitics" as time constants, moment of inertia, masses, capacitances and inductances. Neglecting these small parasitics alleviates the "curse of dimensionality" by lowering the model order and exclusion of the fast states which result in "stiff" models. Approximated models using exclusion of fast states, as in aggregation, may result in an unstable system or a system which is far from its desired optimum. Singular perturbation technique improves this approximation by reintroducing the fast states as a "boundary layer" correction calculated in a separate time scale. An important characteristic of singularly perturbed models is that the structure of the system remains the same for time-varying and nonlinear systems. This is established by a fundamental theorem by Tihonov [1952].

The singular perturbation approach is not only helpful in design procedures but is a powerful tool for analytical investigations of the properties of the system as behavior of optimal controls near singular arcs, stabilizability, systems robustness, etc.

The singular perturbation method has attained a certain maturity in continuous-time control systems [Kokotovic et al. 1976]. The multiple-time-scale property of these systems has been used in deriving the reduced-order models which have been employed in approximation of some desired objectives of the original high-order "stiff" models. More specifically, the analysis and control design of linear time-invariant continuous-time singularly perturbed systems has been well documented [Chow and Kokotovic, 1976, a.b.].

In spite of increasing flow of research directed in the area of singular perturbation theory, many questions are still open as was discussed in [Kokotovic et al. 1976].

One area where on-going research is still in its earlier stages is singularly perturbed difference equations. Hoppensteadt and Miranker [1977] developed a multitime method for difference equations. Phillips [1980] considered the singularly perturbed discrete systems in state variable form and reduced-order models were obtained without considering the initial value lost in the process of order reduction. Blankenship [1980] developed a method of matched asymptotic expansion for a class of singularly perturbed difference equations arising in optimal control problems. Also, different applications of singular perturbation ideas to discrete systems have been investigated by Mahmoud [1982], Naidu and Rao [1981, a,b], Rajagopalan and Naidu [1980], and Sycros and Sannuti [1983].

The objective of this dissertation is to investigate some open problems for the class of linear time-invariant singularly perturbed difference equations and employ the structural properties of singularly perturbed systems to acheive the approximate control design for such systems. The organization of the dissertation is as follows:

In Chapter 2, the continuous-time singularly perturbed systems are, briefly, introduced and a decoupling transformation to separate the slow and fast modes which is applicable to both continuous and discrete systems is studied.

A historical review of singularly perturbed difference equations is performed in Section 2.3 which discusses different model representations and some structural properties of this class of systems. In Section 2.4 different sources of singularly perturbed difference equations are investigated. In the last section of this chapter, Section 2.5, we introduce a useful stability criterion for the class of linear discrete-time systems. Also, an initial value problem, in which solutions of slow and fast problems are used to approximate the solution of the full problem, is investigated.

Chapter 3 deals with the problem of Infinite-Time Optimal Regulators for singularly perturbed difference equations. First, a related background is provided to familiarize the reader with the problem and our motivation. Asymptotic behavior of the optimal solution of linear quadratic regulators is investigated. Conditions for independent design of slow and fast subsystems is studied. A composite feedback control law, which employs the slow and fast controls, is formed and applied to the original system which results in a near-optimal solution.

Also, an iterative technique for solving the discrete-time stiff Riccati equation is presented. This technique, by using the slow and fast subsystems, overcomes the ill-conditioning and provides a fast convergence. An illustrative example which supports the theory is given at the end of this chapter.

Chapter 4 discusses the stabilizability of singularly perturbed difference equations, in view of multirate measurements of the state variables, using a composite feedback control law.

Different design procedures for forming a stabilizing composite feedback control are investigated in this chapter, and it is shown that the application of such control laws results in asymptotic stability of the closed-loop systems and closeness of the trajectories to those predicted by slow and fast subsystems. Two different time-scales, slow and fast, are introduced. The fast-time-scale has a period 1, while the slow-time-scale has a period $N = [\frac{1}{\epsilon}]$ (N is an integer such that $\frac{1}{\epsilon} - 1 < N \leq \frac{1}{\epsilon}$).

The composite feedback control law is formed by using the stabilizing feedback controls of slow and fast subsystems when

i) Both evolve in the fast-time-scale n and their measurements are available for all n (single rate), $n = 0,1,2, \ldots N, \ldots$.

ii) The measurements of slow states are available only at slow-time intervals, but the measurements of fast states are available for all n. In this case we have a multirate measurements and slow and fast controls are designed independently, "Parallel Design". Also, the values for slow states for $n \neq \frac{K}{\epsilon}$, $K = 0,1,2,\ldots$ are predicted using their values at the beginning of the slow periods.

iii) There is a multirate measurement scheme but a pre-conditioning feedback gain which stabilizes the fast states is designed first and based on this gain the slow subsystem is designed, "Sequential Design".

Finally, a numerical example for parallel design illustrates our claims.

Chapter 5 is devoted to the numerical solutions of a more realistic physical model. We have considered the deterministic model of an F-8 aircraft with four state variables, two of which are slow states (incremental velocity, pitch angle) and the other two are fast states (angle of attack, pitch rate). Our claims about the near-optimality of infinite-optimal regulator, iterative technique and multirate stabilization is confirmed using this model.

Chapter 6 is the conclusion which precedes the list of the programs used in solving our numerical examples and application.

CHAPTER 2

STRUCTURAL PROPERTIES AND MODELING OF TWO-TIME-SCALE DISCRETE-TIME SYSTEMS

2.1 Introduction

The main objective of this chapter is to familiarize the reader with two-time-scale linear time-invariant discrete-time systems and their structural properties. Also different sources of such systems are discussed.

In Section 2.2 continuous-time singularly perturbed systems are, briefly, discussed and an important decoupling transformation for separating the slow and fast modes of such systems is introduced. Conditions for existence of this decoupling transformation, which could be applied to both continuous and discrete systems, are given.

Section 2.3 introduces the two-time-scale time-invariant discrete systems. A historical review, which includes different model representations of this class of systems, describes some structural properties as pole clustering, and contains different methods for accomplishing slow-fast decomposition. This, hopefully, provides the reader with a better understanding of the subject.

In Section 2.4 different sources of singularly perturbed difference equations are investigated.

Finally, in Section 2.5, a useful stability criterion for discrete-time systems is introduced. An initial value problem is discussed which reveals an $O(\epsilon)$ approximation between solution of the slow system represented by differential equations and the one obtained from difference equations. Also approximation of the full states using slow and fast states is investigated.

2.2. Continuous-Time Singularly Perturbed Systems and Decoupling Transformation.

Control systems possessing slow and fast phenomena are frequent in applications. A linear continuous-time invariant model of such systems is

$$\dot{x}(t) = A_{11}x(t) + A_{12}z(t) + B_1u(t),\ x(0) = x_0 \qquad (2.1a)$$

$$\epsilon\ \dot{z}(t) = A_{21}x(t) + A_{22}z(t) + B_2u(t),\ z(0) = z_0 \qquad (2.1b)$$

where the state vector comprises the m_1- and m_2- dimensional vectors x and z, the control u is an r-dimentional vector and ϵ is a small positive parameter representing small time constants. All matrices have compatible dimensions. Slow and fast modes correspond to small and large eigenvalues, respectively.

For $\epsilon = 0$ in (2.1) the order $(m_1 + m_2)$ of the system reduces to m_1, that is (2.1) reduces to

$$\dot{\bar{x}} = A_{11}\bar{x} + A_{12}\bar{z} + B_1\bar{u} \qquad (2.2a)$$

$$0 = A_{21}\bar{x} + A_{22}\bar{z} + B_2\bar{u}, \qquad (2.2b)$$

where bar indicates that ϵ is set to zero. If A_{22} is invertible, then

$$\bar{z} = -A_{22}^{-1} A_{21}\bar{x} - A_{22}^{-1}B_2\bar{u}, \qquad (2.3)$$

yielding the reduced model

$$\dot{\bar{x}} = A_0\bar{x} + B_0\bar{u} \,, \tag{2.4}$$

where

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$$

$$B_0 = B_1 - A_{12}A_{22}^{-1}B_2 \,.$$

The use of $\epsilon = 0$ is formal since then

$$\dot{z} = \frac{A_{21}}{\epsilon}x + \frac{A_{22}}{\epsilon} z + \frac{B_2}{\epsilon} u$$

could be unbounded. That is why systems presented in the form (2.1) are called "singularly Perturbed Systems."

Unless x_0 and z_0 are such that $\bar{z}(0) = z_0$, the boundary condition $z(0) = z_0$ will not be met by the approximation (singular perturbation) (2.2).

If quantities on the right hand side of (2.1) are of the same magnitude, $\dot{z}$ will be of the magnitude $\frac{1}{\epsilon}\dot{x}$. For this reason z is considered a "fast" state and (2.4) which neglects the fast dynamics is considered as "slow" system.

System (2.1) is said to possess a two-time scale property if it has m_1 small eigenvalues of magnitude $O(1)$ and m_2 large eigenvalues of magnitude $O(\frac{1}{\epsilon})$. Singular perturbation exploits this property of the system to approximate it with two lower order, slow and fast, subsystems. The approximate slow subsystem is justified by considering that in an asymptotically stable system the fast modes corresponding to large eigenvalues are important only during a short initial period and after this period the behavior of the system could be represented by its slow modes. Neglecting these fast modes is equivalent to assuming that they are infinitely fast, that is pushing $\epsilon \to 0$ in (2.1). [Chow and Kokotovic, 1976]. A fast subsystem is derived by assuming that the slow variables are constant during the fast transients. Now subtracting (2.2 b) from (2.1 b) yields

$$\epsilon \dot{x}_f = A_{22}x_f + B_2u_f, \; x_f(0) = z_0 - \bar{z}(0), \tag{2.5}$$

where

$$x_f = z_2 - \bar{z}_2 \quad \text{and} \quad u_f = u - \bar{u}.$$

To separate the slow and fast modes of the singularly perturbed system (2.1) a state transformation due to Chang [1972] is used which completely decomposes the system (2.1) into slow and fast modes by transforming it into a block diagonalized system as in (2.6)

$$\begin{bmatrix} \dot{\eta}_1 \\ \epsilon \dot{\eta}_2 \end{bmatrix} = \begin{bmatrix} A_0 + O(\epsilon) & 0 \\ 0 & A_{22} + O(\epsilon) \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} B_0 + O(\epsilon) \\ B_2 + O(\epsilon) \end{bmatrix} u. \tag{2.6}$$

From (2.6) it is obvious that as $\epsilon \to 0$, the first m_1 eigenvalues of the original system (2.1) tend to the eigenvalues of the reduced system (2.4), while the remaining m_2 eigenvalues tend to infinity as the eigenvalues of $\frac{A_{22}}{\epsilon}$.

Let these eigenvalues be divided into two distinct sets which are arranged in increasing order

$$N_s = \{\lambda_{s_1}, \ldots, \lambda_{s_{m_1}}\}$$

$$N_f = \{\lambda_{f_1}, \ldots, \lambda_{f_{m_2}}\},$$

where S and f represent slow and fast modes, then we have

$$|\lambda_{s_{m_1}}| / |\lambda_{f_1}| \ll 1 . \qquad (2.7)$$

A schematic representation of (2.7) is shown in Figure 2.1 where the shaded areas indicate the locations for slow and fast eigenvalues.

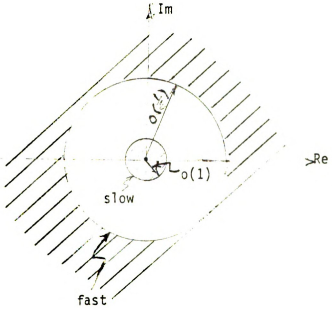


Figure 2.1.

If the system $\dot{X} = AX$ satisfies condition (2.7), then it possesses a two-time-scale property.

The decoupling transformation is a useful tool in decomposing the singularly perturbed systems into slow and fast parts and could be applied to both continuous and discrete systems. A more general case of block-diagonalization of ill-conditioned systems is presented by Kokotovic [1975] which considers systems not necessarily in singularly perturbed form. Due to importance of this transformation throughout our present work, we give a brief explanation of the latter work.

Consider the following free system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} , \qquad (2.8)$$

where x and z are m_1- and m_2-dimensional state vectors.

Let

$$\eta_2 = z + Lx , \qquad (2.9)$$

where L is a real root of

$$A_{22}L - LA_{11} + LA_{12}L - A_{21} = 0 . \qquad (2.10)$$

If L exists, then substitution of (2.9) into (2.8) yields the block triangular form

$$\begin{bmatrix} \dot{x} \\ \dot{\eta}_2 \end{bmatrix} = \begin{bmatrix} B_1 & A_{12} \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} x \\ \eta_2 \end{bmatrix} , \qquad (2.11)$$

where

$$B_1 = A_{11} - A_{12}L \tag{2.12}$$

$$B_2 = A_{22} + LA_{12}. \tag{2.13}$$

If A_{22} is invertible, let

$$L_0 = A_{22}^{-1}A_{21}\ ,\ A_0 = A_{11} - A_{12}L_0. \tag{2.14}$$

Now L is sought in the form

$$L = L_0 + D, \tag{2.15}$$

where D is a real root of

$$DA_0 - (A_{22} + L_0A_{12})D - DA_{12}D + L_0A_0 = 0. \tag{2.16}$$

It is proved by Kokotovic [1975] that a unique real root of (2.16) exists satisfying

$$0 \leq \|D\| \leq \frac{2\|A_0\|\ \|L_0\|}{\|A_0\|+\|A_{12}\|\ \|L_0\|}\ , \tag{2.17}$$

if the following condition on matrix norms is met

$$\|A_{22}^{-1}\| \leq \frac{1}{3}\left(\|A_0\| + \|A_{12}\|\ \|L_0\|\right)^{-1}, \tag{2.18}$$

where $\|\ \|$ is assumed to be a 2-norm.

He also proves that D in (2.16) is an asymptotically stable equilibrium of the difference equation

$$D_{K+1} = A_{22}^{-1}\,(L_0A_0+D_KA_0-L_0A_{12}D_K-D_KA_{12}D_K) \equiv f(D_K)\,. \tag{2.19}$$

Furthermore, using the change of variable

$$\eta_1 = x - M\eta_2\,, \tag{2.20}$$

where M is a real root of

$$B_1M - MB_2 + A_{12} = 0\,, \tag{2.21}$$

and substitution into (2.10) yields

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}. \tag{2.22}$$

It is proved that under the conditon (2.18) the solution M of (2.21) is the asymptotically stable equilibrium of the linear difference equation

$$M_{K+1} = [(A_{11}-A_{12}L)M_K-M_KLA_{12}]A_{22}^{-1} + A_{12}A_{22}^{-1}. \tag{2.23}$$

The above two-stage transformation could be represented by

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} I_1 & M \\ -L & I_2-LM \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}, \tag{2.24}$$

where I_1 and I_2 are the m_1- and m_2-dimensional identity matrices respectively.

It is easy to see that

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} I_1 - ML & -M \\ L & I_2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} . \tag{2.25}$$

The above transformation is particularly convenient for singularly perturbed systems and was introduced by Chang [1972] in the form

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} I_1 - \epsilon ML & -\epsilon M \\ L & I_2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} , \tag{2.25a}$$

and was applied on many control problems of singularly perturbed systems. [Kokotovic, Haddad, 1975], [Chow and Kokotovic, 1976].

In many applications we are dealing with continuous-time models which possess a two-time-scale property, while they are not, explicitly, in the singularly perturbed form. The major problem in converting a given system of equations to a singularly perturbed form is in grouping the state variables into slow and fast states such that (2.18) is satisfied. If the original system possesses a two-time-scale property (2.7) but condition (2.18) is not satisfied, it may still be possible to satisfy (2.18) by either scaling and regrouping the state variables or by allowing linear combinations of certain fast states with the slow state group. Readers who are interested in learning about this modeling process are referred to Chow and Kokotovic [1976-b], Sain et al. [1977], Anderson [1978], Sycros and Sannuti [1983] and Chow [1983].

2.3. Historical Review of Two-Time-Scale Discrete-Time Systems.

The well known difficulties in dealing with high-order models and the class of "stiff" systems plus the recent interest in optimization and control algorithms for discrete systems operating on widely separated time-scales has provided an impetus for applying singular perturbation methods to order reduction and control of discrete systems.

The objective of this section is to describe previous efforts to extend singular perturbation techniques to discrete-time systems having a two-time scale property. Different approaches to characterize two-time-scale discrete-time systems are presented.

The models considered earlier in the literature classify into three groups.

1. Phillips, and Rajagopolan and Naidu
2. Mahmoud
3. Hoppensteadt and Miranker, and Blankenship.

Group I.

Philips [1980] considers linear time invariant discrete-time systems. There is always a basis such that a discrete-time system takes the form

$$\begin{bmatrix} x_s(k+1) \\ \\ x_f(k+1) \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ \\ 0 & A_f \end{bmatrix} \begin{bmatrix} x_s(k) \\ \\ x_f(k) \end{bmatrix} + \begin{bmatrix} B_s \\ \\ B_f \end{bmatrix} u(k) , \qquad (2.26)$$

where

$$\lambda_f < \lambda_s$$

$$\lambda_f \triangleq \max_j |\lambda_j(A_f)|$$

$$\lambda_s \triangleq \min_i |\lambda_i(A_s)| .$$

The system (2.26) is not necessarily in its modal form. However, multiple and complex conjugate eigenvalues are naturally grouped together in either A_s or A_f. System (2.26) is said to possess a two-time-scale property if there is a sufficient gap between the eigenvalues of A_s and A_f, i.e.

$$\lambda_f << \lambda_s . \qquad (2.27)$$

Noting that

$$\min_i \; |\lambda_i(A_s)| \geq \|A_s^{-1}\|^{-1} \quad \text{(lower bound)}$$

$$\max_j \; |\lambda_j(A_f)| \leq \|A_f\| \quad \text{(upper bound).}$$

the two-time-scale property can be expressed as

$$\|A_s^{-1}\|^{-1} >> \|A_f\| . \qquad (2.28)$$

Phillips[1980], then considers a class of discrete-time system of the form

$$x(k+1) = A_{11}x(k) + \epsilon^{1-j}A_{12}z(k) + B_1u(k),\ x(0) = x_0 \qquad (2.29a)$$

$$z(k+1) = \epsilon^{j}A_{21}x(k) + \epsilon A_{22}z(k) + B_2u(k),\ z(0) = z_0 \qquad (2.29b)$$

where x and z are m_1- and m_2-dimensional vector states, u is an r-dimensional input, ϵ is a small positive parameter, $0 \leq j \leq 1$ and A_{11}^{-1} exists. He shows that for sufficiently small ϵ, the system (2.29) possesses a two-time-scale property. In particular, let

$$a = \|A_{22} - A_{21}A_{11}^{-1}A_{12}\|$$

$$b = \|A_{21}A_{11}^{-1}\|\ \|A_{12}\|$$

$$c = \|A_{11}^{-1}\|,$$

and

$$d = a + b.$$

Then, if

$$\epsilon < \frac{d}{c(d^2+8ab)},$$

the system (2.29) can be transformed using the decoupling transformation of Kokotovic [1975] into the form (2.26) with

$$A_s = A_{11} - \epsilon^{1-j} A_{12} L$$

$$A_f = \epsilon A_{22} + \epsilon^{1-j} L\, A_{12}$$

$$B_s = (I - ML)B_1 - MB_2$$

$$B_f = LB_1 + B_2 \; .$$

The matrices L and M satisfy equations (2.10) and (2.21) with A_{12}, A_{21} and A_{22} replaced by $\epsilon^{1-j}A_{12}$, $\epsilon^{j}A_{21}$ and ϵA_{22}, respectively.

Furthermore, he shows that $L = O(\epsilon^{j})$ so that letting $L = \epsilon^{j}\hat{L}$, the matrices A_s and A_f take the form

$$A_s = A_{11} - \epsilon\, A_{12}\hat{L}, \quad A_f = \epsilon(A_{22} + \hat{L}\, A_{12}).$$

Thus, the system satisfies the two-time-scale property (2.28) since for sufficiently small ϵ

$$\|(A_{11} - \epsilon\, A_{12}\hat{L})^{-1}\|^{-1} >> \epsilon\|A_{22} + \hat{L}\, A_{12}\| \; . \tag{2.30}$$

Also, x is the slow state and z is the fast state.

A special case of (2.29) with $j = 0$ has been considered by Rajagopolan and Naidu [1980]. It takes the form

$$x(k+1) = A_{11}x(k) + \epsilon\, A_{12}\, \tilde{z}(k) + B_1\, u(k) \tag{2.31a}$$

$$\tilde{z}(k+1) = A_{21}x(k) + \epsilon\, A_{22}\, \tilde{z}(k) + B_2\, u(k) \, . \tag{2.31b}$$

Although (2.31) is a special case of (2.29), it is seen that in the absence of inputs the two systems are equivalent in the sense that $\tilde{z}(k) = \epsilon^j z(k)$.

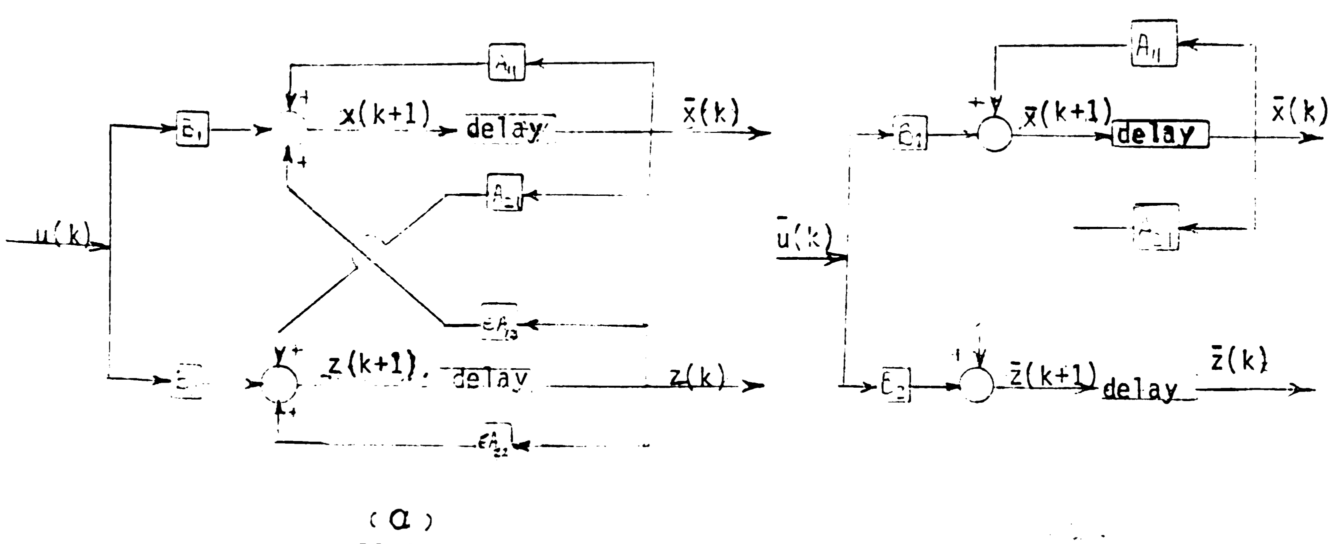
For simplicity let us continue our discussion using the model (2.31). Letting $\epsilon = 0$ in (2.27) yields a reduced (degenerate) system of order of m_1.

$$\bar{x}(K+1) = A_{11}\bar{x}(K) + B_1\bar{u}(K) \tag{2.32a}$$

$$\bar{z}(K+1) = A_{21}\bar{x}(K) + B_2\bar{u}(K) . \tag{2.32b}$$

We note that $\bar{x}(0) = x_0$ and $\bar{z}(0) \neq z_0$. This situation of order reduction and consequent loss of initial conditions is analogous to singular perturbation in differential equations [O'Malley 1971].

The state variables of the full system (2.31) and the reduced system (2.32) are shown in Figure 2.2.



(a) Full System

Reduced System

Figure 2.2

System (2.31) may be regarded as a system in slow-time scale, that is, the slow state is varying on an $O(1)$ time-scale.

Group II

Mahmoud [1982] considers the system

$$x(K+1) = A_{11}x(K) + A_{12}z(K) + B_1u(K),\ x(0) = x_0 \quad (2.33a)$$

$$z(K+1) = A_{21}x(K) + A_{22}z(K) + B_2u(K),\ z(0) = z_0 \quad (2.33b)$$

His work is essentially repetition of the work of Phillips [1980] and Kokotovic [1975], except that ϵ does not appear in his system explicitly. By using the decoupling transformation, mentioned in Section 2.2, he arrives at a similar condition as (2.30) so the system (2.33) possesses a two-time-scale property.

Group III

Hoppensteadt and Miranker [1977] have considered a free system of the general form

$$x(K+1) = Ax(K) + \epsilon f(x(K),\epsilon), x(0) = x_0, \quad (2.34)$$

where A and f are time-invariant.

They assume that there exists an invertible matrix P such that

$$P^{-1}AP = \text{diag}(\theta, S), \quad (2.35)$$

where the matrix θ is oscillatory, that is, has all characteristic roots on the unit circle, $|\lambda| = 1$, and the matrix S is stable, that is, has its eigenvalues inside the unit circle, $|\lambda| < 1$.

The matrix θ is assumed to be diagonalizable. They also assume that f is a smooth function of its arguments. Applying the transformation

$$x(K) = P \begin{bmatrix} \theta^K u(K) \\ v(K) \end{bmatrix}, \tag{2.36}$$

yields

$$u(K+1) = u(K) + \epsilon\, \theta^{-K-1} g(\theta^K u(K), v(K), \epsilon) \tag{2.37a}$$

$$v(K+1) = Sv(K) + \epsilon h(\theta^K u(K), v(K), \epsilon), \tag{2.37b}$$

where

$$f = P\binom{g}{h}.$$

Next it is assumed that there is a smooth function $\phi(u,\epsilon)$ (fast quasi-steady-state) such that

$$\phi(\theta^K u(K), \epsilon) = S\phi(\theta^K u(K), \epsilon) + \epsilon\, h(\theta^K u^K, \phi, \epsilon), \tag{2.38}$$

with $\phi = O(\epsilon)$.

Letting $v(K) = \phi(\theta^K u(K), \epsilon) + V(K)$ yields

$$u(K+1) = u(K) + \epsilon\, \theta^{-K-1} g(\theta^K u(K), \phi + V(K), \epsilon) \tag{2.39a}$$

$$V(K+1) = SV(K) + \epsilon\{h(\theta^K u(K), \phi + V(K), \epsilon) - h(\theta^K u(K), \phi, \epsilon)\}. \tag{2.39b}$$

Due to the location of ϵ, u will change little before V has reached $O(\epsilon)$. Thus the slow behavior is found by solving

$$u(K+1) = u(K) + \epsilon\, \Theta^{-K-1} g(\theta^K u(K), \phi(\Theta^K u(K),\epsilon),\epsilon) \qquad (2.40)$$

By expanding $u(K)$ in ϵ, the solution of (2.40) is found in the form

$$u(K) = U(K,S,\epsilon) = U^0(K,S) + \epsilon\, U^1(K,S) + O(\epsilon^2), \qquad (2.41)$$

where $S = \epsilon K$. Note that U is evolving in slow time scale S.

Solving for the terms in the expansion yields

$$U^0(K+1,S) = U^0(K,S). \qquad (2.42)$$

So U^0 is independent of K, that is, in the slow-time scale the limiting value of u is constant.

Equating coefficients on a fast time-scale and taking the limit as $\epsilon \to 0$ and assuming U^1 is bounded yields

$$\frac{dU^0}{dS} = \lim_{k\to\infty} \frac{1}{K} \sum_{n=0}^{K-1} \theta^{-n-1} g(\theta^n U^0(S),0,0). \qquad (2.43)$$

Thus, it is shown that the solution to (2.34) can be approximated by

$$x(K) = P \begin{bmatrix} \theta^K U^0(\epsilon_K) \\ S^K v_0 \end{bmatrix} + O(\epsilon), \qquad (2.44)$$

where

$$P \begin{bmatrix} U^0(0) \\ v_0 \end{bmatrix} = x_0.$$

And the approximate solution to the original problem is determined by the solution to the reduced equations (2.43) and (2.44).

Blankenship [1980] analyzed the system

$$x(K+1) = x(K) + \epsilon A_{11}x(K) + \epsilon A_{12}z(K) + \epsilon B_1u(K) \qquad (2.45a)$$

$$z(K+1) = A_{22}z(K) + \epsilon A_{21}x(K) + \epsilon D\, z(K) + (B_2 + \epsilon H)u(K). \qquad (2.45b)$$

The above system is a fast time-scale model.

Here the fast eigenvalues are inside the unit circle but of $O(1)$. The eigenvalues corresponding to slow modes are assumed $O(\epsilon)$ away from 1. This model thus assumes that the slow modes are almost constant while the fast modes are approximately given by the boundary-layer system.

$$z(K+1) = A_{22}\, z(K) + B_2u(K). \qquad (2.46)$$

It should be noted that system (2.29) assumes that the fast state z is treated on a time-scale slow enough for its response to be deadbeat while system (2.46) assumes that the slow state is treated on a time-scale fast enough for the slow state x to remain approximately constant.

The presence of ϵ in the above mentioned models classifes them as singularly perturbed systems as the order reduction and separation of time-scales are apparent by setting $\epsilon = 0$.

It should be noted that system (2.37) has the structure of (2.45) where $u(K)$ is the slow state and $v(K)$ is the fast state. The two-time-scale property of the system (2.45) will be shown in Section 2.4. Blankenship examines a linear quadratic regulator problem subject to the system equation (2.45).

In deriving the results in his work, the control input is assumed to consist of two components, one which vanishes as $K \to \infty$ and the other which is bounded. That is

$$u(K) = v(K) + u(\epsilon K) \quad , \quad K = 0,1,2,... \tag{2.47}$$

$$\lim_{K\to\infty} v(K) = 0 .$$

For $u(K)$ of this form a solution of (2.45) is sought in the form

$$x(K) = a(K) + X(\epsilon K), \quad z(K) = b(K) + Y(\epsilon K) \tag{2.48}$$

with

$$\lim_{K\to\infty} a(K) = 0, \quad \lim_{K\to\infty} b(K) = 0 . \tag{2.49}$$

The terms $(X(\epsilon K), Y(\epsilon K))$ are "outer solution" and $(a(K), b(K))$ are "initial boundary layers" similar to those defined for singularly perturbed differential equations [Hoppensteadt, 1971].

The final-value problem

$$a(K+1) = a(K) + \epsilon A_{11}a(K) + \epsilon A_{12}b(K) + \epsilon B_1u(K)$$

$$b(K+1) = A_{22}b(K) + \epsilon A_{21}a(K) + \epsilon D\, b(K) + (B_2+\epsilon H)v(K) \tag{2.50}$$

$$\lim_{K\to\infty} a(K) = 0 , \quad \lim_{K\to\infty} b(K) = 0 \tag{2.51}$$

defines the boundary layer terms, and

$$X(\epsilon K+\epsilon) = (I+\epsilon A_{11})\, X(\epsilon K) + \epsilon A_{12}Y(\epsilon K) + \epsilon B_1U(\epsilon K)$$

$$Y(\epsilon K+\epsilon) = A_{22}Y(\epsilon K) + \epsilon A_{21}\, X(\epsilon K) + \epsilon DY(\epsilon K) + (B_2+\epsilon H)U(\epsilon K) \tag{2.52}$$

$$X(0) = x_0 - a_0 \,, \quad Y(0) = z_0 - b_0$$

defines the outer solution.

Now by assuming that A_{22} is stable and that $u(K)$ is any function which satisfies (2.47) and

$$v^{\epsilon}(K) = \sum_{n=0}^{\infty} \epsilon^n v^{(n)}(K), \; u^{\epsilon}(S) = \sum_{n=0}^{\infty} \epsilon^n u^{(n)}(S), \; \sum_{K=0}^{\infty} |v^{(0)}(K)| < \infty, \tag{2.53}$$

where $S = \epsilon K$ (superindex (n) shows the nth derivative) and by taking asymptotic expansions in a, b, X, and Y and matching the coefficients in ϵ in the fast-time-scale K and slow time-scale S, he shows that the solution of (2.45) satisfies

$$x(K) = X^{(0)}(\epsilon K) + O(\epsilon) \tag{2.54a}$$

$$z(K) = Y^{(0)}(\epsilon K) + b^{(0)}(K) + O(\epsilon), \tag{2.54b}$$

where

$$\frac{dX^0(S)}{dS} = A_{11}X^{(0)}(S) + [A_{12}(I-A_{22})^{-1}B_2 + B_1]u^{(0)}(S), \; X^{(0)}(0) = x_0 \tag{2.55}$$

$$Y^{(0)}(S) = (I-A_{22})^{-1}B_2u^{(0)}(S) \tag{2.56}$$

$$a^{(0)}(K) = 0 \tag{2.57}$$

$$b^{(0)}(K+1) = A_{22}b^{(0)}(K) + B_2v^{(0)}(K), \; b^{(0)}(0) = z_0 - Y^{(0)}(0). \tag{2.58}$$

Note that the system (2.55), (2.56) is the "reduced-order" system corresponding to (2.45). It evolves in the slow time scale $S = \epsilon K$ consistant with the analogous notion of reduced-order in the continuous-time problems and the solution is obtained by solving the differential Equation (2.55). Also it is interesting to note that this solution parallels the expressions derived for the uncontrolled cases by Hoppensteadt and Miranker [1977] since they use the method of matched asymptotic expansion using a multitime method and, like Blankenship, they exhibits a hybrid situation where the slow system is represented by a set of differential equations and boundary-layer (fast) system is given by difference equations, which is different from the case of Mahmoud and Phillips where the slow and fast systems are both given by difference equations.

In this thesis we adopt the model of Blankenship and Hoppensteadt and Miranker. Justification for adopting this model is given in Section 2.4.

2.4. Sources of Singularly Perturbed Difference Equations.

There are four important sources of discrete-time models described by singularly peturbed difference equations of the form (2.45). These sources are presented in this section.

2.4.1. Inherently discrete-time singularly perturbed models.

This class of systems results when the physical system is inherently discrete. Such models are common in economic, biological, and sociological systems. Some examples of this type of discrete-time singularly perturbed systems are given in [Hoppensteadt and Miranker, 1977]. We breifly explain one of these examples.

Example 2.4.1: A population genetics model

In a large population of diploid organisms having discrete generations, the genotypes determined by one locus having two alleles, A and a, divide the population into three groups of type AA, Aa, and aa, respectively. The gene pool carried by this population is assumed to be in proportion P_n of type A in the n^{th} generation. It follows that [Crow and Kimura, 1970]

$$P_{n+1} = P_n + \frac{P_n(1-P_n)[(W_{11}-W_{12})P_n + (W_{12}-W_{22})(1-P_n)]}{W_{11}P_n^2 + 2W_{12}P_n(1-P_n) + W_{22}(1-P_n)^2}, \quad (2.59)$$

where W_{11}, W_{12}, and W_{22} are the relative fitnesses of the genotypes AA, Aa, and aa, respectively. Now if the selective pressures are acting slowly, i.e., if $W_{11} = 1 + \epsilon\alpha$, $W_{12} = 1$, $W_{22} = 1 + \epsilon\beta$, where ϵ is small positive number, then

$$P_{n+1} = P_n + \epsilon \frac{P_n(1-P_n)[(\alpha+\beta)P_n-\beta]}{1 + O(\epsilon)}. \quad (2.60)$$

This model is a special case of (2.45) where there is a slow state only.

2.4.2. Singularly perturbed difference equations obtained by numerical solution of stiff differential equations.

This class of systems is ususally found as a result of numerical soulution of stiff differential equations where they are approximated by corresponding difference equations, usually for the purpose of digital simulation. To clarify this we give two examples; the first one was considered by Hoppensteadt and Miranker [1977] and the second one by Blankenship [1981].

Example 2.4.2.

The two-dimensional stiff linear differential equation

$$\frac{dZ}{dt} = (B + \epsilon C)\, Z \quad , \quad B = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.61}$$

which is written in the fast time-scale is considered. Let $Z = (x,y)^T$. Introducing a mesh with increment h and applying an r-step linear multistep method to the system leads to the following difference equations.

$$\sum_{j=0}^{r} \alpha_j Z_{n-j} - h(\beta + \epsilon C) \sum_{j=1}^{r} \beta_j Z_{n-j} = 0, \; n = r, r+1, \ldots \tag{2.62}$$

Let $\underline{X}_n = (x_n, x_{n-1}, \ldots, x_{n-r+1})^T$ and $Y_n = (y_n, y_{n-1}, \ldots, y_{n-r+1})^T$ and the $r \times r$ matrices

$$R = \begin{bmatrix} -\alpha_1 & \cdots & -\alpha_{r-1} & -\alpha_r \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} \beta_1 & \cdots & \beta_r \\ & 0 & \end{bmatrix}. \tag{2.63}$$

The difference equation for Z_n may be written in the following form

$$\underline{X}_{n+1} = R\,\underline{X}_n + hb\beta\underline{X}_n + \epsilon h\beta(C_{11}\underline{X}_n + C_{12}\,Y_n) \tag{2.64a}$$

$$Y_{n+1} = RY_n \quad + \epsilon h\beta(C_{21}\underline{X}_n + C_{22}Y_n). \tag{2.64b}$$

Equation (2.64) is of the form of (2.34) and by using a transformation similar to (2.36) it is shown [Hoppensteadt and Mirankar 1977] that the system can be brought into the singularly perturbed form.

Example 2.4.3. Blankenship uses the Euler's approximation for the following stiff differential equations,

$$\frac{d\tilde{x}(t)}{dt} = \tilde{A}\,\tilde{x}(t) + \tilde{B}\tilde{z}(t) + \tilde{F}\,\tilde{u}(t) \qquad (2.65a)$$

$$\epsilon\,\frac{d\tilde{z}(t)}{dt} = \tilde{C}\,\tilde{x}(t) + \tilde{D}\tilde{z}(t) + \tilde{G}\,\tilde{u}(t) \qquad (2.65b)$$

$$\tilde{x}(0) = x_0 \in R^{m_1}, \quad \tilde{z}(0) = z_0 \in R^{m_2}, \; 0 \le t \le 1,$$

where the matrices are constant, $\epsilon > 0$ is a small parameter and $\tilde{D}$ is assumed to be nonsingular.

Introducing the stretching time scale $\tau = t/\epsilon$ and

$$y(\tau) = \tilde{z}(\epsilon\tau) + \tilde{D}^{-1}\tilde{C}\tilde{x}(\epsilon\tau), \; x(\tau) = \tilde{x}(\epsilon\tau), \; u(\tau) = \tilde{u}(\epsilon\tau),$$

we obtain

$$\frac{dx(\tau)}{d\tau} = \epsilon\,Ax(\tau) + \epsilon\,B\,y(\tau) + \epsilon\,Fu(\tau), \qquad (2.66a)$$

$$\frac{dy(\tau)}{d\tau} = S\,y(\tau) + \epsilon Cx(\tau) + \epsilon Dy(\tau) + (G + \epsilon H)u(\tau), \qquad (2.66b)$$

$$x(0) = x_0 \,, \; y(0) = y_0 \,, \; 0 \le \tau \le 1/\epsilon \,,$$

where the coefficients A,B, etc., are simple combinations of $\tilde{A}, \tilde{B}$, etc.; $S = \tilde{D}$.

Let $\{0,h,2h,\ldots,Nh\}$ be a mesh on $[0, 1/\epsilon]$ and let

$$x_n = x(nh),\ y_n = y(nh),\ u_n = u(nh)$$

be a numerical approximation to $x(\tau)$, $y(\tau)$, and $u(\tau)$. Using Euler's approximation to the derivative we obtain

$$x_{n+1} = x_n + \epsilon hAx_n + \epsilon hBy_n + \epsilon hFu_n \tag{2.67a}$$

$$y_{n+1} = (I+hS)y_n + \epsilon hCx_n + \epsilon hDy_n + hGu_n + \epsilon hHu_n \tag{2.67b}$$

$$x(0) = x_0,\ y(0) = y_0\ ,\ n = 0,1,\ldots,N-1 \simeq 0\ (1/\epsilon h).$$

We note that system (2.67) is a singularly perturbed system equivalent to (2.45) with $h = 1$ and $I + hS = A_{22}$.

