This is to certify that the

thesis entitled

CALCULATED STRUCTURAL RESPONSE USING
A "REDUCED" FINITE ELEMENT MODEL

presented by

Mark Norman Pickelmann

has been accepted towards fulfillment
of the requirements for

MASTER OF SCIENCE degree in MECHANICAL ENGINEERING

Major professor

Date 5/17/82

O-7639

CALCULATED STRUCTURAL RESPONSE USING
A "REDUCED" FINITE ELEMENT MODEL

By

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A THESIS

Submitted to

Michigan State University

in partial fulfillment of the requirements

for the degree of

MASTER OF SCIENCE

Department of Mechanical Engineering

1982

ABSTRACT

CALCULATED STRUCTURAL RESPONSE USING
A "REDUCED" FINITE ELEMENT MODEL

By

Mark Norman Pickelmann

The use of finite element models for engineering design has grown rapidly in the past few years. These models are useful tools for predicting the behavior of systems long before the system is actually constructed. The resulting models, however, are often quite large, requiring hours of computer time to use.

This thesis demonstrates that a finite element model can be reduced for the purpose of calculating structural response. This reduction is done systematically so that the model is transformed into a set of first order ordinary differential equations. These equations are solved and used to calculate frequency responses.

This reduction offers considerable time and cost savings over computing the response directly from the finite element model.

CALCULATED STRUCTURAL RESPONSE USING

A "REDUCED" FINITE ELEMENT MODEL

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To my loving wife Kristi

AKNOWLEDGEMENTS

I would like to thank Dr. James E. Bernard, my major professor, for his help, friendship, and advice over the last two years.

Also, I wish to thank Dr. Clark Radcliffe for all of his help on the Oldsmobile project and for letting me use his computer to print my thesis.

Many thanks to Don Hine, of Oldsmobile, for all his help on the project.

Finally, I would especially like to thank my mom and dad, Mary and Russell Pickelmann, for having me in the first place.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

The use of finite element models for engineering design has grown rapidly in the past few years. These models are useful tools for predicting the behavior of systems long before the system is actually constructed.

For complex structures, the finite element model can become quite large, requiring a large computer for the calculations. Even with the computing power of large computers the finite element models require a great deal of computing time. However, for the purpose of calculating the frequency response of a structure in a specified frequency range, the finite element model can often be reduced so that the calculation can be done on a minicomputer. This thesis is concerned with the process of reducing a large model to a smaller one for the purpose of such a frequency response calculation.

Chapter 2 explains the formulation of a large finite element model, and its transformation into a modal model. The modal representation has a coupled differential equation and associated mode shape for each degree of freedom in the original finite element model. Chapter 3 presents the rational for the reduction of the modal model. Chapter 4 introduces the forcing functions so that frequency response can be calculated, and the calculation of the response from the reduced model is presented in Chapter 5. Chapter 6 presents some details of a project where structural responses were calculated by this method. A summary of the assumptions used in the analysis are reviewed in chapter 7 along with some of the advantages to this method.

CHAPTER 2

STRUCTURAL MODEL

The goal of the analysis discussed here is to develop an analytical model of a structure which predicts the response of that structure to forces of a given frequency range. The starting point of the analysis is a finite element model of the structure, which yields equations of the form.

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + 1/w\,[D]\{\dot{X}\} + [K]\{X\} = \{F\}\exp(i\ w\ t)$$

Equation 1

Since our main interest is the calculation of frequency response, the forcing function has been assumed to be harmonic. However it could be any forcing function provided it can be expressed as a Fourier series.

In the structural model developed here dissipative forces arise from two different sources, viscous damping and structural damping. The damping forces which are proportional to velocity are classified as viscous damping. Viscous damping occurs when molecules of a viscous fluid rub together, causing a resistive friction force that is proportional to, and opposing, the velocity of an object moving through the fluid.

Damping forces which are proportional to Displacement are classified as structural damping. Structural damping may be viewed as a sliding friction mechanism between molecular layers in a material. The friction force is proportional to the deformation or displacement from some equilibrium point with an orientation opposite the relative velocity. Imagine a rod made up of a bundle of axial fibers. The siding friction force between each fiber and its neighbor will increase as the rod is bent and the fibers are pinched together. This pinching phenomena occurs in most materials as the various molecular layers slide past one another [1] [2].

A complex structure such as an automobile includes several sources of dissipation. The shock absorber, whose design mission is to provide damping, is closely approximated by a viscous model. But important dissipation occurs in mounting elements such as coil springs and rubber mounts as well. Tests indicate that the dissipation of a spring is most closely approximated by a structural damping model. Tests done on rubber mounts indicate a combination of viscous and structural dissipation is needed to adequately model the dissipation.

For the problems of concern here, we will assume the structure is lightly damped, resulting in small but non zero dissipation forces. Since the total dissipation is small, the natural frequencies and mode shapes of the structure can be determined from the mass and stiffness matrices. But the amplitude of the forced response of the structure depends on the damping as well.

