INFINITE GROUPS WITH SUBNORMALITY OR DESCENDANCE CONDITIONS ON THEIR INFINITE SUBGROUPS, AND SOME SPECIAL ČERNIKOV p-GROUPS

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R. E. Phillips
Major professor

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ABSTRACT

INFINITE GROUPS WITH SUBNORMALITY OR DESCENDANCE CONDITIONS ON THEIR INFINITE SUBGROUPS, AND SOME SPECIAL ČERNIKOV p-GROUPS

By

Veril LeRoy Phillips

In [2], R. Phillips provides a characterization of groups in the class $L\mathfrak{F} \cap I\mathfrak{N}^{s} - \mathfrak{N}^{s}$, of infinite, locally finite groups, with all infinite subgroups serial, but with some finite subgroup not serial. This class turned out to be equal to a similarly defined class by Černikov [1].

In this thesis two additional similarly defined classes are studied: $L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}^{snb}$ is the class of all infinite locally nilpotent groups having all infinite subgroups subnormal and with a bound on the defects of infinite subgroups, but with the finite subgroups failing to have that property. $L\mathfrak{N} \cap I\mathfrak{N}^{d} - \mathfrak{N}^{d}$ is the class of infinite locally nilpotent groups having all infinite subgroups descendant but some finite subgroups not descendant.

After reducing the study of groups in these classes to the study of p-groups in these classes, necessary and sufficient conditions for an infinite p-group P to be in the classes are developed. In both instances these conditions include: P has a normal divisible abelian subgroup D of finite rank, whose centralizer C has finite index (not 1) in P and furthermore for all $x \in P-C$, conjugation by x is an automorphism of D which does not normalize any infinite proper subgroup of D. Such automorphisms are called S-I automorphisms.

By using the fact that the automorphisms of D are essentially $r \times r$ matrices over the p-adic integers with determinant a unit, it is further shown for the classes studied that the rank of D is $p - 1$, and the rational canonical form for S-I automorphisms is computed.

Finally, using the study of S-I automorphisms, a structure theorem for direct limits G of p-groups of maximal class is developed which shows G is a semi-direct product of a divisible abelian p-group D of rank $p - 1$ by a cyclic group of order p or of order p^2 with an amalgamated subgroup trivial or of order p, respectively. A relationship between these groups and the class $L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}^{snb}$ is established and examples thereby provided.

All known examples of groups in either $L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}^{snb}$ or $L\mathfrak{N} \cap I\mathfrak{N}^{d} - \mathfrak{N}^{d}$ satisfy Min, hence are Černikov groups. Indeed, it is shown that a group G in the former class necessarily must satisfy Min, but the corresponding question for the latter class remains unsolved. If they also must satisfy Min, then the two classes are shown in fact to be equal.

References

1. Černikov, S. N., Groups with prescribed properties for systems of infinite subgroups, Ukrainian Math. J. 19 (1967), 715-731.

2. Phillips, R. E., Infinite groups with normality conditions on infinite subgroups, Rocky Mountain J. of Math., to appear.

INFINITE GROUPS WITH SUBNORMALITY OR DESCENDANCE CONDITIONS ON THEIR INFINITE SUBGROUPS, AND SOME SPECIAL ČERNIKOV p-GROUPS

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To Pam and RoiAnn

whose patience and understanding have been so supportive

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Chapter I

Introduction and Notation

In this paper we investigate two classes of infinite periodic groups: The first has all infinite subgroups subnormal and with a bound on the subnormal defects but with this condition failing for finite subgroups. The second has all infinite subgroups descendant but has some finite subgroup not descendant. We obtain a structure theorem for the first class in Chapter III and a structure theorem for the second class in Chapter IV. In Chapter V we develop some special notation to discuss the kind of automorphisms of divisible abelian p-groups which arise in Chapters III and IV, and use this notation in Chapter VI to obtain a characterization of direct limits of p-groups of maximal class and also to obtain more information about the first class of groups. In this chapter we list our notation and in Chapter II we state some preliminary results.

Notation 1.1: If F is any set, $|F|$ will denote its cardinality.

Definition 1.2: A class of groups, $\mathfrak{X}$, is any collection of groups such that whenever $G \in \mathfrak{X}$ and G_1 is isomorphic to G then also $G_1 \in \mathfrak{X}$. $G \in \mathfrak{X}$ is also expressed by saying that G is an $\mathfrak{X}$-group. Many authors require that a class of groups also contains a trivial group (group of order 1).

Notation 1.3: If G and G_1 are groups, $G \cong G_1$ will denote that G is isomorphic to G_1.

<u>Notation</u> 1.4: The identity element of a group and a trivial group will both be denoted by 1.

<u>Notation</u> 1.5: If X and Y are any sets (including classes of groups) then X - Y denotes the set of all elements of X which are not elements of Y.

<u>Notation</u> 1.6: If G and G_1 are groups, their direct sum is denoted by $G + G_1$. If $\{G_i | i \in I\}$ is a collection of groups, their direct sum is denoted by $\sum \{G_i | i \in I\}$.

<u>Definition</u> 1.7: If G has a normal subgroup H, and H has a proper supplement K in G (meaning that K is a proper subgroup of G and G = HK), then G is a <u>semi-direct product of H by K with amalgamated subgroup H ∩ K</u>. In case H ∩ K is trivial, then G is simply called the semi-direct product of H by K or a split extension of H by K, and this fact is denoted by $G = H]K$.

Definition 1.7 provides the internal criteria for a group G to be a semi-direct product with an amalgamated subgroup. In Chapter VI we use Gorenstein's construction [24; pp. 27-28] of the external semi-direct product with an amalgamated subgroup. (Gorenstein's term is "partial semi-direct product.")

<u>Notation</u> 1.8: " $\leq$ ", " $<$ ", and " $\not\leq$ " will denote, respectively, "is a subgroup of," "is a proper subgroup of," and "is not a subgroup of." These symbols will also have their usual meanings.

Notation 1.9: If X is any subset of a group G, then $< X >$ denotes the subgroup of G generated by X. $< x, y, \ldots , z >$ will be used instead of $< \{x, y, \ldots , z\} >$.

Notation 1.10: If G is a group, and H is a subgroup of G, then

(i) $|G:H|$ denotes the index of H in G,

(ii) $H \triangleleft G$ ($H \not\triangleleft G$) means that H is (is not) a normal subgroup of G,

(iii) H char G (H c~~h~~ar G) means that H is (is not) a characteristic subgroup of G,

(iv) $N_G(H)$ and $C_G(H)$ denote, respectively, the normalizer in G of H and the centralizer in G of H; $C_G(x)$ will be used in place of $C_G(< x >)$, and

(v) H is a Sylow p-subgroup (Sylow p'-subgroup) if H is maximal in G with respect to all its elements having order a power of p (prime to p).

Definition 1.11: In any group G and for any subgroup H, there is a *natural homomorphism* $f:N_G(H) \to \mathrm{Aut}\, H$ (the automorphism group of H) *with kernel* $C_G(H)$. (See, e.g., [41, Thm. 3.2.3].)

Notation 1.12: If H is a subgroup of a group G, $h \in H$, $g \in G$, and $\alpha \in \mathrm{Aut}\, G$, then

(i) $h^g = g^{-1}hg$,

(ii) h^α is the image of h under α,

(iii) H^g is the conjugate of H by g,

(iv) H^α is the image of the subgroup H under α,

(v) $H^G = < h^g | h \in H$ and $g \in G >$, the normal closure of H in G, and

(vi) $H^{<\alpha>} = < h^\beta | h \in H$ and $\beta \in < \alpha >>$.

Our use of the term "series" corresponds to the definition of "normal system" by Kuroš [31, p. 171].

Definition 1.13: Let H be a subgroup of a group G. A series between H and G is a chain of subgroups $\mathcal{S}$ such that (i) if $X \in \mathcal{S}$, then $H \leq X \leq G$, (ii) $H \in \mathcal{S}$, $G \in \mathcal{S}$, (iii) $\mathcal{S}$ is complete, i.e., contains all unions and intersections of its members, and (iv) if X has an immediate successor Y in the natural ordering of $\mathcal{S}$, then $X \lhd Y$. We write H ser G (H s$\not{e}$r G) if there exists (there does not exist) a series between H and G. A descending series between H and G is a series $\mathcal{S}$ between H and G in which the reverse of the natural ordering of $\mathcal{S}$ is a well ordering. H is (is not) a descendant subgroup of G, written H desc G (H d$\not{e}$sc G) if there exists (there does not exist) a descending series between H and G. If H desc G, then a descending series $\mathcal{S}$ between H and G will be denoted $\mathcal{S} = \{H_\alpha | \alpha < \sigma\}$, where σ is an ordinal number and where $\alpha_1 < \alpha_2 < \sigma \Longrightarrow H_{\alpha_2} \leq H_{\alpha_1}$.

Definition 1.14: A descending chain of subgroups of a group G is a chain $\mathcal{C}$ of subgroups in which the reverse of the natural ordering of $\mathcal{C}$ is a well ordering. Notice that a descending chain differs from a descending series in that the former has no condition of normality and is not necessarily complete.

Definition 1.15: A subgroup H of a group G is subnormal, written $H \lhd\lhd G$, if there exists a finite descending series between H and G.

Notation 1.16: Let H be a subgroup of a group G and α an ordinal number. We define the subgroups $H^{G,\alpha}$ inductively as follows: $H^{G,0} = G$, $H^{G,\alpha+1} = H^{(H^{G,\alpha})}$, and $H^{G,\lambda} = \bigcap_{\alpha < \lambda} H^{G,\alpha}$ for α an ordinal and λ a limit ordinal.

It is well known that H desc G if and only if there exists an α such that $H = H^{G,\alpha}$, and that in this case $\{H^{G,\beta} | \beta < \alpha + 1\}$ is the fastest descending series between H and G. (See [35, Sec. 1.4] or [37, Sec. 3].)

Definition 1.17: If $H \lhd\lhd G$, the subnormal index or defect of H in G is the least nonnegative integer d such that $H^{G,d} = H$. If H is a subgroup of G and s a nonnegative integer we write $H \lhd\lhd_s G$ if H is subnormal with defect at most s.

Definition 1.18: If x and y are elements of a group, the commutator of x with y is $[x,y] = x^{-1}y^{-1}xy$. If X and Y are subsets of a group, the commutator subgroup of X with Y is $[X,Y] = \langle [x,y] : x \in X, y \in Y \rangle$. If $x_1, x_2, \ldots$ are elements of a group, we define more general commutators inductively by $[x_1] = x_1$ and $[x_1, \ldots, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}]$. If $X_1, X_2, \ldots$ are subsets of a group, we define more general commutator subgroups inductively by $[X_1] = < X_1 >$ and $[X_1, \dots, X_{n+1}] = [[X_1, \dots, X_n], X_{n+1}]$. If G is a group, the derived group is $G' = [G,G]$.

Definition 1.19: Let G be any group. The center of G is denoted by $Z(G)$. The upper central series of G is the series $\{Z_\alpha(G)\}$ defined inductively by

$$Z_0(G) = 1,\ Z_{\alpha+1}(G) = Z(G/Z_\alpha) \ \text{ and } \ Z_\lambda(G) = \bigcup_{\alpha < \lambda} Z_\alpha(G) ,$$

for α an ordinal and λ a limit ordinal. The lower central series of G is the series $\{\gamma_\alpha(G)\}$ defined inductively by

$$\gamma_1(G) = G,\ \gamma_{\alpha+1}(G) = [\gamma_\alpha(G),G], \ \text{ and } \ \gamma_\lambda(G) = \bigcap_{\alpha < \lambda} \gamma_\alpha(G) ,$$

for α an ordinal and λ a limit ordinal.

Definition 1.20: Let G be a group. We will use the following notation for classes of groups encountered frequently:

$G \in \mathfrak{F}$ if G is finite.

$G \in \mathfrak{N}$ if G is nilpotent.

$G \in \mathfrak{N}_c$ if G is nilpotent of class no more than c.

$G \in Z$ if G has a central series.

$G \in ZA$ if G is hypercentral; i.e., if $Z_\alpha(G) = G$ for some ordinal α.

$G \in ZD$ if $\gamma_\alpha(G) = 1$ for some ordinal α.

$G \in \mathfrak{N}^{snb}$ if all subgroups are subnormal with bounded defects; i.e., if there is a positive integer n such that $H^{G,n} = H$ for all subgroups H of G.

$G \in \mathfrak{N}^d$ if all subgroups are descendant.

$G \in \mathfrak{N}^s$ if all subgroups are serial.

$G \in I\mathfrak{N}^{sn}$ if G is an infinite group in which all infinite subgroups are subnormal.

$G \in I\mathfrak{N}^{snb}$ if G is an infinite group in which all infinite subgroups are subnormal with bounded defects.

$G \in I\mathfrak{N}^{d}$ if G is an infinite group in which all infinite subgroups are descendant.

$G \in I\mathfrak{N}^{s}$ if G is an infinite group in which all infinite subgroups are serial.

$G \in L\mathfrak{F}$ if G is locally finite.

$G \in L\mathfrak{N}$ if G is locally nilpotent.

$G \in$ Min if G satisfies the minimal condition on subgroups.

G is a Černikov group (extremal in Černikov's works) if G is a finite extension of an abelian group $A \in$ Min.

G is divisible (radicable in [38] and [39]) if for all positive integers n and for all $x \in G$, there is a $y \in G$ such that $x = y^n$.

G is a group of type p^∞ for p a prime if G is a noncyclic abelian p-group all of whose proper subgroups are cyclic. (See [21], p. 65.)

Notation 1.21: If G is a group and p a relevant prime, then $G(p^n)$ denotes the subgroup of G generated by all elements of order p^n.

Definition 1.22: If G is an abelian group satisfying Min then it decomposes into a direct sum of finitely many summands, each of which is finite cyclic or of type p^∞ for some prime p. (See, e.g., [22, Thms. 3.1 and 25.1].) The decomposition is not unique, but the number of summands is an invariant, called the <u>rank of G</u>.