Also, note that for a discrete model obtained in this way, $(I + hS)$ is generally nonsingular since h must be small for a good approximation to a continuous-time system.

2.4.3. Sampled-Data Control of Singularly Perturbed Systems.

Another source of singularly perturbed difference equations comes form study of sampled-data systems or computer-controlled systems where a continuous-time singularly perturbed system is driven by an input specified at discrete-time points and has output and state variables sampled only at discrete-time points.

The standard example of sampled-data system follows when $u(t)$ is a piece-wise constant function of time, i.e.

$$u(t) = u(t_K) \quad , \quad t_K \leq t < t_{K+1} \; ,$$

and the state and output are sampled at discrete time points t_K. Consider the following singularly perturbed linear time-invariant continuous-time system

$$\dot{x} = A_{11}x(t) + A_{12}Z(t) + B_1u(t), \; x(0) = x_0 \tag{2.68a}$$

$$\epsilon \; \dot{Z} = A_{21}x(t) + A_{22}Z(t) + B_2u(t), \; Z(0) = Z_0 \, , \tag{2.68b}$$

where x and Z are m_1- and m_2-dimensional state vectors and all the matrices have compatible dimensions with A_{22} nonsingular.

The solution of the system (2.68) between $(0,t)$ is given by

$$\begin{bmatrix} x(t) \\ Z(t) \end{bmatrix} = e^{\tilde{A}t} \begin{bmatrix} x_0 \\ Z_0 \end{bmatrix} + \int_0^t e^{\tilde{A}(t-\tau)}\tilde{B} \; u(\tau)d\tau \, , \tag{2.69}$$

where

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix} \; , \quad \tilde{B} = \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix} . \tag{2.70}$$

For a piece-wise constant $u(t) = u(t_K)$, $t_K \leq t \leq t_{K+1}$ and the sampling period $\epsilon T = t_{K+1} - t_K$ we have [Levis, et.al., 1971], [Levis, Dorato, 1971]

$$\begin{bmatrix} x(K+1) \\ Z(K+1) \end{bmatrix} = \phi(\epsilon T)\begin{bmatrix} x(K) \\ Z(K) \end{bmatrix} + \Gamma(\epsilon T)u(K) , \tag{2.71}$$

where

$$\phi(\epsilon T) = e^{\tilde{A}\epsilon T} \quad \text{and} \quad \Gamma(\epsilon T) = \int_0^{\epsilon T} e^{\tilde{A}t}\tilde{B}dt . \tag{2.72}$$

To evaluate $e^{\tilde{A}t}$ we use the Chang transformation (2.25a) to block diagonalize $\tilde{A}$ and we get

$$\tilde{A} = \begin{bmatrix} I_1 & \epsilon M \\ -L & I_2-\epsilon LM \end{bmatrix} \begin{bmatrix} A_{11}-A_{12}L & 0 \\ 0 & \frac{A_{22}}{\epsilon} + LA_{12} \end{bmatrix} \begin{bmatrix} I_1-\epsilon ML & -\epsilon M \\ L & I_2 \end{bmatrix} , \tag{2.73}$$

and

$$e^{\tilde{A}t} = \begin{bmatrix} I_1 & \epsilon M \\ -L & I_2-\epsilon LM \end{bmatrix} \begin{bmatrix} e^{(A_{11}-A_{12}L)t} & 0 \\ 0 & e^{(A_{22}+\epsilon LA_{12})t/\epsilon} \end{bmatrix} \begin{bmatrix} I_1-\epsilon ML & -\epsilon M \\ L & I_2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{(A_{11}-A_{12}L)t}(I_1-\epsilon ML)+\epsilon Me^{(A_{22}+\epsilon LA_{12})t/\epsilon}L \\ -Le^{(A_{11}-A_{12}L)t}(I_1-\epsilon ML)+(I_2-\epsilon LM)e^{(A_{22}+\epsilon LA_{12})t/\epsilon}L \end{bmatrix}$$

$$\left.\begin{matrix} -\epsilon e^{(A_{11}-A_{12}L)t}M + \epsilon M e^{(A_{22}+\epsilon LA_{12})t/\epsilon} \\ \epsilon L e^{(A_{11}-A_{12}L)t}M + (I_2-\epsilon LM)e^{(A_{22}+\epsilon LA_{12})t/\epsilon} \end{matrix}\right] .$$

I_1 and I_2 are m_1 and m_2-dimensional identities, respectively, L satisfies (2.10) and M satisfies (2.21) and could be approximated by

$$L = A_{22}^{-1} A_{21} + O(\epsilon) = L_0 + O(\epsilon)$$

$$M = A_{12} A_{22}^{-1} + O(\epsilon) = M_0 + O(\epsilon) .$$

Let $A_0 = A_{11}-A_{12}A_{22}^{-1}A_{21}$. For sufficiently small ϵ, $\phi(\epsilon T)$ is given by the following: (1,1) element is

$$= e^{\epsilon[A_0+O(\epsilon)]T}[I_1-\epsilon M_0L_0 + O(\epsilon^2)] + \epsilon[M_0 + O(\epsilon)]e^{[A_{22}+ O(\epsilon)]T}[L_0 + O(\epsilon)]$$

$$= [I + \epsilon TA_0+O(\epsilon^2)][I_1-\epsilon M_0L_0+O(\epsilon^2)] + \epsilon[M_0+O(\epsilon)][e^{A_{22}T} + O(\epsilon)][L_0+O(\epsilon)]$$

$$= I + \epsilon[TA_0+M_0(e^{A_{22}T}-I_2)L_0] + O(\epsilon^2) \tag{2.74}$$

$$= I + \epsilon A + O(\epsilon^2) ,$$

where

$$A = TA_0 + M_0(e^{A_{22}T}-I_2)L_0$$

(1,2) element is

$$= -\epsilon e^{\epsilon[A_0+O(\epsilon)]T}[M_0 + O(\epsilon)] + \epsilon[M_0 + O(\epsilon)]e^{[A_{22} + O(\epsilon)]T}$$

$$= -\epsilon[I_1 + \epsilon A_0T + O(\epsilon^2)][M_0 + O(\epsilon)] + \epsilon[M_0 + O(\epsilon)][e^{A_{22}T} + O(\epsilon)]$$

$$= \epsilon M_0(e^{A_{22}T}-I_2) + O(\epsilon^2) = \epsilon B + O(\epsilon^2), \tag{2.75}$$

where

$$B = M_0(e^{A_{22}T}-I_2).$$

(2,1) element is

$$=-[L_0+O(\epsilon)]e^{\epsilon[A_0+O(\epsilon)]T}[I_1-\epsilon M_0L_0+O(\epsilon^2)]+[I_2-\epsilon L_0M_0+O(\epsilon^2)]e^{[A_{22}+O(\epsilon)]T}[L_0+O(\epsilon)]$$

$$=-[L_0+O(\epsilon)][I_1+\epsilon TA_0+O(\epsilon^2)][I_1-\epsilon M_0L_0+O(\epsilon^2)]+[I_2-\epsilon L_0M_0+O(\epsilon^2)][e^{A_{22}T}+O(\epsilon)]$$

$$[L_0+O(\epsilon)]$$

$$= (e^{A_{22}T}-I_2)L_0 + O(\epsilon) = C + O(\epsilon), \tag{2.76}$$

where

$$C = (e^{A_{22}T}-I_2)L_0.$$

(2,2) element is

$$= [L_0+O(\epsilon)]e^{\epsilon[A_0+O(\epsilon)]T}\epsilon[M_0+O(\epsilon)]+[I_2-\epsilon L_0M_0+O(\epsilon^2)][e^{A_{22}T}+O(\epsilon)]$$

$$= e^{A_{22}T} + O(\epsilon) = S + O(\epsilon), \tag{2.77}$$

where

$$S = e^{A_{22}T}.$$

We have

$$\Gamma(\epsilon T) = \begin{bmatrix}\Gamma_1(\epsilon T)\\ \Gamma_2(\epsilon T)\end{bmatrix} = \int_0^{\epsilon T} e^{\tilde{A}t}\tilde{B}dt.$$

Let $A_0 = A_{11}-A_{12}L_0$ and $F_{22} = A_{22} + \epsilon L A_{12}$, then

$$\Gamma_1(\epsilon T) = \int_0^{\epsilon T} [e^{[A_0+O(\epsilon)]t} B_1 + Me^{F_{22}\frac{t}{\epsilon}} B_2 - e^{[A_0 + O(\epsilon)]t} MB_2 + O(\epsilon)]\, dt,$$

and

$$\Gamma_2(\epsilon T) = \int_0^{\epsilon T} [\frac{1}{\epsilon} e^{F_{22}\frac{t}{\epsilon}} B_2 - Le^{[A_0+O(\epsilon)]t} B_1 + e^{F_{22}\frac{t}{\epsilon}} LB_1 + Le^{[A_0+O(\epsilon)]t} MB_2 - LMe^{F_{22}\frac{t}{\epsilon}} B_2 + O(\epsilon)]dt.$$

For sufficiently small ϵ, F_{22} is invertible and we have

$$\Gamma_1(\epsilon T) = \epsilon M(e^{F_{22}T} - I_2)F_{22}^{-1}B_2 + \int_0^{\epsilon T} [e^{[A_0+O(\epsilon)]t} (B_1-MB_2)+O(\epsilon)]\, dt.$$

Using Householder theorem (see appendix 2.1) we obtain

$$\Gamma_1(\epsilon T) = \epsilon[M_0(e^{F_{22}T} - I_2)A_{22}^{-1}B_2 + \int_0^{\epsilon T} \frac{1}{\epsilon} e^{[A_0+O(\epsilon)]t} B_0 dt] + O(\epsilon^2),$$

where

$$B_0 = B_1 - M_0 B_2 .$$

Using the power series expansion for $e^{A_0 + O(\epsilon)}$ we can see that $\Gamma_1(\epsilon T)$ can be obtained by

$$\Gamma_1(\epsilon T) = \epsilon[M_0(e^{A_{22}T} - I_2)A_{22}^{-1}B_2 + \int_0^{\epsilon T} \frac{1}{\epsilon} e^{A_0 t} B_0 dt] + O(\epsilon^2).$$

By the same type of approximations we have

$$\Gamma_2(\epsilon T) = (e^{A_{22}T} - I_2)A_{22}^{-1}B_2 + \epsilon(e^{A_{22}T} - I_2)A_{22}^{-1}LB_1$$

$$- \int_0^{\epsilon T} Le^{[A_{0+}O(\epsilon)]t}[B_0 + O(\epsilon)]dt - \epsilon LM(e^{A_{22}T} - I_2)A_{22}^{-1}B_2$$

$$+ O(\epsilon^2).$$

By using power series expansion of $e^{\epsilon At}$ we obtain

$$\Gamma(\epsilon T) = \begin{bmatrix} \epsilon F + O(\epsilon^2) \\ G + O(\epsilon) \end{bmatrix}, \tag{2.78}$$

where

$$G = (e^{A_{22}T} - I_2)A_{22}^{-1}B_2$$

and

$$F = M_0 G + TB_0 .$$

Now system (2.71) can be represented by

$$x(K+1) = [I + \epsilon A + O(\epsilon^2)]\ x(K) + \epsilon[B + O(\epsilon)]Z(K) + \epsilon[F + O(\epsilon)]u(K)$$

$$Z(K+1) = [C + O(\epsilon)]\ x(K) + [S + O(\epsilon)]\ Z(K) + [G + O(\epsilon)]u(K), \tag{2.79}$$

which is a singularly perturbed difference equation.

2.4.4 Two-time-scale discrete-time systems which can be brought into the singularly perturbed form by artificial introduction of ϵ.

In conclusion of this section we give a simple example of discrete-time systems which are not in, so called, "explicit singularly perturbed" form but ϵ can be introduced artifically to transfer the system into singularly perturbed difference equations.

Consider the following discrete-time system:

$$\begin{bmatrix} x_1(K+1) \\ x_2(K+1) \\ Z(K+1) \end{bmatrix} = \begin{bmatrix} 1.025 & .0175 & -.0075 \\ -.0232 & .978 & .0192 \\ -1.325 & .975 & -2.1 \end{bmatrix} \begin{bmatrix} X_1(K) \\ X_2(K) \\ Z(K) \end{bmatrix} + \begin{bmatrix} .0082 \\ -.029 \\ 1 \end{bmatrix} u(K),$$

with eigenvalues $\lambda_1 = 1.009643$, $\lambda_2 = -2.109263$, and $\lambda_3 = 1.002619$. By investigating the numerics of the above system it is seen that the system could be put in the singularly perturbed form as follow:

$$\begin{bmatrix} \underline{\overline{X}}(K+1) \\ \\ Z(K+1) \end{bmatrix} = \begin{bmatrix} I+\epsilon A_{11} & \epsilon A_{12} \\ \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \underline{\overline{X}}(K) \\ \\ Z(K) \end{bmatrix} + \begin{bmatrix} \epsilon B_1 \\ \\ B_2 \end{bmatrix} u(K),$$

where

$$\epsilon = .01 \qquad \underline{\overline{X}} = [x_1 \quad x_2]^T$$

$$A_{11} = \begin{bmatrix} 2.5 & 1.75 \\ \\ -2.32 & -2.2 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -.75 \\ \\ 1.92 \end{bmatrix}, \quad A_{21} = (-1.325 \quad .975)$$

$$A_{22} = -2.1, \quad B_1 = \begin{bmatrix} .82 \\ -2.9 \end{bmatrix} \text{ and } B_2 = 1.$$

For a physical example one can refer to Mahmoud [1982] who has considered ninth-order boiler problem and ϵ could be introduced artifically similar to the above example.

It should be noted that in most physical problems some scaling and regrouping may be needed to obtain ϵ. This is shown in Chapter five where we consider an F-8 aircraft model.

2.5 Stability and Approximation Results

This section addresses some important topics which are useful to understand the two-time-scale nature of systems described by singularly perturbed difference equations. In particular, stability and approximation results are presented.

Consider the following discrete-time system

$$x(n+1) = [I+\epsilon A_{11}(\epsilon)]x(n) + \epsilon A_{12}(\epsilon)Z(n) \tag{2.80a}$$

$$Z(n+1) = A_{21}(\epsilon)\, x(n) + A_{22}(\epsilon)Z(n), \tag{2.80b}$$

where $(I_2-A_{22}(0))$ is invertible, x and Z are m_1- and m_2-dimensional state vectors, respectively, and all the matrices are analytic functions of ϵ with compatible dimensions. Unsing the transformation

$$y(n) = Z(n) + L\, x(n), \tag{2.81}$$

we have

$$x(n+1) = [I + \epsilon A_{11}(\epsilon)-\epsilon A_{12}(\epsilon)L]x(n) + \epsilon A_{12}(\epsilon)y(n) \tag{2.82a}$$

$$y(n+1) = [A_{21}(\epsilon)-A_{22}(\epsilon)L + L + \epsilon LA_{11}(\epsilon)-\epsilon LA_{12}(\epsilon)L]x(n) +$$

$$[A_{22}(\epsilon) + \epsilon LA_{12}(\epsilon)]y(n) \tag{2.82b}$$

Using the implicit function theorem and nonsingularity of $(I_2 - A_{22}(0))$ it can be shown that for sufficiently small ϵ there exists L satisfying

$$A_{21}(\epsilon) - A_{22}(\epsilon)L + L + \epsilon L A_{11}(\epsilon) - \epsilon L A_{12}(\epsilon)L = 0, \qquad (2.83)$$

and it can be approximated by

$$L = -[I - A_{22}(0)]^{-1} A_{21}(0) + 0(\epsilon),$$

which reduces the above system to the following block triangular form

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} I + \epsilon A_0 + 0(\epsilon^2) & \epsilon A_{12}(\epsilon) \\ 0 & A_{22}(0) + 0(\epsilon) \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}, \qquad (2.84)$$

where

$$A_0 = A_{11} + A_{12}(I - A_{22})^{-1} A_{21}.$$

Transformation (2.81) is the same one presented in Section 2.2.

From (2.84) it is seen that the eigenvalues of (2.80) are given by the eigenvalues of $[I + \epsilon A_0 + 0(\epsilon^2)]$ and $[A_{22}(0) + 0(\epsilon)]$. Using the continuous dependence of the eigenvalues of a matrix on its parameters it follows that the eigenvalues of $I + \epsilon A_0 + 0(\epsilon^2)$ are in the neighborhood of the point $z = 1$ in the complex plane and the eigenvalues of $A_{22}(0) + 0(\epsilon)$ are in the neighborhood of the eigenvalues of $A_{22}(0)$. Since $I_2 - A_{22}(0)$ is nonsingular, $A_{22}(0)$ has no eigenvalues at the point

$z = 1$, or in other words, the eigenvalues of $A_{22}(0)$ are $O(1)$ away from the point $z = 1$. Thus, for sufficiently small ϵ the eigenvalues of (2.80) are clustered into slow and fast eigenvalues as shown in Figure 2.3. There are m_1 slow eigenvalues and m_2 fast eigenvalues. Figure 2.3 is the discrete-time (or z-domain) version of Figure 2.1 which shows

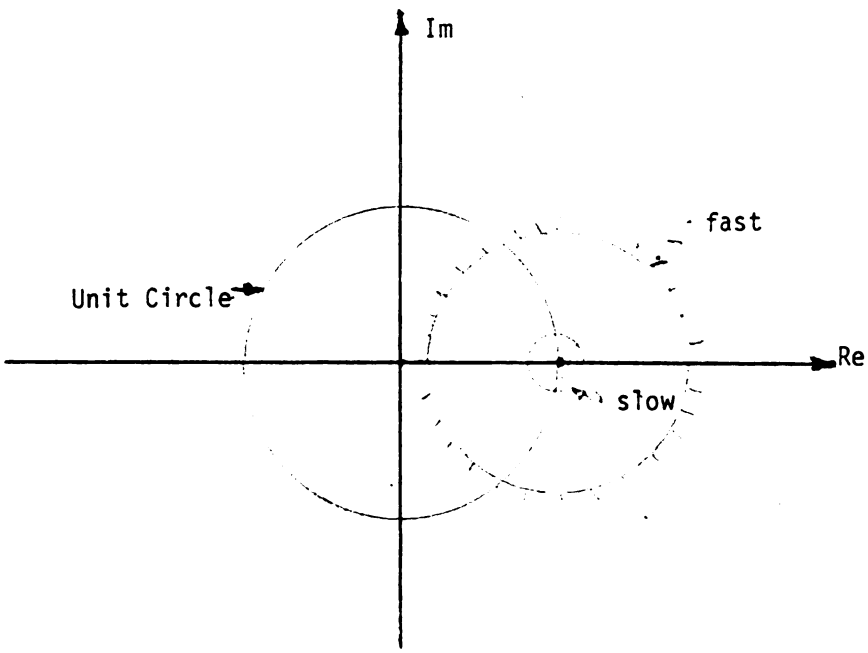


Figure 2.3

the slow-fast clustering of eigenvalues in the continuous-time (or s-domain). If we denote the set of slow eigenvalues by m_s and the set of fast eigenvalues by m_f, then, for sufficiently small ϵ, the eigenvalues of (2.80) satisfy the condition

$$\frac{\min\limits_{i \in m_f} |1-\lambda_i|}{\max\limits_{j \in m_s} |1-\lambda_j|} >> 1 \qquad (2.85)$$

Notice that (2.85) is more general than the eigenvalue separation condition (2.27) which was used by Phillips [1980] to define the two-time-scale property of discrete-time systems. Notice, however, that if (2.80) is asymptotically stable and the fast eigenvalues are well-damped, then (2.85) implies (2.27). Phillips' definition cannot handle the cases of unstable eigenvalues (outside the unit circle) or stable but oscillatory eigenvalues (inside the unit circle but close to it). These cases are important since, in general, we deal with open-loop systems where the eigenvalues could be of any of the above forms. Such eigenvalues will then be stabilized by the use of feedback.

The block triangular form of (2.84) leads to the following stability criterion.

Theorem 2.5.1. If the eigenvalues of A_0 are in the open left-half complex plane, i.e., $\text{Re}\lambda(A_0) < 0$, and the eigenvalues of A_{22} are inside the unit circle, i.e., $|\lambda(A_{22}(\epsilon)| < 1$, then there exists $\epsilon^* > 0$ such that for all $0 < \epsilon \leq \epsilon^*$ the system (2.80) is asymptotically stable.

Proof: From (2.84), the eigenvalues of (2.80) are given by the eigenvalues of $I + \epsilon(A_0 + O(\epsilon)$ and $A_{22}(0) + O(\epsilon)$. Since the eigenvalues of $A_{22}(0)$ are inside the unit circle, it follows that, for sufficiently small ϵ, the eigenvalues of $A_{22}(0) + O(\epsilon)$ will be inside the unit circle. By a well-known theorem [Stewart 1973, pp. 266] the eigenvalues of $I + \epsilon(A_0 + O(\epsilon))$ are given by $1 + \epsilon\lambda_i$ where λ_i are the eigenvalues of $A_0 + O(\epsilon)$. For sufficiently small ϵ the eigenvalues of $A_0 + O(\epsilon)$ have negative real parts. Let $\lambda_i = -\alpha_i + j\beta_i$, $\alpha_i > 0$. Then

$$|1 + \epsilon\lambda_i|^2 = |1 - \epsilon\alpha_i + \epsilon j\beta_i|^2 \leq (1-\epsilon\alpha_i)^2 + \epsilon^2\beta_i^2 = 1-2\epsilon\alpha_i + \epsilon^2(\alpha_i^2 + \beta_i^2),$$

which is less than one for sufficiently small ϵ. Thus all the eigenvalues of (2.80) are inside the unit circle.

Approximation Results:

Consider the linear time invariant discrete-time singularly perturbed system

$$x_1(n+1) = [I + \epsilon A_{11}(\epsilon)]x_1(n) + \epsilon A_{12}(\epsilon)x_2(n) + \epsilon B_1(\epsilon)u(n) \qquad (2.86a)$$

$$x_2(n+1) = A_{21}(\epsilon)\, x_1(n) + A_{22}(\epsilon)x_2(n) + B_2(\epsilon)u(n). \qquad (2.86b)$$

$x_1(0)$ and $x_2(0)$ are given and ϵ is a small positive parameter. All the matrices are analytic functions of ϵ and $|\lambda(A_{22})| < 1$. The control input $u(n)$ is assumed to be constant for $K/\epsilon \leq n < K+1/\epsilon$. Where K indicates the sampling points of slow states (see Figure 2.4). Matrices evaluated at $\epsilon = 0$ are denoted by deleting the argument ϵ, i.e., $A = A(0)$. The solution of (2.86) will be approximated by the solutions of slow and fast subproblems defined to describe the behavior of the slow and fast states, respectively.

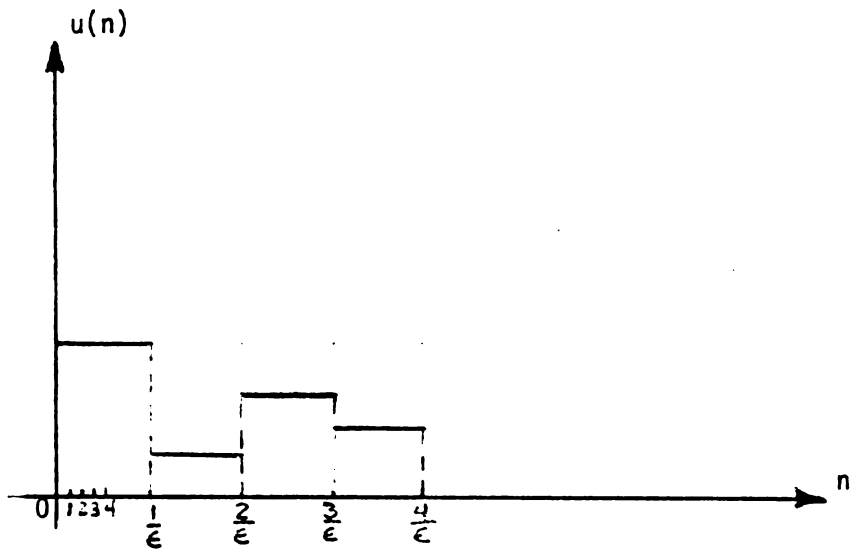


Figure 2.4

Slow subproblem

Assume that $x_2(n)$ has reached its steady state, then system (2.86) reduces to

$$\bar{x}_1(n+1) = [I + \epsilon A_{11}(\epsilon)]\bar{x}_1(n) + \epsilon A_{12}(\epsilon)\bar{x}_2(n) + \epsilon B_1(t)\bar{u}(n) \qquad (2.87a)$$

$$\bar{x}_2(n) = A_{21}(\epsilon)\bar{x}_1(n) + A_{22}(\epsilon)\bar{x}_2(n) + B_2(\epsilon)\bar{u}(n), \qquad (2.87b)$$

where bars show the steady state case and $\bar{u}(n) = U(n)$. From (2.87b) we have

$$\bar{x}_2(n) = [I_2-A_{22}(\epsilon)]^{-1}[A_{21}(\epsilon)\bar{x}_1(n) + B_2(\epsilon)\bar{u}(n)]. \tag{2.88}$$

So

$$\bar{x}_1(n+1) = [I + \epsilon(A_{11}(\epsilon) + A_{12}(\epsilon)(I_2-A_{22}(\epsilon))^{-1}A_{21}(\epsilon)]\bar{X}_1(n)$$
$$+ \epsilon[B_1(\epsilon) + A_{12}(\epsilon)(I_2-A_{22}(\epsilon))^{-1}B_2(\epsilon)]\bar{u}(n), \bar{x}_1(0) = x_0. \tag{2.89}$$

Since the matrices $A_{ij}(\epsilon)$ and $B_i(\epsilon)$ are analytic in ϵ, they can be approximated, up to $O(\epsilon)$, by their values at $\epsilon = 0$. Furthermore $(I_2-A_{22}(\epsilon))^{-1}$ can be approximated as

$$[I_2-A_{22}(\epsilon)]^{-1} = [(I_2-A_{22}) + O(\epsilon)]^{-1} = (I_2-A_{22})^{-1} + O(\epsilon).$$

Employing these approximation in (2.89) we obtain

$$\tilde{x}_1(N+1) = (I + \epsilon A_0)\tilde{x}_1(n) + \epsilon B_0\bar{u}(n),\ \tilde{x}_1(0) = x_1(0), \tag{2.90}$$

where

$$A_0 = A_{11} + A_{12}(I_2-A_{22})^{-1}A_{21}$$
$$B_0 = B_1 + A_{12}(I_2-A_{22})^{-1}B_2.$$

Note that systems (2.86) and the reduced system (2.90) evolve in the fast time-scale n.

Since $\bar{u}_1(n)$ is constant over the cycle $\frac{K}{\epsilon} \leq n < \frac{K+1}{\epsilon}$ we can express $\tilde{x}_1(\frac{K+1}{\epsilon})$ in terms of $\tilde{x}_1(\frac{K}{\epsilon})$ and $\bar{u}(K/\epsilon)$

$$\tilde{x}_1(\tfrac{K+1}{\epsilon}) = (I_1 + \epsilon A_0)^{1/\epsilon}\tilde{x}_1(K/\epsilon) + \sum_{j=K/\epsilon}^{\frac{K+1}{\epsilon}-1} [(I_1 + \epsilon A_0)^{\frac{K+1}{\epsilon}-1-j} \epsilon \cdot B_0]\bar{u}(K/\epsilon). \tag{2.91}$$

Letting $i = \frac{K+1}{\epsilon} -1-j$ we get

$$\tilde{x}_1(\tfrac{K+1}{\epsilon}) = (I_1 + \epsilon A_0)^{1/\epsilon}\tilde{x}_1(K/\epsilon) + \epsilon \sum_{i=0}^{1/\epsilon-1} (I_1 + \epsilon A_0)^{i} B_0\bar{u}(K/\epsilon).$$

Now let

$$x_s(K) = \tilde{x}_1(K/\epsilon), \quad u_s(K) = \bar{u}(K/\epsilon),$$

and

$$A_s = e^{A_0}, \quad B_s = \int_0^1 e^{A_0(1-t)} dt B_0 . \tag{2.92}$$

We define the slow subsystem to be

$$x_s(K+1) = A_s x_s(K) + B_s u_s(K), \quad x_s(0) = x_1(0). \tag{2.93}$$

Fast Subsystem:

Consider $x_1(n)$, $\bar{x}_2(n)$ to be constant during the fast transients. Let

$$x_f(n) = x_2(n) - \tilde{x}_2(n), \tag{2.94}$$

where

$$\tilde{x}_2(n) = (I-A_{22})^{-1}[A_{21}\bar{x}_1(n) + B_2\bar{u}(n)]. \tag{2.95}$$

The fast subproblem is defined to be

$$x_f(n+1) = A_{22}x_f(n) \quad , \quad x_f(K/\epsilon) = x_2(K/\epsilon)-\tilde{x}_2(K/\epsilon), \tag{2.96}$$

or

$$x_f(K/\epsilon) = x_2(K/\epsilon) - (I_2-A_{22})^{-1}[A_2\bar{x}_1(K/\epsilon) + B_2\bar{u}(\tfrac{K}{\epsilon})] \text{ for } K = 0,1,2,\ldots$$

We note that for every period of slow-time-scale the fast subsystem has a different initial condition due to $\tilde{x}_2(K/\epsilon)$ which depends on $\bar{x}_1(K/\epsilon)$. Figure 2.5 shows a typical shape of the response of $\bar{x}_1$ and x_f. At the beginning of every $1/\epsilon$ cycle, the fast modes are excited by the jump in u. Then x_f dacays exponentially towards zero.

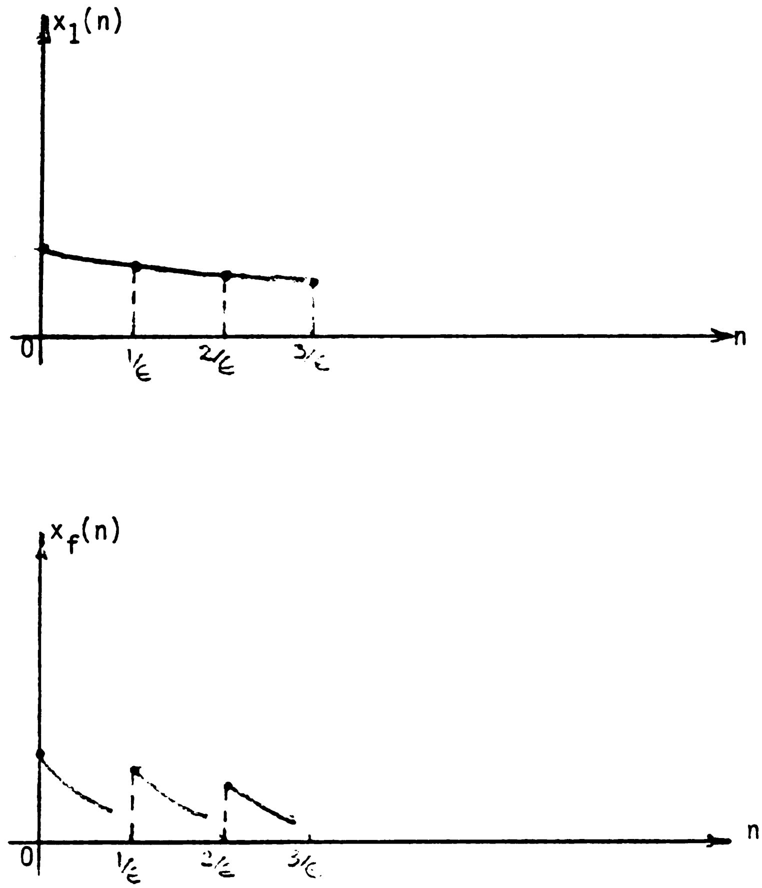


Figure 2.5

Let us apply the Chang transformation (2.25a) to the system (2.86) i.e.,

$$\begin{bmatrix} \eta_1(n) \\ \eta_2(n) \end{bmatrix} = \begin{bmatrix} I_1-\epsilon M(\epsilon)L(\epsilon) & -\epsilon M(\epsilon) \\ L(\epsilon) & I_2 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}, \tag{2.97}$$

where $L(\epsilon)$ and $M(\epsilon)$ satisfy

$$L(\epsilon) = A_{22}(\epsilon)L(\epsilon) - \epsilon L(\epsilon)A_{11}(\epsilon) + \epsilon L(\epsilon)A_{12}(\epsilon)L - A_{21}(\epsilon) \tag{2.98}$$

$$\begin{aligned} 0 = {} & M(\epsilon)A_{22}(\epsilon) - M(\epsilon) - \epsilon[A_{11}(\epsilon) - A_{12}(\epsilon)L(\epsilon)]M(\epsilon) + \epsilon M(\epsilon)L(\epsilon)A_{12}(\epsilon) \\ & - A_{12}(\epsilon), \end{aligned} \tag{2.99}$$

and could be approximated by

$$L(\epsilon) = -(I_2-A_{22})^{-1}A_{21} + 0(\epsilon) \tag{2.100}$$

$$M(\epsilon) = -A_{12}(I - A_{22})^{-1} + 0(\epsilon). \tag{2.101}$$

We get

$$\begin{aligned} \begin{bmatrix} \eta_1(n+1) \\ \eta_2(n+1) \end{bmatrix} = {} & \begin{bmatrix} I_1 + \epsilon A_0 + 0(\epsilon^2) & 0 \\ 0 & A_{22}+0(\epsilon) \end{bmatrix} \begin{bmatrix} \eta_1(n) \\ \eta_2(n) \end{bmatrix} \\ & + \begin{bmatrix} \epsilon B_0 + 0(\epsilon^2) \\ B_2 + 0(\epsilon) \end{bmatrix} u(n), \end{aligned} \tag{2.102}$$

where

$$A_0 = A_{11} + A_{12}(I-A_{22})^{-1}A_{21}$$

$$B_0 = B_1 + A_{12}(I-A_{22})^{-1}B_2$$

The solution for η_1 and η_2 for $K/\epsilon \leq n < \frac{K+1}{\epsilon}$ is

$$\eta_1(n) = [I + \epsilon A_0 + 0(\epsilon^2)]^{n-K/\epsilon} \eta_1(K/\epsilon) + \epsilon \sum_{i=0}^{n-1} [I + \epsilon A_0 + 0(\epsilon^2)]^i [B_0 + 0(\epsilon)]\, u(K/\epsilon) \tag{2.103}$$

$$\eta_2(n) = [A_{22} + 0(\epsilon)]^{n-K/\epsilon} \eta_2(K/\epsilon) + \sum_{i=0}^{n-1} [A_{22} + 0(\epsilon)]^i [B_2 + 0(\epsilon)] u(K/\epsilon). \tag{2.104}$$

From (2.103) we obtain

$$\eta_1(\tfrac{K+1}{\epsilon}) = [I + \epsilon A_0 + 0(\epsilon^2)]^{1/\epsilon} \eta_1(\tfrac{K}{\epsilon}) + \epsilon \sum_{i=0}^{\frac{1}{\epsilon}-1} [I + \epsilon A_0 + 0(\epsilon^2)]^i [B_0 + 0(\epsilon)].\, u(K/\epsilon). \tag{2.105}$$

Using the following identities

$$[I + \epsilon A + 0(\epsilon^2)]^i = [I + \epsilon A]^i + 0(\epsilon)\;,\quad 0 \leq i \leq 1/\epsilon \tag{2.106}$$

$$[I + \epsilon A]^{1/\epsilon} = e^A + 0(\epsilon) \tag{2.107}$$

$$\epsilon \sum_{i=0}^{\frac{1}{\epsilon}-1} (I + \epsilon A)^i = \int_0^1 e^{A(1-t)}dt + 0(\epsilon) \tag{2.108}$$

(for proofs see Appendixes 2.2, 2.3 and 2.4, respectively) and the continuous dependence of the solution of difference equations on parameters (see Appendix 2.5) we get

$$\eta_1(K/\epsilon) = x_s(K) + O(\epsilon). \tag{2.109}$$

Now Chang transformation yields

$$x_1(K/\epsilon) = x_s(K) + O(\epsilon). \tag{2.110}$$

(2.110) shows that the slow trejectories x_1 could be approximated up to $O(\epsilon)$ by the solution of the slow subproblem.

We define the steady state of $\eta_2(n)$ as follow

$$\bar{\eta}_2 = [A_{22}+O(\epsilon)]\bar{\eta}_2 + [B_2+O(\epsilon)]u(\tfrac{K}{\epsilon}), \ \tfrac{K}{\epsilon} \leq n < \tfrac{K+1}{\epsilon} . \tag{2.111}$$

Subtraction of (2.111) from (2.102) (η_2 - equation) and letting $\tilde{\eta}_2 = \eta_2 - \bar{\eta}_2$ yields

$$\tilde{\eta}_2(n+1) = [A_{22} + O(\epsilon)]\tilde{\eta}_2(n) \ , \ \tfrac{K}{\epsilon} \leq n < \tfrac{K+1}{\epsilon} , \tag{2.112}$$

with initial conditions

$$\tilde{\eta}_2(K/\epsilon) = \eta_2(K/\epsilon) - \bar{\eta}_2(K/\epsilon), \ K = 0,1,2,...$$

$$= - (I-A_{22})^{-1}[A_{21}x_1(K/\epsilon) + B_2u(K/\epsilon)] + x_2(K/\epsilon) + O(\epsilon). \tag{2.113}$$

Comparing (2.112), (2.113) and (2.96) and using again the continuous-dependence of solutions on parameters we obtain

$$\tilde{\eta}_2(n) = x_f(n) + O(\epsilon) \ , \ \frac{K}{\epsilon} \leq n < \frac{K+1}{\epsilon} \ . \tag{2.114}$$

But for $K/\epsilon \leq n < \frac{K+1}{\epsilon}$ we have

$$x_2(n) = (I-A_{22})^{-1}\eta_1(n) + \eta_2(n) + O(\epsilon), \tag{2.115}$$

or

$$\begin{aligned} x_2(n) &= (I-A_{22})^{-1}A_{21}\tilde{\eta}_1(n) + x_f(n) + \bar{\eta}_2(n) + O(\epsilon) \\ &= (I-A_{22})^{-1}A_{21}\tilde{\eta}_1(n) + x_f(n) + (I-A_{22})^{-1}B_2u(K/\epsilon) + O(\epsilon), \end{aligned} \tag{2.116}$$

where $\tilde{\eta}_1(n)$ is an $O(\epsilon)$-approximation of $\eta_1(n)$ which is given by

$$\tilde{\eta}_1(n) = (I + \epsilon A_0)^{n-K/\epsilon}\eta_1(K/\epsilon) + \epsilon \sum_{i=0}^{n-1} (I + \epsilon A_0)^i B_0 u(K/\epsilon), \ \frac{K}{\epsilon} \leq n < \frac{K+1}{\epsilon}. \tag{2.117}$$

Thus, we can express $x_2(n)$ in terms of the fast and slow states and control input.

Based on our discussion above, the following theorem holds.

Theorem 2.5.2: If $|\lambda(A_{22})| < 1$, then for all finite $K \geq 0$ the solution of (2.86) can be approximated by

$$x_1(K/\epsilon) = x_s(K) + O(\epsilon)$$

$$x_2(n) = (I-A_{22})^{-1}A_{21}[(I + \epsilon A_0)^{n-K/\epsilon}x_s(K) + \epsilon \sum_{i=0}^{n-1} (I + \epsilon A_0)^i B_0 u(K/\epsilon)]$$

$$+ x_f(n) + (I-A_{22})^{-1}B_2 u(K/\epsilon) + O(\epsilon), \ \frac{K}{\epsilon} \leq n < \frac{K+1}{\epsilon},$$

where x_s and x_f are solutions of the slow subsystem (2.93) and the fast subsystem (2.96), respectively. Moreover, if Re $\lambda(A_0) < 0$, the above theorem holds for all $K \geq 0$.