In general, the matrices in Equation 1 are not diagonal. Therefore the solution of one equation depends on the solution of others and the system of equations is said to be coupled. The size of the matrices depends on the number of elements in the finite element model and the number of degrees of freedom of each element. The structure discussed as an example is modeled by 500 elements, each with six degrees of freedom, thus Equation 1 would include 3000 coupled equations. It is desirable to simplify the model in such a way as to make the response calculation more convenient.

The procedure which leads to a simplified model begins with the equations of undamped free vibration.

$$[M]\,\{\ddot{X}\} + [K]\,\{X\} = \{0\} \qquad (2)$$

Where:

[M] and [K] are n x n matrices

Equation 2 is formulated from Equation 1 by neglecting the damping matrices and setting the force vector to zero. A solution for {X} may be found in the form

$$\{X\} = \{A\}\,\exp(i\ w\ t) \qquad (3)$$

Using Equation 3 in Equation 2 results in

$$(\,-w^2[M] + [K]\,)\,\{A\}\,\exp(i\ w\ t) = \{0\} \qquad (4)$$

Rewriting Equation 4 defines the eigenvalue problem

$$[K]\,\{A\} = \lambda\,[M]\,\{A\} \qquad (5)$$

The solution of Equation 5 results in a set of n eigenvalues λ_i. If these are distinct, as is the usual case, there will be a corresponding unique set of n eigenvectors $\{A\}_i$. Since [M] and [K] are symmetric and positive definite, both the eigenvalues and eigenvectors are real. The eigenvectors are used to form two transformation matrices [U] and $[U]^T$ where the eigenvectors $\{A\}_i$ make up the columns of [U]. The transformation matrix [U] is used to define a modal coordinate Y

$$\{X\} = [U]\ \{Y\} \qquad (6)$$

When the relationship from Equation 6 is substituted into Equation 1, which is then premultipled by $[U]^T$, we get

$$[Mm]\{\ddot{Y}\} + [Cm]\{\dot{Y}\} + 1/w\ [Dm]\{\dot{Y}\} + [Km]\{Y\} = \{Fm\} \qquad (7)$$

This coordinate transformation uncouples the mass and stiffness matrices, but in general does not uncouple the damping matrices [3]. Equation 7 is called the "modal model". The modal model is a set of n coupled second order ordinary differential equations where n is the dimension of Equation 1. Each coordinate Y_i of the modal model is associated with one natural frequency and its corresponding mode shape or eigenvector.

The steps described above are usually done by the finite element programs on large computers. The output from the finite element program would be the transformation matrix [U], the diagonal modal mass matrix [Mm], the diagonal modal stiffness matrix [Km], and the damping matrices [Cm] and [Dm].

The damping matrices can be thought of as a coupling by which energy can flow from one mode to another. The damping can then be thought of as an input force. This can be seen by rewriting Equation 7 in the form

$$[Mm]\ \{\ddot{Y}\} + [Km]\ \{Y\} = \{Fm\} - [Cm]\ \{\dot{Y}\} - 1/w\ [Dm]\ \{\dot{Y}\} \qquad (8)$$

The fact that the {Fm} are harmonic dictates that the velocities $\{\dot{Y}\}$ are also harmonic at the same frequencies. Thus, it is convenient to think of the right hand side of Equation 8 expressed as $\{Fm\}^{*}exp(iwt)$.

$$[Mm]\ \{\ddot{Y}\} + [Km]\ \{Y\} = \{Fm\}^{*}exp(i\ w\ t) \qquad (9)$$

The solution to Equation 9 is shown in Figure 1.

Of course, in order to calculate a response the damping must be included in the left hand side of Equation 8. But the introduction of this small amount of damping will limit the peak amplitude but will not drastically alter the basic chatacter of the frequency response as shown in Figure 2. Thus if the frequency of the force is near a natural frequency, the response of that mode will be large. By knowing the frequency range over which the forcing function is active, the modes which heavily participate in the response can be