<u>Notation</u> 1.23: Let G, G_1 be groups, R a ring with unity, and r a positive integer.

End G is the set of all endomorphisms of G.

Aut G is the set of all automorphisms of G.

Hom (G,G_1) is the set of all homomorphisms from G into G_1.

$GL(r,R)$ is the set of all $r \times r$ matrices over R and with determinant a unit in R.

$\bar{Z}$ is the ring of integers.

$\bar{Z}_p$ is the field of integers modulo p.

$\bar{Q}$ is the field of rational numbers.

R is a <u>principal ideal domain</u> (<u>PID</u>) if R is an integral domain in which all ideals are of the form aR, for some $a \in R$.

R is a <u>unique factorization domain</u> (<u>UFD</u>) if R is an integral domain in which each nonzero element is either a unit or can be written as a product of finitely many irreducible elements of R, uniquely, up to the number of factors and up to associates of the irreducible elements.

<u>Definition</u> 1.24: An ordinal number λ is <u>cofinal</u> with a limit ordinal α if λ is the limit of an increasing α-sequence; i.e., if $\lambda = \lim_{\xi < \alpha} \varphi(\xi)$ for some sequence φ of ordinals satisfying

$$\xi_1 < \xi_2 < \alpha \text{ implies } \varphi(\xi_1) < \varphi(\xi_2) .$$

(See [29, p. 236].)

Chapter II

Preliminary Results

In this chapter we compile some general theorems, most of which are well known, that we will use later. More specialized results will be stated (with or without proofs) as they are needed. We begin with some lemmas involving descending chains of subgroups, for which the first requires the following result about ordinal numbers, which we state without proof.

Proposition 2.1: [29; Thm. 2, p. 243.] Every limit ordinal of the form $\lambda = \lim_{\xi < \alpha} \varphi(\xi)$, where φ is any sequence of ordinal numbers, is cofinal with some ordinal $\gamma \leq \alpha$.

The following is a generalization of [15; Lemma 1].

Lemma 2.2: Let G be any group and $\{H_\alpha | \alpha < \sigma\}$ a descending chain of nontrivial subgroups with $\bigcap \{H_\alpha | \alpha < \sigma\} = 1$ (and hence σ is a limit ordinal). Let F be a subgroup and $\mu = \min\{\beta : |\beta| \geq |F|\}$. We suppose that for all limit ordinals λ such that $\lambda \leq \mu$, σ is not cofinal with λ. (This is satisfied automatically if F is finite.) Then if $H_\alpha F$ is a subgroup for all $\alpha < \sigma$, then $\bigcap \{H_\alpha F | \alpha < \sigma\} = F$.

Proof: Let $F = \{f_\beta | 0 \leq \beta < \mu\}$ be a well ordering of F with $f_0 = 1$. Let $H_\sigma = \bigcap \{H_\alpha | \alpha < \sigma\} = 1$. Now for every β, $0 \leq \beta < \mu$ let $\varphi(\beta)$ be minimal such that

$$\{f_\gamma | 0 \leq \gamma \leq \beta\} \cap H_{\varphi(\beta)} = \{1\} .$$

We claim that there is some ordinal $\xi < \sigma$ such that $F \cap H_\xi = 1$. Let $\xi = \lim_{\beta < \mu} \varphi(\beta)$. Note that $\xi \leq \sigma$, and that $F \cap H_\xi = 1$.

Case 1. F is finite. For every β, $0 \leq \beta < \mu$, let $\theta(\beta)$ be the minimal ordinal such that $f_\beta \notin H_{\theta(\beta)}$. Note that $\theta(\beta) < \sigma$ for all β. Clearly

$$\xi = \max\{\theta(\beta) \mid 0 \leq \beta < \mu\}$$

and thus $\xi < \sigma$ since σ is a limit ordinal.

Case 2. F is infinite. By 2.1, ξ is cofinal with some ordinal $\lambda \leq \mu$. If $\xi = \sigma$, the hypotheses are contradicted, and so we have $\xi < \sigma$, as desired.

Next we claim that $\bigcap_{\xi \leq \alpha < \sigma} (H_\alpha F) \leq F$. Let $x \in \bigcap_{\xi \leq \alpha < \sigma} (H_\alpha F)$. For all α, $\xi \leq \alpha < \sigma$ write $x = h_\alpha f_\alpha$, $h_\alpha \in H_\alpha$, $f_\alpha \in F$. Then $h_\xi^{-1} h_\alpha = f_\xi f_\alpha^{-1} \in H_\xi \cap F = 1$; hence $h_\alpha = h_\xi \in H_\alpha$ for all α, $\xi \leq \alpha < \sigma$. Thus $h_\xi \in \bigcap_{\xi \leq \alpha < \sigma} H_\alpha = \bigcap_{\alpha < \sigma} H_\alpha = H_\sigma = 1$ and $x = h_\xi f_\xi = f_\xi \in F$, as desired. But it is now clear that

$$F \leq \bigcap_{\alpha < \sigma} (H_\alpha F) = \bigcap_{\xi \leq \alpha < \sigma} (H_\alpha F) \leq F .$$

<u>Lemma</u> 2.3: Let G be any group and $\mathcal{H} = \{H_\gamma \mid \gamma \in \Gamma\}$ any nonempty collection of infinite subgroups with finite intersection. Then there is a collection $\mathcal{L}$ of infinite subgroups with finite intersection with the properties: (i) for every $L \in \mathcal{L}$ there is a subset $\Delta \leq \Gamma$ such that $L = \bigcap \{H_\gamma \mid \gamma \in \Delta\}$ and (ii) either $\mathcal{L}$ has exactly two members or $\mathcal{L}$ is an infinite descending chain.

Proof: For every subset $\Delta \leq \Gamma$, define $L_\Delta = \bigcap\{H_\gamma | \gamma \in \Delta\}$. Consider the set of all possible intersections of the members of $\mathcal{H}$: $\mathcal{S} = \{L_\Delta | \Delta \leq \Gamma\}$. By hypothesis, L_Γ is finite. Let $\Delta \leq \Gamma$ have minimal cardinality such that L_Δ is finite.

Case 1. Δ is finite. Let $\Delta = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. Then

$$L_\Delta = H_{\gamma_1} \cap H_{\gamma_2} \cap \cdots \cap H_{\gamma_n}$$

is finite. Let m be minimal such that $H_{\gamma_1} \cap \cdots \cap H_{\gamma_m}$ is finite. Note that $m \geq 2$. Take $\Delta_1 = \{\gamma_1, \gamma_2, \ldots, \gamma_{m-1}\}$ and $\Delta_2 = \{\gamma_m\}$. By the choice of m, L_{Δ_1} is infinite, and clearly $L_{\Delta_2} = H_{\gamma_m}$ is infinite, but $L_{\Delta_1} \cap L_{\Delta_2} = L_\Delta$ is finite. We are finished by taking $\mathcal{L} = \{L_{\Delta_1}, L_{\Delta_2}\}$.

Case 2. Δ is infinite. Let σ be the minimal ordinal of cardinality $|\Delta|$. Let Δ be well ordered with order type σ. Set $\mathcal{L} = \{L_{\Delta'} | \Delta'$ is a proper initial segment of $\Delta\}$. For every proper initial segment Δ' of Δ we have $|\Delta'| < |\Delta|$ by the minimality of σ and hence $L_{\Delta'}$ infinite by the choice of Δ. Furthermore, $\Delta' \leq \Delta'' \leq \Delta$ implies $L_{\Delta''} \leq L_{\Delta'}$ so that $\mathcal{L}$ is a descending chain of infinite subgroups whose intersection $L_\Delta = \bigcap \mathcal{L}$ is finite.

Lemma 2.4: If H and K are infinite normal subgroups of a group G, $H \cap K = 1$, and F is a finite subgroup of G, then $HF \cap KF$ is finite.

Proof: Suppose that $HF \cap KF$ is infinite. Then there is an $f \in F$ and an infinite collection $\{h_i \in H | i = 1, 2, \ldots\}$ such that

$$h_i f \in HF \cap KF, \quad \text{for all} \quad i = 1, 2, \ldots$$

For all $i = 1, 2, \ldots$, let $f_i \in F$, $k_i \in K$ such that $h_i f = k_i f_i$. Since F is finite there is an infinite subcollection of the k_i's for which the corresponding f_i's are all equal. Hence after renumbering, we have $h_i f = k_i f_1$, $i = 1, 2, \ldots$ and $i \neq j$ implies $h_i \neq h_j$. But then, $ff_1^{-1} = h_i^{-1} k_i \in HK = H + K$ and the unique representation of an element in $H + K$ is violated. Hence $HF \cap KF$ is finite.

Next we collect some conditions for subnormality and descendance in groups. The first is due to Robinson [35; Sec. 1.4] or [37; Lemmas 3.11 and 3.12].

<u>Proposition</u> 2.5: If G is any group and $H \leq G$, then $H \triangleleft\triangleleft_r G$ if and only if $[G, \underbrace{H, H, \ldots, H}_{r}] \leq H$.

<u>Sketch of proof</u>: For the full proof, see the reference above. It is easy to show that

$$H^{G,i} = H[G, \underbrace{H, H, \ldots, H}_{i}]$$

for each nonnegative integer i by use of induction and the usual commutator formulas.

It is not generally true that descendant subgroups remain descendant in homomorphic images, as the following example shows. However, when the descendant subgroup contains the kernel, then it remains descendant in the image as seen in 2.7.

Example 2.6: Let G be the dihedral group on an infinite cyclic group. Write $G = \langle x \rangle]\langle y \rangle$, where $\langle x \rangle$ is infinite cyclic and y is an element of order 2 with $x^y = x^{-1}$. Let p be prime and let

$$X_n = \langle x^{p^n} \rangle \leq G \text{ for all } n = 1, 2, \ldots$$

Notice that $X_n \lhd G$ for all n, and $\bigcap_{n=1}^{\infty} X_n = 1$. By 2.2, $\bigcap_1^{\infty} X_n \langle y \rangle = \langle y \rangle$. Now

$$\langle y \rangle = \bigcap_1^{\infty} X_n \langle y \rangle \lhd \cdots \lhd X_{n+1} \langle y \rangle \lhd X_n \langle y \rangle \lhd \cdots \lhd X_1 \langle y \rangle = G$$

showing that $\langle y \rangle$ desc G. For all $n = 1, 2, \ldots$ let $\theta_n : G \to G/X_n$ be the natural map. For $n \geq 1$, $\langle y \rangle \theta_n = X_n \langle y \rangle < G/X_n$. We claim that for odd p, $X_n \langle y \rangle / X_n$ is proper and self normalizing in G/X_n. To see this, suppose otherwise. Then

$$X_n \langle y \rangle < N_G(X_n \langle y \rangle) .$$

Let $z \in N_G(X_n \langle y \rangle) - X_n \langle y \rangle$. Write $z = x^m y^j$ where m is an integer and $j = 0$ or 1.

Case 1. $j = 0$. Then $y^{x^m} = x^{-2m} y \in X_n \langle y \rangle$. Hence $x^{2m} \in X_n$, giving $2m = kp^n$, for some integer k. But p^n is odd, so that $k = 2\ell$ for some ℓ and hence $m = \ell \cdot p^n$. But then $z = x^m = (x^{p^n})^{\ell} \in X_n \leq X_n \langle y \rangle$, contradicting the choice of z.

Case 2. $j = 1$. Then $y^{x^m y} = (x^{-2m} y)^y = x^{2m} y \in X_n \langle y \rangle$. Thus $x^{2m} \in X_n$, and we complete the argument exactly as in Case 1. Thus, $X_n \langle y \rangle / X_n$ is indeed proper and self normalizing in G/X_n for p

odd. Now if $\frac{X_n < y >}{X_n}$ desc $\frac{G}{X_n}$ then since G/X_n is finite, $\frac{X_n < y >}{X_n} \lhd\lhd_r \frac{G}{X_n}$, for some positive integer r, and hence

$$X_n < y > \lhd\lhd_r G ,$$

whence $X_n < y >$ is distinct from its normalizer in G. Thus we are assured that

$$\frac{X_n < y >}{X_n} \text{ d}\not{\text{e}}\text{sc } \frac{G}{X_n} .$$

Thus, although $< y >$ desc G, we have $< y > \theta_n$ d$\not{e}$sc $G\theta_n$.

Proposition 2.7: Let G be a group and H, K, and L subgroups.

(i) If $K \lhd G$ and H desc G with $K \leq H$, then

$$\frac{H}{K} \text{ desc } \frac{G}{K} .$$

(ii) If H desc K and K desc L, then H desc L.

(iii) If H desc K, then $H \cap L$ desc $K \cap L$.

(iv) If H desc G and $H \leq K$, then H desc K.

(v) If H desc G and K desc G, then $H \cap K$ desc G.

(vi) The class $I\mathfrak{N}^d$ is closed under the taking of infinite subgroups.

Proof: (i) Let $\mathcal{H} = \{H_\alpha | \alpha < \sigma\}$ be a descending series between H and G. Then for all $\alpha < \sigma$ we have $K \leq H \leq H_\alpha$, so that also $\frac{H}{K} \leq \frac{H_\alpha}{K}$. Thus $\mathcal{L} = \{H_\alpha/K | \alpha < \sigma\}$ is a descending chain containing $\frac{H}{K}$ and satisfying $H_{\alpha+1}/K \lhd H_\alpha/K$ for all $\alpha < \sigma$. Let λ be a limit ordinal, $\lambda \leq \sigma$. Then there exists some $L \in \mathcal{H}$ such that $L = \bigcap\{H_\alpha | \alpha < \lambda\}$. Thus $L/K \in \mathcal{L}$ and it is clear that $L/K = \bigcap\{H_\alpha/K | \alpha < \lambda\}$. Thus $\mathcal{L}$ contains all intersections of its members and since $\mathcal{L}$ is a descending chain, it contains all unions of its members; i.e., $\mathcal{L}$ is complete. Thus $\mathcal{L}$ is a descending series between H/K and G/K.

