

CHIRAL DYNAMICS CALCULATIONS
OF SINGLE PION PRODUCTION IN
PION-NUCLEON INELASTIC SCATTERING

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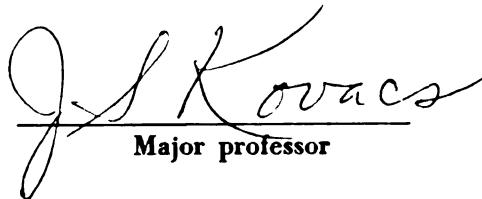


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ABSTRACT

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By

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The current algebra program initiated by Murray Gell-Mann has been incorporated by dynamical models by several people. Weinberg and Schwinger, among others, have constructed so-called "chiral dynamics" Lagrangians which describe the interaction between nucleons and "soft" pions. A common difficulty of these formulations is an ambiguity in the pi-pi interaction. The most straightforward way of eliminating this ambiguity would be measurement of pi-pi scattering lengths, but that is a very difficult experiment. Olsson and Turner have attempted to resolve the difficulty by calculating the threshold cross section for the process $\pi^- p \rightarrow \pi^+ \pi^- n$ in which process the disputed pi-pi interaction strongly contributes. But near threshold the breaking of isospin symmetry is reflected in large differences in phase space volumes, depending on which mass of the supposedly degenerate isomultiplets is used. For this reason, it is desirable to extend the cross section calculation off threshold to where the isospin symmetry incorporated in

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the model is a more realistic approximation and conclusions about the validity of different models can be based on a larger set of experimental data. Furthermore, such a calculation would give some idea of the maximum energies for which the model, derived for soft pions, could be applied.

The calculation was done for five different charge channels retaining all Feynman diagrams and computing the integrals necessary to obtain total and differential cross sections by means of a Monte Carlo method to within four (4) per cent. The results indicate that the best fit to low energy total cross sections was given by a model in which chiral symmetry was broken by a term which transforms as a rank two chiral tensor. For incident pion energies much greater than 300 MeV none of the chiral dynamics models employed fit the total cross sections well. Comparatively little relevant data exists for differential cross sections, but what there is indicates poor agreement between experiment and chiral dynamics predictions.

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NOTATION

Conventions for metric and gamma matrices, as well as most of the other notation, has been taken from Relativistic Quantum Fields by James Bjorken and Sidney Drell. (See especially appendix A, p. 377.) Some specifics of notation are given below.

Metric and Four-Vectors

A covariant notation is used with metric matrix

$g_{\alpha\beta}$ defined by

$$\begin{aligned} g_{00} &= 1 \\ g_{11} &= g_{22} = g_{33} = -1 \\ g_{\alpha\beta} &= 0 ; \alpha \neq \beta \end{aligned}$$

The summation convention is used, repeated Latin letters indicating a sum over three indices, repeated Greek letters indicating a sum over four indices. Vectors are denoted by a symbol with an arrow, e.g. $\vec{x}$, and four vectors are defined by

$$\begin{aligned} x^\alpha &= (x^0, \vec{x}), \\ x_\alpha &= g_{\alpha\beta} x^\beta = (x_0, -\vec{x}) \end{aligned}$$

Products of two four-vectors are of the form

$$xy = x_\alpha y^\alpha = x_0 y_0 - \vec{x} \cdot \vec{y}$$

We define

$$\partial_\tau \equiv \frac{\partial}{\partial x_\tau} \quad , \quad \partial^\tau \equiv \frac{\partial}{\partial x_\tau}$$

Integrals

We use the notation

$$\int d^3x = \int dx_1 dx_2 dx_3$$

$$\int d^4x = \int dx_1 dx_2 dx_3 dx_0$$

Operators

Operators are denoted by capital letters. Some operators frequently encountered are:

$$\begin{array}{ll} Q & = \text{general operator} \\ T & = \text{isospin operator} \\ X & = \text{generator of axial vector current} \end{array}$$

Currents

Currents are denoted by script letters. Some currents often used are:

$$\begin{array}{ll} J(x) & = \text{general current} \\ J^\mu(x) & = \text{hadron vector current} \\ J_5^\mu(x) & = \text{hadron axial current} \end{array}$$

Particle Field Operators

Particle field operators are denoted by Greek letters, except for photon fields. Field operators used are:

$$\begin{aligned}\varphi(x) &= \text{general field} \\ \Phi(x) &= \text{general particle field} \\ \phi(x) &= \text{general boson field} \\ \Psi(x), \bar{\Psi}(x) &= \text{general fermion field and its adjoint} \\ \phi(x) &= \text{pion field} \\ \Psi(x), \bar{\Psi}(x) &= \text{nucleon field and its adjoint} \\ A^\tau(x) &= \text{photon field}\end{aligned}$$

Constants

Constants are written in various ways, as the examples below show.

$$\begin{aligned}m &= \text{general boson mass} \\ M &= \text{general fermion mass} \\ \mu &= \text{pion mass} \\ M &= \text{nucleon mass} \\ G &= \text{weak coupling constant} \\ g &= \text{strong coupling constant, taken such that} \\ &\quad g^2/4\pi = 14.6 \\ g_A/g_V &= \text{ratio of axial current coupling constant} \\ &\quad \text{to vector current coupling constant} \\ \delta_{ab} &= \text{Kronecker delta} \\ \epsilon_{abc} &= \text{antisymmetric tensor}\end{aligned}$$

Functional dependence and subscripts are frequently suppressed to simplify notation.

SECTION I

INTRODUCTION

Recent interest in the application of the algebra of currents to particle physics grew out of studies of the weak interaction. It was found that the effective Lagrangian density for the weak interaction, $\mathcal{L}_w$, could be written as a weak current, $J_w(x)$, coupled to itself, i.e.

$$\mathcal{L}_w(x) = \frac{G}{\sqrt{2}} J_{w\mu}(x)^\dagger J_w^\mu(x) \quad (1.1)$$

where G is a coupling constant which is the same for all weak interactions. The weak current may be written as the sum of a leptonic part, $j_l(x)$, and a hadronic part, $J_h(x)$. Each of these may be broken up into the difference between a vector current invariant under spatial inversion, and an axial vector current which changes sign under spatial inversion. For the leptons,

$$j_l(x) = j(x) - j_5(x)$$

and for the hadrons,

$$J_h(x) = J(x) - J_5(x).$$

In the case of beta decay, for example,

$$n \rightarrow p + e^- + \bar{\nu}, \quad (1.2)$$

the lepton currents take the form

$$\begin{aligned} j_V(\alpha) &= \bar{\Psi}_e(\alpha) \gamma_\alpha \Psi_\nu(\alpha), \\ j_{S_V}(\alpha) &= \bar{\Psi}_e(\alpha) \gamma_\alpha \gamma_5 \Psi_\nu(\alpha). \end{aligned} \quad (1.3)$$

Assuming small momentum transfer, the hadron currents take the form

$$\begin{aligned} j_V(\alpha) &= \bar{\Psi}_p(\alpha) \gamma_\alpha \Psi_n(\alpha), \\ j_{S_V}(\alpha) &= \frac{g_A}{g_V} \bar{\Psi}_p(\alpha) \gamma_\alpha \gamma_5 \Psi_n(\alpha). \end{aligned} \quad (1.4)$$

The quantity g_A/g_V is the ratio of the axial vector coupling constant to the vector coupling constant. It is measured experimentally to be 1.18 ± 0.02 . If we use the currents given by (1.3) and (1.4) with the Lagrangian (1.1) we get a four fermion interaction which is not renormalizable, but which gives satisfactory agreement with experiment if used naively in perturbation theory to lowest order in G .

Two aspects of the interaction (1.1) and the structure of the weak currents are particularly interesting. First of all, the coupling constant G is the same whether the interaction is leptonic, semi-leptonic, or non-leptonic. Second, the hadronic and leptonic weak currents are constructed from their respective vector and axial

vector currents in a remarkably similar fashion. In the leptonic case, the weak current assumes a symmetric "vector current - axial vector current" form, and in the hadronic case the weak current deviates from this form only because of the factor g_A/g_V in the axial vector current, and g_A/g_V is tantalizingly close to unity. Taken together, these two observations mean that leptonic and hadronic vector currents couple in the same way in the weak interaction, and leptonic and hadronic axial vector currents couple in nearly the same way. Let's consider first the implications of this for the vector current.

The identical coupling of leptonic and hadronic vector currents comes as a surprise because we would expect a current of the strongly interacting particles to be modified by the pion clouds which surround such particles. In terms of Feynman diagrams, this means that as far as the vector current is concerned, the interaction of Figure 1 consists entirely of the diagrams of Figure 2 with no contribution by diagrams like Figure 3.

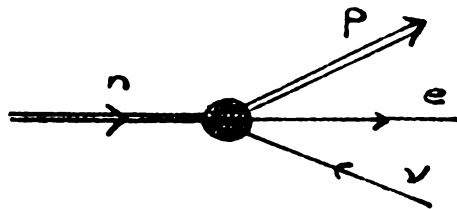


Figure 1. Diagram of beta decay process (1.2).

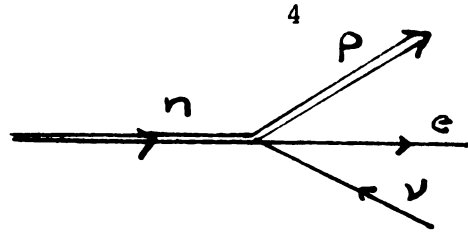


Figure 2. Diagram of beta decay with unrenormalized vertex.

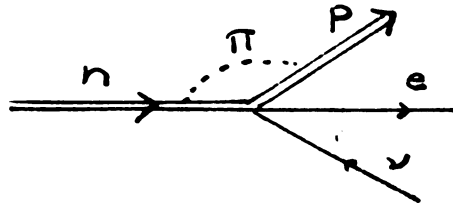


Figure 3. Diagram of beta decay with renormalized vertex.

In other words, the vector current is unrenormalized by the strong interaction.

A very similar situation exists in the case of the electromagnetic interaction. The interaction Lagrangian density for the electromagnetic interaction may be written

$$\mathcal{L}_{em}(\alpha) = e j_{em}^\alpha(\alpha) A_\alpha(\alpha) \quad (1.5)$$

where $A_\alpha(\alpha)$ is the photon field operator and $j_{em}^\alpha(\alpha)$ is the four-vector electromagnetic current of either a hadron or a lepton. Here again we have a current participating in a direct way in an interaction, and here again we have a coupling constant, this time the fundamental unit of charge, e , which is the same if the current in the interaction is associated with leptons or with strongly interacting particles. The fact that electric charge is unrenormalized by the strong interaction indicates that the strong interaction Hamiltonian commutes with the charge operator Q which generates the

electromagnetic current, and therefore the electromagnetic current is conserved.* This is expressed by the equation

$$\frac{\partial j_{em}^{\alpha}(\alpha)}{\partial \alpha^{\alpha}} = 0.$$

Returning to the discussion of the vector current part of the weak current, we see that application of arguments similar to those employed for the electromagnetic current imply that $J(\alpha)$ is generated by some operator which commutes with the strong interaction Hamiltonian, and

$$\frac{\partial J^{\alpha}(\alpha)}{\partial \alpha^{\alpha}} = 0. \quad (1.6)$$

We'd like to be able to identify the operator generating the vector current. A clue comes from rewriting the hadronic vector current for beta decay in the following way:

$$J_{\alpha} = \bar{\Psi}_p \gamma_{\alpha} \Psi_n = \frac{1}{2} \bar{\Psi} \gamma_{\alpha} (\tau_1 + i \tau_2) \Psi \quad (1.7)$$

where Ψ is the eight component spinor

$$\Psi = \begin{pmatrix} \Psi_p \\ \Psi_n \end{pmatrix}$$

and the τ 's are the Pauli matrices. Now the isospin current for nucleons may be written

$$\vec{J}_I(\alpha) = \frac{1}{2} \bar{\Psi}(\alpha) \gamma_{\alpha} \vec{\tau} \Psi(\alpha). \quad (1.8)$$

*The formalism relating gauge transformation and currents is reviewed in section II.

Comparing (1.7) and (1.8), we see that the hadronic vector current for beta decay equals the "plus" component of the isospin current. Since, moreover, the isospin operator $\vec{T}$ commutes with the strong interaction Hamiltonian and the isospin current satisfies the conservation law

$$\frac{\partial \vec{g}_i^{\mu}(\alpha)}{\partial \alpha^{\mu}} = 0,$$

we postulate the identity of the vector current and isospin current, and identify $\vec{T}$ as the generator of both. The identification of the vector current as a component of the isospin current together with equation (1.6) is called the "conserved vector current hypothesis", which alphabetizes to CVC. A consequence of the conserved vector current hypothesis is that form factors of the nucleon weak current and the nucleon isospin current are, because of the Wigner-Eckart theorem, proportional to one another, the proportionality constant being just a Clebsch-Gordon coefficient. This has been verified experimentally in analyses of the beta spectra of decays of B^{12} and N^{12} into C^{12} .¹

The case of the axial vector current of strongly interacting particles is somewhat more complicated. The success of CVC makes it tempting to postulate that the axial current is also conserved, but two facts militate against this hypothesis. First, the axial current contribution to the hadronic weak current enters not with a factor of unity but a factor $g_A/g_V = 1.18 \pm 0.02$.

This is close enough to one to suggest that the renormalization of the axial current is slight, but far enough from one to indicate that the renormalization may be not ignored. Second, a conserved axial vector hypothesis

$$\frac{\partial g_5^\mu(\alpha)}{\partial \alpha^\mu} = 0 \quad (1.9)$$

would forbid the process

$$\pi \longrightarrow \mu + \nu. \quad (1.10)$$

This decay is governed by the matrix element

$$\langle 1 | g_5^\mu(\alpha) | \pi(q) \rangle \quad (1.11)$$

where $| \rangle$ is the vacuum state and $|\pi(q)\rangle$ is a state with one pion of four-momentum q . If (1.9) holds, then

$$\begin{aligned} \langle 1 | \partial_\mu g_5^\mu(\alpha) | \pi(q) \rangle &= -i q_\mu \langle 1 | g_5^\mu(\alpha) | \pi(q) \rangle \\ &= 0 \end{aligned} \quad (1.12)$$

so that the matrix element (1.11) vanishes and the decay amplitude is zero. Since the pion decay process (1.10) does take place, equation (1.9) cannot hold. However, (1.12) contains an important clue to the resolution of the dilemma. If the pion is at rest when it decays, (1.11) becomes

$$\mu \langle 1 | g_5^0(\alpha) | \pi \rangle = 0$$

where μ is pion mass. We see that if $\mu = 0$,

(1.9) could hold without forbidding pion decay by the process (1.10). Therefore we say that the axial symmetry of the strong interaction is broken by the pion mass.

The simplest way of accounting for the symmetry breaking is to write the divergence of the axial current as follows:

$$\partial_\mu \vec{J}_5^\mu = \text{constant} \times \mu \times \text{some pseudoscalar operator}$$

This has the correct parity and reduces to (1.9) in the limit as $\mu \rightarrow 0$. The simplest pseudoscalar operator we could choose would simply be the pion field operator, $\vec{\phi}$. The operator $\vec{\phi}$ is an isotriplet, so we incorporate the axial vector current in an isotriplet and write

$$\partial_\mu \vec{J}_5^\mu(x) = \text{constant} \times \mu \times \vec{\phi}(x). \quad (1.13)$$

It was shown by Gell-Mann and Levy² that the constants in (1.13) must be chosen such that

$$\partial_\mu \vec{J}_5^\mu(x) = \frac{M}{g} \frac{g_A}{g_V} \mu^2 \vec{\phi}(x), \quad (1.14)$$

and that it is possible to use this formula to derive by a field theoretic approach a relation between the axial vector form factors and the pion decay rate which was originally derived from dispersion theory by Goldberger and Treiman.^{*,3} Here, g is the strong

*The Goldberger-Treiman relation is discussed in more detail in Appendix A.

coupling constant and M is the nucleon mass. Equation (1.14) forms the "partially conserved axial vector current hypothesis", which alphabetizes to PCAC. However, PCAC by itself gives no clue about the nature of the operator generating the axial vector current.

In an early attempt to discover the group properties of the generator of the axial vector current, Gell-Mann and Levy proposed three Lagrangian models incorporating PCAC and CVC;⁴ a gradient coupling model, the sigma model, and a variant of the sigma model called the non-linear model. All these models were unsatisfactory by the criteria applied, the first and third models being unrenormalizable and the second requiring the existence of the never-to-be-discovered sigma meson. Despite the drawbacks of these models, there were some useful results of this line of inquiry. The most important result was the Lie algebra of the generators of the vector current and the axial vector current for the sigma model and the non-linear model. If we let $\vec{X}$ be the operator which generates the axial vector current, these two models yield the commutators

$$\begin{aligned} [T_a, T_b] &= i \epsilon_{abc} T_c \\ [T_a, X_b] &= i \epsilon_{abc} X_c \\ [X_a, X_b] &= i \epsilon_{abc} T_c \end{aligned} \quad (1.15)$$

If we define two new triplets of operators by

$$\begin{aligned}\vec{R} &= (\vec{T} + \vec{X})/2 \\ \vec{L} &= (\vec{T} - \vec{X})/2\end{aligned}\tag{1.16}$$

routine algebra reduces (1.15) to the symmetric forms

$$\begin{aligned}[R_a, R_b] &= i\epsilon_{abc} R_c \\ [L_a, L_b] &= i\epsilon_{abc} L_c \\ [L_a, R_b] &= 0\end{aligned}\tag{1.17}$$

Equation (1.17) shows that the operators $\vec{R}$ and $\vec{L}$ generate two independent SU(2) groups, and together they generate the symmetry group $SU(2)_R \otimes SU(2)_L$ with subscripts referring to the operator generating each SU(2) algebra. This group is called chiral $SU(2) \otimes SU(2)$ because of the similarity of the forms in (1.16) to the right and left hand chirality operators of field theory.

It turned out that (1.15) was correct as far as it went, but it remained to be explained how to incorporate strange particles into the theory. Clues to the solution of the problem were furnished by the success of SU(3) as a symmetry group of the strong interaction and by the Cabbibo theory of the weak interaction⁵ which placed the weak currents in SU(3) multiplets. These discoveries set the stage for Gell-Mann's current algebra hypothesis.

Gell-Mann's hypothesis made the logical extension of chiral $SU(2) \otimes SU(2)$ to chiral $SU(3) \otimes SU(3)$. This may be done by changing the ϵ_{abc} 's of equation (1.17) to the structure constants of SU(3), conventionally denoted f_{abc} , and allowing a , b , and c to run

over the eight indices labeling SU(3) operators. Equation (1.17) becomes

$$\begin{aligned} [R_a, R_b] &= i f_{abc} R_c \\ [L_a, L_b] &= i f_{abc} L_c \\ [L_a, R_b] &= 0 \end{aligned} \quad (1.18)$$

The operators are related to the eight generators of vector currents, F_a , and the eight generators of axial vector currents, F_{5a} , by

$$\begin{aligned} R_a &= (F_a + F_{5a})/2 \\ L_a &= (F_a - F_{5a})/2. \end{aligned} \quad (1.19)$$

And finally, (1.15) becomes

$$\begin{aligned} [F_a, F_b] &= i f_{abc} F_c \\ [F_a, F_{5b}] &= i f_{abc} F_{5c} \\ [F_{5a}, F_{5b}] &= i f_{abc} F_c. \end{aligned} \quad (1.20)$$

Gell-Mann went one step further than this, however. If we denote vector and axial vector currents by $\mathcal{F}(\alpha)$ and $\mathcal{F}_5(\alpha)$ respectively, Gell-Mann postulates the following equal time commutation relations for the octet charge

densities:

$$\begin{aligned} [\mathcal{F}_a^\circ(\alpha), \mathcal{F}_b^\circ(\gamma)]_{\alpha^0=\gamma^0} &= i \delta^3(\vec{\alpha}-\vec{\gamma}) f_{abc} \mathcal{F}_c^\circ(\alpha) \\ [\mathcal{F}_a^\circ(\alpha), \mathcal{F}_{5b}^\circ(\gamma)]_{\alpha^0=\gamma^0} &= i \delta^3(\vec{\alpha}-\vec{\gamma}) f_{abc} \mathcal{F}_{5c}^\circ(\alpha) \\ [\mathcal{F}_{5a}^\circ(\alpha), \mathcal{F}_{5b}^\circ(\gamma)]_{\alpha^0=\gamma^0} &= i \delta^3(\vec{\alpha}-\vec{\gamma}) f_{abc} \mathcal{F}_c^\circ(\alpha). \end{aligned} \quad (1.21)$$

Since $F_a = \int d^3x \bar{\psi}_a^\circ(x)$, and $F_{5a} = \int d^3x \bar{\psi}_{5a}^\circ(x)$, (1.21) integrates to (1.20) and is a stronger hypothesis than (1.20). Equation (1.21) is referred to as the current algebra hypothesis and it forms the foundation of most recent work in current algebra, along with PCAC and some version of PCAC for strange particles.

One interesting aspect of the current algebra hypothesis is that it is free of specific assumptions about the dynamics of the strong interaction, and therefore freed of the limitations inherent in the perturbation expansions of quantum field theory. Yet, the hypothesis is not sterile and permits calculation of many interesting physical quantities through manipulations of matrix elements of the currents. One particularly interesting class of calculations is that of so-called "soft pion processes". These calculations express the matrix element for a process in which one or more pions of small four-momentum are emitted or absorbed in terms of certain equal time commutators of currents and the matrix element of the same process without soft pions. Soft pion calculations have been applied to $K_{\ell 4}$ decay,⁶ multiple pion production,^{7,8} pi-pi and pion-nucleon scattering lengths,⁹ pion photoproduction,¹⁰ pion production in pion-nucleon interaction near threshold,¹¹ and many other problems.

But the model independence of the current algebra hypothesis was eventually compromised by the introduction of specific Lagrangian densities which were vastly easier to work with, if somewhat less elegant. The first of these "chiral dynamics" Lagrangians was introduced by Weinberg.¹² His model was developed by transforming the sigma model in such a way that it would reproduce the results of current algebra if used to lowest order in the coupling constant, g . The problem of renormalizing higher order contributions of perturbation theory was overcome by fiat: one just ignored terms leading to divergences. Weinberg's model included only pions and nucleons. Soon Schwinger¹³ introduced a Lagrangian model for pions and nucleons which also satisfied the restrictions imposed by the current algebra hypothesis, but which differed from Weinberg's model. Since then the chiral dynamics approach has enjoyed increasing popularity. Several different papers have been published on different methods of constructing pion-nucleon Lagrangians,¹⁴ and many authors have published applications involving extensions to other particle fields.¹⁵

As useful as chiral dynamics Lagrangians have been found to be, they still cannot be considered a true field theory for the strong interaction in the sense that (1.5) forms the basis of quantum electrodynamics. The reason for this has already been alluded to, namely, we avoid using these Lagrangians to more than lowest order in the coupling

constants, though higher order terms are manifestly divergent and therefore non-negligible. The algorithm for using chiral dynamics Lagrangians reduces to this: Use the Lagrangian with the ordinary Feynman-Dyson rules to lowest order in g , but include only contributions from tree diagrams. Tree diagrams are diagrams which contain no loops and no internal momenta to be integrated over.

Two main points of view are held about chiral dynamics theories. They are exemplified by the attitudes of Schwinger and Weinberg, two of the first to participate in the Lagrangian revival. Schwinger feels that Lagrangians represent a suitable way to investigate strong interaction phenomenology and to remove the weak interaction orientation of current algebra. His rebuttal to anyone who objects to the chiral dynamics algorithm explained above is,

It is not meaningful to question the use of coupling terms 'in lowest order'. That is the nature of a numerical effective Lagrange function, which gives a direct description of the phenomena.¹⁶

Weinberg, on the other hand, is less sanguine about such procedures, and clings to the primacy of current algebra. He says,

Opinions differ as to whether any fundamental significance resides in the Lagrangians which have been used... [I] myself remain uneasy at using a symmetry on the phenomenological level, when it is not clear how any fundamental Lagrangian could give rise to the supposed symmetry of

phenomena. From this point of view, chirality is in good shape because we have current algebra to underwrite it...¹⁷

But no matter what philosophy is adopted, the utility of chiral dynamics in practical calculations cannot be denied. Besides, strong precedent exists for the use of a Lagrangian with shaky underpinnings in the case of the weak interaction Lagrangian (1.1). Had investigators had too many scruples about employing such an unrenormalizable Lagrangian, current algebra might never have been discovered.

SECTION II

CURRENTS AND GAUGE TRANSFORMATIONS

If a system may be described by a Lagrangian density $\mathcal{L}$, which is a function of n fields $\varphi_n(x)$ and their derivatives $\partial_\alpha \varphi_n(x)$, the equations of motion of the system are given by the usual Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \varphi_n(x)} - \frac{\partial}{\partial x^\alpha} \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \varphi_n(x))} = 0; n=1, \dots, m \quad (2.1)$$

We are interested in the effect of a small change in each field $\varphi_n(x)$, of the form

$$\varphi_n(x) \rightarrow \varphi_n(x) - i\epsilon \lambda_{na} \varphi_a(x) \quad (2.2)$$

where ϵ is some small constant and λ_{na} is a constant matrix or, at most, a function of boson fields.

Equation (2.2) is called a gauge transformation. Using the chain rule, along with (2.1) and (2.2), we find for $\delta \mathcal{L}$, the change in the Lagrangian, after all the have undergone gauge transformations,

$$\begin{aligned} \delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \varphi_n} \delta \varphi_n + \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \varphi_n)} \delta (\partial_\alpha \varphi_n) \\ &= -i\epsilon \lambda_{na} \left\{ \left[\frac{\partial}{\partial x^\alpha} \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \varphi_n)} \right] \varphi_a + \left[\frac{\partial \mathcal{L}}{\partial (\partial_\alpha \varphi_n)} \right] \frac{\partial \varphi_a}{\partial x^\alpha} \right\} \end{aligned}$$

$$\delta \mathcal{L} = \epsilon \frac{\partial}{\partial x^\mu} \left[-i \frac{\partial \mathcal{L}}{\partial (\partial^\mu \psi_n(x))} \lambda_n \psi_n(x) \right] \quad (2.3)$$

Let us now define the four vector current generated

by the gauge transformation (2.2) as

$$j^\mu(x) = -i \frac{\partial \mathcal{L}}{\partial (\partial^\mu \psi_n(x) / \partial x^\mu)} \lambda_n \psi_n(x), \quad (2.4)$$

and define its "charge" as

$$Q(t) = \int_{\text{all space}} d^3x \, j_0(x). \quad (2.5)$$

If $\mathcal{L}$ is invariant under the transformation (2.2),

$\delta \mathcal{L} = 0$ and from (2.3) we see that $j^\mu(x)$ satisfies the conservation equation

$$\partial_\mu j^\mu(x) = 0. \quad (2.6)$$

Furthermore, for a conserved current, Q is a constant of motion since

$$\begin{aligned} \frac{\partial Q}{\partial t} &= \int_{\text{all space}} d^3x \frac{\partial j_0(x)}{\partial t} = - \int d^3x \, \vec{\nabla} \cdot \vec{j}(x) \\ &= - \int_S d\vec{S} \cdot \vec{j}(x) = 0 \end{aligned}$$

where we've used Gauss' Law and assumed that $\vec{j}(x)$ has finite extent and therefore vanishes on a surface S at infinity.

In canonical field theory, we write that the operator $\pi_n(x)$ conjugate to the operator $\psi_n(x)$ is given by

$$\pi_n(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_n(x)}. \quad (2.7)$$

Using (2.4), (2.5), and (2.7), we get for Q

$$Q = \int d^3x \left[-i \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} \right] \lambda_{ns} \epsilon_n$$

$$Q = -i \int d^3x \pi_n(x) \lambda_{ns} \epsilon_n(x). \quad (2.8)$$

We shall need to know the commutator of the operator Q with the field operator $\phi_n(x)$. Assume first that we want to know

$$[Q(t), \phi_p(\vec{x}, t)] \quad (2.9)$$

where $\phi_p(x)$ is the p th boson field. The usual quantization procedures require the following equal-times commutators:

$$\begin{aligned} [\phi_n(x), \phi_p(y)]_{x_0=y_0} &= 0 \\ [\pi_n(x), \pi_p(y)]_{x_0=y_0} &= 0 \\ [\pi_n(x), \phi_p(y)]_{x_0=y_0} &= -i \delta_{np} \delta^3(\vec{x} - \vec{y}). \end{aligned} \quad (2.10)$$

Applying (2.8) and (2.10) in the evaluation of (2.9), we get

$$\begin{aligned} [Q(t), \phi_p(\vec{x}, t)] &= [-i \int d^3y \pi_n(\vec{y}, t) \lambda_{ns} \phi_s(\vec{y}, t), \phi_p(\vec{x}, t)] \\ &= -i \int d^3y [\pi_n(\vec{y}, t), \phi_p(\vec{x}, t)] \lambda_{ns} \phi_s(\vec{y}, t) \\ &= -i \int d^3y \{-i \delta_{np} \delta^3(\vec{x} - \vec{y})\} \lambda_{ns} \phi_s(\vec{y}, t) \end{aligned}$$

$$[Q(t), \phi_p(\vec{x}, t)] = -\lambda_{ps} \phi_s(\vec{x}, t). \quad (2.11)$$

Assume instead we wanted the commutator

$$[Q(t), \Psi_{p,r}(\vec{x}, t)] \quad (2.12)$$

where $\Psi_{p,r}(\vec{x})$ is the r th component of the p th fermion field. This time, quantization procedures yield the anticommutators

$$\begin{aligned} \{\Psi_{i,\alpha}(\vec{x}), \Psi_{j,\beta}(\vec{y})\}_{\alpha_0=\beta_0} &= 0 \\ \{\Psi_{i,\alpha}^\dagger(\vec{x}), \Psi_{j,\beta}^\dagger(\vec{y})\}_{\alpha_0=\beta_0} &= 0 \\ \{\Psi_{i,\alpha}(\vec{x}), \Psi_{j,\beta}^\dagger(\vec{y})\}_{\alpha_0=\beta_0} \\ &= \delta_{\alpha\beta} \delta_{ij} \delta^3(\vec{x}-\vec{y}) \end{aligned} \quad (2.13)$$

The operator conjugate to $\Psi_{n,\alpha}$ is

$$\frac{\partial \mathcal{L}}{\partial \Psi_{n,\alpha}} = i \Psi_{n,\alpha}^\dagger,$$

so the charge operator becomes

$$Q(t) = \int d^3y \Psi_{n,\alpha}^\dagger(\vec{y}, t) \lambda_{n\alpha} \Psi_{\alpha,\alpha}(\vec{y}, t). \quad (2.14)$$

Evaluating (2.12) using (2.13) and (2.14), we find

$$\begin{aligned} [Q(t), \Psi_{p,r}(\vec{x}, t)] &= \left[\int d^3y \Psi_{n,\alpha}^\dagger(\vec{y}, t) \lambda_{n\alpha} \Psi_{\alpha,\alpha}(\vec{y}, t), \Psi_{p,r}(\vec{x}, t) \right] \\ &= \int d^3y \lambda_{n\alpha} \left[\Psi_{n,\alpha}^\dagger(\vec{y}, t) \{ \Psi_{\alpha,\alpha}(\vec{y}, t), \Psi_{p,r}(\vec{x}, t) \} \right. \\ &\quad - 2 \Psi_{n,\alpha}^\dagger(\vec{y}, t) \Psi_{p,r}(\vec{x}, t) \Psi_{\alpha,\alpha}(\vec{y}, t) \\ &\quad + \{ \Psi_{n,\alpha}^\dagger(\vec{y}, t), \Psi_{p,r}(\vec{x}, t) \} \Psi_{\alpha,\alpha}(\vec{y}, t) \\ &\quad \left. - 2 \Psi_{p,r}(\vec{y}, t) \Psi_{n,\alpha}^\dagger(\vec{x}, t) \Psi_{\alpha,\alpha}(\vec{x}, t) \right] \\ &= \int d^3y \lambda_{n\alpha} \left[-\delta^3(\vec{x}-\vec{y}) \delta_{n\alpha} \Psi_{\alpha,r}(\vec{y}, t) \right] \end{aligned}$$

$$[Q(t), \Psi_{p,r}(\vec{x}, t)] = -\lambda_{p0} \Psi_{a,r}(\vec{x}, t). \quad (2.15)$$

Since (2.15) is in the same form as (2.11), we may write for any field $\psi_p(x)$,

$$[Q(t), \psi_p(\vec{x}, t)] = -\lambda_{p0} \psi_a(\vec{x}, t). \quad (2.16)$$

Now let us define an operator $\hat{G}$ which generates the gauge transformation (2.2). If we transform some field $\psi_n(x)$, $\hat{G}$ is defined by

$$e^{i\epsilon \hat{G}} \psi_n(x) e^{-i\epsilon \hat{G}} = \psi_n(x) - i\epsilon \lambda_{n0} \psi_a(x)$$

But, since ϵ is small,

$$\begin{aligned} e^{i\epsilon \hat{G}} \psi_n(x) e^{-i\epsilon \hat{G}} &\cong (1 + i\epsilon \hat{G}) \psi_n(x) (1 - i\epsilon \hat{G}) \\ &= \psi_n(x) + i\epsilon [\hat{G}, \psi_n(x)] \end{aligned}$$

and so

$$[\hat{G}, \psi_n(x)] = -\lambda_{n0} \psi_a(x)$$

By comparison with (2.16), we see that effectively $\hat{G} = Q$, and hence the charge Q is the generator of the gauge transformation. It is important to realize that (2.2) will give a different form of $\mathcal{L}(x)$ for every different Lagrangian, but in every case the charge Q obtained by integrating $\mathcal{L}_0(x)$ will generate the gauge transformation which we started with.