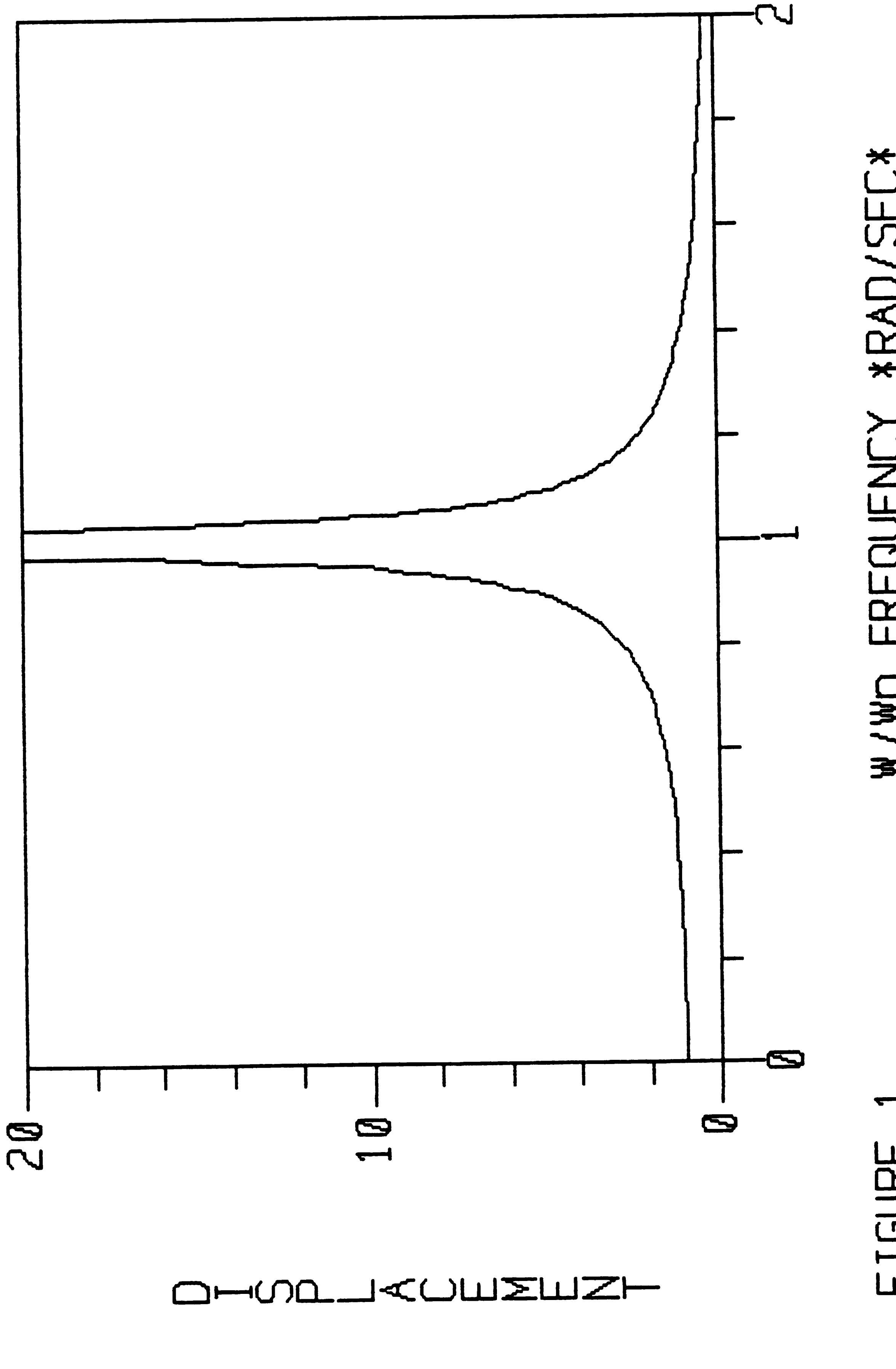

FIGURE 1

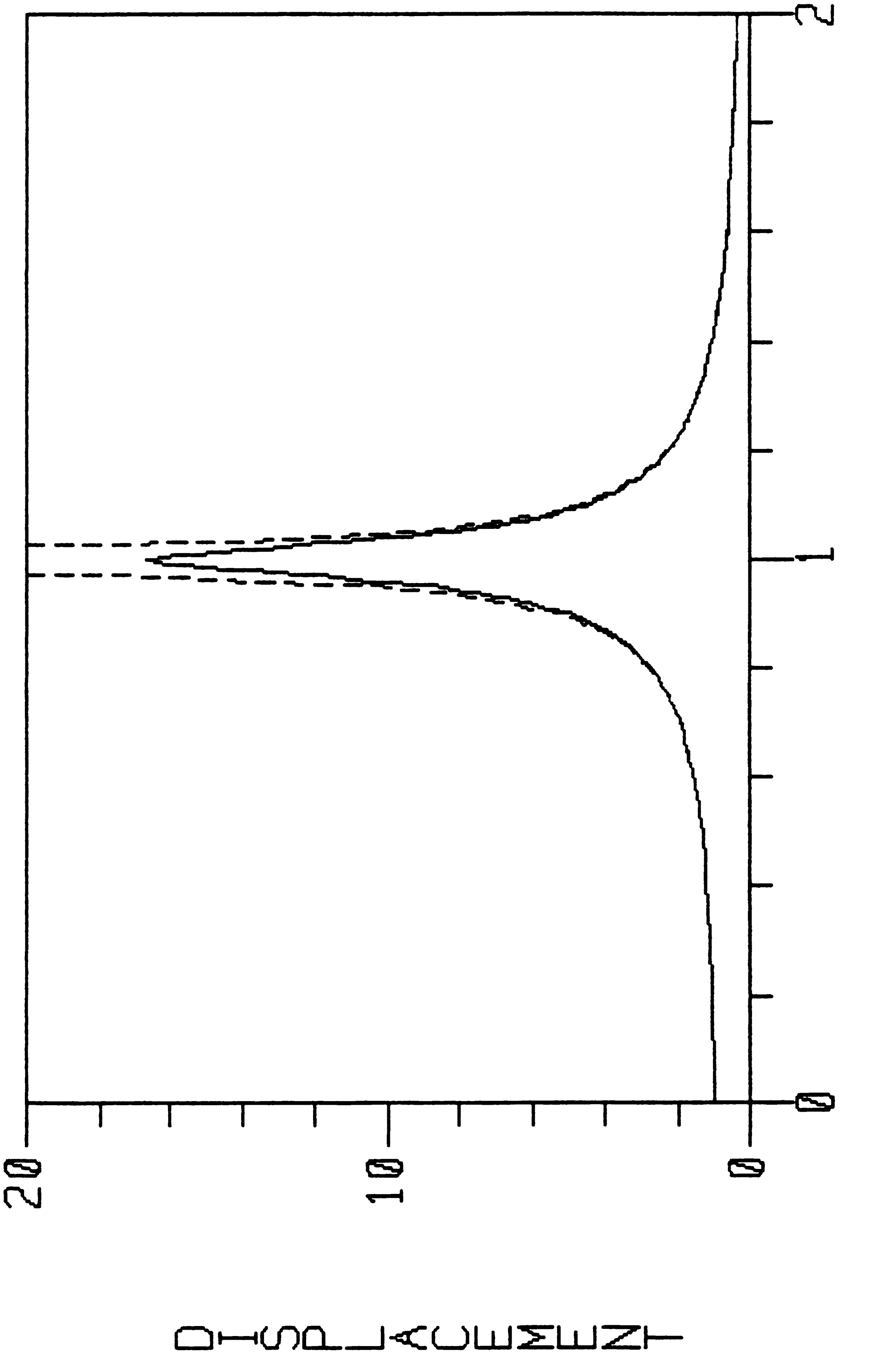

FIGURE 2

identified. Those which do not participate are eliminated by dropping the associated modal coordinate from Equation 7 and the mode shape vector from the transformation matrix [U].

The number of modes has been reduced by detemining which modes fall significantly outside of the frequency range of the forcing function. A rule which is often used is to keep modes whose associated natural frequency is less than twice the maximum frequency of the force [4].

Since the number of modes which have meaningful participation in the response may be a great deal smaller than the number of degrees of freedom, the size of Equation 7 and the transformation matrix [U] can often be substantially reduced. In our previous example there were 500 elements each with six degrees of freedom resulting in 3000 equations. If, for example, only 100 of the 3000 natural frequencies are determined to meaningfully participate in the response, we can reduce the size of the matrices from 3000 x 3000 to 100 x 100 without significant loss in accuracy.

At this point the problem has been substantially reduced. But the equations are still coupled in the damping matrices and thus the response calculation is not in a convenient form. needs further attention.

CHAPTER 3

UNCOUPLING OF MODAL EQUATIONS

Chapter 2 showed that the n degree of freedom finite element model could be cast in the form of the modal model of Equation 7 and that the modal model could be reduced by eliminating the modes whose participation in the response was determined to be insignificant. The reduced modal model is then a set of k coupled second order differential equations. The fact that k<<n facilitates the solution. However, since the equations are still coupled in the damping matrices, they are not in a convenient form for solution. In this chapter a coordinate transformation will be introduced to uncouple the modal equations.