APPENDIX 2.1

<u>Householder Theorem</u>:

If A is a nonsingular matrix then [Householder, 1964]

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(I + CDA^{-1}B)^{-1}CDA^{-1}.$$

For $C = V^{-1}$ we have

$$(A + BV^{-1}D)^{-1} = A^{-1} - A^{-1}B[V + DA^{-1}B]^{-1}DA^{-1} .$$

In particular when $B = \epsilon I$ and $V = I$ we have

$$(A + \epsilon D)^{-1} = A^{-1} - \epsilon A^{-1}[I + \epsilon DA^{-1}]^{-1} DA^{-1} .$$

If all the matrices are $O(1)$, then we obtain

$$(A + \epsilon D)^{-1} = A^{-1} + O(\epsilon) .$$

APPENDIX 2.2

Wish to prove that for any positive integer $j \leq 1/\epsilon$

$$\|(I + \epsilon A + \epsilon^2 B(\epsilon))^j - (I + \epsilon A)^j\| = O(\epsilon). \qquad (1)$$

Let L.H.S. of (1) = T_j , $\|A\| = a$ and $\|B(\epsilon)\| \leq b$. Then

$$T_j \leq \sum_{i=1}^{j} \epsilon^i \binom{j}{i} (1 + \epsilon a)^{j-i} (\epsilon b)^i. \qquad (2)$$

(2) can be shown by mathematical induction. Apparently, $T_1 \leq \epsilon^2 b$.

Suppose that (2) is valid for (j-1). Then

$$T_j = \|(I + \epsilon A + \epsilon^2 B)(I + \epsilon A + \epsilon^2 B)^{j-1} - (I + \epsilon A)(I + \epsilon A)^{j-1}\|$$

$$\leq (1 + \epsilon a)T_{j-1} + \epsilon^2 b(1 + \epsilon a + \epsilon^2 b)^{j-1}$$

$$\leq (1 + \epsilon a) \sum_{i=1}^{j-1} \epsilon^i \binom{j-1}{i} (1 + \epsilon a)^{j-1-i} (\epsilon b)^i$$

$$+ \epsilon^2 b \sum_{r=0}^{j-1} \epsilon^r \binom{j-1}{r} (1 + \epsilon a)^{j-1-r} (\epsilon b)^r$$

$$= \sum_{i=1}^{j-1} \epsilon^i \binom{j-1}{i} (1 + \epsilon a)^{j-i} (\epsilon b)^i + \sum_{r=0}^{j-1} \epsilon^{r+1} \binom{j-1}{r} (1+\epsilon a)^{j-1-r} (\epsilon b)^{r+1}$$

$$= \sum_{i=1}^{j-1} \epsilon^i \binom{j-1}{i} (1 + \epsilon a)^{j-1} (\epsilon b)^i + \sum_{i=1}^{j} \epsilon^i \binom{j-1}{i-1} (1+\epsilon a)^{j-i} (\epsilon b)^i$$

$$= \sum_{i=1}^{j-1} \epsilon^i [\binom{j-1}{i} + \binom{j-1}{i-1}](1 + \epsilon a)^{j-i}(\epsilon b)^i + \epsilon^j(\epsilon b)^j$$

$$= \sum_{i=1}^{j-1} \epsilon^i \binom{j}{i}(1 + \epsilon a)^{j-i}(\epsilon b)^i + \epsilon^j(\epsilon b)^j = \sum_{i=1}^{i} \epsilon^i \binom{j}{i}(1 + \epsilon a)^{j-i} (\epsilon b)^i,$$

which proves (2). Now, using (2) we have

$$T_j \le \sum_{i=1}^{j} \epsilon^i \binom{j}{i} (1 + \epsilon a)^{j-i}(\epsilon b)^i$$

$$= (1 + \epsilon a)^j \sum_{i=1}^{j} \epsilon^i \binom{j}{i} \left(\frac{\epsilon b}{1+\epsilon a}\right)^i$$

$$\le (1 + \epsilon a)^{1/\epsilon} \sum_{i=1}^{j} \epsilon^i \binom{j}{i} (\epsilon b)^i.$$

Since $\lim_{\epsilon \to 0} (1 + \epsilon a)^{1/\epsilon} = e^a$, there exists $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$

$$(1 + \epsilon a)^{1/\epsilon} \le K_1 e^a \triangleq K .$$

So

$$T_j \le K \sum_{i=1}^{j} \epsilon^i \binom{j}{i} (\epsilon b)^i \qquad j \le 1/\epsilon$$

$$= K \sum_{i=1}^{j} (\epsilon^2 b)^i \binom{j}{i}$$

$$= K[\sum_{i=0}^{j} (\epsilon^2 b)^i \binom{j}{i} - \binom{j}{0}]$$

$$= K[(1 + \epsilon^2 b)^j - 1]$$

$$\le K[(1 + \epsilon^2 b)^{1/\epsilon} - 1]$$

$$= K[1 + \frac{1}{\epsilon} \epsilon^2 b + \frac{\frac{1}{\epsilon}(\frac{1}{\epsilon} - 1)}{2!} \epsilon^4 b^2 + \dots -1]$$

$$= K[1 + \epsilon b + \frac{(1-\epsilon)}{2!} \epsilon^2 b^2 + \frac{(1-\epsilon)(1-2\epsilon)}{3!} \epsilon^3 b^3 + \cdots - 1]$$

$$\leq K[1 + \epsilon b + \epsilon^2 b^2 + \epsilon^3 b^3 + \cdots \quad -1]$$

$$= K \epsilon b[1 + (\epsilon b) + \epsilon^2 b^2 + \cdots]$$

$$= K \frac{\epsilon b}{1-\epsilon b} \leq K_2 \epsilon .$$

Q.E.D.

APPENDIX 2.3

Wish to prove $\|(I + \epsilon A)^{1/\epsilon} - e^{A}\| = O(\epsilon)$.

Let $\|A\| = a$,

and let $$Y = Ln(I + \epsilon A)^{1/\epsilon} = \frac{1}{\epsilon} Ln(I + \epsilon A)$$

$$= \frac{1}{\epsilon}[\epsilon A - \frac{1}{2} (\epsilon A)^2 + \frac{1}{3} (\epsilon A)^3 - \cdots]$$

$$= A - \frac{1}{2} \epsilon A^2 + \frac{1}{3} \epsilon^2 A^3 - \cdots$$

$$= A + \epsilon[- \frac{1}{2} A^2 + \frac{1}{3} \epsilon A^3 - \frac{1}{4} \epsilon^2 A^4 + \ldots] .$$

Inside the bracket, by ratio test, is convergent series and bounded since

$$\| - \frac{1}{2} A^2 + \frac{\epsilon A^3}{3} - \frac{\epsilon^2 A^4}{4} + \cdots\| \leq \frac{1}{2} a^2 + \frac{1}{3} \epsilon a^3 + \frac{1}{4} \epsilon^2 a^4 + \cdots$$

$$< a^2 + \epsilon a^3 + \epsilon^2 a^4 + \cdots = a^2 (1 + \epsilon a + (\epsilon a)^2 + \cdots)$$

$$= a^2 \frac{1}{1-\epsilon a} = \frac{a^2}{1-\epsilon a} .$$

So we have

$$Y = A + O(\epsilon),$$

and

$$(I + \epsilon A)^{1/\epsilon} = e^{Y} = e^{A+0(\epsilon)} = I + (A + 0(\epsilon) + \frac{1}{2!} (A + 0(\epsilon))^2 + \dots .$$

Applying norms we obtain

$$\|(I + \epsilon A)^{1/\epsilon} - e^{A}\|$$

$$= \| I + A + 0(\epsilon) + \frac{(A+0(\epsilon))^2}{2!} + \frac{(A+0(\epsilon))^3}{3!} + \dots - (I + A + \frac{A^2}{2!} + \frac{A^3}{3} + \dots)\|$$

$$\leq 0(\epsilon)[1 + \frac{2a}{2!} + \frac{3a^2}{3!} + \frac{4a^3}{4!} + \cdots]$$

$$+ 0(\epsilon^2)[\frac{1}{2!} + \frac{1}{3!}\binom{3}{2} a + \frac{1}{4!}\binom{4}{2} a^2 + \cdots + \frac{1}{K!}\binom{K}{2} a^{K-2} + \cdots]$$

$$+ 0(\epsilon^3)[\frac{1}{3!}\binom{3}{3} + \frac{1}{4!}\binom{4}{3} a + \frac{1}{5!}\binom{5}{3} a^2 + \cdots + \frac{1}{K!}\binom{K}{3} a^{K-3} + \cdots]$$

$$\vdots$$

$$+ 0(\epsilon^i)[\frac{1}{i!} + \frac{1}{(i+1)!}\binom{i+1}{i} + \frac{1}{(i+2)!}\binom{i+2}{i} + \cdots]$$

$$+$$

$$\vdots$$

$$= 0(\epsilon)[1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots]$$

$$+ 0(\epsilon^2)\frac{1}{2!}[1 + a + \frac{a^2}{2!} + \cdots \frac{a^{K-2}}{(K-2)!} + \cdots]$$

$$+ 0(\epsilon^3)\frac{1}{3!}[1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots]$$

$$+$$

$$\vdots$$

$$+ 0(\epsilon^i)\frac{1}{i!}[1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots + \frac{a^{K-i}}{(K-i)!} + \cdots]$$

$$+ \\ \vdots \\ = O(\epsilon)\, e^{a} + O(\epsilon^{2})\, \frac{e^{a}}{2!} + O(\epsilon^{3})\, \frac{e^{a}}{3!} + \cdots = O(\epsilon).$$

Q.E.D.

APPENDIX 2.4

Wish to prove that for sufficiently small ϵ

$$\epsilon \sum_{i=0}^{1/\epsilon -1} (I + \epsilon A)^i = \int_0^1 e^{(1-\tau)A} d\tau + O(\epsilon) . \qquad (1)$$

We have

$$\int_0^1 e^{A(1-\tau)} d\tau = \sum_{i=1}^{1/\epsilon} \int_{\epsilon(i-1)}^{\epsilon i} e^{A(1-\tau)} d\tau. \qquad (2)$$

Let $t = \epsilon i - \tau$, then the above is equal to

$$\sum_{i=1}^{1/\epsilon} \int_0^{\epsilon} e^{A(1-\epsilon i + t)} dt = \sum_{i=1}^{1/\epsilon} e^{A(1-\epsilon i)} \int_0^{\epsilon} e^{At} dt. \qquad (3)$$

But

$$\int_0^{\epsilon} e^{At} dt = \int_0^{\epsilon} [I + At + \frac{(At)^2}{2!} + \cdots]dt = It + \frac{At^2}{2!} + \frac{A^2t^3}{3!} + \cdots \Big|_0^{\epsilon}$$

$$= \epsilon I + \epsilon^2 \frac{A}{2!} + \epsilon^3 \frac{A^2}{3!} + \cdots$$

$$= \epsilon I + \epsilon^2 [\frac{A}{2!} + \frac{\epsilon A^2}{3!} + \cdots].$$

Using the ratio test on the norms and letting $\|A\| = a$ yields

$$\left\|\frac{A}{2!} + \frac{\epsilon A^2}{3!} + \cdots \right\| \leq \frac{a}{2!} + \frac{\epsilon a^2}{3!} + \frac{\epsilon^2 a^3}{4!} + \cdots < a[1 + \epsilon a + (\epsilon a)^2 + \cdots]$$

$$= \frac{a}{1-\epsilon a} .$$

Thus

$$\int_0^{\epsilon} e^{At}dt = \epsilon I + 0(\epsilon^2). \tag{4}$$

Also,

$$\sum_{i=1}^{1/\epsilon} e^{A(1-\epsilon i)} = I + e^{\epsilon A} + e^{2\epsilon A} + \ldots + e^{\epsilon(1/\epsilon - 1)A} . \tag{5}$$

Thus

$$\int_0^1 e^{A(1-\tau)}d\tau = \epsilon[I + e^{\epsilon A} + \cdots + e^{\epsilon(\frac{1}{\epsilon} - 1)A}][I + 0(\epsilon)],$$

$$= \epsilon \sum_{i=0}^{1/\epsilon - 1} e^{\epsilon i A}[I + 0\ (\epsilon)] . \tag{6}$$

On the other hand [Hoppensteadt and Miranker, 1977]

$$(I + \epsilon A)^i = e^{\epsilon i A}[I + 0(\epsilon)] . \tag{7}$$

So

$$\epsilon \sum_{i=0}^{\frac{1}{\epsilon}-1} (I + \epsilon A)^i = \epsilon \sum_{i=0}^{1/\epsilon - 1} e^{\epsilon i A}[I + 0(\epsilon)] . \tag{8}$$

Thus

$$\|(8) - (6)\| = 0(\epsilon) .$$

Q.E.D.

APPENDIX 2.5

Lemma: Consider the difference equations

$$x(k+1) = A\,x(k),\ x(0) = x_0, \qquad (1)$$

and

$$y(K+1) = (A+E)\,y(k),\ y(0) = y_0, \qquad (2)$$

where $\|E\| = O(\varepsilon)$ and $\|x_0 - y_0\| - O(\varepsilon)$.

The following assertions are true

(i) $\|x(k) - y(k)\| = O(\varepsilon)$ for all finite K

(ii) If $|\lambda(A)| \leq \beta < 1$, then $\|x(k) - y(k)\| \leq C \in \alpha^K$ for all K. where $0 < \alpha < 1$, Hence $\|x(K) - y(K)\| \to 0$ as $K \to \infty$.

Proof: The assertion is obvious in the case of finite time. We prove it only for the infinite-time case. From the continuous dependence of the eigenvlaues of a matrix on its parameters there exists $\varepsilon^* > 0$ such that for all $\varepsilon < \varepsilon^*$ the eigenvalues of $(A + E)$ satisfy

$$|\lambda(A+E)| \leq \beta_1 < 1 .$$

Moreover, for any $\delta > 0$ there exists a matrix norm $|\cdot|$ such that

$$|(A+E)| \leq |\lambda(A+E)| + \delta \leq \beta_1 + \delta .$$

Choosing $\delta = (1-\beta_1)/2$ we get

$$|(A + E)| \leq \beta_2 < 1 , \qquad (3)$$

where $\beta_2 = (1 + \beta_1)/2$

Subtracting (1) from (2), letting $g_k = |y(k) - x(k)|$ and using the equivalence of norms we get

$$g_{K+1} = |y(K+1) - x(K+1)| = |(A+E)y(K) - Ax(K)|$$

$$= |(A+E)(y(K) - x(K)) + E\, x(K)|$$

$$\leq \beta_2\, g_K + |E|\ |x(K)|$$

$$\leq \beta_2\, g_K + C_1 \epsilon\, |x_0| \beta^K .$$

So we have

$$g_{K+1} \leq \beta_2\, g_K + C_2 \epsilon\, \beta^K \quad ; \qquad \beta < 1,\ \beta_2 < 1 .$$

From above we get

$$g_1 \leq \beta_2 g_0 + C_2 \epsilon$$

$$g_2 \leq \beta_2 g_1 + C_2 \epsilon \beta \leq \beta_2^2 g_0 + \beta_2 C_2 \epsilon + C_2 \epsilon \beta$$

$$\vdots$$

$$g_n \leq \beta_2^n g_0 + C_2 \epsilon \sum_{K=0}^{n-1} \beta_2^K \beta^{n-1-K} .$$

But

$$\sum_{K=0}^{n-1} \beta_2^K \beta^{n-1-K} = \beta^{n-1} \sum_{K=0}^{n-1} \left(\frac{\beta_2}{\beta}\right)^K$$

$$= \beta^{n-1} \frac{1-(\beta_2/\beta)^n}{1-\frac{\beta_2}{\beta}} \,. \tag{4}$$

If $\beta_2 < \beta$, then we have

$$\beta^{n-1} \frac{1-(\beta_2/\beta)^n}{1-\beta_2/\beta} = \frac{\beta^n - \beta_2^n}{\beta - \beta_2} \leq K \beta^n \,.$$

If $\beta_2 > \beta$, Then

$$(4) = \beta^{n-1} \frac{(\frac{\beta_2}{\beta})^n - 1}{\frac{\beta_2}{\beta} - 1} = \frac{\beta_2^n - \beta^n}{\beta_2 - \beta}$$

$$= \beta_2^{n-1} \frac{1-(\beta/\beta_2)^n}{1-\beta/\beta_2} \leq K \beta_2^n \,.$$

If $\beta_2 = \beta$, then

$$(4) = \sum_{K=0}^{n-1} \beta^{n-1} = n\, \beta^{n-1} = n(\sqrt{\beta})^{n-1}(\sqrt{\beta})^{n-1} \leq K\,(\sqrt{\beta})^{n-1} \quad \text{since} \quad \beta < 1.$$

So we have

$$g_n \leq \beta_2^n g_0 + \epsilon C' \alpha^n \quad , \qquad \alpha < 1,$$

where C is a constant of $O(1)$.

So $g_n \to 0$ as $n \to \infty$.

Q.E.D.

CHAPTER 3

INFINITE-TIME OPTIMAL REGULATORS FOR SINGULARLY PERTURBED DIFFERENCE EQUATIONS

3.1. Introduction

Optimal control of infinite-time regulators for singularly perturbed continuous-time systems have been discussed by Chow and Kokotovic [1976] and the role of slow - fast decomposition and composite control in approximating the performance index are thoroughly investigated. Also, Blankenship has considered the optimal control of finite-time linear quadratic regulators of singularly perturbed difference equations and showed that the basic features of singular perturbation theory can be extended to the class of discrete systems as well.

Section 3.2 is devoted to discussing their work and, hopefully, provide the reader with a better understanding of the problem and our motivation.

In Section 3.3 the problem statement is introduced and, more specifically, the source of difficulty in applying Blankenship's approach is addressed.

Section 3.4 deals with asymptotic behavior of the optimal solution of linear quadratic regulators and some improtant theorems are proved which removes the difficulty in dealing with the problem mentioned in Section 3.3.

In Section 3.5 it is shown that a near-optimal solution can be obtained by applying slow-fast decompositions as in the continuous-time work of Chow and Kokotovic [1976]. In our case the slow optimal control problem will be a continuous-time problem, while the fast optimal control problem will be a discrete-time problem.

In Section 3.6 we present an interative technique to slove the discrete-time stiff Riccati equations which avoids the ill-conditioning problem and provides a high convergence rate.

Finally, our claims are illustrated, by considering a simple example, in Section 3.7.

3.2. Related Background

Chow and Kokotovic [1976] investigated a near-optimum state regulator for singularly perturbed continuos-time systems which is composed of the slow and fast subsystem regulators and showed that a second-order approximation of the optimal performance is achieved. Also, they formulated a complete separation of slow and fast regulator designs.

Due to close anology of our problem with the continuous-time problem we give a brief explanation of their work.

They consider the continuous-time singularly perturbed system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u, \; x_1(0) = x_{10} \tag{3.1a}$$

$$\epsilon\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u, \; x_2(0) = x_{20} \tag{3.1b}$$

$$y = C_1x_1 + C_2x_2 \, , \tag{3.1c}$$

where ϵ is a small positive scalar, the state x is formed by m_1 and m_2 vectors x_1, x_2. The control u is an r vector and the output y is a K vector.

The performance index is

$$\min J = \frac{1}{2}\int_0^\infty (y^T y + u^T Ru)dt, \quad R > 0. \tag{3.2}$$

Optimal control is given by

$$u_{opt} = -R^{-1}B^T Kx, \tag{3.3}$$

where K is the stabilizing solution of the Riccati equation

$$0 = -KA-A^T K + KSK-C^T C \tag{3.4}$$

with $C = [C_1 \; C_2]$, $S = BR^{-1}B^T$, and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix}.$$

By assuming A_{22} is nonsingular, the slow and fast subsystems are defined (see Section 2.2) with their performance indexes.

Slow regulator problem:

For the slow subsystem

$$\dot{x}_s = A_0 x_s + B_0 u_s, \qquad x_s(0) = x_{10} \tag{3.5a}$$

$$y_s = C_0 x_s + D_0 u_s, \tag{3.5b}$$

where

$$A_0 = A_{11}-A_{12}A_{22}^{-1}A_{21} , \quad B_0 = B_1-A_{12}A_{22}^{-1}B_2$$

$$C_0 = C_1-C_2A_{22}^{-1}A_{21} , \quad D_0 = -C_2A_{22}^{-1}B_2, \qquad (3.5c)$$

find u_s to minimize

$$J_s = \frac{1}{2}\int_0^\infty (y_s^T y_s + u_s^T R u_s)dt, \quad R > 0 . \qquad (3.6)$$

In terms of x_s and u_s, (3.6) becomes

$$J_s = \frac{1}{2}\int_0^\infty [x_s^T C_0^T C_0 x_s + 2u_s^T D_0^T C_0 x_s + u_s^T R_0 u_s]dt , \qquad (3.7)$$

where

$$R_0 = R + D_0^T D_0 .$$

They prove that if the triple (A_0,B_0,C_0) is stabilizable-detectable, then the Riccati equation

$$0 = - K_s(A_0-B_0R_0^{-1}D_0^TC_0) - (A_0-B_0R_0^{-1}D_0^TC_0)^TK_s + K_sB_0R_0^{-1}B_0^TK_s$$
$$- C_0^T(I-D_0R_0^{-1}D_0^T)C_0 \qquad (3.8)$$

has a positive semidefinite stabilizing solution K_s and the optimal control for (3.5) and (3.6) is

$$u_s = -R_0^{-1}(D_0^TC_0 + B_0^TK_s)x_s . \qquad (3.9)$$

Fast regulator problem:

For the fast subsystem

$$\epsilon\dot{x}_f = A_{22}x_f + B_2u_f \quad , \quad x_f(0) = x_{20} - \bar{x}_2(0) \tag{3.10a}$$

$$y_f = C_2x_f \ , \tag{3.10b}$$

where

$$X_f = x_2 - \bar{x}_2 \quad , \quad u_f = u - u_s \quad \text{and} \quad y_f = y - y_s,$$

find u_f to minimize

$$J_f = \frac{1}{2}\int_0^\infty (y_f^T y_f + u_f^T R u_f)dt, \ R > 0. \tag{3.11}$$

It is also shown that if the triple (A_{22},B_2,C_2) is stabilizable-detectable, then the Riccati equation

$$0 = -K_fA_{22}-A_{22}^TK_f + K_fB_2R^{-1}B_2^TK_f-C_2^TC_2 \tag{3.12}$$

has a positive semidefinite stabilizing solution K_f and the optimal control for (3.10) and (3.11) is

$$u_f = -R^{-1}B_2^TK_fx_f. \tag{3.13}$$

It is shown that, under the stabilizability-detectability of slow and fast subsystems, the solution of the Riccati Equation (3.4) possesses a power series expansion at $\epsilon = 0$ that is,

$$K = \begin{bmatrix} K_1 & \epsilon K_2 \\ \epsilon K_2^T & \epsilon K_3 \end{bmatrix} + \sum_{i=1}^{\infty} \frac{\epsilon^i}{i!} \begin{bmatrix} K_1^{(i)} & \epsilon K_2^{(i)} \\ \epsilon K_2^{(i)^T} & \epsilon K_3^{(i)} \end{bmatrix}, \tag{3.14}$$

and

$$K_1 = K_s, \ K_3 = K_f \ \text{ and } \ K_2 = f(K_s,K_f). \tag{3.15}$$

Furthermore, it is proved that for the composite feedback control

$$u_C = -[(I-R^{-1}B_2^TK_fA_{22}^{-1}B_2)R_0^{-1}(D_0^TC_0 + B_0^TK_s) + R^{-1}B_2^TK_fA_{22}^{-1}A_{21}]x_1 - R^{-1}B_2^TK_fx_2, \tag{3.16}$$

$$u_{opt} = u_C + O(\epsilon), \tag{3.17}$$

and

$$J = J_C + O(\epsilon^2), \tag{3.18}$$

where J_C is the value of performance index J of system (3.1) with u_C and hence the composite feedback control (3.16) is an $O(\epsilon^2)$ near-optimal solution to the complete regulator problem (3.1), (3.2).

Blankenship [1981] studied linear quadratic optimal control problems for singularly perturbed difference equations when the cost function is defined on a finite-time period. In particular he considered the system

$$x(n+1) = x(n) + \epsilon Ax(n) + \epsilon BZ(n) + \epsilon Fu(n) \tag{3.19a}$$

$$Z(n+1) = SZ(n) + \epsilon CX(n) + \epsilon DZ(n) + (G+\epsilon H)u(n), \tag{3.19b}$$

where

$$x(0) = x_0 \quad \text{and} \quad Z(0) = Z_0.$$

x and Z are m_1- and m_2-dimensional state vectors and the control input u is r-dimensional. All the matrices are constant matrices of appropriate dimensions. $\epsilon > 0$ is a parameter and $n = 0,1,2,\dots,N-1$. The performance index to be minimized is

$$J_r(u) = X^T(N)K_1X(N) + 2X^T(N)K_2Z(N) + Z^T(N)K_3Z(N) + \sum_{K=r}^{N-1}[u^T(K)Ru(K) + X^T(K)Q_1X(K) + 2X^T(K)Q_2Z(K) + Z^T(K)Q_3Z(K)],\ 0 \le r \le N-1, \tag{3.20}$$

where

$$R = R^T > 0, \quad K = \begin{bmatrix} K_1 & K_2 \\ K_2^T & K_3 \end{bmatrix} = K^T \geq 0 .$$

$Q = Q^T \geq 0$ with Q defined from Q_i, $i = 1,2,3$ in the same way.

Slow-fast decomposition of system (3.1) was discussed in Chapter 2. He, essentially, showed that the basic features of the singular perturbation approach to continuous-time can be extended to discrete-time ones. We extend on his work and consider the infinite-time optimal control problem and study the asymptotic behavior of the resulting algebraic Riccati equations as the perturbation parameter tends to zero.

It is shown in Section 3.3 that the asymptotic behavior of the infinite-time problem does not follow as a limiting case of the finite-time problem. A special scaling of the solution of the Riccati equation is employed and is shown to be appropriate to expand solution as a series in the perturbation parameter.

3.3. Problem Statement

Consider the linear time-invariant discrete-time system

$$x(n+1) = [illegible] \, n) + \epsilon \, Bz(n) + \epsilon \, Fu(n) \tag{3.21a}$$

$$z(n+1) = [illegible] - Sz(n) + Gu(n) \tag{3.21b}$$

$$y(n) = [illegible] - D_2 z(n) + Mu(n), \tag{3.21c}$$

where $\epsilon > 0$ is a small positive parameter, the state vector comprises the m_1 and m_2 dimensional vectors x and z, the control u is an r-dimensional vector and the output y is a k vector. The initial states $x(0)$ and $z(0)$ are given. The controls $u(n)$ are to be selected to minimize the performance index

$$J = \varepsilon \sum_{n=0}^{\infty} [y^T(n)y(n) + u^T(n)R\,u(n)], \quad R = R^T > 0. \qquad (3.22)$$

For simplicity the matrices A, B, C, F, G, S, D_1, D_2, M and R are taken to be independent of ϵ but they could be analytic functions of ε and the problem would be treated in the same way. The finite-time version of this problem was considered by Blankenship who investigated the asymptotic behavior of the optimal solution as $\epsilon \to 0$. The asymptotic behavior of the infinite-time problem we are discussing here does not follow form Blankenship's study as a limiting case when the terminal time N tends to ∞. To see this observe that he gives the solution to the Riccati equation by defining

$$V(x(n), Z(n), n, \epsilon) = \min_u [J_n(u)] ,$$

where V is the "cost to go" from the point $(X(n),Z(n))$ at time n in the problem (3.19), (3.20).

He proves that V has a Hamilton-Jacobi equation which has solution

$$V(X,Z,n) = X^T P_n^1 X + 2X^T P_n^2 Z + Z^T P_n^3 Z , \qquad (3.23)$$

where $P_n = (P_n^1, P_n^2, P_n^3)$ satisfies

$$P_n^3 = Q_3 + F^3(P_{n+1}^3) + O(\epsilon) \qquad (3.24)$$

$$P_n^2 = Q_2 + F^2(P_{n+1}^2, P_{n+1}^3) + O(\epsilon) \qquad (3.25)$$

$$P_n^1 = P_{n+1}^1 + Q_1 + F^1(P_{n+1}^2, P_{n+1}^3) + O(\epsilon) \qquad (3.26)$$

$$P_N = K_i, \; i = 1,2,3 \; , \; n = 0,1,\ldots,N-1. \qquad (3.27)$$

Under proper conditions, he gives the solution to P_n up to $O(\epsilon^2)$ for sufficiently small ϵ. In particular the solution for P_n^1 is given by

$$P_n^1 = P^{10}(\epsilon n) + \ell_n^{10} + (N-n)[Q_1 + F^1(\bar{P}^2, \bar{P}^3)] + O(\epsilon), \qquad (3.28)$$

where P^{10} indicates the zero order term of P^1 and ℓ_n^{10} represents the zero order term of the boundary layer solution. Note that (3.28) gives an asymptotic formula for the solution of the associated Riccati equation which is proportional to N, so it blows up as $N \to \infty$. By appropriately scaling the solution of the Riccati equaiton, similar to (3.14), in the next section we will be able to overcome this difficulty.

The infinite-time regulator problem (3.21) (3.22) could be a result of discretization or sampled-data control of infinite-time regulators for singularly perturbed continuous-time systems using the method of [Levis and Dorato, 1971]; the details are similar to the finite-time example presented by Blankenship [1981] and explained in Section 2.4. The form of the performance index is a little bit more complicated than the one studied by Blankenship because of the presence of the matrix M. When $M = 0$, the performance index J reduces to the one studied by Blankenship. The current form is chosen to accommodate the sampled-data control case where J is obtained by discretizing an integral performance index of a continuous-time system.

It was shown by [Levis and Dorato, 1971], (See Appendix 3.1), that discretizing an integral performance index with quadratic terms in the state and control (and no cross product terms) results in a discrete-time performance index of the chosen form with $M \neq 0$.

3.4. Asymptotic Behavior of the Optimal Solution

The optimal control of the system (3.21) with performance index (3.22) is given by [Levis and Dorato, 1971]

$$u_{opt}(n) = -[R+M^TM+\mathbb{B}^TP\mathbb{B}]^{-1}[\mathbb{B}^TP\mathbb{A}+M^TD] \begin{bmatrix} x(n) \\ z(n) \end{bmatrix}, \tag{3.29}$$

where P is a stabilizing solution of the discrete-time algebraic Riccati equation

$$P = D^TD+\mathbb{A}^TP\mathbb{A}-[\mathbb{A}^TP\mathbb{B}+D^TM][R+M^TM+\mathbb{B}^TP\mathbb{B}]^{-1}[\mathbb{B}^TP\mathbb{A}+M^TD] , \tag{3.30}$$

and where

$$\mathbb{A} = \begin{bmatrix} I+\epsilon A & \epsilon B \\ C & S \end{bmatrix}, \qquad \mathbb{B} = \begin{bmatrix} \epsilon F \\ G \end{bmatrix}, \quad D = D_1, D_2 .$$

In studying the asymptotic behavior of the Riccati equation (3.30), we seek the matirx P in the form

$$P = \begin{bmatrix} P_1/\epsilon & P_2 \\ P_2^T & P_3 \end{bmatrix} . \tag{3.31}$$

The form (3.40) plays a crucial role in studying the solution of the Riccati equation (3.30). It is different from the form used by Blankenship since he used

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}. \tag{3.32}$$

Substituting (3.31) into (3.30) and partitioning the Riccati equaiton yields

$$0 = f_1(P_1,P_2,P_3\epsilon), \tag{3.33}$$

$$P_2 = f_2(P_1,P_2,P_3\epsilon), \tag{3.34}$$

$$P_3 = f_3(P_1,P_2,P_3,\epsilon), \tag{3.35}$$

where the functions f_1, f_2 and f_3 are defined in Appendix 3.2. To study the solution of (3.33)-(3.35) near $\epsilon = 0$, it is natural to start by setting $\epsilon = 0$ in (3.33)-(3.35). This yields

$$\begin{aligned} 0 = {} & P_1(0)A+P_2(0)C+A^TP_1(0)+C^TP_2^T(0)+C^TP_3(0)C+D_1^TD_1 \\ & - [P_1(0)F+P_2(0)G+C^TP_3(0)G+D_1^TM][R+M^TM+G^TP_3(0)G]^{-1} \\ & \times [F^TP_1(0)+G^TP_2^T(0)+G^TP_3(0)C+M^TD_1], \end{aligned} \tag{3.36}$$

$$\begin{aligned} P_2(0) = {} & P_1(0)B+P_2(C)S+C^TP_3(0)S+D_1^TD_2 \\ & -[P_1(0)F+P_2(0)G+C^TP_3(0)G+D_1^TM][R+M^TM+G^TP_3(0)G]^{-1} \\ & [G^TP_3(0)S+M^TD_2], \end{aligned} \tag{3.37}$$

$$P_3(0) = S^TP_3(0)S+D_2^TD_2-[S^TP_3(0)G+D_2^TM][R+M^TM+G^TP_3(0)G]^{-1}$$
$$[G^TP_3(0)S+M^TD_2]. \qquad (3.38)$$

Equation (3.38) is a discrete-time algebraic Riccati equation. It is well known [Kwakernaak and Sivan, 1972] that if the pair $[S-G(R+M^TM)^{-1}M^TD_2,G]$ is stabilizable and the pair $[S-G(R+M^TM)^{-1}M^TD_2, \sqrt{D_2^TD_2-D_2^TM(R+M^TM)^{-1}M^TD_2}\,]$ is detectable then (3.38) has a unique positive semidefinite solution. It is obvious that the stabilizability of $[S-G(R+M^TM)^{-1}M^TD_2,G]$ is equivalent to the stabilizability of $[S,G]$. Moreover, using the matrix identity $I-M(R+M^TM)^{-1}M^T = (I+MR^{-1}M^T)^{-1}$ (For proof, see Appendix 3.3), it can be shown that the detectability of $[S-G(R+M^TM)^{-1}M^TD_2, \sqrt{D_2^TD_2-D_2^TM(R+M^TM)^{-1}M^TD_2}\,]$ is equivalent to the detectability of $[S-G(R+M^TM)^{-1}M^TD_2,D_2]$ which is equivalent to the detectability of $[S,D_2]$. Note that

$$\sqrt{D_2^TD_2-D_2^TM(R+M^TM)^{-1}M^TD_2} = \sqrt{D_2^T[I-M(R+M^TM)^{-1}M^T]D_2}$$
$$= \sqrt{D_2^T(I+MR^{-1}M^T)^{-1}D_2} = \sqrt{Q^TQ}, \text{ for some matrix } Q.$$

Thus we assume that the triple $[S,G,D_2]$ is stabilizable-detectable which guarantees the existence of $P_3(0) \geq 0$. Furthermore, from the properties of Riccati equations [Kwakernaak and Sivan, 1972] we have the stability property

$$|\lambda(\alpha_3)| < 1 \,, \qquad (3.39)$$

where $\alpha_3 = S-G[R+M^TM+G^TP_3(0)G]^{-1}[G^TP_3(0)S+M^TD_2]$.

We turn now to equation (3.37) and notice that $P_2(0)$ can be expressed in terms of $P_1(0)$ and $P_3(0)$ as

$$P_2(0) = L_1 + P_1(0)L_2 , \tag{3.40}$$

where

$$L_1 = \{D_1^TD_2+C^TP_3(0)S-[C^TP_3(0)G+D_1^TM][R+M^TM+G^TP_3(0)G]^{-1}$$
$$[G^TP_3(0)S+M^TD_2]\}L_3^{-1},$$

$$L_2 = \{B-F[R+M^TM+G^TP_3(0)G]^{-1}[G^TP_3(0)S+M^TD_2]\}L_3^{-1},$$

$$L_3 = I-\alpha_3 ,$$

and where the nonsingularity of L_3 follows from the stability property (3.39). Substituting (3.40) into (3.36) yields

$$0 = P_1(0)\hat{A}+\hat{A}^TP_1(0)+\hat{Q}-P_1(0)\hat{B}\hat{R}^{-1}\hat{B}^TP_1(0), \tag{3.41}$$

where

$$\hat{B} = F+L_2G,$$

$$\hat{R} = R+M^TM+G^TP_3(0)G,$$

$$\hat{A} = A+L_2C-\hat{B}\hat{R}^{-1}[G^TL_1^T+G^TP_3(0)C+M^TD_1],$$

and

$$\hat{Q} = D_1^TD_1+L_1C+C^TL_1^T+C^TP_3(0)C$$
$$- [L_1G+C^TP_3(0)G+D_1^TM]\hat{R}^{-1}[G^TL_1^T+G^TP_3(0)C+M^TD_1].$$

Equation (3.41) is a continuous-time algebraic Riccati equaiton. We assume that the triple $(\hat{A},\hat{B},\sqrt{\bar{Q}})$ is stabilizable-detectable. This guarantees [Kwakwenaak and Sivan, 1972] that (3.41) has a unique positive semidefinite solution $P_1(0) \geq 0$ and

$$\text{Re}\,\{\lambda(\alpha_1)\} < 0, \tag{3.42}$$

where

$$\alpha_1 = \hat{A}-\hat{B}\hat{R}^{-1}\hat{B}^T P_1(0).$$

Thus we have established the existence of the solution of (3.33)-(3.35) at $\epsilon = 0$. For ϵ near zero, let

$$P_i = P_i(0) + \epsilon\, E_i \quad \text{for} \quad i = 1,2,3\,, \tag{3.43}$$

where E_i indicates the non-zero-order terms. The existenc of E_i, $i = 1,2,3$ is established by applying the implicit function theorem where the nonsingularity of $I-\alpha_3$ and α_1, (which follow from the stability properties (3.39) and (3.42), respectively), are used to show that the Frechet derivatives of E_i for $i = 1,2,3$ at $\epsilon = 0$ is invertible. This is shown in Section 3.6 and the existence of P_i for $i = 1,2,3$ follows immediately.

It remains now to show that this solution is stabilizing. THis, however, follows immediately by applying the stability criterion, derived in Appendix 2.1, to the closed-loop system where the stability properties (3.39) and (3.42) are used. Our conclusion is summarized in the following theorem.

Theorem 3.1: Assume that

Condition a: The triple $(\hat{A},\hat{B},\hat{Q})$ is stabilizable-detectable in the continuous-time sense; i.e., every eigenvalue of $\hat{A}$ which lies in the closed right-half complex plane is controllable and obserable,

Condition b: The triple (S,G,D_2) is stabilizable-detectable in the descrete-time sense; i.e., every eigenvalue of S which has modulus greater or equal to one is controllable and observable.

Then there exists $\epsilon^* > 0$ such that for all $0 < \epsilon < \epsilon^*$, the Riccati equation (3.30) has a unique positive semidefinite stabilizing solution. Furthermore, the solution has a power series expansion at $\epsilon = 0$, that is

$$P = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \begin{bmatrix} P_1^{(i)}/\epsilon & P_2^{(i)} \\ P_2^{(i)T} & P_3^{(i)} \end{bmatrix} .$$

One unpleasant feature of Theorem 3.1 is that Condition a is dependent on $P_3(0)$, the solution of the discrete-time Riccati Equation (3.38). It will be shown later that the matrices $\hat{A},\hat{B}$ and $\hat{Q}$ are indeed independent of $P_3(0)$. For the time-being, however, let us assume that Conditions a and b hold so that Theorem 3.1 provides us with a reasonable way to approximate the optimal control for small ϵ. An approximate state feedback control is defined by

$$u(n) = -[R+M^TM+\mathbf{B}^T\tilde{P}\,\mathbf{B}]^{-1}[\mathbf{B}^T\tilde{P}\,\mathbf{A}+M^TD] \begin{pmatrix} x(n) \\ z(n) \end{pmatrix}, \qquad (3.44)$$

where $\tilde{P}$ is obtained by truncating the expansion of P; i.e.,

$$\tilde{P} = \sum_{i=0}^{N-1} \frac{\epsilon^i}{i!} \begin{bmatrix} P_1^{(i)}/\epsilon & P_2^{(i)} \\ P_2^{(i)T} & P_3^{(i)} \end{bmatrix}. \tag{3.45}$$

The near optimality of the control law (3.44), (3.45) is established in the following theorem.

Theorem 3.2: Under conditions (a) and (b), the use of feedback control (3.44), (3.45) is $O(\epsilon^{2N})$ near-optimal in the sense that

$$J-J_{opt} = O(\epsilon^{2N}), \tag{3.46}$$

where J is the value of the performance criterion when (3.44), (3.45) is used, while J_{opt} is its optimal value.