Since Q generates the transformation of all operators, we may apply it to the Lagrangian density.

Hence, under the transformation (2.2),

$$\begin{aligned} \mathcal{L} \rightarrow \mathcal{L}' &= e^{i\epsilon Q} \mathcal{L} e^{-i\epsilon Q} \\ &\cong \mathcal{L} + i\epsilon [Q, \mathcal{L}]. \end{aligned}$$

The change in the Lagrangian induced by the gauge transformation is

$$\delta \mathcal{L} = \mathcal{L}' - \mathcal{L} = i\epsilon [Q, \mathcal{L}].$$

But from (2.3) and (2.4),

$$\delta \mathcal{L} = \epsilon \partial_\alpha \mathcal{J}^\alpha(x),$$

so

$$i [Q(t), \mathcal{L}] = \partial_\alpha \mathcal{J}^\alpha(x). \quad (2.17)$$

Equation (2.17) is a convenient way of finding the divergence of a current without actually constructing the current or the equations of motion of the system.

What we usually know from experiments is that some quantum number q , such as charge, strangeness, or baryon number is conserved, hence some charge operator Q commutes with the Hamiltonian. Next we try to deduce the commutator

$$[Q(t), \psi_n(x)]$$

for whatever fields $\psi_n(x)$ are involved. Then we construct the Lagrangian from the fields in such a way that

$$[Q(t), \mathcal{L}(\psi_n(x), \partial_\alpha \psi_n(x))] = 0.$$

If q is not conserved, but we know $\partial_\alpha g^\alpha(\kappa)$, the divergence of the current, it is trivial to use (2.17) to generalize this procedure.

One common situation is when we have a Lagrangian invariant under transformations generated by a set of operators $\{Q_a\}$ which form a Lie algebra with structure constants q_{abc} ,

$$[Q_a, Q_b] = i q_{abc} Q_c.$$

Particle fields $\Phi_a(\kappa)$ are composed of matrices involving one or more fields $\varphi_n(\kappa)$, boson fields being composed of one field, fermion fields of a spinor with four fields, etc. If then the particle field operators are tensors of the n -dimensional representation of the Lie group of the operators Q_a , then

$$[Q_a, \Phi_b(\kappa)] = -(\lambda_a)_{bc} \Phi_c(\kappa) \quad (2.18)$$

where λ_a is the a th matrix of the $n \times n$ representation of the operator Q_a . Equation (2.2) generalizes to

$$\begin{aligned} \Phi_a(\kappa) &\rightarrow \Phi_a(\kappa) - i \epsilon_b (\lambda_b)_{ac} \Phi_c(\kappa) \\ &= \Phi_a(\kappa) + i \epsilon_b [Q_b, \Phi_a(\kappa)], \end{aligned} \quad (2.19)$$

ϵ_b being the b th member of a set of small numbers.

An illustration of the formalism discussed above is the case of the isospin gauge transformation generated by the operator $\vec{T}$ which satisfies the Lie algebra

$$[T_a, T_b] = i \epsilon_{abc} T_c$$

The nucleons form an isodoublet, and since the 2×2 representation of the isospin generator T_a is

$$(t_a)_{bc} = \left(\frac{\tau_a}{2} \right)_{bc}, \quad (2.18) \text{ gives}$$

$$[T_a, \Psi_b] = - \left(\frac{\tau_a}{2} \right)_{bc} \Psi_c \quad (2.20)$$

The pions form an isotriplet, and the 3×3 representation of T_a is

$$(t_a)_{bc} = -i \epsilon_{abc}$$

so (2.18) gives

$$[T_a, \phi_b] = i \epsilon_{abc} \phi_c. \quad (2.21)$$

If in (2.19) we let $\epsilon_b = \epsilon (\hat{\phi})_b$, where $\hat{\phi}$ is a unit vector in isospin space, (2.19), (2.20), and (2.21) give for the isotopic gauge transformations of nucleon and pion fields

$$\begin{aligned} \Psi(x) &\rightarrow \Psi(x) - i \epsilon \frac{1}{2} \vec{\tau} \cdot \hat{\phi} \Psi(x) \\ \vec{\phi}(x) &\rightarrow \vec{\phi}(x) - \epsilon (\vec{\phi}(x) \times \hat{\phi}) \end{aligned} \quad (2.22)$$

If we take for our Lagrangian density the free field Lagrangian

$$\mathcal{L}_{\text{free}} = \bar{\Psi}(i \not{\partial} - M) \Psi + \frac{1}{2} (\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \mu^2 \vec{\phi} \cdot \vec{\phi}) \quad (2.23)$$

the current generated is

$$\vec{J}_T(x) = \frac{1}{2} \bar{\Psi}(x) \gamma_4 \vec{\tau} \Psi(x) + \vec{\phi}(x) \times \partial_4 \vec{\phi}(x).$$

Since $\mathcal{L}_{\text{free}}$ is an isotopic scalar, (2.17) shows

that $\partial^\alpha \vec{g}_\alpha(x) = 0$. If we went on to evaluate $\vec{T} = \int d^3x \vec{g}_0(x)$ and used the second quantizations (2.10) and (2.13) to evaluate $[T_a, T_b]$, we would recover the Lie algebra $[T_a, T_b] = i\epsilon_{abc} T_c$ which we started with.

SECTION III

CONSTRUCTION OF CHIRAL DYNAMICS LAGRANGIANS

Since Weinberg's first paper on chiral dynamics, several different methods have been put forward for construction of chiral dynamics Lagrangians. The method we shall use, that of covariant derivatives, was developed in a somewhat later paper by Weinberg.¹⁶ The discussion here follows closely the discussion in Weinberg's paper.

We shall restrict ourselves to Lagrangians containing only pion and nucleon fields, hence the symmetry group we'll be concerned with is chiral $SU(2) \otimes SU(2)$. The program will be to construct the gauge transformation of chiral $SU(2) \otimes SU(2)$, then to construct a chiral invariant Lagrangian, and finally to add to this chiral invariant Lagrangian a symmetry breaking term which reproduces PCAC.

As discussed in Section I, the generators of vector and axial vector currents, $\vec{T}$ and $\vec{X}$ respectively, satisfy the Lie algebra

$$\begin{aligned} [T_a, T_b] &= i \epsilon_{abc} T_c \\ [T_a, X_b] &= i \epsilon_{abc} X_c \\ [X_a, X_b] &= i \epsilon_{abc} T_c. \end{aligned} \tag{3.1}$$

We have already written down the commutator of $\vec{T}$ with the nucleon field operator and the pion field operator in equations (2.20) and (2.21), and used them to derive the gauge transformation (2.22) which generates the vector current. We could immediately write down the gauge transformation generated by $\vec{X}$ in the same way as we've done for $\vec{T}$, if we knew the commutators $[\vec{X}, \vec{\phi}(\alpha)]$ and $[\vec{X}, \Psi(\alpha)]$. However, attempts to make physical fields tensors of the operator $\vec{X}$ have not been successful,* so we have to use other means of evaluating these commutators.

First, let's consider the transformation of the pion field. We wish to evaluate the commutator

$[X_a, \phi_b]$. Define a function $f_{ab}(\vec{\phi})$ by

$$-i f_{ab}(\vec{\phi}) = [X_a, \phi_b] \quad (3.2)$$

Our problem now reduces to evaluating $f_{ab}(\vec{\phi})$. We do this by deriving a pair of simultaneous differential equations for $f_{ab}(\vec{\phi})$.

We will need two identities to derive these differential equations. One is the well-known Jacobi identity

$$[A, [B, C]] = [B, [A, C]] + [[A, B], C]. \quad (3.3)$$

The other identity is

$$[Q_a, h(\varphi_n)] = \frac{\partial h}{\partial \varphi_n} [Q_a, \varphi_n] \quad (3.4)$$

where Q_a is some operator and $h(\varphi_n)$ is some

*The sigma model was one such attempt.

general function of the set of n fields $\{\varphi_n\}$.

To prove (3.4), expand $h(\varphi_n)$ in a Taylor's series around some set of constant fields ξ_n very close to φ_n . Our commutator becomes

$$\begin{aligned} [Q_a, h(\varphi_n)] \\ = [Q_a, h(\xi_n) + \frac{\partial h}{\partial \varphi_n} \Big|_{\varphi_n = \xi_n} (\varphi_n - \xi_n) + \dots] \end{aligned}$$

Because $h(\xi_n)$ and $\frac{\partial h}{\partial \varphi_n} \Big|_{\varphi_n = \xi_n}$ are constants, this becomes

$$[Q_a, h(\varphi_n)] = \frac{\partial h}{\partial \varphi_n} \Big|_{\varphi_n = \xi_n} [Q_a, \varphi_n].$$

But $\xi_n \cong \varphi_n$, so we arrive at (3.4).

Apply the Jacobi identity (3.3) in the following form:

$$[T_a, [X_b, \phi_c]] = [X_b, [T_a, \phi_c]] + [[T_a, X_b], \phi_c].$$

Using (2.21), (3.1), and (3.2), this becomes

$$[T_a, f_{bc}(\vec{\phi})] = i \epsilon_{acd} f_{bd}(\vec{\phi}) + i \epsilon_{abc} f_{dc}(\vec{\phi}). \quad (3.5)$$

Using (2.21) and (3.4), (3.5) reduces to

$$\frac{\partial f_{bc}(\vec{\phi})}{\partial \phi_d} \epsilon_{ade} \phi_e = f_{bd}(\vec{\phi}) \epsilon_{acd} + f_{dc}(\vec{\phi}) \epsilon_{abd}. \quad (3.6)$$

This is our first differential equation for $f_{bc}(\vec{\phi})$.

Now apply the Jacobi identity in the form

$$[X_a, [X_b, \phi_c]] - [X_b, [X_a, \phi_c]] = [[X_a, X_b], \phi_c].$$

Using (3.1) and (3.2) with this identity, we obtain

$$[X_a, f_{bc}(\vec{\theta})] - [X_b, f_{ac}(\vec{\theta})] = -i \epsilon_{abcd} \epsilon_{dce} \theta_e. \quad (3.7)$$

The tensor identity

$$\epsilon_{abcd} \epsilon_{dce} = \delta_{ac} \delta_{be} - \delta_{ae} \delta_{bc} \quad (3.8)$$

simplifies the right hand side of (3.7), and using (3.2)

and (3.4) to simplify the left hand side, we get

$$\frac{\partial f_{bc}(\vec{\theta})}{\partial \theta_d} f_{ad}(\vec{\theta}) - \frac{\partial f_{ac}(\vec{\theta})}{\partial \theta_d} f_{bd}(\vec{\theta}) = \delta_{ac} \theta_b - \delta_{bc} \theta_a. \quad (3.9)$$

This is our second differential equation for $f_{bc}(\vec{\theta})$.

We must now solve equation (3.6) and (3.9) for $f_{bc}(\vec{\theta})$.

We note that (3.5) indicates $f_{bc}(\vec{\theta})$ is an isotropic tensor. Since both $\vec{x}$ and $\vec{\theta}$ have odd parity,

$[X_a, \theta_b]$ must have even parity, and (3.2) indicates $f_{bc}(\vec{\theta})$ must also have even parity. Thus we take as a solution for f_{bc}

$$f_{bc}(\vec{\theta}) = \delta_{bc} f(\vec{\theta}^2) + \theta_b \theta_c g(\vec{\theta}^2) \quad (3.10)$$

where $f(\vec{\theta}^2)$ and $g(\vec{\theta}^2)$ are functions to be determined.

If we substitute (3.10) into (3.6) we get no restrictions on f and g . This is not the case if we substitute (3.10) into (3.9). From the chain rule, for a general function $h(\vec{\theta}^2)$, we get

$$\frac{\partial h}{\partial \theta_d} = 2 \theta_d h' \quad (3.11)$$

where a prime denotes differentiation with respect to $\vec{\theta}^2$.

Using the form (3.10) and the identity (3.11),

$$\begin{aligned} \frac{\partial f_{bc}}{\partial \bar{\phi}^2} f_{ad} &= \delta_{ab} \phi_c f_g + \delta_{ac} \phi_b f_g \\ &+ 2 \delta_{bc} \phi_a (ff' + \bar{\phi}^2 g f') \\ &+ 2 \phi_a \phi_b \phi_c (g^2 + \bar{\phi}^2 g g'). \end{aligned}$$

From this,

$$\frac{\partial f_{bc}}{\partial \bar{\phi}^2} f_{ad} - \frac{\partial f_{ac}}{\partial \bar{\phi}^2} f_{bd} = (fg - 2ff' - 2\bar{\phi}^2 g f')(\delta_{ac} \phi_b - \delta_{bc} \phi_a)$$

Substituting this in our second differential equation

(3.9) gives

$$(fg - 2ff' - 2\bar{\phi}^2 g f')(\delta_{ac} \phi_b - \delta_{bc} \phi_a) = \delta_{ac} \phi_b - \delta_{bc} \phi_a$$

which is satisfied if f and g are related by

$$g = \frac{1 + 2ff'}{f - 2\bar{\phi}^2 f'}. \quad (3.12)$$

Equation (3.12) gives the only relationship between $f(\bar{\phi}^2)$ and $g(\bar{\phi}^2)$. This means the commutator

$$[X_a, \phi_b] \quad (3.13)$$

is not unique, since $f(\bar{\phi}^2)$ may be chosen to be any isoscalar function of the pion field, and each different choice of $f(\bar{\phi}^2)$ gives a different function $f_{ab}(\bar{\phi})$ which is proportional to the commutator (3.13). What we do when we choose $f(\bar{\phi}^2)$ is to choose what isovector quantity we'll call the pion field. Changing $f(\bar{\phi}^2)$ is equivalent to redefining the pion field by $\bar{\phi} \rightarrow \bar{\phi}^* = \bar{\phi} \Phi(\bar{\phi}^2)$ where $\Phi(\bar{\phi}^2)$ is some isoscalar function determined by the new choice of $f(\bar{\phi}^2)$.

Once we have settled on a definition, we can calculate

(3.13) and determine the axial gauge transformation for the pion field as we've defined it.

We shall restrict ourselves to the form of $f(\vec{\theta}^2)$ used by Weinberg¹⁹ and Schwinger,²⁰ namely

$$f(\vec{\theta}^2) = -\frac{1}{2\lambda}(1 - \lambda^2 \vec{\theta}^2) \quad (3.14)$$

where λ is a constant which we'll determine later.

Using (3.12) we find the function $g(\vec{\theta}^2)$ corresponding to our choice (3.14) for $f(\vec{\theta}^2)$ to be simply

$$g(\vec{\theta}^2) = -\lambda. \quad (3.15)$$

Our result for the commutator (3.13) is then

$$[X_a, \theta_b] = \frac{i}{\lambda} \left\{ \frac{1}{2}(1 - \lambda^2 \vec{\theta}^2) \delta_{ab} + \lambda^2 \theta_a \theta_b \right\}. \quad (3.16)$$

We now turn our attention to the nucleon field and the commutator $[X_a, \Psi_b]$ which determines its axial gauge transformation. Define a function $U_{ab}(\vec{\theta})$ by

$$[X_a, \Psi_c] = U_{ab}(\vec{\theta}) \left(\frac{\tau_b}{2} \right)_{cd} \Psi_d. \quad (3.17)$$

Our problem is now reduced to evaluation of $U_{ab}(\vec{\theta})$ and again we solve it by employing the Jacobi identity twice in order to obtain a pair of differential equations for $U_{ab}(\vec{\theta})$.

The first form of the Jacobi identity we use is

$$[\tau_a, [X_b, \Psi_c]] = [X_b, [\tau_a, \Psi_c]] + [[\tau_a, X_b], \Psi_c]. \quad (3.18)$$

Using (2.20), (3.1), (3.4), and (3.17), equation (3.18)

yields our first differential equation

$$\frac{\partial U_{be}}{\partial \Phi_d} \epsilon_{ade} \Phi_e = U_{bd} \epsilon_{acd} + U_{dc} \epsilon_{abd}. \quad (3.19)$$

The second Jacobi identity we use is

$$[X_a, [X_b, \Psi_c]] - [X_b, [X_a, \Psi_c]] = [[X_a, X_b], \Psi_c]. \quad (3.20)$$

Using (3.4) and (3.17), the left hand side of (3.20)

becomes (suppressing matrix indices on Ψ and τ)

$$\begin{aligned} [X_a, [X_b, \Psi]] - [X_b, [X_a, \Psi]] \\ = [X_a, U_{be}] \frac{\tau_e}{2} \Psi - [X_b, U_{ae}] \frac{\tau_e}{2} \Psi \\ + \left[\frac{\tau_c}{2}, \frac{\tau_d}{2} \right] U_{bd} U_{ac} \Psi. \end{aligned}$$

From (2.20) and (3.1), the right hand side of (3.20) becomes

$$[[X_a, X_b], \Psi] = i \epsilon_{abc} [\tau_c, \Psi] = -i \epsilon_{abc} \frac{\tau_c}{2} \Psi.$$

Equating right and left hand sides, we obtain

$$\begin{aligned} [X_a, U_{be}] \frac{\tau_e}{2} - [X_b, U_{ae}] \frac{\tau_e}{2} \\ + U_{bd} U_{ac} \left[\frac{\tau_d}{2}, \frac{\tau_c}{2} \right] = -i \epsilon_{abc} \frac{\tau_e}{2}. \end{aligned} \quad (3.21)$$

Using (3.2), (3.4), and the commutation rule

$$\left[\frac{\tau_d}{2}, \frac{\tau_c}{2} \right] = i \epsilon_{dce} \frac{\tau_e}{2}$$

equation (3.21) becomes

$$\frac{\partial U_{be}}{\partial \Phi_d} f_{ad} - \frac{\partial U_{ae}}{\partial \Phi_d} f_{bd} = -U_{ac} U_{bd} \epsilon_{cde} + \epsilon_{abe}. \quad (3.22)$$

This is our second differential equation for $U_{ab}(\vec{\Phi})$.

We see that (3.19), our first differential equation for $U_{ab}(\vec{\Phi})$, has the same form as equation (3.6) for $f_{ab}(\vec{\Phi})$. Since (3.6) just arose because $f_{ab}(\vec{\Phi})$ is an isotopic tensor, $U_{ab}(\vec{\Phi})$ must also be an isotopic tensor. Since $\vec{X}$ has odd parity, (3.17) shows that U_{ab} must have odd parity, so we take as our solution the form

$$U_{ab}(\vec{\Phi}) = \epsilon_{abc} \Phi_c U(\vec{\Phi}^2) \quad (3.23)$$

where $U(\vec{\Phi}^2)$ is a scalar to be determined. Substitution of (3.23) into (3.19) gives no more information about $U(\vec{\Phi}^2)$, but substitution into (3.22) does.

Substituting (3.23) in (3.22) and simplifying the results using (3.8) and (3.11), we get

$$\begin{aligned} \Phi_d [\epsilon_{bcd} \Phi_a - \epsilon_{acd} \Phi_b] \times \{ U g + 2 U' [f + \vec{\Phi}^2 g] \} \\ + 2 \epsilon_{abc} U_f = \Phi_d \Phi_e \epsilon_{bad} U^2 + \epsilon_{abc} \end{aligned} \quad (3.24)$$

Using the identity

$$\vec{\Phi}^2 \epsilon_{abc} = \epsilon_{abc} \Phi_c \Phi_e + \epsilon_{bec} \Phi_a \Phi_c + \epsilon_{eac} \Phi_c \Phi_d \quad (3.25)$$

to eliminate the term $\Phi_d \Phi_e \epsilon_{bad} U^2$ from (3.24), we thereby obtain

$$\begin{aligned} \Phi_d [\epsilon_{bcd} \Phi_a - \epsilon_{acd} \Phi_b] \times \{ U g + 2 U' [f + \vec{\Phi}^2 g] - U^2 \} \\ + \epsilon_{abc} [2 U_f + \vec{\Phi}^2 U^2 - 1] = 0. \end{aligned}$$

(3.26)

To satisfy (3.26) we must have

$$u g + 2 u' (f + \vec{\phi}^2 g) - u^2 = 0 \quad (3.27)$$

$$2 u f + \vec{\phi}^2 u^2 - 1 = 0. \quad (3.28)$$

The latter equation has the solutions

$$u = \frac{-f \pm \sqrt{f^2 + \vec{\phi}^2}}{\vec{\phi}^2} = (f \pm \sqrt{f^2 + \vec{\phi}^2})^{-1}. \quad (3.29)$$

A long but straightforward calculation shows that

$$u = (f \pm \sqrt{f^2 + \vec{\phi}^2})^{-1} \text{ is also a solution of (3.27).}$$

Substituting (3.29) back into (3.23), we have

$$u_{ab}(\vec{\phi}) = \epsilon_{abc} \phi_c [f(\vec{\phi}^2) \pm \sqrt{f^2(\vec{\phi}^2) + \vec{\phi}^2}]^{-1} \quad (3.30)$$

From this and (3.17) we see that once the pion field is defined, i.e. $f(\vec{\phi}^2)$ is chosen, the nucleon transformation is also determined.*

In particular, let's go back to the form of $f(\vec{\phi}^2)$ we've chosen in (3.14). This gives for u the two solutions

$$u(\vec{\phi}^2) = \frac{1}{\lambda \vec{\phi}^2}, \quad -\lambda$$

The first of these yields a non-linear gauge transformation with unpleasant properties near where $\vec{\phi} = 0$, so we take the second solution. With this solution our commutator $[X_a, \Psi_G]$ becomes

$$[\vec{X}, \Psi] = -\frac{\lambda}{2} (\vec{\tau} \times \vec{\phi}) \Psi. \quad (3.31)$$

*It is interesting to note that the above discussion may be generalized from the nucleon field Ψ to a general field Φ by simply replacing $\tau_{a/2}$ by the matrix representation of $\vec{T}$ appropriate to Φ .

With the commutators (3.16) and (3.31), it is easy to construct the axial gauge transformation using (2.19). Taking $\epsilon_G = \epsilon(\hat{n})_G$ where $\hat{n}$ is a unit vector in isospin space, the axial gauge transformation is

$$\begin{aligned}\Psi(x) &\rightarrow \Psi(x) - i\epsilon\lambda \frac{\vec{\tau}}{2} [\vec{\phi}(x) \times \hat{n}] \Psi(x) \\ \vec{\phi}(x) &\rightarrow \vec{\phi}(x) - \frac{\epsilon\hat{n}}{2\lambda} [1 - \lambda^2 \vec{\phi}(x) \cdot \vec{\phi}(x)] \\ &\quad - \frac{\epsilon}{\lambda} [\hat{n} \cdot \vec{\phi}(x)] \vec{\phi}(x). \quad (3.32)\end{aligned}$$

Now that we have the gauge transformations generating vector and axial vector currents, we wish to construct a Lagrangian invariant under these transformations, to which we'll later add a symmetry breaking term. Because the vector current is conserved, we know our Lagrangian must be an isoscalar. But we must further limit our Lagrangian to isoscalars which are invariant under the axial gauge transformation. From (3.17) and (2.20) we can show that any isoscalar function of the nucleon field Ψ must also be invariant under the axial gauge transformation. To show this, consider some isoscalar function of the nucleon field $h(\Psi)$. Because it is an isoscalar

$$[\vec{T}, h(\Psi)] = 0.$$

But from (2.20) and (3.4)

$$[T_a, h(\Psi)] = -(\frac{\tau_a}{2})_{cd} \Psi_d \frac{\partial h}{\partial \Psi_c}$$

and so

$$(\tau_a)_{cd} \Psi_d \frac{\partial h}{\partial \Psi_c} = 0.$$

(3.33)

The commutator of $h(\Psi)$ and the generator $\vec{X}$ is, using (3.4) and (3.17),

$$[X_b, h(\Psi)] = \frac{v_{ba}(\tau_a)}{2} \Psi_d \frac{\partial h}{\partial \Psi_c}.$$

But using (3.33), this just becomes

$$[\vec{X}, h(\Psi)] = 0,$$

so $h(\Psi)$ is invariant under axial gauge transformations as asserted.

The form of the commutator (3.2) makes it impossible to go through a parallel argument for isoscalar functions of the pion field. However, it is possible to construct a function proportional to the derivative of the pion field, for which isoscalars are also chiral invariant, just as was the case for Ψ . Define this function to be

$$D_\alpha \phi_a = d_{ab}(\vec{\phi}) \partial_\alpha \phi_b. \quad (3.34)$$

The quantity $D_\alpha \vec{\phi}$ is called a covariant derivative. We assure that isoscalar functions of the covariant derivative are chiral invariants by choosing d_{ab} such that $D_\alpha \vec{\phi}$ transforms like Ψ , i.e.

$$[X_a, D_\alpha \phi_c] = -i v_{ab}(\vec{\phi}) \epsilon_{acd} D_\alpha \phi_d \quad (3.35)$$

$$[T_a, D_\alpha \phi_c] = i \epsilon_{acd} D_\alpha \phi_d \quad (3.36)$$

where we've let $\frac{(\tau_b)_c}{2} \rightarrow -i \epsilon_{bca}$.

The first step in obtaining $d_{ab}(\vec{\phi})$ is to substitute (3.34) into (3.36). We shall need the identity

$$[Q_a, \partial_r \varphi_n(x)] = \partial_r [Q_a, \varphi_n(x)]. \quad (3.37)$$

To derive this, use the usual laws governing commutators to obtain

$$[Q_a, \partial_r \varphi_n(x)] = \partial_r [Q_a, \varphi_n(x)] + [Q_a, \partial_r] \varphi_n(x)$$

and note that $[Q_a, \partial_r] = 0$ since Q_a and ∂_r operate in different spaces. Using (3.34), (3.36), and (3.37), we obtain

$$\begin{aligned} [T_a, D_r \phi_c] &= [T_a, d_{cb} \partial_r \phi_b] \\ &= d_{cb} i \in abd (\partial_r \phi_d) + [T_a, d_{cb}] (\partial_r \phi_b) \\ &= i \in acd d_{de} (\partial_r \phi_e). \end{aligned} \quad (3.38)$$

A little rearrangement of (3.38) gives

$$[T_a, d_{cb}] = i \in abd d_{cd} + i \in acd d_{db}. \quad (3.39)$$

By this time we recognize that (3.39) indicates d_{cb} is an isotopic tensor. This suggests that it may be expanded like f_{ab} , i.e.

$$d_{cb}(\vec{\theta}) = d_{cb} d(\vec{\theta}^2) + \theta_c \theta_b w(\vec{\theta}^2), \quad (3.40)$$

where $d(\vec{\theta}^2)$ and $w(\vec{\theta}^2)$ are scalar functions.

We determine d and w much as we did f and g , that is we form differential equations from (3.35) and (3.36) and use them to deduce $d(\vec{\theta}^2)$ and $w(\vec{\theta}^2)$.

If we do this we find, as we've come to expect, that (3.36) gives no further information about $d(\vec{\theta}^2)$ and $w(\vec{\theta}^2)$.

All restrictions on these functions are obtained by substituting (3.34) into (3.35).