In some cases the modal equations are uncoupled by neglecting the off diagonal terms in the damping matrices. This assumes that the off diagonal terms have a small effect on the response of the system. To uncouple the modal equations without having to make this assumption, a more general approach will be taken. To do this the reduced form of Equation 7 is written.

$$[Mm]\{\ddot{Y}\} + [Cm]\{\dot{Y}\} + 1/w\ [Dm]\{\dot{Y}\} + [Km]\{Y\} = \{Fm\} \quad (10)$$

The matrices of Equation 10 are of dimension k. Before this set of equations can be uncoupled the 1/w which multiplies the structural damping matrix must be eliminated. This is done by assuming a solution for {Y} of the form[5]

$$\{Y\} = \{B\}\ \exp(i\ w\ t) \quad (11)$$

Taking the first derivative with respect to time yields

$$\{\dot{Y}\} = i\ w\ \{B\}\ \exp(i\ w\ t) \quad (12)$$

substituting Equation 11 into Equation 12 yields

$$\{\dot{Y}\} = i\ w\ \{Y\} \qquad (13)$$

Following the procedure of [5], Equation 13 is then substituted into Equation 10 which results in

$$[Mm]\{\ddot{Y}\} + [Cm]\{\dot{Y}\} + (\ [Km] + i\ [Dm]\ )\{Y\} = \{Fm\} \qquad (14)$$

Next, consider the homogenous solution i.e., $\{Fm\} = \{0\}$. The following solution for $\{Y\}$ is assumed

$$\{Y\} = \{C\}\ \exp(\lambda\ t) \qquad (15)$$

Taking the first and second derivatives with respect to time of Equation 15 and substituting them into Equation 14 will result in

$$(\lambda^2[Mm] + \lambda [Cm] + \{[Km] + i [Dm]\}) \{C\} = \{0\} \qquad (16)$$

Equation 16 has a nontrivial solution only if the determinant of the coefficient matrix is zero. This results in an algebraic equation of order 2k in λ (k being the dimension of the matrices). This equation will result in a set of 2k λ solutions. With each eigenvalue λ_i there is an associated eigenvector $\{C\}_i$, and both are complex. In the case where the system is modeled with viscous damping only, the eigenvalues and eigenvectors would appear as complex conjugates, but with structural damping in the system the pairs are rotated so they are no longer conjugates [7]. This means that the eigenvalues and eigenvectors cannot be used to solve the transient problem but the solution must be of the form of Equation 15 [6]. The forced vibration problem however can be solved by uncoupling Equation 14.