Proof

Let us represent the optimal feedback control as

$$u_{opt} = -(F_1^0 \; F_2^0)X(n) \triangleq -\mathcal{F}^0 X(n); \; X(n) = \begin{bmatrix} x(n) \\ z(n) \end{bmatrix}, \tag{3.47}$$

then

$$J_{opt} = \epsilon X^T(0)PX(0), \tag{3.48}$$

where P satisfies equation (3.30). Similarly, the approximate feedback control is represented as

$$u = -(F_1 \; F_2)X(n) \triangleq -\mathcal{F}X(n). \tag{3.49}$$

It can be easily verified that

$$F_i^0 - F_i = O(\epsilon^N) \ , \ i = 1,2 . \tag{3.50}$$

The closed-loop system under the feedback control (3.49) is

$$X(n+1) = (\mathbb{A}-\mathbb{B}\mathbb{F})X(n), \tag{3.51}$$

and the performance index is

$$J = \epsilon X^T(0)P'X(0), \tag{3.52}$$

where P' satisfies the Lyapunov equation

$$P' = (\mathbb{A}-\mathbb{B}\mathbb{F})^T P'(\mathbb{A}-\mathbb{B}\mathbb{F})+D^TD+\mathbb{F}^T(R+M^TM)\mathbb{F}-D^TM\mathbb{F}-\mathbb{F}^TM^TD . \tag{3.53}$$

Subtracting (3.30) from (3.53) and letting $\tilde{R} = (R+M^TM+\mathbb{B}^TP\mathbb{B})$ yields

$$\begin{aligned}
P'-P &= (\mathbb{A}-\mathbb{B}\mathbb{F})^T(P'-P)(\mathbb{A}-\mathbb{B}\mathbb{F}) + (\mathbb{A}-\mathbb{B}\mathbb{F})^TP(\mathbb{A}-\mathbb{B}\mathbb{F}) \\
&\quad - \mathbb{A}^TP\mathbb{A}+\mathbb{F}^T(R+M^TM))\mathbb{F} - D^TM\mathbb{F} - \mathbb{F}^TM^TD \\
&\quad +(\mathbb{A}^TP\mathbb{B}+D^TM)\ \tilde{R}^{-1}\ (\mathbb{B}^TP\mathbb{A}+M^TD) \\
&= (\mathbb{A}-\mathbb{B}\mathbb{F})^T(P'-P)(\mathbb{A}-\mathbb{B}\mathbb{F}) - \mathbb{F}^T\mathbb{B}^TP\mathbb{A}-\mathbb{A}^TP\mathbb{B}\mathbb{F} + \mathbb{F}^T\mathbb{B}^TP\mathbb{B}\mathbb{F} \\
&\quad + \mathbb{F}^T(R+M^TM)\mathbb{F} - D^TM\mathbb{F}-\mathbb{F}^TM^TD + (\mathbb{A}^TP\mathbb{B}+D^TM)\tilde{R}^{-1}(\mathbb{B}^TP\mathbb{A}+M^TD) \\
&= (\mathbb{A}-\mathbb{B}\mathbb{F})^T(P'-P)(\mathbb{A}-\mathbb{B}\mathbb{F})-\mathbb{F}^T\ \tilde{R}\ \tilde{R}^{-1}\mathbb{B}^TP\mathbb{A} \\
&\quad - \mathbb{A}^TP\mathbb{B}\ \tilde{R}^{-1}\tilde{R}\ \mathbb{F} + \mathbb{F}^T\tilde{R}\ \mathbb{F} \\
&\quad -D^TM\mathbb{F}-\mathbb{F}^TM^TD + (\mathbb{A}^TP\mathbb{B}+D^TM)\ \tilde{R}^{-1}\ (\mathbb{B}^TP\mathbb{A}+M^TD)
\end{aligned}$$

$$= (\mathbb{A}-\mathbb{B}\mathcal{F})^T(P'-P)(\mathbb{A}-\mathbb{B}\mathcal{F}) - \mathcal{F}^T\tilde{R}\,\tilde{R}^{-1}\,(\mathbb{B}^TP\mathbb{A}+M^TD)$$

$$- (\mathbb{A}^TP\mathbb{B}+D^TM)\,\tilde{R}^{-1}\,\tilde{R}\,\mathcal{F} + \mathcal{F}^T\,\tilde{R}\mathcal{F} + (\mathbb{A}^TP\mathbb{B}+D^TM)\tilde{R}^{-1}$$

$$(\mathbb{B}^TP\mathbb{A}+M^TD)$$

$$= (\mathbb{A}-\mathbb{B}\mathcal{F})^T(P'-P)(\mathbb{A}-\mathbb{B}\mathcal{F}) + [\mathcal{F}^T-(\mathbb{A}^TP\mathbb{B}+D^TM)\tilde{R}^{-1}]$$

$$\tilde{R}[\mathcal{F}-\tilde{R}^{-1}(\mathbb{B}^TP\mathbb{A}+M^TD)].$$

So we obtain

$$P'-P = (\mathbb{A}-\mathbb{B}\mathcal{F})^T(P'-P)(\mathbb{A}-\mathbb{B}\mathcal{F})+[\mathcal{F}-\mathcal{F}^0]^T(R+M^TM+\mathbb{B}^TP\mathbb{B})[\mathcal{F}-\mathcal{F}^0]. \qquad (3.54)$$

But the term inside the last bracket is

$$[\mathcal{F}-\mathcal{F}^0] = [F_1-F_1^0,\ F_2-F_2^0] = O(\epsilon^N).$$

Letting $P'-P = V$, (3.54) becomes

$$V = (\mathbb{A}-\mathbb{B}\mathcal{F})^TV(\mathbb{A}-\mathbb{B}\mathcal{F})+O(\epsilon^{2N}). \qquad (3.55)$$

By application of the Implicit Function Theorem we can show that P' possesses a power series expansion at $\epsilon = 0$. So V can be expanded as follows:

$$V = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \begin{bmatrix} V_1^{(i)}/\epsilon & V_2^{(i)} \\ V_2^{(i)T} & V_3^{(i)} \end{bmatrix}. \qquad (3.56)$$

Partioning (3.55) and matching the zero order terms yields

$$0 = V_1^{(0)}(A - FR_0^{-1}N_0^T) + V_2^{(0)}W + (A - FR_0^{-1}N_0^T)^T V_1^{(0)}$$

$$+ W^T V_2^{(0)T} + W^T V_3^{(0)} W, \qquad (3.57a)$$

$$V_2^{(0)} \cdot = V_1^{(0)} J + V_2^{(0)} \alpha_3 + W^T V_3^{(0)} \alpha_3 \ , \qquad (3.57b)$$

$$V_3^{(0)} = \alpha_3^T V_3^{(0)} \alpha_3 \ , \qquad (3.57c)$$

where

$$R_0 = R + M^T M + G^T P_3(0) G \ ,$$

$$D_0 = S^T P_3(0) G + D_2^T M \ ,$$

$$N_0 = P_1(0) F + P_2(0) G + C^T P_3(0) G + D_1^T M \ ,$$

$$J = B - FR_0^{-1} D_0^T \ ,$$

$$W = C - GR_0^{-1} N_0^T \ .$$

Stability of α_3 implies $V_3^{(0)} = 0$.

Evaluating $V_2^{(0)}$ from (3.57b) and substituting into (3.57a) and using (3.57c) yields

$$V_1^{(0)} \alpha_1 + \alpha_1^T V_1^{(0)} = 0 \ .$$

But α_1 is stable which implies $V_1^{(0)} = 0$.

For higher order terms up to $(2N-1)$ we prove, by induction, that $V_j^{(i)} = 0$ for $j = 1,2,3$ and $i = 1,\ldots,2N-1$.

Let $A - FR_0^{-1}N_0^T = \beta$, then partitioning (3.55) yields

$$0 = V_1\beta' + V_2W' + \beta'^TV_1 + W'^TV_2 + W'^TV_3W' + \epsilon(\beta'^T V_2W'+\beta'^TV_1\beta'+W'^TV_2^T\beta')$$
$$+ O(\epsilon^{2N}),$$

$$V_2 = V_1J' + V_2\alpha_3' + W'^TV_3\alpha_3' + \epsilon(\beta'^TV_1J' + \beta'^TV_2\alpha_3' + W'^TV_2^TJ') + O(\epsilon^{2N}),$$

$$V_3 = \alpha_3'^TV_3\alpha_3' + \epsilon(J'^TV_1J' + J'^TV_2\alpha_3' + \alpha_3'^TV_2^TJ') + O(\epsilon^{2N}), \qquad (3.58)$$

where β', W', J', and α_3' are matrices analytical on ϵ with their zero-order terms given by β, W, J, and α_3, respectively.

Now assume that for $1 \leq i \leq K-1$ we have

$$V_j^{(i)} = 0 \ , \quad j = 1,2,3, \qquad (3.59)$$

and we prove that $V_j^{(K)} = 0$, $j = 1,2,3$.
Note that we already proved that (3.59) holds for $K = 1$.
Matching the K^{th}-order terms in equation (3.58) yields

$$0 = V_1^{(K)}\beta + V_2^{(K)}W + \beta^TV_1^{(K)} + g_1(V_j^{(i)}) \ , \qquad (3.60a)$$

$$V_2^{(K)} = V_1^{(K)}J + V_2^{(K)}\alpha_3 + W^TV_3^{(K)}\alpha_3 + g_2(V_j^{(i)}) \ , \qquad (3.60b)$$

$$V_3^{(K)} = \alpha_3^TV_3^{(K)}\alpha_3 + g_3(V_j^{(i)}) \ , \qquad (3.60c)$$

where g_1, g_2, and g_3 are functions of $V_j^{(i)}$ for $0 \leq i \leq K-1$ and $j = 1,2,3$.

In view of (3.59), (3.60) reduces to

$$0 = V_1^{(K)}\beta + V_2^{(K)}W + \beta^T V_1^{(K)}, \tag{3.61a}$$

$$V_2^{(K)} = V_1^{(K)}J + V_2^{(K)}\alpha_3 + W^T V_3^{(K)}\alpha_3 , \tag{3.61b}$$

$$V_3^{(K)} = \alpha_3^T V_3^{(K)}\alpha_3 . \tag{3.61c}$$

But (3.61) is similar to (3.57) and we can repeat the same argument which completes the proof of theorem 3.2.

3.5. Slow-Fast decompostion and composite control

System (3.21) is singular as a function of ϵ ; i.e., we can observe order reduction and separation of time scales as $\epsilon \to 0$. Blankenship [1981] showed that for small ϵ the variables can be decomposed into slow and fast variables. Using Blankenship's time decompostions, slow and fast subproblems are defined in a way similar to that of Chow and Kokotovic [1976].

Slow Subproblem: The slow variables, evolving in slow time scale ϵn, satisfy the outer solution

$$X(\epsilon n+\epsilon)-X(\epsilon n) = \epsilon AX(\epsilon n) + \epsilon BZ(\epsilon n) + \epsilon FU(\epsilon n) \tag{3.62a}$$

$$Z(\epsilon n) = CX(\epsilon n) + SZ(\epsilon n) + GU(\epsilon n) \tag{3.62b}$$

$$Y(\epsilon n) = D_1X(\epsilon n) + D_2Z(\epsilon n) + MU(\epsilon n). \tag{3.62c}$$

Dividing (3.62a) by ϵ and letting $\epsilon \to 0$ yields

$$\frac{d\bar{x}}{dt} = A\,\bar{x}(t) + B\,\bar{z}(t) + F\,\bar{u}(t) \tag{3.63a}$$

$$\bar{z}(t) = C\,\bar{x}(t) + S\,\bar{z}(t) + G\,\bar{u}(t) \tag{3.63b}$$

$$\bar{y}(t) = D_1\,\bar{x}(t) + D_2\,\bar{z}(t) + M\,\bar{u}(t)\,. \tag{3.63c}$$

Assuming that (I-S) is nonsingular, (3.63b) can be used to eliminate $\bar{z}(t)$ from (3.63a) and (3.63c) resulting in the slow subsystem

$$\frac{dx_s}{dt} = A_s x_s(t) + B_s u_s(t) \tag{3.64a}$$

$$y_s(t) = C_s x(t) + D_s u\,(t), \tag{3.64b}$$

where $x_s = \bar{x}$, $y_s = \bar{y}$, $u_s = \bar{u}$ and $x_s(0) = x(0)$, and where

$$A_s = A+B(I-S)^{-1}C, \quad B_s = F+B(I-S)^{-1}G,$$

$$C_s = D_1+D_2(I-S)^{-1}C, \; D_s = M+D_2(I-S)^{-1}G.$$

We define a slow performance index J_s as

$$J_s = \int_0^\infty (y_s^T(t)y_s(t) + u_s^T(t)R\,u_s(t))\,dt. \tag{3.65}$$

where in obtaining (3.65) we used the limiting relation

$$\lim_{\epsilon\to 0} \epsilon \sum_{n=0}^{\infty} \bar{U}^T(\epsilon n)R\,\bar{U}(\epsilon n) = \int_0^\infty \bar{u}^T(t)R\bar{u}(t)dt.$$

The slow problem defined by (3.64), (3.65) is a continuous-time regulator problem identical to the slow problem of Chow and Kokotovic [1976]. Following their work we assume that

<u>Condition a'</u>: the triple (A_s,B_s,C_s) is stabilizable-detectable in the continuous-time sense; i.e., every eigenvalue of A_s which lies in the closed right-half complex plane is controllable and observable. Under Condition a', the optimal feedback control law of the slow problem

(3.64), (3.65) is given by

$$u_s(t) = -R_s^{-1}[D_s^T C_s + B_s^T P_s]x_s(t) \triangleq -F_s x_s(t)\,, \tag{3.66}$$

where $R_s = R+D_s^T D_s$ and P_s is the unique positive semidefinite stabilizing solution of the continuous-time algebraic Riccati equation

$$0 = P_s(A_s-B_sR_s^{-1}D_s^TC_s) + (A_s-B_sR_s^{-1}D_s^TC_s)^TP_s$$
$$- P_sB_sR_s^{-1}B_s^TP_s + C_s^T(I-D_sR_s^{-1}D_s^T)C_s. \tag{3.67}$$

<u>Fast Subproblem</u>: Assume that the variables $x(n)$, $z(n)$, $y(n)$ and $u(n)$ decompose as

$$x(n) = x_f(n) + x_s(t) \tag{3.68a}$$

$$z(n) = z_f(n) + z_s(t) \tag{3.68b}$$

$$y(n) = y_f(n) + y_s(t), \tag{3.68c}$$

and

$$u(n) = u_f(n) + u_s(t). \tag{3.68d}$$

Substituting (3.68) in (3.21), taking the limit $\varepsilon \to 0$ and using (3.63b) and (3.63c) we get

$$z_f(n+1) = Sz_f(n) + Gu_f(n), \tag{3.69a}$$

$$y_f(n) = D_2z_f(n) + Mu_f(n). \tag{3.69b}$$

We define a fast performance index J_f as

$$J_f = \sum_{n=0}^{\infty} [y_f^T(n)y_f(n) + u_f^T(n)Ru_f(n)]. \tag{3.70}$$

The fast discrete-time regulator problem (3.69), (3.70) is a standard problem and it is well-known [Kwakwenaak and Sivan, 1972] that if the triple (S,G,D_2) is stabilizable-detectable (condition b), then the optimal solution is given by

$$u_f(n) = -[R+M^TM+G^TP_fG]^{-1}[G^TP_fS+M^TD_2]z_f(n) \triangleq -F_fz_f(n), \quad (3.71)$$

where P_f is the unique positive semidefinite stabilizing solution of the discrete-time algebraic Riccati equation

$$P_f = D_2^TD_2+S^TP_fS-[G^TP_fS+M^TD_2]^T[R+M^TM+G^TP_fG]^{-1}[G^TP_fS+M^TD_2]. \quad (3.72)$$

Inspection of (3.38) and (3.72), together with the uniqueness of the solution of the Riccati equation, shows that

$$P_3(0) = P_f \ . \quad (3.73)$$

Motivated by the results of Chow and Kokotovic [1976] in the continuous-time case it is natural to ask the question: Is there a similar relation between $P_1(0)$ and P_s? The answer is yes, as it can be seen from the following lemma.

Lemma 3.1. If $(I-S)$ is nonsingular, then the matrices $\hat{A}$, $\hat{B}$ and $\hat{Q}$ appearing in (3.41) are given by

$$\hat{A} = A_s-B_sR_s^{-1}D_s^TC_s \ , \quad (3.74)$$

$$\hat{B}\hat{R}^{-1}\hat{B}^T = B_sR_s^{-1}B_s^T \ , \quad (3.75)$$

$$\hat{Q} = C_s^T(I-D_sR_s^{-1}D_s^T)C_s. \quad (3.76)$$

The lemma is proved in Appendix (3.4)

As a consequence of the lemma we have

$$P_1(0) = P_s . \qquad (3.77)$$

Also, condition (a) is equivalent to condition (a') which is independent of $P_3(0)$. Therefore Theorems 3.1 and 3.2 hold under conditions (a') and (b).

Composite Control: With the solutions of the slow and fast problems in hand, a composite feedback control is formed as

$$u_c = u_s + u_f$$
$$= -F_s x_s(t) - F_f z_f(n).$$

Approximating $z_f(n)$ by $z(n)-\bar{z}(t)$, expressing $\bar{z}(t)$ in terms of $x_s(t)$ and $u_s(t)$ and approximating $x_s(t)$ by $x(n)$, we get

$$u_c = -F_s x(n) - F_f[z(n)-(I-S)^{-1}(C-GF_s)x(n)]$$
$$= -[F_s - F_f(I-S)^{-1}(C-GF_s)]x(n) - F_f z(n) . \qquad (3.78)$$

The composite feedback control law (3.78) is near-optimal as established in the following theorem.

Theorem 3.3: Under Conditions (a') and (b) the composite control (3.78) is $O(\epsilon^2)$ near-optimal in the sense that

$$J - J_{opt} = O(\epsilon^2) .$$

Proof: The main step in the proof is verifying that the feedback coefficients in (3.78) are $O(\epsilon)$ perturbations of the feedback coefficients in the optimal control (3.29). To see this, note that from (3.29) we have

$$F^0 = [R_0 + O(\epsilon)]^{-1}[N_0^T + O(\epsilon),\ D_0^T + O(\epsilon)]\ .$$

Using Householder Theorem we obtain

$$F^0 = [R_0^{-1} + O(\epsilon)][N_0^T + O(\epsilon)\ ,\ D_0^T + O(\epsilon)]$$

$$= [R_0^{-1}N_0^T + O(\epsilon)\ ,\ R_0^{-1}D_0^T + O(\epsilon)]$$

$$\triangleq [F_1^0\ ,\ F_2^0].$$

On the other hand

$$F_f = R_0^{-1}D_0^T\ .$$

So by comparing with (3.29) when it is partioned we get

$$F_2^0 = F_f + O(\epsilon),$$

Let

$$F_1 = F_s - F_f(I-S)^{-1}(C-GF_s)$$

$$= [I + F_f(I-S)^{-1}G]F_s - F_f(I-S)^{-1}C\ .$$

Using

$$F_s = R_s^{-1}(D_s^T(C_s + B_s^TP_s),$$

yields

$$F_1 = [I + R_0^{-1}D_0^T(I-S)^{-1}G]R_s^{-1}(D_sTC_s + B_s^TP_s) - R_0^{-1}D_0^T(I-S)^{-1}C.$$

Using (1) from Appendix 3.4 and defintion of H^{-1} we get

$$F_1 = H^{-1}R_s^{-1}(D_s^TC_s + B_s^TP_s) - R_0^{-1}H^TH^{-T}D_0^T(I-S)^{-1}C$$

$$= R_0^{-1}H^TB_s^TP_s + R_0^{-1}H^T\ [D_s^TC_s - (I+R_0^{-1}D_0^T(I-S)^{-1}G)^T$$

$$(G^TP_fS + M^TD_2)(I-S)^{-1}C]\ .$$

Inside the bracket is

$$D_s^T C_s - G^T P_f S(I-S)^{-1} C - M^T D_2 (I-S)^{-1} C - G^T (I-S)^{-T} D_0 R_0^{-1} D_0^T (I-S)^{-1} C.$$

Using (3.72) yields

$$D_s^T C_s - G^T (I-S)^{-T} [(I-S)^T P_f S - P_f + D_2^T D_2 + S^T P_f S](I-S)^{-1} C$$
$$- M^T D_2 (I-S)^{-1} C$$
$$= D_s^T C_s + G^T (I-S)^{-T} P_f C - G^T (I-S)^{-T} D_2^T D_2 (I-S)^{-1} C - M^T D_2 (I-S)^{-1} C.$$

Substituting for $D_s^T C_s$ yields

$$F_1 = R_0^{-1} H^T B_s^T P_s + R_0^{-1} H^T [M^T D_1 + G^T (I-S)^{-T} D_2^T D_1 + G^T (I-S)^{-T} P_f C].$$

Using (2), and (5) of Appendix 3.4 we obtain

$$F_1 = R_0^{-1} [(F + L_2 G)^T P_s + (L_1 G + C^T P_f G + D_1^T M)^T]$$
$$= R_0^{-1} [F^T P_s + G^T (L_1 + L_2 P_s)^T + G^T P_f C + M^T D_1].$$

So

$$F_1 = R_0^{-1} N_0^T .$$

By comparing with (3.29) we obtain

$$F_1^0 = F_1 + O(\epsilon).$$

The rest of the proof is similar to the proof of Theorem 3.2.

3.6 An Iterative solution of Riccati Equation for Linear Quadratic Singularly Perturbed Systems.

One of the main difficulties in dealing with optimal control of infinite-time regulators of singularly perturbed systems is solving the stiff Riccati equation arising in this class of systems. In this section an efficient iterative technique for solving the stiff algebraic difference Riccati equation (3.30) is developed and it is shown that the accuracy of $O(\epsilon^i)$ can be obtained by performing only $(i-1)$ iterations. Also, since only the lower order systems are employed, the algorithm is very efficient from the computational point of view.

Let $P_i = P_i(0) + \epsilon E_i$ for $i = 1,2,3$ as in (3.43), where $P_i(0)$ indicates the zero-order terms and let

$$N_1 = A^T P_1 A + A^T P_2 C + C^T P_2^T A,$$

$$N_2 = A^T P_1 B + A^T P_2 S + C^T P_2^T B,$$

$$N_3 = B^T P_1 B + B^T P_2 S + S^T P_2^T B,$$

$$N_4 = F^T P_1 F + F^T P_2 G + G^T P_2^T F,$$

$$N_5 = A^T P_1 F + C^T P_2^T F + A^T P_2 G,$$

$$N_6 = B^T P_1 F + S^T P_2^T F + B^T P_2 G,$$

and

$$N_e = E_1 F + E_2 G + C^T E_3 G.$$

Note that

$$\tilde{R}^{-1} = (R + M^T M + \mathcal{B}^T P \mathcal{B})^{-1} = [R + M^T M + G^T P_3(0)G + \epsilon(G^T E_3 G + N_4)]^{-1},$$

and by Householder theorem (see Appendix 2.1) we obtain

$$\tilde{R}^{-1} = (R + M^T M + \mathcal{B}^T P \mathcal{B})^{-1} = R_0^{-1} + \epsilon\, K\,,$$

where

$$K = -[R_0 + \epsilon\,(G^T E_3 G + N_4)]^{-1}(G^T E_3 G + N_4)R_0^{-1}.$$

Dividing K into zero-order and non-zero-order terms we get

$$K = K_0 + \epsilon\, K_1,$$

where

$$K_0 = -\,R_0^{-1}[G^T E_3 G + F^T P_1(0)F + F^T P_2(0)G + G^T P_2^T(0)F]R_0^{-1}\,,$$

and

$$K_1 = -K(G^T E_3 G + N_4\,)R_0^{-1} - R_0^{-1} N_{4e} R_0^{-1},$$

where

$$N_{4e} = \frac{N_4 - N_4(0)}{\epsilon}\,.$$

Now subtracting (3.38) from (3.35) yields

$$\begin{aligned}\epsilon E_3 = {} & \epsilon(S^T E_3 S + N_3) - \epsilon[D_0 R_0^{-1}(G^T E_3 S + N_6^T) + (S^T E_3 G + N_6) \\ & R_0^{-1} D_0^T + D_0 K D_0^T\,] - \epsilon^2[(S^T E_3 G + N_6)\tilde{R}^{-1}(G^T E_3 S + N_6^T) \\ & + D_0 K(G^T E_3 S + N_6^T) + (S^T E_3 G + N_6)K D_0^T]\,.\end{aligned}$$

Eliminating ϵ and factoring yields the discrete Lyapunov equation

$$E_3 - \alpha_3^T E_3 \alpha_3 = C_3 + \epsilon \gamma_3 (E_1, E_2, E_3, \epsilon) \tag{3.79}$$

where

$$C_3 = N_3(0) - D_0 R_0^{-1} N_6^T(0) - N_6(0) R_0^{-1} D_0^T + D_0 R_0^{-1} N_4(0) R_0^{-1} D_0^{-1}$$

$$\gamma_3 = N_{3e} - D_0 R_0^{-1} N_{6e}^T - N_{6e} R_0^{-1} D_0^T - D_0 K_1 D_0^T - [(S^T E_3 G + N_6)\tilde{R}^{-1}$$
$$(G^T E_3 S + N_6^T) + D_0 K (G^T E_3 S + N_6^T) + (S^T E_3 G + N_6) K D_0^T].$$

Note that

$$N_{ie} = \frac{N_i - N_i(0)}{\epsilon}, \quad i = 1, \ldots, 6.$$

By the same way, subtracting (3.37) from (3.34) and (3.36) from (3.33), respectively yields

$$E_2 - E_1 B - E_2 S - C^T E_3 S + N_0 R_0^{-1} G^T E_3 S + N_e R_0^{-1} D_0^T - N_0 R_0^{-1} G^T E_3 G R_0^{-1} D_0^T$$
$$= C_2 + \epsilon \gamma_2 (E_1, E_2, E_3, \epsilon), \tag{3.80}$$

and

$$E_1 A + A^T E_1 + E_2 C + C^T E_2^T + C^T E_3 C - N_0 R_0^{-1} N_e^T - N_e R_0^{-1} N_0^T + N_0 R_0^{-1} G^T E_3 G R_0^{-1} N_0^T$$
$$= C_1 + \epsilon \gamma_1 (E_1, E_2, E_3, \epsilon), \tag{3.81}$$

where

$$C_2 = N_2(0) - N_0 R_0^{-1} N_6^T(0) - N_5(0) R_0^{-1} D_0^T + N_0 R_0^{-1} N_4(0) R_0^{-1} D_0^T$$

$$\gamma_2 = N_{2e} - N_0 R_0^{-1} N_{6e}^T - N_{5e} R_0^{-1} D_0^T - N_0 K_1 D_0^T - (N_e + N_s)\tilde{R}^{-1}(G^T E_3 S + N_6^T)$$
$$- N_0 K (G^T E_3 S + N_6^T) - (N_e + N_5) K D_0^T$$

$$C_1 = -N_1(0) + N_0R_0^{-1}N_5^T(0) + N_5(0)R_0^{-1}N_0^T - N_0R_0^{-1}N_4(0)R_0^{-1}N_0^T ,$$

and

$$\gamma_1 = -N_{1e} + N_0R_0^{-1}N_{5e}^T + N_{5e}R_0^{-1}N_0^T + N_0K_1N_0^T + (N_e + N_5)\tilde{R}^{-1}(N_e + N_5)^T$$

$$+ N_0K(N_e + N_5)^T + (N_e + N_5)KN_0^T.$$

Letting $W = C-GR_0^{-1}N_0^T$ and $J = B-FR_0^{-1}D_0^T$ from (3.80) we get

$$E_2 = (E_1J + W^TE_3\alpha_3 + C_2 + \epsilon\, \gamma_2)L_3^{-1}. \tag{3.82}$$

Substituting (3.82) in (3.81) and using (3.79) yields the continuous Lyapunov equation

$$E_1(A-FR_0^{-1}N_0^T + L_2W) + (A-FR_0^{-1}N_0^T + L_2W)^TE_1$$

$$= -(C_2 + \epsilon\gamma_2)L_3^{-1}W-W^TL_3^{-T}(C_2 + \epsilon\gamma_2)^T-W^TL_3^{-T}(E_3-\alpha_3^TE_3\alpha)L_3^{-1}W$$

$$+ C_1 + \epsilon\, \gamma_1. \tag{3.83}$$

But

$$A-FR_0^{-1}N_0^T + L_2W = A + L_2C - FR_0^{-1}N_0^T - L_2GR_0^{-1}N_0^T$$

$$= A + L_2C - (F + L_2G)R_0^{-1}N_0^T .$$

Using the proof of Theorme 3.3 we have

$$N_0^T = (F + L_2G)^TP_1(0) + (L_1G + C^TP_3(0)G + D_1^TM)^T,$$

and by recalling that $\hat{B} = F + L_2G$ we obtain

$$A-FR_0^{-1}N_0^T + L_2W = A + L_2C - \hat{B}R_0^{-1}\hat{B}^TP_1(0) - \hat{B}R_0^{-1}(G^TL_1^T$$

$$+ G^TP_3(0)C + M^TD_1) = \hat{A} - \hat{B}R_0^{-1}\hat{B}^TP_1(0) = \alpha_1.$$

Now ,(3.83) reduces to

$$E_1\alpha_1 + \alpha_1^TE_1 = -(C_2 + \epsilon\gamma_2)L_3^{-1}W - W^TL_3^{-T}(C_2 + \epsilon\gamma_2)^T$$

$$- W^TL_3^{-T}(C_3 + \epsilon\gamma_3)L_3^{-1}W + C_1 + \epsilon\gamma_1,$$

or

$$E_1\ \alpha_1 + \alpha_1^TE_1 = \phi + \epsilon\ \gamma\ , \tag{3.84}$$

where

$$\phi = -C_2L_3^{-1}W-W^TL_3^{-T}C_2^T-W^TL_3^{-T}C_3L_3^{-1}W + C_1$$

$$\gamma = -\gamma_2L_3^{-1}W-W^TL_3^{-T}\gamma_2^T-W^TL_3^{-T}\gamma_3L_3^{-1}W + \gamma_1.$$

Equations (3.79), (3.82), and (3.84) have an interesting form since all non-linear terms and cross-coupling terms are multiplied by a small parameter ϵ. This suggests that a successive approximation algorithm can be efficient for their solution.

Let us propose the following algorithm:

$$E_1^{(i+1)}\alpha_1 + \alpha_1^TE_1^{(i+1)} = \phi + \epsilon\ \gamma^{(i)}, \tag{3.85a}$$

$$E_3^{(i+1)} - \alpha_3^TE_3^{(i+1)}\alpha_3 = C_3 + \epsilon\ \gamma_3^{(i)}, \tag{3.85b}$$

$$E_2^{(i+1)} = E_1^{(i+1)}L_2 + W^TE_3^{(i+1)}\alpha_3L_3^{-1} + C_2L_3^{-1} + \epsilon\ \gamma_2^{(i)}L_3^{-1}, \tag{3.85c}$$

for all non-negative integers i and

$$E_1^{(0)} = E_2^{(0)} = E_3^{(0)} = 0 . \tag{3.86}$$

Theorem 3.4:

The algorithm (3.85), (3.86) converges to the exact solution E with the rate of convergence of $O(\epsilon)$, i.e.,

$$\|E-E^{(i+1)}\| = O(\epsilon)\|E-E^{(i)}\| , \quad i = 0,1,2,... \tag{3.87}$$

or equivalently

$$\|E-E^{(i)}\| = O(\epsilon^i), \quad i = 1,2,... . \tag{3.88}$$

Proof:

Let us represent equation (3.85) by

$$Q_i(E_1,E_2,E_3,\epsilon) = 0 \quad , \quad i = 1,2,3. \tag{3.89}$$

As a starting point we need the existence of bounded solutions of E_1,E_2, and E_3 in the neighborhood of $\epsilon = 0$. This is established by applying the implicit function theorem to show that the Frechet derivatives of Q_1, Q_2, and Q_3 with respect to E_1, E_2, and E_3 at $\epsilon = 0$ are invertible. i.e.,

$$d\,Q_i(E,\epsilon)\Big|_{\substack{\epsilon=0\\ E=0}} \quad \text{is invertible for } i = 1,2,3 ,$$

where

$$dQ_i = \lim_{\lambda\to 0} \frac{Q_i(E+\lambda\delta E)-Q_i(E)}{\lambda} . \tag{3.90}$$

But

$$Q_1 = E_1\alpha_1 + \alpha_1^T E_1 - \phi - \epsilon\gamma(E_1,E_2,E_3,\epsilon).$$

So

$$dQ_1\Big|_{\substack{\epsilon=0\\ E=0}} = \lim_{\lambda\to 0}\frac{1}{\lambda}[(E_1 + \lambda\delta E_1)\alpha_1 + \alpha_1^T(E_1 + \lambda\delta E_1) - \phi$$

$$- \epsilon\gamma(E_1 + \lambda\delta E_1, E_2 + \lambda\delta E_2, E_3 + \lambda\delta E_3) + \phi$$

$$- E_1\alpha_1 - \alpha_1^T E_1$$

$$= (\delta E_1)\alpha_1 + \alpha_1^T(\delta E_1). \qquad (3.91)$$

α_1 is stable matrix in the continuous-time sense, so given dQ_1, there exists a unique E_1 satisfying the continuous-time Lyapanov equation (3.91).

By the same way we obtain

$$dQ_3 = \delta E_3 - \alpha_3^T(\delta E_3)\alpha_3. \qquad (3.92)$$

α_3 is stable matrix in the discrete-time sence, so given dQ_3, there exists a unique δE_3 satisfying the discrete-time Lyapanov equation (3.92). For Q_2, using the same approach, we get

$$dQ_2 = \delta E_2\, L_3 - (\delta E_1)J - W^T(\delta E_3)\alpha_3. \qquad (3.93)$$

L_3 is invertible, so given dQ_2, there is a unique δE_2 satisfying (3.93). The existence and uniqueness of δE_1, δE_2 and δE_3 for any given δQ_1, δQ_2 and δQ_3 establishes the invertibility of the Frechet derivative.

For $i = 0$, subtracting (3.85b) from (3.79) yields

$$(E_3-E_3^{(1)}) - \alpha_3^T(E_3-E_3^{(1)})\alpha_3 = \epsilon[\gamma_3(E_1,E_2,E_3,\epsilon) - \gamma_3(0,0,0,\epsilon)].$$

By stability of α_3 and existence of bounded solutions of E_1,E_2, and E_3 we obtain

$$\|E_3-E_3^{(1)}\| = O(\epsilon). \tag{3.94}$$

Similarly, subtracting (3.85a) from (3.84) yields

$$(E_1-E_1^{(1)})\alpha_1 + \alpha_1^T(E_1-E_1^{(1)}) = \epsilon\,\gamma\,(E,\epsilon).$$

By stability of α_1 and existence of bounded solutions E_i, $i = 1,2,3$, we get

$$\|E_1-E_1^{(1)}\| = O(\epsilon), \tag{3.95}$$

and by subtracting (3.85c) from (3.82) and using the same approach we have

$$\|E_2-E_2^{(1)}\| = O(\epsilon). \tag{3.96}$$

For the next iteration step we have

$$E_3^{(2)}\alpha_3 + \alpha_3^T E_3^{(2)} = C_3 + \epsilon\gamma_3(E_1^{(1)},\ E_2^{(1)},\ E_3^{(1)},\epsilon),$$

and subtracting the above from (3.79) yields

$$(E_3-E_3^{(2)}) + \alpha_3^T(E_3-E_3^{(2)})\alpha_3 = \epsilon[\gamma_3(E,\epsilon)-\gamma_3(E^{(1)},\epsilon)].$$

The term $[\gamma_3(E,\epsilon)-\gamma_3(E^{(1)},\epsilon)]$ satisfies a Lipschitz condition uniformly in ϵ for ϵ sufficiently small. Hence its order of magnitude is the same as $(E_j-E_j^{(1)})$, $j = 1,2,3$. So from (3.94) we obtain

$$\|E_3-E_3^{(2)}\| = \epsilon \cdot O(\epsilon) = O(\epsilon^2). \qquad (3.97)$$

Repeating the same arguments we can conclude that

$$\|E_3-E_3^{(i)}\| = O(\epsilon^i),$$

and by analogy

$$\|E_1-E_1^{(i)}\| = O(\epsilon^i) \ ,$$

$$\|E_2-E_2^{(i)}\| = O(\epsilon^i),$$

or

$$\|E-E^{(i)}\| = O(\epsilon^i),$$

which proves the theorem.

Clearly, since we are doing iterations of the same system of equations, this algorithm is efficient not only for obtaining required accuracy, but for finding an exact solution.

It should be pointed out that another way to overcome the stiffness problem in solving the Riccati equation (3.30) is by using a power series expansion with respect to small parameter ϵ and matching the corresponding coefficients. This provides us with a family of well-defined problems for which standard techniques are applicable. However, if we are interested in high order of accuracy the amount of required computations can be substantial, even though we are faced with solving low order problems.

The steps in our alogrithm are as follow:

Algorithm

Step 1.

Solve (3.36)-(3.38) i.e. find $P_1(0)$, $P_2(0)$ and $P_3(0)$.

Step 2.

Find α_1, α_3, C_1, C_2, C_3 and ϕ.

Step 3.

Let $i = 0$, $E_j^{(0)} = 0$ for $j = 1,2,3$.

Step 4.

Set $P_j^{(i)} = P_j(0) + \epsilon E_j^{(i)}$, $j = 1,2,3$.

Step 5.

Find $\gamma_1^{(i)}$, $\gamma_2^{(i)}$, $\gamma_3^{(i)}$, and $\gamma^{(i)}$.

Step 6.

Solve (3.85).

Step 7.

Check the required accuracy. If it is not satisfied, set $i = i + 1$ and go to step 4, otherwise stop.

Thus, the overall solution of $(m_1 + m_2)$ dimensional algebraic Riccati equation (3.30) can be found by solving two lower-order Riccati and two lower-order Lyapanov equations which can bring considerable saving in computations and high rate of convergence.

3.7. Numerical example

In this section by means of a simple example we illustrate the points in Theorem 3.2 and the iterative technique mentioned earlier.

Consider the difference equations

$$x(K+1) = (1-2\epsilon)x(K) + \epsilon\, z(K) + 1.5\epsilon\, u(K) \tag{3.98a}$$

$$z(K+1) = -.7x(K) + .45z(K) + .8\, u(K) \tag{3.98b}$$

$$y(K) = .6x(K) + .75z(K) + .4\, u(K), \tag{3.98c}$$

with the performance index

$$\min J = \epsilon \sum_{K=0}^{\infty} [y^2(K) + u^2(K)] \tag{3.99}$$

<u>Slow subsystem</u>:

$$\frac{dx_s}{dt} = -3.272727x_s(t) + 2.954545 \tag{3.100a}$$

$$y_s(t) = -.3545x_s(t) + 1.490909\, u_s(t), \tag{3.100b}$$

with the performance index

$$J_s = \int_0^{\infty} [y_s^2(t) + u_s^2(t)]dt\ .$$

<u>Fast subsystem</u>:

$$z_f(K+1) = .45z_f(K) + .8\, u_f(K) \tag{2.101a}$$

$$y_f(K) = .75z_f(K) + .4\, u_f(K), \tag{3.101b}$$

with the performance index

$$J_f = \sum_{K=0}^{\infty} [y_f^2(K) + u_f^2(K)]. \tag{3.102}$$

Using "LAS package" [Bingulac et al., 1982] the programs for solving the example are written and run on Prime Computer at Michigan State University.

Solving for slow and fast Riccati equations yields

$$P_f = .508377,$$

$$P_s = .006971,$$

and

$$P_2(0) = L_1 + P_s L_2 = .380134.$$

Solution of the approximate feedback $\mathfrak{F}$ is found to be

$$\mathfrak{F} = (.191039,\ .325184).$$

In the next table values of $V = P-P'$ and $|\mathfrak{F}-\mathfrak{F}^0|$, for different values of ϵ, are given.