Using (3.2), (3.34), (3.35), and (3.37), we get

$$[X_a, d_{ce}] \partial_r \Phi_e - i d_{ce} \frac{\partial f_{ad}}{\partial \Phi_a} \partial_r \Phi_e = -i U_{ab} \epsilon_{bcd} d_{de} \partial_r \Phi_e.$$

For this to be satisfied,

$$[X_a, d_{cb}] = i d_{cd} \frac{\partial f_{ad}}{\partial \Phi_b} - i U_{ae} \epsilon_{ecd} d_{de} \quad (3.41)$$

Straightforward, but tedious, manipulations using (3.2), (3.4), (3.11), and (3.40) give for the left hand side of (3.41)

$$[X_a, d_{cb}] = -i \{ \delta_{bc} \Phi_a (2 f_{cd}' + 2 \vec{\Phi}^2 g_{cd}') + \delta_{ac} \Phi_b f_{cw} + \delta_{ab} \Phi_c f_{cw} + \Phi_a \Phi_b \Phi_c (2 f_{cw}' + 2 \vec{\Phi}^2 g_{cw}' + 2 g_{cw}) \}. \quad (3.42)$$

Equally straightforward, but even more tedious manipulations using (3.10), (3.11), and (3.40) give for the first term on the right of (3.41)

$$i d_{cd} \frac{\partial f_{ad}}{\partial \Phi_b} = i \{ \delta_{ac} \Phi_b 2 f_{cd}' + \delta_{ab} \Phi_c (g_{cd} + \vec{\Phi}^2 g_{cw}) + \delta_{bc} \Phi_a g_{cd} + \Phi_a \Phi_b \Phi_c (2 d g_{cw}' + 2 w f_{cw}' + g_{cw} + 2 \vec{\Phi}^2 g_{cw}) \}. \quad (3.43)$$

Finally, use (3.8), (3.23), and (3.40) to obtain for the final term on the right hand side of (3.41)

$$-i U_{ae} \epsilon_{ecd} d_{de} = -i \{ \delta_{ab} \Phi_d d_{de} - \delta_{ac} \Phi_b (d_{de} + \vec{\Phi}^2 v_{de}) + \Phi_a \Phi_b \Phi_c v_{de} \}. \quad (3.44)$$

Substituting (3.42), (3.43), and (3.44) back into (3.41) and doing some rearranging we finally obtain

$$\begin{aligned}
& \delta_{ac} \phi_a (2 f d' + 2 \bar{\phi}^2 q d' + q d) \\
& + \delta_{ac} \phi_b (f w + 2 d f' + h w + \bar{\phi}^2 v w) \\
& + \delta_{ab} \phi_c (f w + q d + \bar{\phi}^2 q w - d v) \\
& + \phi_a \phi_b \phi_c (2 f w' + 2 \bar{\phi}^2 q w' + 3 q w + 2 d q' \\
& \quad + 2 w f' + 2 \bar{\phi}^2 w q' - v w) = 0.
\end{aligned} \tag{3.45}$$

For (3.45) to hold, $\mathcal{Q}$ and w must satisfy the four coupled differential equations

$$2 f d' + 2 \bar{\phi}^2 q d' + q d = 0 \tag{3.46}$$

$$f w + 2 d f' + d v + \bar{\phi}^2 v w = 0 \tag{3.47}$$

$$f w + q d + \bar{\phi}^2 q w - d v = 0 \tag{3.48}$$

$$\begin{cases} 2 f w' + 2 \bar{\phi}^2 q w' + 3 q w + d q' \\ + 2 w f' + 2 \bar{\phi}^2 w q' - v w = 0. \end{cases} \tag{3.49}$$

Incredibly enough, a closed solution exists to this

intractable looking set of equations. But since we're

only going to be interested in the form of $f(\bar{\phi}^2)$

chosen in (3.14), we will only obtain the solution for

$\mathcal{Q}(\bar{\phi}^2)$ and $w(\bar{\phi}^2)$ where

$$f = -\frac{(1 - \lambda^2 \bar{\phi}^2)}{2\lambda}, \quad q = v = -\lambda \tag{3.50}$$

Direct substitution of these forms in (3.48) shows that

$$w(\bar{\phi}^2) = 0. \tag{3.51}$$

Substitution of (3.50) in (3.46) yields the differential equation

$$(1 + \lambda^2 \bar{\phi}^2) \mathcal{Q}' + \lambda^2 \mathcal{Q} = 0$$

which has as its solution

$$\mathcal{Q}(\vec{\phi}^2) \propto (1 + \lambda^2 \vec{\phi}^2)^{-1} \quad (3.52)$$

Solutions (3.51) along with (3.50) satisfy our remaining two equations, (3.47) and (3.49). For our covariant derivative we have, using (3.34), (3.40), (3.51), and (3.52),

$$D_\tau \vec{\phi} \propto (1 + \lambda^2 \vec{\phi}^2)^{-1} \partial_\tau \vec{\phi}.$$

For convenience we choose the proportionality constant such that

$$D_\tau \vec{\phi} = \partial_\tau \vec{\phi} + \dots$$

so

$$D_\tau \vec{\phi} = (1 + \lambda^2 \vec{\phi}^2)^{-1} \partial_\tau \vec{\phi}. \quad (3.53)$$

Much the same procedure is necessary to construct the covariant derivative of the nucleon field, $D_\tau \Psi$.

It must satisfy the equations

$$[T_a, D_\tau \Psi] = -\frac{\tau_a}{2} D_\tau \Psi \quad (3.54)$$

$$[X_a, D_\tau \Psi] = \omega_{ab} \frac{\tau_b}{2} D_\tau \Psi. \quad (3.55)$$

For the form of $D_\tau \Psi$ we take

$$D_\tau \Psi = \partial_\tau \Psi + i M_c(\vec{\phi}) (\partial_\tau \vec{\phi}_c) \Psi \quad (3.56)$$

where $M_c(\vec{\phi})$ is a 2×2 matrix function to be determined.

Substituting (3.56) in (3.55) and simplifying the result using (3.2), (3.4), (3.17), and (3.37),

$$\begin{aligned}
[X_a, D_\alpha \Psi] &= \frac{\partial U_{ab}}{\partial \theta_c} \frac{\tau_b}{2} (\partial_\alpha \theta_c) \Psi + U_{ab} \frac{\tau_b}{2} (\partial_\alpha \Psi) \\
&\quad + i M_c (\partial_\alpha \theta_c) U_{ab} \frac{\tau_b}{2} \Psi + M_c \frac{\partial f_{ad}}{\partial \theta_c} (\partial_\alpha \theta_d) \Psi \\
&\quad + \frac{\partial M_c}{\partial \theta_d} f_{ad} (\partial_\alpha \theta_c) \Psi = U_{ab} \frac{\tau_b}{2} D_\alpha \Psi \\
&= U_{ab} \frac{\tau_b}{2} \partial_\alpha \Psi + i U_{ab} \frac{\tau_b}{2} M_c \partial_\alpha \theta_c \Psi.
\end{aligned}$$

After some rearrangement this becomes

$$i U_{ab} \left[\frac{\tau_b}{2}, M_c \right] = \frac{\partial U_{ab}}{\partial \theta_c} \frac{\tau_b}{2} + M_d \frac{\partial f_{ad}}{\partial \theta_c} + \frac{\partial M_c}{\partial \theta_d} f_{ad}.$$

Let us take as a form of the solution to (3.57)

$$M_c(\vec{\theta}) = \epsilon_{abc} \frac{\tau_d}{2} \theta_d \gamma(\vec{\theta}^2) \quad (3.58)$$

where we must determine $\gamma(\vec{\theta}^2)$. Using the forms

(3.8) and (3.58) along with the identities

$$\begin{aligned}
\epsilon_{abc} \theta_b \theta_c &= (\vec{\theta} \times \vec{\theta})_a = 0, \\
\left[\frac{\tau_a}{2}, \frac{\tau_b}{2} \right] &= i \epsilon_{abc} \frac{\tau_c}{2}
\end{aligned}$$

the left hand side of (3.57) becomes

$$i U_{ab} \left[\frac{\tau_b}{2}, M_c \right] = \epsilon_{abc} \theta_b \theta_d \frac{\tau_d}{2} \gamma'.$$

Use of straightforward differentiation and the chain rule

gives for the terms on the right hand side of (3.57)

$$\begin{aligned}
\frac{\partial U_{ab}}{\partial \theta_c} \frac{\tau_b}{2} &= -\epsilon_{acd} \frac{\tau_d}{2} U - 2\epsilon_{abd} \theta_b \theta_c \frac{\tau_d}{2} U' \\
\frac{\partial M_c}{\partial \theta_d} f_{ad} &= \epsilon_{acd} \frac{\tau_d}{2} f_\gamma + \epsilon_{cdb} \theta_a \theta_b \frac{\tau_d}{2} \gamma\gamma \\
&\quad + 2\epsilon_{cdb} \theta_a \theta_b \frac{\tau_d}{2} (f_\gamma' + \vec{\theta}^2 \gamma\gamma') \\
M_d \frac{\partial f_{ad}}{\partial \theta_c} &= -2\epsilon_{abd} \theta_b \theta_c \frac{\tau_d}{2} f_\gamma' \\
&\quad + \epsilon_{bcd} \theta_a \theta_b \frac{\tau_d}{2} \gamma\gamma.
\end{aligned}$$

Using these last four equations in (3.57), after some simplification we arrive at

$$\begin{aligned}
 & -\epsilon_{abc} \partial_b \partial_d u_z + \epsilon_{acd} (\frac{1}{2} z - u) \\
 & + \epsilon_{bcd} \partial_a \partial_b (2 q_z + 2 z' f + 2 \vec{\partial}^2 z' z) \\
 & - 2 \epsilon_{abd} \partial_b \partial_c f' z = 0.
 \end{aligned}
 \tag{3.59}$$

Comparison with the identity (3.25) shows (3.59) is satisfied only if

$$\frac{1}{2} z - u = -u_z = -2 q_z - 2 z' f - 2 \vec{\partial}^2 z' z = 2 f' z.
 \tag{3.60}$$

From the first part of (3.60),

$$z = u (\frac{1}{2} + u \vec{\partial}^2)^{-1}.
 \tag{3.61}$$

Using $f = - (1 - \lambda^2 \vec{\partial}^2) / 2\lambda$ and $u = -\lambda$, (3.61)

gives

$$z = 2\lambda^2 (1 + \lambda^2 \vec{\partial}^2)^{-1}$$

and the nucleon covariant derivative becomes, using (3.56) and (3.58),

$$D_\tau \Psi = \partial_\tau \Psi + i \lambda^2 (1 + \lambda^2 \vec{\partial}^2)^{-1} \vec{\tau} \cdot \vec{\partial} \times \partial_\tau \vec{\partial} \Psi.$$

We now have three covariant quantities from which we may construct our chiral symmetric Lagrangian, namely

$$\begin{aligned}
 D_\tau \Psi &= \partial_\tau \Psi + i \lambda^2 (1 + \lambda^2 \vec{\partial}^2)^{-1} \vec{\tau} \cdot \vec{\partial} \times \partial_\tau \vec{\partial} \Psi \\
 D_\tau \vec{\partial} &= (1 + \lambda^2 \vec{\partial}^2)^{-1} \partial_\tau \vec{\partial}.
 \end{aligned}$$

(3.62)

Any isotopic scalar function of these quantities will be a chiral invariant. But we are not free to choose just

any isoscalar, because we want the Lagrangian to contain the usual free field Lagrangian

$$\mathcal{L}_{\text{free}} = \bar{\Psi} (i \not{\partial} - M) \Psi + \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{\mu^2}{2} \vec{\phi} \cdot \vec{\phi} \quad (3.63)$$

so our Lagrangian will take the form

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_I. \quad (3.64)$$

In addition, $\mathcal{L}_I$ should contain a term corresponding to the usual $\text{ps}(\text{ps})$ or $\text{ps}(\text{pv})$ coupling.

We can arrive at a Lagrangian of form (3.64) by writing (3.63) with the following substitutions to make it chiral invariant:

$$\begin{aligned} \Psi &\rightarrow \Psi \\ \partial_\mu \Psi &\rightarrow D_\mu \Psi \\ \partial_\mu \vec{\phi} &\rightarrow D_\mu \vec{\phi}. \end{aligned}$$

But we immediately encounter difficulty with the pion mass term since there is no covariant analogue of $\vec{\phi} \cdot \vec{\phi}$. This is not too surprising since we showed in Section I that chiral symmetry breaking was intimately connected with the non-zero mass of the pion. Thus, no chiral symmetric Lagrangian may be constructed for pions with finite mass and we shall have to pretend, for the time being, that $\mu = 0$. Thus

$$\mathcal{L}_{\text{free}}^{\mu=0} = \bar{\Psi} (i \not{\partial} - M) \Psi + \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi}$$

is the free field Lagrangian which belongs in (3.64).

Now we may proceed with the recipe outlined above and write

$$\mathcal{L}_{\text{chiral symmetric}} = \bar{\Psi}(i\not{\partial} - M)\Psi - \lambda^2 \bar{\Psi}(1 + \lambda^2 \vec{\phi}^2)^{-1} \vec{\tau} \cdot \vec{\phi} \times \not{\partial} \vec{\phi} \Psi + \frac{1}{2} (1 + \lambda^2 \vec{\phi}^2)^{-2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi}.$$

However, we still haven't fulfilled our second requirement, which was that the Lagrangian contain the usual pion-nucleon coupling terms

$$ig \bar{\Psi} \gamma_5 \vec{\tau} \cdot \vec{\phi} \Psi \quad \text{or} \quad \frac{g}{2M} \bar{\Psi} \gamma_5 \vec{\tau} \cdot \not{\partial} \vec{\phi} \Psi.$$

The pseudoscalar coupling involves the same problem with the pion field which the pion mass term had. So we're forced to use the pseudovector coupling. Adding on this gradient coupling term, our chiral symmetric Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{\text{chiral symmetric}} = & \bar{\Psi}(i\not{\partial} - M)\Psi \\ & - \lambda^2 \bar{\Psi}(1 + \lambda^2 \vec{\phi}^2)^{-1} \vec{\tau} \cdot \vec{\phi} \times \not{\partial} \vec{\phi} \Psi \\ & + \frac{1}{2} (1 + \lambda^2 \vec{\phi}^2)^{-2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} \\ & + \frac{g}{2M} \bar{\Psi} \gamma_5 (1 + \lambda^2 \vec{\phi}^2)^{-1} \vec{\tau} \cdot \not{\partial} \vec{\phi} \Psi \end{aligned} \quad (3.65)$$

To the chiral symmetric Lagrangian (3.65) we shall now add a symmetry breaking term $\mathcal{L}_{\text{s.B.}}$ so that our final Lagrangian will have the form

$$\mathcal{L} = \mathcal{L}_{\text{chiral symmetric}} + \mathcal{L}_{\text{s.B.}}$$

We shall see that we recover our pion mass terms and fix the heretofore undetermined constant λ by constructing

this symmetry breaking term in such a way that PCAC is satisfied.

From (1.14) we have for the PCAC condition

$$\partial_\tau \vec{g}^{\vec{s}}(\pi) = \frac{M}{g} \frac{q_A}{q_V} \mu^2 \vec{\phi}(\pi). \quad (1.14)$$

In appendix B it is shown that this form is not unique

and in fact the general form of PCAC is

$$\partial_\tau \vec{g}^{\vec{s}} = \frac{M}{g} \frac{q_A}{q_V} \mu^2 \vec{\phi} \left[1 + \sum_{n=1}^{\infty} a_n (\vec{\phi}^2)^n \right] \quad (3.66)$$

where $\{a_n\}$ is a set of arbitrary constants. From

(2.17) we see that the divergence of the axial current

is given by

$$i [\vec{X}, \mathcal{L}_{s.B.}] = \partial_\tau \vec{g}^{\vec{s}}. \quad (3.67)$$

Combining (3.66) and (3.67), we obtain

$$i [\vec{X}, \mathcal{L}_{s.B.}] = \frac{M}{g} \frac{q_A}{q_V} \mu^2 \vec{\phi} \left[1 + \sum_{n=1}^{\infty} a_n (\vec{\phi}^2)^n \right]. \quad (3.68)$$

Equation (3.68) is the equation $\mathcal{L}_{s.B.}$ must satisfy.

Because the set $\{a_n\}$ is arbitrary, the solution of (3.68) is not unique and many different models may be used as solutions.

The simplest solution is just to restore the pion mass term, that is, take

$$\mathcal{L}_{s.B.} = -\frac{\mu^2}{2} \vec{\phi}^2. \quad (3.69)$$

From (3.16),

$$[\vec{X}, \vec{\phi}^2] = i \lambda^{-1} (1 + \lambda^2 \vec{\phi}^2) \vec{\phi} \quad (3.70)$$

so

$$i \left[\vec{X}, -\frac{\mu^2}{2} \vec{\phi}^2 \right] = \frac{\mu^2}{2} \lambda^{-1} (1 + \lambda^2 \vec{\phi}^2) \vec{\phi}. \quad (3.71)$$

This satisfies (3.68) if we take

$$\lambda = \frac{g}{2M} \frac{q_V}{q_A}. \quad (3.72)$$

We shall call the model defined by (3.69) the minimal coupling model, for want of a better name.

We can see from (3.70) and (3.71) that any model we wish to use must be of the form

$$\mathcal{L}_{s.B.} = -\frac{\mu^2}{2} \Phi^2 + \sum_{n=2}^{\infty} b_n (\Phi^2)^n,$$

where $\{b_n\}$ may be any set of numbers. In the minimal coupling model we've taken $b_n = 0$ for all n . But another alternative is to take $a_n = 0$ for all n . This puts PCAC in the simplest form, that of equation (1.14). To obtain $\mathcal{L}_{s.B.}$ in this model we must solve

$$i [\vec{X}, \mathcal{L}_{s.B.}] = \frac{M}{g} \frac{q_A}{q_V} \mu^2 \Phi. \quad (3.73)$$

Using (3.4), (3.11), and (3.16), equation (3.73) becomes the differential equation

$$(1 + \lambda^2 \Phi^2) \mathcal{L}'_{s.B.} = -\frac{\lambda^2 \mu^2}{2}$$

which has the solution

$$\mathcal{L}_{s.B.} = -\frac{\mu^2}{2} \lambda^{-2} \ln(1 + \lambda^2 \Phi^2).$$

But $\lambda = \frac{g}{2M} \frac{q_V}{q_A}$, so the final form of the solution is

$$\mathcal{L}_{s.B.} = -\frac{\mu^2}{2} \left(\frac{g}{2M} \frac{q_V}{q_A} \right)^{-2} \ln \left[1 + \left(\frac{g}{2M} \frac{q_V}{q_A} \right)^2 \Phi^2 \right]. \quad (3.74)$$

This is the model proposed by Schwinger.²¹

The final solution of (3.68) which we shall consider will be Weinberg's which consists of making $\mathcal{L}_{s.B.}$ an

$SU(2) \otimes SU(2)$ tensor. If $t_{\alpha\beta}^{(N)} \dots \gamma$ designates a traceless symmetric chiral $SU(2) \otimes SU(2)$ tensor of rank N , we let

$$\mathcal{L}_{S.B.} = \mathcal{L}_N = t_{00}^{(N)} \dots 0.$$

In order to derive a form for $\mathcal{L}_N$, we must first make a short digression into the properties of $SU(2) \otimes SU(2)$ tensors. Let Π_α be such a tensor of the fundamental four dimensional representation where α runs over the indices 1, 2, 3, 0. Equation (2.18) tells us that

$$\begin{aligned} [X_a, \Pi_\alpha] &= -(\kappa_a)_{\alpha\beta} \Pi_\beta \\ [T_a, \Pi_\alpha] &= -(t_a)_{\alpha\beta} \Pi_\beta \end{aligned} \quad (3.75)$$

where κ_a and t_a are the 4 x 4 representations of X_a and T_a and therefore satisfy

$$\begin{aligned} [t_a, t_b] &= i\epsilon_{abc} t_c \\ [t_a, \kappa_b] &= i\epsilon_{abc} \kappa_c \\ [\kappa_a, \kappa_b] &= i\epsilon_{abc} t_c. \end{aligned}$$

A specific form of the 4 x 4 representation is

$$\begin{aligned} (t_a)_{bc} &= -i\epsilon_{abc} \\ (t_a)_{b0} &= (t_a)_{0b} = (t_a)_{00} = 0 \\ (\kappa_a)_{b0} &= -(\kappa_a)_{0b} = i\delta_{ab} \\ (\kappa_a)_{bc} &= (\kappa_a)_{00} = 0. \end{aligned} \quad (3.76)$$

From (3.76) we see that the first three components of Π_α form an isotopic triplet, and Π_0 is an isotopic singlet. In the sigma model the triplet was identified with the

pion field $\vec{\Phi}(x)$ and the zeroth component with the sigma field.

Let us temporarily restrict ourselves to

$\mathcal{L}_{s.b.} = \mathcal{L}_1 = t_0$. From (3.75) and (3.76)

$$\begin{aligned} [X_a, t_0] &= i t_a \\ [X_a, t_b] &= -i \delta_{ab} t_0. \end{aligned} \quad (3.77)$$

From (3.77),

$$[X_a, [X_a, \mathcal{L}_1]] = \delta_{aa} t_0 = 3 t_0 = 3 \mathcal{L}_1, \quad (3.78)$$

To generalize (3.78), note that a rank N tensor is constructed from the product of N rank 1 tensors. Hence

$$\begin{aligned} [X_a, t_{00\dots 0}^{(N)}] &= N t_{00\dots 0}^{(N-1)} [X_a, t_0] \\ &= i N t_{00\dots 0}^{(N-1)} t_a \\ &= i N t_{a0\dots 0}^{(N)} \end{aligned} \quad (3.79)$$

where the last step uses the symmetric property of $t_{\alpha\beta\dots\gamma}$.

For the generalization of (3.78), using (3.77) and (3.79) gives

$$\begin{aligned} [X_b, [X_a, \mathcal{L}_N]] &= [X_b, i N t_{a0\dots 0}^{(N)}] \\ &= -N(N-1) t_{ab0\dots 0}^{(N)} + N \delta_{ab} t_{00\dots 0}^{(N)}. \end{aligned} \quad (3.80)$$

Since $t_{\alpha\beta\dots\gamma}^{(N)}$ is traceless

$$t_{\alpha\alpha 0\dots 0}^{(N)} = t_{aa0\dots 0}^{(N)} + t_{000\dots 0}^{(N)} = 0,$$

so setting $a = b$ in (3.80),

$$\begin{aligned}
[X_a, [X_a, \mathcal{L}_N]] &= -N(N-1)t_{\infty 0 \dots 0}^{(N)} + 3N t_{0 \dots 0}^{(N)} \\
&= \{N(N-1) + 3N\} t_{0 \dots 0}^{(N)} = N(N+2) \mathcal{L}_N
\end{aligned}$$

We derive from the double commutator

$$[X_a, [X_a, \mathcal{L}_N]] = N(N+2) \mathcal{L}_N \quad (3.81)$$

a differential equation for $\mathcal{L}_N$ using (3.4), (3.11), (3.14), and (3.15). This equation is

$$\begin{aligned}
(1 + \lambda^2 \vec{\phi}^2)^2 \vec{\phi}^2 \mathcal{L}_N'' + \frac{1}{2}(1 + \lambda^2 \vec{\phi}^2)(3 + \lambda^2 \vec{\phi}^2) \mathcal{L}_N' \\
+ N(N+2) \lambda^2 \mathcal{L}_N = 0.
\end{aligned} \quad (3.82)$$

William Sollfrey²² has shown that, in accord with the condition that $\mathcal{L}_N$ contain the usual pion mass term $-\frac{\mu^2}{2} \vec{\phi}^2$, (3.82) has the general solution

$$\mathcal{L}_N = \frac{3\mu^2}{4N(N+2)\lambda^2} \times \frac{(1 + \lambda^2 \vec{\phi}^2) \sin[2(N+1) \tan^{-1} \lambda \vec{\phi}]}{2(N+1) \lambda \vec{\phi}}.$$

For rank one and rank two tensors this reduces to

$$\begin{aligned}
\mathcal{L}_1 &= -\frac{\mu^2}{2} (1 + \lambda^2 \vec{\phi}^2)^{-1} \vec{\phi}^2 \\
\mathcal{L}_2 &= -\frac{\mu^2}{2} (1 + \lambda^2 \vec{\phi}^2)^{-2} \vec{\phi}^2
\end{aligned}$$

where constant terms have been omitted since they have no physical effect. Using Sollfrey's solution or solving (3.82) by series as Weinberg actually did, we obtain the useful solution

$$\begin{aligned}
\mathcal{L}_N &= -\frac{\mu^2}{2} \vec{\phi}^2 + \frac{\mu^2}{10} [N(N+2) + 2] \lambda^2 (\vec{\phi}^2)^2 \\
&+ \dots
\end{aligned} \quad (3.83)$$

Weinberg chose as his hypothesis that the symmetry breaking term transforms as a rank one $SU(2) \otimes SU(2)$ tensor, that is

$$\mathcal{L}_{s.B.} = -\frac{\mu^2}{2} \left[1 + \left(\frac{g}{2M} \frac{\partial_\mu \chi}{\partial_\mu \chi} \vec{\phi}^2 \right)^{-1} \vec{\phi}^2 \right] \quad (3.84)$$

where, as usual, $\lambda = \frac{g}{2M} \frac{\partial_\mu \chi}{\partial_\mu \chi}$. This will be referred to as Weinberg's²³ model.

We see that the minimal coupling hypothesis that

$$\mathcal{L}_{s.B.} = -\frac{\mu^2}{2} \vec{\phi}^2$$

and Schwinger's hypothesis

$$\partial^\mu \vec{\phi} \partial_\mu \vec{\phi} = \frac{M}{g} \frac{\partial_\mu \chi}{\partial_\mu \chi} \mu^2 \vec{\phi}^2$$

are dependent on the definition of $\vec{\phi}$ implicit in our choice of $f(\vec{\phi}^2)$. Weinberg justifiably objects to these hypotheses for that reason. His own hypothesis as incorporated in (3.81) has no reference to the form of $f(\vec{\phi}^2)$. However, in order to derive the differential equation form of (3.81) from which $\mathcal{L}_N$ is actually obtained, we do have to employ a specific form of $f(\vec{\phi}^2)$, so it appears that the dependence of Weinberg's model on the definition of the pion field is merely camouflaged. It would seem that the only way to remedy the ambiguity in the model is to compare the model's predictions with experiments.

To sum up the results of this section, our chiral dynamics Lagrangian takes the form

$$\begin{aligned}
\mathcal{L} = & \bar{\Psi} \left[i \not{\partial} - M + \left\{ 1 + \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2 \vec{\phi}^2 \right\}^{-1} \right. \\
& \times \left\{ \left(\frac{g}{2M} \right) \gamma_5 \vec{\tau} \cdot \vec{\partial} \vec{\phi} - \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2 \vec{\tau} \cdot \vec{\phi} \times \vec{\partial} \vec{\phi} \right\} \left. \right] \Psi \\
& + \frac{1}{2} \left\{ 1 + \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2 \vec{\phi}^2 \right\}^{-2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} + \mathcal{L}_{\text{s.B.}}
\end{aligned}
\tag{3.85}$$

The term $\mathcal{L}_{\text{s.B.}}$ breaks the chiral symmetry of the rest of the Lagrangian, and the exact form of $\mathcal{L}_{\text{s.B.}}$ depends on specific assumptions about the model. Specific models we'll consider are these:

1. Minimal Coupling Model

$$\mathcal{L}_{\text{s.B.}} = -\frac{\mu^2}{2} \vec{\phi}^2$$

2. Schwinger's Model

$$\begin{aligned}
\mathcal{L}_{\text{s.B.}} &= -\frac{\mu^2}{2} \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2 \ln \left[1 + \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2 \vec{\phi}^2 \right] \\
&= -\frac{\mu^2}{2} \vec{\phi}^2 + \frac{1}{4} \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2 (\vec{\phi}^2)^2 + \dots
\end{aligned}$$

3. Weinberg's Model

$$\begin{aligned}
\mathcal{L}_{\text{s.B.}} &= \mathcal{L}_1 = -\frac{\mu^2}{2} \left[1 + \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2 \vec{\phi}^2 \right]^{-1} \vec{\phi}^2 \\
&= -\frac{\mu^2}{2} \vec{\phi}^2 + \frac{1}{2} \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2 (\vec{\phi}^2)^2 + \dots
\end{aligned}$$

4. Tensor Rank Two Model

$$\begin{aligned}
\mathcal{L}_{\text{s.B.}} &= \mathcal{L}_2 = -\frac{\mu^2}{2} \left[1 + \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2 \vec{\phi}^2 \right]^{-2} \vec{\phi}^2 \\
&= -\frac{\mu^2}{2} \vec{\phi}^2 + \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2 (\vec{\phi}^2)^2 + \dots
\end{aligned}$$

5. Tensor Rank Three Model

$$\begin{aligned}
\mathcal{L}_{\text{s.B.}} &= \mathcal{L}_3 \\
&= -\frac{\mu^2}{2} \vec{\phi}^2 + \frac{17}{10} \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2 (\vec{\phi}^2)^2 + \dots
\end{aligned}$$

SECTION IV

PION PRODUCTION PROCESS

The process

$$\pi + N \longrightarrow \pi + \pi + N \quad (4.1)$$

has long been of interest to theorists. The first papers which attempted to calculate theoretical cross sections for it appeared in the mid-fifties. These first calculations were based on the static model with pseudovector coupling and yielded total cross sections an order of magnitude too small.²⁴ This model was later modified by consideration of final state interactions²⁵ and pi-pi scattering effects.²⁶ The model in this form was fairly successful, but was still unable to explain final state mass distributions. Another approach was the ρ exchange model.²⁷ This model proved inadequate below 1 BeV because it favored the wrong isospin channel and forbade N^* production. More recently, Olsson and Yodh obtained a very good fit to total and differential cross sections employing a phenomenological model with seven free parameters.²⁸

The process (4.1) is of interest in the study of chiral dynamics because it may be used to discriminate among the various Lagrangians discussed in the previous section. To see why this is true, consider the general

form of the symmetry breaking term,

$$\mathcal{L}_{\text{s.b.}} = -\frac{\mu^2}{2} \phi^2 - \frac{1}{2} \sum_{m=1}^{\infty} (-1)^m \left(\frac{g}{2M}\right)^{2m} \left(\frac{g}{f_\pi}\right)^{2m} (\phi^2)^m n_m$$

where $\{n_m\}$ is a set of model dependent parameters.

In order to make an experimental determination of n_m , we need to look at processes involving vertices at which $2n$ pion lines intersect. In particular, if we want n_1 , we need a process with Feynman diagrams containing the vertex of Figure 4.*

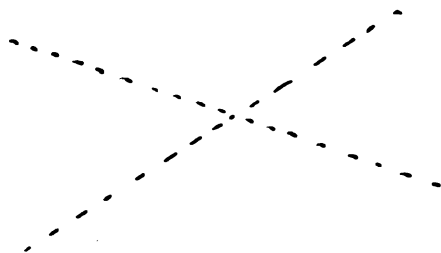
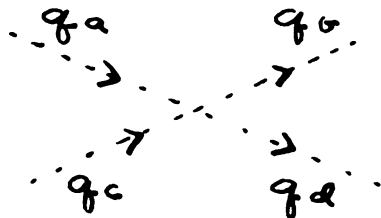


Figure 4. Vertex dependent on n_1 .