UNCOUPLING OF MODAL EQUATIONS

The eigenvectors can be used to form the $k \times 2k$ rectangular modal matrix [V]. But since, this modal matrix [V] cannot be used as a transformation matrix of the form

$$\{Y\} = [V]\{Z\} \qquad (17)$$

to obtain a solution to the nonhomogeneous problem. The reason is that there are 2k modes and consequently 2k coordinates z_i, and only k coordinates y_i. This difficulty can be overcome by introducing a set of auxiliary variables and converting the set of k second order ordinary differential equations into an equivalent set of 2k first order ordinary differential equations known as Hamilton's Canonical Equations [8]. The auxiliary variables are the modal velocities $\{\dot{Y}\}$. The modal coordinate Y and the modal velocities $\dot{Y}$ define a set of new variables P in the following way

$$P_i = \dot{Y}_i \qquad i = 1,k$$

$$P_{i+k} = Y_i \qquad i = 1,k \tag{18}$$

P is then used into Equation 14 to formulate the following:

$$\begin{bmatrix} [0] & [Mm] \\ [Mm] & [Cm] \end{bmatrix} \{\dot{P}\} + \begin{bmatrix} -[Mm] & [0] \\ [0] & ([Km]+i[Dm]) \end{bmatrix} \{P\} = \begin{bmatrix} \{0\} \\ \{Fm\} \end{bmatrix}$$

Equation 19

which can be written as

$$[Mp]\ \{\dot{P}\} + [Kp]\ \{P\} = \{Fp\} \tag{20}$$

UNCOUPLING OF MODAL EQUATIONS

Setting {Fp} = {O} defines the homogeneous problem which is similar to Equation 2 and can be solved in a similar fashion. The result is a set of 2k complex eigenvalues and a corresponding set of 2k eigenvectors. As before, the eigenvectors are used as the columns of the transformation matrix [V] and as the rows of $[V]^T$. Equation 21 defines a new coordinate Z

$$\{P\} = [V]\,\{Z\} \qquad (21)$$

Which is substituted into Equation 20 and the result is premultiplied by $[V]^T$, leading to

$$[Mz]\,\{\dot{Z}\} + [Kz]\,\{Z\} = \{Fz\} \qquad (22)$$

This coordinate transformation uncouples both the mass and stiffness matrices. Equation 22 is now a set of 2k uncoupled ordinary first order differential equations.

CHAPTER 4

FORCING FUNCTION

In the preceeding chapters a finite element model was reduced first to a coupled second order modal model and then to a first order uncoupled modal model, which is the left hand side of Equation 22. Before Equation 22 can be solved the right hand side of the equation must be defined. Two ways of doing this will be discussed in this chapter.

In order to calculate a response to a forcing function first the function must be defined. In the analysis here the forcing function will first be defined for the structure and then undergo the same transformations as the finite element model. The forces which act on the structure can be either measured or analytically determined forces.

FORCING FUNCTION

Analytical functions can be any periodic forcing function limited only by the frequency restrictions used to reduce the equations. They could be obtained by simulating components of the system and calculating the forces which would be transmitted to the structure, or the frequency range of interest may be spanned for the purposes of computing frequency response. Once the functions have been calculated or defined they are transformed into the frequency domain.

Test data can be obtained by measuring the forces which would be transmitted to the structure. This testing can be carried out in two ways: 1) measuring the forces on a prototype of the structure or 2) measuring forces from components which transmit forces to the structure being studied. If the second approach is chosen care must be taken to assure that the boundary conditions of the components are the same as in the total system. Once the forces have been measured the Fast Fourier Transform (FFT) can be taken to put the forces into the frequency domain.

CHAPTER 4

At this point the forces which act on the structure are defined in the frequency domain. A check must be made see that they fall within the original specified frequency range. Forces having large magnitudes and oscillation frequencies outside of the specifed range would violate the original assumption which was used to reduce the modal model. If the assumption is not valid then the calculated response will be inaccurate.

Each of the force time histories has been broken up into discreet frequencies. At a given frequency w there is an amplitude {D} and a phase angle ϕ.

$$\{F\} = \{D\} e(i\ \phi) \qquad (23)$$

Letting $\exp(i\,\Phi)$ = Cos(Fee) + i Sin(Fee) we have

$$\{F\} = \{R\} + i\ \{I\} \qquad (24)$$

$$\{R\} = \{D\}\ \mathrm{Cos}(\Phi)$$

$$\{I\} = \{D\}\ \mathrm{Sin}(\Phi)$$

{F} is then premultiplied by the reduced $[U]^T$ which results in {Fm}. {Fm} is then used to create the {Fp} vector which is premultipled by $[V]^T$ resulting in {Fz}. Now that {Fz} is defined Equation 22 can be solved.

CHAPTER 5

RESPONSE CALCULATION

Thus far a finite element model has been reduced to a set of uncoupled first order ordinary differential equations. Chapter 4 defined the forcing function for these equations. This chapter will discuss the solution of the uncoupled equations.