Table 3.1

	$F_1-F_1^0$	$F_2-F_2^0$	$P_1'-P_1$	$P_2'-P_2$	$P_3'-P_3$
$\epsilon = .1$	.062091	.019034	-.020479	.001304	-.000626
$\epsilon = .05$	.035648	.009489	-.011813	.000323	-.000148
$\epsilon = .025$	.022486	.004739	-.008277	.000080	-.000036

As we see the numerical values agree with the theoretical results.

$$\mathfrak{F}_i-\mathfrak{F}_i^0 = O(\epsilon)\ , \quad i = 1,2\ ,$$

and

$$P_i-P_i' = O(\epsilon^2)\ , \quad i = 1,2,3$$

and consequently

$$J - J_{opt} = 0(\epsilon^2).$$

On the next table we give the numerical results using the iterative technique for ϵ equal to .1, .05, and .025, respectively. Also the exact solution is evaluated for comparison.

Table 3.2

# of iteration	$\epsilon = .025$			$\epsilon = .05$			$\epsilon = .1$		
	P_1	P_2	P_3	P_1	P_2	P_3	P_1	P_2	P_3
1	.016971	.375847	.510311	.026655	.371657	.512107	.045149	.363768	.51576
2	.017281	.375743	.510353	.027857	.371248	.512324	.049660	.362293	.516363
3	.017291	.375739	.510355	.027937	.371256	.512339	.050239	.362079	.516443
4	.017291	.375739	.510355	.027942	.371254	.512340	.050310	.362052	.516453
5	"	"	"	.027942	.371254	.512340	.050319	.362049	.516454
6	"	"	"	"	"	"	.050320	.362048	"
7	"	"	"	"	"	"	"	"	"
Exact Solution	.017291	.375739	.510355	.027943	.371263	.512360	.050320	.362048	.516455

By investigating the results in Table 3.2 we observe that the iterative technique has a convergence rate of ϵ, or even less, at each iteration and tends to the exact solution. It is interesting to note that this algorithm gives better result as the value of ϵ decreases while the standard methods for solving discrete-time Riccati equations exhibit worse results for smaller values of ϵ. In Table (3.3) the number of iterations to obtain convergence in our iterative technique is given for different values of ϵ.

ϵ	No. of iterations for Convergence up to 6^{th} digit after the decimal point.
.025	3
.05	4
.1	6

It should be noted that, due to the round-off errors of the computer system, there are some small errors on the 5^{th} or 6^{th} digit after the decimal point. (As in the case of values of P_i, i = 1,2,3, for ϵ = .05).

APPENDIX 3.1

Consider the sampled-data control of the regulator problem

$$\dot{x}(t) = Ax(t) + B\,u(t) \quad , \quad X(t_0) = x_0$$

$$y(t) = C\,x(t),$$

with the performance index

$$J = \int_{t_0}^{\infty} (y^T y + u^T R u)dt$$

$$= \int_{t_0}^{\infty} [x^T C^T C x + u^T R u]dt \quad , \quad R > 0.$$

Let $u(t)$ be a piecewise constant function of time, i.e.,

$$u(t) = u(t_i) = u_i \quad , \quad t_i \leq t < t_{i+1}.$$

By sampling the above system with period T, see Section 2.4, we obtain

$$x_{i+1} = \phi(T)x_i + \Gamma(T)u_i \quad , \quad X_0 = X(t_0)$$

$$y_i = C\,X_i + D\,u_i\,,$$

Where

$$\phi(T) = e^{AT}\,,$$

and

$$\Gamma(T) = \int_0^T e^{A(T-t)}dt\ B\ ,$$

with the performance index

$$J = \sum_{i=0}^{\infty} \{[\phi(T)x_i + \Gamma(T)u_i]^T C^T C[\phi(T)x_i + \Gamma(T)u_i] + u_i^T R u_i\}\ .$$

So, J will take the form

$$J = \sum_{i=0}^{\infty} [x_i^T Q x_i + 2x_i^T M u_i + u_i^T \tilde{R}\ u_i]\ ,$$

where

$$Q = \int_0^T \phi^T(T)C^T C\phi(T),$$

$$M = \int_0^T \phi^T(T)C^T C\Gamma(T),$$

and

$$\tilde{R} = \int_0^T [R + \Gamma^T(T)C^T C\Gamma(T)]dt\ .$$

We note the appearance of the cross product matrix M when a continuous-time regulator without a cross product is sampled.

APPENDIX 3.2

The functions f_1, f_2 and f_3 in (3.33)-(3.35) are defined by

$$
\begin{aligned}
f_1 = {} & P_1A+P_2C+A^TP_1 + C^TP_2^T + C^TP_3C+D_1^TD_1 \\
& + \varepsilon[A^TP_1A+A^TP_2C+C^TP_2^TA] \\
& - [P_1F+P_2G+C^TP_3G+D_1^TM+\varepsilon(A^TP_1F+A^TP_2G+C^TP_2^TF)]\times \\
& [R+M^TM+G^TP_3G+\varepsilon(F^TP_1F+F^TP_2G+G^TP_2^TF)]^{-1} \times \\
& [F^TP_1+G^TP_2^T+G^TP_3C+M^TD_1+\varepsilon(F^TP_1A+G^TP_2^TA+F^TP_2C)] ,
\end{aligned}
$$

$$
\begin{aligned}
f_2 = {} & P_1B+P_2S+C^TP_3S+D_1^TD_2+\varepsilon[A^TP_1B+A^TP_2S+C^TP_2^TB] \\
& - [P_1F+P_2G+C^TP_3G+D_1^TM+\varepsilon(A^TP_1F+A^TP_2G+C^TP_2^TF)]\times \\
& [R+M^TM+G^TP_3G+\varepsilon(F^TP_1F+F^TP_2G+G^TP_2^TF)]^{-1} \times \\
& [G^TP_3S+M^TD_2+\varepsilon(F^TP_1B+G^TP_2^TB+F^TP_2S)] ,
\end{aligned}
$$

and

$$
\begin{aligned}
f_3 = {} & S^TP_3S+D_2^TD_2+\varepsilon[B^TP_1B+B^TP_2S+S^TP_2^TB] \\
& - [S^TP_3G+D_2^TM+\varepsilon(B^TP_1F+B^TP_2G+S^TP_2^TF)] \times \\
& [R+M^TM+G^TP_3G+\varepsilon(F^TP_1F+F^TP_2G+G^TP_2^TF)]^{-1} \times \\
& [G^TP_3S+M^TD_2+\varepsilon(F^TP_1B+G^TP_2^TB+F^TP_2S] .
\end{aligned}
$$

APPENDIX 3.3

Wish to prove that

$$I-M(R + M^TM)^{-1}M^T = (I + MR^{-1}M^T)^{-1} .$$

Note that

$$[I-M(R + M^TM)^{-1}M^T][I + MR^{-1}M^T]$$

$$= I + MR^{-1}M^T - M(R + M^TM)^{-1}M^T - M(R + M^TM)^{-1}M^TMR^{-1}M^T$$

$$= I + M R^{-1}M^T - M(R + M^TM)^{-1}M^T - M(R + M^TM)^{-1}(R + M^TM-R)R^{-1}M^T$$

$$= I + M R^{-1}M^T - M(R + M^TM)^{-1}M^T - MR^{-1}M + M(R + M^TM)^{-1}M^T$$

$$= I .$$

By the same way

$$[I + MR^{-1}M^T][I - M(R + M^TM)^{-1}M^T] = I .$$

Q.E.D.

APPENDIX 3.4

Proof of the Lemma: Define

$$H = I-R_0^{-1}D_0^T(I-S+GR_0^{-1}D_0^T)^{-1}G,$$

then

$$H^{-1} = I+R_0^{-1}D_0^T(I-S)^{-1}G.$$

H and H^{-1} are well defined if $(I-S)$ is nonsingular. Consider

$$H^{-T}R_0H^{-1} = R_0+G^T(I-S)^{-T}D_0+D_0^T(I-S)^{-1}G+G^T(I-S)^{-T}D_0R_0^{-1}D_0^T(I-S)^{-1}G.$$

Using equaiton (3.38) (or equivalently, equation (3.72), to eliminate $D_0R_0^{-1}D_0^T$ we get

$$H^{-T}R_0H^{-1} = R_0+G^T(I-S)^{-T}D_0+D_0^T(I-S)^{-1}G$$

$$+ G^T(I-S)^{-T}[D_2^TD_2+S^TP_fS-P_f]\ (I-S)^{-1}G.$$

Substituting for R_0 and D_0 using their defining expressions, given after (3.57c), we have

$$H^{-T}R_0H^{-1} = R+M^TM+G^TP_fG+G^T(I-S)^{-T}(S^TP_fG+D_2^TM)+(M^TD_2+G^TP_fS)(I-S)^{-1}G$$

$$+ G^T(I-S)^{-T}D_2^TD_2(I-S)^{-1}G+G^T(I-S)^{-T}(S^TP_fS-P_f)(I-S)^{-1}G$$

$$= R+M^TM+G^T(I-S)^{-T}D_2^TM+M^TD_2(I-S)^{-1}G+G^T(I-S)^{-T}D_2^TD_2(I-S)^{-1}G$$

$$+ G^T(I-S)^{-T}(I-S)^T P_f(I-S)(I-S)^{-1}G+G^T(I-S)^{-T}S^T P_f(I-S)(I-S)^{-1}G+G^T(I-S)^{-T}$$

$$(I-S)^T P_f S(I-S)^{-1} + G^T(I-S)^{-T}(S^T P_f S-P_f)(I-S)^{-1}G$$

$$= R+D_s^T D_s+G^T(I-S)^{-T}[(I-S)^T P_f(I-S)+S^T P_f(I-S)+(I-S)^T P_f S + S^T P_f S-P_f](I-S)^{-1}G.$$

Inside the bracket is zero, so we obtain

$$H^{-T}R_0H^{-1} = R+D_s^T D_s .$$

Hence

$$H^{-T}R_0H^{-1} = R_s . \qquad (1)$$

Consider next

$$\hat{B} = F+L_2G = F+(B-FR_0^{-1}D_0^T)(I-S+GR_0^{-1}D_0^T)^{-1}G$$

$$= F+(B-FR_0^{-1}D_0^T)[I-(I-S+GR_0^{-1}D_0^T)^{-1}GR_0^{-1}D_0^T](I-S)^{-1}G$$

$$= F+B(I-S)^{-1}G-FR_0^{-1}D_0^T(I-S)^{-1}G-L_2GR_0^{-1}D_0^T(I-S)^{-1}G$$

$$= B_s-(F+L_2G)(H^{-1}-I) = B_s-\hat{B}(H^{-1}-I).$$

Hence

$$\hat{B} = B_sH. \qquad (2)$$

Using (1) and (2) we get

$$\hat{B}\ \hat{R}^{-1}\ \hat{B}^T = B_s H R_0^{-1} H^T B_s^T = B_s\ R_s^{-1}\ B_s^T ,$$

which proves (3.75).

Consider now

$$\hat{A} = A+L_2C-\hat{B}\ R_0^{-1}[G^T L_1^T+G^T P_f C+M^T D_1] . \tag{3}$$

The second term L_2C is given by

$$\begin{aligned} L_2C &= (B-FR_0^{-1}D_0^T)(I-S+GR_0^{-1}D_0^T)^{-1}C \\ &= (B-FR_0^{-1}D_0^T)(I-S+GR_0^{-1}D_0^T)^{-1}(I-S+GR_0^{-1}D_0^T-GR_0^{-1}D_0^T)(I-S)^{-1}C \\ &= (B-FR_0^{-1}D_0^T)[I-(I-S+GR_0^{-1}D_0^T)^{-1}GR_0^{-1}D_0^T](I-S)^{-1}C \\ &= B(I-S)^{-1}C-(FR_0^{-1}D_0^T+L_2GR_0^{-1}D_0^T)(I-S)^{-1}C \\ &= B(I-S)^{-1}C-\hat{B}\ R_0^{-1}D_0^T(I-S)^{-1}C , \end{aligned} \tag{4}$$

and the third term of (3) is given by

$$\begin{aligned} L_1G+C^TP_fG+D_1^TM &= [D_1^TD_2+C^TP_fS-(C^TP_fG+D_1^TM)R_0^{-1}D_0^T](I-S+GR_0^{-1}D_0^T)^{-1}G \\ &\quad + C^TP_fG+D_1^TM. \end{aligned}$$

Using the matrix identity

$$(I-S+GR_0^{-1}D_0^T)^{-1}G = (I-S)^{-1}G[I-R_0^{-1}D_0^T(I-S+GR_0^{-1}D_0^T)^{-1}G] ,$$

and the definition of H, we get

$$\begin{aligned} L_1G+C^TP_fG+D_1^TM &= (D_1^TD_2+C^TP_fS)(I-S)^{-1}GH+(C^TP_fG+D_1^TM)H \\ &= D_1^TD_2(I-S)^{-1}GH+D_1^TMH+C^TP_f(I-S)^{-1}GH. \end{aligned} \tag{5}$$

Now (2), (4) and (5) yield

$$\hat{A} = A_s - B_s H R_0^{-1} H^T [H^{-T} D_0^T (I-S)^{-1} C + G^T (I-S)^{-T} D_2^T D_1 + M^T D_1 + G^T (I-S)^{-T} P_f C].$$

Substituting for H^{-T} and D_0 and using (1) yield

$$\hat{A} = A_s - B_s R_s^{-1} [G^T (I-S)^{-T} D_0 R_0^{-1} D_0^T (I-S)^{-1} C + M^T D_2 (I-S)^{-1} C + G^T P_f S (I-S)^{-1} C$$

$$+ G^T (I-S)^{-T} D_2^T D_1 + M^T D_1 + G^T (I-S)^{-T} P_f C].$$

Using (3.72) to eliminate P_f, we get

$$\hat{A} = A_s - B_s R_s^{-1} [G^T (I-S)^{-T} D_2^T D_2 (I-S)^{-1} C + M^T D_2 (I-S)^{-1} C + M^T D_1$$

$$+ G^T (I-S)^{-T} D_2^T D_1]$$

$$= A_s - B_s R_s^{-1} D_s^T C_s ,$$

which proves (3.74)

Finally, for proving (3.76) we write L_1 as

$$L_1 = [D_1^T D_2 + C^T P_f S - (C^T P_f G + D_1^T M) R_0^{-1} D_0^T][I - S + G R_0^{-1} D_0^T]^{-1} [I - S + G R_0^{-1} D_0^T$$

$$- G R_0^{-1} D_0^T](I-S)^{-1}$$

$$= [D_1^T D_2 + C^T P_f S - (C^T P_f G + D_1^T M) R_0^{-1} D_0^T][I - (I - S + G R_0^{-1} D_0^T)^{-1} G R_0^{-1} D_0^T](I-S)^{-1}$$

$$= (D_1^T D_2 + C^T P_f S](I-S)^{-1} - (L_1 G + C^T P_f G + D_1^T M) R_0^{-1} D_0^T (I-S)^{-1}.$$

Now we get

$$\hat{Q} = D_1^T D_1 + D_1^T D_2 (I-S)^{-1} C + C^T (I-S)^{-T} D_2^T D_1 + C^T P_f S (I-S)^{-1} C + C^T (I-S)^{-T} S^T P_f C$$

$$+ C^T P_f C - (L_1 G + C^T P_f G + D_1^T M) R_0^{-1} D_0^T (I-S)^{-1} C - C^T (I-S)^{-T} D_0 R_0^{-1} (L_1 G$$

$$+ C^TP_fG+D_1^TM)^T-(L_1G+C^TP_fG+D_1^TM)R_0^{-1}(L_1G+C^TP_fG+D_1^TM)^T .$$

Using (5) and simple algebraic manipulations we get

$$\hat{Q} = C_s^TC_s-C^T(I-S)^{-T}D_2^TD_2(I-S)^{-1}C+C^T(I-S)^{-T}[I-S)^TP_fS+S^TP_f(I-S)$$

$$+ (I-S)^TP_f(I-S)](I-S)^{-1}C-[(D_1^TD_2+C^TP_fS)(I-S)^{-1}G+D_1^TM]R_s^{-1}H^{-T}D_0^T$$

$$(I-S)^{-1}C-C^T(I-S)^{-T}D_0H^{-1}R_s^{-1}[(D_1^TD_2+C^TP_fS)(I-S)^{-1}G+D_1^TM]^T$$

$$- [(D_1^TD_2+C^TP_fS)(I-S)^{-1}G+D_1^TM]R_s^{-1}[(D_1^TD_2+C^TP_fS)(I-S)^{-1}G+D_1^TM]^T .$$

Using (3.72) we obtain

$$\hat{Q} = C_s^TC_s-C^T(I-S)^{-T}D_2^TD_2(I-S)^{-1}C+C^T(I-S)^{-T}[D_2^TD_2-D_0R_0^{-1}D_0^T](I-S)^{-1}C$$

$$- [D_1^TD_2(I-S)^{-1}G+C^TP_fS(I-S)^{-1}G+D_1^TM+C^T(I-S)^{-T}D_0H^{-1}]R_s^{-1}$$

$$[D_1^TD_2(I-S)^{-1}G+C^TP_fS(I-S)^{-1}G+D_1^TM+C^T(I-S)^{-T}D_0H^{-1}]^T$$

$$+ C^T(I-S)^{-T}D_0H^{-1}R_s^{-1}H^{-T}D_0^T(I-S)^{-1}C .$$

After eliminating similar terms, the manipulations are just repetitions of what was done to prove (3.74) and (3.75). First the term $(L_1G+C^TP_fG+D_1^TM)$ is substituted, using (5). Second, (3.72) is used to eliminate P_f. The remaining expression, which is independent of P_f, can be easily shown to be $C_s^T(I-D_sR_s^{-1}D_s^T)C_s$.

CHAPTER 4

COMPOSITE CONTROL AND MULTIRATE MEASUREMENT

4.1. Introduction

In singular perturbation theory the use of feedback control input which is obtained by composing the control inputs of slow and fast subsystems is considered frequently. By employing such feedback controls on the full system a close approximation of the design objective has been acheived. [Chow and Kokotovic, 1976], [Phillips, 1980].

In chapter three an infinite-time regulator problem for difference equations was studied and the role of composite feedback control in achieving an $O(\epsilon^2)$ near-optimality was discussed. In this chapter we extend on our discussion on the role of composite feedback control in the context of stabilization. Also, the problem of stabilization in view of multirate measurements of the state variables using a composite feedback control is investigated and different designs are proposed.

In section 4.2 we consider a discrete linear-time-invariant system and by using the stabilizing feedback controls of slow and fast subsystems, which evolve in fast time-scale, we introduce a composite feedback control for stabilizing the full system and we show the closeness of trajectories.

Section 4.3 deals with the same problem as section 4.2 but the composite feedback control employs multirate measurements. By letting $\{0,\epsilon,2\epsilon,\ldots,\bar{N}\epsilon\}$ to be a mesh on $[0, \frac{1}{\epsilon}]$, the fast states are measured for every n, while the slow states are measured periodically with a period of $\frac{1}{\epsilon}$. To compute the slow states for any n, we solve their dynamic equations in terms of their measurements at the beginning of the slow measurement periods.

This way of forming the composite control is quite natural because one, usually, expects to acquire the measurements of slow states at slow-time intervals, while for fast states the measurements are obtained for each mesh point n.

A parallel design procedure for slow and fast subsystems is introduced and a composite feedback control is formed, using these subsystems. By applying this composite feedback control to the full system, the aysmptotic stability and closeness of trajectories is shown. Also, a numerical example to illustrate the claims is given at the end of this section.

Finally, in Section 4.4 a sequential design is studied where a pre-conditioning feeback gain is designed first. The role of this gain is to stabilize the fast modes and allocates the corresponding eigenvalues, appropriately. Based on this pre-conditioning gain the slow subsystem is designed. A composite feedback control is formed and a similar investigation as in Section 4.3 is performed.

4.2. A Stabilizing Composite Control with State Measurements in Fast Time Scale

In this section we discuss the application of composite feedback control in the context of stabilizability and closeness of trajectories for the case that slow and fast subsystems evolve in the fast time-scale (n)

and the measurements of states are available at all values of n. Consider the linear time-invariant discrete-time singularly perturbed system as in (2.86)

$$x_1(n+1) = [I_1+\epsilon A_{11}(\epsilon)]x_1(n) + \epsilon A_{12}(\epsilon)x_2(n) + \epsilon B_1(\epsilon)u(n) \qquad (4.1a)$$

$$x_2(n+1) = A_{21}(\epsilon)x_1(n) + A_{22}(\epsilon)x_2(n) + B_2(\epsilon)u(n)\,. \qquad (4.1b)$$

The initial values are given and all the matrices are analytic functions of ϵ and (I_2-A_{22}) is nonsingular.

We assume that the control input decompses as

$$u(n) = u_1(n) + u_2(n),\ n = 0,1,2,\ldots$$

where $u_2(n)$ is exponetially stable i.e.,

$$|u_2(n)| \leq K\alpha^n\ ,\quad \alpha < 1\,.$$

The procedure for finding the slow and fast subproblems is similar to the "approximation result" discussed in Section 2.5. So in this section we briefly reintroduce them.

<u>Slow Subsystem</u>

Assuming $x_2(n)$ has reached its steady state $(u_2(n) \simeq 0)$ and repeating the same steps as in Section 2.5, we arrive at

$$\tilde{x}_1(n+1) = (I+\epsilon A_0)\tilde{x}_1(n) + \epsilon B_0\bar{u}_1(n),\ \tilde{x}_1(0) = x_1(0) \qquad (4.2)$$

where

$$A_0 = A_{11} + A_{12}(I_2-A_{22})^{-1}A_{21}$$

$$B_0 = B_1 + A_{12}(I_2-A_{22})^{-1}B_2 \,.$$

Note again that matrices evaluated at $\epsilon = 0$ are denoted by deleting the argument ϵ, This will be used through the text. Also, bars indicate the steady state case. Now letting $x_s(n) = \tilde{x}_1(n)$ and $u_s(n) = \bar{u}_1(n)$ we define the slow subsystem to be

$$x_s(n+1) = (I_1+\epsilon A_0)x_s(n) + \epsilon B_0 u_s(n),\ x_s(0) = x_1(0)\,. \tag{4.3}$$

<u>Condition a</u>.

Suppose that the state feedback control law for $u_s(n)$ is designed as

$$u_s(n) = F_s x_s(n) \tag{4.4}$$

where F_s is chosen such that the $\operatorname{Re} \lambda(A_0 + B_0F_s) < 0$ or equivalently the closed-loop system

$$x_s(n+1) = [I+\epsilon(A_0+B_0F_s)]x_s(n) \tag{4.5}$$

is asymptotically stable in the discrete-time sense and meets some design objectives as pole-placement, linear quadratic, etc.

<u>Fast subsystem</u>

Following the same procedure as in Section 2.5 and letting $x_f(n) = x_2(n) - \tilde{x}_2(n)$ and $u_f(n) = u_2(n)$ and

$$\tilde{x}_2(n) = (I-A_{22})^{-1}[A_{21}\tilde{x}_1(n) + B_2\bar{u}_1(n)] . \tag{4.6}$$

The fast subsystem is defined to be

$$x_f(n+1) = A_{22}x_f(n) + B_2u_f(n), x_f(0) = x_2(0) - \tilde{x}_2(0) . \tag{4.7}$$

Condition b.

Suppose that the feedback control law $u_f(n)$ is designed as

$$u_f(n) = F_f x_f(n) \tag{4.8}$$

where F_f is chosen such that the closed-loop system

$$x_f(n+1) = [A_{22}+B_2F_f]x_f(n) \tag{4.9}$$

is asymptotically stable and meets some design objectives. Again the design method is not crucial.

Composite Control

With the solutions of slow and fast problems, a composite feedback control is formed as

$$u_c(n) = u_s(n) + u_f(n) = F_s x_s(n) + F_f x_f(n). \tag{4.10}$$

Substituting for $x_f(n) = x_2(n) - \tilde{x}_2(n)$, using (4.6) and (4.4), and approximating $x_s(n)$ by $x_1(n)$ yields

$$u_c(n) = [F_s - F_f(I_2-A_{22})^{-1}(A_{21} + B_2F_s)]x_1(n) + F_f x_2(n) \tag{4.11}$$

$$\triangleq F_1x_1(n) + F_2x_2(n).$$

Applying the composite feedback control (4.11) to the full system (4.1) yields

$$x_1(n+1) = [I_1+\epsilon\bar{A}_{11}(\epsilon)]x_1(n) + \epsilon\bar{A}_{12}(\epsilon)x_2(n) \tag{4.12a}$$

$$x_2(n+1) = \bar{A}_{21}(\epsilon)x_1(n) + \bar{A}_{22}(\epsilon)x_2(n) \tag{4.12b}$$

where

$$\bar{A}_{ij}(\epsilon) = A_{ij}(\epsilon) + B_i(\epsilon)F_j \ , \quad i,j = 1,2. \tag{4.13}$$

Using a decoupling transformation similar to (2.25a), i.e.,

$$\eta = \begin{bmatrix} I_1-\epsilon M_1(\epsilon)L_1(\epsilon) & -\epsilon M_1(\epsilon) \\ L_1(\epsilon) & I_2 \end{bmatrix} x \tag{4.14}$$

where the matrices L_1 and M_1 satisfy

$$0 = \bar{A}_{21}(\epsilon) + L_1(\epsilon)-\bar{A}_{22}(\epsilon)L(\epsilon) + \epsilon L_1(\epsilon)[\bar{A}_{11}(\epsilon)-\bar{A}_{12}(\epsilon)L_1(\epsilon)] \tag{4.15}$$

$$0 = \bar{A}_{12}(\epsilon) + M_1(\epsilon)-M_1(\epsilon)\bar{A}_{22}(\epsilon) + \epsilon[\bar{A}_{11}(\epsilon)-\bar{A}_{12}(\epsilon)L_1(\epsilon)]\, M_1 - \epsilon M_1(\epsilon)L_1(\epsilon)\bar{A}_{12}(\epsilon) \tag{4.16}$$

In fact $L_1(\epsilon)$ and $M_1(\epsilon)$ can be approximated by

$$L_1(\epsilon) = -\,(I_2-\bar{A}_{22})^{-1}\bar{A}_{21} + 0(\epsilon) = -[I_2-A_{22}-B_2F_2]^{-1}[A_{21} + B_2F_1] + 0(\epsilon) \tag{4.17}$$

$$M_1(\epsilon) = -\,\bar{A}_{12}(I_2-\bar{A}_{22})^{-1} + 0(\epsilon) = -[A_{12} + B_1F_2][I_2-A_{22}-B_2F_2]^{-1} + 0(\epsilon). \tag{4.18}$$

The system (4.12) becomes

$$\eta_1(n+1) = [I_1 + \epsilon(\bar{A}_{11}+\bar{A}_{12}(I-\bar{A}_{22})^{-1}\bar{A}_{21}) + O(\epsilon^2)]\eta_1(n) \quad (4.19a)$$

$$\eta_1(n+1) = [\bar{A}_{22} + O(\epsilon)]\eta_2(n) \quad (4.19b)$$

where we have used Householder Theorem to show $O(\epsilon)$ approximation of matrices.

But (see Appendix 4.1)

$$\bar{A}_{11} + \bar{A}_{12}(I_2-\bar{A}_{22})^{-1}\bar{A}_{21} = A_0 + B_0F_s. \quad (4.20)$$

Now the asymptotic stability of (4.19) and consequently (4.12) follows by using

$$\text{Re}\ \lambda(A_0 + B_0F_s) < 0$$

$$|\lambda(A_{22} + B_2F_f)| < 1$$

and Theorem 2.5.1.

By comparing (4.5) and (4.19a) and using (4.20) and (4.14) it is obvious that

$$x_1(n) = x_s(n) + O(\epsilon). \quad (4.21)$$

Similarly, comparing (4.9) and (4.19b) and using (4.14) it follows that

$$x_2(n) = x_f(n) - L_1\ x_s(n) + O(\epsilon)$$

$$= x_f(n) + (I_2-A_{22}-B_2F_2)^{-1}(A_{21}+B_2F_1)x_s(n) + O(\epsilon). \quad (4.22)$$

But

$$(I_2-A_{22}-B_2F_2)^{-1}(A_{21}+B_2F_1)$$

$$= (I_2-A_{22}-B_2F_f)^{-1}[A_{21}+B_2F_s-B_2F_f(I_2-A_{22})^{-1}(A_{21}+B_2F_s)]$$

$$= (I_2-A_{22}-B_2F_f)^{-1}(I_2-A_{22}-B_2F_f)(I_2-A_{22})^{-1}A_{21}$$

$$+ (I_2-A_{22}-B_2F_f)^{-1}(I_2-A_{22}-B_2F_f)(I_2-A_{22})^{-1}B_2F_s$$

$$= (I_2-A_{22})^{-1}(A_{21}+B_2F_s) .$$

Hence

$$x_2(n) = x_f(n) + (I_2-A_{22})^{-1}(A_{21}+B_2F_s)x_s(n) + O(\epsilon) \qquad (4.23)$$

which agrees with the intuitive decomposion of x_2 as the sum of x_2 and $\tilde{x}_2$, where $\tilde{x}_2$ is given by (4.6).

Based on our discussions above we conclude the following theorem.

Theorem 4.2.1

Under the conditions a and b, and for sufficiently small ϵ, the application of the composite control (4.11) to the system (4.1) results in an asymptotically stable closed-loop system. Moreover, the solution of (4.1) can be approximated by

$$x_1(n) = x_s(n) + O(\epsilon)$$

$$x_2(n) = x_f(n) + (I-A_{22})^{-1}(B_2F_s+A_{21})x_s(n) + O(\epsilon)$$

where x_f and x_s are solutions of the fast subsystem (4.9) and the slow subsystem (4.5), respectively.

4.3. Multirate stabilization with slow state measurements in slow time-scale.

The practical need for multirate measurement is a basic consequence of the finite computing capabilities of onboard digital computers and the common goal of reducing the operating cost. Functions associated with fast modes typically demand measurements an order of magnitude higher than the rate which is necessary for suitable control of slow modes of the system. Faced with widely varying measurement requirements among the dynamic modes of the system, a multirate feedback control structure is the solution to computational and cost limitations.

Synthesizing a multirate control system to meet desired objectives has been a difficult task. As an example, the problem of multirate sampled-data control of optimal regulators for singularly perturbed systems has been an open subject.

In this section we investigate the application of a multirate stabilizing composite feedback control on system (4.1) when the measurements of the slow states are available only at slow time-scale $(\frac{K}{\varepsilon})$, $K = 0,1,2,\ldots$, and the control input consists of slow and fast parts. A parallel design procedure for designing control inputs of slow and fast subsystems is introduced. Also, an overall control input which is composed of control inputs of slow and fast subsystems is evaluated which results in stabilizing control for the full system.

Again, consider system (4.1). We assume that the control input decomposes as

$$u(n) = u_1(n) + u_2(n), \quad n = 0,1,2,\ldots$$

where $u_1(n)$ is constant for $\frac{K}{\epsilon} \leq n < \frac{K+1}{\epsilon}$ and $u_2(n)$ is exponentially stable, i.e.,

$$|u_2(n)| \leq K\alpha^n \ , \quad \alpha < 1$$

Slow subsystem

Assume that $x_2(n)$ has reached its steady state $(u_2(n) \simeq 0)$. Repeating the same steps as in Section 2.5, we arrive at (4.2) which is

$$\tilde{x}_1(n+1) = (I_1+\epsilon A_0)\tilde{x}_1(n) + \epsilon B_0\bar{u}_1(n), \ \tilde{x}_1(0) = x_1(0)$$

and A_0, B_0 are given as before and $\bar{u}_1(n) = u_1(n)$. Since $\bar{u}_1(n)$ is constant over the cycle $\frac{K}{\epsilon} \leq n < \frac{K+1}{\epsilon}$ we can express $\tilde{x}_1(\frac{K+1}{\epsilon})$ in terms of $\tilde{x}_1(K/\epsilon)$ and $\bar{u}_1(K/\epsilon)$.

$$\tilde{x}_1(\tfrac{K+1}{\epsilon}) = (I_1+\epsilon A_0)^{\frac{1}{\epsilon}} \tilde{x}_1(\tfrac{K}{\epsilon}) + \epsilon \sum_{j=K/\epsilon}^{\frac{K+1}{\epsilon}-1} (I_1+\epsilon A_0)^{\frac{K+1}{\epsilon}-1-j} B_0\bar{u}_1(\tfrac{K}{\epsilon}) .$$

Letting $i = \frac{K+1}{\epsilon} - 1 - j$ we obtain

$$\tilde{x}_1(\tfrac{K+1}{\epsilon}) = (I_1+\epsilon A_0)^{1/\epsilon} \tilde{x}_1(K/\epsilon) + \epsilon \sum_{i=0}^{1/\epsilon-1} (I_1+\epsilon A_0)^i B_0\bar{u}_1(K/\epsilon). \quad (4.24)$$

Now let

$$u_s(K) = \bar{u}_1(\tfrac{K}{\epsilon}) \quad (4.25)$$

$$x_s(K) = \tilde{x}_1(K/\epsilon) \quad (4.26)$$

$$A_s = e^{A_0} \quad (4.27)$$

and

$$B_s = \int_0^1 e^{A_0(1-t)} dtB_0 \tag{4.28}$$

We define the slow subsystem to be

$$x_s(K+1) = A_s\, x_s(K) + B_s\, u_s(K),\ x_s(0) = x_1(0). \tag{4.29}$$

Suppose that the state feedback control law for $u_s(K)$ is designed as

$$u_s(K) = F_s\, x_s(K) \tag{4.30}$$

where F_s is chosen such that the closed-loop system

$$x_s(K+1) = (A_s+B_sF_s)x_s(K) \tag{4.31}$$

is asymptotically stable and meets some design objective like pole-placement, linear quadratic, etc.

<u>Fast subsystem</u>

Following the same procedure as previous section we define the fast subsystem to be

$$x_f(n+1) = A_{22}x_f(n) + B_2u_f(n),\ x_f(\tfrac{K}{\epsilon}) = x_2(\tfrac{K}{\epsilon}) - \tilde{x}_2(\tfrac{K}{\epsilon}) \tag{4.32}$$

where

$$\tilde{x}_2(n) = (I_2-A_{22})^{-1}[A_{21}\tilde{x}_1(n) + B_2\bar{u}_1(n)]. \tag{4.33}$$

From (4.32) and (4.33) the initial conditions for fast subsystem are

$$x_f(\tfrac{K}{\epsilon}) = x_2(\tfrac{K}{\epsilon}) - (I_2-A_{22})^{-1}[A_{21}x_1(\tfrac{K}{\epsilon}) + B_2u_s(K)] \tag{4.34}$$

where we have used (4.25)

From (4.34) we obtain

$$x_f(0) = x_2(0) - (I_2-A_{22})^{-1}[A_{21}x_1(0) + B_2u_s(0)]. \qquad (4.35)$$

Now, let us rewrite the equation for x_2. Applying (4.33) during the (k-1)th cycle we get

$$x_2(n) \simeq (I_2-A_{22})^{-1}[A_{21}\tilde{x}_1(n) + B_2u_s(K-1)] \; , \; \frac{K-1}{\epsilon} < n \leq \frac{K}{\epsilon} \; .$$

Thus

$$x_2(\tfrac{K}{\epsilon}) = (I_2-A_{22})^{-1}[A_{21}x_1(\tfrac{K}{\epsilon}) + B_2u_s(K-1)].$$

Now, the initial condition of equation (4.34) reduces to

$$x_f(\tfrac{K}{\epsilon}) = (I_2-A_{22})^{-1}B_2[u_s(K-1) - u_s(K)] \; , \; K > 0 \; . \qquad (4.35)$$

Again, assume that condition 'b', as in section 4.2, is satisfied so we have the asymptotically stable closed-loop system

$$x_f(n+1) = (A_{22} + B_2F_f)x_f(n) \qquad (4.37a)$$

with

$$x_f(0) = x_2(0) - (I_2-A_{22})^{-1}[A_{21}x_1(0) + B_2u_s(0)] \qquad (4.37b)$$

$$x_f(\tfrac{K}{\epsilon}) = (I_2-A_{22})^{-1}B_2[u_s(K-1) - u_s(K)] \; , \; K > 0 \; . \qquad (4.37c)$$

We notice that the initial conditons for the fast subsystem depend on the slow control. This means that any abrupt change in the slow control will excite the fast modes and causes fast transients for a short period.

This observation is particularly important in our scheme since for every $\frac{1}{\epsilon}$ time-intervals there is an abrupt change in the slow control so that the fast subsystem has to be resolved at the beginning of each cycle of the slow control.

Composite Control:

With the solutions of slow and fast subsystems in hand, a composite feedback control is formed as

$$u_c(n) = u_s(n) + u_f(n) = F_s x_s(K) + F_f x_f(n), \; \frac{K}{\epsilon} \leq n < \frac{K+1}{\epsilon}. \quad (4.38)$$

So we have

$$u_c(n) = F_s x_s(K) + F_f[x_2(n) - \tilde{x}_2(n)]$$

$$= F_s x_s(K) + F_f x_2(n) - F_f(I_2 - A_{22})^{-1}[A_{21}\tilde{x}_1(n) + B_2\bar{u}_1(n)] .$$

We approximate $\bar{u}_1(n)$ by $F_s x_1(\frac{K}{\epsilon})$ and also approximate $\tilde{x}_1(n)$ by interpolating (4.2) for $\frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon}$, i.e.

$$\tilde{x}_1(n) = [(I_1 + \epsilon A_0)^{n-\frac{K}{\epsilon}} + \epsilon \sum_{j=\frac{K}{\epsilon}}^{n-1} (I_1 + \epsilon A_0)^{n-1-j} B_0 F_s] x_1(\tfrac{K}{\epsilon}) . \quad (4.39)$$

Note, for $n = \frac{K}{\epsilon}$ we define $\tilde{x}_1(n) = x_1(\frac{K}{\epsilon})$.

Thus we have the following form for the composite control law

$$u_c(n) = \hat{F}(n) x_1(\tfrac{K}{\epsilon}) + F_f x_2(n) \quad , \quad \frac{K}{\epsilon} \leq n < \frac{K+1}{\epsilon} \quad (4.40)$$

where

$$\hat{F}(n) = \begin{cases} F_s - F_f(I_2 - A_{22})^{-1}[A_{21}V(n) + B_2 F_s], & K/\epsilon < n < \frac{K+1}{\epsilon} \\ F_s - F_f(I_2 - A_{22})^{-1}(A_{21} + B_2 F_s) \; , & n = K/\epsilon \end{cases} \quad (4.41)$$

and $V(n)$ satisfies

$$V(n+1) = (I_1+\epsilon A_0)V(n) + \epsilon B_0 F_s, \; V(\tfrac{K}{\epsilon}) = I_1 \qquad (4.42)$$

The use of the composite feedback control law (4.42) is justified by the following theorems.

Theorem 4.3.1: Stability Result

If the feedback control input u_c, defined by (4.40), is applied to the system (4.1) and if the closed-loop systems (4.31) and (4.37) are asymptotically stable in the discrete-time sence, i.e., all their eigenvalues are withing the unit circle, then for sufficiently small ϵ, $X(n)$, $X^T(n) = [x_1^T(n), x_2^T(n)]$, is asymptotically stable, i.e.,

$$X(n) \to 0 \quad \text{as} \quad n \to \infty.$$

We note that while the measurements of x_1 are available at only slow cycles $(\frac{K}{\epsilon})$, K integer, Theorem 4.3.1 gives the asymptotic stability of x_1 at all fast mesh points n.

Theorem 4.3.2: Approximation Result

If the conditions of Theorem 4.3.1 are met and if the initial conditions

$$x_s(0) = x_1(0), \qquad (4.43a)$$

$$x_f(0) = x_2(0) - (I_2-A_{22})^{-1}(A_{21}+B_2F_s)x_1(0) \qquad (4.43b)$$

$$x_f(\tfrac{K}{\epsilon}) = (I_2-A_{22})^{-1}B_2F_s[x_s(K-1) - x_s(K)], \; K > 0 \qquad (4.43c)$$

are satisfied, then, for sufficiently small ϵ, the state trajectories could be approximated as

$$x_1(K/\epsilon) = x_s(K) + O(\epsilon)$$

$$x_2(n) = x_f(n) + (I_2-A_{22})^{-1}[A_{21}V(n) + B_2F_s]x_s(K) + O(\epsilon)$$

We note that

(1) Theorem 4.3.2 gives an approximation result for x_1, only, at the points K/ϵ while it gives an approximation result for x_2 for all n, which is the best we can expect in view of using multirate measurements.