(ii) Let $\mathcal{H} = \{H_\alpha | \alpha < \sigma\}$ be a descending series between H and K and $\mathcal{K} = \{K_\beta | \beta < \lambda\}$ a descending series between K and L. Then $\mathcal{H} \cup \mathcal{K}$ is clearly a descending series between H and L, showing H desc L.

(iii) Let $\{H_\alpha | \alpha < \sigma\}$ be a descending series between H and K. Then $\{H_\alpha \cap L | \alpha < \sigma\}$ is a descending series between $H \cap L$ and $K \cap L$.

(iv) Let H desc G and $H \leq K \leq G$. By (iii) we have

$$H \cap K = H \text{ desc } G \cap K = K .$$

(v) Let H desc G and K desc G. By (iii) we have $H \cap K$ desc $G \cap K = K$. Thus by (ii) it follows that $H \cap K$ desc G.

(vi) follows from (iv).

Proposition 2.8: (i) If $G \in I\mathfrak{N}^d$ and $K \triangleleft G$, then $G/K \in I\mathfrak{N}^d$ if G/K is infinite. (ii) If $G \in I\mathfrak{N}^d$ and H is an infinite normal subgroup of G, then $G/H \in \mathfrak{N}^d$.

Proof: (i) Let H/K be an infinite subgroup of G/K. Then H is an infinite subgroup of G and hence by hypothesis, H desc G. By 2.7(i) we have $\frac{H}{K}$ desc $\frac{G}{K}$. Thus $\frac{G}{K} \in I\mathfrak{N}^d$.

(ii) Let L/H be a subgroup of G/H. Then L is an infinite subgroup of G and by hypothesis we have L desc G. By 2.7(i),

$$\frac{L}{H} \text{ desc } \frac{G}{H}, \quad \text{showing } \frac{G}{H} \in \mathfrak{N}^{d} .$$

Remark 2.9: We will use the ideas of 2.7 and 2.8 in later chapters without reference.

Next we collect some elementary properties of Černikov groups, since they play a prominent role in our investigations. Recall a Černikov group is a finite extension of an abelian group with Min. In particular all finite groups are Černikov groups.

Proposition 2.10: (i) Every group G has a maximal normal divisible abelian subgroup D (not necessarily nontrivial). (ii) If G is an infinite Černikov group, then G has a maximal normal divisible abelian subgroup D which is nontrivial, has finite rank, has finite index in G, and is characteristic in G.

Proof: (i) Notice the trivial subgroup is a normal divisible abelian subgroup and hence the collection

$$\mathfrak{S} = \{A \lhd G \mid A \text{ is divisible and abelian}\}$$

is nonempty. If $\mathfrak{T}$ is a nonempty totally ordered subset of $\mathfrak{S}$ and $D = \bigcup \mathfrak{T}$, then D is a normal divisible abelian subgroup of G; i.e., $D \in \mathfrak{S}$ so that D is an upper bound for $\mathfrak{T}$. Thus by Zorn's lemma $\mathfrak{S}$ has maximal elements.

(ii) Let G be an infinite Černikov group, and A a normal abelian subgroup of finite index with $A \in$ Min. By 1.22, we write

$A = A_1 + A_2 + \cdots + A_s$ for s a positive integer with A_i a group of type p^∞ (for various primes p) or finite cyclic for each i. Since A^* has finite index and G is infinite, there is some i, $1 \leq i \leq s$ such that A_i is a group of type p^∞. Let $I \leq \{1,2, \ldots , s\}$ be defined by $i \in I$ if and only if A_i is a group of type p^∞ for some prime p. Let $D = \sum \{A_i | i \in I\}$. $|G:D| = |G:A||A:D| < \infty$. Hence D is a maximal divisible abelian subgroup which is nontrivial, has finite rank and has finite index.

To see that it is characteristic it suffices to show that it is the unique minimal subgroup of G of finite index. Suppose H is any subgroup of finite index in G. Then $|D:H \cap D| = |DH:H| < \infty$. Since D has no proper subgroups of finite index, $H \cap D = D$; i.e., $D \leq H$. Thus D is the unique minimal subgroup of finite index and as such is characteristic in G.

Notation 2.11: If G is an infinite Černikov group, then the subgroup D of 2.10(ii) will be referred to in the sequel as the minimal subgroup of finite index, and denoted $D(G)$.

When we investigate groups G in the classes $L\mathfrak{X} \cap I\mathfrak{X}^{snb} - \mathfrak{X}^{snb}$ and $L\mathfrak{X} \cap I\mathfrak{X}^{d} - \mathfrak{X}^{d}$ we will find that the centralizer of a normal divisible abelian subgroup D plays a significant role. We sharpen some results of Robinson in the two lemmas below to obtain desired information.

Lemma 2.12: If D is a normal divisible abelian subgroup of a group G and H a subgroup of G such that (i) $[D,H] \neq 1$ implies $[D,H,H] < [D,H]$ and (ii) H/H' is periodic, then $[D,H] = 1$.

Proof: The statement and the proof are only a slight sharpening of [38; Lemma 3.13]. Let $D_1 = [D,H]$ and $D_2 = [D,H,H]$. Define a mapping φ from the set $D \times H/H'$ to D_1/D_2 by

$$(d,hH')\varphi = [d,h]D_2 .$$

We claim φ is well defined: Let $h_1, h_2 \in H$ with $h_1H' = h_2H'$. Then since $H' \triangleleft H$ we have $h_2h_1^{-1} \in H'$ and

$$[d,h_2]^{-1}[d,h_1] = h_2^{-1}d^{-1}h_2h_1^{-1}dh_1$$

$$= h_1^{-1}[h_2h_1^{-1},d]h_1 \in [H',D] .$$

But by the 3-subgroup lemma, $[H',D] \leq [D,H,H] = D_2$. Hence $(d,h_1H')\varphi = (d,h_2H')\varphi$. Now we claim that φ is bilinear: Let $d_1, d_2 \in D$, $h \in H$. Note that $[D,H]$ is a subgroup of D and hence centralized by D. Thus

$$(d_1d_2,hH')\varphi = [d_1d_2,h]D_2$$

$$= [d_1,h]^{d_2}[d_2,h]D_2$$

$$= [d_1,h][d_2,h]D_2$$

$$= (d_1,hH')\varphi(d_2,hH')\varphi .$$

Let $d \in D$ and $h_1, h_2 \in H$. Then

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\begin{aligned}
(d,(h_1H')(h_2H'))\varphi &= [d,h_1h_2]D_2 \\
&= [d,h_2][d,h_1]^{h_2}D_2 \\
&= [d,h_2][d^{h_2},h_1^{h_2}]D_2 \\
&= [d,h_2][d^{h_2},h_1]D_2 ,
\end{aligned}
$$

since φ is well defined. Now since D is abelian,

$$(d,h_1H')\varphi(d,h_2H')\varphi = [d,h_2][d,h_1]D_2 .$$

Thus $(d,(h_1H')(h_2H'))\varphi = (d,h_1H')\varphi(d,h_2H')\varphi$ if and only if $[d^{h_2},h_1]^{-1}[d,h_1] \in D$. Again, since D is abelian,

$$
\begin{aligned}
[d^{h_2},h_1]^{-1}[d,h_1] &= (d^{-1})^{h_2h_1}d^{h_2}d^{-1}d^{h_1} \\
&= d^{h_2}d^{-1}d^{h_1}(d^{-1})^{h_2h_1} \\
&= [d^{-1},h_2]^{-1}h_1^{-1}[d^{-1},h_2]h_1 \\
&= [d^{-1},h_2,h_1] \in D_2 .
\end{aligned}
$$

Thus φ is indeed bilinear, and there exists a unique homomorphism $\theta:D\otimes H/H' \to D_1/D_2$ ($\otimes$ denotes tensor product) making the diagram

$$
\begin{array}{ccc}
D \times \frac{H}{H'} & \xrightarrow{\text{identity}} & D\otimes\frac{H}{H'} \\
& \varphi \searrow \quad \swarrow \theta & \\
& D_1/D_2 &
\end{array}
$$

commute. [22; Thm. 59.1.] Since φ is onto D_1/D_2, so is θ. But since D is divisible and H/H' is periodic, $D\otimes H/H'$ is **trivial.** Hence $D_1 = D_2$. By condition (i) we conclude that $[D,H] = 1$ as desired.

Lemma 2.12 allows us to sharpen another result of Robinson [36; Lemma 2.1(iii)] which states some conditions under which a subgroup H of a group G will centralize a normal divisible abelian subgroup D of G.

Lemma 2.13: If G is a group with a normal divisible abelian subgroup D and H is any descendant ZD-subgroup of G with H/H' periodic, then $[D,H] = 1$.

We postpone the proof of 2.13 until proving 2.15, but we do note the immediate

Corollary 2.14: If G is a periodic $\mathfrak{N}^d$ group and possesses a normal divisible abelian subgroup D, then $D \leq Z(G)$.

The following lemma is probably well known.

Lemma 2.15: A group G is a ZD group if and only if for every nontrivial normal subgroup K of G we have $[G,K] < K$.

Proof: Sufficiency. We suppose that for every nontrivial normal subgroup K of G we have $[G,K] < K$ and that $\gamma_\alpha(G) \neq 1$ for all ordinals α. Then the lower central series must stabilize at some nontrivial normal subgroup: i.e., there is an ordinal β such that $\gamma_\beta(G) \neq 1$ and $\alpha \geq \beta$ implies $\gamma_\alpha(G) = \gamma_\beta(G)$. But by hypothesis we have $\gamma_{\beta+1}(G) = [G, \gamma_\beta(G)] < \gamma_\beta(G)$, a contradiction as desired.

Necessity. Let G be a ZD-group and

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_\alpha \geq \cdots \geq G_\sigma = 1$$

be the lower central series for G. Let K be a nontrivial normal subgroup of G and choose α minimal such that $K \not\leq G_\alpha$. Suppose α is a limit ordinal. Then for all $\beta < \alpha$ we have $K \leq G_\beta$ by choice of α and hence $K \leq \bigcap_{\beta < \alpha} G_\beta = G_\alpha$, a contradiction. Thus α is not a limit ordinal and we have $K \leq G_{\alpha-1}$ and

$$[G,K] \leq [G, G_{\alpha-1}] = G_\alpha .$$

Since $K \triangleleft G$, we have $[G,K] \leq K$. If $[G,K] = K$, then $K \leq G_\alpha$, contradicting the choice of α. Hence $[G,K] < K$ and the lemma is proved.

Proof of 2.13: Let G have a normal divisible abelian subgroup D and let H be a descendant ZD-subgroup of G, with H/H' periodic.

Case 1. $D \leq N_G(H)$. Then $H \triangleleft DH$ and hence $[D,H]$ is a normal subgroup of the ZD-group H. By 2.15 we have $[D,H] \neq 1 \Longrightarrow [D,H,H] < [D,H]$. Hence by 2.12, $[D,H] = 1$ as desired.

Case 2. $D \not\leq N_G(H)$. Then $H \ntriangleleft DH$. Since H desc G, we have H desc $DH \leq G$. Thus $H^{DH,2} < H^{DH,1}$. Now

$$H^{DH,1} = H^{HD} = H^D = H[D,H]$$

and

$$H^{DH,2} = H^{(H^{DH,1})} = H^{H[D,H]} = H^{[D,H]} = H[D,H,H] .$$

Hence $[D,H,H] < [D,H]$, and by 2.12 we have $[D,H] = 1$. But then $D \leq C_G(H) \leq N_G(H)$, a contradiction. Thus Case 2 cannot occur.

Chapter III

Structure of $L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}^{snb}$

In a number of papers, [9-13] and [20], Černikov studies infinite groups with particular subgroup properties satisfied by all those infinite subgroups contained in a specified class of groups. For example, [11] and [13] study three classes of infinite groups: those with all infinite subgroups normal, ascendant, or complemented, respectively. In [34], R. Phillips studies the second of these classes under the name $I\mathfrak{N}^{s} - \mathfrak{N}^{s}$.

If $\mathfrak{X}$ is a subgroup theoretic property, let $\overline{\mathfrak{X}}$ denote the class of groups all of whose subgroups are $\mathfrak{X}$-subgroups, and let $I\overline{\mathfrak{X}}$ denote the class of all infinite groups, all of whose infinite subgroups are $\mathfrak{X}$-subgroups, (temporary notation). Belonging to all classes of the form $I\overline{\mathfrak{X}}$ would be an infinite nonabelian group with all proper subgroups finite, if such a group exists. The question whether such groups exist has remained open since its formulation by Schmidt in 1938. (See [19] for an expository account of Schmidt's problem; also [38; Sec. 3.4] provides an excellent discussion). Thus, in studying groups of type $I\overline{\mathfrak{X}}$, one must either solve Schmidt's problem or impose additional restrictions to avoid the problem.

In [11] and [34] the additional restriction of local finiteness is placed on the groups studied, which avoids the problem because of a theorem discovered independently by Kargapolov [28] and by P. Hall and Kulatilaka [25] (see also [38; Thm. 3.43]), which says that an infinite locally finite group always possesses an infinite abelian subgroup. By using the known results on the class $L\mathfrak{F} \cap I\mathfrak{N}^{s} - \mathfrak{N}^{s}$

we find it more natural to use local nilpotence than local finiteness in the present discussion. Thus, this paper may be considered an investigation into the structure of the classes $L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}^{snb}$ and $L\mathfrak{N} \cap I\mathfrak{N}^{d} - \mathfrak{N}^{d}$.

We will make considerable use of the fact that $\mathfrak{N}^{snb} = \mathfrak{N}$, a result due to Roseblade, which we state without proof:

Theorem 3.1: There is a function f with domain and range the positive integers, such that if every subgroup of a group G is subnormal with subnormal defect at most s, then G is nilpotent of class not exceeding $f(s)$.

Reference for proof: [40; Thm. 1] or [39; Thm. 7.42 and Corollary].