The obvious candidate for such a process would be simply pi-pi scattering which at low energies would be dominated by the diagram



It is easy to calculate low energy parameters for pi-pi

*Dashed lines are used for pions, solid lines for nucleons.

scattering using the chiral dynamics,^{*} but unfortunately it is difficult to perform the experiments necessary for comparison.

Another diagram containing the four pion vertex in question may be formed by simply attaching a nucleon line to Figure 4. The resultant diagram is

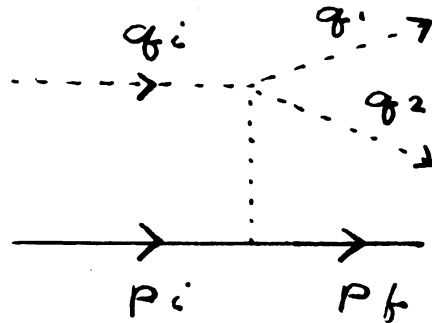
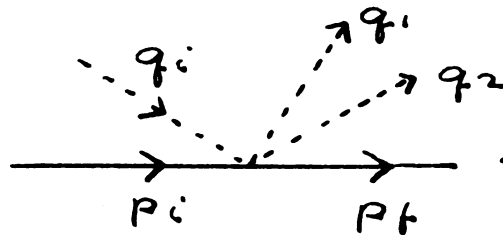


Figure 5. Diagram contributing to process (4.1)

which contributes to the pion production process (4.1). As luck would have it, this diagram usually dominates the process at low energies and is very sensitive to the choice of η_1 , in this region.

The utility of chiral dynamics models in pion production processes was first noted by Olsson and Turner.²⁹ Their calculation included, in addition to the diagram of Figure 5, the contact term



^{*}See appendix E for a calculation of s-wave scattering lengths.

Other terms generated by the Lagrangian contained one or more nucleon pole terms, and were therefore neglected as being relatively small. Actual cross sections were calculated by approximating all final state momenta to be zero in the transition amplitude $\mathcal{M}$ so that the cross section was essentially $|\mathcal{M}|^2$ times phase space. This approximation circumvented a laborious integration over final state phase space, as well as greatly simplifying spin averaging $\mathcal{M}$, but it restricted the calculation to total cross sections very near threshold which made conclusions drawn from the calculation very dependent on the accuracy of a handful of low energy experimental data. Olsson and Turner concluded from their calculation that, of the two, Weinberg's model agreed with the data better than Schwinger's model.

The procedure employed by Olsson and Turner may be objected to on two main grounds. First of all, even at the lowest energies for which experimental cross sections exist (incident pion lab energy $T_\pi = 210$ MeV versus threshold energy of about 180 MeV), approximating the final state momenta to be zero causes an error of 20% or so, and the discrepancy could only increase if the method were applied to higher energies. Secondly, it is not really clear that we should restrict our attention to cross sections very near threshold. We have built into our model isospin invariance. We know that isospin symmetry is not quite good since proton

and neutron masses differ by about 0.1% and there is about 3% difference in the masses of the charged and neutral pions. These small differences made no difference in high energy calculations, but near threshold small differences in final state masses may cause enormous differences in the magnitude of the phase space integral since the rest masses of the particles take up a large fraction of the energy available. In fact, at threshold the error becomes infinite! Even at $T_\pi = 210$ MeV, the choice of masses within the pion multiplet can cause a difference of about 85% in the size of the phase space integral. So the problem is, we claimed when we made isospin a good quantum number that within isomultiplets such as the pions or nucleons, all masses are equal for all practical purposes. However, in the kinematic region we are considering, phase space integrals, and therefore cross sections, are strongly dependent on just which of the masses we choose from the supposedly degenerate isomultiplets. Hence, the breaking of isospin symmetry which we neglected when forming our models turns out to have an important effect. Fortunately, the effect of this symmetry breaking on phase space diminishes rapidly with increasing energy. At $T_\pi = 300$ MeV, for example, phase space only varies about 20% with different choices of mass within isomultiplets, and this is within the experimental errors in this energy region. But it is clear that it is dangerous to make any judgments on the

relative merits of different models without examining a range of energies above threshold.

The work done for this thesis is an attempt to make a more accurate determination of cross sections predicted by chiral dynamics Lagrangians under consideration by retaining the momentum dependence of the final state within the amplitude and performing the necessary integrations by Monte Carlo methods. Because we wished to do this calculation for energies above threshold, it was no longer possible to neglect diagrams with nucleon poles, so all diagrams generated by the Lagrangians were retained in forming the transition amplitude. It was hoped that by comparing theoretical and experimental total cross sections over a range of energies we could get an idea which of the competing models was most satisfactory, and incidentally find out at what energies the exact form of the symmetry breaking assumptions are important. Differential cross sections were also calculated in order to gain further insight into the structure of the invariant amplitudes obtained from the models.

SECTION V

FEYNMAN RULES FOR CHIRAL DYNAMICS

The chiral dynamics algorithm tells us to apply the usual Wick-Dyson reduction methods to the chiral dynamics Lagrangian and retain only tree diagrams of lowest order in the coupling constant g . But in actual practice it is best to use the Wick-Dyson reduction methods only enough to be able to deduce the Feynman rules and then to construct transition amplitudes directly from the diagrams.* The procedure is to draw all the tree diagrams for the process of interest and associate with the n th graph an amplitude $\mathcal{M}^{(n)}$ which is the product of factors associated with the various topological elements of the graph. The final invariant amplitude $\mathcal{M}$ is just the sum of all these $\mathcal{M}^{(n)}$'s. The following rules specify factors which are independent of the form of the interaction Lagrangian:

1. For each internal nucleon line (called a nucleon propagator) of momentum p , there is a factor

$$\frac{i \mathbf{1}}{p - M + i\epsilon} = \frac{i(p + M)\mathbf{1}}{p^2 - M^2 + i\epsilon}$$

where $\mathbf{1}$ is a 2 x 2 unit matrix in the space of the

*For a review of the reduction methods of quantum field theory, see appendix B.

τ matrices, and ϵ is a small number which is allowed to vanish after all integrations over internal momenta are completed.*

2. For each internal pion line (pion propagator) of momentum q there is a factor

$$\frac{i \delta_{ab}}{q^2 - \mu^2 + i\epsilon}$$

where δ_{ab} is a Kronecker delta connecting isospin indices at the vertices joined by the internal pion line and is used as with the internal nucleon line.

3. Define two column matrices

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These matrices operate in the space of the τ matrices and serve to keep track of nucleon isospin. The rule is that for an external nucleon line of momentum p , spin α , and isospin z-component $T_z = w$, there is a spinor and isospin factor as follows:

incoming nucleon: $\chi_w u(p, \alpha)$

outgoing nucleon: $\chi_w^\dagger \bar{u}(p, \alpha)$

incoming anti-nucleon: $\chi_{-w} v(p, \alpha)$

outgoing anti-nucleon: $\chi_{-w}^\dagger \bar{v}(p, \alpha)$

Here, $u(p, \alpha)$ and $v(p, \alpha)$ are particle and anti-particle spinors, respectively, and $\bar{u}(p, \alpha)$ and $\bar{v}(p, \alpha)$ are their conjugates.

4. Define three unit vectors in isospin space by

*Because there will be no such integrations in tree diagrams, we can omit this ϵ altogether in chiral dynamics calculations.

$$\begin{aligned}
\hat{\phi}_+ &= \frac{1}{\sqrt{2}} (1, i, 0) \\
\hat{\phi}_- &= \frac{1}{\sqrt{2}} (1, -i, 0) \\
\hat{\phi}_0 &= (0, 0, 1).
\end{aligned}$$

These vectors keep track of the pion isospin. The rule is that for an external pion line with $T_z = t$ and isospin index a , there is a factor as follows:

$$\begin{aligned}
\text{incoming pion: } & (\hat{\phi}_t)_a \\
\text{outgoing pion: } & (\hat{\phi}_{-t})_a = (\hat{\phi}_t^*)_a
\end{aligned}$$

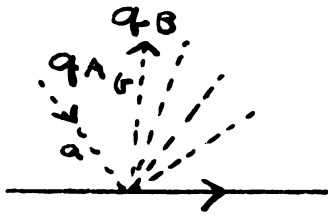
The following rules for vertices apply only to the Lagrangian (3.85). Let us adopt the notation that $\sum^P$ means to sum over all permutations of unrepeated indices in whatever expression follows it, e.g.

$$\begin{aligned}
\sum^P \delta_{ab} &= \delta_{ab} + \delta_{ba}, \\
\sum^P \epsilon_{abc} (\hat{\phi})_c &= \epsilon_{abc} (\hat{\phi})_c + \epsilon_{bac} (\hat{\phi})_c.
\end{aligned}$$

For a pion line of momentum q , define Q to be $-q$ if the line enters a vertex, $+q$ if the line leaves a vertex. If a pion line is labelled by isospin index a , its momentum is designated q_A . These capital indices go in tandem with small indices when they stand to the right of $\sum^P$, e.g.

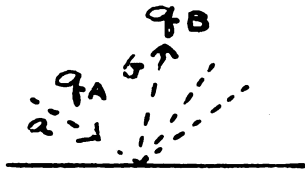
$$\sum^P \delta_{ab} q_B = \delta_{ab} q_B + \delta_{ba} q_A.$$

With this notation in mind, the factors for the three kinds of vertices generated by chiral dynamics Lagrangians are as follows:



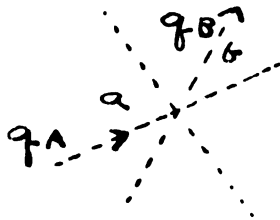
} $2m+1$ pion lines

$$\text{vertex: } -(-1)^m \left(\frac{g}{2M} \right)^{2m+1} \left(\frac{g_V}{g_A} \right)^{2m} \\ \times \sum^P \underbrace{\delta_{ab} \dots \delta_{pq}}_{m \text{ Kronecker deltas}} \tau_n \gamma_5 Q_R$$



} $2m+2$ pion lines

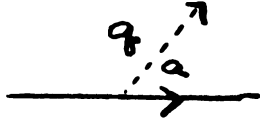
$$\text{vertex: } (-1)^m \left(\frac{g}{2M} \right)^{2m+2} \left(\frac{g_V}{g_A} \right)^{2m+2} \\ \times \sum^P \underbrace{\delta_{ab} \dots \delta_{pq}}_{m \text{ Kronecker deltas}} \tau_t \epsilon_{rst} Q_s$$



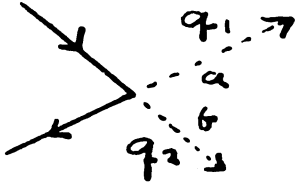
} $2m+2$ pion lines

$$\text{vertex: } -i(-1)^m \left(\frac{g}{2M} \right)^{2m} \left(\frac{g_V}{g_A} \right)^{2m} \\ \times \sum^P \underbrace{\delta_{ab} \dots \delta_{rs}}_{m+1 \text{ Kronecker deltas}} \left[\frac{n_m \mu^2}{2} + \frac{(m+1)}{2} Q_R Q_S \right]$$

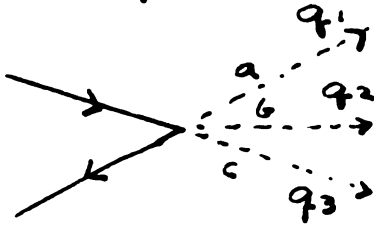
These somewhat forbidding general forms become much simpler for special cases. For our purposes we shall actually only need four vertices, and they are



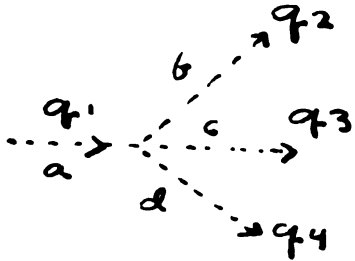
$$\text{vertex: } -\frac{g}{2M} \tau_a \gamma_5 \phi$$



$$\text{vertex: } -\left(\frac{g}{2M}\right)^2 \left(\frac{g_V}{g_A}\right)^2 \tau_c \epsilon_{abc} (\phi_1 - \phi_2)$$



$$\text{vertex: } 2\left(\frac{g}{2M}\right)^3 \left(\frac{g_V}{g_A}\right)^2 \\ \times [\delta_{ab} \tau_c \gamma_5 \phi_3 + \delta_{ac} \tau_b \gamma_5 \phi_2 + \delta_{bc} \tau_a \gamma_5 \phi_1]$$



$$\text{vertex: } 4i\left(\frac{g}{2M}\right)^2 \left(\frac{g_V}{g_A}\right)^2 \\ \times [\delta_{ab} \delta_{cd} (n_1 \mu^2 + q_3 q_4 - q_1 q_2) \\ + \delta_{ac} \delta_{bd} (n_1 \mu^2 + q_2 q_4 - q_1 q_3) \\ + \delta_{ad} \delta_{bc} (n_1 \mu^2 + q_2 q_3 - q_1 q_4)]$$

The first of these is just the familiar vertex of $ps(pv)$ coupling. The latter three vertices are unique to the chiral dynamics theory, and the last vertex contains the model dependent parameter n_1 . The values of n_1 for the different models we'll consider are given in the table below.

MODEL	n_1
Minimal Coupling	0
Schwinger Model	1/2
Weinberg Model (Tensor Rank 1)	1
Tensor Rank 2	2
Tensor Rank 3	17/5

As an example of how to use the rules enumerated above, let's evaluate the amplitude corresponding to Figure 5. Figure 5 is redrawn in Figure 6 with each nucleon line labeled by its four momentum, each pion line labeled by its four momentum and isospin index, and the factors given by the Feynman rules encircled next to the appropriate parts of the graph.

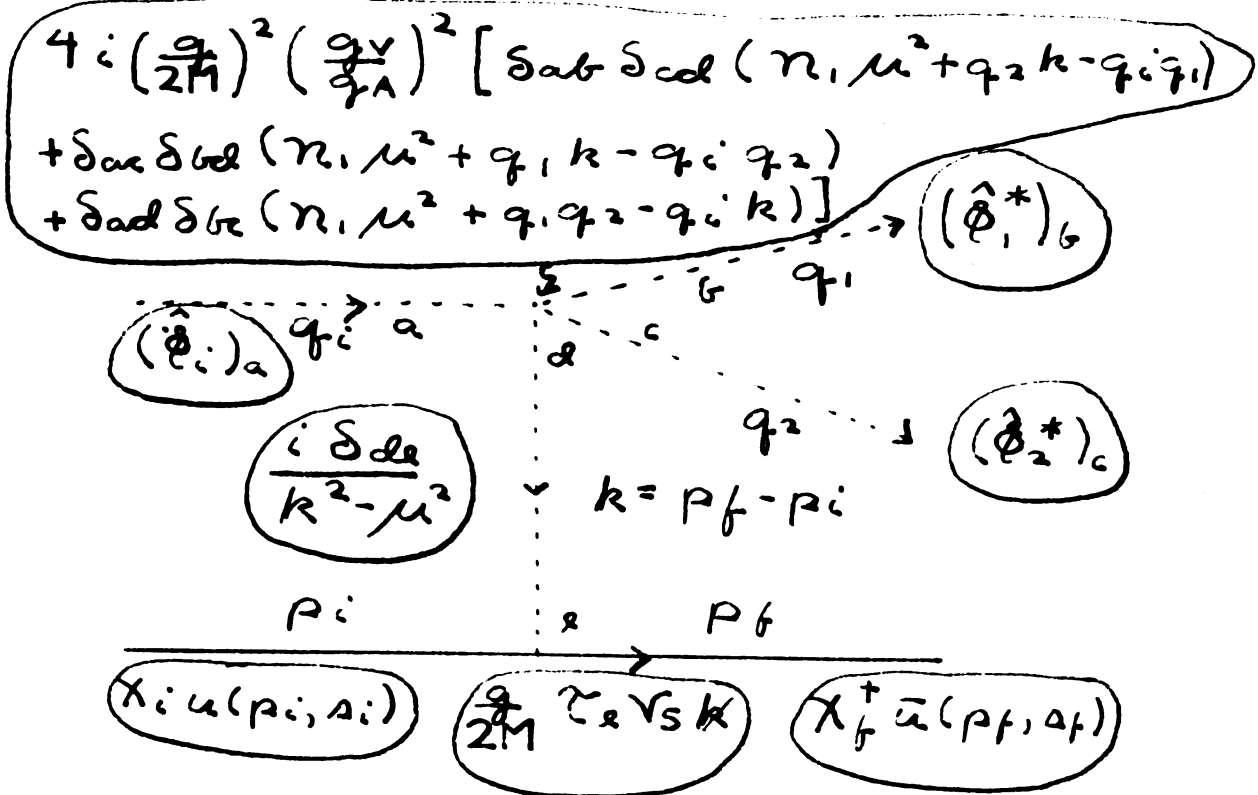


Figure 6. Diagram of Figure 5 with factors given by Feynman rules for chiral dynamics encircled next to appropriate topological parts.

To form the invariant amplitude $\mathcal{M}$ for this diagram, multiply the circled factors, reading along the nucleon line to keep matrix multiplications in the right order.

The result is

$$\begin{aligned}
 \mathcal{M} = & \{ \bar{\chi}_f^\dagger \bar{u}(p_f, \alpha_f) \} \left\{ \frac{g}{2M} \tau_z \gamma_5 k \right\} \{ \chi_i u(p_i, \alpha_i) \} \\
 & \times \left\{ \frac{i \delta_{ab}}{k^2 - \mu^2} \right\} \{ (\hat{\phi}_i)_a \} \{ (\hat{\phi}_1^*)_b \} \{ (\hat{\phi}_2^*)_c \} \\
 & \times \left\{ 4 i \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2 [\delta_{ab} \delta_{cd} (n_1 \mu^2 + q_2 k - q_i \cdot q_1) \right. \\
 & \quad \left. + \delta_{ac} \delta_{bd} (n_1 \mu^2 + q_1 k - q_i \cdot q_2) \right. \\
 & \quad \left. + \delta_{ad} \delta_{bc} (n_1 \mu^2 + q_1 q_2 - q_i \cdot k) \right] \} \\
 = & -4 \left(\frac{g}{2M} \right)^3 \left(\frac{g_V}{g_A} \right)^2 \frac{\bar{u}(p_f, \alpha_f) \gamma_5 k u(p_i, \alpha_i)}{k^2 - \mu^2} \\
 & \times \left\{ [\bar{\chi}_f^\dagger \hat{\phi}_i \cdot \hat{\phi}_1^* \vec{\tau} \cdot \hat{\phi}_2^* \chi_i] (n_1 \mu^2 + q_2 k - q_i \cdot q_1) \right. \\
 & \quad + [\bar{\chi}_f^\dagger \hat{\phi}_i \cdot \hat{\phi}_2^* \vec{\tau} \cdot \hat{\phi}_1^* \chi_i] (n_1 \mu^2 + q_1 k - q_i \cdot q_2) \\
 & \quad \left. + [\bar{\chi}_f^\dagger \hat{\phi}_i \cdot \hat{\phi}_2^* \vec{\tau} \cdot \hat{\phi}_1^* \chi_i] (n_1 \mu^2 + q_1 q_2 - q_i \cdot k) \right\}.
 \end{aligned}$$

Using various kinematic identities and Dirac algebra, this may be simplified still further. For a specific charge channel use the appropriate forms of $\chi_f, \chi_i, \hat{\phi}_i, \hat{\phi}_1$, and $\hat{\phi}_2$ and the correct isotopic factors result from a simple matrix multiplication.

After the total $\mathcal{M}$ is obtained for a given process by summing the $\mathcal{M}^{(m)}$'s for all contributing diagrams, cross sections are obtained by integrating over all unobserved final state kinematic variables in the formula

$$dG = \frac{1}{|\vec{U}_1 - \vec{U}_2|} \left(\frac{1}{2\omega_{p_1}} \right) \left(\frac{1}{2\omega_{p_2}} \right) \overline{|M|^2} \\ \times \frac{d^3k_1}{2\omega_1(2\pi)^3} \dots \frac{d^3k_m}{2\omega_m(2\pi)^3} \times (2\pi)^4 \delta^4(p_1 + p_2 - \sum_{i=1}^m k_i) S$$

(5.1)

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$, and $\vec{U}_1$ and $\vec{U}_2$ are velocities of the incident particles. The incident particles have four momenta p_1 and p_2 and the n final state particles have four momenta $k_1, k_2, k_3, \dots, k_m$. $\overline{|M|^2}$ is the spin average of $M^+ M$. The factor S is obtained from

$$S = \prod_i \frac{1}{m_i!}$$

where m_i is the number of identical final state particles of the i th type.* The factor S is only included in calculations of total cross sections, not differential cross sections. For fermions of mass m , the factors $\frac{1}{2\omega}$ which appear in (f.1) are replaced by $\frac{m}{E}$.

*By identical particles we mean particles and experimentalist cannot distinguish. Thus a π^+ and a π^- are not identical particles even though they are members of the same isomultiplet.

SECTION VI

MATRIX ELEMENTS FOR $\pi + N \rightarrow \pi + \pi + N$

Let us now restrict our discussion to the pion production process

$$\pi(q_i) + N(p_i) \rightarrow \pi_1(q_1) + \pi_2(q_2) + N'(p_f) \quad (6.1)$$

The quantities in parenthesis are the four momenta of the particles. Letting $u(i) \equiv u(p_i, \Delta_i)$ and $\bar{u}(f) \equiv \bar{u}(p_f, \Delta_f)$, the invariant amplitude for (6.1) may be written in the form

$$\mathcal{M} = \bar{u}(f)(W + X q_1 + Y q_2 + Z q_2 q_1) \gamma_5 u(i). \quad (6.2)$$

Here W , X , Y , and Z are functions of the kinematics. Other forms such as $\bar{u}(f) q_i \gamma_5 u(i)$, $\bar{u}(f) q_i q_1 q_2 \gamma_5 u(i)$, or $\bar{u}(f)(p_i - p_f) \gamma_5 u(i)$ which may appear in the course of applying the Feynman rules to the diagrams may always be brought into the form (6.2) by using the identities of Dirac algebra.*

Before evaluating cross sections, we must spin average $|\mathcal{M}|^2 = \mathcal{M}^\dagger \mathcal{M}$. If we write

$$\mathcal{M} = \bar{u}(f) \Gamma u(i)$$

where of course $\Gamma = (W + X q_1 + Y q_2 + Z q_2 q_1) \gamma_5$, and define $\bar{\Gamma} = \gamma_0 \Gamma^\dagger \gamma_0$, the usual methods yield

*See Drell and Bjorken, Relativistic Quantum Mechanics, appendix A.

for $\overline{|m|^2}$, the spin average of $|m|^2$,

$$\overline{|m|^2} = \frac{1}{2} \sum_{\mathbf{a}_f} \sum_{\mathbf{a}_i} m^\dagger m = \frac{1}{2} \text{trace } \Gamma\left(\frac{\mathbf{a}_i + \mathbf{M}}{2M}\right) \overline{\Gamma\left(\frac{\mathbf{a}_f + \mathbf{M}}{2M}\right)}$$

Now the usefulness of reducing m to the form (6.2)

becomes clear, because expressions of the form

$$\text{trace } \mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n$$

become very difficult to evaluate for large n . If we

apply our spin averaging theorem to (6.2), we arrive at

$$\begin{aligned} \overline{|m|^2} = \frac{1}{2M^2} \{ & |W|^2 [(\mathbf{p}_i \mathbf{p}_f) - M^2] \\ & + |X|^2 [2(\mathbf{p}_i \mathbf{q}_1)(\mathbf{p}_f \mathbf{q}_1) - \mu^2(\mathbf{p}_i \mathbf{p}_f) - \mu^2 M^2] \\ & + |Y|^2 [2(\mathbf{p}_i \mathbf{q}_2)(\mathbf{p}_f \mathbf{q}_2) - \mu^2(\mathbf{p}_i \mathbf{p}_f) - \mu^2 M^2] \\ & + |Z|^2 [4(\mathbf{p}_i \mathbf{q}_1)(\mathbf{p}_f \mathbf{q}_2)(\mathbf{q}_1 \mathbf{q}_2) - 2\mu^2(\mathbf{p}_i \mathbf{q}_2)(\mathbf{p}_f \mathbf{q}_2) \\ & \quad - 2\mu^2(\mathbf{p}_i \mathbf{q}_1)(\mathbf{p}_f \mathbf{q}_1) - \mu^4(\mathbf{p}_i \mathbf{p}_f) - \mu^4 M^2] \\ & + 2M \text{Re } W^* X [(\mathbf{p}_i \mathbf{q}_1) - (\mathbf{p}_f \mathbf{q}_1)] \\ & + 2M \text{Re } W^* Y [(\mathbf{p}_i \mathbf{q}_2) - (\mathbf{p}_f \mathbf{q}_2)] \\ & \quad + 2 \text{Re } W^* Z [(\mathbf{p}_i \mathbf{p}_f)(\mathbf{q}_1 \mathbf{q}_2) - (\mathbf{p}_i \mathbf{q}_2)(\mathbf{p}_f \mathbf{q}_1) \\ & \quad \quad + (\mathbf{p}_i \mathbf{q}_1)(\mathbf{p}_f \mathbf{q}_2) - M^2(\mathbf{q}_1 \mathbf{q}_2)] \\ & + 2 \text{Re } X^* Y [(\mathbf{p}_i \mathbf{q}_2)(\mathbf{p}_f \mathbf{q}_1) - (\mathbf{p}_i \mathbf{p}_f)(\mathbf{q}_1 \mathbf{q}_2) \\ & \quad \quad + (\mathbf{p}_i \mathbf{q}_1)(\mathbf{p}_f \mathbf{q}_2) - M^2(\mathbf{q}_1 \mathbf{q}_2)] \\ & + 2M \text{Re } X^* Z [2(\mathbf{p}_i \mathbf{q}_1)(\mathbf{q}_1 \mathbf{q}_2) - \mu^2(\mathbf{p}_f \mathbf{q}_2) \\ & \quad \quad - \mu^2(\mathbf{p}_i \mathbf{q}_2)] \\ & + 2M \text{Re } Y^* Z [\mu^2(\mathbf{p}_i \mathbf{q}_1) + \mu^2(\mathbf{p}_f \mathbf{q}_1) \\ & \quad \quad - 2(\mathbf{p}_f \mathbf{q}_2)(\mathbf{q}_1 \mathbf{q}_2)] \}. \end{aligned}$$

At threshold, this rather involved expression takes the simple form

$$|\overline{m}|^2_{\text{threshold}} = \left(\frac{E_i - M}{2M} \right) |W + X_\mu + Y_\mu + Z_\mu|^2,$$

where of course W , X , Y , and Z are calculated at threshold. It is easy to see that at threshold $|\overline{m}|^2$ is real and positive, as it must be.

Chiral dynamics Lagrangians generate five general types of tree diagrams for the process (6.1). These are shown in Figure 7.

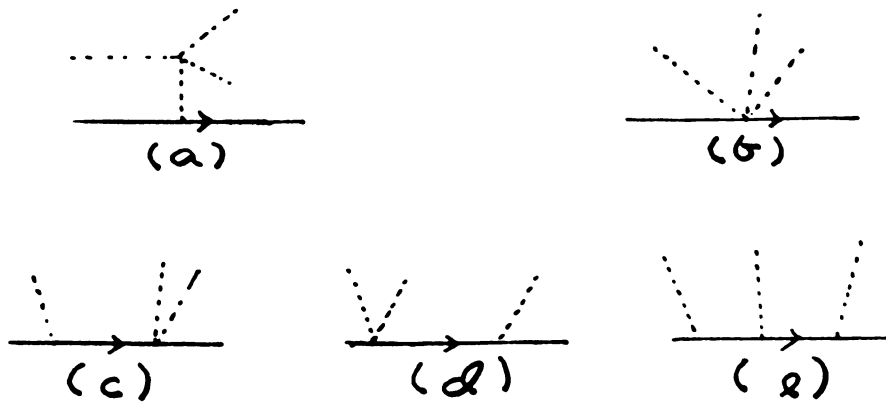
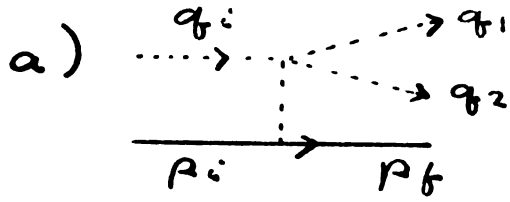


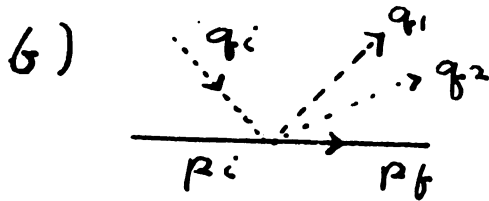
Figure 7. The five diagrams contributing to the process (6.1)

For different labeling of the pion lines, these five general types of diagrams yield altogether fourteen different Feynman graphs. These graphs along with the amplitudes they generate upon application of the Feynman rules of the previous section are tabulated on pp. 69-74. For specific charge channels, the isospin

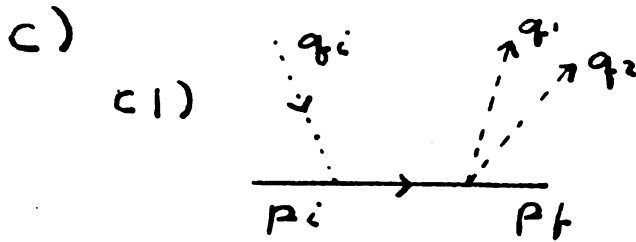
factors cause about half these diagrams to vanish identically. Note that if we include only diagrams (e1) through (e6), we obtain the results of pseudo-vector coupling theory.