(2) The value of the theorem as a design tool can be seen if we desire a specific case, pole-placement design say. A designer would choose F_s to locate the poles of the closed-loop system (4.31) at certain locations inside the unit circle. Next, F_f is chosen to locate the poles of the closed-loop system (4.37) at certain locations inside the unit circle. Finally, (4.31) and (4.37) are solved for the initial conditons (4.43) to obtain $x_s(K)$ and $x_f(n)$. The actual response of the system is predicted using the relations

$$x_1(\frac{K}{\epsilon}) = x_s(K)$$

and

$$x_2(n) = x_f(n) + (I_2-A_{22})^{-1}[A_{21}V(n) + B_2F_s]x_s(K).$$

If the designer is not satisfied with the response, the choice of F_s and F_f is iterated until a satisfactory choice is reached.

(3) The solutions of (4.31) with initial conditions (4.43a), and of (4.37) with initial conditons (4.43b,c) can be obtained simultaneously. At $K = 0$, given $x_1(0)$ and $x_2(0)$ we can compute $x_f(n)$ for all $0 \leq n < \frac{1}{\epsilon}$.

At $K = 1$, given $x_s(0)$ we compute $x_s(1)$ which together with $x_s(0)$ provides the initial conditions $x_f(\frac{1}{\epsilon})$ so that we can compute $x_f(n)$ for $\frac{1}{\epsilon} \leq n < \frac{2}{\epsilon}$. In general, at any K, $x_s(K)$ and $x_s(K-1)$ will be available so that (4.37) can be solved in the K^{th} cycle.

Proof of the Theorem 4.3.1

First, we prove the asymptotic stability for $x_1(\frac{K}{\epsilon})$ as $K \to \infty$. Second, we prove that $x_1(n)$ and $x_2(n)$ are linear combinations of $x_1(\frac{K}{\epsilon})$ and $x_2(\frac{K}{\epsilon})$ with bounded coefficients and so they are asymptotically stable as $n \to \infty$.

Applying the composite feedback control (4.38) to the full system (4.1) yields

$$x_1(n+1) = [I_1 + A_{11}(\epsilon)]x_1(n) + \epsilon[A_{12}(\epsilon) + B_1(\epsilon)F_f]x_2(n) + \epsilon B_1(\epsilon)\hat{F}(n)x_1(\tfrac{K}{\epsilon}) \tag{4.44a}$$

$$x_2(n+1) = A_{21}(\epsilon)x_1(n) + [A_{22}(\epsilon) + B_2(\epsilon)F_f]x_2(n) + B_2(\epsilon)\hat{F}(n)x_1(\tfrac{K}{\epsilon}). \tag{4.44b}$$

Using the decoupling transformation

$$\eta = \begin{bmatrix} I_1 - \epsilon M_2(\epsilon)L_2(\epsilon) & -M_2(\epsilon) \\ L_2(\epsilon) & I_2 \end{bmatrix} X \tag{4.45}$$

where the matrices L_2 and M_2 satisfy

$$0 = A_{21}(\epsilon) + L_2(\epsilon) - [A_{22}(\epsilon) + B_2(\epsilon)F_f]L_2(\epsilon) + \epsilon L_2(\epsilon)[A_{11}(\epsilon) - (A_{12}(\epsilon) + B_1(\epsilon)F_f)L_2(\epsilon)] \tag{4.46}$$

$$0 = A_{12}(\epsilon) + B_1(\epsilon)F_f + M_2(\epsilon) - M_2(\epsilon)[A_{22}(\epsilon) + B_2(\epsilon)F_f]$$
$$+ [A_{11}(\epsilon) - (A_{12}(\epsilon) + B_1(\epsilon)F_f)L_2(\epsilon)]$$
$$- M_2(\epsilon)L_2(\epsilon)[(A_{12}(\epsilon) + B_1(\epsilon)F_f)L_2(\epsilon)] \quad (4.47)$$

yields

$$\eta_1(n+1) = [I_1+\epsilon\tilde{A}+O(\epsilon^2)]\eta_1(n) + \epsilon[\tilde{B}+O(\epsilon)]\hat{F}(n)x_1(K/\epsilon) \quad (4.48a)$$

$$\eta_2(n+1) = [A_{22}+B_2F_f+O(\epsilon)]\eta_2(n) + [B_2+O(\epsilon)]\hat{F}(n)x_1(K/\epsilon), \quad (4.48b)$$

where

$$\tilde{A} = A_{11} + (A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)^{-1}A_{21} \quad (4.49)$$

$$\tilde{B} = B_1 + (A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)^{-1}B_2 . \quad (4.50)$$

From (4.48a) we have

$$\eta_1(n) = [I_1+\epsilon\tilde{A}+O(\epsilon^2)]^{n-K/\epsilon}\,\eta_1(K/\epsilon) + \epsilon\sum_{j=K/\epsilon}^{n-1}[I_1+\tilde{A}+O(\epsilon^2)]^{n-1-j}[\tilde{B}+O(\epsilon)]\hat{F}(j)\,x_1(K/\epsilon),$$
$$\frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon}. \quad (4.51)$$

In particular

$$\eta_1\left(\frac{K+1}{\epsilon}\right) = [I_1+\epsilon\tilde{A}+O(\epsilon^2)]^{1/\epsilon}\,\eta_1(K/\epsilon) + \epsilon\sum_{j=K/\epsilon}^{\frac{K+1}{\epsilon}-1}[I_1+\epsilon\tilde{A}+O(\epsilon^2)]^{\frac{K+1}{\epsilon}-1-j}[\tilde{B}+O(\epsilon)]\hat{F}(j)x_1(K/\epsilon) \quad (4.52)$$

using the inverse of transformation (4.45) we get

$$\eta_1(\tfrac{K+1}{\epsilon}) = \{[I_1+\epsilon\tilde{A}+O(\epsilon^2)]^{\frac{1}{\epsilon}} + \epsilon \sum_{j=\frac{K}{\epsilon}}^{\frac{K+1}{\epsilon}-1} [I_1+\epsilon\tilde{A}+O(\epsilon^2)]^{\frac{K+1}{\epsilon}-1-j} [\tilde{B}+O(\epsilon)]\hat{F}(j)\}\eta_1(\tfrac{K}{\epsilon})$$

$$+ O(\epsilon)\eta_2(\tfrac{K}{\epsilon}).$$

It is shown in Appendix 4.2 that

$$(I_1+\epsilon\tilde{A})^{\frac{1}{\epsilon}} + \epsilon \sum_{j=\frac{K}{\epsilon}}^{\frac{K+1}{\epsilon}-1} (I_1+\epsilon\tilde{A})^{\frac{K+1}{\epsilon}-1-j}\tilde{B}\hat{F}(j) = (I_1+\epsilon A_0)^{\frac{1}{\epsilon}} + \epsilon\sum_{\ell=0}^{\frac{1}{\epsilon}-1}(I_1+\epsilon A_0)^{\frac{1}{\epsilon}-1-\ell} B_0F_s.$$

On the other hand, using (2.107) and (2.108) we have

$$(I_1+\epsilon A_0)^{1/\epsilon} = e^{A_0} + O(\epsilon) = A_s + O(\epsilon) \tag{4.53}$$

$$\epsilon \sum_{j=0}^{\frac{1}{\epsilon}-1} (I_1+\epsilon A_0)^j B_0F_s = \int_0^1 e^{A_0(1-t)} B_0 dt\ F_s + O(\epsilon)$$

$$= B_sF_s + O(\epsilon). \tag{4.54}$$

Noting

$$\sum_{\ell=0}^{\frac{1}{\epsilon}-1} (I_1+\epsilon A_0)^{\frac{1}{\epsilon}-1-\ell} = \sum_{j=0}^{\frac{1}{\epsilon}-1} (I_1+\epsilon A_0)^j \ , \tag{4.54}$$

we get

$$\eta_1(\tfrac{K+1}{\epsilon}) = [A_s+B_sF_s+O(\epsilon)]\eta_1(\tfrac{K}{\epsilon}) + O(\epsilon)\eta_2(\tfrac{K}{\epsilon}). \tag{4.55}$$

Similarly, from (4.48b) we have

$$\eta_2(n) = [A_{22}+B_2F_f+O(\epsilon)]^{n-K/\epsilon}\eta_2(K/\epsilon)$$

$$+ \sum_{j=K/\epsilon}^{n-1} [A_{22}+B_2F_f+O(\epsilon)]^{n-1-j}[B_2+O(\epsilon)]\hat{F}(j)x_1(K/\epsilon),\ \frac{K}{\epsilon} < n \le \frac{K+1}{\epsilon} \tag{4.56}$$

and

$$\eta_2(\tfrac{K+1}{\epsilon}) = [A_{22}+B_2F_f+O(\epsilon)]^{\frac{1}{\epsilon}}\eta_2(\tfrac{K}{\epsilon}) + \sum_{j=\frac{K}{\epsilon}}^{\frac{K+1}{\epsilon}-1} [A_{22}+B_2F_f+O(\epsilon)]^{\frac{K+1}{\epsilon}-1-j}[B_2+O(\epsilon)]$$

$$\hat{F}(j)x_1(\tfrac{K}{\epsilon}). \qquad (4.57)$$

Using the asymptotic stability of $(A_{22}+B_2F_f)$ it follows that

$$[A_{22}+B_2F_f+O(\epsilon)]^{\frac{1}{\epsilon}} = O(\epsilon)$$

Using above and (4.41) yields

$$\eta_2(\tfrac{K+1}{\epsilon}) = O(\epsilon)\eta_2(\tfrac{K}{\epsilon}) + \sum_{j=\frac{K}{\epsilon}}^{\frac{K+1}{\epsilon}-1} [A_{22}+B_2F_f+O(\epsilon)]^{\frac{K+1}{\epsilon}-1-j}[I_2-B_2F_f(I_2-A_{22})^{-1}]B_2F_sx_1(\tfrac{K}{\epsilon})$$

$$- \sum_{j-\frac{K}{\epsilon}}^{\frac{K+1}{\epsilon}-1} [A_{22}+B_2F_f+O(\epsilon)]^{\frac{K+1}{\epsilon}-1-j}B_2F_f(I_2-A_{22})^{-1}A_{21}V(n)x_1(K/\epsilon)$$

$$+ O(\epsilon)x_1(K/\epsilon), \qquad (4.58)$$

where $V(n)$ satisfies

$$V(n+1) = (I+\epsilon A_0)V(n) + \epsilon B_0F_s, \quad V(\tfrac{K}{\epsilon}) = I. \qquad (4.59)$$

Noting that

$$\sum_{j=0}^{\frac{1}{\epsilon}-1} (A_{22}+B_2F_f)^j = (I_2-A_{22}-B_2F_f)^{-1}[I_2-(A_{22}+B_2F_f)^{1/\epsilon}]$$

$$= (I_2-A_{22}-B_2F_f)^{-1} + O(\epsilon),$$

the second term on the right hand side of (4.58) simplifies to

$$[(I_2-A_{22})^{-1}B_2F_s + O(\epsilon)]x_1(K/\epsilon) \qquad (4.60)$$

To simplify the third term on the R.H.S. of (4.58) we need the following lemma

Lemma 4.3.1:

$$\text{Let} \quad Y(n) = \sum_{j=\frac{K}{\epsilon}}^{n-1} A^{n-1-j} B\, V(j), \quad \frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon}, \tag{4.61}$$

where $V(n)$ satifies (4.59) and $\|A\| = \alpha < 1$(A is an asymptotically stable matrix).

Then ,

$$Y(n) = \sum_{j=\frac{K}{\epsilon}}^{n-1} A^{n-1-j} B\, V(n) + O(\epsilon), \quad \frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon}, \tag{4.62}$$

or, equivalently,

$$Y(n) = (I-A)^{-1}(I-A^{n-\frac{K}{\epsilon}})BV(n) + O(\epsilon). \tag{4.63}$$

Proof:

(4.61) can be written as

$$Y(n) = \sum_{j=\frac{K}{\epsilon}}^{n-1} A^{n-1-j}BV(n) + \sum_{j=\frac{K}{\epsilon}}^{n-1} A^{n-1-j}B[V(j)-V(n)], \quad \frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon}. \tag{4.64}$$

From (4.59) we observe that

$$V(n+1) - V(n) = O(\epsilon).$$

So we obtain

$$\|V(n) - V(j)\| \leq c\, \ell\, \epsilon,$$

where $\ell = n-j$.

Now, for the second summation in (4.64) we have

$$\left\| \sum_{j=\frac{K}{\epsilon}}^{n-1} A^{n-1-j} B[V(j)-V(n)] \right\| \leq \sum_{j=\frac{K}{\epsilon}}^{n-1} \|A^{n-1-j}\| \, \|B\| \, \|V(j)-V(n)\|$$

$$= C' \sum_{j=\frac{K}{\epsilon}}^{n-1} \alpha^{n-1-j} \, \ell \, \epsilon = C' \sum_{\ell=1}^{n-\frac{K}{\epsilon}} \alpha^{\ell-1} \, \ell \, \epsilon, \qquad (4.65)$$

where

$$C' = C\|B\|.$$

Let

$$S = \sum_{\ell=1}^{n-\frac{K}{\epsilon}} \alpha^{\ell-1} \, \ell = 1 + 2\alpha + 3\alpha^2 + \cdots + (n-\tfrac{K}{\epsilon})\alpha^{n-1-\frac{K}{\epsilon}},$$

then

$$S - S\alpha = 1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1-\frac{K}{\epsilon}} - (n-\tfrac{K}{\epsilon})\alpha^{n-\frac{K}{\epsilon}}$$

$$= \frac{1-\alpha^{n-\frac{K}{\epsilon}}}{1-\alpha} - (n-\tfrac{K}{\epsilon})\,\alpha^{n-\frac{K}{\epsilon}}.$$

So

$$S = \frac{1}{1-\alpha}\left[\frac{1-\alpha^{n-\frac{K}{\epsilon}}}{1-\alpha} - (n-\tfrac{K}{\epsilon})\alpha^{n-\frac{K}{\epsilon}}\right], \quad \frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon},$$

which is bounded.

Now using (4.65), equation (4.64) reduces to

$$Y(n) = \sum_{j=\frac{K}{\epsilon}}^{n-1} A^{n-1-j} BV(n) + O(\epsilon), \quad \frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon}.$$

Q.E.D.

Using (4.60) and Lemma 4.3.1, (4.58) can be rewritten as

$$\eta_2(\tfrac{K+1}{\epsilon}) = [(I_2-A_{22})^{-1}B_2F_s - \sum_{j=\frac{K}{\epsilon}}^{\frac{K+1}{\epsilon}-1} (A_{22}+B_2F_f)^{\frac{K+1}{\epsilon}-1-j} B_2F_f(I_2-A_{22})^{-1}A_{21}V(\tfrac{K+1}{\epsilon})$$

$$+ O(\epsilon)]x_1(K/\epsilon) + O(\epsilon)\eta_2(\tfrac{K}{\epsilon}). \tag{4.66}$$

But

$$\sum_{j=\frac{K}{\epsilon}}^{\frac{K+1}{\epsilon}-1} (A_{22}+B_2F_f)^{\frac{K+1}{\epsilon}-1-j} = \sum_{\ell=0}^{\frac{1}{\epsilon}-1} (A_{22}+B_2F_f)^{\ell}$$

$$= (I_2-A_{22}-B_2F_f)^{-1} + O(\epsilon) \tag{4.67}$$

and

$$V(\tfrac{K+1}{\epsilon}) = (I+\epsilon A_0)^{1/\epsilon} + \sum_{j=\frac{K}{\epsilon}}^{\frac{K+1}{\epsilon}-1} (I+\epsilon A_0)^{\frac{K+1}{\epsilon}-1-j} \epsilon B_0F_s$$

$$= (I+\epsilon A_0)^{1/\epsilon} + \sum_{\ell=0}^{\frac{1}{\epsilon}-1} (I+\epsilon A_0)^{\ell} \epsilon B_0F_s,$$

which, in view of (2.106)-(2.108), is given by

$$V(\tfrac{K+1}{\epsilon}) = A_s + B_sF_s + O(\epsilon). \tag{4.68}$$

Using (4.67), (4.68) and $x_1(n) = \eta_1(n) + O(\epsilon)\eta_2(n)$ in (4.66) yields

$$\eta_2(\tfrac{K+1}{\epsilon}) = [H + O(\epsilon)]\eta_1(\tfrac{K}{\epsilon}) + O(\epsilon)\eta_2(\tfrac{K}{\epsilon}), \tag{4.69}$$

where

$$H = (I_2-A_{22})^{-1}[A_{21}(A_s+B_sF_s) + B_2F_s] + (I_2-A_{22}-B_2F_f)^{-1}A_{21}(A_s+B_sF_s). \tag{4.70}$$

Combining (4.55) and (4.69) and using transformation (4.45) yields

$$\begin{bmatrix} x_1(\frac{K+1}{\epsilon}) \\ \\ x_2(\frac{K+1}{\epsilon}) \end{bmatrix} = \begin{bmatrix} (A_s+B_sF_s) + O(\epsilon) & O(\epsilon) \\ \\ (I_2-A_{22})^{-1}[A_{21}(A_s+B_sF_s)+B_2F_s]+ O(\epsilon), & O(\epsilon) \end{bmatrix} \begin{bmatrix} x_1(\frac{K}{\epsilon}) \\ \\ x_2(\frac{K}{\epsilon}) \end{bmatrix}. \tag{4.71}$$

The above system can be represented as

$$x(\tfrac{K+1}{\epsilon}) = (A+E)x(\tfrac{K}{\epsilon}) \quad , \qquad \|E\| = O(\epsilon), \tag{4.72}$$

where

$$A = \begin{bmatrix} A_s + B_sF_s & 0 \\ \\ (I_2-A_{22})^{-1}[A_{21}(A_s+B_sF_s) + B_2F_s] & 0 \end{bmatrix}. \tag{4.73}$$

Using the asymptotic stability of $(A_s+B_sF_s)$ and the continuous dependence of eigenvalues of a matrix on its parameters, it follows that for sufficiently small ϵ the system (4.71) is asymptotically stable. Hence

$$x_1(\tfrac{K}{\epsilon}) \to 0 \quad \text{and} \quad x_2(\tfrac{K}{\epsilon}) \to 0 \quad \text{as} \quad K \to \infty . \tag{4.74}$$

To show that the asymptotic attractivity holds for every n, let us rewrite (4.51) as

$$\eta_1(n) = [(I_1+\epsilon\tilde{A})^{n-K/\epsilon} + \epsilon \sum_{j=\frac{K}{\epsilon}}^{n-1} (I_1+\epsilon\tilde{A})^{n-1-j}\tilde{B}\hat{F}(j) + O(\epsilon)]\eta_1(\tfrac{K}{\epsilon})$$

$$+ O(\epsilon)\eta_2(K/\epsilon), \tfrac{K}{\epsilon} < n \leq \tfrac{K+1}{\epsilon} , \tag{4.75}$$

where (2.106) has been used.

Using Appendix 4.2, (4.75) can be written as

$$\eta_1(n) = [(I_1+\epsilon A_0)^{n-\frac{K}{\epsilon}} + \epsilon \sum_{j=\frac{K}{\epsilon}}^{n-1} (I_1+\epsilon A_0)^{n-1-j} B_0 F_s + 0(\epsilon)]\eta_1(\tfrac{K}{\epsilon}) + 0(\epsilon)\eta_2(\tfrac{K}{\epsilon}),$$

$$\frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon}. \qquad (4.76)$$

Inside the bracket above is bounded using (2.106)-(2.108). Now, asymptotic stability of $\eta_1(\frac{K}{\epsilon})$ an $\eta_2(\frac{K}{\epsilon})$ implies $\eta_1(n) \to 0$ as $n \to \infty$.

Using the inverse of transformation (4.43), equation (4.56) can be written as

$$\eta_2(n) = [A_{22}+B_2F_f+0(\epsilon)]^{n-\frac{K}{\epsilon}}\eta_2(\tfrac{K}{\epsilon}) + \sum_{j=K/\epsilon}^{n-1} [A_{22}+B_2F_f+0(\epsilon)]^{n-1-j}[B_2+0(\epsilon)]\hat{F}(j)$$

$$(\eta_1(\tfrac{K}{\epsilon}) + 0(\epsilon)\eta_2(\tfrac{K}{\epsilon})), \quad \frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon}. \qquad (4.77)$$

Using Lemma 4.3.1 it can be shown that

$$\sum_{j=\frac{K}{\epsilon}}^{n-1} [A_{22}+B_2F_f+0(\epsilon)]^{n-1-j}[B_2+0(\epsilon)]\hat{F}(j)$$

is bounded while the boundedness of $[A_{22} + B_2F_f + 0(\epsilon)]^{n-\frac{K}{\epsilon}}$ follows from the stability of $[A_{22} + B_2F_f]$.

Again, by asymptotic stability of $\eta_1(\frac{K}{\epsilon})$ and $\eta_2(\frac{K}{\epsilon})$ we can see from (4.77) that $\eta_2(n) \to 0$ as $n \to \infty$.

Q.E.D.

<u>Proof of the Theorem 4.3.2</u>:

From (4.71) we have

$$x_1(\tfrac{K+1}{\epsilon}) = [A_s+B_sF_s+0(\epsilon)]x_1(\tfrac{K}{\epsilon}) + 0(\epsilon)x_2(\tfrac{K}{\epsilon}).$$

Comparing above with (4.31) it is obvious that

$$x_1(\tfrac{K}{\epsilon}) = x_s(K) + O(\epsilon).$$

To prove the rest of the claim, first we establish a relationship between $\eta_2(n)$ and $x_f(n)$.

We rewrite (4.48b) as

$$\eta_2(n+1) = [(A_{22}+B_2F_f) + O(\epsilon)]\eta_2(n) + [B_2+ O(\epsilon)]\{[(I_2-F_f(I_2-A_{22})^{-1}B_2)$$

$$F_s + O(\epsilon)]x_1(\tfrac{K}{\epsilon})-F_f(I_2-A_{22})^{-1}A_{21}\}\tilde{x}_1(n),$$

where $\tilde{x}_1(n)$ is given by (4.39).

Define $\bar{\eta}_2(K)$ as

$$\bar{\eta}_2(K) = [A_{22}+B_2F_f+O(\epsilon)]\bar{\eta}_2(K)+[B_2+O(\epsilon)]\{[I_2-F_f(I_2-A_{22})^{-1}B_2]F_s-F_f(I_2-A_{22})^{-1}$$

$$A_{21}+O(\epsilon)\}x_1(\tfrac{K}{\epsilon}). \tag{4.78}$$

Subtracting $\bar{\eta}_2(K)$ from η_2 and letting $\gamma(n) = \eta_2(n)-\bar{\eta}_2(K)$ yields

$$\gamma(n+1) = [A_{22}+B_2F_f + O(\epsilon)]\gamma(n)-[B_2+ O(\epsilon)]F_f(I_2-A_{22})^{-1}A_{21}[\tilde{x}_1(n)-\bar{x}_1(\tfrac{K}{\epsilon})],$$

$$\gamma(\tfrac{K}{\epsilon}) = \eta_2(\tfrac{K}{\epsilon})-\bar{\eta}_2(K), \ \frac{K}{\epsilon} \leq n < \frac{K+1}{\epsilon}. \tag{4.79}$$

Now, we represent $\gamma(n)$ as the sum of a zero-input response γ_1 and a zero-state response γ_2.

$$\gamma(n) = \gamma_1(n) + \gamma_2(n), \tag{4.80}$$

where

$$\gamma_1(n+1) = [A_{22}+B_2F_f+ O(\epsilon)]\gamma_1(n), \quad \frac{K}{\epsilon} \leq n < \frac{K+1}{\epsilon} \tag{4.81a}$$

with the initial conditions

$$\gamma_1(0) = \gamma(0) = \eta_2(0) - \bar{\eta}_2(0) = -(I_2-A_{22})^{-1}(A_{21}+B_2F_s)x_1(0) + x_2(0) + O(\epsilon)$$

$$\gamma_1(\tfrac{K}{\epsilon}) = \eta_2(\tfrac{K}{\epsilon}) - \bar{\eta}_2(K) = (I_2-A_{22})^{-1}B_2F_s[x_s(K-1) - x_s(K)] + O(\epsilon), \qquad (4.81b)$$

where (4.71) has been used to substitute for $x_2(\frac{K+1}{\epsilon})$ and

$$\gamma_2(n+1) = [A_{22}+B_2F_f + O(\epsilon)]\gamma_2(n) - [B_2 + O(\epsilon)]F_f(I_2-A_{22})^{-1}A_{21}[\tilde{x}_1(n) - \bar{x}_1(\tfrac{K}{\epsilon})],$$

$$\gamma_2(\tfrac{K}{\epsilon}) = 0, \; K \geq 0, \qquad \tfrac{K}{\epsilon} \leq n < \tfrac{K+1}{\epsilon}. \qquad (4.82)$$

From (4.82) for $\frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon}$ we get

$$\gamma_2(n) = \sum_{j=\frac{K}{\epsilon}}^{n-1} [A_{22}+B_2F_f + O(\epsilon)]^{n-1-j}[B_2 + O(\epsilon)]F_f(I_2-A_{22})^{-1}A_{21}x_1(\tfrac{K}{\epsilon})$$

$$- \sum_{j=\frac{K}{\epsilon}}^{n-1} [A_{22}+B_2F_f + O(\epsilon)]^{n-1-j}[B_2 + O(\epsilon)]F_f(I_2-A_{22})^{-1}\tilde{x}_1(j), \qquad (4.83)$$

$$\tfrac{K}{\epsilon} < n \leq \tfrac{K+1}{\epsilon}.$$

Using the asymptotic stability of $(A_{22}+B_2F_f)$ and Lemma 4.3.1 (Note that from (4.39) $\tilde{x}_1(j)$ satisfies a similar equation as (4.59)), (4.83) reduces to

$$\gamma_2(n) = (I_2-A_{22}-B_2F_f)^{-1}[I_2-(A_{22}+B_2F_f)^{n-\frac{K}{\epsilon}}]B_2F_f(I_2-A_{22})^{-1}A_{21}x_1(\tfrac{K}{\epsilon})$$

$$- \sum_{j=\frac{K}{\epsilon}}^{n-1} [A_{22}+B_2F_f + O(\epsilon)]^{n-1-j}[B_2 + O(\epsilon)]F_f(I_2-A_{22})^{-1}A_{21}\tilde{x}_1(n) + O(\epsilon),$$

where we have used the Householder Theorem.

Or, equivalently,

$$\gamma_2(n) = (I_2-A_{22}-B_2F_f)^{-1}[I_2-(A_{22}+B_2F_f)^{n-\frac{K}{\epsilon}}]B_2F_f(I_2-A_{22})^{-1}A_{21}\cdot$$

$$[x_1(\tfrac{K}{\epsilon})-\tilde{x}_1(n)], \qquad \tfrac{K}{\epsilon} < n \leq \tfrac{K+1}{\epsilon}. \qquad (4.84)$$

From (4.39) we have $\tilde{x}_1(\frac{K}{\epsilon}) = x_1(\frac{K}{\epsilon})$ and $\tilde{x}_1(n+1) = \tilde{x}_1(n) + O(\epsilon)$ and we obtain

$$\|x_1(\tfrac{K}{\epsilon}) - \tilde{x}_1(n)\| = \ell \cdot O(\epsilon) \leq C\,\ell\,\epsilon ,$$

where C is a constant and $\ell = n - \frac{K}{\epsilon}$.

Now,(4.84) reduces to

$$\begin{aligned}\gamma_2(n) = {} & (I_2-A_{22}-B_2F_f)^{-1}B_2F_f(I_2-A_{22})^{-1}A_{21}[x_1(\tfrac{K}{\epsilon}) - \tilde{x}_1(n)] \\ & - (I_2-A_{22}-B_2F_f)^{-1}(A_{22}+B_2F_f)^{n-\frac{K}{\epsilon}} B_2F_f(I_2-A_{22})^{-1}A_{21}[x_1(\tfrac{K}{\epsilon}) - \tilde{x}_1(n)] \\ & + O\,(\epsilon)\,. \end{aligned} \tag{4.85}$$

We also note that

$$\|(A_{22}+B_2F_f)^{n-\frac{K}{\epsilon}} B_2F_f(I_2-A_{22})^{-1}A_{21}[x_1(\tfrac{K}{\epsilon})-\tilde{x}_1(n)]\| \leq C\alpha^{n-\frac{K}{\epsilon}}(n- \tfrac{K}{\epsilon}) = O(\epsilon) \tag{4.86}$$

where

$$\|A_{22}+B_2F_f\| = \alpha < 1\ .$$

Using (4.86), (4.85) reduces to

$$\begin{aligned}\gamma_2(n) &= (I_2-A_{22}-B_2F_f)^{-1}B_2F_f(I_2-A_{22})^{-1}A_{21}[x_1(\tfrac{K}{\epsilon}) - \tilde{x}_1(n)] + O(\epsilon) \\ &= [(I_2-A_{22}-B_2F_f)^{-1}A_{21} - (I_2-A_{22})^{-1}A_{21}][x_1(\tfrac{K}{\epsilon}) - \tilde{x}_1(n)] + O(\epsilon). \end{aligned} \tag{4.87}$$

Now, by comparing (4.81) with (4.37) we observe that

$$\gamma_1(n) = x_f(n) + O(\epsilon)\ . \tag{4.88}$$

From the inverse of the transformation (4.45) we get

$$x_2(n) = -L_2\eta_1(n) + \eta_2(n) + O(\epsilon)$$

$$= (I_2-A_{22}-B_2F_f)^{-1}A_{21}\eta_1(n) + \eta_2(n) + O(\epsilon). \quad (4.89)$$

But

$$\eta_2(n) = \gamma(n) + \bar{\eta}_2(K) = \gamma_1(n) + \gamma_2(n) + \bar{\eta}_2(K), \quad (4.90)$$

where $\bar{\eta}_2(K)$ is obtained from (4.78) to be

$$\bar{\eta}_2(K) = (I_2-A_{22}-B_2F_f)^{-1}\{[I_2-B_2F_f(I_2-A_{22})^{-1}]B_2F_s-B_2F_f(I_2-A_{22})^{-1}A_{21}\}x_1(\tfrac{K}{\epsilon})+O(\epsilon)$$

$$= [(I_2-A_{22})^{-1}(B_2F_s+A_{21}) - (I_2-A_{22}-B_2F_f)^{-1}A_{21}]x_1(\tfrac{K}{\epsilon}) + O(\epsilon) \quad (4.91)$$

In view of (2.106)-(2.108) and Appendix 4.2, comparison of (4.39) with (4.51) yields

$$\eta_1(n) = \tilde{x}_1(n) + O(\epsilon) . \quad (4.92)$$

Now, using (4.87), (4.88), (4.91) and (4.92), equation (4.89) reduces to

$$x_2(n) = (I_2-A_{22}-B_2F_f)^{-1}A_{21}\tilde{x}_1(n) + x_f(n) +$$

$$[(I_2-A_{22}-B_2F_f)^{-1}A_{21}-(I_2-A_{22})^{-1}A_{21}][x_1(\tfrac{K}{\epsilon}) - \tilde{x}_1(n)]$$

$$+ [(I_2-A_{22})^{-1}(B_2F_s+A_{21}) - (I_2-A_{22}-B_2F_f)^{-1}A_{21}]x_1(\tfrac{K}{\epsilon}) +$$

$$O(\epsilon), \qquad \frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon} \quad (4.93)$$

Or

$$x_2(n) = x_f(n) + (I_2-A_{22})^{-1}A_{21}\tilde{x}_1(n) + (I_2-A_{22})^{-1}B_2F_sx_1(\tfrac{K}{\epsilon}) + O(\epsilon) \ . \quad (4.94)$$

Approximating $x_1(\frac{K}{\epsilon})$ by $x_s(K)$, using (4.39) and (4.42) we obtain

$$x_2(n) = X_f(n) + (I_2-A_{22})^{-1}[B_2F_s+A_{21}V(n)]x_s(K) + O(\epsilon), \ \frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon} . \quad (4.95)$$

Q.E.D.

Equation (4.95) states that $x_2(n)$ can be approximated for all n, using the solution of slow and fast subsystems.

<u>Example</u>: To illustrate our claims, we apply the above design procedure to the example of Section 3.7 when the design criterion is the pole-placement.

Consider the difference equations

$$x_1(n+1) = (1-2\epsilon)x_1(n) + \epsilon x_2(n) + 1.5\epsilon u(n), \ x_1(0) = .5 \qquad (4.96a)$$

$$x_2(n+1) = -.7 \ x_1(n) + .45x_2(n) + .8 \ u(n), \ x_2(0) = -.5 \qquad (4.96b)$$

<u>Slow subsystem</u>

$$x_s(K+1) = A_s \ x_s(K) + B_s \ u_s(K), \ x_s(0) = .5 \ , \qquad (4.97)$$

where

$$A_s = .038073 \ \text{ and } \ B_s = .868406$$

The gain F_s in $u_s(K) = F_sx_s(K)$ in chosen such that the closed-loop slow subsystem has eigenvalue located at .5, i.e., $A_s + B_sF_s = .5$ which results in $F_s = .532026$.

Fast subsystem

$$x_f(n+1) = A_{22}x_f(n) + B_2u_f(n)$$

$$\text{where} \quad A_{22} = .45 \quad \text{and} \quad B_2 = .8.$$

The gain F_f in $u_f(n) = F_f x_f(n)$ is chosen such that the closed-loop fast subsystem has eigenvalue located at .5, i.e., $A_{22} + B_2F_f = .5.$ which yields

$$F_f = .0625.$$

The programs are written, using 'LAS package' [Bingular et al., 1982], and run on the Prime computer at Michigan State University.

The results are evaluated for three different values of ϵ (.1, .05, .025) and four slow periods.

The results are tabulated in following tables. For the sake of compactness we do not give all the values of $x_2(n)$ and $x_2'(n)$ (which is the predicted value of $x_2(n)$ given by (4.95) and should be within $O(\epsilon)$ from $x_2(n)$ for each n), although these values are available.

$\epsilon = .1$

Table 4.1

n	$x_2(n)$	$x_2'(n)$
0	-.5	-.5
1	-.374718	-.266483
5	.031076	.027885
10	.098466	.074759
11	-.012021	.026102
16	-.004897	.028341
20	.038196	.037697
21	-.008541	.013053
26	-.001639	.014173
30	.018002	.018352
31	-.003950	.006528
36	-.000779	.007088
40	.008434	.009428

Table 4.2

K	$x_1(\frac{K}{\epsilon})$	$x_s(K)$	$x_2(\frac{K}{\epsilon})$	$x_2'(\frac{K}{\epsilon})$
0	.5	.5	-.5	-.5
1	.236551	.250044	.098966	.074759
2	.110806	.125044	.038196	.037697
3	.051924	.062533	.018002	.018352
4	.024331	.031272	.008434	.009428

$\epsilon = .05$

Table 4.3

n	$x_2(n)$	$x_2'(n)$
0	-.5	-.5
1	-.374718	-.320600
10	.031778	-.000245
20	.076225	.072006
21	-.022511	-.000961
30	-.000947	.013139
40	.035731	.036010
41	-.011732	-.000481
50	.066265	.006570
60	.017445	.018008
61	-.005718	-.000240
70	.032341	.003286
80	.008514	.009006

Table 4.4

K	$x_1(\frac{K}{\epsilon})$	$x_s(K)$	$x_2(K/\epsilon)$	$x_2'(\frac{K}{\epsilon})$
0	.5	.5	-.5	-.5
1	.243044	.250044	.076225	.072006
2	.118622	.125044	.035731	.036010
3	.057894	.062533	.017445	.018008
4	.028255	.031272	.008514	.009006

$\epsilon = .025$

Table 4.5

n	$x_2(n)$	$x_2'(n)$	n	$x_2(n)$	$x_2'(n)$
0	-.5	-.5	81	-.012994	-.007248
1	-.374718	-.347659	90	-.023584	-.014802
10	-.067701	-.059808	100	.002864	.005327
20	.022807	.021298	110	.013188	.013939
30	.058087	.055736	120	.017200	.017607
40	.072016	.070402			
41	-.025451	-.014493	121	-.006423	-.003625
50	-.047775	-.029598	130	-.011660	-.007402
60	.005765	.010651	140	.001416	.002664
70	.026662	.027873	150	.006520	.006971
80	.034785	.035207	160	.008504	.008805

Table 4.6

K	$x_1(\frac{K}{\epsilon})$	$x_s(K)$	$x_2(K/\epsilon)$	$x_2'(K/\epsilon)$
0	.5	.5	-.5	-.5
1	.246391	.250044	.072016	.070402
2	.121821	.125044	.034785	.035207
3	.060230	.062533	.017200	.017607
4	.029778	.031272	.008504	.008805

By noting the results in Tables 4.1-4.6 we can observe the following:

1 - Asymptotic stability of $x_1(\frac{K}{\epsilon})$ and $x_2(\frac{K}{\epsilon})$ as K increases.

2 - Asymptotic stability of $x_2(n)$ as n increase.

3 - The closeness of $x_1(\frac{K}{\epsilon})$ and $x_s(K)$, $K = 0,1,2,\ldots$, up to $O(\epsilon)$.

4 - The closeness of $x_2(n)$ and $x_2'(n)$, which is the approximated value of $x_2(n)$, up to $O(\epsilon)$.

5 - The abrupt change in $x_2(n)$ and $x_2'(n)$ at the beginning of each slow cycle which is a result of the abrupt change in the slow control that excites the fast modes.

4.4. Sequential Design

The composite control of Section 4.3 has a slow component, which stabilizes the slow modes, and a fast component, which stablizes the fast modes. Suppose, however, that the open-loop fast modes are already asymptotically stable with acceptable transient response, i.e., the eigenvalues of A_{22} are appropriately located inside the unit circle, then the fast component of the composite control may be omitted and only the slow control is used. If this is possible, a considerable reduction in the on-line computations will be achieved since implementation of the slow control does not require the solution of the slow equations.

Even if A_{22} is not asymptotically stable, or it is so but its eigenvalues are not sufficiently well damped, the above idea might still be useful by using feedback from the fast variable to pre-condition the matrix A_{22} to have the desirable stability property and then a slow control can be designed as in Section 4.3. Such a design procedure will be sequential since the design of the slow control will be dependent on the pre-conditioning feedback gains.

In this section we investigate this sequential design procedure and give a similar approximation results as Section 4.3.

Again, consider the singularly perturbed difference equation (4.1) where (I_2-A_{22}) is nonsingular and let the input control decomposes into two parts as in (4.98).

$$u(n) = u_1(n) + u_2(n) \tag{4.98}$$

where $u_1(n)$ is constant over the cycle $\frac{K}{\epsilon} \leq n < \frac{K+1}{\epsilon}$. Let us choose the feedback control

$$u_2(n) = F_2x_2(n) \tag{4.99}$$

such that the matrix $A_{22}+B_2F_2$ is asymptotically stable and meets some desired objectives or has appropriate eigenvalue locations. Now, system (4.1) becomes

$$x_1(n+1) = [I_1+\epsilon A_{11}(\epsilon)]x_1(n) + \epsilon[A_{12}(\epsilon)+B_1(\epsilon)F_2]x_2(n) + \epsilon B_1(\epsilon)u_1(n) \tag{4.100a}$$

$$x_2(n+1) = A_{21}(\epsilon)x_1(n) + [A_{22}(\epsilon)+B_2(\epsilon)F_2]x_2(n) + B_2u_1(n) \tag{4.100b}$$

Slow subsystem

Following the same method as in previous section the slow subsystem is defined to be

$$x_s(K+1) = A_sx_s(K) + B_su_s(K), \; x_s(0) = x_1(0) \tag{4.101}$$

where $A_s = e^{\tilde{A}}$, $B_s = \int_0^1 e^{(1-t)\tilde{A}} dt \tilde{B}$, $\tilde{A}$ and $\tilde{B}$ are defined by (4.49) and (4.50)

Suppose that the state feedback control law for $u_s(K)$ is designed as

$$u_s(K) = F_s x_s(K), \tag{4.102}$$

where F_s is chosen such that the closed-loop system

$$x_s(K+1) = (A_s + B_s F_s) x_s(K) \tag{4.103}$$

is asymptotically stable and meets some design criteria.