Since the equations represented by Equation 22 are uncoupled, each can be solved independently. If the equations to be solved are written in the form

$$m_j \dot{z}_j + k_j z_j = f_j \exp (i w t) \qquad (25)$$

RESPONSE CALCULATION

Following the procedure given in [9], the solution can be expressed as

$$z_j = f_j \exp(i w t)/[k_j + i w m_j] \qquad (26)$$

With all of the 2k z_j known, Equation 21 can be used to find all 2k p_j. The definition of P in Equation 18 is then used to find the k second order modal coordinates y_j. The reduced form of Equation 6 can then be used to find the response at any point in the original finite element model.

These steps are repeated for each discreet frequency in the forcing function. The result is a complex amplitude for each frequency. The magnitude of the response is found by taking the magnitude of the complex number. The phase angle with respect to the forcing function can be computed based on the real and imaginary parts of the amplitude. A time response can be computed from the frequency response by taking the inverse Fourier Transform.

CHAPTER 5

The main goal of the analysis was to find a simple and fast way to calcualte predicted structural response using a large finite element model. This was done by reducing the finite element model to a small set of complex modes. The response was then calculated by summing the modes. The next chapter will give some highlights of a project in which this analysis was used.

CHAPTER 6

OLDSMOBILE PROJECT

In the preceeding chapters it was shown that a finite element model could be reduced to a set of uncoupled first order ordinary differential equations. These equations were then solved for the response to a given forcing input. This chapter gives some of the details of a project in which this analysis was used.

Currently the Albert H. Case Center for Computer Aided Design at Michigan State University is involved in a joint project with the Oldsmobile Division of General Motors. One goal of this project is to limit the forces transmitted from the engine to the passenger compartment of an automobile. As part of this project a method was developed to quickly and inexpensively calculate the response of an automobile to a given set of forces from the engine.

The finite element model for this project was created by General Motors Engineering Staff, using the finite element program Nastran. The model consisted of 408 nodes, each of which was allowed six degrees of freedom, yielding 2448 equations to be solved for frequencies and mode shapes. The Nastran program solves this second order eigenvalue problem and uses a post processor to create the modal model. Of the 2448 modes found, 60 fell within the frequency range of interest. The diagonal [Mm] and [Km] matrices and the 60 x 60 [Cm] and [Dm] matrices as well as the 2448 x 60 transformation matrix [U] were sent to the Case Center.

The reduced modal matrices were then used to make up the two first order [Mp] and [Kp] matrices. Initially the Case Center's minicomputer, a thirty two bit Prime 750 with one megabyte of memory, was used to solve the 120 by 120 first order eigenvalue problem. It took the computer four hours to compute the eigenvalues and eigenvectors. The resulting eigenvectors were used as the transformation matrix [V]. But the resulting [Mz] and [Kz] matrices were not diagonal. Since the same software had been sucessful in solving smaller problems, the indication was that the trouble was in the size of the problem.

The size of the problem can affect the accuracy of the solution because computers use floating-point arithmetic [10]. This means that after each operation in floating-point arithmetic the result is rounded off to a fixed number of digits. The resulting number is then an approximation of the actual result. The larger the size of the problem the greater the number of operations required for the solution. This, along with the relative size of the numbers in the problem, can cause error due to round off, and have detrimental effects on the accuracy of the solution.

CHAPTER 6

With single precision, the number of digits kept after each operation on the Prime 750 is seven digits. More digits were needed for accuracy in this problem. The number of digits kept can be increased to fourteen by in double precision. However the library subroutines we used to solve the eigenvalue problem were not available using double precision.

The round off problem was solved through the use of a Control Data Corporation Cyber 170 series model 750, at the Michigan State University Computer Center. Single precision on the Cyber is fourteen digits and the same library subroutines were available. The problem took less than fifteen minutes to compute and the resulting eigenvectors produced diagonal [Mz] and [Kz] matrices.

With the first order eigenvalue problem properly solved, the diagonalized [Mz] and [Kz] matrices and the transformation matrix [V] formed from the eigenvectors were put up on the minicomputer where the response would be computed.

With the equations uncoupled the next thing needed was the force input to the model. In the Oldsmobile project the forces that drive the finite element model came from two sources. These were test data taken from an engine and forces derived from a rigid body simulation of the engine. The testing phase of the project is well under way and producing results. The analytical phase is still under development and as such no results have yet been calculated using this type of data.

The tests fall into three catagories: 1) engine in a prototype vehicle, 2) engine on a rigid test stand, and 3) engine in a test buck (the test buck having the front suspension and part of the structure of the automobile). Data was taken in a prototype vehicle to establish a base line for later tests on the test stand. This also gave the opportunity to collect some vehicle response data for correlation purposes. In each case an engine was run and the time histories of the forces which the engine transmitted were recorded on a FM tape recorder. This consisted of recording forces in the X, Y and Z directions of each of six mounts, yielding eighteen forces. These forces are then postprocessed using a Hewlett Packard (HP 5423A) Structural Analyzer. The FFT's of the the forces are then transferred to the minicomputer and transformed so they can be used in Equation 26.