Corollary 3.2: $\mathfrak{N}^{snb} = \mathfrak{N}$.

Corollary 3.3: If $G \in I\mathfrak{N}^{snb}$ and H is an infinite normal subgroup of G, then $G/H \in \mathfrak{N}^{snb} = \mathfrak{N}$.

Notation 3.4: Throughout the remainder of this chapter, f will denote the function of 3.1.

Lemma 3.5: If $G \in L\mathfrak{N} \cap I\mathfrak{N}^{snb}$ and M is a finite normal subgroup of G with $G/M \in \mathfrak{N}_r$, then $G \in \mathfrak{N}_c$, where c depends only on r, $|M|$, and the bound b for defects of infinite subgroups.

Proof: Let F be a finite subgroup of G. Then MF is nilpotent since $G \in L\mathfrak{N}$ and hence

$$F \lhd\lhd_{|M|} MF \lhd\lhd_{r} G .$$

That is, every subgroup H of G satisfies $H \lhd \lhd_s G$, where $s = \max\{b, |M| + r\}$. By 3.1, we have $G \in \mathfrak{N}_{f(s)}$.

Lemma 3.6: Let $G \in L\mathfrak{N} \cap I\mathfrak{N}^{snb}$. If G has any nonempty collection $\mathfrak{H} = \{H_\gamma | \gamma \in \Gamma\}$ of infinite normal subgroups such that $\cap \mathfrak{H}$ is finite, then $G \in \mathfrak{N}_c$, where c depends only on the bound for the defects of infinite subgroups, the order of $\cap \mathfrak{H}$, and the class of $\cap \mathfrak{H}$.

Proof: First we suppose that $\cap \mathfrak{H} = 1$. Let b be the bound on subnormal defects of infinite subgroups of G. Then for all γ, $G/H_\gamma \in \mathfrak{N}_{f(b)}$. Since $\cap \{H_\gamma | \gamma \in \Gamma\} = 1$, we have G isomorphically contained in the direct product of the G/H_γ, which is nilpotent of class at most $f(b)$. Thus $G \in \mathfrak{N}_{f(b)}$. The desired conclusion for the general case, $\cap \mathfrak{H} \neq 1$, now follows from 3.5.

Theorem 3.7: Let $G \in I\mathfrak{N}^{snb}(I\mathfrak{N}^d \cap L\mathfrak{N})$. If G is nonperiodic, then $G \in \mathfrak{N}^{snb}(\mathfrak{N}^d)$.

Reference for proof: [34; Sec. VI; in particular Thm. G].

The first major theorem of this chapter is 3.12 below, during the course of whose proof we need the following four known theorems which we state without proof.

Theorem 3.8: Let G be an abelian by finite p-group. Then G is a Černikov group if and only if its center satisfies Min.

Reference for proof: [7; Thm. 3] or [39; Lemma 10.21].

Theorem 3.9: The locally nilpotent groups which satisfy the minimal condition on normal subgroups are precisely the hypercentral Černikov groups.

Reference for proof: [6], [32], or [38; Thm. 5.27, Corollary 2].

Theorem 3.10: A minimal normal subgroup of a locally nilpotent group G is contained in the center of G and is of prime order.

Reference for proof: [32].

Theorem 3.11: A periodic hypercentral group is a Černikov group if and only if each upper central factor with finite ordinal type satisfies Min.

Reference for proof: [33; Thm. 8] or [39; Thm. 10.23, Corollary 1].

The following theorem obtains $(L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}) \leq \text{Min}$ and shows that we may limit our attention to p-groups in $(L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N})$.

Theorem 3.12: $G \in L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$ if and only if $G = P + K$, where P is a Sylow p-subgroup of G, P is an infinite Černikov group, $P \in L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$ and K is a finite nilpotent group.

Proof: Necessity. Let $G \in L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$. Let

$$A = \bigcap \{H \triangleleft G \mid H \text{ is infinite}\} .$$

By 3.6, A is infinite. Since $A \in L\mathfrak{N}$ and since by 3.7, A is periodic, we may write $A = \sum \{A_i \mid i \in I\}$, each A_i a Sylow p_i-subgroup of A. Notice that A_i char A; thus $A_i \triangleleft G$ for all $i \in I$. If $|I| > 1$, A_i is finite for all $i \in I$ since A can have no proper infinite subgroup which is normal in G. Hence $|I| > 1$ implies that I is infinite since A is infinite; but then A has a proper infinite subgroup which is normal in G. Hence $|I| = 1$; i.e., A is a p-group. Let P be a Sylow p-subgroup of G containing A. Since $G \in L\mathfrak{N}$ and periodic write $G = P + K$, where K is a Sylow p'-group of G. If K is infinite, then by definition of A and P we have $A \leq P \cap K = 1$. Hence K is finite and $K \cong G/P \in \mathfrak{N}$ by 3.3. Clearly $P \notin \mathfrak{N}$ since $G = P + K \notin \mathfrak{N}$. However, $P \in L\mathfrak{N} \cap I\mathfrak{N}^{snb}$ and hence it remains only to show that P is a Černikov p-group. To this end we claim first that A' is finite. Suppose A' is infinite. Since A' char A char G, $A' \lhd G$ and hence $A' = A$. Since $P \in L\mathfrak{N} \leq Z$, we have for all x, $1 \neq x \in A$, $[P, x^P] < x^P$. Hence $x^P < A$. Since A has no proper infinite subgroups which are normal in P, x^P is finite. Thus $P/C_P(x^P)$ is finite and hence $A \leq C_P(x^P)$. Since $x \in A$ was arbitrary we have $A' = 1$, a contradiction. Thus A' is a finite normal subgroup of P.

Next we claim that A is a hypercentral Černikov group. Let

$$\frac{B}{A'} = \frac{A}{A'}(p)$$

be the maximal elementary abelian subgroup of A/A'. Suppose that B/A' is infinite. Then $A \leq B$; i.e., $A = B$ and A/A' is infinite elementary abelian. Let $F \leq P$ be finite. $P/A \in \mathfrak{N}_s$, for some s. Thus

$$(1) \qquad [P, \underbrace{F, \ldots, F}_{s}] \leq [P, \underbrace{AF, \ldots, AF}_{s}] \leq A .$$

Now AF/A' is abelian by finite and is not a Černikov group. Thus by 3.8 its center is infinite. Hence $Z(\frac{AF}{A'}) \cap \frac{A}{A'}$ is also infinite, and an elementary abelian p-group; i.e., it is an infinite direct sum of cyclic groups of order p and contained in the center of AF/A'. Thus $AF/A' \in L\mathfrak{N} \cap I\mathfrak{N}^{snb}$ (bound b) and has an infinite descending chain of normal subgroups with trivial intersection. By 3.6, $AF/A' \in \mathfrak{N}_c$, for some c. Thus

$$[P, \underbrace{F, \ldots, F}_{s+c}] \leq [A, \underbrace{AF, \ldots, AF}_{c}] \leq A'$$

where we have used (1). Hence every subgroup of P/A' is subnormal with defect at most $\max\{b, s+c\}$. Thus $P/A' \in \mathfrak{N}$. By 3.5, $P \in \mathfrak{N}$, a contradiction. Thus B/A' is finite; i.e., A/A' has Min and so $A \in$ Min. By 3.9, A is hypercentral Černikov p-group.

Next we claim that $P \in ZA$. Since $A \triangleleft P$ and $A \in$ Min we may choose $1 \neq B \leq A$, B minimal with respect to $B \triangleleft P$. By 3.10, $B \leq Z(P)$. Hence $Z(P) \neq 1$. Now $P/Z(P) \in L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$ and so by induction

$$Z_n(P) < Z_{n+1}(P), \quad n = 0, 1, 2, \ldots$$

whence $Z_\omega(P) = \bigcup_{n<\omega} Z_n(P)$ is infinite. Thus $P/Z_\omega(P) \in \mathfrak{N}$ by 3.3 and we have $P \in ZA$.

Next note that $Z_n(P)$ is finite for all $n < \omega$ since otherwise $P/Z_n(P)$ is nilpotent for some n, implying that P is nilpotent. Thus by 3.11, P is a Černikov group. This completes the necessity.

Sufficiency. Let $G = P + K$, P an infinite p-group, $P \in L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$ and K a finite nilpotent p'-group. Then $G \in L\mathfrak{N} - \mathfrak{N}$ and we claim furthermore that $G \in I\mathfrak{N}^{snb}$. Let r be the bound on subnormal defects of infinite subgroups of P and $K \in \mathfrak{N}_c$. Let H be an infinite subgroup of G. Since $H \in L\mathfrak{N}$, write

$H = H_p + H_{p'}$, where H_p is the Sylow p-subgroup of H and $H_{p'}$ is the Sylow p'-subgroup of H. Then $H_p \leq P$ and $H_{p'} \leq K$; hence H_p is infinite. But then we have

$$H = H_p + H_{p'} \lhd\lhd_c H_p + K \lhd\lhd_r P + K ,$$

whence every infinite subgroup of G is subnormal of defect no more than $c + r$, as desired.

For the next theorem about the structure of p-groups in the class $L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$, we need the following lemma which we state without proof:

Lemma 3.13: Let D be a divisible abelian subgroup and F a finite subgroup of a group. If $D^F = D$, then $D = C_D(F)[D,F]$.

Reference for proof: [38; Lemma 3.29.1].

Now we restrict our attention to p-groups in the class $L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$. The next theorem gives a version of their structure which is further pursued, together with examples, in Chapter VI.

Theorem 3.14: Let P be an infinite p-group. Then $P \in L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$ if and only if P is a Černikov p-group satisfying the following: Let $D = D(P)$ and $C = C_P(D)$. (i) $C < P$ with $C \in \mathfrak{N}$, and (ii) $x \in P - C$ implies x does not normalize any infinite proper subgroup of D.

<u>Proof:</u> Necessity. P is Černikov by 3.12; by 2.10(ii), D is a divisible abelian p-group of finite index in P. Thus $D \leq Z(C)$ and hence $C/Z(C)$ is a finite p-group and thus nilpotent. Thus $C \in \mathfrak{N}$ and we conclude that $C < P$. Now let $x \in P$ normalize an infinite proper subgroup H of D. Then $D = C_D(x)[D, \langle x \rangle]$ by 3.13. If $C_D(x)$ is finite, then $[D, \langle x \rangle]$ has finite index in D. Since D has no proper subgroups of finite index, we have for all $n \geq 1$, $[\underbrace{D, \langle x \rangle, \langle x \rangle, \ldots, \langle x \rangle}_{n}] = D$. But, $H \lhd D\langle x \rangle$ and H is infinite. Thus by hypothesis and using 3.3

$$\frac{D\langle x \rangle}{H} \in \mathfrak{N}.$$

Hence there exists an r such that

$$[\underbrace{D, \langle x \rangle, \langle x \rangle, \ldots, \langle x \rangle}_{r}] \leq H < D,$$

a contradiction. Thus we have $C_D(x)$ infinite. Thus $\langle x \rangle \lhd C_P(x) \lhd\lhd_s P$, where s is the bound on subnormal defects of infinite subgroups of P, and by 2.13 we have $[D,x] = 1$; i.e., $x \in C = C_P(D)$.

Sufficiency. Let H be an infinite subgroup of P. We claim that either $D \leq H$ or $H \leq C$. For suppose otherwise. Then $H \cap D$ is a proper infinite subgroup of D and there is some $x \in H - C$. By condition (ii) we have $x \notin N_P(H \cap D)$. But $H \cap D \lhd H$ and $x \in H$, a contradiction, which establishes the claim. But now clearly $P \in I\mathfrak{N}^{snb}$ because $D \leq H$ implies $H \lhd\lhd_d P$ where $P/D \in \mathfrak{N}_d$ (P/D being a finite p-group) and $H \leq C$ implies $H \lhd\lhd_c C \lhd P$, where $C \in \mathfrak{N}_c$. Hence $H \lhd\lhd_s P$, where $s = \max\{d, c + 1\}$. Now let $x \in P - C$ (we use condition (i)). By 2.13 we have $< x >$ is not descendant in P. Thus $P \notin \mathfrak{N}$. A Černikov p-group is locally finite and hence in $L\mathfrak{N}$ and so $P \in L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$.

It is primarily because of [34] that we have restricted our attention to locally nilpotent groups instead of locally finite groups. We now consider the relationship of 3.14 to [34; Thm. D].

<u>Lemma</u> 3.15: ([34; Lemma 3.1]) $\mathfrak{N}^s \cap L\mathfrak{F} = L\mathfrak{N} \cap L\mathfrak{F}$.

<u>Lemma</u> 3.16: $(L\mathfrak{F} \cap I\mathfrak{N}^{snb} - \mathfrak{N}) - (L\mathfrak{F} \cap I\mathfrak{N}^s - \mathfrak{N}^s) = L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$.

<u>Proof</u>: Clearly $L\mathfrak{F} \cap I\mathfrak{N}^{snb} \leq L\mathfrak{F} \cap I\mathfrak{N}^s$. Therefore

$$\begin{aligned}(L\mathfrak{F} \cap I\mathfrak{N}^{snb} - \mathfrak{N}) - (L\mathfrak{F} \cap I\mathfrak{N}^s - \mathfrak{N}^s) &= L\mathfrak{F} \cap I\mathfrak{N}^{snb} \cap \mathfrak{N}^s - \mathfrak{N} \\ &= L\mathfrak{F} \cap L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}\end{aligned}$$

where we have also used 3.15. But $G \in L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$ implies G is a Černikov group (3.12) and hence $G \in L\mathfrak{F}$. The desired result follows.