Composite Control

With the pre-conditioning feedback gains and slow control in hand, a composite feedback control is formed as

$$u_c(n) = F_2 x_2(n) + u_s(n) = F_2 x_2(n) + F_s x_s(K). \tag{4.104}$$

By approximating $x_s(K)$ with $x_1(\frac{K}{\epsilon})$ we obtain

$$u_c(n) = F_2 x_2(n) + F_s x_1(\tfrac{K}{\epsilon}) \ , \ \tfrac{K}{\epsilon} \le n < \tfrac{K+1}{\epsilon} . \tag{4.105}$$

Applying (4.105) to system (4.1) yields

$$x_1(n+1) = [I_1 + \epsilon A_{11}(\epsilon)] x_1(n) + \epsilon [A_{12}(\epsilon) + B_1(\epsilon) F_2] x_2(n) + \epsilon B_1(\epsilon) F_s x_1(\tfrac{K}{\epsilon}) \tag{4.106a}$$

$$x_2(n+1) = A_{21}(\epsilon) x_1(n) + [A_{22}(\epsilon) + B_2(\epsilon) F_2] x_2(n) + B_2(\epsilon) F_s x_1(\tfrac{K}{\epsilon}). \tag{4.106b}$$

We define the fast subsystem to be

$$x_f(n+1) = (A_{22}+B_2F_2)x_f(n) \tag{4.107a}$$

$$x_f(0) = x_2(0)-(I_2-A_{22}-B_2F_2)^{-1}(A_{21}+B_2F_s)x_1(0) \tag{4.107b}$$

$$x_f(\tfrac{K}{\epsilon}) = (I_2-A_{22}-B_2F_2)^{-1}B_2F_s[x_s(K-1)-x_s(K)], \ K = 1,2,... \tag{4.107c}$$

Theorem 4.4.1

If the feedback control input u_c, defined by (4.105), is applied to the system (4.1), if the matrices $(A_{22}+B_2F_2)$ and $(A_s+B_sF_s)$ are asymptotically stable in the discrete-time sence, i.e., all their eigenvalues are inside the unit circle, and if the initial conditions

$$x_s(0) = x_1(0)$$

$$x_f(0) = x_2(0)-(I_2-A_{22}-B_sF_2)^{-1}(A_{21}+B_2F_s)x_1(0)$$

$$x_f(\tfrac{K}{\epsilon}) = (I_2-A_{22}-B_2F_2)^{-1}B_2F_s[x_s(K-1)-x_s(K)], \ K = 1,2,...$$

are satisfied, then for sufficiently small ϵ, $x_1(n)$ and $x_2(n)$ are asymptotically stable solutions and

$$x_1(\tfrac{K}{\epsilon}) = x_s(K) + O(\epsilon) \ , \ K = 0,1,2,...$$

and

$$x_2(n) = x_f(n) + (I_2-A_{22}-B_2F_2)^{-1}[B_2F_s+A_{21}\tilde{V}(n)]x_s(K) + O(\epsilon),$$

where

$$\tilde{V}(n+1) = (I_1+\epsilon\tilde{A})\tilde{V}(n) + \epsilon\tilde{B}F_s \ , \ \tilde{V}(\tfrac{K}{\epsilon}) = I_1.$$

Proof

This is a special case of Theorem 4.3.1 with A_{22} and A_{12} replaced by $A_{22}+B_2F_2$ and $A_{12}+B_1F_2$, respectively and $F_f = 0$.

Q.E.D.

To point out the computational difficulties for this type of design procedure let us, for example, choose the design criterion to be pole-placement.

The designer first evaluates the pre-conditioning gain F_2. Using F_2, the poles for slow subsystem are selected. On applying the composite feedback control, if the designer is not satisfied with the response of the system, the whole procedure should be repeated. If the design of the pre-conditioning gain is costly, then the sequential design procedure is not desirable. Based on the specific problem, the designer may wish to choose the parallel design, discussed in previous section.

APPENDIX 4.1

By letting $F_2 = F_f$ and $F_1 = \tilde{F}$ we have

$$\bar{A}_{11}+\bar{A}_{12}(I_2-\bar{A}_{22})^{-1}\bar{A}_{21} = A_{11}+B_1\tilde{F}+(A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)^{-1}(A_{21}+B_2\tilde{F})$$

$$= A_{11}+(A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)\ \ A_{21}+[B_1+(A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)^{-1}B_2]\tilde{F} = \tilde{A}+\tilde{B}\tilde{F}.$$

Now wish to prove

$$\tilde{A}+\tilde{B}\tilde{F} = A_0+B_0F_s$$

$$\text{L.H.S.} = A_{11}+(A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)^{-1}A_{21}+[B_1+(A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)^{-1}B_2]\cdot$$

$$\{[I_2-F_f(I_2-A_{22})^{-1}B_2]F_s-F_f(I_2-A_{22})^{-1}A_{21}\}$$

$$= A_{11}+(A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)^{-1}[I_2-B_2F_f(I_2-A_{22})^{-1}]A_{21}-B_1F_f(I_2-A_{22})^{-1}$$

$$A_{21}+(A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)^{-1}(I_2-A_{22}-B_2F_f)(I_2-A_{22})^{-1}B_2F_s$$

$$+ B_1[I_2-F_f(I_2-A_{22})^{-1}B_2]F_s$$

$$= A_{11}+A_{12}(I_2-A_{22})^{-1}A_{21}+B_1F_f(I_2-A_{22})^{-1}A_{21}-B_1F_f(I_2-A_{22})^{-1}A_{21}$$

$$+ A_{12}(I_2-A_{22})^{-1}B_2F_s+B_1F_s$$

$$= A_{11}+A_{12}(I_2-A_{22})^{-1}A_{21}+[B_1+A_{12}(I_2-A_{22})^{-1}B_2]F_s$$

$$= A_0+B_0F_s$$

Q.E.D.

APPENDIX 4.2.

Wish to prove by induction that

$$(I_1+\epsilon\tilde{A})^{n-K/\epsilon}+ \epsilon\sum_{j=K/\epsilon}^{n-1}(I_1+\epsilon\tilde{A})^{n-1-j}\tilde{B}\hat{F}(j) = (I_1+\epsilon A_0)^{n-\frac{K}{\epsilon}}+ \epsilon\sum_{j=\frac{K}{\epsilon}}^{n-1}(I_1+\epsilon A_0)^{n-j-1}B_0F_s$$

for $\frac{K}{\epsilon} < n \leq \frac{K+1}{\epsilon}$. (1)

For $n = \frac{K}{\epsilon} + 1$, (1) reduces to

$$I_1+\epsilon(\tilde{A}+\tilde{B}\tilde{F}) = I_1+\epsilon(A_0+B_0F_s),$$

which is true (see Appendix 4.1) where

$$\tilde{F} = F_s-F_f(I_2-A_{22})^{-1}(A_{21}+B_2F_s) .$$

Now let the assertion be true for $n = \frac{K}{\epsilon} + m$, $0 < m \leq \frac{1}{\epsilon}$, i.e.

$$(I_1+\epsilon\tilde{A})^m + \epsilon\sum_{j=\frac{K}{\epsilon}}^{\frac{K}{\epsilon}+m-1}(I_1+\epsilon A)^{m+\frac{K}{\epsilon}-1-j}\tilde{B}\hat{F}(j) = (I_1+\epsilon A_0)^m + \epsilon\sum_{j=\frac{K}{\epsilon}}^{\frac{K}{\epsilon}+m-1}(I_1+\epsilon A_0)^{\frac{K}{\epsilon}+m-1-j}B_0F_s.$$

Or by $\ell = j-\frac{K}{\epsilon}$ we have the following equality

$$(I_1+\epsilon\tilde{A})^m + \epsilon\sum_{\ell=0}^{m-1}(I_1+\epsilon\tilde{A})^{m-1-\ell}\tilde{B}\hat{F}((\ell+\frac{K}{\epsilon}) = (I_1+\epsilon A_0)^m + \epsilon\sum_{\ell=0}^{m-1}(I_1+\epsilon A_0)^{m-1-\ell}B_0F_s.$$

(2)

For $n = m + 1 + \frac{K}{\epsilon}$ we should have

$$(I_1+\epsilon\tilde{A})^{m+1} + \epsilon\sum_{j=\frac{K}{\epsilon}}^{m+\frac{K}{\epsilon}}(I_1+\epsilon\tilde{A})^{m+\frac{K}{\epsilon}-j}\tilde{B}\hat{F}(j) = (I_1+\epsilon A_0)^{m+1} + \epsilon\sum_{j=\frac{K}{\epsilon}}^{m+\frac{K}{\epsilon}}(I_1+\epsilon A_0)^{m+\frac{K}{\epsilon}-1}B_0F_s.$$

Or by $\ell = j - K/\epsilon$ we get

$$(I_1+\epsilon\tilde{A})^{m+1} + \epsilon \sum_{\ell=0}^{m} (I_1+\epsilon\tilde{A})^{m-\ell}\tilde{B}\hat{F}(\ell+\frac{K}{\epsilon}) = (I_1+\epsilon A_0)^{m+1} + \epsilon \sum_{\ell=0}^{m} (I_1+\epsilon A_0)^{m-\ell} B_0 F_s \, , \tag{3}$$

where

$$\hat{F}(\ell+\frac{K}{\epsilon}) = \begin{cases} [F_s - F_f(I_2-A_{22})^{-1}B_2F_s] - F_f(I_2-A_{22})^{-1}A_{21}[(I_1+\epsilon A_0)^{\ell} + \epsilon \sum_{j=0}^{\ell-1}(I_1+\epsilon A_0)^{\ell-1-j}B_0F_s]\} & \ell \neq 0 \\ \tilde{F} & \text{for } \ell = 0 \, . \end{cases}$$

Let

$$Q(i) = (I+\epsilon A_0)^i + \epsilon \sum_{\ell=0}^{i-1} (I+\epsilon A_0)^{i-1-\ell} B_0 F_s \, , \quad 0 < i \leq \frac{K}{\epsilon} \, .$$

So

$$\hat{F}(\ell+\frac{K}{\epsilon}) = C - F_f(I_2-A_{22})^{-1}A_{21}Q(\ell) \, , \quad \ell > 0 \, , \tag{4}$$

where

$$C = F_s - F_f(I_2-A_{22})^{-1}B_2F_s \, .$$

The L.H.S. of (3) is

$$(I_1+\epsilon\tilde{A})^{m+1} + \epsilon \sum_{\ell=0}^{m-1} (I_1+\epsilon\tilde{A})^{m-\ell}\tilde{B}\hat{F}(\ell+\frac{K}{\epsilon}) + \epsilon\tilde{B}\hat{F}(m+\frac{K}{\epsilon})$$

$$= (I_1+\epsilon\tilde{A})[(I_1+\epsilon\tilde{A})^m + \epsilon \sum_{\ell=0}^{m-1}(I_1+\epsilon\tilde{A})^{m-\ell-1}\tilde{B}\hat{F}(\ell+\frac{K}{\epsilon})] + \epsilon\tilde{B}[C-F_f(I_2-A_{22})^{-1}A_{21}Q(m)],$$

where we have used (4). Inside the first bracket above is equal to $Q(m)$ by (1), so we get

$$\text{L.H.S. of (3)} = [I_1+\epsilon\tilde{A}-\epsilon\tilde{B}F_f(I_2-A_{22})^{-1}A_{21}]Q(m) + \epsilon\tilde{B}C \; . \tag{5}$$

But

$$I_1+\epsilon[\tilde{A}-\tilde{B}F_f(I_2-A_{22})^{-1}A_{21}] = I_1+\epsilon[A_{11}+(A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)^{-1}A_{21}$$

$$- (B_1+(A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)^{-1}B_2)F_f(I_2-A_{22})^{-1}A_{21}]$$

$$= I_1+\epsilon\{A_{11}+(A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)^{-1}[I_2-B_2F_f(I_2-A_{22})^{-1}]A_{21}$$

$$- B_1F_f(I_2-A_{22})^{-1}A_{21}\}$$

$$= I_1+\epsilon[A_{11}+(A_{12}+B_1F_f)(I_2-A_{22})^{-1}A_{21}-B_1F_f(I_2-A_{22})^{-1}A_{21}]$$

$$= I_1+\epsilon A_0 \; . \tag{6}$$

Also

$$\tilde{B}C = \tilde{B}[I_2-F_f(I_2-A_{22})^{-1}B_2]F_s$$

$$= [B_1+(A_{12}+B_1F_f)(I_2-A_{22}-B_2F_f)^{-1}B_2][I_2-F_f(I_2-A_{22})^{-1}B_2]F_s$$

$$= B_1[I_2-F_f(I_2-A_{22})^{-1}B_2]F_s+(A_{12}+B_1F_f)(I_2-A_{22})^{-1}B_2F_s$$

$$= [B_1+A_{12}(I_2-A_{22})^{-1}B_2]F_s = B_0F_s \; . \tag{7}$$

Using (6) and (7), the R.H.S. of equation (5) becomes

$$(I_1+\epsilon A_0)Q(m) + \epsilon B_0F_s = (I_1+\epsilon A_0)^{m+1} + \epsilon\sum_{\ell=0}^{m}(I_1+\epsilon A_0)^{m-\ell}B_0F_s$$

$$= \text{R.H.S. of (3)} \; .$$

Q.E.D.

APPENDIX 4.3

Wish to prove

$$\sum_{j=\frac{K}{\epsilon}}^{n-1}[A_{22}+B_2F_f+0(\epsilon)]^{n-1-j}[B_2+0(\epsilon)]\hat{F}(j) = 0(1). \qquad (1)$$

$$\text{L.H.S. of (1)} = \sum_{j=\frac{K}{\epsilon}}^{n-1}[A_{22}+B_2F_f+0(\epsilon)]^{n-1-j}[B_2+0(\epsilon)]$$

$$\cdot[F_s-F_f(I_2-A_{22})^{-1}(A_{21}V(j)+B_2F_s)],$$

where $\hat{F}(j)$ has been substituted for using (4.41).

Using Lemma 4.3.1 we get

$$\text{L.H.S. of (1)} = \sum_{j=\frac{K}{\epsilon}}^{n-1}[A_{22}+B_2F_f+0(\epsilon)]^{n-1-j}[B_2+0(\epsilon)].$$

$$[F_s-F_f(I_2-A_{22})^{-1}(A_{21}V(n)+B_2F_s)]+0(\epsilon)$$

$$= [I_2-A_{22}-B_2F_f+0(\epsilon)]^{-1}[I_2-(A_{22}+B_2F_f+0(\epsilon))^{n-\frac{K}{\epsilon}}].$$

$$[B_2+0(\epsilon)][F_s-F_f(I_2-A_{22})^{-1}(A_{21}V(n)+B_2F_s)]+0(\epsilon)$$

$$= 0(1).$$

Q.E.D.

CHAPTER 5

APPLICATION

5.1 Introduction

The main objective of this chapter is to use a more realistic physical model to demonstrate our results about near-optimality of the composite feedback control for infinite-time regulators and the iterative technique for solving the discrete-time stiff Riccati equations which were presented in Chapter 3, and also the asymptotic stability of the solutions and closeness of trajectories in multirate stabilization presented in Chapter 4. For this purpose, we consider the deterministic model of an F-8 aircraft. Different control problems of this aircraft are investigated by different authors [IEEE Transaction on Automatic Control, Mini issue on the F-8 aircraft, Oct., 1977].

In particular, we consider the model considered by J. Elliott [1977].

5.2. Longitudinal Equations of Motion for an F-8 Aircraft

The linearized aircraft equations of motion is given by

$$\frac{d}{dt}\begin{bmatrix} u \\ \theta \\ \alpha \\ q \end{bmatrix} = \begin{bmatrix} X_u & -g & X_\alpha & 0 \\ 0 & 0 & 0 & 1 \\ Z_u & 0 & Z_\alpha & 1 \\ M_u & 0 & M_\alpha & M_q \end{bmatrix}\begin{bmatrix} u \\ \theta \\ \alpha \\ q \end{bmatrix} + \begin{bmatrix} X_{\delta_e} & X_{\delta_T} \\ 0 & 0 \\ Z_{\delta_e} & 0 \\ M_{\delta_e} & 0 \end{bmatrix}\begin{bmatrix} \delta_e \\ \delta_T \end{bmatrix}, \tag{5.1}$$

where

u = incremental velocity, ft/s.

θ = incremental pitch angle, rad.

α = incremental angle of attack, rad.

q = incremental pitch rate rad/s.

δ_e = incremental elevator position, rad.

δ_T = incremental throttle position, nondimensional.

g = gravity acceleration

$M_{(\ )}$, $X_{(\ )}$, $Z_{(\ )}$ are longitudinal dimensional stability derivatives reffered to stability or wind axes.

By experience with this model, it is known that u and θ are slow while α and q are the fast variables.

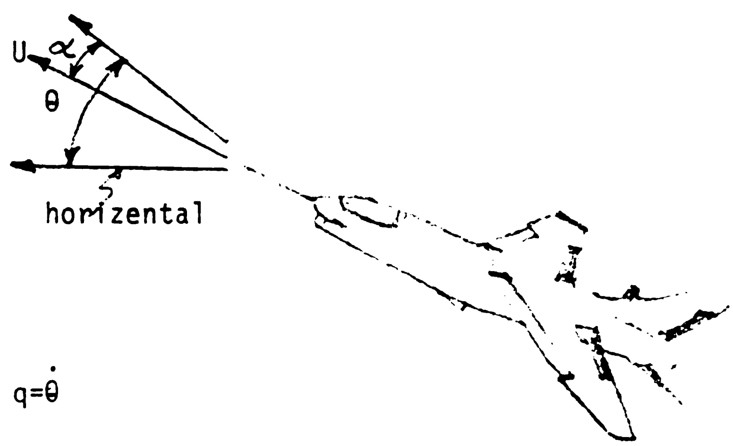


Figure 5.1. Aircraft longitudinal variables.

Representative numbers for flight of the F-8 at 20,000 ft. with total equilibrium velocity = 620 ft/s (Mach number = .6) and α_0 = .078 rad. are

$$\frac{d}{dt}\begin{bmatrix} u \\ \theta \\ \alpha \\ q \end{bmatrix} = \begin{bmatrix} -.015 & -32.2 & -14.0 & 0 \\ 0 & 0 & 0 & 1 \\ -.00019 & 0 & -.84 & 1 \\ .00005 & 0 & -4.8 & -.49 \end{bmatrix}\begin{bmatrix} u \\ \theta \\ \alpha \\ q \end{bmatrix} + \begin{bmatrix} -1.1 & 8.9 \\ 0 & 0 \\ -.11 & 0 \\ -8.7 & 0 \end{bmatrix}\begin{bmatrix} \delta_e \\ \delta_T \end{bmatrix} \tag{5.2}$$

Scaling

First scaling is to bring the system into the normal singularly perturbed form. This system takes the form

$$\begin{bmatrix} A_{11} & A_{12/\epsilon} \\ A_{12} & A_{22/\epsilon} \end{bmatrix}$$

The transformation $\begin{bmatrix} \epsilon I_1 & 0 \\ 0 & I_2 \end{bmatrix}$ brings the system into the form

$$\begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{bmatrix}$$

We take $\epsilon = \frac{1}{30}$ which is the ratio of the magnitude of slow to fast poles. The first scaling is

$$S = \text{diag}\ [\frac{1}{30}\ ,\ \frac{1}{30}\ ,\ 1,\ 1]. \tag{5.3}$$

Second scaling is to balance the outer diagonal elements of $(A_{11} - A_{12}A_{22}^{-1}A_{21})$

$$S_2 = \text{diag}\ [\frac{1}{400}\ ,\ 1,\ 1,\ 1]. \tag{5.4}$$

Now, total scaling is

$$S = S_2S_1 = \text{diag}\ [\frac{1}{12000}\ ,\ \frac{1}{30}\ ,\ 1,\ 1], \tag{5.5}$$

and we have

$$A_c = \begin{bmatrix} -.015 & -.0805 & -.0011666 & 0 \\ 0 & 0 & 0 & .03333 \\ -2.28 & 0 & -.84 & 1 \\ .6 & 0 & -4.8 & -.49 \end{bmatrix} \tag{5.6a}$$

$$B_c = \begin{bmatrix} -.0000916 & .0007416 \\ 0 & 0 \\ -.11 & 0 \\ -8.7 & 0 \end{bmatrix}. \tag{5.6b}$$

The initial values are $[-1,\ 0,\ .08,\ 0]^T$.

We use the sampled-data to discretize the system. Two different sampling rates are chosen, $T = .05$, which is a typical value [Elliott, 1977], and $T = 1$ which is much larger, and we observe that our claims hold for both choices. The choice of sampling period T is based on a theorem [Kalman et al. 1963] which states that:

If the continuous time-invariant system is completely controllable, then the time-invariant discrete-time system is completely controllable if

$$I_m\{\lambda_i(A) - \lambda_j(A)\} \neq n \frac{2\pi}{T}, \tag{5.7}$$

whenever

$$R_e\{\lambda_i(A) - \lambda_j(A)\} = 0 \quad \text{and} \quad n = \pm 1, \pm 2, \ldots .$$

The open-loop eigenvalues of system (5.2) are

$$-.006852 \pm j\ .076519\ , \quad -.665648 \pm j\ 2.182122,$$

so that a choice of sampling period which satisfies $T < 1.44$ will preserve controllability.

Sampling the system (5.2) yields

$$x(n+1) = e^{A_c T}\, x(n) + \int_0^T e^{A_c(T-t)} B_c dt\ u(n)$$

or

$$x(n+1) = A\, x(n) + B\, u(n)\ . \tag{5.8}$$

By equating (5.8) with our form of system

$$x_1(n+1) = (I_1+\epsilon A_{11})x_1(n) + \epsilon A_{12}x_2(n) + \epsilon B_1 u(n)$$

$$x_2(n+1) = A_{21}x_1(n) + A_{22}x_2(n) + B_2 u(n) \ , \qquad (5.9)$$

the matrices A_{ij}, B_i $(i,j = 1,2)$ are determined.

We have evaluated the eigenvalues for slow and fast subsystems, i.e., eigenvalues of $A_0 = A_{11} + A_{12}(I_2-A_{22})^{-1}A_{21}$ and A_{22}. It is seen that the eigenvalues of A_0 are close to the slow eigenvalues of A and the eigenvalues of A_{22} are close to the fast eigenvalues of A which guarantees the existence of two-time-scale property of the full system.

All the programs for performing the computations are written using the Prime Computer and "LAS" package [Bingulac et al., 1982] at Michigan State University. The programs are attached at the end of this chapter.

5.3 Results for Infinite-Time Regulator:

For computational purpose of this part we choose the output to be

$$y(n) = DX(n),$$

where

$$D = \text{diag}\,[.1,.1,.1,.1]\,,$$

with the performance index

$$\min J = \epsilon \sum_{n=1}^{\infty} [y^T(n)y(n) + u^T(n)Ru(n)]\,,$$

where

$$R = R^T = I_{4x4}\,.$$

Slow subsystem:

For $T = 1$ we have

$$\frac{dx_s}{dt} = \begin{bmatrix} -.530438 & -2.393496 \\ 2.178240 & -.087888 \end{bmatrix} x_s(t) + \begin{bmatrix} .107593 & .022050 \\ -1.297632 & .000810 \end{bmatrix} u_s(t)$$

$$y_s(t) = \begin{bmatrix} 1 & 0 \\ 0 & .1 \\ -.009481 & .002009 \\ .217693 & -.009422 \end{bmatrix} x_s(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -.168058 & -.000019 \\ -.129740 & .000087 \end{bmatrix} u_s(t),$$

and for $T = .05$ we have

$$\frac{dx_s}{dt} = \begin{bmatrix} -.022539 & -.120705 \\ -.109791 & -.000221 \end{bmatrix} x_s(t) + \begin{bmatrix} .002932 & .001112 \\ -.065042 & .000002 \end{bmatrix} u_s(t)$$

$$y_s(t) = \begin{bmatrix} .1 & 0 \\ 0 & .1 \\ -.009920 & .000024 \\ -.219581 & -.000443 \end{bmatrix} x_s(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -.167970 & 0 \\ -.130084 & .000004 \end{bmatrix} u_s(t) ,$$

with the performance index

$$\min J_s = \int_0^\infty [y_s^T(t) y_s(t) + u_s^T(t) u_s(t)] dt.$$

<u>Fast subsystem</u>:

For $T = 1$ we have

$$z_f(K+1) = \begin{bmatrix} -.329907 & .193177 \\ -.924546 & -.263252 \end{bmatrix} z_f(K) + \begin{bmatrix} -1.984393 & -.000414 \\ -3.192726 & .000925 \end{bmatrix} u_f(K)$$

$$y_f(K) = D_2 \, z_f(K) ,$$

where

$$D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ .1 & 0 \\ 0 & .1 \end{bmatrix}$$

For $T = .05$ we have

$$z_f(K+1) = \begin{bmatrix} .953092 & .048269 \\ -.231691 & .969983 \end{bmatrix} z_f(K) + \begin{bmatrix} -.016002 & -.000002 \\ -.428220 & .000001 \end{bmatrix} u_f(K),$$

$$y_f(K) = D_2 z_f(K),$$

with the performance index

$$\min J_f = \sum_{K=0}^{\infty} [y_f^T(K) y_f(K) + u_f^T(K) u_f(K)] .$$

In the following two tables we give the values of $V = P-P'$ for two sampling periods $T = 1$ and $T = .05$. It is observed that the numerical values agree with the theoretical results (Theorem 3.2).

Table 5.1

T = .05

V = P-P'

-0.068998	-0.008956	0.000213	-0.000029
-0.008956	-0.094044	0.002944	-0.000406
0.000213	0.002944	-0.000162	0.000011
-0.000029	-0.000406	0.000011	-0.000009

Table 5.2

T = 1

V = P-P'

-0.027394	-0.005159	0.000174	-0.000010
-0.005159	-0.031458	0.000988	-0.000177
0.000174	0.000988	-0.000036	0.000007
-0.000010	-0.000177	0.000007	-0.000002

On the next two tables we give the numerical resutls using the iterative technique and also the exact solution for $T = .05$ and $T = 1$.

Table 5.3

$T = .05$

Solution of the Riccati equation

Iteration	P_1		P_2		P_3	
1	1.405989	-.001167	-.295408	-.156022	.398268	-.048487
	-.001167	1.552002	-1.361687	.250796	-.048487	.094758
2	1.413724	-0.001904	-0.297776	-0.157317	0.401409	-0.049061
	-0.001904	1.558631	-1.367825	0.252564	-0.049061	0.094061
3	1.414150	-0.001984	-0.297944	-0.157413	0.401629	-0.049103
	-0.001984	1.558902	-1.368078	0.252672	-0.049103	0.094980
4	1.414172	-0.001989	-0.297953	-0.157418	0.401640	-0.049105
	-0.001989	1.558913	-1.368088	0.252678	-0.049105	0.094981
5	1.414173	-0.001989	-0.297954	-0.157418	0.401641	-0.049105
	-0.001989	1.558913	-1.368089	0.252678	-0.049105	0.094982
Exact	1.414093	-.001990	-.297974	-.157426	.401665	-.049107
Solution	-.001990	1.558813	-1.367995	.252662	-.049107	.094990

Table 5.4

T = 1

Solution of the Riccati equation

iteration	P_1		P_2		P_3	
1	.065598	-.000306	-.012264	-.010470	.022148	.000990
	-.000306	.072273	-.063944	.012474	.000990	.011486
2	0.069828	-0.000140	-0.013135	-0.010497	0.022555	0.000914
	-0.000140	0.076873	-0.067946	0.013283	0.000914	0.011499
3	0.070894	-0.000126	-0.013362	-0.010509	0.022676	0.000891
	-0.000126	0.078008	-0.068924	0.013483	0.000891	0.011504
4	0.071153	-0.000124	-0.013417	-0.010512	0.022706	0.000886
	-0.000124	0.078278	-0.069158	0.013531	0.000886	0.011505
5	0.071215	-0.000123	-0.013431	-0.010512	0.022713	0.000884
	-0.000123	0.078343	-0.069215	0.013542	0.000884	0.011505
6	0.071230	-0.000123	-0.013434	-0.010512	0.022715	0.000884
	-0.000123	0.078358	-0.069228	0.013545	0.000884	0.011505
7	0.071234	-0.000123	-0.013435	-0.010512	0.022715	0.000884
	-0.000123	0.078362	-0.069231	0.013546	0.000884	0.011505
8	0.071245	-0.000123	-0.013435	-0.010513	0.022715	0.000884
	-0.000123	0.078363	-0.069232	0.013546	0.000884	0.011505
Exact	.071240	-.000123	-.013435	-.010513	.022716	.000884
Solution	.000123	.078369	-.069237	.013547	.000884	.011505

By investigating the results in Tables 5.3, 5.4 we observe that the iterative technique has a convergence rate of $O(\epsilon) \approx \frac{1}{30}$, or even less, at each iteration and tends to the exact solution.

It is important to mention that the saving in computational efforts for this fourth-order example was quite substantial. Solving the Riccati equation using iterative technique was very much faster than solving the full Riccati equation. This confirms the excellent efficiency of iterative technique, specially for high-order stiff systems.

It should be noted that due to round-off errors there are very small differences in some of the numerical figures.

5.4. Multirate stabilization:

Our criterion for this part is pole-placement. The eigenvalues of slow and fast subsystems for the continuous-time system are given as

slow, $-.007442 \pm j\ .076413$

fast $-.665 \pm j\ 2.18389$

Real parts of slow eigenvalues are small which results in very oscillatory response. To avoid this problem, we locate our closed-loop slow eigenvalues at $-.1 \pm j\ .1$ with magnitude .141. Also, we locate the closed-loop eigenvalues of fast subsystem at $-1 \pm j\ 2$ with magnitude 2.236. As we can see, the ratio of magnitudes of slow and fast is about 16 which provides a proper gap between two sets of eigenvalues and has the desired two-time-scale property (2.7).

Now, the mapping $z = e^{ST}$ from S-domain to Z-domain [Franklin and Powell, 1980] will provide us with discrete-time eigenvalues.

Closed-loop fast eigenvalues

$$z = e^{ST} = e^{(-1 \pm j2)T} = r_f\ e^{\pm j\theta_f} ,$$

where $r_f = e^{-T}$ and $\theta_f = 2T$ (rad) .

For $T = 1$, we get E.V. $\simeq -.36 \pm j.78$

For $T = 0.05$, we get E.V. $\simeq .95 \pm j.095$

Closed-loop slow eigenvalues

$$z = e^{ST} = e^{(-.1 \pm j.1)T} = r_s\ e^{\pm j\theta_s} ,$$

where $r_s = e^{-.1T}$ and $\theta_s = .1T$ (rad).

For $T = 1$, we get E.V. $\simeq .9 \pm j.09$

For $T = .05$, we get E.V. $\simeq .995 \pm j.005$

In view of (2.85), we observe the two-time-scale property of our system.

It should be noted that the eigenvalues of the slow subsystem, evaluated above, are found in fast-time scale, while for our design procedure they should be determined in slow-time scale. Thus, the actual locations of slow eigenvalues are found by raising their values to the power $\frac{1}{\epsilon}$.

This is equivalent to the mapping

$$z = e^{ST/\epsilon} = e^{30(-.1 \pm j.1)T} .$$

And the locations of slow eigenvalues are as follow:

For $T = 1$, we obtain E.V. $\simeq .05 \pm j.007$

For $T = .05$, we obtain E.V. $\simeq .85 \pm j.13$.

Table 5.5

T = .05

K	$x_1(K/\epsilon)$	$x_s(K)$	The values of x_2 and x_2' at the end of the slow period $x_2(\frac{K+1}{\epsilon})$	$x_2'(\frac{K+1}{\epsilon})$	The values of x_2 and x_2' at the beginning of each slow period $x_2(\frac{K}{\epsilon}+1)$	$x_2'(\frac{K}{\epsilon}+1)$
0	-.1	-.1	-.592194	-.585282	.083056	.083045
	0	0	-.769791	-.801221	-.136725	-.136494
1	-.095030	-.095192	-.075732	-.211630	-.593010	-.465931
	-.043437	-.029220	-.193325	-.344752	-.665538	-.529259
2	-.086633	-.087887	-.097644	-.057633	-.072933	-.258245
	-.049281	-.049674	-.272114	-.209182	-.200351	-.361502
3	-.077950	-.079022	.057672	.058503	-.097439	-.094507
	-.062357	-.062841	-.095404	-.100700	-.240697	-.223220
4	-.068453	-.069354	.106949	.142069	.058769	.030285
	-.066155	-.070100	-.052787	-.016520	-.086139	-.112179
5	-.059030	-.059473	.167037	.198260	.107321	.121362
	-.068390	-.072706	.020077	.046373	-.040502	-.025655
6	-.049827	-.049824	.196989	.231996	.167449	.183923
	-.067180	-.071768	.059421	.091049	.026283	.039331
7	-.041164	-.040726	.217028	.247800	.197162	.222935
	-.064227	-.068247	.093410	.120495	.063894	.085832
8	-.033179	-.032395	.223482	.249722	.217118	.242996
	-.059729	-.062954	.113773	.137520	.095218	.116833
9	-.025988	-.024958	.221422	.241304	.223464	.248255
	-.054327	-.056561	.126335	.144690	.113949	.135152
10	-.019640	-.018476	.212184	.225573	.221340	.242362
	-.048374	-.049605	.131292	.144290	.125091	.143372
11	-.014149	-.012955	.198085	.205053	.212048	.228456
	-.042219	-.042507	.130771	.138310	.129074	.143802
12	-.009494	-.012955	.180653	.181802	.197915	.209173
	-.036112	-.042507	.125918	.128439	.127862	.138453
13	-.005633	-.008363	.161315	.157447	.180464	.186673
	-.030250	-.035584	.118002	.116079	.122591	.129043
14	-.002505	-.004638	.141162	.133236	.160749	.162682
	-.024770	-.029064	.107996	.102368	.111169	.117001

Table 5.6

T = 1

K	$x_1(\frac{K}{\epsilon})$	$x_s(K)$	The values of x_2 and x_2' at the end of the slow period $x_2(\frac{K+1}{\epsilon})$	$x_2'(\frac{K+1}{\epsilon})$	The values of x_2 and x_2' at the beginning of the slow period $x_2(\frac{K}{\epsilon}+1)$	$x_2'(\frac{K}{\epsilon}+1)$
0	-.1	-.1	.155724	.156616	.292168	.291966
	0	0	.065083	.069128	.084391	.087832
1	-.023462	-.025672	.031551	.035121	-.036552	-.027735
	-.017459	-.013225	.048898	.037227	-.237570	-.228585
2	.010426	.002822	-.014661	-.004359	-.041094	-.027036
	.006065	.001322	-.014912	-.003887	-.070729	-.067993
3	-.001649	-.000217	.002591	.000346	.012494	.002774
	-.000042	-.000099	.000526	.000294	.028602	.007382
4	-.000611	.000014	.000794	-.000024	-.000198	-.000209
	-.000565	.000006	.001385	-.000019	-.003709	-.000565
5	.000333	-.000001	-.000473	.000001	-.001206	.000014
	.000176	-0.000000	-.000458	.000001	-.001899	.000038
6	-.000043	-.000000	.000068	-.000000	.000377	-.000001
	.000002	-.000000	.000005	-.000000	.000904	-.000002

Table 5.7

T = .05

n	$x_2(n)$	$x_2'(n)$
0	.08	.08
	.0	.0
1	.083056	.083045
	-.136725	-.136494
10	-.111319	-.111161
	-.983884	-.981730
20	-.441535	-.440763
	-1.089893	-1.086384
30	-.592194	-.590487
	-.769791	-.766151
31	-.593010	-.465931
	-.665538	-.529259
40	-.436467	-.378445
	-.027074	-.241178
50	-.185518	-.258244
	.046237	-.225835
60	-.075732	-.211630
	-.193325	-.344752
61	-.072933	-.258245
	-.200351	-.361502
70	-.067100	-.189223
	-.258973	-.131780
80	-.083026	-.094362
	-.284971	-.117392
90	-.097644	-.057633
	-.272114	-.209182

n	$x_2(n)$	$x_2'(n)$
91	-.097439	-.094507
	-.240697	-.223220
100	-.048251	-.041858
	-.049784	-.045698
110	.026399	.030527
	-.027544	-.032584
120	.057672	.058503
	-.095404	-.100700
121	.058769	.030285
	-.086139	-.112179
130	.079053	.068754
	-.035078	.019751
140	.100778	.121672
	-.034331	.031406
150	.106949	.142069
	-.052787	-.016520
151	.107321	.121362
	-.040502	-.025655
160	.127713	.147831
	.033415	.067366
170	.156057	.184268
	.042931	.077483
180	.167037	.198260
	.020077	.046373
181	.167449	.183923
	.026283	.039331

n	$x_2(n)$	$x_2'(n)$
190	.178667	.200476
	.063017	.099918
200	.192377	.223292
	.068320	.108500
210	.196989	.231996
	.059421	.091049
211	.197162	.222935
	.063894	.085832
220	.204229	.231503
	.091511	.120050
230	.213610	.243348
	.097744	.127159
240	.217028	.247800
	.093410	.120495
241	.217118	.242996
	.095218	.116833
250	.219593	.245323
	.107060	.130206
260	.222570	.248590
	.112065	.135945
270	.223482	.249722
	.113773	.137520
271	.223464	.248255
	.113949	.135152
280	.222807	.245876
	.116424	.132585

n	$x_2(n)$	$x_2'(n)$
290	.221877	.242672
	.121006	.137086
300	.221422	.241304
	.126335	.144690
301	.221340	.242362
	.125091	.143372
310	.218156	.236597
	.119201	.129120
320	.213880	.228734
	.123042	.132527
330	.212184	.225573
	.131292	.144290

n	$x_2(n)$	$x_2'(n)$
331	.212048	.228456
	.129074	.143802
340	.207110	.220415
	.117455	.121471
350	.200586	.209417
	.120705	.123936
360	.198085	.205053
	.130771	.138310
361	.197915	.209173
	.127862	.138453
370	.191787	.199765
	.112111	.111030
380	.183715	.186882
	.114762	.112700
390	.180653	.181802
	.125918	.128439

n	$x_2(n)$	$x_2'(n)$
391	.180464	.186673
	.122591	.129043
400	.173644	.176626
	.104289	.098934
410	.164687	.162857
	.106410	.099952
420	.161315	.157447
	.118002	.116079
421	.160233	.162682
	.108133	.117001
430	.153998	.152558
	.094816	.086093
440	.144661	.138676
	.096457	.086588
450	.141162	.133236
	.107996	.102368

Table 5.8

T = 1.0

n	$x_2(n)$	$x_2'(n)$
0	.08	.08
	.0	.0
1	.292168	.291966
	.084391	.08782
10	.176842	.179724
	.00644	.019437
20	.162823	.165221
	.018140	.029970
30	.155724	.156616
	.065083	.069128
31	-.027735	-.036552
	-.228585	-.237570
40	.036730	.040747
	-.019060	-.030204
50	.027406	.031095
	.038055	.021736
60	.031551	.035121
	.048898	.037227
61	-.041094	-.027036
	-.070729	-.067993
70	-.012468	-.002888
	-.014228	-.013399
80	-.015736	-.005756
	-.015116	-.006912
90	-.014661	-.004359
	-.014912	-.003887
91	.012494	.002774
	.028602	.007382

n	$x_2(n)$	$x_2'(n)$
100	.001886	.000185
	.003536	.001417
110	.003093	.000496
	.000696	.000636
120	.002591	.000346
	.000526	.000294
121	-.000198	-.000209
	-.003709	-.000565
130	.000852	.000011
	-.000003	-.000108
140	.000719	-.000035
	.001238	-.000046
150	.000794	-.000024
	.001385	-.000019
151	-.001206	.000014
	-.001899	.000038
160	-.000415	.000001
	-.000382	.000007
170	-.000503	.000002
	-.000444	.000003
180	-.000473	.000001
	-.000458	.000001
181	.000377	-.000001
	.000904	-.000001
190	.000046	-0.0
	.000111	-0.0
200	.000084	-0.0
	.000013	-0.0

n	$x_2(n)$	$x_2'(n)$
210	.000068	-0.0
	.000005	-0.0

By investigating the results in Tables 5.5-5.8 we can observe the following:

1. Asymptotic stability of $x_1(\frac{K}{\epsilon})$ and $x_2(\frac{K}{\epsilon})$ as K increases.
2. Asymptotic stability of $x_2(n)$ as n increase.
3. The closeness of $x_1(\frac{K}{\epsilon})$ and $x_s(K)$, $K = 0,1,2,\ldots$, up to $O(\epsilon)$.
4. The closeness of $x_2(n)$ and $x_2'(n)$, which is the approximated value of $x_2(n)$, up to $O(\epsilon)$.
5. The abrupt change in $x_2(n)$ and $x_2'(n)$ at the beginning of each slow cycle which is a result of the abrupt change in the slow control that excites the fast modes.
6. The settling-time of $x_2(n)$ is longer for the sampling period $T = .05$ than $T = 1$. This is expected by the following estimate for the settling-time

$$S.T. = \max_i \frac{4.6}{|Re(\lambda_i)|},$$

where for the case $T = .05$ our settling-time is equal to

$$S.T. = \frac{4.6}{.1} = 46 \text{ (sec)}.$$

CHAPTER 6

CONCLUSION AND RECOMMENDATION

In this dissertation we investigated some problems which have been open for the class of linear time-invariant singularly perturbed difference equations. Indeed, we arrived at approximate control designs by employing the two-time-scale property of such systems.