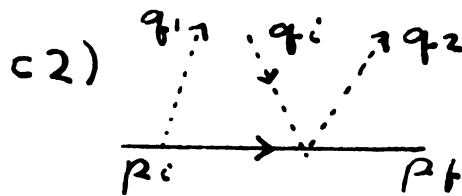
$$\mathcal{M}^{(a)} = -8M \left(\frac{q_i}{2M} \right)^3 \left(\frac{q_v}{q_\Lambda} \right)^2 \frac{\bar{u}(f) V_S u(i)}{2p_i p_f - 2M^2 + \mu^2} \\ \times \{ [\chi_f^\dagger \hat{\phi}_i \cdot \hat{\phi}_1^* \vec{\tau} \cdot \hat{\phi}_2^* \chi_i] (\eta, \mu^2 + p_f q_2 - p_i q_2 - q_i q_1) \\ + [\chi_f^\dagger \hat{\phi}_i \cdot \hat{\phi}_2^* \vec{\tau} \cdot \hat{\phi}_1^* \chi_i] (\eta, \mu^2 + p_f q_1 - p_i q_1 - q_i q_2) \\ + [\chi_f^\dagger \hat{\phi}_1^* \cdot \hat{\phi}_2^* \vec{\tau} \cdot \hat{\phi}_i \chi_i] (\eta, \mu^2 + q_1 q_2 - p_f q_i + p_i q_i) \}$$



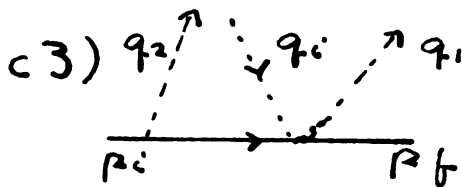
$$\mathcal{M}^{(b)} = -2 \left(\frac{q_i}{2M} \right)^3 \left(\frac{q_v}{q_\Lambda} \right)^2 \bar{u}(f) \{ \\ [\chi_f^\dagger \hat{\phi}_i \cdot \hat{\phi}_2^* \vec{\tau} \cdot \hat{\phi}_1^* \chi_i] q_1 \\ + [\chi_f^\dagger \hat{\phi}_i \cdot \hat{\phi}_1^* \vec{\tau} \cdot \hat{\phi}_2^* \chi_i] q_2 \\ - [\chi_f^\dagger \hat{\phi}_1^* \cdot \hat{\phi}_2^* \vec{\tau} \cdot \hat{\phi}_i \chi_i] (2M + q_1 + q_2) \} V_S u(i)$$



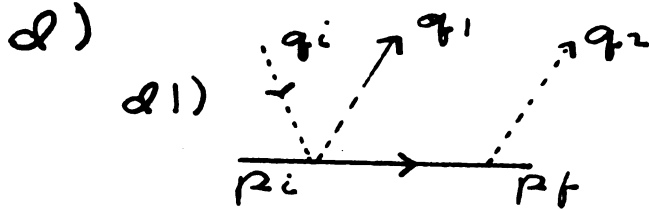
$$\mathcal{M}^{(c1)} = -i \left(\frac{g}{2M} \right)^3 \left(\frac{g_V}{g_A} \right)^2 [\chi_f^\dagger \bar{\tau} \cdot \hat{\phi}_1^* \times \hat{\phi}_2^* \bar{\tau} \cdot \hat{\phi}_i \chi_i] \\ \times \bar{u}(f) \left\{ \frac{4M(2p_f q_2 + \mu^2)}{2p_i q_i + \mu^2} - 2M - q_1 + q_2 \right. \\ \left. + \frac{4M q_2 q_1}{2p_i q_i + \mu^2} \right\} V_S u(i)$$



$$\mathcal{M}^{(c2)} = -i \left(\frac{g}{2M} \right)^3 \left(\frac{g_V}{g_A} \right)^2 [\chi_f^\dagger \bar{\tau} \cdot \hat{\phi}_i \times \hat{\phi}_2^* \bar{\tau} \cdot \hat{\phi}_1^* \chi_i] \\ \times \bar{u}(f) \left\{ q_1 + 2q_2 + \frac{4M q_2 q_1}{2p_i q_i - \mu^2} \right\} V_S u(i)$$

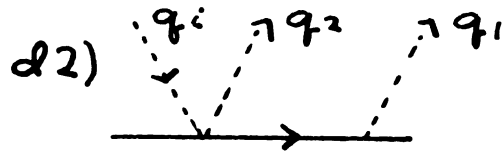


$$\mathcal{M}^{(c3)} = -i \left(\frac{g}{2M} \right)^3 \left(\frac{g_V}{g_A} \right)^2 [\chi_f^\dagger \bar{\tau} \cdot \hat{\phi}_i \times \hat{\phi}_1^* \bar{\tau} \cdot \hat{\phi}_2^* \chi_i] \\ \times \bar{u}(f) \left\{ \frac{8M q_1 q_2}{2p_i q_2 - \mu^2} + 2q_1 + q_2 \right. \\ \left. - \frac{4M q_2 q_1}{2p_i q_2 - \mu^2} \right\} V_S u(i)$$



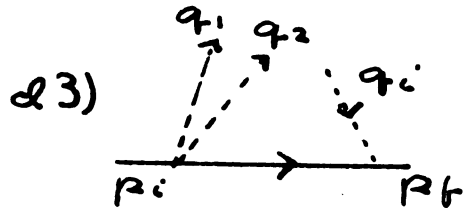
$$M^{(d1)} = -i \left(\frac{q}{2M} \right)^3 \left(\frac{q_V}{q_A} \right)^2 [\chi_f^\dagger \vec{\tau} \cdot \hat{\theta}_2^* \vec{\tau} \cdot \hat{\theta}_i \times \hat{\theta}_1^* \chi_i]$$

$$\times \bar{u}(f) \left\{ -2q_1 - q_2 + \frac{4M q_2 q_1}{2p_f q_2 + \mu^2} \right\} V_S u(i)$$



$$M^{(d2)} = -i \left(\frac{q}{2M} \right)^3 \left(\frac{q_V}{q_A} \right)^2 [\chi_f^\dagger \vec{\tau} \cdot \hat{\theta}_1^* \vec{\tau} \cdot \hat{\theta}_i \times \hat{\theta}_2^* \chi_i]$$

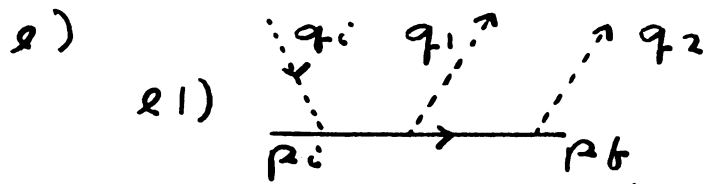
$$\times \bar{u}(f) \left\{ \frac{8M q_1 q_2}{2p_f q_1 + \mu^2} - q_1 - 2q_2 - \frac{4M q_2 q_1}{2p_f q_1 + \mu^2} \right\} V_S u(i)$$



$$M^{(d3)} = -i \left(\frac{q}{2M} \right)^3 \left(\frac{q_V}{q_A} \right)^2 [\chi_f^\dagger \vec{\tau} \cdot \hat{\theta}_i \vec{\tau} \cdot \hat{\theta}_1^* \times \hat{\theta}_2^* \chi_i]$$

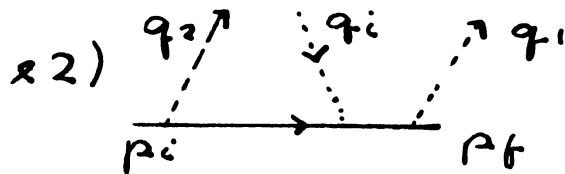
$$\times \bar{u}(f) \left\{ \frac{4M(2p_i q_i - \mu^2)}{2p_f q_i - \mu^2} - 2M + q_1 - q_2 \right.$$

$$\left. - \frac{4M q_2 q_1}{2p_f q_i - \mu^2} \right\} V_S u(i)$$



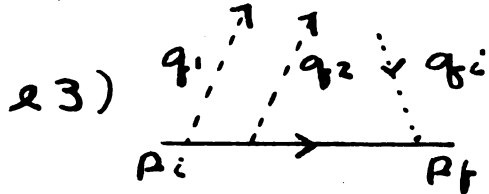
$$\mathcal{M}^{(21)} = \left(\frac{g}{2M}\right)^3 [\chi_f^\dagger \vec{\tau} \cdot \hat{\theta}_2^* \vec{\tau} \cdot \hat{\theta}_1^* \vec{\tau} \cdot \hat{\theta}_i \chi_i]$$

$$\times \bar{u}(f) \left\{ 2M \left[1 - \frac{2p_f q_2 + m^2}{2p_i q_i + m^2} \right] \right. \\ + \left[1 + \frac{4M^2}{2p_i q_i + m^2} \right] q_1 - 4M^2 \left[\frac{1}{2p_f q_2 + m^2} - \frac{1}{2p_i q_i + m^2} \right] q_2 \\ \left. - 2M \left[\frac{1}{2p_f q_2 + m^2} + \frac{1}{2p_i q_i + m^2} + \frac{1}{(2p_f q_2 + m^2)(2p_i q_i + m^2)} \right] q_2 q_1 \right\} \gamma_5 u(i)$$

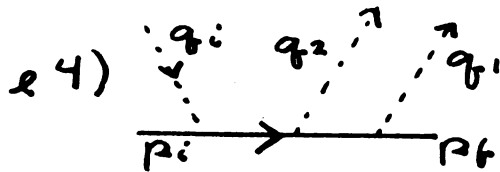


$$\mathcal{M}^{(22)} = \left(\frac{g}{2M}\right)^3 [\chi_f^\dagger \vec{\tau} \cdot \hat{\theta}_1^* \vec{\tau} \cdot \hat{\theta}_i \vec{\tau} \cdot \hat{\theta}_2^* \chi_i]$$

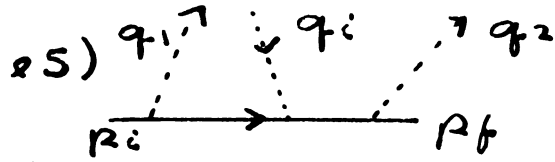
$$\times \bar{u}(f) \left\{ 2M \left[1 + \frac{2q_1 q_2}{2p_f q_1 + m^2} - \frac{2q_1 q_2}{2p_i q_2 - m^2} \right. \right. \\ - \left. \frac{8M^2 q_1 q_2}{(2p_f q_1 + m^2)(2p_i q_2 - m^2)} \right] - \left[1 + \frac{4M^2}{2p_f q_1 + m^2} \right] q_1 \\ - \left[1 - \frac{4M^2}{2p_i q_2 - m^2} \right] q_2 + 2M \left[\frac{1}{2p_i q_2 - m^2} - \frac{1}{2p_f q_1 + m^2} \right. \\ \left. \left. + \frac{4M^2}{(2p_i q_2 - m^2)(2p_f q_1 + m^2)} \right] q_2 q_1 \right\} \gamma_5 u(i)$$



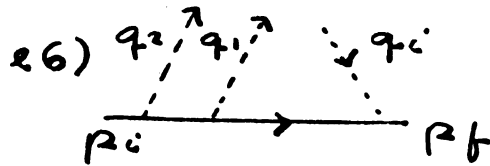
$$\begin{aligned}
 \mathcal{M}^{(23)} &= \left(\frac{g}{2M}\right)^3 [\chi_f^\dagger \vec{\tau} \cdot \hat{\theta}_i \vec{\tau} \cdot \hat{\theta}_2^* \vec{\tau} \cdot \hat{\theta}_1^* \chi_i] \\
 &\times \bar{u}(f) \left\{ 2M \left[1 - \frac{2p_i q_1 - \mu^2}{2p_f q_i - \mu^2} \right] \right. \\
 &+ 4M^2 \left[\frac{1}{2p_i q_1 - \mu^2} - \frac{1}{2p_f q_i - \mu^2} \right] q_1 + \left[1 - \frac{4M^2}{2p_f q_i - \mu^2} \right] q_2 \\
 &+ 2M \left[\frac{1}{2p_f q_i - \mu^2} + \frac{1}{2p_i q_1 - \mu^2} - \frac{4M^2}{(2p_f q_i - \mu^2)(2p_i q_1 - \mu^2)} \right] q_2 q_1 \\
 &\left. \right\} V_S u(i)
 \end{aligned}$$



$$\begin{aligned}
 \mathcal{M}^{(24)} &= \left(\frac{g}{2M}\right)^3 [\chi_f^\dagger \vec{\tau} \cdot \hat{\theta}_1^* \vec{\tau} \cdot \hat{\theta}_2^* \vec{\tau} \cdot \hat{\theta}_i \chi_i] \\
 &\times \bar{u}(f) \left\{ 2M \left[1 - \frac{2p_f q_1 + \mu^2}{2p_i q_i + \mu^2} - \frac{2q_1 q_2}{2p_i q_i + \mu^2} - \frac{2q_1 q_2}{2p_f q_1 + \mu^2} \right. \right. \\
 &- \left. \frac{8M^2 q_1 q_2}{(2p_f q_1 + \mu^2)(2p_i q_i + \mu^2)} \right] + 4M^2 \left[\frac{1}{2p_i q_i + \mu^2} \right. \\
 &- \left. \frac{1}{2p_f q_1 + \mu^2} \right] q_1 + \left[1 + \frac{4M^2}{2p_i q_i + \mu^2} \right] q_2 \\
 &+ 2M \left[\frac{1}{2p_f q_1 + \mu^2} + \frac{1}{2p_i q_i + \mu^2} - \frac{4M^2}{(2p_f q_1 + \mu^2)(2p_i q_i + \mu^2)} \right] q_2 q_1 \\
 &\left. \right\} V_S u(i)
 \end{aligned}$$



$$\begin{aligned}
 \mathcal{M}^{(25)} &= \left(\frac{g}{2M}\right)^3 [\chi_f^\dagger \vec{\tau} \cdot \hat{\theta}_2^* \vec{\tau} \cdot \hat{\theta}_i \vec{\tau} \cdot \hat{\theta}_1^* \chi_i] \\
 &\times \bar{u}(f) \left\{ 2M - \left[1 - \frac{4M^2}{2p_i q_1 - m^2} \right] q_1 \cdot \left[1 + \frac{4M^2}{2p_f q_2 + m^2} \right] q_2 \right. \\
 &+ 2M \left[\frac{1}{2p_f q_2 + m^2} - \frac{1}{2p_i q_1 - m^2} - \frac{4M^2}{(2p_f q_2 + m^2)(2p_i q_1 - m^2)} \right] q_2 q_1 \\
 &\left. \right\} \gamma_5 u(i)
 \end{aligned}$$



$$\begin{aligned}
 \mathcal{M}^{(26)} &= \left(\frac{g}{2M}\right)^3 [\chi_f^\dagger \vec{\tau} \cdot \hat{\theta}_i \vec{\tau} \cdot \hat{\theta}_1^* \vec{\tau} \cdot \hat{\theta}_2^* \chi_i] \\
 &\times \bar{u}(f) \left\{ 2M \left[1 - \frac{2p_i q_2 - m^2}{2p_f q_i - m^2} + \frac{2q_1 q_2}{2p_f q_i - m^2} \right. \right. \\
 &+ \left. \frac{2q_1 q_2}{2p_i q_2 - m^2} - \frac{8M^2 q_1 q_2}{(2p_f q_i - m^2)(2p_i q_2 - m^2)} \right] \\
 &+ \left[1 - \frac{4M^2}{2p_f q_i - m^2} \right] q_1 + 4M^2 \left[\frac{1}{2p_i q_2 - m^2} \right. \\
 &- \left. \frac{1}{2p_f q_i - m^2} \right] q_2 - 2M \left[\frac{1}{2p_f q_i - m^2} \right. \\
 &+ \left. \frac{1}{2p_i q_2 - m^2} - \frac{4M^2}{(2p_f q_i - m^2)(2p_i q_2 - m^2)} \right] q_2 q_1 \left. \right\} \gamma_5 u(i)
 \end{aligned}$$

SECTION VII
NUMERICAL CALCULATIONS

The previous section showed how to obtain $\overline{|m|^2}$ for processes of the form

$$\pi(q_i) + N(p_i) \rightarrow \pi_1(q_1) + \pi_2(q_2) + N'(p_f) \quad (6.1)$$

All we have to do now is substitute $\overline{|m|^2}$ in (5.1) and integrate over the appropriate quantities. For the process (6.1), we note that in center of mass

$$\vec{U}_1 = \frac{\vec{q}_i}{\omega_i}, \quad \vec{U}_2 = \frac{\vec{p}_i}{E_i}, \quad \text{and} \quad \vec{p}_i = -\vec{q}_i \quad \text{and}$$

$$\text{so } \frac{1}{|\vec{U}_1 - \vec{U}_2|} = \frac{E_i \omega_i}{E |\vec{q}_i|}, \quad \text{where } E = E_i + \omega_i$$

is total center of mass energy. Using this and the appropriate normalization factors for initial and final nucleons, (5.1) becomes

$$\begin{aligned} d\sigma = & \left\{ \frac{1}{8(2\pi)^5} \frac{M^2}{E |\vec{q}_i|} \overline{|m|^2} S \right\} \\ & \times \frac{d^3 q_1}{\omega_1} \frac{d^3 q_2}{\omega_2} \frac{d^3 p_f}{E_f} \\ & \times \delta(E - \omega_1 - \omega_2 - \omega_f) \delta^3(\vec{q}_1 + \vec{q}_2 + \vec{p}_f). \end{aligned} \quad (7.1)$$

Equation (7.1) is what we must integrate. However, this integration is no trivial matter, mainly because of the number of variables to be integrated over. Even after the

delta function is eliminated there still remain five variables. Analytic forms might be possible if the quantity in curly brackets took on a simple form. But a glance at the amplitudes pp. 69-74 should be adequate to crush any hopes that this might be the case for chiral dynamics. So it seems clear that we must resort to numerical methods. However, even the usual numerical methods like Simpson's rule or Weddle's rule fail us because of the number of variables to integrate over. Ordinary quadratures become impractical for more than two independent variables. So we'll have to employ a method often used for numerical integration over large numbers of variables, the Monte Carlo method.

The basic idea of Monte Carlo quadrature methods is that instead of making a systematic sampling of the integrand over a grid of the independent variables, a random sample over the grid of independent variables is taken and statistical criteria are used to decide when the sample is large enough for the accuracy desired. The difference between the two methods is rather like the difference between an election and a Gallup poll. The drawback of the Monte Carlo method is that it severely limits accuracy, and must be used only on slowly varying functions. Its great advantage is that bad as it is, it doesn't get worse as the number of independent variables increases.

Let us define a covariant phase space for three particles with masses μ_1 , μ_2 , and M and total center of mass energy E , by

$$P_3(E; \mu_1, \mu_2, M) = \iiint \frac{d^3\vec{q}_1}{\omega_1} \frac{d^3\vec{q}_2}{\omega_2} \frac{d^3\vec{p}_f}{E_f} \delta(E - \omega_1 - \omega_2 - E_f) \delta^3(\vec{q}_1 + \vec{q}_2 + \vec{p}_f)$$

where $\omega_1 = \sqrt{\vec{q}_1^2 + \mu_1^2}$, $\omega_2 = \sqrt{\vec{q}_2^2 + \mu_2^2}$, and $E_f = \sqrt{\vec{p}_f^2 + M^2}$.

We can prove that

$$G(E) = A(E) P_3(E; \mu_1, \mu_2, M) \quad (7.2)$$

where $A(E)$ is the average of the quantity

$$\left\{ \frac{1}{8(2\pi)^5} \frac{M^2}{E |\vec{q}_1|} \frac{1}{|\vec{m}|^2} S \right\} \quad (7.3)$$

over phase space.* It turns out that we can reduce

calculation of $P_3(E; \mu_1, \mu_2, M)$ to a

Simpson's rule quadrature, and thus we can obtain values

of the phase space integral with great accuracy. It is

in calculation of $A(E)$ that the Monte Carlo approach

becomes necessary. The method used employs subroutines

of a computer program called FAKE which was developed for

use by high energy experimentalists. FAKE generates

random events uniformly distributed in covariant phase

space, i.e. if we consider n particles of masses m_1 ,

$m_2, \dots, m_n$ in the center of mass with total mass

energy E , FAKE will generate random sets of vectors

$\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n$ for which

*See appendix D for a proof of (7.2) and a more thorough discussion of several points mentioned in this section, especially the Monte Carlo technique.

$$\sum_{i=1}^3 \vec{k}_i = 0, \quad \sum_{i=1}^3 \sqrt{\vec{k}_i^2 + m_i^2} = E \quad .$$

To evaluate $A(E)$, we have FAKE generate a three particle state $\{q_1, q_2, p_f\}$ and a two particle state $\{q_i, p_i\}$. These states are generated independently of one another and are oriented randomly with respect to one another. The four-momenta p_f , p_i , q_1 , q_2 , and q_i are now substituted into (7.3). This process is repeated many times and the resultant values for (7.3) averaged until statistical criteria tell us that our average is sufficiently close to the true average. This resultant average is designated $A(E)$.

A similar technique is used to obtain differential cross sections. Let's orient a coordinate system with its z-axis parallel to $\vec{k}$ where $\vec{k}$ is one of the vectors $\vec{q}_1$, $\vec{q}_2$, or $\vec{p}_f$. Let the polar and aximuthal angles of $\vec{q}_i$ be θ and ϕ in this coordinate system. We can prove that

$$\left. \frac{dG(E)}{d\Omega} \right|_{\Omega=\Omega_0} = A(E, \theta=\theta_0) \frac{P_3(E; \mu_1, \mu_2, M)}{4\pi} \quad (7.4)$$

where $A(E, \theta=\theta_0)$ is the average of (7.3) over that part of phase space where θ is constrained to be equal to θ_0 .* Program FAKE is used as above, but with the random events restricted to that region of covariant

*See appendix D for proof.

phase space where

$$\frac{\vec{k} \cdot \vec{q}_i}{|\vec{k}| |\vec{q}_i|} = \cos \theta_0.$$

Doubtless, ways of applying the Monte Carlo method to (7.1) exist besides application of (7.2) and (7.3). But aside from practical considerations, formulae (7.2) and (7.3) have the advantage of separating the calculation of the phase space $P_s(E; \mu, \mu_2, M)$ and the model dependent factor $\alpha(E)$ or $\alpha(E, \theta)$. As we've discussed, the phase space factor is very sensitive to the exact choices of μ , μ_2 , and M , so in its calculation we use experimental values of the three masses involved. The factors $\alpha(E)$ and $\alpha(E, \theta)$ are much less sensitive to the mass values used, and so, in the spirit of the isospin symmetry built in our Lagrangians, we use some average mass for the pion mass μ and nucleon mass M . To be exact, we take

$$\mu = \frac{\mu_i + \mu_1 + \mu_2}{3}, \quad M = \frac{M_i + M_f}{2}$$

though this choice is pretty arbitrary. We see that we've taken the isospin symmetry breaking into account solely in the phase integral, but it is in this integral where it's most important.

In obtaining the results of the next section Monte Carlo errors were kept equal to or less than 4%. This means that there is a 68% probability that they are within 8% of the right answer, and a 99.95% probability they are

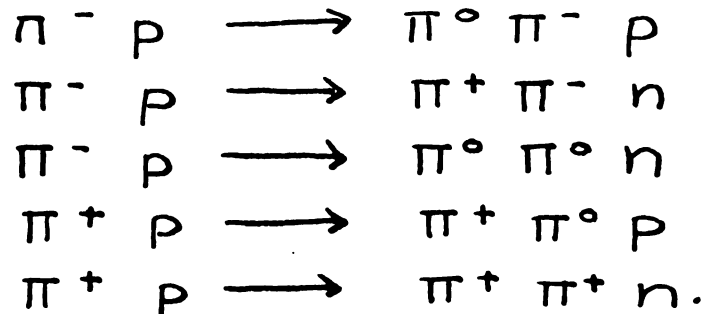
within 12% of the right answer. The Monte Carlo errors are always smaller than the experimental errors for the cross sections, and a great deal smaller than the experimental errors below $T_{\pi} = 300$ MeV or so, a region of special interest.

SECTION VIII

RESULTS

Theoretical calculations of total and differential cross sections were carried out using the five different chiral dynamics models listed on page 50. Cross sections were calculated with an error of 4% or less. The results were compared with the data compilation of Olsson and Yodh.³⁰ Experimental points and theoretical curves are shown in Figures 8-15.

Data for total cross sections was available for five different charge channels which are



Certain features are common to the predictions for all these processes. In each case we have generally good agreement between chiral dynamics predictions and experimental data for energies below 300 MeV. Above this region a discrepancy develops between theory and experiment, but around 900 MeV the gap between the two again

narrows. The disagreement above 300 MeV is probably due to the effects of resonances which the models cannot take into account. The agreement around 900 MeV must be considered a coincidence which stems from the fact that the theoretical model gives cross sections which increase without bound, whereas the experimental cross section begins to decrease around 1 BeV. Since the theoretical cross sections are initially smaller, the two curves are bound to intersect. But in the region below 300 MeV, Chang has shown that resonances make only small contributions to the total amplitude,³¹ so it is in this energy region where we would expect the chiral dynamics models to be successful. It is interesting to note that though the chiral dynamics models we're dealing with are essentially chiral symmetric versions of the ps(pv) model, the ps(pv) model gives cross sections which are vastly different from the chiral dynamics predictions and the experimental data. This discrepancy is at its worst in the energy region where we expect the greatest validity of the chiral dynamics approach.

In the case of the processes $\pi^- p \rightarrow \pi^0 \pi^- p$ and $\pi^+ p \rightarrow \pi^+ \pi^0 p$ we see that all five chiral dynamics models give good fits to the data, even above the low energy region, whereas the ps(pv) model is much too small. This gives support to the chiral dynamics approach, but is of no help in discriminating among the models. For the process $\pi^- p \rightarrow \pi^0 \pi^0 n$ no

data exists below 374 MeV, so it would be futile to try to draw any conclusions from it.

Two experimental cross sections exist for the process $\pi^+ p \rightarrow \pi^+ \pi^+ n$ at moderately low energies and these are tabulated in Table 1. The experimental cross section at 300 MeV is rather poorly determined. None of the models are within the error bars, but the tensor rank 3 model and--surprisingly--the ps(pv) model come closest with the tensor rank 2 model somewhat farther away. But the models of Weinberg and Schwinger and the minimal coupling model disagree badly with the data. The data at 357 MeV is somewhat above the region where we expect good results from chiral dynamics, but the data here seems to confirm the above conclusions about the different models.

A great deal of total cross section data exists for the process $\pi^- p \rightarrow \pi^+ \pi^- n$, much of it at low energies. Some of the better determined experimental points are tabulated in Table 2, along with the corresponding theoretical predictions. None of the models fits the cross sections at 210 MeV or 222 MeV, although Weinberg's model and the tensor rank 2 model come close. From 233 MeV to 290 MeV, however, the tensor rank 2 model fits the data quite well. Weinberg's model is consistently too small in this region, and the tensor rank 3 model is generally too large. The remaining two chiral dynamics models, Schwinger's model and the

Table 1. Experimental and theoretical total cross sections for $\pi^+p \rightarrow \pi^+\pi^+\eta$.
All cross sections in millibarns.

T_{π} (MeV)	Experiment	Tensor Rank 3 Model	Tensor Rank 2 Model	Weinberg Model	Schwinger Model	Minimal Coupling Model	ps(pv) Model
300	0.025 ± 0.018	0.053	0.065	0.110	0.145	0.187	0.047
357	0.12 ± 0.01	0.15	0.21	0.31	0.37	0.45	0.17

Table 2. Experimental and theoretical total cross sections for $\pi^-\rho \rightarrow \pi^+\pi^-\eta$.
All cross sections in millibarns.

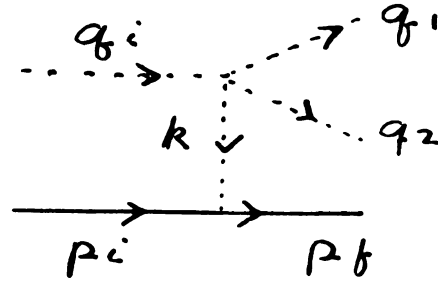
T π (MeV)	Experiment	Tensor Rank 3 Model	Tensor Rank 2 Model	Weinberg Model	Schwinger Model	Minimal Coupling Model	ps(pv) Model
210	0.015+0.003	0.064	0.024	0.008	0.004	0.002	0.0003
222	0.027+0.005	0.104	0.040	0.015	0.008	0.004	0.0008
233	0.053+0.013	0.147	0.059	0.025	0.011	0.008	0.001
245	0.10 +0.04	0.21	0.09	0.03	0.02	0.01	0.003
246	0.125+0.028	0.217	0.086	0.035	0.022	0.014	0.004
264	0.16 +0.06	0.29	0.13	0.06	0.04	0.03	0.008
288	0.28 +0.09	0.48	0.23	0.09	0.08	0.06	0.02
290	0.61 +0.13	0.50	0.21	0.11	0.08	0.07	0.02

minimal coupling model, are much too small throughout this energy region, and the ps(pv) model is completely negligible.

What little data there is for differential cross sections exists in the form of histograms of the angular distribution of the various final state particles. These histograms are exhibited along with the theoretical predictions in Figures 13 through 15. Unfortunately, statistical fluctuations in the histograms sometimes make it difficult to determine the angular dependency clearly. The experimental ordinate which was number of events has been converted to microbarns per steradian for easy comparison of theoretical and experimental numbers.

Differential cross sections have been measured for $\pi^- p \rightarrow \pi^0 \pi^- p$ at 450 MeV. This energy is rather high, but the angular dependence of the theoretical predictions is at least plausible. At 357 MeV there is differential cross section data for $\pi^+ p \rightarrow \pi^+ \pi^+ n$. The experimental data exhibits unmistakable angular dependencies for this process which are not matched very well by any of the chiral dynamics models, though the ps(pv) model gives a fairly good fit. But 357 MeV is still a bit above the region where we can expect chiral dynamics models to be valid, so we should refrain from making any judgments on the basis of these results.

Once again it appears we shall have to depend on data for the process $\pi^- p \rightarrow \pi^+ \pi^- n$, for which experimental differential cross sections have been measured at 290 MeV. The fact that the experimental total cross section data leaps from 0.28 ± 0.09 millibarns at 288 MeV to 0.61 ± 0.13 millibarns at 290 MeV, indicates that the experimental situation is in some doubt, so we would probably be well advised to concentrate mainly on the angular dependence of the cross sections, experimental and theoretical, rather than their actual magnitudes. Comparing theory and experiment for $d\sigma/d\Omega_{\pi^+}$, we see that just about any of the chiral dynamics models exhibit the right angular dependence. In the case of $d\sigma/d\Omega_{\pi^-}$, none of the chiral dynamics models predict the large backward scattering, but in general they exhibit the proper dependence for positive values of $\cos \theta_{\pi^-}$. But it is for the cross section $d\sigma/d\Omega_n$ that the most unmistakable angular dependence exists, and it is here that chiral dynamics has its most clearcut failure. While the experimental data shows strong forward peaking, the chiral dynamics models predict backward peaking, or at best, isotropy. The reason for this prediction is clear upon examination of the Feynman diagram below which makes the dominant contribution to the invariant amplitude.



Conservation of momentum at the vertex gives $\vec{p}_i + \vec{k} = \vec{p}_f$.

For low energies the exchanged pion will carry off only small momentum so that $\vec{p}_i \simeq \vec{p}_f$. By definition,

$$\cos \theta_n = \frac{\vec{q}_i \cdot \vec{p}_f}{|\vec{q}_i| |\vec{p}_f|} \simeq \frac{\vec{q}_i \cdot \vec{p}_i}{|\vec{q}_i| |\vec{p}_i|}$$

But in center of mass, $\vec{q}_i = -\vec{p}_i$ and $\cos \theta_n \simeq -1$,

hence this diagram will favor negative values of $\cos \theta_n$.