It is an unsettled question whether $(L\mathfrak{F} \cap I\mathfrak{N}^s - \mathfrak{N}^s) - (L\mathfrak{F} \cap I\mathfrak{N}^{snb} - \mathfrak{N})$ is empty; equivalently whether $L\mathfrak{F} \cap I\mathfrak{N}^s - \mathfrak{N}^s \leq I\mathfrak{N}^{snb}$. To answer affirmatively, it suffices to show that $L\mathfrak{F} \cap I\mathfrak{N}^s - \mathfrak{N}^s \leq I\mathfrak{N}^d$ because $L\mathfrak{F} \cap I\mathfrak{N}^s - \mathfrak{N}^s \leq \mathrm{Min}$ (see 3.17 below) and it is known that if G satisfies Min, then there is a bound on the subnormal defects of subnormal subgroups (see [38; Corollary to Thm. 5.49]). Therefore $I\mathfrak{N}^d \cap \mathrm{Min} \leq I\mathfrak{N}^{sn} \cap \mathrm{Min} \leq I\mathfrak{N}^{snb}$. Another equivalent formulation of this question is whether a group in the class $L\mathfrak{F} \cap I\mathfrak{N}^s - \mathfrak{N}^s$ must satisfy the conditions (i) - (iii) of Theorem 3.19 below. But first some preliminaries.

Theorem 3.17: ([34; Theorem D]) $G \in L\mathfrak{F} \cap I\mathfrak{N}^s - \mathfrak{N}^s$ if and only if G has a normal Sylow p-subgroup P satisfying

(i) P is a Černikov p-group,

(ii) $G = P]K$ where $G/D(P)$ is a finite nilpotent group,

(iii) If $x \in K$ normalizes an infinite subgroup of $D(P)$, then $x \in C(P)$, and

(iv) There is an $x \in K$ such that $[x,P] \neq 1$.

Lemma 3.18: ([34; 2.4.1 and 2.4.2]) (i) If $G \in L\mathfrak{F}$ and P is a normal Sylow p-subgroup of G with G/P countable, then P has a complement. (ii) If $G \in L\mathfrak{F}$ and locally solvable and $G = P]K$ where P is a Sylow p-subgroup of G and K is finite, then any Sylow p'-subgroup L of G is conjugate to K.

Theorem 3.19: $G \in L\mathfrak{F} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$ if and only if G has a normal Sylow p-subgroup P such that P is an infinite Černikov group and G satisfies either conditions (i) - (iii) or conditions (i') - (iii') depending on whether $G \notin L\mathfrak{N}$ or $G \in L\mathfrak{N}$, respectively. Let $D = D(P)$ and $C = C_P(D)$.

(i) $G = P]K$, $G/D \in \mathfrak{N} \cap \mathfrak{F}$	(i') $G = P + K$, $G/D \in \mathfrak{N} \cap \mathfrak{F}$
(ii) $C_G(D) \in \mathfrak{N}$ and K s$\not$r G	(ii') $C < P$ with $C \in \mathfrak{N}$

(iii) If $x \in G$ normalizes an infinite proper subgroup of D, then $x \in C_G(D)$. If, in addition, $x \in K$, then $x \in C_G(P)$.	(iii') $x \in P - C$ implies x does not normalize any infinite proper subgroup of D.

Proof: Sufficiency. Let G satisfy (i) - (iii). Clearly $G \in L\mathfrak{F}$, and since K s$\not$r G, we have $G \notin \mathfrak{N}$. We need only show that $G \in I\mathfrak{N}^{snb}$. Notice that G is locally solvable since every finitely generated subgroup is an extension of a finite p-group by a finite nilpotent group, hence is solvable. Let H be an infinite subgroup of G. Then $H \cap P$ is an infinite normal Sylow p-subgroup of H and by 3.18 we may write

$$H = (H \cap P)]L ,$$

where L is a p'-group, and for some $t \in G$, $L^t \leq K$. Now

$$L \leq N_G(D) \cap N_G(H \cap P)$$

and hence

(*) $\qquad H \cap P \cap D = H \cap D$ is an infinite subgroup of D ,

normalized by L.

Case 1. $H \cap D = D$. Then $D \leq H$ and hence

$$\frac{H}{D} \lhd \lhd \frac{G}{D} \in \mathfrak{N}$$

giving $H \lhd \lhd G$.

Case 2. $H \cap D < D$. We claim that $L \leq C_G(P)$. On account of (*) and condition (iii) we have that $L \leq C_G(D)$. Let $x \in L$. Then x^t is an element of K which centralizes the infinite proper subgroup of D, $H \cap D$. By the second part of condition (iii), $x^t \in C_G(P)$ and hence $x \in C_G(P)$. Thus $L \leq C_G(P)$. Hence, $H = (H \cap P) + L$. Now we claim that $H \cap P \lhd \lhd P$. We assume without loss of generality that $P \notin \mathfrak{N}$. Then $C = C_P(D) \in \mathfrak{N}$ since $C \leq C_G(D) \in \mathfrak{N}$ by condition (ii). Thus $C < P$. Furthermore, $x \in P - C$ implies that x does not normalize any infinite proper subgroup of D by condition (iii). Thus, P satisfies conditions (i) and (ii) of 3.14, so that $P \in I\mathfrak{N}^{snb}$. Hence, $H \cap P \lhd \lhd P$, and we have

$$H = (H \cap P) + L \lhd \lhd P + L \lhd \lhd G,$$

where the last relation follows from the fact that G/P is nilpotent, being an image of G/D.

Thus, in both Case 1 and Case 2 we have $H \lhd \lhd G$, and we have shown $G \in I\mathfrak{N}^{snb}$. Hence also $G \in L\mathfrak{F} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$.

Now we show $G \notin L\mathfrak{N}$. Since G satisfies conditions (i) - (iii), G also satisfies conditions (i) - (iii) of 3.17. If $[K,P] = 1$, then $G = P + K$, and $K \lhd G$, violating (ii). Hence G satisfies (iv) of 3.17 as well, showing $G \in L\mathfrak{F} \cap I\mathfrak{N}^s - \mathfrak{N}^s$. By 3.16, $G \notin L\mathfrak{N}$.

That conditions (i') - (iii') imply $G \in L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$ follows from 3.12 and 3.14. Furthermore, $G \in L\mathfrak{F}$, since G is a Černikov group. This completes sufficiency.

Necessity. Let $G \in L\mathfrak{F} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$. If $G \in L\mathfrak{N}$, we are done, by 3.12 and 3.14. Suppose $G \notin L\mathfrak{N}$. By 3.16, $G \in L\mathfrak{F} \cap I\mathfrak{N}^s - \mathfrak{N}^s$. Hence, by 3.17, G has a normal Sylow p-subgroup P which is a Černikov group with $D = D(P)$, $C = C_P(D)$ and furthermore, $G = P]K$, $G/D \in \mathfrak{N} \cap \mathfrak{F}$.

We claim that condition (ii) is satisfied. Now $D \leq Z(C_G(D))$ and hence $\overline{C} = \dfrac{C_G(D)}{Z(C_G(D))}$ is a subgroup of an image of G/D. Hence $\overline{C}$ is nilpotent, giving $C_G(D) \in \mathfrak{N}$. Since a serial Sylow p'-subgroup of any group is normal, K ser G. Thus we have (ii).

To show (iii), let $x \in G$ normalize an infinite proper subgroup H of D. We argue exactly as in 3.14 that $C_D(x)$ is infinite. Then since $G \in I\mathfrak{N}^{snb}$, we have

$$\langle x \rangle \triangleleft C_G(x) \triangleleft\triangleleft G$$

and by 2.13, $[D,x] = 1$; i.e., $x \in C_G(D)$. If $x \in K$ as well, then by 3.17(iii) we have $x \in C_G(P)$. Thus we have condition (iii).

Chapter IV

Structure of $L\mathfrak{N} \cap I\mathfrak{N}^d - \mathfrak{N}^d$

In this section we present a structure theorem for $L\mathfrak{N} \cap I\mathfrak{N}^d - \mathfrak{N}^d$ analogous to 3.12 and 3.14. It is still unsettled, however, whether such groups satisfy Min. If they do, then $L\mathfrak{N} \cap I\mathfrak{N}^d - \mathfrak{N}^d = L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$. Constructing an example without Min seems to be a difficult problem--the Heineken-Mohammed group seems invited into the construction, but it cannot be used in the obvious place, as we shall see. Since the study of $L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$ made considerable use of the fact that $\mathfrak{N}^{snb} = \mathfrak{N}$, it may be that the question will only be settled after significant information is obtained about p-groups, all of whose subgroups are descendant, not satisfying Min, and possessing a finite non-trivial abelian image (see 4.7).

<u>Lemma</u> 4.1: If $G \in L\mathfrak{N} \cap I\mathfrak{N}^d$ and $M \triangleleft G$, M a finite subgroup with $G/M \in \mathfrak{N}^d$, then $G \in \mathfrak{N}^d$.

<u>Proof</u>: Let F be finite. Then MF is nilpotent and hence

$$F \triangleleft\triangleleft MF \text{ desc } G .$$

Thus F desc G.

<u>Lemma</u> 4.2: Let $G \in L\mathfrak{N} \cap I\mathfrak{N}^d$. If G has any nonempty collection $\mathfrak{C} = \{H_\gamma | \gamma \in \Gamma\}$ of infinite normal subgroups such that $\cap \mathfrak{C}$ is finite, then $G \in \mathfrak{N}^d$.

<u>Proof</u>: By 2.3 we assume without loss of generality either $|\Gamma| = 2$ or $\mathfrak{C}$ is a descending chain. By 4.1 we further assume that $\cap \mathfrak{C} = 1$.

Case 1. $|\Gamma| = 2$. Let $\mathcal{C} = \{H,K\}$. Then $H, K \triangleleft G$, $H \cap K = 1$. Let F be a finite subgroup of G. By 2.4, we have $HF \cap KF$ finite, hence nilpotent. Thus $F \triangleleft\triangleleft HF \cap KF$ desc G since the intersection of descendant subgroups is descendant.

Case 2. $\mathcal{C}$ is a descending chain, say $\mathcal{C} = \{H_\alpha | \alpha < \sigma\}$. Let β be an ordinal such that for all infinite subgroups H of G, we have $H^{G,\beta} \leq H$. Let F be a finite subgroup of G. Then

$$(H_\alpha F)^{G,\beta} \leq H_\alpha F ,$$

for all $\alpha < \sigma$. Thus $F^{G,\beta} \leq \bigcap_{\alpha < \sigma} (H_\alpha F) = F$ where the last equality comes from 2.2.

The following theorem shows we may limit our attention to p-groups in the class $L\mathfrak{N} \cap I\mathfrak{N}^d - \mathfrak{N}^d$.

Theorem 4.3: $G \in L\mathfrak{N} \cap I\mathfrak{N}^d - \mathfrak{N}^d$ if and only if $G = P + K$, where P is a p-group in the class $L\mathfrak{N} \cap I\mathfrak{N}^d - \mathfrak{N}^d$ and K is a finite nilpotent p'-group.

Proof: Necessity. Let $A = \bigcap \{H \triangleleft G | H$ is infinite$\}$. By 4.2 A is infinite. We prove exactly as in 3.12 that A is a p-group and $G = P + K$, where P is a Sylow p-subgroup of G and K a Sylow p'-subgroup of G. Furthermore K is finite. That $P \in L\mathfrak{N} \cap I\mathfrak{N}^d$ is clear. Suppose $P \in \mathfrak{N}^d$. Let H be any subgroup of G. Since G is periodic (3.7) and locally nilpotent, elements of relative prime order commute, and we write

$$H = (H \cap P) + L, \quad L^t \leq K, \quad \text{for some} \quad t \in G .$$

Now by supposition, $H \cap P$ desc P and $G/P \cong K \in \mathfrak{M}$ and hence

$$H = (H \cap P) + L \text{ desc } P + L \lhd\lhd P + K = G .$$

Thus $G \in \mathfrak{M}^d$, a contradiction, assuring that $P \in L\mathfrak{M} \cap I\mathfrak{M}^d - \mathfrak{M}^d$.

Sufficiency. Let $G = P + K$ be as in the statement. Clearly $G \in L\mathfrak{M} - \mathfrak{M}^d$. We claim $G \in I\mathfrak{M}^d$ as well. Let H be an infinite subgroup of G. As above, write

$$H = (H \cap P) + L, \quad \text{where } [P,L] = 1, \quad L^t \leq K .$$

Then L is finite and hence $H \cap P$ is infinite. Thus

$$H = (H \cap P) + L \text{ desc } P + L \lhd\lhd P + K = G .$$

<u>Theorem</u> 4.4: Let P be a locally nilpotent p-group. Then $P \in I\mathfrak{M}^d - \mathfrak{M}^d$ if and only if P has a normal divisible abelian subgroup D of finite rank with $C = C_P(D)$ satisfying

(i) $P/D \in \mathfrak{M}^d$,

(ii) $C < P$ and $C \in \mathfrak{M}^d$,

(iii) $x \in P - C$ implies x does not normalize any proper infinite subgroup of D, and

(iv) if H is an infinite subgroup of P, then either $H \leq C$ or $D \leq H$.

<u>Proof</u>: Necessity. Let $A = \bigcap\{H \lhd P \mid H \text{ is infinite}\}$. By 4.2, A is infinite and by construction contains no proper infinite subgroups which are normal in P. We argue exactly as in 3.12 that A' is a finite normal subgroup of P. Let $\frac{B}{A'} = \frac{A}{A'}(p)$ be the maximal elementary

abelian subgroup of A/A'. Suppose that B is infinite. Then $A = B$; i.e., A/A' is an infinite elementary abelian p-group. Let F be a finite subgroup of P. Then AF/A' is abelian by finite and not a Černikov group, so it has infinite center by 3.8. Thus

$$Z(\frac{AF}{A'}) \cap \frac{A}{A'}$$

is also infinite. Thus $AF/A' \in L\mathfrak{N} \cap I\mathfrak{N}^d$ and has an infinite descending chain of normal subgroups with trivial intersection. By 4.2, $AF/A' \in \mathfrak{N}^d$ and by 4.1, $AF \in \mathfrak{N}^d$. Hence F desc AF desc P; but then $P \in \mathfrak{N}^d$, a contradiction. Thus B is finite and $A/A' \in$ Min and hence $A \in$ Min, meaning that A is a hypercentral Černikov group (3.9).