The frequency response corresponding to each set of forces can be calculated in approximantly three minutes on the Prime 750 minicomputer.

The force data measured from the prototype vehicle was used to calculate a response. This response could then be compared to the response data taken from the same vehicle. Figure 3 is the FFT of one of the eighteen forces which came from the testing. In this case it is the Z direction of mount number one while the engine was running at 750 RPM. This and the other seventeen forces were then used as input to calculate a response. In Figure 4, this response is compared to the measured response of the vehicle.

Figure 4 indicates that the model did not exactly predict the measured response. There are many possible reasons for the response difference, the most important one being that the results are limited by the original finite element model. The measured data presented in Figure 4 was measured from a prototype vehicle, while the finite element model is of a production car which has been structurally up-dated from the prototype stage.

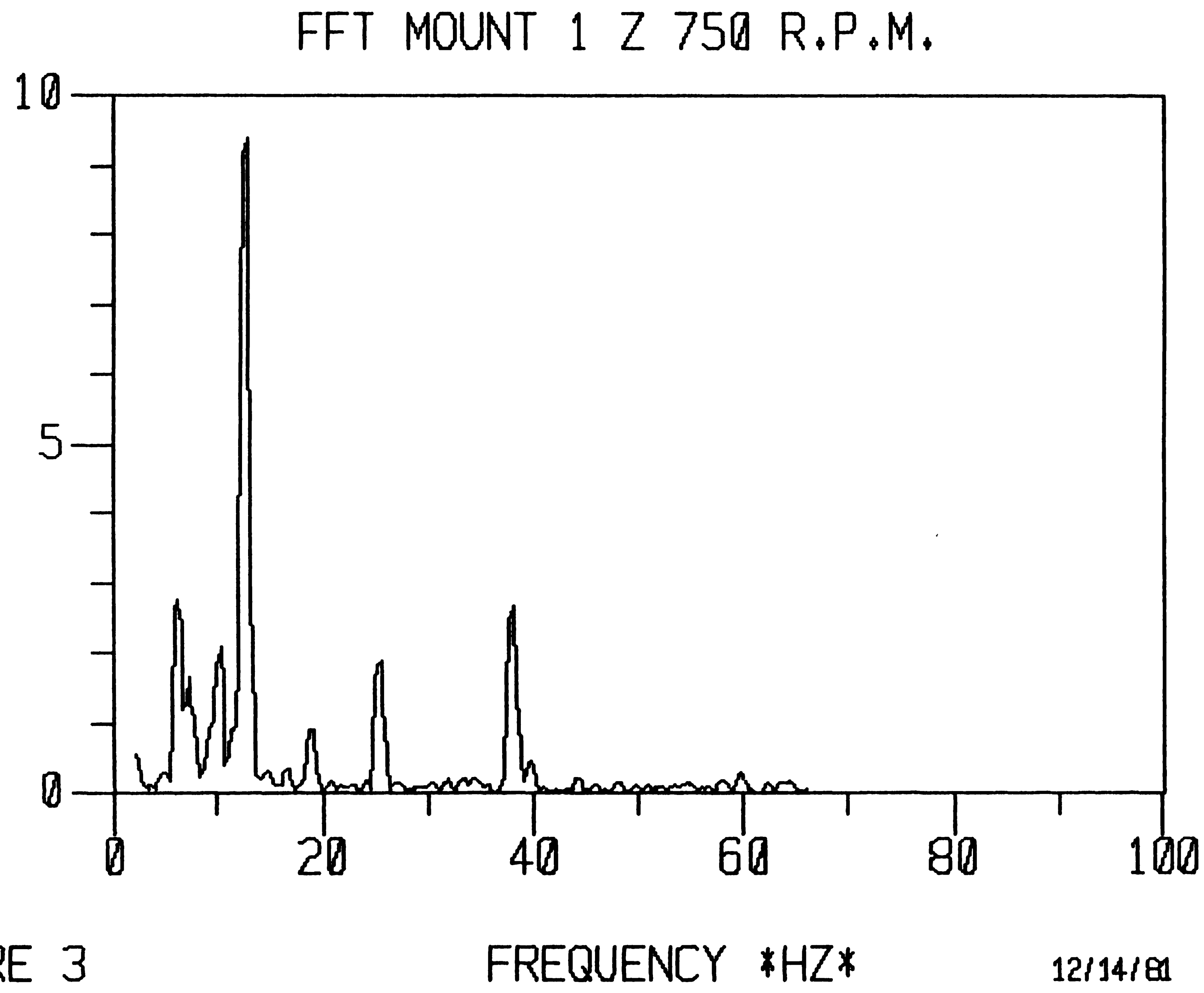
FFT MOUNT 1 Z 750 R.P.M.
10
5
0
FORCE *LBS*
0
20
40
60
80
100
FREQUENCY *HZ*
12/14/81

FIGURE 3

RESPONSE Z STEERING COLUMN 750 R.P.M.