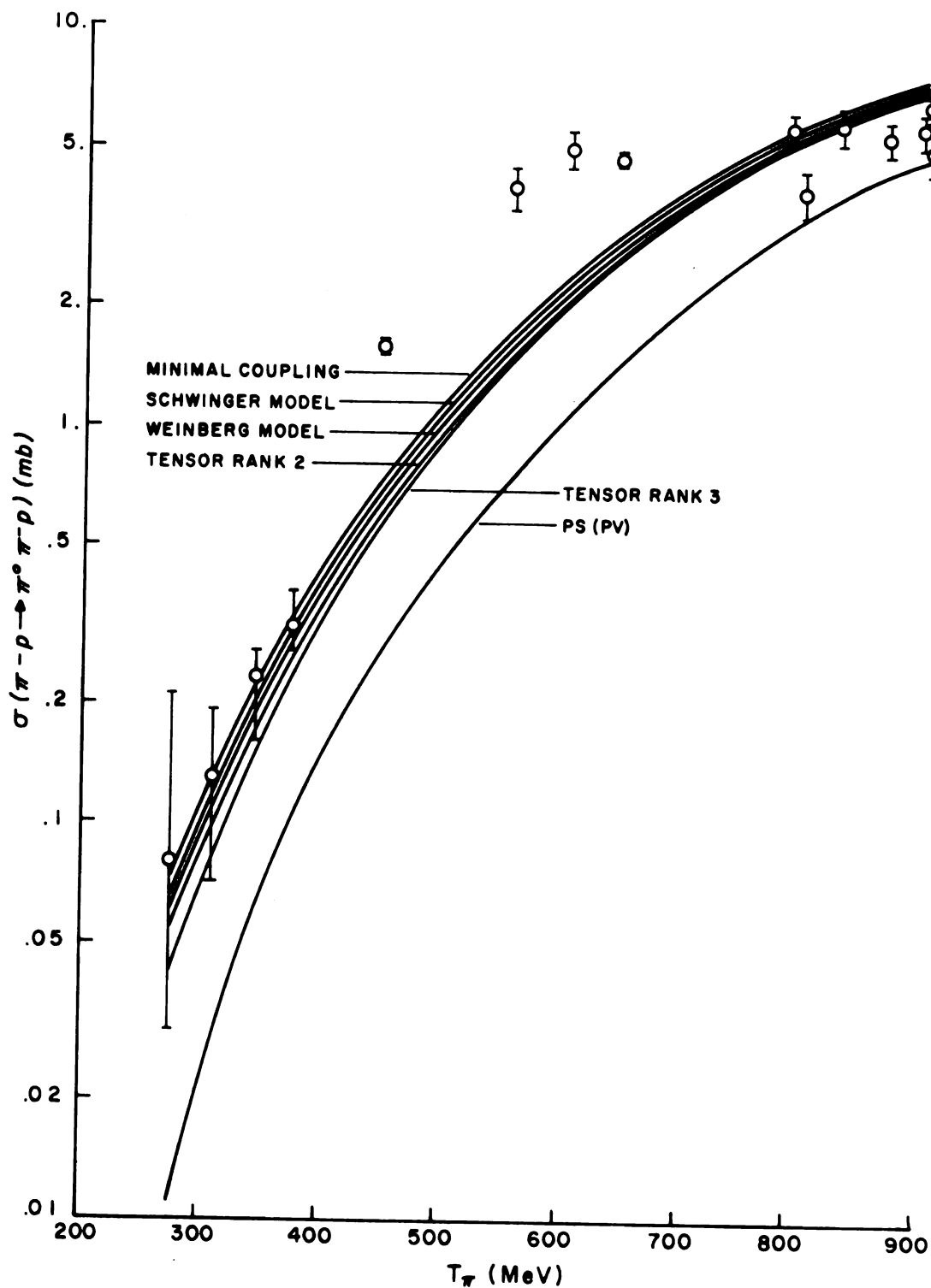


Figure 8. Total cross sections for $\pi^- p \rightarrow \pi^0 \pi^- p$

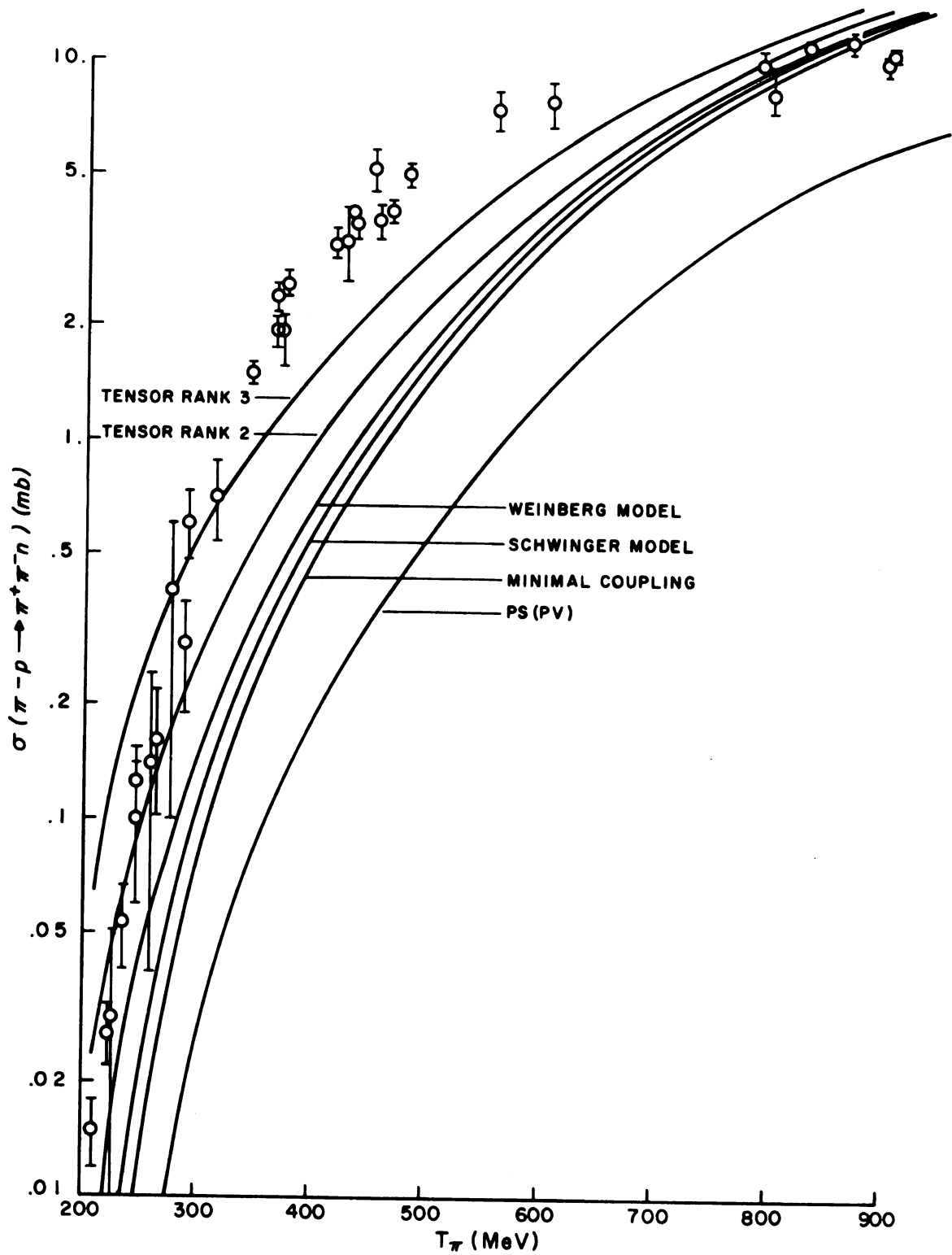


Figure 9. Total cross sections for $\pi^- p \rightarrow \pi^+ \pi^- n$

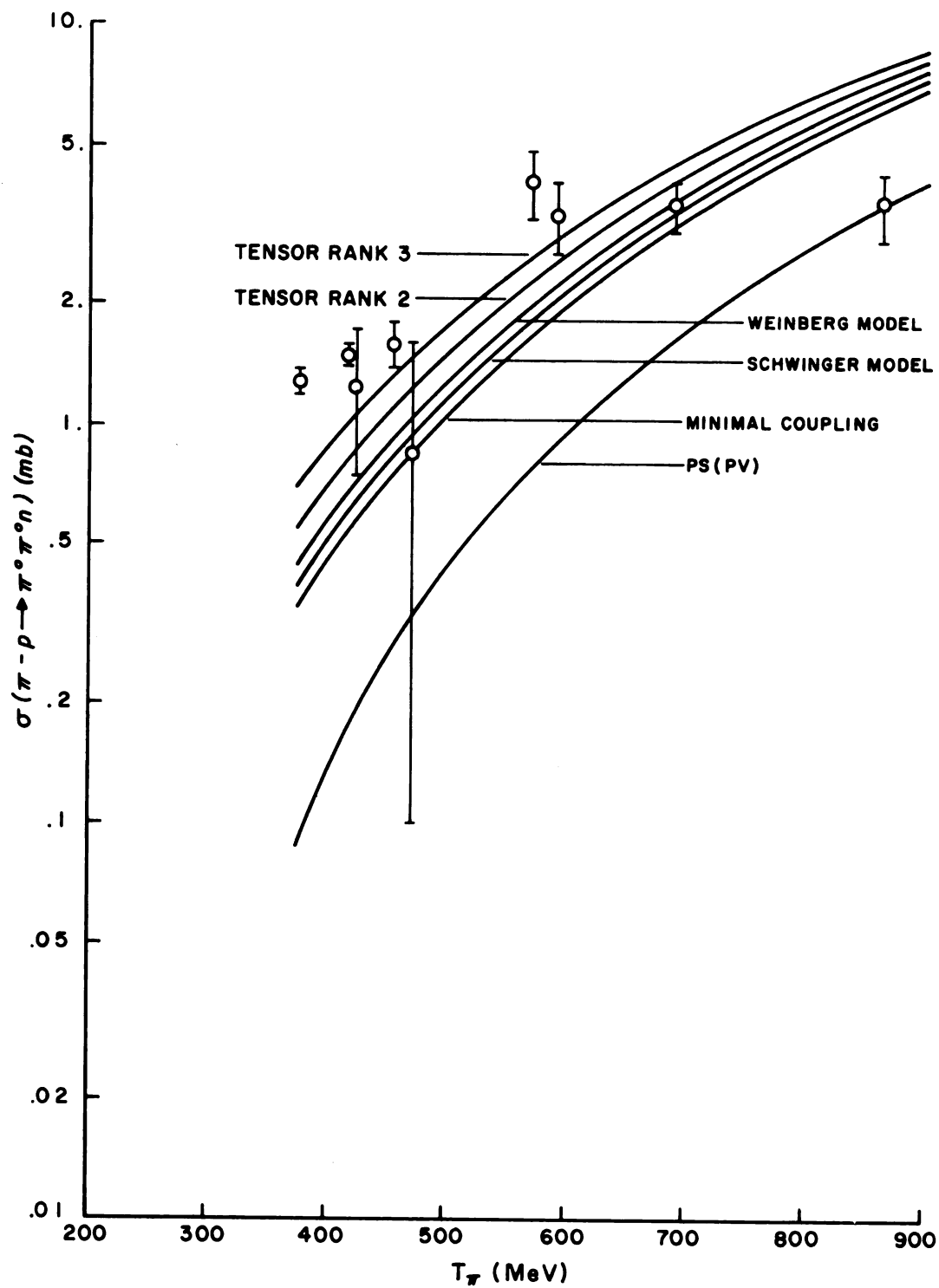


Figure 10. Total cross sections for $\pi^- p \rightarrow \pi^0 \pi^0 n$

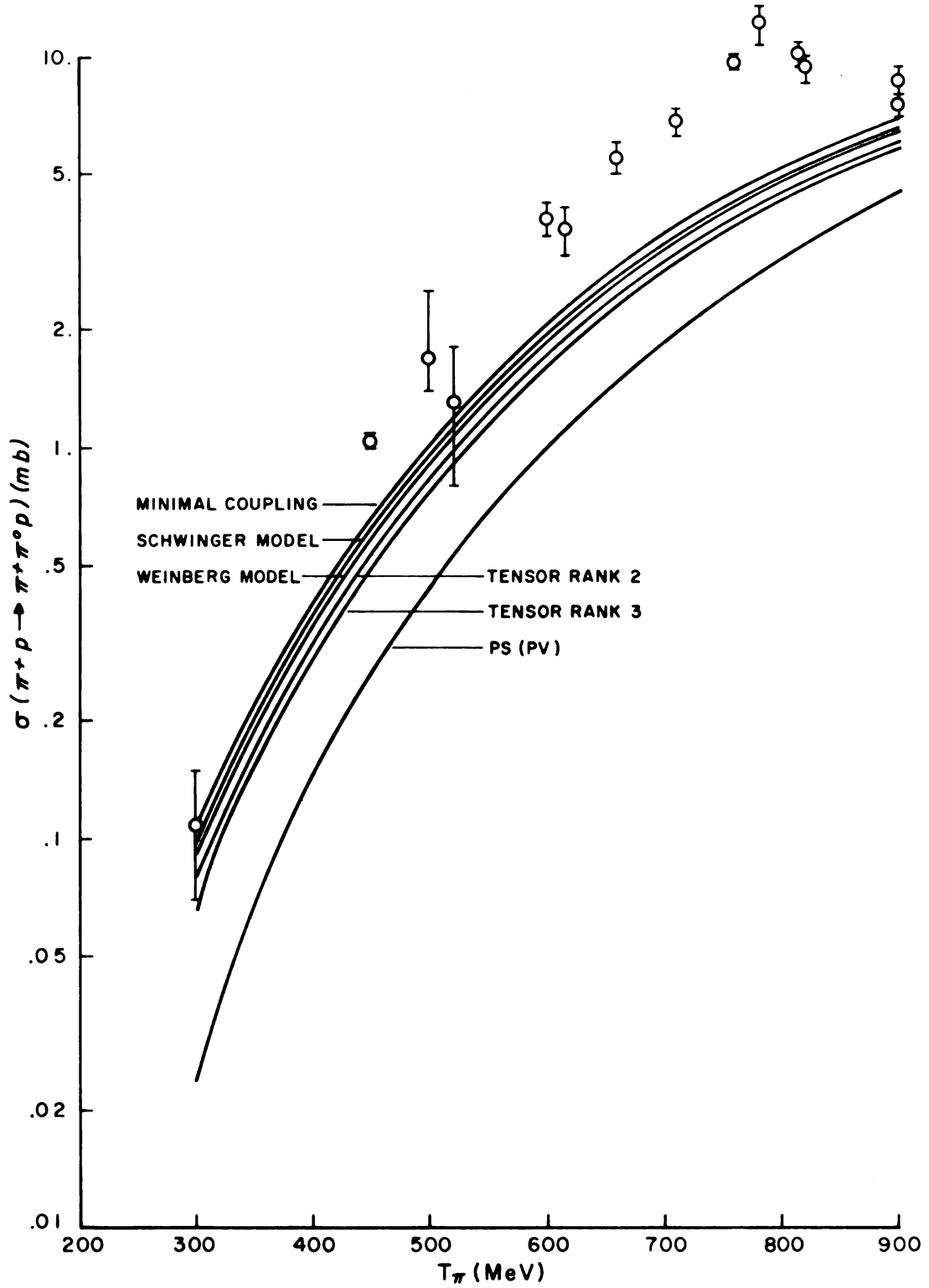


Figure 11. Total cross sections for $\pi^+ p \rightarrow \pi^+ \pi^0 p$

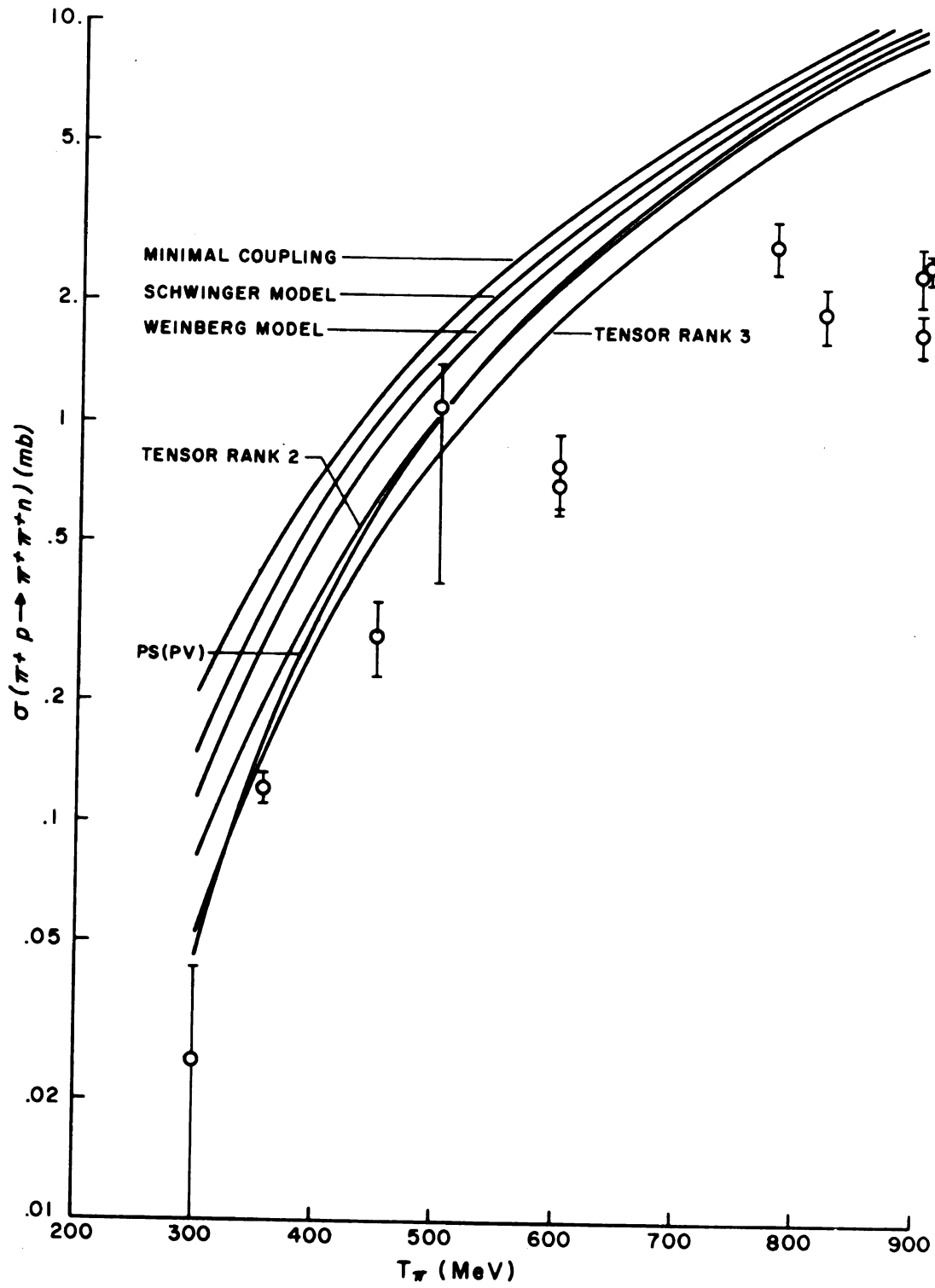


Figure 12. Total cross sections for $\pi^+ p \rightarrow \pi^+ \pi^+ n$

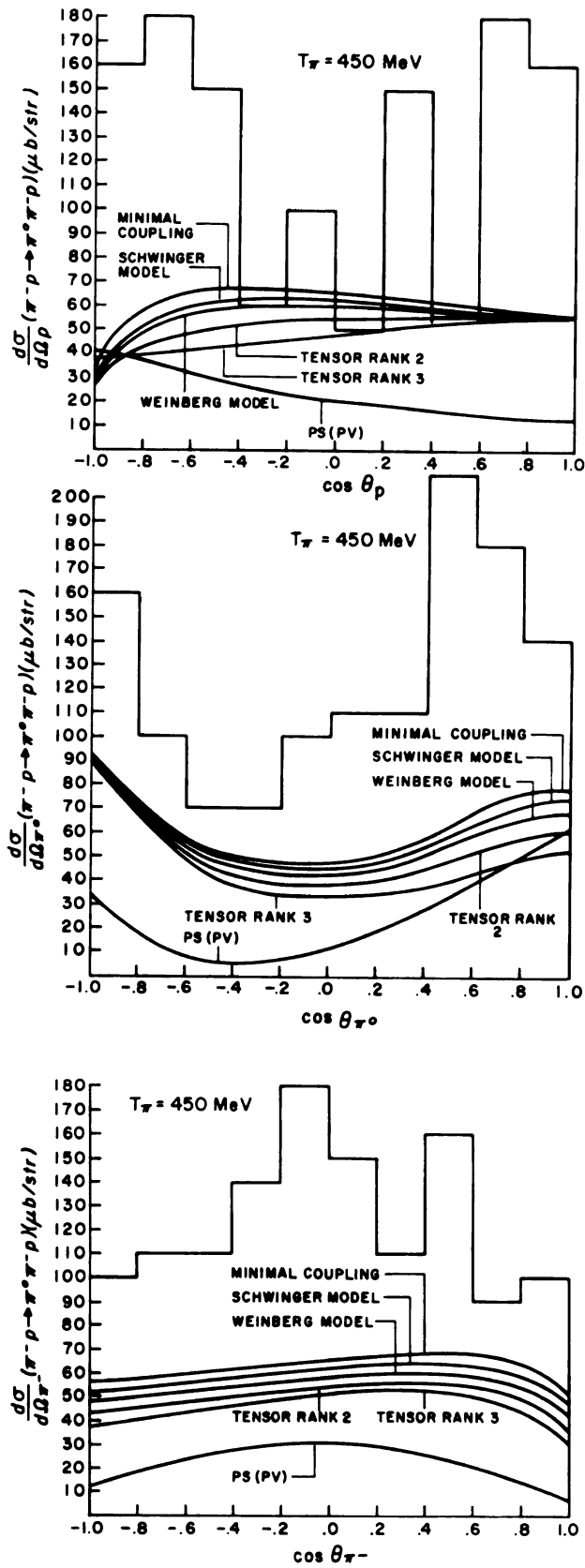


Figure 13. Differential cross sections for
 $\pi^- p \rightarrow \pi^0 \pi^- p$ at $T_\pi = 450$ MeV.

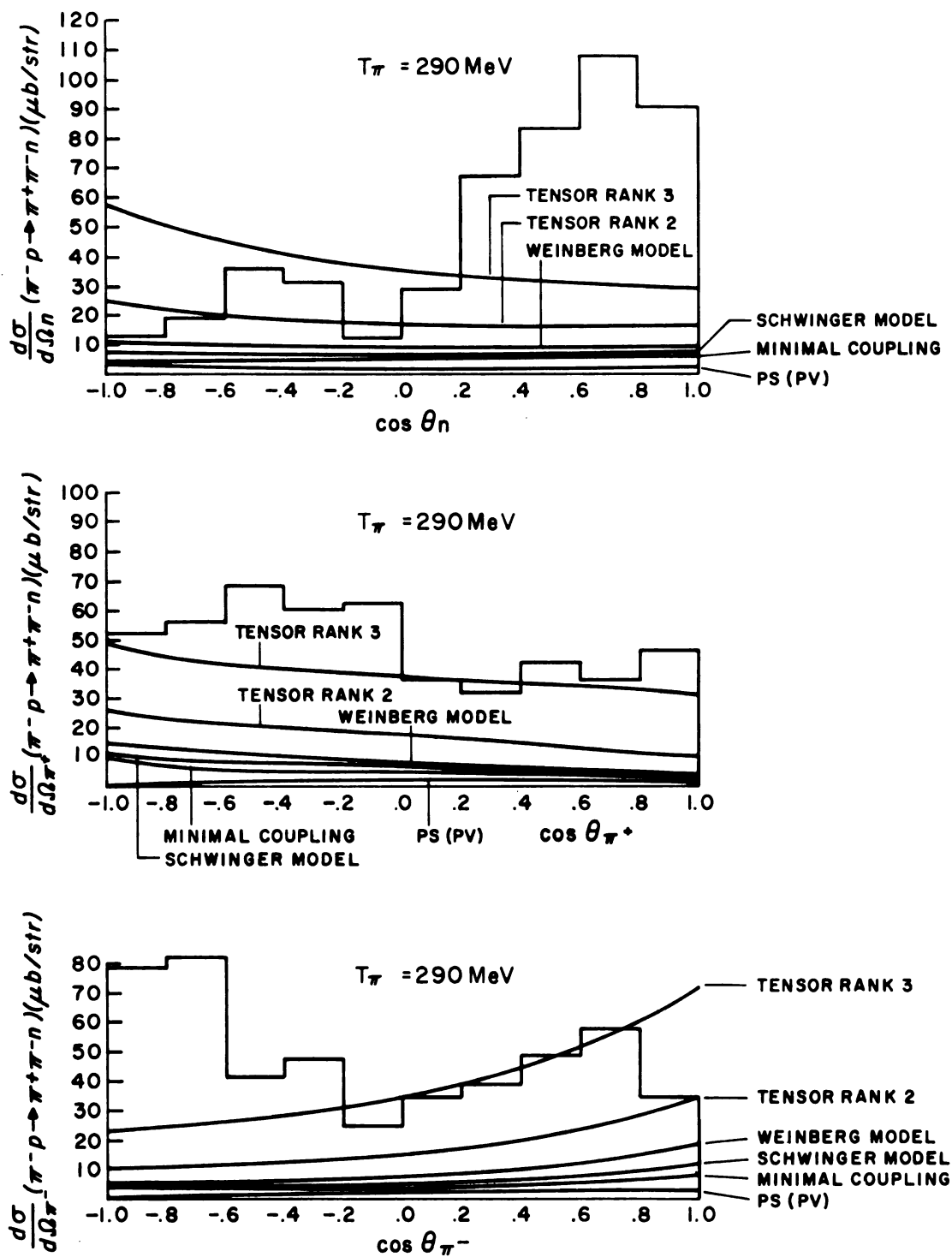


Figure 14. Differential cross sections for
 $\pi^- p \rightarrow \pi^+ \pi^- n$ at $T_\pi = 290 \text{ MeV}$.

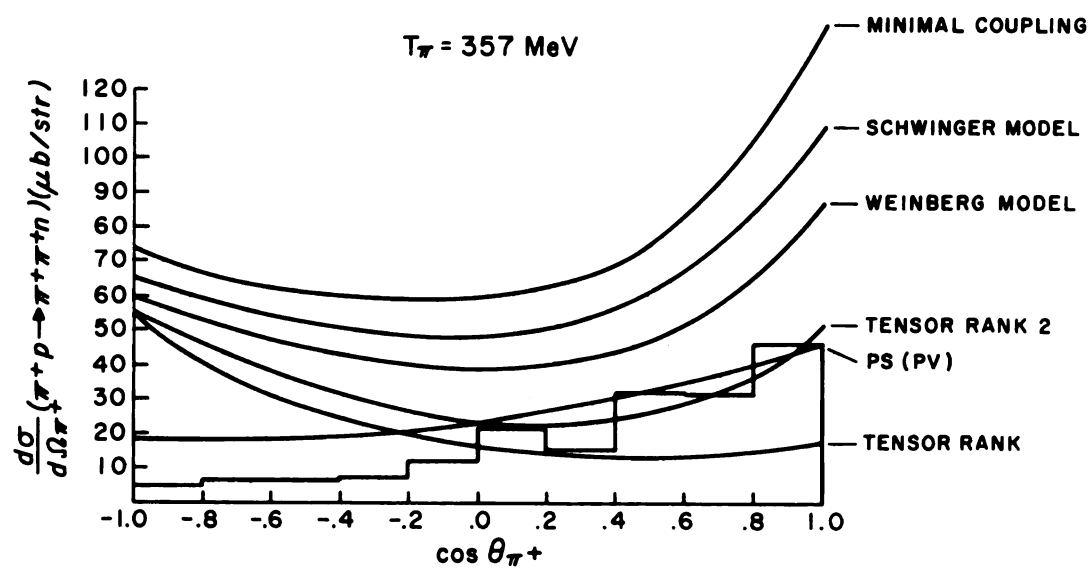
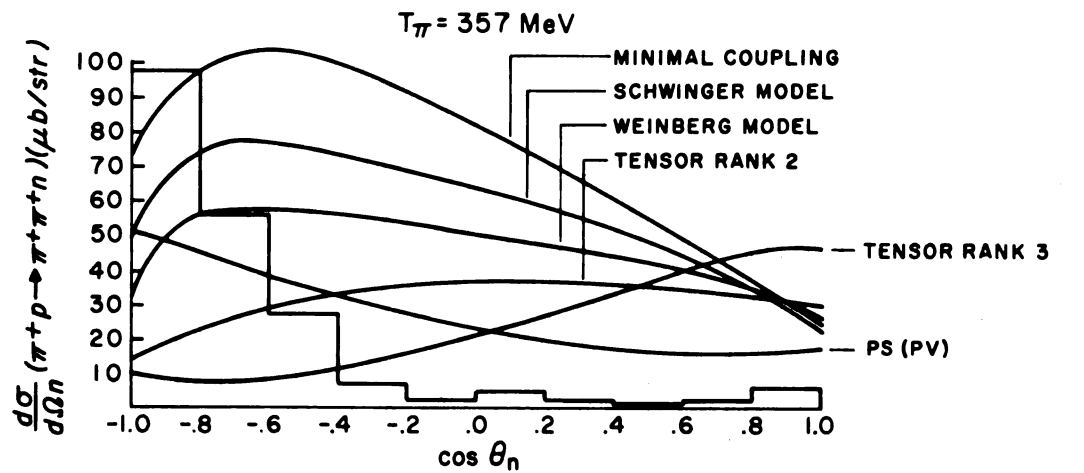


Figure 15. Differential cross sections for
 $\pi^+p \rightarrow \pi^+\pi^+n$ at $T_\pi = 357 \text{ MeV}$.

SECTION IX

CONCLUSIONS

It is difficult to know what to say about the various chiral dynamics models or the chiral dynamics method in general on the basis of the pion production predictions discussed in section VIII. Total cross section data for different charge channels does not unequivocally favor any of the models, and the differential cross section data seems to damn them all.

If anything is clear, though, it is that the evidence Olsson and Turner found for Weinberg's model is not convincing on closer examination. While they were correct in asserting that Weinberg's model works better than Schwinger's model in reproducing low energy total cross sections for $\pi^- p \rightarrow \pi^+ \pi^- n$, more careful calculations show that Weinberg's model is not much better than the tensor rank 2 model at the lowest energies, and is inferior to it throughout a range of higher energies. Because of this and because of the relatively good agreement the tensor rank 2 model gives for total cross sections of the process $\pi^+ p \rightarrow \pi^+ \pi^+ n$, not to mention the processes $\pi^+ p \rightarrow \pi^+ \pi^0 p$ and $\pi^- p \rightarrow \pi^0 \pi^- p$ where all the models give good results,

the tensor rank two model appears to be the single model giving the best all round description of total cross sections.