Now let D be a maximal normal divisible abelian subgroup of P (use 2.10(i)). Clearly D is nontrivial. If rank D is infinite, then $D(p)$ is infinite and normal in P; hence $A \leq D(p)$, a contradiction. Thus D has finite rank. Let $C = C_P(D)$. Condition (i) is satisfied since D is infinite and $P \in I\mathfrak{N}^d$. Now $C \in \mathfrak{N}^d$, because any group in $I\mathfrak{N}^d$ with infinite center is in $\mathfrak{N}^d$. (Let F be any subgroup of C. Then $F \lhd Z(C)F$ desc C.) Condition (ii) is now apparent.

Now let $x \in P$ normalize an infinite proper subgroup H of D. Suppose $C_D(x)$ is finite. Since $D = C_D(x)[D, < x >]$ by 3.13, we have $[D, < x >]$ a subgroup of D of finite index; thus $[D, < x >]$ $= D$. But then

$$(H < x >)^{D < x >} = H < x >[D < x >, H < x >] \geq H < x > D = D < x > ;$$

i.e., $H < x >$ d~~e~~sc $D < x >$ so that $D < x > \notin I\mathfrak{N}^d$. The contradiction assures that $C_D(x)$ is infinite. Then

$$< x > \lhd C_P(x) \text{ desc } P$$

and by 2.13 we have $[D,x] = 1$; i.e., $x \in C$. Thus we have condition (iii) as well. To obtain condition (iv), let H be any infinite subgroup of P and assume that $H \not\leq C$. Let $x \in H - C$. If $C_D(x)$ is infinite, then x centralizes an infinite subgroup of D so that by condition (iii) it must centralize D, contradicting $x \notin C$. Thus $C_D(x)$ is finite and so by 3.13 $[D, < x >]$ is a subgroup of D of finite index. Hence $[D, < x >] = D$. Now

$$H^{DH} = H[DH,H] \geq H[D, < x >] = HD = DH .$$

But since H desc DH, we have $H = DH$, whence $D \leq H$, as desired.

Sufficiency. Let H be an infinite subgroup of P. If $H \leq C$, then H desc $C \lhd P$ and hence H desc P. Otherwise $D \leq H$ by condition (iv) and

$$\frac{H}{D} \text{ desc } \frac{P}{D}$$

by condition (i), and hence H desc P. Thus $P \in I\mathfrak{N}^d$. By condition (ii) there is some $x \in P - C$; by condition (iii) and by 2.13 $< x >$ d̸esc P. Thus $P \notin \mathfrak{N}^d$.

Because of the similarity of 4.4 with 3.14, the question arises whether conditions (i) - (iii) in 4.4 actually imply condition (iv). The following example for $p = 2$ shows that this is not the case. In fact the construction can be carried out for arbitrary primes p, for one only needs to know the existence of a divisible abelian p-group of finite rank and an automorphism of order p which satisfies condition (iii). We show the existence of these in Chapter V.

Example 4.5: Let D be a 2^{∞} group, E an infinite elementary abelain 2-group, $\theta : E \to \langle \alpha \rangle$ where $\alpha \in \operatorname{Aut} D$ is the automorphism sending every element to its inverse. Let $P = D]_{\theta} E$. Let $K = \operatorname{Ker} \theta$. Then $C_P(D) = D]K = D + K$ is abelian, hence in $\mathfrak{N}^d$, $P/D \cong E$ is also in $\mathfrak{N}^d$, and $x \in P - C$ implies that x normalizes no proper infinite subgroup of D. Yet if $P \in I\mathfrak{N}^d$, then $P \in \mathfrak{N}^d$ because $K \leq Z(P)$, and K is infinite elementary abelian so that P has an infinite descending chain of normal subgroups intersecting in 1.

We next collect miscellaneous information about the groups in 4.4, some of which could be helpful in deciding whether such groups must satisfy Min. First, we make the following

Definition 4.6: A group G is a Heineken-Mohamed group if G is a metabelian p-group with the following properties: (i) G' is elementary abelian, (ii) every proper subgroup of G is subnormal and nilpotent, and (iii) $Z(G) = 1$. Examples of such groups were discovered by Heineken and Mohamed [27]. It is known [27; Lemma 1] that such a group G also satisfies (iv) G/G' is a group of type p^{∞}.

Lemma 4.7: Let P be a p-group, $P \in L\mathfrak{N} \cap I\mathfrak{N}^d - \mathfrak{N}^d$. Let D and C be as in 4.4. Then the following additional properties hold:

(i) for all $n \geq 0$, $Z_n(P) < Z_{n+1}(P)$ and $Z_n(P)$ is finite.

(ii) $Z(C) \in$ Min and P/C is finite.

(iii) $P \in$ Min if and only if $C \in \mathfrak{N} \cap$ Min if and only if $C/Z(C)$ is finite.

(iv) P has a local system of nondescendant subgroups.

(v) P/D has a finite nontrivial abelian image; thus P/D is not a Heineken-Mohamed group.

(vi) If $P \notin$ Min, then D has no proper supplement; thus in particular, P is not a split extension of D.

The proofs of portions of the above lemma require the following three results due to Baer.

Theorem 4.8: $G \in \mathfrak{N} \cap$ Min if and only if $Z(G) \in$ Min and $G/Z(G)$ is a finite nilpotent group.

References for proof: [1; Sec. 6, Satz 2] or [38; Theorem 3.14].

Lemma 4.9: Let D be a divisible abelian p-group of finite rank, and α an automorphism of D which fixes every element of order p and, when $p = 2$, every element of order 4. Then α has infinite order.

References for proof: [2; p. 525] or [38; Lemma 3.28].

Corollary 4.10: Let D be a divisible abelian p-group of finite rank. Then periodic subgroups of Aut D are finite.

Reference for proof: [38; Corollary to Lemma 3.28].

Proof of 4.7: (i) Since $D \in$ Min and $D \triangleleft P$, we may choose B, $1 \neq B \triangleleft D$, minimal with respect to $B \triangleleft P$. By 3.10, $1 \neq B \leq Z(P)$.

Thus $Z_0(P) < Z_1(P)$. If $Z_1(P)$ is infinite, then for all finite subgroups F of P, we have $F \triangleleft Z_1(P)F$ desc P; hence $P \in \mathfrak{N}^d$. Thus $Z_1(P)$ is finite. Suppose for induction that $Z_{n-1}(P) < Z_n(P)$ and $Z_n(P)$ is finite. Then by 4.1, we have $P/Z_n(P) \in L\mathfrak{N} \cap I\mathfrak{N}^d - \mathfrak{N}^d$ and so by the case already established, $P/Z_n(P)$ has finite nontrivial center. Thus $Z_n(P) < Z_{n+1}(P)$ and $Z_{n+1}(P)$ is finite.

(ii) As in the proof of 4.4, let A be the intersection of all infinite normal subgroups of P. A is infinite and $A \in$ Min. If $Z(C) \notin$ Min then $[Z(C)](p)$ is an infinite normal subgroup of P. Thus $A \leq [Z(C)](p)$, a contradiction. Thus $Z(C) \in$ Min. Now P/C is isomorphic to a periodic subgroup of Aut D and hence is finite by 4.10.

(iii) First we show $P \in$ Min implies $C \in \mathfrak{N} \cap$ Min. By 4.4 we clearly have $P \in$ Min implies $C \in \mathfrak{N}^d \cap$ Min. But $\mathfrak{N}^d \cap$ Min $\leq \mathfrak{N}^{sn} \cap$ Min is clear, and since a group satisfying Min has a bound on the subnormal defects of its subnormal subgroups [38; Corollary to 5.49] we have $\mathfrak{N}^{sn} \cap \text{Min} \leq \mathfrak{N}^{snb} = \mathfrak{N}$. Next we note that $C \in \mathfrak{N} \cap$ Min implies $C/Z(C)$ is finite, by using (ii) and 4.8. Finally, we show that $C/Z(C)$ finite implies $P \in$ Min. Since $Z(C) \in$ Min by (ii), we have $C \in$ Min, and since P/C is finite also by (ii), we have $P \in$ Min.

(iv) This fact was pointed out to me by my advisor. Let F be a nondescendant subgroup of P, and L any finitely generated subgroup. If $\langle F,L \rangle$ desc P, then $F \triangleleft \triangleleft \langle F,L \rangle$ desc P, since $\langle F,L \rangle \in \mathfrak{N}$ because F is necessarily finite. Hence $\langle F,L \rangle$ d$\not{e}$sc P. Hence $\{\langle F,L \rangle | L$ is finitely generated, $L \leq P\}$ is a local system of nondescendant subgroups of P.

(v) Since $D \leq C$, P/C is a homomorphic image of P/D. Thus it suffices to show that P/C has nontrivial abelian image. But P/C is a finite nontrivial p-group, and hence is nilpotent. Let $F = P/C$. Then $F' < F$ and F/F' is a finite nontrivial abelian image of P/C.

Now let H be a Heineken-Mohamed group. Suppose H has a finite nontrivial abelian image. Then H/H' has a proper subgroup of finite index. But H/H' is a group of type p^{∞} by 4.6(iv), which has no such subgroups.

(vi) Suppose $P \notin$ Min, and let H be any supplement of D; i.e., let $P = DH$. If $|P:D|$ is finite, then $P \in$ Min. Hence H is infinite and by 4.4(iv) we have either $H \leq C$ or $D \leq H$. But $H \leq C$ implies $DH \leq C < P$. Thus $D \leq H$ and we have $P = DH = H$.

We remarked at the beginning of this chapter that deciding the question of whether the groups studied in this chapter satisfy Min might depend on knowing the structure of infinite groups, all of whose subgroups are descendant, not satisfying Min, and which have a finite nontrivial abelian image. For if $P \notin$ Min, then P/D is such a group. Now we list some elementary results related to these groups.

<u>Lemma</u> 4.11: (i) Let G be a periodic $\mathfrak{N}^d$ group. If G has a normal divisible abelian subgroup D, then $D \leq Z(G)$.

(ii) $\mathfrak{N}^d \cap \text{Min} = ZD \cap \text{Min} = \mathfrak{N} \cap \text{Min}$

(iii) $G \in \mathfrak{N}^d$ if and only if for every $H < K \leq G$ we have $H^K < K$.

(iv) $ZD \not\leq \mathfrak{N}^d$ and $L\mathfrak{N} \cap \mathfrak{N}^d \not\leq ZD$.

Proof: (i) Let $1 \neq x \in G$. Then $\langle x \rangle$ is a descendant abelian subgroup which is periodic. Thus by 2.13, we have $[D,x] = 1$; hence $D \leq Z(G)$.

(ii) $\mathfrak{N} \cap \text{Min} \leq ZD \cap \text{Min} \cap \mathfrak{N}^d$ is clear. $ZD \cap \text{Min} \leq \mathfrak{N} \cap \text{Min}$ is also clear since the lower central series must stop after finitely many steps. All that remains is $\mathfrak{N}^d \cap \text{Min} \leq \mathfrak{N}$. But this fact we already noted in the proof of 4.7(iii).

(iii) This is obvious.

(iv) Suppose $ZD \leq \mathfrak{N}^d$. Let F be a free group. Then F is residually a finite p-group; i.e., F is a ZD group since all residually nilpotent groups are ZD groups. By hypothesis every subgroup of F is descendant. If G is a finite image of G, then G is nilpotent; i.e., every finite group is nilpotent. Thus $ZD \not\leq \mathfrak{N}^d$.

Let G be the Heineken-Mohamed group displayed by Hartley in [26] as a subgroup of the restricted wreath product W of a group $C = \langle c \rangle$ of order p by a group $U = (u_1, u_2, \ldots | u_1^p = 1, u_{i+1}^p = u_i, i = 1, 2, \ldots)$ of type p^∞. The base group B of W is naturally a right module for the group ring $\bar{Z}_pU$. Hartley obtains certain elements $a_i \in B^{u_i - 1}$ and sets $z_i = u_i a_i$, $(i = 2, 3, \ldots)$. Then $G = \langle z_2, z_3, \ldots \rangle$ is the Heineken-Mohamed group. Every Heineken-Mohamed group is an extension of an elementary abelian p-group by a p-group and hence is locally a finite p-group. Furthermore every subgroup is subnormal. Thus $G \in L\mathfrak{N} \cap \mathfrak{N}^d$. We claim that $G \notin ZD$. Let $a = c^{u_2 - 1} = c^{u_2}c^{-1}$. Now from the equation two lines below (3) of [26], we have

$$(*) \qquad \left[a^{(u_{i+1}-1)^{p-1}}, z_{i+1}\right] = \left[z_{i+1}, a^{(u_{i+1}-1)^{p-1}}\right]^{-1}$$
$$= [z_{i+1}, z_i]$$
$$= a^{u_i - 1} .$$

From equation (5) of [26] $G' = a^{\Delta}$, where Δ is the augmentation ideal of $\overline{Z}_pU$. Hence the left-hand side of (*) is in $\gamma_3(G) = [G,G,G]$ and hence for all $i = 1, 2, \dots ,$ we have

$$a^{u_i - 1} \in \gamma_3(G) .$$

But $\gamma_3(G)$ is a $\overline{Z}_pU$ submodule of B (since G' is a $\overline{Z}_pU$ submodule of B) so that

$$a^{u_i^m - 1} \in \gamma_3(G) \quad \text{for all} \quad i = 1, 2, \dots , \quad \text{and} \quad 0 \leq m < p^i .$$

Thus $a^{\Delta} = G' = \gamma_3(G)$. Now $G' = \gamma_2(G)$ is nontrivial (in fact infinite) and hence $G \notin ZD$.

We make one final remark regarding the question of whether $L\mathfrak{N} \cap I\mathfrak{N}^d - \mathfrak{N}^d \leq Min$.

Remark 4.12: If $L\mathfrak{N} \cap I\mathfrak{N}^d - \mathfrak{N}^d \leq Min$, then $L\mathfrak{N} \cap I\mathfrak{N}^d - \mathfrak{N}^d = L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$.