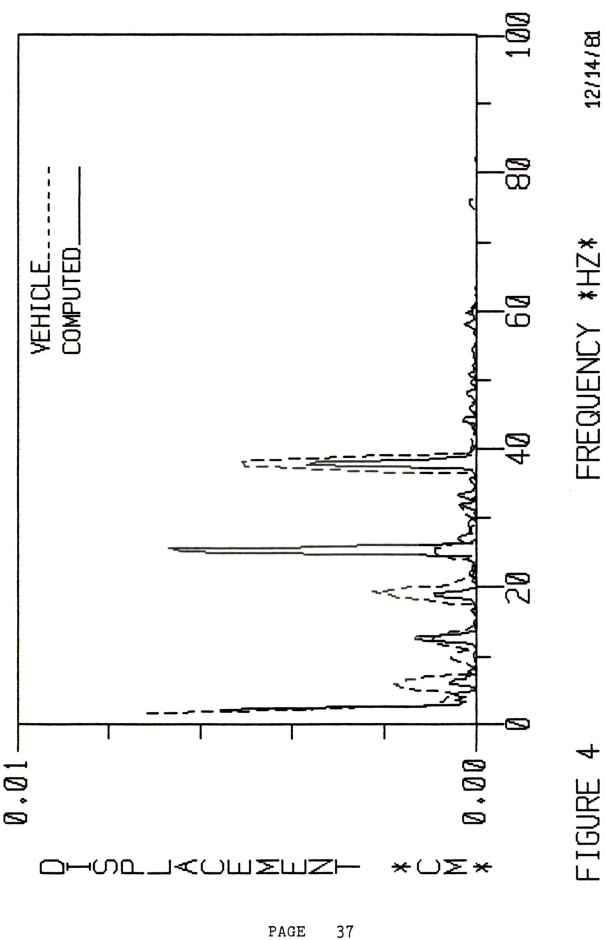

FIGURE 4

CHAPTER 7

CONCLUSIONS

In the preceeding chapters an analysis was developed whereby a finite element model of a complex structure could be reduced to a set of uncoupled ordinary first order differential equations. These equations could then be solved and the response of the structure calculated. In the last chapter the analysis was put to use and the results compared to the measured data.

The analysis is based on four assumptions, 1) the finite element model is an accurate model of the structure, 2) the damping forces which occur in the structure are small, 3) the frequency of oscillation of the forcing function are known to be in a given range, 4) modes whose natural frequencies are not near the range of the forcing frequencies do not significantly affect the response of the structure.

The goal of this analysis is to facilitate the calculation of the response of a structure based on a finite element model of the structure. The analysis met this goal offering considerable cost savings over computing the response directly from the finite element model.

APPENDIX A

NOMENCLATURE

NOMENCLATURE

$[M]$ = Matrix of inertia coefficients (mass matrix)

$\{\ddot{X}\}$ = The acceleration vector

$[C]$ = Matrix of viscous damping coefficients

$[D]$ = Matrix of structural damping coefficients

w = The frequency of oscillation

$\{\dot{X}\}$ = The velocity vector

$[K]$ = Matrix of stiffness coefficients

$\{X\}$ = The displacement vector

$\{F\}$ = The force vector

i = $\sqrt{-1}$

t = Time

exp = The exponential function

λ = w^2 is the eigenvalue and the square of the undamped natural frequency

$\{A\}$ = The associated eigenvector

$[U]$ = The transformation matrix of eigenvectors

$[U]^T$ = The transpose of the $[U]$ matrix

$[Mm]$ = The diagonal modal mass matrix $[U]^T[M]\,[U]$

$\{\ddot{Y}\}$ = The second order modal acceleration vector

$[Cm]$ = The coupled modal viscous damping matrix $[U]^T[C]\,[U]$

$\{\dot{Y}\}$ = The second order modal velocity vector

APPENDIX A

$[Dm]$ = The coupled modal structural damping matrix

$[U]^T[D]\ [U]$

$[Km]$ = The diagonal modal stiffness matrix $[U]^T[K]\ [U]$

$\{Y\}$ = The second order modal coordinate vector

$\{Fm\}$ = The modal force vector $[U]^T\{F\}\ \exp\ (i\ w\ t)$

$\{B\}$ = A solution vector

λ = The first order eigenvalue

$\{C\}$ = The first order eigenvector

$[V]$ = The first order transformation matrix made up of the first order eigenvectors

$[V]^T$ = The transpose of the $[V]$ matrix

$\{P\}$ = The first order transformation coordinate

$\{\dot{P}\}$ = The first order transformation velocity

$[Mp]$ = The first order mass matrix

$[Kp]$ = The first order stiffness matrix

$\{Fp\}$ = The first order force vector

$[Mz]$ = The diagonal first order modal mass matrix

$[V]^T[Mp]\ [V]$

$\{\dot{Z}\}$ = The first order modal velocity

$[Kz]$ = The diagonal first order modal stiffness matrix

$[V]^T[Kp]\ [V]$

$\{Z\}$ = The first order modal coordinate

NOMENCLATURE

{Fz} = The first order modal force vector $[V]^T\{Fp\}$

Φ = The phase angel with respect to time

{D} = The amplitude of the force

{R} = The real part of the force

{I} = The imaginary part of the force

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