After providing a historical review which reveals different model representations and sources of singularly perturbed difference equations along with some structural properties for this class of systems, we introduced a stability criterion for our system and obtained an initial value result which approximates the solution of the full system by using the solutions of slow and fast subproblems.

We, also, investigated the asymptotic behaviour of infinite-time optimal regulators (linear quadratic) and showed that it does not follow as a limiting case of the finite-time problem considered by Blankenship [1980] and a special scaling was employed to remove this difficulty. Furthermore, conditions for independent design of slow and fast subsystems were derived and by applying slow-fast decomposition, as in continuous-time work of Chow and Kokotovic [1976], we acheived an $O(\epsilon^2)$ near-optimal solution.

In contrast with continuous-time work, a priori knowledge of perturbation parameter ϵ is necessary for our design procedure.

The well-known difficulty in solving discrete-time "stiff" Riccati equations was overcome by providing an iterative technique with a fast convergence rate which avoids the ill-conditioning by dealing with lower-order, slow and fast, subsystems. The efficiency and value of this technique is more appreciated as the order of the system increases and ϵ gets smaller. We achieve off-line computational savings by solving two lower-order, slow and fast, models and also, by avoiding the ill-conditioned numerical problems.

Multirate control design of singularly perturbed systems to meet desired objectives is an important problem. We studied the stabilization of this class of systems using single rate and multirate measurements of the state variables. Different design procedures for forming a stabilizing composite feedback control were investigated and we showed that the application of such controls results in asymptotic stability of the closed-loop system and closeness of trajectories to those predicted by slow and fast subsystems.

The proposed scheme has several computational advantages. The off-line computational effort is reduced because design and simulations are performed for two lower-order models instead of the full model. There is computational savings because of the order reduction and avoidance of stiff numerical problems.

The on-line computational effort is reduced because the slow feedback signal has to be processed only at slow-time intervals rather than fast-time intervals as in the single rate case. Of course, we have the computational cost of predicting slow states, between the slow-time intervals, in terms of their values at the beginning of each slow-period. By means of numerical examples, our claims were confirmed.

The results obtained in this dissertation show that many of the phenomena normally associated with continuous singularly perturbed systems are also present in discrete systems. These results provide a foundation for further research. In particular, our multirate stabilization results can be extended to the class of output feedback control systems. Also, with regard to the nonlinear continuous work [Peponides, et al., 1982] and along the lines of multitime method of Hoppensteadt and Miranker [1977] for difference equations, the extension of multirate stabilization results to nonlinear case is feasible. Furthermore, it seems evident that our results could be extended to time-varying case in view of usual features of time-varying systems.

An interesting research topic is the establishment of bounds on ϵ and deriving sufficient conditions, usually using matrix norms, for validity of approximation results as in multirate stabilization and infinite-time regulator.

As was mentioned earlier, many systems possess a two-time-scale property, while they are not, explicitly, in the singularly perturbed form. In spite of some efforts for converting a given system of equations into a singularly perturbed form as in [Phillip's, 1980], [Sain et.al., 1977], and recently Sycros and Sannuti [1983], more work is still needed in that direction.

List of the Programs

CTD: Sampled-data of the continuous system.

PRT: Partions the full matrices A and B to find the block matrices.

CHK: Evaluates the eigenvalues of slow and fast matrices A_s and A_f.

FULMAT: Knowing the block matrices finds the full matrix.

INIT: Initilizes the matrix Riccati P to its zero order terms and sets the errors to zero for the iterative technique.

CIT: Finds the constant matrices of the iterative techniques.

CITC: Continuation of CIT.

DISRIC: Solves a discrete Riccati equation by fixed point method and evaluates the gain F_0.

SRIC: Solves the slow subsystem Riccati equation and evaluates P_s.

SF STAB: Demonstrates the stability of slow and fast subsystems and evaluates $x_2'(n)$ which is O( ) close to $x_2(n)$.

STFUL: Demonstrates the stability of the full system.

FRIC: Solves the fast subsystem Riccati equation and evaluates the values of α_3, L_1,L_2,L_3, and P_f (fast Riccati matrix). Note that this program has to read the slow Riccati matrix P_s.

FF3: Finds the matrix γ_3 used in iterative technique, equation (3.79).

F12: Finds the matrics γ_1 and γ_2 in (3.80) and (3.81).

FES: Finds the errors E_1,E_2, and E_3 at each iteration.

FXS: Finds the full system Riccati matrix (3.33)-(3.35) using fixed point method.

THM2: Demonstrates theorem 2 of chapter 3.

Remarks:

i) In the iterative technique the sequence of execution of the programs should be as follows:

CTD, PRT, INIT, CIT, CITC, SRIC, FRIC, FF3, F12, FES, and FXS.

ii) For obtaining an extra $O(\epsilon)$ accuracy in each iteration the sequence FF3, F12, FES, and FXS should be executed.

```
OK, ED CTD
EDIT
$ P999
.NULL.
00001: (RDF)=AC,[AC]
00002: (RDF)=BC,[BC]
00003: (DSC)=DT
00004: DT.AC,BC(DRL,T)=A,B
00005: A,[A](WDF)=
00006: B,[B](WDF)=
BOTTOM
$ Q
OK, SPOOL L
```

```
OK, ED PRT
EDIT
$ P999
.NULL.
00001: (RDF)=A, [A]
00002: (RDF)=B, [B]
00003: (RDF)=EPS, [EPS]
00004: (RDF)=NN, [NN]
00005: A(CTR)=Z1, Z2
00006: Z1(CTC)=Z3, Z4
00007: Z2(CTC, T)=A21, A22
00008: (INP)=I
00009: Z3, I(-)=Z1
00010: Z1, NN(S*)=A11
00011: Z4, NN(S*)=A12
00012: B(CTR)=Z1, B2
00013: Z1, NN(S*)=B1
00014: I, A22(-)=Z1
00015: Z1(-1, T)=INV
00016: A11, [A11](WDF)=
00017: A12, [A12](WDF)=
00018: A21, [A21](WDF)=
00019: A22, [A22](WDF)=
00020: B1, [B1](WDF)=
00021: B2, [B2](WDF)=
00022: INV, [INV](WDF)=
00023: A11, A12, A21, A22, B1, B2(OUT)=
BOTTOM
$ Q
OK, SPOOL L
```

```
OK, ED FULMAT
EDIT
$ P999
.NULL.
00001: (RDF)=A11, [A11]
00002: (RDF)=A12, [A12]
00003: (RDF)=A21, [A21]
00004: (RDF)=A22, [A22]
00005: (RDF)=B1, [B1]
00006: (RDF)=B2, [B2]
00007: (RDF)=EPS, [EPS]
00008: (RDF)=I, [I]
00009: (RDF)=D1, [D1]
00010: (RDF)=D2, [D2]
00011: A11, EPS(S*)=Z1
00012: I, Z1(+)=A1
00013: A12, EPS(S*)=A2
00014: A1, A2(CTI)=Z1
00015: A21, A22(CTI)=Z2
00016: Z1, Z2(RTI, T)=A
00017: A, [A](WDF)=
00018: B1, EPS(S*)=Z1
00019: Z1, B2(RTI, T)=B
00020: B, [B](WDF)=
00021: D1, D2(CTI, T)=D
00022: D, [D](WDF)=
BOTTOM
$ Q
OK, SPOOL L
```

```
OK, ED CHX
EDIT
$ P???
.NULL.
00001: (RDF)=A, [A]
00002: (RDF)=B, [B]
00003: (RDF)=EPS, [EPS]
00004: (RDF)=NN, [NN]
00005: A(CTR)=Z1, Z2
00006: Z1(CTC)=A1, A2
00007: Z2(CTC, T)=A21, A22
00008: (INP)=I
00009: I, A22(-)=Z1
00010: Z1(-1, T)=INV
00011: A2, INV, A21(*)=Z1
00012: A1, Z1(+, T)=AS
00013: AS(EGV, T)=Z1, Z2
00014: A22, (EGV, T)=Z1, Z2
00015: A(EGV, T)=Z1, Z2
00016: A21, [A21](WDF)=
00017: A22, [A22](WDF)=
00018: A1, [A1](WDF)=
00019: A2, [A2](WDF)=
00020: INV, [INV](WDF)=
BOTTOM
$ Q
OK, SPOOL L
```

```
CALLED DFSTAB
EDIT
[illegible]
L ALL
(RDF)=A11,[A11]
(RDF)=A12,[A12]
(RDF)=A21,[A21]
(RDF)=A22,[A22]
(RDF)=B1,[B1]
(RDF)=B2,[B2]
(RDF)=AS,[AS]
(RDF)=BS,[BS]
(RDF)=FF,[FF]
(RDF)=FS,[FS]
(RDF)=EPS,[EPS]
(RDF)=I,[I]
(RDF)=INV,[INV]
(RDF)=X10,[X10]
(RDF)=X20,[X20]
A11,A12,A21,A22,B1,B2,AS,BS(OUT)=
FF,FS,EPS,X10,X20(OUT)=
(INP)=Z
I,Z(MCP)=ASHAT,BSHAT
(DSC)=IND,ONE,K,II
BS,FS(*)=D
AS,D(+,T)=SCL
A12,INV,A21(*)=D
A11,D(+,T)=AST
A12,INV,B2(*)=D
B1,D(+,T)=BST
AST,EPS(S*)=D
I,D(+,T)=IAS
BST,EPS(S*)=D
D,FS(*,T)=EBFS
(ELM)=AST,BST,A11,A12,B1
INV,A21,X10(*)=D
X20,D(-)=DO
INV,B2,FS,X10(*)=D
DO,D(-,T)=XFO
(ELM)=D,DO
X10(MCP)=XSK
B2,FF(*)=D
A22,D(+,T)=FCL
XFO(MCP,T)=XFO
(ELM)=A22,A21
Y:FCL,XFO(*)=XFN
IAS,ASHAT(*)=D
D(MCP)=ASHAT
IAS,BSHAT(*)=D
D,EBFS(+)=BSHAT
ASHAT,BSHAT(+)=DI
(RDF)=A21,[A21]
A21,DI(*)=D
(ELM)=DI,A21
B2,FS(*)=D2
D,D2(+)=D3
(ELM)=D,D2
INV,D3,XSK(*)=C
(ELM)=D3
XFN,C(-,T)=X20
XFN(MCP)=XFO
(RDF)=ONE,[ONE]
II,ONE(+)=D
(ELM)=ONE
```

```
D(MCP,T)=II
(RDF)=NN,[NN]
II,NN(IFJ)=Y,Y,X
X:(ELM)=NN,D
SCL,X10(*,T)=XSK
X10,XSK(-,T)=DIF
I,Z(MCP)=ASHAT,BSHAT
INV,B2,FS,DIF(*,T)=XFO
(ELM)=DIF
(RDF)=ONE,[ONE]
IND,ONE(+)=D
D,XSK,ONE(MCP,T)=IND,X10,II
(ELM)=ONE
IND,K(IFJ)=Y,Y,S
S:(STO)=
END
```

```
OK, ED STFUL
EDIT
$ P990
.NULL.
(RDF)=A11,[A11]
(RDF)=A12,[A12]
(RDF)=A21,[A21]
(RDF)=A22,[A22]
(RDF)=B1,[B1]
(RDF)=B2,[B2]
(RDF)=AS,[AS]
(RDF)=BS,[BS]
(RDF)=FF,[FF]
(RDF)=FS,[FS]
(RDF)=INV,[INV]
(RDF)=I,[I]
(RDF)=EPS,[EPS]
(RDF)=X10,[X10]
(RDF)=X20,[X20]
(RDF)=NN,[NN]
A11,A12,A21,A22,B1,B2,AS,BS(OUT)=
FF,FS(OUT)=
(INP)=Z
A12,INV,A21(*)=D
A11,D(+,T)=AST
A12,INV,B2(*)=D
B1,D(+,T)=BST
X10,X20(MCP,T)=X10,X20
(ELM)=X20
(DSC)=K
(DSC)=ONE,IND,CNT
AST,EPS(S*)=D
BST,EPS(S*)=EBST
EBST,FS(*,T)=EBFS
I,D(+,T)=IAS
(ELM)=EBST,AST,BST
A11,EPS(S*)=D
I,D(+,T)=C2
B1,FF(*)=D
A12,D(+)=DO
DO,EPS(S*,T)=C3
(ELM)=DO
B2,FF(*)=D
A22,D(+,T)=C4
(ELM)=A11,A12,A22
V:ONE(MCP,T)=IND
I,I,Z(MCP)=DI.ASHAT,BSHAT
Y:IND,NN(IFJ)=Z,Z,W
Z:A21,DI(*)=D
(ELM)=DI
B2,FS(*)=D2
D,D2(+)=D3
(ELM)=D2
FF,INV,D3(*)=D
(ELM)=D3
FS,D(-)=FHAT
B1,EPS(S*)=D
D,FHAT,X10(*)=C1
B2,FHAT,X10(*)=C5
(ELM)=FHAT
C2,X10(*)=D
C3,X20(*)=DO
D,DO,C1(+,T)=X1N
```

```
(ELM)=C1
A21,X1D(*)=D
C4,X2D(*)=DO
D,DO,C5(+,T)=X2N
(ELM)=D,DO,C5
X1N,X2N(MCP)=X1D,X2D
IAS,ASHAT(*)=D
D(MCP)=ASHAT
IAS,BSHAT(*)=D
D,EBFS(+)=BSHAT
ASHAT,BSHAT(+)=DI
IND,ONE(+)=D
D(MCP,T)=IND
(JMP)=Y
 W:CNT,ONE(+)=D
X1N,D(MCP,T)=X1D,CNT
CNT,K(IFJ)=V,V,S
```

```
OK, ED INIT
EDIT
$ P99?
.NULL.
00001: (RDF)=PS, [PS]
00002: (RDF)=P20, [P20]
00003: (RDF)=PF, [PF]
00004: (INP)=E1, E2, E3
00005: E1, [E1](WDF)=
00006: E2, [E2](WDF)=
00007: E3, [E3](WDF)=
00008: PS, [P10](WDF)=
00009: P20, [P20](WDF)=
00010: PF, [P30](WDF)=
00011: (RDF)=P10, [P10]
00012: (RDF)=P20, [P20]
00013: (RDF)=P30, [P30]
00014: P10, [P1](WDF)=
00015: P20, [P2](WDF)=
00016: P30, [P3](WDF)=
00017: (RDF)=P1, [P1]
00018: (RDF)=P2, [P2]
00019: (RDF)=P3, [P3]
00020: PS, P20, PF, P1, P2, P3(OUT)=
BOTTOM
$ Q
OK, SPOOL L
```

```
OK, ED CIT
EDIT
$ P999
.NULL
(RDF)=A,[A11]
(RDF)=B,[A12]
(RDF)=C,[A21]
(RDF)=S,[A22]
(RDF)=F,[B1]
(RDF)=G,[B2]
(RDF)=M,[M]
(RDF)=R,[R]
(RDF)=P10,[P10]
(RDF)=P20,[P20]
(RDF)=P30,[P30]
P20(T)=P2TO
F(T)=FT
FT,P20,G(*)=Z1
Z1(T)=Z2
FT,P10,F(*)=Z3
Z1,Z2,Z3(+,T)=C10
C(T)=CT
M(T)=MT
G(T)=GT
MT,M(*)=Z1
GT,P30,G(*)=Z2
R,Z1,Z2(+)=RO
RO(-1,T)=ROR
A(T)=AT
AT,P20,C(*)=Z1
Z1(T)=Z2
AT,P10,A(*)=Z3
Z3,Z1,Z2(+,T)=C20
P10,F(*)=Z1
P20,G(*)=Z2
CT,P30,G(*)=Z3
Z1,Z2,Z3(+,T)=C30
AT,P10,F(*)=Z1
AT,P20,G(*)=Z2
CT,P2TO,F(*)=Z3
Z1,Z2,Z3(+,T)=C40
S(T)=ST
B(T)=BT
BT,P10,F(*)=Z1
ST,P2TO,F(*)=Z2
BT,P20,G(*)=Z3
Z1,Z2,Z3(+,T)=C50
AT,P10,B(*)=Z1
AT,P20,S(*)=Z2
CT,P2TO,B(*)=Z3
Z1,Z2,Z3(+,T)=C60
BT,P20,S(*)=Z1
Z1(T)=Z2
BT,P10,B(*)=Z3
Z1,Z2,Z3(+,T)=C70
C10,[C10](WDF)=
C20,[C20](WDF)=
C30,[C30](WDF)=
C40,[C40](WDF)=
C50,[C50](WDF)=
C60,[C60](WDF)=
C70,[C70](WDF)=
ROR,[ROR](WDF)=
```

```
(ELM)=AT, BT, CT, RO, GT, FT, A, B, C
ST, P30, S(*)=Z1
(ELM)=ST, S, F, G, MT, R
(RDF)=D2, [D2]
D2(T)=D2T
D2T, M(*)=Z2
Z1, Z2(+, T)=DO
DO(T)=DOT
C50, ROR, DOT(*)=Z1
Z1(T)}=Z2
DO, ROR, C10, ROR, DOT(*)=Z3
C70, Z3(+)=Z4
Z1, Z2(+)=Z3
Z4, Z3(-, T)=S3
(RDF)=D1, [D1]
D1(T)=D1T
D1T, M(*)=Z1
C30, Z1(+, T)=NO
C50(T)=C5TO
NO, ROR, C5TO(*)=Z1
C40, ROR, DOT(*)=Z2
Z1, Z2(+)=Z3
NO, ROR, C10, ROR, DOT(*)=Z1
C60, Z1(+)=Z2
Z2, Z3(-, T)=S2
NO(T)=NOT
C40, ROR, NOT(*)=Z1
Z1(T)=Z2
Z1, Z2(+)=Z3
NO, ROR, C10, ROR, NOT(*)=Z1
C20, Z1(+)=Z2
Z3, Z2(-, T)=S1
S1, [S1](WDF)=
S2, [S2](WDF)=
S3, [S3](WDF)=
NO, [NO](WDF)=
DO, [DO](WDF)=
(ELM)=S1, S2, S3, C10, C20, C30, C40, C50, C60.
(RDF)=S, [A22]
(RDF)=G, [B2]
G, ROR, DOT(*)=Z1
S, Z1(-, T)=AL3
AL3, [AL3](WDF)=
(INP)=I
I, AL3(-)=H
(ELM)=I, AL3, Z4
(RDF)=C, [A21]
G, ROR, NOT(*)=Z1
C, Z1(-, T)=W
(ELM)=S, C
W, [W](WDF)=
(RDF)=B, [A12]
(RDF)=F, [B1]
F, ROR, DOT(*)=Z1
B, Z1(-, T)=J
J, [J](WDF)=
H, [H](WDF)=
H(-1, T)=HR
HR, [HR](WDF)=
BOTTOM
$ Q
OK, SPOOL L
```

```
OK, ED CITC
EDIT
$ P999
.NULL.
00001: (RDF)=A, [A11]
00002: (RDF)=F, [B1]
00003: (RDF)=J, [J]
00004: (RDF)=W, [W]
00005: (ROF)=HR, [HR]
00006: (ROF)=NO, [NO]
00007: (RDF)=ROR, [ROR]
00008: NO(T)=NOT
00009: J, HR, W(*)=Z1
00010: A, Z1(+)=Z2
00011: F, ROR, NOT(*)=Z1
00012: Z2, Z1(-, T)=AL1
00013: AL1, [AL1](WDF)=
BOTTOM
$ Q
OK, SPOOL L
```

```
OK, ED DISRIC
EDIT
$ P970
.NULL
00001: (RDF)=A22,[A]
00002: (RDF)=B2,[B]
00003: (RDF)=D2,[D]
00004: (RDF)=PF,[P]
00005: (RDF)=M,[M]
00006: (RDF)=R,[R]
00007: (RDF)=ERR,[ERR]
00008: A22(T)=AT22
00009: B2(T)=B2T
00010: D2(T)=D2T
00011: M(T)=MT
00012: Y: AT22,PF,B2(*)=D
00013: D2T,M(*)=DP
00014: D,DP(+)=C
00015: C(T)=CT
00016: MT,M(*)=D
00017: B2T,PF,B2(*)=DP
00018: R,D,DP(+)=DPP
00019: DPP(-1)=C1R
00020: DPP,[RFUL](WDF)=
00021: (ELM)=DPP
00022: AT22,PF,A22(*)=D
00023: D2T,D2(*)=DP
00024: D,DP(+)=DPP
00025: C,C1R,CT(*)=D
00026: DPP,D(-)=PFN
00027: (ELM)=D,DP,DPP
00028: PF,PFN(-)=DF
00029: DF(NRR)=NDF
00030: (ELM)=DF
00031: PFN(MCP)=PF
00032: NDF,ERR(IFJ)=X,Y,Y
00033: X: C1R,CT(*,T)=FO
00034: PFN(MCP,T)=P
00035: P,[P](WDF)=
00036: FO,[FO](WDF)=
00037: END
BOTTOM
$ Q
OK, SPOOL L
```

SRIC

```
(RDF)=A11, [A11]
(RDF)=A12, [A12]
(RDF)=A21, [A21]
(RDF)=A22, [A22]
(RDF)=B1, [B1]
(RDF)=B2, [B2]
(RDF)=I, [I]
(RDF)=D1, [D1]
(RDF)=D2, [D2]
(RDF)=M, [M]
(RDF)=INV, [INV]
(RDF)=R, [R]
A11, A12, A21, A22, B1, B2, D1, D2, M(OUT)=
A12, INV, A21(*)=D
A11, D(+, T)=AS
A12, INV, B2(*)=D
B1, D(+, T)=BS
D2, INV, A21(*)=D
D1, D(+, T)=CS
D2, INV, B2(*)=D
M, D(+, T)=DS
DS(T)=DST
BS(T)=BST
DST, DS(*)=D
R, D(+, T)=RS
RS(-1)=RSR
BS, RSR, DST, CS(*)=D
AS, D(-, T)=A
(INP)=I
DS, RSR, DST(*)=D
I, D(-)=DP
CS(T)=CST
CST, DP, CS(*, T)=G
A, BS, G, RS(RIK, T)=PS
DST, CS(*)=D
BST, PS(*)=DP
D, DP(+)=DPP
RSR, DPP(*, T)=FS
PS, [PS](WDF)=
FS, [FS](WDF)=
```

```
OK, ED FRIC
EDIT
$ P990
.NULL.
(RDF)=A11,[A11]
(RDF)=A12,[A12]
(RDF)=A21,[A21]
(RDF)=A22,[A22]
(RDF)=B1,[B1]
(RDF)=B2,[B2]
(RDF)=EPS,[EPS]
(RDF)=I,[I]
(RDF)=D1,[D1]
(RDF)=D2,[D2]
(RDF)=M,[M]
(RDF)=ERR,[ERR]
(RDF)=PS,[PS]
(RDF)=R,[R]
(INP)=PF
A22(T)=AT22
B2(T)=B2T
D2(T)=D2T
M(T)=MT
Y:AT22,PF,B2(*)=D
D2T,M(*)=DP
D,DP(+)=C
C(T)=CT
MT,M(*)=D
B2T,PF,B2(*)=DP
R,D,DP(+)=DPP
DPP(-1)=C1R
(ELM)=DPP
AT22,PF,A22(*)=D
D2T,D2(*)=DP
D,DP(+)=DPP
C,C1R,CT(*)=D
DPP,D(-)=PFN
(ELM)=D,DP,DPP
PF,PFN(-)=DF
DF(NRR)=NDF
(ELM)=DF
PFN(MCP)=PF
NDF,ERR(IFJ)=X,Y,Y
X:B2,C1R,CT(*)=D
NDF,PF(OUT)=
C1R,CT(*,T)=FF
FF,[FF](WDF)=
A22,D(-,T)=ALF3
I,ALF3(-,T)=L3
(ELM)=ALF3
B2T,PF,A21(*)=D
MT,D1(*)=DP
D,DP(+,T)=C2
L3(-1)=L3R
(ELM)=DP,L3
D2T,D1(*)=D
AT22,PF,A21(*)=DP
D,DP(+)=DPP
C,C1R,C2(*)=D
DPP,D(-)=DP
(ELM)=DPP
DP(T)=D
D,L3R(*,T)=L1
```

```
00060:B1,C1R,CT(*)=D
00061:A12,D(-)=DP
00062:DP,L3R(*,T)=L2
00063:PS,L2(*)=D
00064:L1,D(+,T)=P20
00065:P20,[P20](WDF)=
00066:PF,[PF](WDF)=
BOTTOM
$ Q
OK, SPOOL L
```

```
CY. SD FF3
EDIT
$ PSYC
.NULL.
(RDF)=B, [A12]
(RDF)=S, [A22]
(RDF)=F, [B1]
(RDF)=G, [B2]
(RDF)=EPS, [EPS]
(RDF)=C10, [C10]
(RDF)=ROR, [ROR]
(RDF)=C50, [C50]
(RDF)=D0, [D0]
(RDF)=E1, [E1]
(RDF)=E2, [E2]
(RDF)=E3, [E3]
B(T)=BT
BT, E1, B(*)=Z1
BT, E2, S(*)=Z2
Z2(T)=Z3
Z1, Z2, Z3(+)=C7E
C7E, [C7E](WDF)=
S(T)=ST
E2(T)=E2T
BT, E1, F(*)=Z1
ST, E2T, F(*)=Z2
BT, E2, G(*)=Z3
Z1, Z2, Z3(+)=C5E
C5E, [C5E](WDF)=
F(T)=FT
FT, E1, F(*)=Z1
FT, E2, G(*)=Z2
Z2(T)=Z3
Z1, Z2, Z3(+)=C1E
C1E, [C1E](WDF)=
C1E, EPS(S*)=Z1
Z1, C10(+)=C1
C1, [C1](WDF)=
G(T)=GT
GT, E3, G(*)=Z1
Z1, C1(+)=Z2
Z2, EPS(S*)=Z3
ROR(-1)=RO
RO, Z3(+)=Z1
Z1(-1)=Z3
Z3(MCP)=RR
Z3, Z2, ROR(*)=Z1
(RDF)=MYK, [MYK]
Z1, MYK(S*)=L
L, [L](WDF)=
GT, E3, G(*)=Z1
Z1, C10(+)=Z2
ROR, Z2, ROR(*)=Z1
Z1, MYK(S*)=L0
L0, [L0](WDF)=
(ELM)=MYK
(RDF)=NN, [NN]
L, L0(-)=Z1
Z1, NN(S*)=L1
(ELM)=NN
L1, [L1](WDF)=
RR, [RR](WDF)=
ST, E3, G(*)=Z1
```

```
C5E,EPS(S+)=Z2
Z1,C50,Z2(+)=Z4
Z4(T)=Z2
Z4,RR,Z2(*)=Z3
DO(T)=DOT
(ELM)=RR,C1E,C1,LO
Z4,L,DOT(*)=Z2
Z2(T)=Z1
Z1,Z2,Z3(+)=Z4
DO,L1,DOT(*)=Z1
C5E,ROR,DOT(*)=Z2
Z2(T)=Z3
Z1,Z2,Z3,Z4(+)=Z5
(ELM)=Z1,Z2,Z3,Z4
C7E,Z5(-)=F3
F3,[F3](WDF)=
BOTTOM
```

```
OK, ID F12
EDIT
S P470
.NULL.
00001. (RDF)=B, [A12]
00002: (RDF)=S, [A22]
00003: (RDF)=F, [B1]
00004: (RDF)=G, [B2]
00005: (RDF)=EPS, [EPS]
00006: (RDF)=ROR, [ROR]
00007: (RDF)=C50, [C50]
00008: (RDF)=DO, [DO]
00009: (RDF)=E1, [E1]
00010: (RDF)=E2, [E2]
00011: (RDF)=E3, [E3]
00012: (RDF)=A, [A11]
00013: (RDF)=C, [A21]
00014: (RDF)=NO, [NO]
00015: (RDF)=C5E, [C5E]
00016: (RDF)=C40, [C40]
00017: (RDF)=RR, [RR]
00018: (RDF)=L, [L]
00019: (RDF)=L0, [L0]
00020: (RDF)=L1, [L1]
00021: A(T)=AT
00022: C(T)=CT
00023: AT, E2, S(*)=Z1
00024: AT, E1, B(*)=Z2
00025: E2(T)=E2T
00026: CT, E2T, B(*)=Z3
00027: Z1, Z2, Z3(+)=C6E
00028: C6E, [C6E](WDF)=
00029: E1, F(*)=Z1
00030: E2, G(*)=Z2
00031: CT, E3, G(*)=Z3
00032: Z1, Z2, Z3(+)=C3E
00033: C3E, [C3E](WDF)=
00034: AT, E1, F(*)=Z1
00035: CT, E2T, F(*)=Z2
00036: AT, E2, G(*)=Z3
00037: Z1, Z2, Z3(+)=C4E
00038: C4E, [C4E](WDF)=
00039: S(T)=ST
00040: ST, E3, G(*)=Z1
00041: C3E, EPS(S*)=Z2
00042: Z1, C50, Z2(+)=Z3
00043: Z3(T)=Z1
00044: C4E, EPS(S*)=Z2
00045: C40, Z2(+)=Z3
00046: Z3, C3E(+)=Z2
00047: Z2, RR, Z1(*)=Z3
00048: DO(T)=DOT
00049: Z2, L, DOT(*)=Z4
00050: (ELM)=DO, B, S, ST, F, G
00051: NO, L, Z1(*)=Z2
00052: Z2, Z3, Z4(+)=Z1
00053: C5E(T)=C5TE
00054: NO, ROR, C5TE(*)=Z2
00055: C4E, ROR, DOT(*)=Z3
00056: NO, L1, DOT(*)=Z4
00057: Z1, Z2, Z3, Z4(+)=Z5
00058: C6E, Z5(-)=F2
00059: F2, [F2](WDF)=
```

```
(ELM)=Z4, Z5
AT, E1, A(*)=Z1
AT, E2, C(*)=Z2
Z2(T)=Z3
Z1, Z2, Z3(+)=C2E
C2E, [C2E](WDF)=
NO(T)=NOT
(ELM)=A, C
C4E, ROR, NOT(*)=Z1
Z1(T)=Z2
NO, L1, NOT(*)=Z3
Z1, Z2, Z3(+)=Z4
C4E, EPS(S*)=Z1
C40, Z1(+)=Z2
Z2, C3E(+)=Z3
Z3, L, NOT(*)=Z1
Z1(T)=Z2
Z1, Z2, Z4(+)=Z5
Z3(T)=Z1
Z3, PR, Z1(*)=Z2
Z2, Z3(+)=Z3
Z3, C2E(-)=F1
F1, [F1](WDF)=
BOTTOM
$ 0
OK, SPOOL L
```

```
OK, ED FES
EDIT
$ P999
.NULL.
(RDF)=S3,[S3]
(RDF)=EPS,[EPS]
(RDF)=A,[AL3]
(RDF)=F3,[F3]
(RDF)=E3,[E3]
(RDF)=ERROR,[ERR]
F3,EPS(S*)=Z1
S3,Z1(+)=Q1
A(T)=AT
Y:AT,E3,A(*)=Z1
Z1,Q1(+)=NE3
NE3,E3(-)=DF
DF(NRR)=DFN
NE3(MCP)=E3
DFN,ERROR(IFJ)=X,Y,Y
X:E3,[E3](WDF)=
(ELM)=DFN,NE3
(RDF)=AL1,[AL1]
(RDF)=S2,[S2]
(RDF)=F2,[F2]
(RDF)=S1,[S1]
(RDF)=F1,[F1]
(RDF)=E1,[E1]
(RDF)=W,[W]
(RDF)=HR,[HR]
HR,W(*)=Z1
F2,EPS(S*)=Z2
S2,Z2(+)=Z5
Z5,Z1(*)=Z2
Z2(T)=Z3
Z1(T)=Z4
Z4,Q1,Z1(*)=Z6
Z2,Z3,Z6(+)=Z1
F1,EPS(S*)=Z2
S1,Z2(+)=Z3
Z1,Z3(-)=Q
AL1,Q(LAP)=E1
E1,[E1](WDF)=
(RDF)=J,[J]
E1,J(*)=Z1
W(T)=WT
WT,E3,A(*)=Z2
Z1,Z2,Z5(+)=Z3
Z3,HR(*)=E2
E2,[E2](WDF)=
(ELM)=Z2,Z3,Z4,Z5,Z6,Q,Q1,W
(ELM)=J,WT,S1,F1,S2,F2,S3,F3
(RDF)=P10,[P10]
(RDF)=P20,[P20]
(RDF)=P30,[P30]
E1,EPS(S*)=Z1
E2,EPS(S*)=Z2
E3,EPS(S*)=Z3
P10,Z1(+,T)=P1
P20,Z2(+,T)=P2
P30,Z3(+,T)=P3
P1,[P1](WDF)=
P2,[P2](WDF)=
P3,[P3](WDF)=
BOTTOM
$ Q
```

```
D[illegible] ED FXS
ED[illegible]
[illegible] P990
.NULL
00001: (RDF)=P1, [P1]
00002: (RDF)=P2, [P2]
00003: (RDF)=P3, [P3]
00004: (RDF)=A, [A11]
00005: (RDF)=C, [A21]
00006: (RDF)=RR, [RR]
00007: (RDF)=EPS, [EPS]
00008: (RDF)=C20, [C20]
00009: (RDF)=C2E, [C2E]
00010: (RDF)=C3E, [C3E]
00011: (RDF)=NO, [NO]
00012: (RDF)=D1, [D1]
00013: P1, A(*)=Z1
00014: P2, C(*)=Z2
00015: C(T)=CT
00016: CT, P3, C(*)=Z3
00017: Z1(T)=Z4
00018: Z2(T)=Z5
00019: Z1, Z2, Z3, Z4, Z5(+)=Z6
00020: D1(T)=Z1
00021: Z1, D1(*)=Z2
00022: C2E, EPS(S*)=Z1
00023: C20, Z1(+)=C2
00024: C2, EPS(S*)=Z3
00025: Z6, Z2, Z3(+)=Z1
00026: (RDF)=C40, [C40]
00027: (RDF)=C4E, [C4E]
00028: C4E, EPS(S*)=Z2
00029: Z2, C40(+)=C4
00030: C4, EPS(S*)=Z2
00031: C3E, EPS(S*)=Z3
00032: Z2, Z3, NO(+)=Z5
00033: Z5(T)=Z2
00034: Z5, RR, Z2(*)=Z4
00035: Z1, Z4(-, T)=X1
00036: (ELM)=A, C, C20, C2E, C2
00037: (RDF)=B, [A12]
00038: (RDF)=S, [A22]
00039: (RDF)=C60, [C60]
00040: (RDF)=C6E, [C6E]
00041: C6E, EPS(S*)=Z1
00042: C60, Z1(+)=C6
00043: (RDF)=C50, [C50]
00044: (RDF)=C5E, [C5E]
00045: C5E, EPS(S*)=Z1
00046: C50, Z1(+)=C5
00047: (ELM)=C60, C6E, C50, C5E
00048: (RDF)=M, [M]
00049: (RDF)=G, [B2]
00050: (RDF)=D2, [D2]
00051: S(T)=ST
00052: ST, P[illegible](*)=Z1
00053: D2(T[illegible]2T
00054: D2T, [illegible])=Z2
00055: C5, EP[illegible](S*)=Z3
00056: Z1, Z2, Z3(+)=Z4
00057: Z4(T)=Z1
00058: Z5, RR, Z1(*)=Z2
00059: P1, B(*)=Z1
```

```
P2,S(*)=Z3
CT,P3,S(*)=Z5
D1(T)=Z6
Z6,D2(*)=Z7
C6,EPS(S*)=Z6
Z1,Z3,Z5,Z7,Z6(+)=Z8
Z8,Z2(-,T)=X2
X2,[X2](WDF)=
(ELM)=Z1,Z2,Z3,Z5,Z6,Z7,Z8,C6O,C6E
(ELM)=C6
ST,P3,S(*)=Z1
D2T,D2(*)=Z2
(RDF)=C7O,[C7O]
(RDF)=C7E,[C7E]
C7E,EPS(S*)=Z3
C7O,Z3(+)=C7
C7,EPS(S*)=Z3
Z1,Z2,Z3(+)=Z5
Z4(T)=Z1
Z4,RR,Z1(*)=Z2
Z5,Z2(-,T)=X3
X3,[X3](WDF)=
BOTTOM
$ Q
OK, SPOOL L
```

```
OK, ED THM2
EDIT
$ P999
.NULL.
00001: (RDF)=C, [A2I]
00002: (RDF)=INV, [INV]
00003: (RDF)=A, [A]
00004: (RDF)=B, [B]
00005: (RDF)=G, [B2]
00006: (RDF)=ERR, [ERR]
00007: (RDF)=R, [R]
00008: (RDF)=M, [M]
00009: (RDF)=D, [D]
00010: (RDF)=FO, [FO]
00011: (RDF)=FF, [FF]
00012: (RDF)=FS, [FS]
00013: (RDF)=P, [P]
00014: G, FS(*)=Z1
00015: C, Z1(-)=Z2
00016: FF, INV, Z2(*)=Z1
00017: FS, Z1(-, )=F1AP
00018: (ELM)=C, G
00019: FF(MCP)=F2AP
00020: F1AP, F2AP(CTI, T)=FAP
00021: FF, FS, FO(OUT)=
00022: (RDF)=PP, [Z]
00023: B, FAP(*)=Z1
00024: A, Z1(-)=CLF
00025: CLF(T)=CLT
00026: D(T)=DT
00027: M(T)=MT
00028: DT, D(*)=Z1
00029: MT, M(*)=Z2
00030: R, Z2(+)=Z4
00031: FAP(T)=Z3
00032: Z3, Z4, FAP(*)=Z2
00033: DT, M, FAP(*)=Z4
00034: Z4(T)=Z3
00035: Z3, Z4(+)=Z5
00036: Z1, Z2(+)=Z3
00037: Z3, Z5(-, T)=G
00038: (ELM)=Z1, Z2, Z3, Z4
00039: Y: CLT, PP, CLF(*)=Z1
00040: Z1, G(+)=PPN
00041: PPN, PP(-)=N1
00042: N1(NRR)=DFN
00043: PPN(MCP)=PP
00044: DFN, ERR(IFJ)=X, Y, Y
00045: X: P, PP, DFN(OUT)=
00046: FO, FAP(-)=Z1
00047: Z1(T)=Z2
00048: (RDF)=RFUL, [RFUL]
00049: Z2, RFUL, Z1(*, T)=LB
00050: P, PP(-, T)=DIF
BOTTOM
$ Q
OK, SPOOL L
```

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