But though the chiral dynamics models seem fairly adequate to explain total cross sections, they are unable to account for differential cross sections which are more dependent on the detailed form of the invariant amplitude. This suggests that the models we've used can only be applied safely near threshold, where the small final state momenta make the exact structure of the invariant amplitude unimportant. This is a severe limitation.

The obvious first step in attempting to make the model more realistic above threshold would be the inclusion of resonances, especially the N^* resonance which is known to play an important part in the pion production process. Another possibility would be the inclusion of final state interactions. Only if these things succeeded in improving differential cross sections at low energies would it be reasonable to extend the calculation to higher energies.

Finally, we might hope for more low energy data to help determine the validity of the conclusions drawn above.

APPENDICES

APPENDIX A

GOLDBERGER TREIMAN RELATION

In investigations of the symmetry groups of the strong interaction, the usual goal has been to find unbroken symmetries. In fact, we have always had to settle for symmetries which are only approximate. But such broken symmetries may be just as useful as unbroken symmetries if one understands why they are broken. For chiral $SU(2) \otimes SU(2)$, information about the symmetry breaking comes from the PCAC hypothesis. To understand the raison d'etre of this hypothesis and its limitations, it is necessary to understand how it is used in derivation of the Goldberger-Treiman relation.

The PCAC hypothesis may be written

$$\partial^\mu \vec{J}_{5\mu}(\alpha) = C \mu^2 \vec{\phi}(\alpha) \quad (A.1)$$

where C is a proportionality constant. The technique used in deriving the Goldberger-Treiman relation is to take the matrix element of (A.1) between nucleon states to evaluate C , and then between a one pion state and a vacuum state to relate C to the pion decay amplitude.

First take the matrix element of the "plus" component of (A.1) between a neutron and a proton state:

$$\begin{aligned}
& \langle p(p_p) | \partial^\tau \vec{g}_{5\tau} \cdot \hat{\Phi}_+ | n(p_n) \rangle \\
&= \langle p(p_p) | \frac{\partial^\tau (g_{5\tau}^{(1)} + i g_{5\tau}^{(2)})}{\sqrt{2}} | n(p_n) \rangle \\
&= \langle p(p_p) | C \mu^2 \vec{\Phi} \cdot \hat{\Phi}_+ | n(p_n) \rangle.
\end{aligned} \tag{A.2}$$

The matrix element of the left hand side becomes, taking into account the form factors $F_A(q^2)$ and $F_P(q^2)$

$$\begin{aligned}
& \langle p(p_p) | \frac{\partial^\tau (g_{5\tau}^{(1)} + i g_{5\tau}^{(2)})}{\sqrt{2}} | n(p_n) \rangle \\
&= \frac{i(\vec{p}_p - \vec{p}_n) \bar{u}(p_p) [\gamma_\tau \gamma_5 F_A(q^2) + q_\tau \gamma_5 F_P(q^2)] u(p_n)}{(2\pi)^3 \sqrt{2}} \\
&= 2M \frac{[F_A(q^2) + q^2 F_P(q^2)]}{(2\pi)^3} \frac{\bar{u}(p_p) i \gamma_5 u(p_n)}{\sqrt{2}}
\end{aligned}$$

where $q^\tau \equiv p_p^\tau - p_n^\tau$ and we've used the identity $\not{p} u(p) = M u(p)$. Now take the limit as $q \rightarrow 0$ to obtain

$$\begin{aligned}
& \lim_{q \rightarrow 0} \langle p(p_p) | \frac{\partial^\tau (g_{5\tau}^{(1)} + i g_{5\tau}^{(2)})}{\sqrt{2}} | n(p_n) \rangle \\
&= 2M F_A(0) \frac{\bar{u}(p_p) i \gamma_5 u(p_n)}{\sqrt{2}}.
\end{aligned}$$

But $F_A(0) = \frac{g_A}{g_V}$, so finally

$$\begin{aligned}
& \lim_{q \rightarrow 0} \langle p(p_p) | \partial^\tau (g_{5\tau}^{(1)} + i g_{5\tau}^{(2)}) | n(p_n) \rangle \\
&= \frac{2M (g_A/g_V)}{(2\pi)^3} \frac{\bar{u}(p_p) i \gamma_5 u(p_n)}{\sqrt{2}}.
\end{aligned}$$

(A.3)

Now evaluate the matrix element of the pion field.

If we define

$$\vec{g}_\pi(x) \equiv (\square + \mu^2) \vec{\phi}(x)$$

then

$$\begin{aligned} & \langle p(p_p) | C \mu^2 \hat{\phi}_+ \cdot \vec{\phi} | n(p_n) \rangle \\ &= \frac{C \mu^2}{\mu^2 - q^2} \langle p(p_p) | \hat{\phi}_+ \cdot \vec{g}_\pi | n(p_n) \rangle \\ &= \frac{C \mu^2}{\mu^2 - q^2} \frac{\sqrt{2} g K_{NN\pi}(q^2)}{(2\pi)^3} \bar{u}(p_p) i \gamma_5 u(p_n) \end{aligned}$$

where $K_{NN\pi}(q^2)$ is the form factor of the pion-nucleon vertex. Again take the limit of $q \rightarrow 0$,

and obtain

$$\begin{aligned} & \lim_{q \rightarrow 0} \langle p(p_p) | C \mu^2 \hat{\phi}_+ \cdot \vec{\phi} | n(p_n) \rangle \\ &= \sqrt{2} C g \frac{K_{NN\pi}(0)}{(2\pi)^3} \bar{u}(p_p) i \gamma_5 u(p_n). \end{aligned}$$

Now $K_{NN\pi}(q^2)$ is normalized such that $K_{NN\pi}(\mu^2) = 1$, but we assume $K_{NN\pi}(0) \cong K_{NN\pi}(\mu^2)$ and so

$$\begin{aligned} & \lim_{q \rightarrow 0} \langle p(p_p) | C \mu^2 \hat{\phi}_+ \cdot \vec{\phi} | n(p_n) \rangle \\ & \cong \sqrt{2} C g \frac{\bar{u}(p_p) i \gamma_5 u(p_n)}{(2\pi)^3}. \end{aligned}$$

Using (A.3) and (A.4) in (A.2),

$$\begin{aligned} \frac{2M(q_A/q_V)}{(2\pi)^3 \sqrt{2}} \bar{u}(p_p) i \gamma_5 u(p_n) \\ = \frac{\sqrt{2} C q}{(2\pi)^3} \bar{u}(p_p) i \gamma_5 u(p_n) \end{aligned}$$

from which we obtain

$$C = \frac{M}{g} \frac{q_A}{q_V},$$

and (A.1) becomes

$$\partial^\sigma \vec{g}_{5\sigma}(x) = \frac{M}{g} \frac{q_A}{q_V} \mu^2 \vec{\phi}(x). \quad (\text{A.5})$$

Now take the matrix element of the "minus" component of (A.5) between a one pion state and a vacuum state.

$$\begin{aligned} \langle 1 | \partial^\sigma \vec{g}_{5\sigma} \cdot \hat{\phi}_- | \pi(q) \rangle \\ = \langle 1 | \partial^\sigma \frac{(\vec{g}_{5\sigma}^{(1)} - i \vec{g}_{5\sigma}^{(2)})}{\sqrt{2}} | \pi(q) \rangle \\ = \frac{M}{g} \frac{q_A}{q_V} \mu^2 \langle 1 | \hat{\phi}_- \cdot \vec{\phi} | \pi(q) \rangle \end{aligned} \quad (\text{A.6})$$

Define the pion decay amplitude $F_\pi(q^2)$ by

$$\langle 1 | \vec{g}_5^\sigma | \pi(q) \rangle = \frac{i M q^\sigma F_\pi(q^2)}{\sqrt{(2\pi)^3} 2\omega_q}.$$

Evaluating the left hand side of (A.6), then

$$\begin{aligned} \langle 1 | \partial^\sigma \frac{(\vec{g}_{5\sigma}^{(1)} - i \vec{g}_{5\sigma}^{(2)})}{\sqrt{2}} | \pi(q) \rangle \\ = -i q^\sigma \langle 1 | \frac{(\vec{g}_{5\sigma}^{(1)} - i \vec{g}_{5\sigma}^{(2)})}{\sqrt{2}} | \pi(q) \rangle \\ = \frac{M \mu^2 F_\pi(\mu^2)}{\sqrt{2} \sqrt{(2\pi)^3} 2\omega_q}. \end{aligned} \quad (\text{A.7})$$

For the right hand side of (A.6),

$$\frac{M}{g} \frac{g_A}{g_v} \mu^2 \langle |\vec{\phi} \cdot \vec{\phi}| \pi(q) \rangle = \frac{M}{g} \frac{g_A}{g_v} \frac{\mu^2}{\sqrt{(2\pi)^3 2\omega_q}}.$$

Substituting (A.7) and (A.8) in (A.6), we get

$$\frac{M \mu^2 F_\pi(\mu^2)}{\sqrt{2} \sqrt{(2\pi)^3 2\omega_q}} = \frac{M}{g} \frac{g_A}{g_v} \frac{\mu^2}{\sqrt{(2\pi)^3 2\omega_q}}$$

which reduces to

$$F_\pi(\mu^2) = \frac{\sqrt{2}}{g} \frac{g_A}{g_v}. \quad (\text{A.9})$$

Equation (A.9) is the Goldberger-Treiman relation. The key assumption in its derivation is that $K_{NN\pi}(q^2)$

is a slowly enough varying function that

$K_{NN\pi}(0) \cong K_{NN\pi}(\mu^2) = 1$. The most important thing

for our purposes is that none of the matrix elements

would be altered if instead of (A.5) we'd used

$$\partial^\mu \vec{J}_{5q}(\pi) = \frac{M}{g} \frac{g_A}{g_v} \mu^2 \vec{\phi}(\pi) \left[1 + \sum_{m=1}^{\infty} a_m (\vec{\phi}^2(\pi))^m \right]$$

where the a_m 's are arbitrary constants. This is true

because matrix elements of $\vec{\phi}(\vec{\phi}^2)^m, m \geq 1$, will

always vanish for the states used in the derivation.

Thus PCAC tells us what the divergence of the axial

vector current is to lowest order in the pion field, but

additional postulates must be made to determine higher

order terms.

APPENDIX B

WICK-DYSON REDUCTION TECHNIQUES

For actual calculations of scattering amplitudes, the well-known Wick-Dyson technique is generally much too cumbersome. It is useful, however, in approaching an unfamiliar Lagrangian because it takes care of symmetrizations, normalization factors, etc. quite mechanically. Because of this and because its use is necessary in obtaining the Feynman rules for a given interaction Lagrangian, the salient results of perturbation theory are summarized here.

Our object is to obtain the matrix element of the operator S between the initial and final states of the system in question. To do this, we employ the matrix expansion

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \cdots \int d^4x_1 \cdots d^4x_n \\ \times P\{\mathcal{H}_I(x_1) \cdots \mathcal{H}_I(x_n)\}$$

(B.1)

where P indicates a Dyson chronological product and $\mathcal{H}_I(x)$ is the interaction Hamiltonian density. The Dyson chronological product is defined

$$P\{A(x)B(y)\} = \begin{cases} A(x)B(y), & \text{if } x_0 > y_0 \\ B(y)A(x), & \text{if } y_0 > x_0. \end{cases}$$

The generalization of this definition for any number of operators is obvious. It is clear from the formula

$$\mathcal{K} = \sum_n \frac{\partial \mathcal{L}}{\partial \dot{\psi}_n} \dot{\psi}_n - \mathcal{L}$$

that if the interaction Lagrangian density contains no derivatives, $\mathcal{K}_I(x)$ may be replaced by $-\mathcal{L}_I(x)$ in (B.1). It is not clear that this replacement may be made if $\mathcal{L}_I(x)$ does contain derivatives, but in fact it can.* So we may write (B.1) as

$$S = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \dots \int d^4x_1 \dots d^4x_n \times P\{\mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n)\}. \quad (\text{B.2})$$

If we are dealing with a theory involving fermion fields, our formalism must use the Wick chronological product instead of the Dyson chronological product. The Wick product is defined by

$$T\{A(x_1)B(x_2)\dots\} = (-1)^P P\{A(x_1)B(x_2)\dots\}$$

where P is the number of interchanges of pairs of fermion fields necessary to change $\{A(x_1)B(x_2)\dots\}$ to chronological order. Since physical theories seem to always require Hamiltonians bilinear in fermion

*See appendix C.

operators, the Wick chronological product of any physical Hamiltonian will involve the interchange of even numbers of fermions. So the Wick product in (B.2) may be replaced by a T product and the S matrix expansion takes the form

$$S = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \cdots \int d^4x_1 \cdots d^4x_n \times T\{L_1(x_1) \cdots L_n(x_n)\}. \quad (B.3)$$

The form (B.3) is useful because of a theorem due to Wick relating the Wick chronological product of a set of operators and the normal product of the set of operators is defined by

$$N\{AB \cdots L\} = (-1)^P \{QR \cdots W\}$$

where $QR \cdots W$ are the operators $AB \cdots L$ reordered so that all annihilation operators stand to the right of all creation operators, and P is the number of interchanges of fermion operators. Let us further define the contraction of two operators A and B , by

$$\underline{AB} = \langle |N(AB)| \rangle.$$

Wick's theorem may now be written

$$\begin{aligned} T\{ABCD \cdots WXYZ\} &= N\{ABCD \cdots WXYZ\} \\ &+ N\{\underline{AB}CD \cdots WXYZ\} + \text{all other single contraction terms} \\ &+ N\{\underline{AB}\underline{CD} \cdots WXYZ\} + \text{all other double contraction terms} \\ &+ \cdots \end{aligned}$$

(B.4)

The contraction $\underline{\psi(x)\psi(y)}$ relates to a propagator connecting the points x and y . The contraction over boson fields is given by

$$\underline{\phi(x)\phi(y)} = i\Delta_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{1}{k^2 - m^2 + i\epsilon}$$

where m is the boson mass. The contraction over fermion fields is given by

$$\underline{\psi_\beta(x)\bar{\psi}_\alpha(y)} = -\underline{\bar{\psi}_\alpha(y)\psi_\beta(x)} = iS_{F\beta\alpha}(x-y)$$

where for the matrix $S_F(x-y)$, we have

$$\begin{aligned} S_F(x-y) &\equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{k + M}{k^2 - M^2 + i\epsilon} \\ &= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{1}{k - M + i\epsilon} \end{aligned}$$

M being the fermion mass. Contractions over unlike fields vanish. We may assume that the contraction of a derivative is the derivative of the contraction, e.g.

$$\underline{\psi(x)\frac{\partial}{\partial y}}\psi(y) = \frac{\partial}{\partial y}\left[\underline{\psi(x)\psi(y)}\right]^* \quad \text{The various terms in}$$

the Wick expansion may be placed in correspondence with Feynman diagrams. In applying the expansion, terms with equal time commutators are omitted, as well as terms corresponding to unconnected diagrams.

All we lack at this point in order to have all the equipment necessary to find $\langle f | S | i \rangle$ is the second quantization of the particle fields we'll use.

*See appendix C.

In our case this means the nucleon and pion fields. The nucleon fields $\Psi(x)$ and $\bar{\Psi}(x)$ have the quantization

$$\Psi(x) = \sum_{\mathbf{p}} \sum_{\omega} \int \frac{d^3 p}{(2\pi)^3} \chi_{\omega} \sqrt{\frac{M}{E_{\mathbf{p}}}} \\ \times [b(\mathbf{p}, \omega) u(\mathbf{p}, \omega) e^{-i p x} + d^\dagger(\mathbf{p}, \omega) v(\mathbf{p}, \omega) e^{i p x}]$$

(B.5)

$$\bar{\Psi}(x) = \sum_{\mathbf{p}} \sum_{\omega} \int \frac{d^3 p}{(2\pi)^3} \chi_{\omega}^\dagger \sqrt{\frac{M}{E_{\mathbf{p}}}} \\ \times [b^\dagger(\mathbf{p}, \omega) \bar{u}(\mathbf{p}, \omega) e^{i p x} + d(\mathbf{p}, \omega) \bar{v}(\mathbf{p}, \omega) e^{-i p x}].$$

(B.6)

Notation for spinors, isospin factors, etc. was established in section III. The creation and annihilation operators have the properties:

$b^\dagger(\mathbf{p}, \omega)$ creates a nucleon of four momentum $\mathbf{p}$,
with spin of z-component $S_z = \omega$, and
isospin of z-component $T_z = \omega$.

$b(\mathbf{p}, \omega)$ annihilates a fermion of momentum $\mathbf{p}$,
with spin of z-component $S_z = \omega$, and
isospin of z-component $T_z = \omega$.

$d^\dagger(\mathbf{p}, \omega)$ creates an anti-fermion of four momentum $\mathbf{p}$,
with spin of z-component $S_z = \omega$, and
isospin of z-component $T_z = \omega$.

$d(\mathbf{p}, \omega)$ annihilates an anti-fermion of four momentum
 $\mathbf{p}$, with spin of z-component $S_z = \omega$,
and isospin of z-component $T_z = \omega$.

These operators have the anti-commutators

$$\{b(\rho, \alpha, \omega), b^\dagger(\rho', \alpha', \omega')\} = \delta_{\alpha\alpha'} \delta_{\omega\omega'} \delta^3(\vec{\rho} - \vec{\rho}')$$

$$\{d(\rho, \alpha, \omega), d^\dagger(\rho', \alpha', \omega')\} = \delta_{\alpha\alpha'} \delta_{\omega\omega'} \delta^3(\vec{\rho} - \vec{\rho}')$$

all other anti-commutators vanishing. If we let Latin letters designate isospin state and Greek letters spinor components, the relevant propagator is

$$\underline{\Psi_{\alpha, \beta}(\kappa) \bar{\Psi}_{\tau, \sigma}(\gamma)} = i \delta_{\alpha\tau} S_{F\beta\sigma}(\kappa - \gamma).$$

The pion field $\vec{\phi}(\kappa)$ has quantization

$$\vec{\phi}(\kappa) = \sum_{t=-1,0,1} \hat{\phi}_t \int \frac{d^3 q}{\sqrt{(2\pi)^3 2\omega_q}} \times [a^\dagger(q, -t) e^{i q \cdot \kappa} + a(q, t) e^{-i q \cdot \kappa}]$$

(B.7)

where the unit vectors $\hat{\phi}_t$ are defined in section III.

The creation and annihilation operators have the properties:

$a^\dagger(q, t)$ creates a pion with four momentum q and isospin of z-component $T_z = t$.

$a(q, t)$ annihilates a pion with four momentum q and isospin of z-component $T_z = t$.

These operators have the commutator

$$[a(q, t), a^\dagger(q', t')] = \delta_{tt'} \delta^3(\vec{q} - \vec{q}')$$

all other commutators vanishing. Let $\hat{\phi}$ and $\hat{n}$ be unit vectors in isospin space. The contraction we'll need is then

$$\hat{\phi} \cdot \vec{\phi}(x) \hat{n} \cdot \vec{\phi}(y) = i \hat{\phi} \cdot \hat{n} \Delta_F(x-y)$$

We are now equipped to find the matrix elements of S for any nucleon-pion Lagrangian by means of a mechanical and foolproof procedure. First insert the interaction Lagrangian $\mathcal{L}_I(x)$ in the S matrix expansion (B.3). Replace $\Psi(x)$, $\bar{\Psi}(x)$, and $\vec{\phi}(x)$ by (B.5) through (B.7). Expand the Wick chronological product by means of Wick's theorem. Now sandwich the S operator between a bra $\langle f |$ and a ket $| i \rangle$ representing final and initial states. For the process

$$\pi^-(q_i) + p(p_i, a_i) \rightarrow \pi^+(q_1) + \pi^-(q_2) + n(p_f, a_f)$$

for example, $\langle f |$ and $| i \rangle$ are written

$$| i \rangle = | \pi^-(q_i), p(p_i, a_i) \rangle = a^+(q_i, -1) b^+(p_i, a_i, \frac{1}{2}) | \rangle$$

$$\begin{aligned} \langle f | &= \langle \pi^+(q_1), \pi^-(q_2), n(p_f, a_f) | \\ &= \langle | a(q_1, 1) a(q_2, -1) b(p_f, a_f, -\frac{1}{2}) \end{aligned}$$

From this point on, just simplify the resultant expression by use of the commutation relations of the creation and annihilation operators.

The matrix element $S_{fi} \equiv \langle f | S | i \rangle$ is related to the invariant amplitude $\mathcal{M}$ by

$$S_{fi} = \frac{(2\pi)^{\frac{2-3n}{2}} \delta^4(p_1 + p_2 - \sum_{i=1}^n k_i) \mathcal{M}}{\left[\frac{1}{2\omega_{p_1}} \frac{1}{2\omega_{p_2}} \frac{1}{2\omega_1} \cdots \frac{1}{2\omega_n} \right]}$$

where we scatter from an initial state of two bosons of four momenta p_1 and p_2 to a state of n bosons of four momenta $k_1, \dots, k_n$. For fermions of mass $\mathcal{M}$, $\frac{1}{2\omega}$ is replaced by $\frac{\mathcal{M}}{E}$. The method of going from the amplitude to cross sections is discussed in section VII.

APPENDIX C

DERIVATIVE COUPLINGS

Most modern books on field theory slight the topic of derivative couplings, probably because they are not renormalizable and do not fit easily in the usual canonical formalism. As an illustration of the inadequacy of the canonical formalism, let us specialize our considerations to a $\psi(pv)$ interaction Lagrangian density

$$\mathcal{L}_I(x) = \frac{f}{\mu} \bar{\Psi}(x) \gamma_5 \gamma^\mu \partial_\mu \phi(x) \Psi(x). \quad (C.1)$$

Here f is a unitless coupling constant. For simplicity, we've omitted the complications introduced by isospin. Now the canonical formalism tells us that the interaction Hamiltonian density corresponding to the above Lagrangian is obtained from the formula

$$\mathcal{H}_I = \frac{\partial \mathcal{L}_I}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L}_I$$

The result when this is applied to (C.1) is

$$\mathcal{H}_I(x) = \frac{f}{\mu} \bar{\Psi}(x) \gamma_5 \vec{\nabla} \cdot \vec{\nabla} \phi(x) \Psi(x) \quad (C.2)$$

Now the Lagrangian of (C.1) is Lorentz covariant, as it must be, but the Hamiltonian obtained from it and given

by (C.2) is not covariant. If employed as it stands, in Wick-Dyson reductions (C.2) will lead to formulae for the invariant amplitude which are not, in fact, invariant. The reason for this paradox lies in the canonical formalism. The formalism is itself not completely covariant because it treats time and space components of four vectors on different footings.

To remedy this difficulty, field theory must be formulated in terms of spacelike surfaces, as Dyson³³ or Umezawa³⁴ do. Such a procedure greatly complicates the formalism and will not be discussed here. But the result of applying this formalism to our $ps(pv)$ Lagrangian is to show the interaction Hamiltonian must be written in the covariant form

$$\begin{aligned} \mathcal{H}_I(\alpha) = & - \frac{1}{\hbar} \bar{\Psi}(\alpha) \gamma_5 \gamma^\tau \phi(\alpha) \Psi(\alpha) \\ & + \frac{1}{2} \left(\frac{1}{\hbar} \right)^2 \left[\bar{\Psi}(\alpha) \gamma_5 \gamma^\tau n_\tau \Psi(\alpha) \right]^2 \end{aligned} \quad (C.3)$$

where n_τ is the normal to a family of spacelike surfaces.³⁵ P.J. Matthews³⁶ has shown that we can use this Hamiltonian in perturbation theory omitting the term dependent on n_τ so that even for derivative couplings we may take $\mathcal{H}_I = - \mathcal{L}_I$ provided we also assume the contraction of a derivative equals the derivative of a contraction.

Matthews proved this for any process generated by the Hamiltonian (C.3). For clarity, let's restrict our

attention to fermion-fermion elastic scattering

$$F_1(p_{i1}) + F_2(p_{i2}) \rightarrow F_3(p_{f1}) + F_4(p_{f2})$$

Our Hamiltonian generates three diagrams for this process which are illustrated in Figure 16.

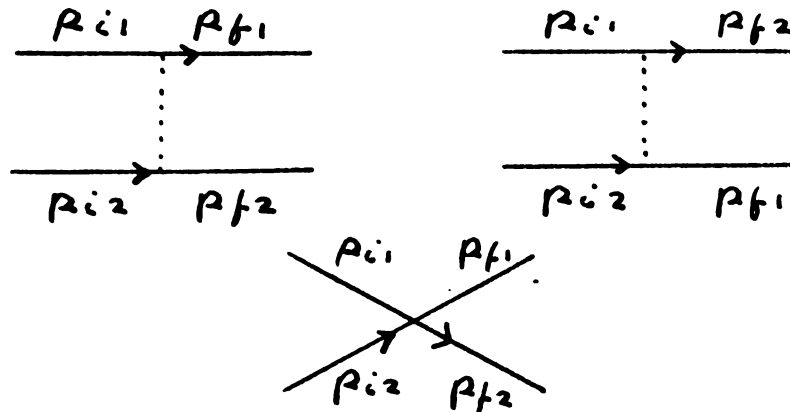


Figure 16. Diagrams for elastic scattering of two fermions in ps(pv) theory.

Taking matrix elements of the S matrix

$$\begin{aligned}
 \langle f | S | i \rangle &= \langle f | \left\{ -i \int d^4x T \left[\frac{1}{2} \left(\frac{f}{\hbar} \right)^2 (\bar{\Psi}(x) \gamma_5 \Psi(x))^2 \right] \right. \\
 &\quad + \frac{(-i)^2}{2} \iint d^4x d^4y T \left[\left(-\frac{f}{\hbar} \right) \bar{\Psi}(x) \gamma_5 \Psi^* \frac{\partial \phi(x)}{\partial x^\alpha} \Psi(x) \right. \\
 &\quad \left. \left. \times \left(-\frac{f}{\hbar} \right) \bar{\Psi}(y) \gamma_5 \Psi^* \frac{\partial \phi(y)}{\partial y^\beta} \Psi(y) \right] \right\} | i \rangle
 \end{aligned}$$

$$\begin{aligned}
\langle f | S | i \rangle &= \langle f | \left\{ -i \int d^4x N \left[\frac{1}{2} \left(\frac{1}{f} \right)^2 (\bar{\Psi}(x) \gamma_5 \Psi(x))^2 \right] \right. \\
&\quad \left. - \frac{1}{2} \iint d^4x d^4y \frac{\partial \phi(x)}{\partial x^\alpha} \frac{\partial \phi(y)}{\partial y^\beta} \right. \\
&\quad \left. \times N \left[\left(\frac{1}{f} \right)^2 \Psi(x) \gamma_5 \gamma^\alpha \Psi(x) \bar{\Psi}(y) \gamma_5 \gamma^\beta \Psi(y) \right] \right\} | i \rangle.
\end{aligned} \tag{C.4}$$

We know how to evaluate the normal products in this expression, but it is not clear how to evaluate the contraction. To evaluate it, we start from the definition

$$\underline{\frac{\partial \phi(x)}{\partial x^\alpha} \frac{\partial \phi(y)}{\partial y^\beta}} = \langle | P \left(\frac{\partial \phi(x)}{\partial x^\alpha} \frac{\partial \phi(y)}{\partial y^\beta} \right) | \rangle. \tag{C.5}$$

Let's define a function $\varepsilon(z)$ by

$$\varepsilon(z) = \begin{cases} 1 & \text{for } z_0 > 0 \\ -1 & \text{for } z_0 < 0. \end{cases}$$

With this function we may write a Dyson chronological product as

$$\begin{aligned}
P(A(x)B(y)) &= \left[\frac{1 + \varepsilon(x-y)}{2} \right] A(x) B(y) \\
&\quad + \left[\frac{1 - \varepsilon(x-y)}{2} \right] B(y) A(x).
\end{aligned}$$

With this identity, (C.5) becomes

$$\begin{aligned}
\underline{\frac{\partial \phi(x)}{\partial x^\alpha} \frac{\partial \phi(y)}{\partial y^\beta}} &= \langle | \left(\frac{1}{2} \left\{ \frac{\partial \phi(x)}{\partial x^\alpha}, \frac{\partial \phi(y)}{\partial y^\beta} \right\} \right. \right. \\
&\quad \left. \left. + \frac{\varepsilon(x-y)}{2} \left[\frac{\partial \phi(x)}{\partial x^\alpha}, \frac{\partial \phi(y)}{\partial y^\beta} \right] \right) | \rangle
\end{aligned}$$

$$\begin{aligned} \frac{\partial \phi(x)}{\partial x_\alpha} \frac{\partial \phi(y)}{\partial y_\beta} &= \frac{1}{2} \frac{\partial^2}{\partial x_\alpha \partial y_\beta} \langle | \{ \phi(x), \phi(y) \} | \rangle \\ &\quad + \frac{\epsilon(x-y)}{2} \frac{\partial^2}{\partial x_\alpha \partial y_\beta} \langle | [\phi(x), \phi(y)] | \rangle. \end{aligned}$$

(C.6)

It was possible to bring the derivatives out in front of the vacuum ket because the derivative doesn't operate in occupation number space. Using the identities of Drell and Bjorken,³⁷

$$\begin{aligned} \langle | [\phi(x), \phi(y)] | \rangle &= i \Delta(x-y) \\ \langle | \{ \phi(x), \phi(y) \} | \rangle &= \Delta_1(x-y) \end{aligned}$$

and (C.6) becomes

$$\begin{aligned} \frac{\partial \phi(x)}{\partial x_\alpha} \frac{\partial \phi(y)}{\partial y_\beta} &= \frac{1}{2} \frac{\partial^2 \Delta_1(x-y)}{\partial x_\alpha \partial y_\beta} \\ &\quad + \frac{i \epsilon(x-y)}{2} \frac{\partial^2 \Delta(x-y)}{\partial x_\alpha \partial y_\beta}. \end{aligned} \quad (C.7)$$

Using the identity

$$2 \Delta_F(x-y) = -i \Delta_1(x-y) + \epsilon(x-y) \Delta(x-y)$$

where $\Delta_F(x-y) = \underline{\phi(x) \phi(y)}$, to eliminate $\Delta_1(x-y)$ from (C.7), we get

$$\begin{aligned}
\frac{\partial \phi(x)}{\partial x_\alpha} \frac{\partial \phi(y)}{\partial y_\beta} &= i \frac{\partial^2 \Delta_F(x-y)}{\partial x_\alpha \partial y_\beta} \\
&\quad - \frac{i}{2} \left\{ \Delta(x-y) \frac{\partial^2 \epsilon(x-y)}{\partial x_\alpha \partial y_\beta} \right. \\
&\quad \left. + \frac{\partial \epsilon(x-y)}{\partial x_\alpha} \frac{\partial \Delta(x-y)}{\partial y_\beta} + \frac{\partial \epsilon(x-y)}{\partial y_\beta} \frac{\partial \Delta(x-y)}{\partial x_\alpha} \right\}.
\end{aligned}$$

Now we must evaluate the expression in curly brackets.