Proof: Clearly, $I\mathfrak{N}^d \cap Min \leq I\mathfrak{N}^{sn} \cap Min$. Since a group satisfying Min has a bound on the subnormal defects of its subnormal subgroups [38; Corollary to 5.49] we have $I\mathfrak{N}^{sn} \cap Min \leq I\mathfrak{N}^{snb}$, and the desired result follows.

Chapter V

Strongly Irreducible Automorphisms

If P is an infinite p-group and either $P \in L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$ or $P \in L\mathfrak{N} \cap I\mathfrak{N}^{d} - \mathfrak{N}^{d}$, then P has a normal divisible abelian subgroup D of finite rank, with $C = C_P(D) < P$ and P/C a finite group of automorphisms of D with the property that nontrivial elements of P/C normalize no proper infinite subgroup of D (3.14 and 4.4). Further investigation of such automorphisms in this chapter leads to some interesting relationships between the classes studied in Chapters III and IV, and direct limits of p-groups of maximal class. Those relationships will be investigated in Chapter VI.

<u>Notation</u> 5.1: Throughout this chapter, p will denote a fixed prime and D a divisible abelian p-group of finite rank, r.

<u>Lemma</u> 5.2: Let $D = D_1 + D_2 + \cdots + D_r$, where D_i has generators $x_{i,1}, x_{i,2}, \ldots$ and defining relations $px_{i,1} = 0$, $px_{i,n+1} = x_{i,n}$, $n = 1, 2, \ldots$. Then every $a \in D$, $a \neq 0$, has unique representation in the form

$$(*) \qquad a = m_1 x_{1,n} + m_2 x_{2,n} + \cdots + m_r x_{r,n} ,$$

where $|a| = p^n$ and the m_i are integers, $0 \leq m_i < p^n$, $i = 1, 2, \ldots , r$. Equivalently, multiplying as matrices we may write

$$(**) \qquad a = (m_1 \ m_2 \cdots m_r)\mathrm{col}(x_{1,n} \ x_{2,n} \cdots x_{r,n}) .$$

<u>Proof:</u> It is clear that $a \neq 0$ has a representation $(*)$.

Suppose

$$a = s_1x_{1,t} + s_2x_{2,t} + \cdots + s_rx_{r,t}$$

is also of the form (*). Then $|a| = p^n = p^t$. Hence $n = t$. Furthermore, since D is a direct sum of the D_i, we have

$$m_ix_{i,n} = s_ix_{i,n} \quad \text{for all} \quad i = 1, 2, \ldots, r .$$

Write $m_i = p^{k_i}\ell_i$, $(\ell_i, p) = 1$ and $s_i = p^{j_i}t_i$, $(t_i, p) = 1$. Then

$$m_ix_{i,n} = p^{k_i}\ell_ix_{i,n} = \ell_ix_{i,n-k_i}$$

and

$$s_ix_{i,n} = p^{j_i}t_ix_{i,n} = t_ix_{i,n-j_i} .$$

Now

$$|m_ix_{i,n}| = |s_ix_{i,n}| \quad \text{implies} \quad n - k_i = n - j_i$$
$$\text{implies} \quad k_i = j_i .$$

Furthermore,

$$m_i \equiv s_i(p^{k_i}) \quad \text{and} \quad k_i < n .$$

Thus

$$m_i \equiv s_i(p^n)$$

and since $0 \leq m_i < p^n$ and $0 \leq s_i < p^n$, we have $m_i = s_i$.

As we shall soon see, the above representation lends itself well to describing the action of automorphisms of D by use of $r \times r$ matrices over the p-adic integers. Our following description of the p-adic integers is taken from [5; pp. 20 ff.] except that we write our sequences on the set of positive integers instead of on the set of nonnegative integers.

Definition 5.3: A sequence of integers $\{x_n\}$ satisfying $x_{n+1} \equiv x_n(p^n)$ for all $n \geq 1$ determines a p-adic integer, where two such sequences $\{x_n\}$ and $\{x_n'\}$ determine the same p-adic integer if and only if $x_n \equiv x_n'(p^n)$, for all $n \geq 1$. The set of p-adic integers, denoted by R_p, form a commutative ring under componentwise addition and multiplication.

Lemma 5.4: Let $\alpha \in R_p$. Then

(i) α has a unique representation as a sequence of integers $\{x_n\}$ such that $x_{n+1} \equiv x_n(p^n)$ and $0 \leq x_n < p^n$ for all $n \geq 1$.

(ii) If $\alpha = \{x_n\}$ is the representation of (i), then α is a unit of R_p if and only if $x_1 \not\equiv 0$.

(iii) If $\alpha \neq 0$, then α has a unique representation of the form $\alpha = p^m\xi$, where m is a nonnegative integer and ξ is a unit. (See Lemma 5.5(ii).)

Proof: (i) is obtained simply by reducing the representation of 5.3 modulo p^n. (ii) and (iii) are [5; Thm. 1] and [5; Thm. 2] respectively.

Lemma 5.5:

(i) R_p is a PID and hence also a UFD.

(ii) $\overline{Q}_p = \{\frac{m}{n} \in \overline{Q} \mid (n,p) = 1\}$ is a subring of R_p. In particular $\overline{Z} \leq R_p$. Positive integers have representation 5.4(i) as eventually constant sequences and negative integers have

representations computed from

$$-1 = \{p^n - 1\} .$$

Proof: (i) follows from [5; Corollary 2 of Thm. 2] and 5.4(iii). (ii) is [5; Corollary 1 of Thm. 1].

Definition 5.6: Denote by F_p the quotient field of R_p. For q a prime (not necessarily distinct from p), denote by Φ_q the (cyclotomic) polynomial with coefficients in R_p given by

$$\Phi_q = X^{q-1} + X^{q-2} + \cdots + X + 1 .$$

Lemma 5.7:

(i) Every $\alpha \in F_p$, $\alpha \neq 0$, has unique representation $\alpha = p^m \xi$, m an integer, ξ a unit of R_p.

(ii) The characteristic of F_p is zero.

(iii) Φ_p is irreducible over F_p.

(iv) If $q \neq p$ is prime and Φ_q factors into relatively prime polynomials f and h of degrees m and n respectively over $\overline{Z}_p$, then Φ_q factors into polynomials of degrees m and n over R_p.

Proof: (i) is [5; Thm. 4].

(ii) The quotient field of a domain always has characteristic zero.

(iii) It is easy to show that over any domain R, $f \in R[X]$ (the polynomial ring over R) is irreducible if and only if $f(X+1)$ is irreducible. Then $\Phi_p(X+1)$ is irreducible by Eisenstein's criterion:

$$\Phi_p(X+1) = \frac{(X+1)^p - 1}{(X+1) - 1} = X^{p-1} + pX^{p-2} + \frac{p(p-1)}{1\cdot 2} X^{p-3} + \cdots + p .$$

(iv) Since $R_p/(p) \cong \bar{Z}_p$, this follows from Hensel's lemma (see [42; p. 279]).

Notation 5.8: Let $GL(r, R_p)$ be the ring of all $r \times r$ matrices over R_p with determinant a unit of R_p.

Definition 5.9: For every $\alpha \in GL(r, R_p)$, define $\alpha\varphi : D \to D$ as follows:

$$\text{Let} \qquad \alpha = (a_{ij}) ,$$

$$a_{ij} = \{b_m^{ij}\}_{m=1}^{\infty} \in R_p ,$$

$$d = (m_1 \; m_2 \; \cdots \; m_r) \begin{pmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{r,n} \end{pmatrix} ,$$

$$\text{where} \qquad |d| = p^n .$$

Then

$$(d)(\alpha\varphi) = \left[(m_1 \; m_2 \; \cdots \; m_r) \begin{pmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{r,n} \end{pmatrix} \right] [(a_{ij})\varphi]$$

$$= (m_1 \; m_2 \; \cdots \; m_r)(b_n^{ij}) \begin{pmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{r,n} \end{pmatrix} ,$$

where $(m_1\ m_2\ \cdots\ m_r)(b_n^{ij})$ denotes usual matrix multiplication.

Lemma 5.10: (i) The endomorphism ring of a group of type p^{∞} is isomorphic to R_p. (ii) For every $\alpha \in GL(r, R_p)$, $\alpha\varphi \in$ Aut D. (iii) φ is a ring isomorphism of $GL(r, R_p)$ onto Aut D.

References for proof: [21; Thm. 55.1] or [30; pp. 154-157].

Remark 5.11: In view of 5.10(i), D is a left module over the PID, R_p, in a natural way. Unfortunately D is not finitely generated as an R_p-module; for if X is any finite subset of D, then there exists a positive integer n for which X is contained in $D(p^n)$, and $D(p^n)$ is clearly a proper submodule of D. Furthermore, D is not a vector space over F_p; for suppose it is. Using the notation of 5.2, for $p^{-1} \in F_p$ we have

$$p^{-1}(x_{1,1}) = p^{-1}(px_{1,2}) = (p^{-1}p)x_{1,2} = x_{1,2} .$$

Thus p^{-1} is not an endomorphism of D since it carries an element of order p into an element of order p^2.

Following [34], we make the following

Definition 5.12: Let $\alpha \in$ Aut D. We call α a strongly irreducible automorphism (abbreviated S-I automorphism) if no proper infinite subgroup of D is α-invariant; i.e., if for all $H < D$, H infinite, we have

$$H^{<\alpha>} = D .$$

Notice that elements of P-C in 3.14 and 4.4 are essentially S-I automorphisms of order a power of p.

Lemma 5.13: Let $\alpha \in \text{Aut } D$. Then α is an S-I automorphism if and only if for every proper nontrivial divisible subgroup D_1 of D we have

$$D_1^{\alpha} \neq D_1 .$$

Proof: Sufficiency. Suppose α is not an S-I automorphism. By 5.12, choose $H < D$, H infinite such that $H^{< \alpha >} < D$. Let D_1 be a maximal divisible subgroup of $H^{< \alpha >}$. Note $D_1 < D$ and $D_1 \neq 1$ since H is infinite. Furthermore D_1 char $H^{< \alpha >}$ and hence

$$D_1^{\alpha} = D_1 .$$

Necessity. Suppose there exists a proper nontrivial divisible subgroup D_1 of D such that

$$D_1^{\alpha} = D_1 .$$

If $D_1^{\alpha^n} = D_1$ for some positive integer n, then

$$D_1^{\alpha^{n+1}} = (D_1^{\alpha^n})^{\alpha} = D_1^{\alpha} = D_1 .$$

Hence, by induction $D_1^{< \alpha >} = D_1 < D$. Hence α is not an S-I automorphism of D.

Lemma 5.14: If α is an S-I automorphism and β any automorphism of D, then α^{β} is an S-I automorphism of D.

Proof: Suppose α^β is not an S-I automorphism. By 5.13, there exists divisible D_1, $1 < D_1 < D$, such that

$$D_1^{\beta^{-1}\alpha\beta} = D_1 .$$

Thus

$$(D_1^{\beta^{-1}})^\alpha = D_1^{\beta^{-1}}$$

and clearly rank $(D_1^{\beta^{-1}})$ = rank D_1 < rank D. Thus by 5.13, α is not an S-I automorphism.

The following result is due to Černikov [11; Thm. 3.2].

Lemma 5.15: Let A be a finite group of S-I automorphisms of D. Let q be the smallest prime divisor of $|A|$ (q not necessarily distinct from p). Then $r < q$.

Proof: Without loss of generality, $r \geq 2$. Write $D = P_1 + P_2$, P_1 a group of type p^∞ and $P_2 \neq 1$. Let $\alpha \in A$, $|\alpha| = q$. Now

$$P_1^{<\alpha>} = D$$

and hence $\text{rank}(P_1^{<\alpha>}) = r$. Let P_1 have generators $a_1, a_2, \ldots$ and defining relations $pa_1 = 0$, $pa_{n+1} = a_n$. Let $s_n = a_n + a_n^\alpha + \cdots + a_n^{\alpha^{q-1}}$. If only finitely many of the s_n are trivial, $< s_n | n = 1, 2, \ldots >$ is an α-invariant subgroup of D of type p^∞. Thus infinitely many of the s_n are zero; but since $ps_{n+1} = s_n$ we have $s_n = 0$ for all n. Hence

$$P_1^{\alpha^i} \leq < P_1^{\alpha^j} | 0 \leq j \leq q - 1, j \neq i > .$$

Thus $r \leq q - 1$, as desired.

We obtain a complete description of S-I automorphisms of order p which depends on the following two lemmas.

Lemma 5.16: Let $r < p - 1$. Then D has no nontrivial automorphisms of order a power of p.

Proof: Suppose otherwise. Then D has an automorphism α of order p. α satisfies

$$X^p - 1 = (X-1)\varphi_p(X) .$$

Viewing α as an $r \times r$ matrix over F_p, α has minimal polynomial, f, over F_p and

$$f \mid (X-1)\varphi_p(X) .$$

Since $X - 1$ and $\varphi_p(X)$ are irreducible over F_p,

$$f \in \{X-1, \varphi_p(X), (X-1)\varphi_p(X)\} ;$$

i.e., $\deg f \in \{1, p-1, p\}$. Let g be the characteristic polynomial of α.

$$\deg g = r < p - 1 ,$$

and we have $f \mid g$ since $g \in F_p[X]$. Thus $\deg f = 1$; i.e., $f = X - 1$ and α is trivial, a contradiction.

Corollary 5.17: Let P be a Černikov p-group. If D, the minimal subgroup of finite index, has rank $r < p - 1$, then $D \leq Z(P)$; in particular $P \in \mathfrak{N}$. (This result is attributed to Černikov by Blackburn [4; Introduction to Sec. 5], but his reference is spurious.)

Proof: $D \leq Z(P)$ follows from 5.16. Then $P/Z(P)$ is an image of the finite p-group P/D and hence is nilpotent. Thus $P \in \mathfrak{N}$.