First find the derivative of $\epsilon(x-y)$. Because $\epsilon(x-y)$ only depends on the time component of $x-y$,

$$\frac{\partial \epsilon(x-y)}{\partial x_\alpha} = g_{\alpha 0} \frac{\partial \epsilon(x-y)}{\partial x_0}.$$

Because of the definition of $\epsilon(x-y)$,

$$\frac{\partial \epsilon(x-y)}{\partial x_0} = 0 \quad \text{if} \quad x_0 \neq y_0.$$

But for any positive number a ,

$$\int_{y_0-a}^{y_0+a} dx_0 \frac{\partial \epsilon(x-y)}{\partial x_0} = \epsilon(a) - \epsilon(-a) = 2$$

hence

$$\frac{\partial \epsilon(x-y)}{\partial x_0} = 2 \delta(x_0 - y_0)$$

and

$$\frac{\partial \epsilon(x-y)}{\partial x_\alpha} = 2 g_{\alpha 0} \delta(x_0 - y_0).$$

(C.8)

Using (C.8) and the fact that $\Delta(z) = -\Delta(-z)$, we get

$$\begin{aligned} \frac{\partial \mathcal{E}(\kappa - \gamma)}{\partial \kappa_\alpha} \frac{\partial \Delta(\kappa - \gamma)}{\partial \gamma_\beta} &= 2g_{\alpha 0} \delta(\kappa_0 - \gamma_0) \frac{\partial \Delta(\kappa - \gamma)}{\partial \gamma_\beta} \\ &= -2g_{\alpha 0} \delta(\kappa_0 - \gamma_0) \left[\frac{\partial \Delta(\gamma - \kappa)}{\partial \gamma_\beta} \right]_{\kappa_0 = \gamma_0}. \end{aligned}$$

(C.9)

But $\left[\frac{\partial \Delta(\gamma - \kappa)}{\partial \gamma_\beta} \right]_{\kappa_0 = \gamma_0} = 0 ; \beta = 1, 2, 3$

and $\left[\frac{\partial \Delta(\gamma - \kappa)}{\partial \gamma_0} \right]_{\kappa_0 = \gamma_0} = -\delta^3(\vec{\gamma} - \vec{\kappa})$

as may be shown using the explicit expression

$$\Delta(\gamma) = -i \int \frac{d^3 k}{(2\pi)^3 2\omega_k} (e^{-ik\gamma} - e^{ik\gamma}).$$

So in general,

$$\frac{\partial \Delta(\gamma - \kappa)}{\partial \gamma_\beta} = -g_{\beta 0} \delta^3(\vec{\gamma} - \vec{\kappa})$$

and combining (C.8) and (C.9)

$$\frac{\partial \mathcal{E}(\kappa - \gamma)}{\partial \kappa_\alpha} \frac{\partial \Delta(\kappa - \gamma)}{\partial \gamma_\beta} = 2g_{\alpha 0} g_{\beta 0} \delta^4(\kappa - \gamma). \quad (C.10)$$

Likewise

$$\frac{\partial \mathcal{E}(\kappa - \gamma)}{\partial \gamma_\beta} \frac{\partial \Delta(\kappa - \gamma)}{\partial \kappa_\alpha} = 2g_{\alpha 0} g_{\beta 0} \delta^4(\kappa - \gamma). \quad (C.11)$$

Finally, using (C.8),

$$\Delta(\kappa - \gamma) \frac{\partial^2 \mathcal{E}(\kappa - \gamma)}{\partial \kappa_\alpha \partial \gamma_\beta} = -2g_{\beta 0} \Delta(\kappa - \gamma) \frac{\partial \delta(\gamma_0 - \kappa_0)}{\partial \kappa_\alpha}.$$

But in general

$$f(x) \frac{d}{dx} \delta(x-a) = -\delta(x-a) \frac{df(x)}{dx}$$

and so

$$\Delta(x-y) \frac{\partial^2 \varepsilon(x-y)}{\partial x_\alpha \partial y_\beta} = g_{\beta 0} \delta(y_0 - x_0) \frac{\partial \Delta(x-y)}{\partial x_\alpha}$$

and our result is

$$\Delta(x-y) \frac{\partial^2 \varepsilon(x-y)}{\partial x_\alpha \partial y_\beta} = -2g_{\alpha 0} g_{\beta 0} \delta^4(x-y). \quad (C.12)$$

Substituting (C.10), (C.11), and (C.12) in (C.7) and simplifying, we arrive at

$$\frac{\partial \phi(x)}{\partial x_\alpha} \frac{\partial \phi(y)}{\partial y_\beta} = i \frac{\partial^2 \Delta_F(x-y)}{\partial x_\alpha \partial y_\beta} - i g_{\alpha 0} g_{\beta 0} \delta^4(x-y).$$

Generalizing from the "normal" spacelike surface implicit in the identities used to derive (C.9), (C.10), and (C.11), our final expression for the contraction in question is

$$\frac{\partial \phi(x)}{\partial x_\alpha} \frac{\partial \phi(y)}{\partial y_\beta} = i \frac{\partial^2 \Delta_F(x-y)}{\partial x_\alpha \partial y_\beta} - i m_\alpha m_\beta \delta^4(x-y). \quad (C.13)$$

Now, at long last, we can return to our expression for the S matrix, (C.4). Substituting (C.13) into (C.4), we get

$$\begin{aligned}
\langle f | S | i \rangle &= \langle f | \left\{ -i \int d^4x N \left[\frac{1}{2} \left(\frac{1}{f_A} \right)^2 \left(\bar{\Psi}(x) \gamma_5 \not{x} \Psi(x) \right)^2 \right] \right. \\
&\quad \left. - \frac{1}{2} \iint d^4x d^4y \left(i \frac{\partial^2 \Delta_F(x-y)}{\partial x_\alpha \partial x_\beta} - i m_\alpha m_\beta \delta^4(x-y) \right) \right. \\
&\quad \left. \times N \left[\left(\frac{1}{f_A} \right)^2 \bar{\Psi}(x) \gamma_5 \gamma_\alpha \Psi(x) \bar{\Psi}(y) \gamma_5 \gamma_\beta \Psi(y) \right] \right\} | i \rangle \\
&= \langle f | \left\{ -\frac{i}{2} \iint d^4x d^4y \frac{\partial^2 \Delta_F(x-y)}{\partial x_\alpha \partial y_\beta} \right. \\
&\quad \left. \times N \left[\left(\frac{1}{f_A} \right)^2 \bar{\Psi}(x) \gamma_5 \gamma_\alpha \Psi(x) \bar{\Psi}(y) \gamma_5 \gamma_\beta \Psi(y) \right] \right\} | i \rangle.
\end{aligned}$$

We see that terms dependent on m_α have cancelled one another. In terms of the diagrams of Figure 16, terms in the propagators of the first two diagrams have just cancelled the last diagram. We could have obtained the same result by assuming

$$\underbrace{\frac{\partial \phi(x)}{\partial x_\alpha} \frac{\partial \phi(y)}{\partial y_\beta}} = \frac{\partial^2}{\partial x_\alpha \partial y_\beta} [\underbrace{\phi(x) \phi(y)}] \quad (\text{C.14})$$

$$\mathcal{K}_I(x) = - \mathcal{I}_I(x) \quad (\text{C.15})$$

Neither (C.14) nor (C.15) is actually correct, but as long as we are using perturbation theory we may pretend they are. It is a case where two wrongs do make a right. Though we've only dealt with ps(pv) coupling, we may assume (C.14) and (C.15) to hold quite generally.

APPENDIX D MONTE CARLO CALCULATIONS

Integration by means of Monte Carlo methods rests on two theorems which are here stated without proof.³⁸

THEOREM I. (The Strong Law of Large Numbers)

If $\rho(x_1, \dots, x_m)$ is a probability density function on a region R such that $\rho(x_1, \dots, x_m) \geq 0$, and $\int_R \rho(x_1, \dots, x_m) dx_1 \dots dx_m = 1$, and $\bar{f}$ is the integral

$$\bar{f} = \int_R f(x_1, \dots, x_m) \rho(x_1, \dots, x_m) dx_1 \dots dx_m,$$

and $\hat{f}_N$ is a random number defined by

$$\hat{f}_N = \frac{1}{N} \sum_{i=1}^N f(x_{1i}, \dots, x_{mi})$$

the set of numbers $(x_{1i}, \dots, x_{mi})$ being the i th set of values chosen at random from R according to the probability density $\rho(x_1, \dots, x_m)$, then

$$\lim_{N \rightarrow \infty} \hat{f}_N = \bar{f}.$$

THEOREM II. (Central Limit Theorem)

For large N the probability that $\bar{f} - \delta \leq \hat{f}_N \leq \bar{f} + \delta$, is independent of the exact nature of $f(x_1, \dots, x_m)$

and $\rho(\alpha_1, \dots, \alpha_m)$, but depends only on N and the variance $G^2 = \overline{f^2} - \bar{f}^2$, in fact

$$\text{Probability} \left\{ \hat{f}_N \leq \bar{f} + \delta \right\} = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\frac{\delta \sqrt{N}}{G}} e^{-\frac{x^2}{2}} dx + \text{terms of order } \frac{1}{\sqrt{N}}.$$

In other words, the probability that $|\hat{f}_N - \bar{f}| \leq G$ is 68%, the probability that $|\hat{f}_N - \bar{f}| \leq 2G$ is 95%, and the probability that $|\hat{f}_N - \bar{f}| \leq 3G$ is 99.95%.

The advantage of the algorithm indicated by Theorem I is that it is independent of the dimensionality of the integral. The purpose of Theorem II is to show what size sample need be used in applying Theorem I in order to obtain a given required accuracy. In applying Theorem II, one must be a bit cautious because for actual applications involving finite samples, the predictions of Theorem II are usually optimistic and should only be considered suggestive. In applying Theorem II we approximate G^2 by $\hat{V}_N$ where

$$\hat{V}_N = \frac{N}{N-1} (\hat{f}^2 - \hat{f}^2)$$

and

$$\hat{f}^2 = \frac{1}{N} \sum_{i=1}^N [f(\alpha_{1i}, \dots, \alpha_{mi})]^2$$

$$\hat{f} = \frac{1}{N} \sum_{i=1}^N f(\alpha_{1i}, \dots, \alpha_{mi}).$$

Our method for evaluating an integral $\bar{f}$ to within a relative error ϵ is to calculate $\hat{f}_N$ for larger and larger values of N until $\sqrt{\hat{V}_N} < \epsilon \hat{f}_N$, and then $\bar{f} \cong \hat{f}_N$, and probably $|\bar{f} - \hat{f}_N| < \epsilon \bar{f}$.

The integrals we must perform in evaluating cross

sections contain Dirac delta functions in the argument, so we need to find a way of evaluating integrals over delta functions. Let's consider the very general case where the integral is

$$\mathcal{I} = \int_R f(x_1, \dots, x_m) z(x_1, \dots, x_m) \times \prod_{k=1}^L \delta[g_k(x_1, \dots, x_m)] dx_1 \dots dx_m. \quad (D.1)$$

Here there are L delta functions with arguments containing functions $g_k(x_1, \dots, x_m)$. The rest of the integrand is broken up into the product of a general function $f(x_1, \dots, x_m)$ and a positive definite function $z(x_1, \dots, x_m)$. Let's define a probability density to be

$$\rho(x_1, \dots, x_m) = \frac{1}{V} z(x_1, \dots, x_m) \times \prod_{k=1}^L \delta[g_k(x_1, \dots, x_m)] \quad (D.2)$$

where V is a normalization constant such that

$$V = \int_R z(x_1, \dots, x_m) \prod_{k=1}^L \delta[g_k(x_1, \dots, x_m)] \times dx_1 \dots dx_m. \quad (D.3)$$

Rewriting our integral $\mathcal{I}$ as

$$\mathcal{I} = \int_R f(x_1, \dots, x_m) \rho(x_1, \dots, x_m) V \times dx_1 \dots dx_m \quad (D.4)$$

we see that it is in a form where we may apply the strong law of large numbers. Doing so gets us

$$\mathcal{Q} = V \times \lim_{N \rightarrow \infty} \left[\frac{1}{N} \sum_{m=1}^N f(\alpha_{1m}, \dots, \alpha_{nm}) \right] \quad (\text{D.5})$$

where $(\alpha_{1m}, \dots, \alpha_{nm})$ is the m th n -tuple chosen according to the probability density (D.2). What this probability density tells us is to generate sets of points $(\alpha_{1m}, \dots, \alpha_{nm})$ which satisfy

$$g_k(\alpha_1, \dots, \alpha_n) = 0; \quad k = 1, \dots, L$$

and which are distributed according to the (usually unnormalized) probability function $z(\alpha_1, \dots, \alpha_n)$.

Now let's apply these results to the problem of finding cross sections. First consider total cross sections.

To find a total cross section we must evaluate an integral of general form

$$G(E) = \iiint \frac{d^3 q_1}{\omega_1} \frac{d^3 q_2}{\omega_2} \frac{d^3 p_f}{E_f} \times f(q_i, p_i; q_1, q_2, p_f) \delta(E - \omega_1 - \omega_2 - E_f) \delta^3(\vec{q}_1 + \vec{q}_2 + \vec{p}_f).$$

The function $f(q_i, p_i; q_1, q_2, p_f)$ will be the square of the transition amplitude times some kinematic factors. The integral is in the form of (D.1), so we apply (D.4) and (D.5) to obtain

$$G(E) = P_S(E; \mu_1, \mu_2, M) \times \left[\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(q_{ik}, p_{ik}; q_{1k}, q_{2k}, p_{fk}) \right]$$

where

$$P_3(E; \mu_1, \mu_2, M) = \iiint \frac{d^3 q_1}{\omega_1} \frac{d^3 q_2}{\omega_2} \frac{d^3 p_f}{E_f} \\ \times \delta(E - \omega_1 - \omega_2 - E_f) \delta^3(\vec{q}_1 + \vec{q}_2 + \vec{p}_f).$$

The sets $(q_{ik}, p_{ik}; q_{1k}, q_{2k}, p_{fk})$ are chosen from the probability distribution $\frac{1}{\omega_1 \omega_2 E_f}$ such that they satisfy

$$\vec{q}_i + \vec{p}_i = \vec{q}_1 + \vec{q}_2 + \vec{p}_f = 0 \\ \omega_i + E_i = \omega_1 + \omega_2 + E_f = E.$$

In other words, we average $f(q_i, p_i; q_1, q_2, p_f)$ over points randomly selected from covariant phase space.

If we denote this average by $A(E)$, that is

$$A(E) \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(q_{ik}, p_{ik}; q_{1k}, q_{2k}, p_{fk})$$

then our theorem is

$$G(E) = P_3(E; \mu_1, \mu_2, M) A(E).$$

Now let's consider the problem of evaluating differential cross sections. Noting that

$$\left. \frac{d}{d\alpha} \int f(\alpha) d\alpha \right|_{\alpha=\alpha_0} = f(\alpha_0) = \int f(\alpha) \delta(\alpha - \alpha_0) d\alpha,$$

we write our differential cross section integral as

$$\left. \frac{dG(E)}{d\Omega} \right|_{\Omega=\Omega_0} = \iiint \frac{d^3 q_1}{\omega_1} \frac{d^3 q_2}{\omega_2} \frac{d^3 p_f}{E_f} \\ \times f(q_i, p_i; q_1, q_2, p_f) \delta(\phi - \phi_0) \delta(\cos \theta - \cos \theta_0) \\ \times \delta(E - \omega_1 - \omega_2 - E_f) \delta^3(\vec{q}_1 + \vec{q}_2 + \vec{p}_f).$$

Applying (D.2), (D.3), and (D.4) as before

$$\left. \frac{dG(E)}{d\Omega} \right|_{\Omega=\Omega_0} = \left. \frac{dP_s(E; \mu_1, \mu_2, M)}{d\Omega} \right|_{\Omega=\Omega_0} \times \left[\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(q_{ik}, p_{ik}; q_{1k}, q_{2k}, p_{fk}) \right].$$

The points in phase space $(q_{ik}, p_{ik}; q_{1k}, q_{2k}, p_{fk})$ now are picked at random from that region of phase space where $\Omega = \Omega_0$, i.e. $\Phi = \Phi_0$ and $\omega\theta = \omega\theta_0$. If we define $A(E, \Omega)$ by

$$A(E, \Omega_0) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(q_{ik}, p_{ik}; q_{1k}, q_{2k}, p_{fk})$$

our theorem becomes

$$\left. \frac{dG(E)}{d\Omega} \right|_{\Omega=\Omega_0} = \left. \frac{dP_s(E; \mu_1, \mu_2, M)}{d\Omega} \right|_{\Omega=\Omega_0} A(E, \Omega_0)$$

In particular, set up a coordinate system with the z-axis along the vector $\vec{k}$ where $\vec{k}$ is one of the vectors $\vec{q}_1$, $\vec{q}_2$, or $\vec{p}_f$. Let θ and ϕ be the polar and azimuthal angles of $\vec{q}_i$ in this system. A little reflection will show $A(E, \Omega_0)$ independent of ϕ_0

and so

$$\left. \frac{dG(E)}{d\Omega} \right|_{\Omega=\Omega_0} = \left. \frac{dP_s(E; \mu_1, \mu_2, M)}{d\Omega} \right|_{\Omega=\Omega_0} A(E, \omega\theta_0).$$

Use of the program FAKE in generating random events for use with these theorems is discussed in section V.

In evaluating cross sections by the method described above, it is necessary to calculate the phase space integral

$$P_s(E; \mu_1, \mu_2, M) = \iiint \frac{d^3 q_1}{\omega_1} \frac{d^3 q_2}{\omega_2} \frac{d^3 p_f}{E_f} \\ \times \delta(E - \omega_1 - \omega_2 - E_f) \delta^3(\vec{q}_1 + \vec{q}_2 + \vec{p}_f)$$

and its derivative $\frac{dP_s(E; \mu_1, \mu_2, M)}{d\Omega}$.

Actual integration over the delta functions gives

$$P_s(E; \mu_1, \mu_2, M) = \int d^3 p_f d\phi_{q_2} \\ \times \left[\frac{1}{E^2} \sqrt{\frac{\xi^4 - 2(\mu_1^2 + \mu_2^2)\xi^2 + (\mu_1^2 - \mu_2^2)^2}{\vec{p}_f^2 + M^2}} \right]$$

(D.6)

where $\xi^2 = E^2 + M^2 + 2E\sqrt{\vec{p}_f^2 + M^2}$. From

(D.6), it is clear that

$$\frac{dP_s(E; \mu_1, \mu_2, M)}{d\Omega} = \frac{P_s(E; \mu_1, \mu_2, M)}{4\pi}.$$

So our formula for differential cross sections becomes

$$\left. \frac{d\sigma(E)}{d\Omega} \right|_{\Omega=\Omega_0} = \frac{\alpha(E, \cos\theta_0) P_s(E; \mu_1, \mu_2, M)}{4\pi}.$$

So for both total and differential cross sections we only

need to find $P_s(E; \mu_1, \mu_2, M)$. Going through the

rest of the integration of (D.6) gives

$$P_s(E; \mu_1, \mu_2, M) = 8\pi^2 \int_0^{p_{\max}} dp \\ \times \left[\frac{p^2}{E^2} \sqrt{\frac{\xi^4 - 2(\mu_1^2 + \mu_2^2)\xi^2 + (\mu_1^2 - \mu_2^2)^2}{p^2 + M^2}} \right]$$

(D.7)

where $p_{\max} = \sqrt{\frac{[E^2 - (\mu_1 + \mu_2 + M)^2][E^2 - (\mu_1 + \mu_2 - M)^2]}{2E}}$.

The form (D.7) contains only one integration variable

and so is amenable to any of the many variants of Simpson's

rule. Because the integrand is a slowly varying function, $P_s(E; \mu_1, \mu_2, M)$ may be evaluated quite accurately by numerical methods.

It is not easy to find trial integrands with which to check the technique outlined above because even fairly simple functions are difficult to integrate over phase space. Nevertheless, two trial functions were discovered and tested. The first of these is the function

$$f(q_i, p_i; q_1, q_2, p_f) = \left(\frac{\vec{p}_f \cdot \vec{q}_i}{|\vec{p}_f| |\vec{q}_i|} \right)^2 \equiv \cos^2 \theta_{pf}.$$

For this function

$$\begin{aligned} A(E) &= \frac{\int \cos^2 \theta_{pf} \delta(E - \omega_1 - \omega_2 - E_f) \delta^3(\vec{p}_f + \vec{q}_1 + \vec{q}_2) \frac{d^3 q_1}{\omega_1} \frac{d^3 q_2}{\omega_2} \frac{d^3 p_f}{E_f}}{\int \delta(E - \omega_1 - \omega_2 - E_f) \delta^3(\vec{p}_f + \vec{q}_1 + \vec{q}_2) \frac{d^3 q_1}{\omega_1} \frac{d^3 q_2}{\omega_2} \frac{d^3 p_f}{E_f}} \\ &= \frac{\int_{\cos \theta_{pf} = -1}^1 \cos^2 \theta_{pf} d(\cos \theta_{pf})}{\int_{\cos \theta_{pf} = -1}^1 d(\cos \theta_{pf})} = \frac{1}{3}. \end{aligned}$$

So for this case, $A(E) = 1/3$ independent of energy.

Results of a series of trials are shown in Table 3.

If we take

$$f(q_i, p_i; q_1, q_2, p_f) = \left(\frac{\vec{q}_1 \cdot \vec{q}_i}{|\vec{q}_1| |\vec{q}_i|} \right)^2 \equiv \cos^2 \theta_{q_1}$$

we should still get $A(E) = 1/3$. Trials for this integrand are shown in Table 4. From the combined results of Table 3 and Table 4, we see that the Monte Carlo result is within the predicted error 80% of the time versus the 68% we'd expect, within twice the predicted

Table 3. Trial run of Monte Carlo program with

$$f(q_i, p_i; q_1, q_2, p_f) = \cos^2 \theta_{pf}.$$

T π (MeV)	Calculated $\alpha(E)$	Predicted error (%)	Actual error (%)
210	0.3311	± 2.8	-0.6
288	0.3311	± 2.8	-0.6
377	0.3542	± 2.8	6.3
466	0.3293	± 3.0	-1.2
905	0.3358	± 2.8	0.7

Table 4. Trial run of Monte Carlo program with

$$f(q_i, p_i; q_1, q_2, p_f) = \cos^2 \theta_{q_1}.$$

T π (MeV)	Calculated $\alpha(E)$	Predicted error (%)	Actual error (%)
210	0.3324	± 3.4	-0.2
288	0.3196	± 2.7	-4.1
377	0.3424	± 3.0	2.7
466	0.3409	± 3.2	2.3
905	0.3377	± 2.7	1.3

error 90% of the time versus the 95% we'd expect, and within three times the predicted error 100% of the time versus the 99.95% we'd expect. Moreover, the calculated answer is greater than the exact answer half the time and less than the exact answer half the time, as we'd expect.

The other trial function was

$$f(q_i, p_i; q_f, q_2, p_f) = \frac{\omega_1 \omega_2 E_f}{\mu_1 \mu_2 M}.$$

If we define $\rho(E)$ and $\rho_s(E)$ by

$$\rho(E) = \int \delta(E - \omega_1 - \omega_2 - E_f) \delta^3(\vec{q}_1 + \vec{q}_2 + \vec{p}_f) \times d^3q_1 d^3q_2 d^3p_f,$$

$$\rho_s(E) = \int \frac{\mu_1 \mu_2 M}{\omega_1 \omega_2 E_f} \delta(E - \omega_1 - \omega_2 - E_f) \delta^3(\vec{q}_1 + \vec{q}_2 + \vec{p}_f) \times d^3q_1 d^3q_2 d^3p_f$$

then for the function defined above,

$$A(E) = \frac{\rho(E)}{\rho_s(E)}.$$

In Table 5 we tabulate the results of trial runs for this integrand. For comparison, $\rho(E)$ and $\rho_s(E)$ were calculated by a Simpson's rule method to within 0.1% or less. For this case the predicted error was smaller than the actual error every time. On the other hand, only one of the Monte Carlo answers was larger than the correct answer, though we'd expect this to happen four or five times. This might be because of some systematic error in calculation of $\rho(E)$ and $\rho_s(E)$ or

some slight preference in the choice of random events by FAKE. At any rate, for such small errors the discrepancy is not significant. The general level of accuracy is particularly impressive for the smaller values of T_π . The overall accuracy of the Monte Carlo results, especially in the low energy region, comes from the fact that there is not much phase space to sample over at these energies, particularly near threshold. This is a happy coincidence for the calculation of chiral dynamics models since it is the region near threshold which is of most interest.

Table 5. Trial run of Monte Carlo program with

$$f(q_i, p_i; q_f, p_f) = \frac{\omega_1 \omega_2 E_f}{m_1 m_2 M}$$

T_π (MeV)	$\frac{\rho(E)}{\rho_s(E)}$	Calculated $\alpha(E)$	Predicted error (%)	Actual error (%)
210	1.200	1.200	0.1	0.0
222	1.263	1.263	0.1	0.0
233	1.321	1.319	0.2	0.2
310	1.746	1.742	0.3	0.2
371	2.103	2.091	0.5	0.6
377	2.138	2.137	0.5	0.5
454	2.613	2.611	0.5	0.1
590	3.512	3.526	1.1	-0.4
800	5.033	5.003	1.1	0.6
905	5.847	5.837	0.7	0.2

APPENDIX E

PION SCATTERING LENGTHS

The most natural place to compare the different versions of chiral dynamics is in their predictions of low energy pi-pi scattering parameters, especially the s-wave scattering lengths. Scattering length a is defined in terms of the scattering amplitude $f(\theta, \phi)$ by

$$a = \lim_{\vec{q} \rightarrow 0} f(\theta, \phi)$$

where $\vec{q}$ is the momentum in center of mass of one of the particles. For identical particles, like pions, the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = |f(\theta, \phi) + f(\pi - \theta, \phi + \pi)|^2$$

But in terms of the invariant amplitude $\mathcal{M}$, the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2^6 (2\pi)^2} \frac{|\mathcal{M}|^2}{\omega^2}$$

so the scattering length is related to $\mathcal{M}$ by

$$a = \lim_{\vec{q} \rightarrow 0} \frac{|\mathcal{M}| e^{i\varphi}}{32\pi\mu} \tag{E.1}$$

the phase factor φ being arbitrary. In our case it is chosen to match the convention of Gürsey and Chang.³⁹

Chiral dynamics Lagrangians give the diagram of Figure 17 for pi-pi scattering.

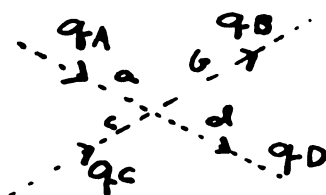


Figure 17. Diagram for pi-pi scattering.

From the Feynman rule

$$\mathcal{M} = 4i \left(\frac{g}{2M} \right)^2 \left(\frac{q_A^\nu}{q_A} \right)^2 \times \left\{ \begin{aligned} & \hat{\phi}_a \cdot \hat{\phi}_c \hat{\phi}_b^* \cdot \hat{\phi}_d^* [\eta_1 \mu^2 + (q_A q_C + q_B q_D)] \\ & + \hat{\phi}_a \cdot \hat{\phi}_b^* \hat{\phi}_c \cdot \hat{\phi}_d^* [\eta_1 \mu^2 - (q_A q_B + q_C q_D)] \\ & + \hat{\phi}_a \cdot \hat{\phi}_d^* \hat{\phi}_b \cdot \hat{\phi}_c^* [\eta_1 \mu^2 - (q_A q_D + q_B q_C)] \end{aligned} \right\}.$$

Go to the Cartesian representation of the vectors $\hat{\phi}_a$, $\hat{\phi}_b$, $\hat{\phi}_c$, and $\hat{\phi}_d$ and take the threshold limit of q_A ,

$$q_B, q_C, \text{ and } q_D \text{ to obtain}$$

$$\mathcal{M} = 4i \left(\frac{g}{2M} \right)^2 \left(\frac{q_A^\nu}{q_A} \right)^2 [\delta_{ab} \delta_{cd} (\eta_1 - 2) \mu^2 + \delta_{ad} \delta_{bc} (\eta_1 - 2) \mu^2 + \delta_{ac} \delta_{bd} (\eta_1 + 2) \mu^2].$$

From (E.1),

$$a = \frac{\mu}{8\pi} \left(\frac{g}{2M} \right)^2 \left(\frac{q_A^\nu}{q_A} \right)^2 [\delta_{ab} \delta_{cd} (\eta_1 - 2) + \delta_{ad} \delta_{bc} (\eta_1 - 2) + \delta_{ac} \delta_{bd} (\eta_1 + 2)]. \quad (E.2)$$

We wish to separate the various isotopic parts of (E.2). Consider some sort of operator in isospin space, Q , which operates on isotopic triplets. We may

expand Q in terms of projection operators $P^T = \sum_{T_1, T_2} |T, T_2\rangle \langle T, T_2|$, as

$$Q = Q_0 P^0 + Q_1 P^1 + Q_2 P^2.$$

If we take a matrix element of Q between Cartesian states $|a, c\rangle$ and $|b, d\rangle$, we get

$$\langle b, d | Q | a, c \rangle = A \delta_{ac} \delta_{bd} + B \delta_{ab} \delta_{cd} + C \delta_{ad} \delta_{bc}.$$

It can be shown that

$$Q_0 = 3A + B + C$$

$$Q_1 = B - C$$

$$Q_2 = B + C.$$

If we apply this result to the scattering length

a , the scattering lengths for different isospin channels are

$$a_0 = \left(\frac{5n_1 + 2}{8\pi} \right) \mu \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2$$

$$a_1 = 0$$

$$a_2 = \left(\frac{n_1 - 2}{4\pi} \right) \mu \left(\frac{g}{2M} \right)^2 \left(\frac{g_V}{g_A} \right)^2.$$

(E.3)

If good experimental numbers existed for a_0 , a_1 , and a_2 , it would now be easy to determine n_1 , and thereby choose one of the chiral dynamics models over the others. Unfortunately, pi-pi scattering experiments are difficult to perform, and all the available results can do is indicate that the expressions of (E.3) are plausible.

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