Theorem 5.18: Let α be an automorphism of D of order p. Then α is an S-I automorphism if and only if $r = p - 1$.

Proof: Necessity. By 5.15, $r \leq p - 1$. If $p = 2$, we are done. Assume $p > 2$. Suppose for contradiction that $r < p - 1$. Then by 5.17 $D]\!< \alpha > \in \mathfrak{N}$ and by 4.8, $Z = Z(D]\!< \alpha >)$ is infinite. Hence $Z \cap D$ is infinite and clearly

$$(Z \cap D)^{< \alpha >} = Z \cap D .$$

Since α is an S-I automorphism, $Z \cap D$ is not proper in D, so $D \leq Z$. But this means α acts trivially on D, contradicting $|\alpha| = p$. Thus $r = p - 1$, as desired.

Sufficiency. Let α be an automorphism of order p and let $r = p - 1$. Choose D_1 minimal such that D_1 is a divisible subgroup of D and $D_1^\alpha = D_1$. By the minimality of D_1 and by 5.13, α is an S-I automorphism of D_1. By the necessity just proved, rank D_1 = $p - 1$ = rank $D = r$. Thus $D_1 = D$.

It is easy to verify that if $\alpha \in GL(p\text{-}1, R_p)$ has rational canonical form

$$(*) \qquad \alpha = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{bmatrix}_{(p-1)\times(p-1)}$$

then α^p is the identity matrix. By the theorem just proved we have that $\alpha\varphi$ is an S-I automorphism of D in the case $r = p - 1$, where φ is the mapping of 5.9 and 5.10. The converse is also true:

Theorem 5.19: Let $\alpha \in GL(p-1, R_p)$, with $\alpha\varphi$ an S-I automorphism of D of order p. Then α has rational canonical form (*).

Proof: Since φ is an isomorphism, α has order p and thus satisfies the polynomial $X^p - 1 = (X-1)\Phi_p(X)$. Since α does not satisfy $X - 1$, α does satisfy $\Phi_p(X)$. By 5.7(iii), Φ_p is irreducible over F_p so that Φ_p is the minimal polynomial for α. Since the degree of Φ_p is $p - 1$, Φ_p is also the characteristic polynomial for α; the desired form (*) now follows.

Remark 5.20: Using essentially the same proof as for 5.7(iii) it can be shown that the polynomial

$$\Phi_{p^2}(X) = \frac{X^{p^2} - 1}{X^p - 1}$$

is irreducible over F_p. Thus if α is an automorphism of D of order p^2, then α^p is not an S-I automorphism of D. For $\Phi_{p^2}(\alpha) = 0$ and by the irreducibility of Φ_{p^2} we have Φ_{p^2} divides the characteristic polynomial, f, of α. But then

$$\text{rank } D = \deg f \geq \deg \Phi_{p^2} = p(p-1) > p - 1$$

and by 5.18, α is not an S-I automorphism. A consequence of this observation is that for P, D, and C as in 3.14 or 4.4 we have $|P:C| = p$. We have recorded this fact for the situation of 3.14 in 6.2.

Chapter VI

Relationship to Direct Limits of p-Groups of Maximal Class

There are precisely two nontrivial direct limits of p-groups of maximal class and presentations for them are known. Both are Černikov p-groups G, satisfying

$$[G:Z_{\omega}(G)] = [Z_{n+1}(G):Z_{n}(G)] = p$$

and the rank of $D(G)$ is $p - 1$. (See [4; Sec. 5] and [3].) In this chapter we use some characterizations of Blackburn for such groups to show a relationship with the groups we studied in Chapter III; then we use the results of Chapter V to obtain another characterization of direct limits of p-groups of maximal class. Finally we provide some examples using the external semi-direct product with an amalgamated subgroup.

Theorem 6.1: [4; Thm. 5.1]. Let P be a Černikov p-group for which $D(P)$ has rank $r \leq p - 1$. Then either $G/Z(G)$ is finite or G has a finite normal subgroup N such that G/N is a direct limit of p-groups of maximal class.

Theorem 6.2: Let P be an infinite p-group. Then $P \in L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$ if and only if there exists a finite normal subgroup $N \lhd P$ and a normal divisible abelian p-group D of rank $p - 1$ such that P/N is a direct limit of p-groups of maximal class and $C_{P}(D) = DN$. Furthermore, in this case, $|P:DN| = p$.

<u>Proof</u>: Necessity. By 3.14, $D = D(P)$ has finite index in P and has finite rank and P/C (where $C = C_P(D)$) is isomorphic to a nontrivial p-group of S-I automorphisms of D. By 5.18 rank $D = p - 1$. Now $Z(P)$ is finite since $P \in I\mathfrak{N}^{snb} - \mathfrak{N}$ so that by 6.1 there exists a finite normal subgroup $N \triangleleft P$ such that P/N is a direct limit of p-groups of maximal class. By 2.13 $[D,N] = 1$; i.e., $DN \leq C$. Now DN/N is a normal subgroup of P/N and since

$$\frac{DN}{N} \cong \frac{D}{D \cap N}$$

DN/N is a divisible abelian p-group of rank $p - 1$. Thus

$$\frac{DN}{N} = Z_\omega(\frac{P}{N})$$

and hence $|P:DN| = p$. But $C < P$ so that

$$p \leq |P:C| \leq |P:DN| = p$$

and we have $DN = C$.

Sufficiency. Let N and D be as in the statement and let $C = C_P(D) = DN$. P/N is Černikov and N is finite; hence P is a Černikov p-group. Now

$$\frac{C}{N} = \frac{DN}{N} \cong \frac{D}{D \cap N}$$

shows that C/N is a normal divisible abelian subgroup of P/N of rank $p - 1$. Thus

$$\frac{C}{N} = Z_\omega(\frac{P}{N})$$

and we have

$$\left|\frac{P}{N} : \frac{DN}{N}\right| = \left|\frac{P}{N} : \frac{C}{N}\right| = \left|\frac{P}{N} : Z_\omega\left(\frac{P}{N}\right)\right| = p .$$

Notice that $|P:D| = |P:DN||DN:D| < \infty$ so that $D = D(P)$. By a well known theorem of Fitting, $C = DN \in \mathfrak{N}$, and hence condition 3.14(i) is satisfied. But P/C is naturally isomorphic to a subgroup of Aut D and has order p. Thus by 5.18, P/C is a group of S-I automorphisms of D and thus condition 3.14(ii) is also satisfied. Hence $P \in L\mathfrak{N} \cap I\mathfrak{N}^{snb} - \mathfrak{N}$.

Now we obtain a characterization of direct limits of p-groups of maximal class which leads to the construction of examples of the groups studied in Chapters III and IV.

Theorem 6.3: Let G be a p-group, $p \geq 3$. The following are equivalent:

(i) G is a direct limit of p-groups of maximal class.

(ii) G is a semi-direct product of a divisible abelian group D of rank $p - 1$ by a cyclic group of order p or p^2 with an amalgamated subgroup of order 1 or p, respectively, and is nonabelian.

Proof: (i) implies (ii): Let G be a direct limit of p-groups of maximal class. Then $|G:Z_\omega(G)| = |Z_{n+1}(G):Z_n(G)| = p$, $n = 0,1,2,\ldots$. Furthermore, $Z_\omega(G)$ is a direct sum of $p - 1$ groups of type p^∞. Let $D = Z_\omega(G)$. Let $x \in G - D$. Since $|G:D| = p$, we have $G = D < x >$. Now $|G:D| = |D < x >:D| = |< x >:D \cap < x >| = p$. Thus $D \cap < x > = < x^p >$. But now notice that

$$< x^p > = D \cap < x > \leq C_G(D) \cap C_G(x) = Z_1(G) .$$

Thus $|\langle x^p \rangle|$ divides $|Z_1(G)| = p$; i.e., $|\langle x^p \rangle| = |D \cap \langle x \rangle|$ is 1 or p. If $D \cap \langle x \rangle = \langle x^p \rangle = 1$, then $|x| = p$; if $D \cap \langle x \rangle = \langle x^p \rangle$ has order p, then $|x| = p^2$.

(ii) implies (i): Let G be as in (ii). Let $\langle x \rangle$ be of order p or p^2, with $D \cap \langle x \rangle$ trivial or of order p, respectively. Then $D \triangleleft G$ and

$$|G:D| = |D\langle x \rangle : D| = |\langle x \rangle : D \cap \langle x \rangle| = p .$$

The proof is complete by a theorem of Blackburn:

<u>Theorem</u> 6.4: [4; Lemma 5.2]. If G is a nonabelian p-group with a normal subgroup D of index p, and D is the direct sum of at most $p - 1$ groups of type p^∞, then G is a direct limit of p-groups of maximal class.

In order to construct examples for 6.3 (and hence also for 6.2) we make use of the following construction.

<u>Lemma</u> 6.5: Let M, D, and K be groups, $M \leq D$ and let $\psi: K \to \operatorname{Aut} D$ be a homomorphism and $\theta: M \to K$ an isomorphism into K, with M invariant under $K\psi$ and satisfying

(i) $d^{(m\theta)\psi} = d^m$ for all $d \in D$, $m \in M$, and

(ii) $(m^{k\psi})\theta = (m\theta)^k$ for all $m \in M$, $k \in K$.

Then there exists a group $\overline{G}$ which is a semi-direct product of $\overline{D}$ by $\overline{K}$ with an amalgamated subgroup $\overline{M}$ such that D, K, and M are isomorphic respectively to $\overline{D}$, $\overline{K}$, and $\overline{M}$, and the action of $\overline{K}$ on $\overline{D}$ corresponds to the action of $K\psi$ on D.

<u>Proof</u>: Let G be the semi-direct product of D by K with respect to ψ. Let $N = \{m\theta m^{-1} \in G | m \in M\}$. We argue exactly as in Gorenstein [24; pp. 27-28] that N is a normal subgroup of G. Set $\overline{G} = G/N$ and $\overline{D}$, $\overline{K}$, and $\overline{M}$ the images of D, K, and M, respectively. Gorenstein's arguments show that $\overline{D} \cong D$, $\overline{M} \cong M$, $\overline{K} \cong K$, and $\overline{M} \leq \overline{D} \cap \overline{K}$. He then concludes that $\overline{M} = \overline{D} \cap \overline{K}$ from the finiteness of the groups by noting that $|\overline{M}| = |\overline{D} \cap \overline{K}|$. A routine argument works for the infinite groups considered here:

Let $xN \in \overline{D} \cap \overline{K} = \frac{DN}{N} \cap \frac{KN}{N}$. Write $xN = dN = kN$, for some $d \in D$, and $k \in K$. Then $k^{-1}d \in N$. Let $m \in M$, such that $k^{-1}d = (m\theta)m^{-1}$. By the uniqueness of representation in KD, we have $k^{-1} = m\theta$, $d = m^{-1} \in M$. Hence $xN = dN \in MN/N = \overline{M}$. Thus we have $\overline{D} \cap \overline{K} \leq \overline{M}$ as well so that $\overline{M} = \overline{D} \cap \overline{K}$. That the action of $\overline{K}$ on $\overline{D}$ corresponds to the action of $K\psi$ on D follows easily.

<u>Example</u> 6.6: Let D be a divisible abelian p-group of rank $r = p - 1$, and α an automorphism of D of order p. Then $G = D]\langle \alpha \rangle$ is a direct limit of p-groups of maximal class by 6.4.

<u>Example</u> 6.7: Let $D = D_1 + D_2 + \cdots + D_{p-1}$ be a divisible abelian p-group of rank $r = p - 1$, with D_i generated by $x_{i,n}$, $n = 1, 2, \ldots$ satisfying the relations $px_{i,1} = 0$, $px_{i,n+1} = x_{i,n}$, $n = 1, 2, \ldots$. Let $\alpha \in GL(p-1, R_p)$ be the matrix

$$\alpha = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ -1 & -1 & -1 & \cdots & & -1 \end{bmatrix}_{(p-1)\times(p-1)}$$

View α as an element of Aut D, with respect to the decomposition above. Note $|\alpha| = p$. Let $K = \langle y \rangle$ be cyclic of order p^2 and $\psi : K \to \langle \alpha \rangle$ the homomorphism for which $y\psi = \alpha$. Notice $y^p\psi$ is the trivial automorphism. Now let

$$M = \left\langle ((p-1)\ (p-2)\ \cdots\ 2\ \ 1)\begin{pmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{p-1,1} \end{pmatrix} \right\rangle \leq D .$$

Notice that M is cyclic of order p. We claim that M is invariant under α:

$$\left[((p-1)\ (p-2)\ \cdots\ 2\ \ 1)\begin{pmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{p-1,1} \end{pmatrix} \right]\alpha$$

$$= (-1\ (p-2)\ (p-3)\ \cdots\ 2\ \ 1)\begin{pmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{p-1,1} \end{pmatrix}$$

$$= ((p-1)\ (p-2)\ (p-3)\ \cdots\ 2\ \ 1)\begin{pmatrix} x_{1,1} \\ \vdots \\ x_{p-1,1} \end{pmatrix}$$

Thus α in fact acts trivially on M. Now let $\theta : M \to \langle y \rangle$ be the homomorphism defined by

$$\left[((p-1)\ (p-2)\ \cdots\ 2\ \ 1)\begin{pmatrix} x_{1,1} \\ \vdots \\ x_{p-1,1} \end{pmatrix}\right]\theta = y^p .$$

Notice that $M\theta\psi$ is trivial so that condition (i) of 6.5 is satisfied. Since α acts trivially on M, $K\psi$ does also, and since K is abelian, condition (ii) of 6.5 is also satisfied. Thus the semi-direct product G of D by K with M amalgamated exists; by 6.3, G is a direct limit of p-groups of maximal class.

Example 6.8: Let G be the group of 6.6 or 6.7. Let N be any finite p-group. Set $P = G + N$. By 6.2, $P \in L\mathfrak{N} \in I\mathfrak{N}^{snb} - \mathfrak{N}$.

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