ABSTRACT

AN INVESTIGATION OF CHILDREN'S LEARNING OF SOME CONCEPTS AND PRINCIPLES WHICH ENABLE THEM TO PERFORM EXAMPLES OF ADDITION OF COMMON FRACTIONS

by Marjorie Pickering

This study investigated the order in which children learned some concepts and principles which enabled them to perform examples of addition of common fractions. To delineate the concepts and principles, a hierarchy was developed which had as its base the understanding of operations with whole numbers and as its apex the performance of the example 3 3/10 + 2 5/6. Two classes were instructed in the concepts of fractions and tested at regular intervals. One of the classes used commercial materials and was instructed as a group with everyone working on the same material at the same time (Treatment A). The other class used a specially developed set of materials which maximized individual work and allowed some free choice of the order in which certain of the principles were studied (Treatment B). The test results were analyzed to determine invariances in the order in which students

developed an understanding of the concepts and principles of the hierarchy. The data was examined for patterns of learning, the relative performances of the two classes were compared, and contrasts in the two methods were reported.

The main criterion for determining order were eight surveys, each containing 26 examples, one for each concept or principle of the hierarchy. The items from the surveys were considered in pairs (a,b). For each class the number of students who performed a on an earlier survey than b, who performed b on an earlier survey than a, and who performed a and b simultaneously were tabulated. This tabulation was analyzed and where applicable an order $a < b$ or $b < a$ was established. The results for each class were compiled into a projected hierarchy. Comparison of the two hierarchies indicated that with the exception of the concepts of least common multiple and the principle of multiplication of fractions all of the orders under Treatment B also applied under Treatment A. It appeared that prescribing the order of instruction has a direct effect upon the order of learning. More students seemed to have an understanding of the partition and the rational number interpretation of fraction if the partition interpretation was taught first and drill provided. More students seemed to have an understanding of the addition of fractions having the same denominators if they approached addition through the rational number interpretation of fraction using concrete aids than if they approached addition through the partition interpretation.

The average pretest-posttest gain of students who understood the partition interpretation of fraction at the time of the pretest was greater than the class average. Several students individual histories showed that they performed only examples which had easy algorithms involving whole numbers. An understanding of equivalent fractions was acquired under Treatment A without an understanding of multiplication of fractions or of fractional names for one. An understanding of mixed numeral seemed to aid in the understanding of fractional names for one.

Although the difference in gains was not statistically significant, students receiving Treatment B showed greater gains in performance and better retention than students who used the textbook materials. Informal observation in later mathematics lessons seemed to indicate that the students who had received Treatment B were more enthusiastic than students who had received Treatment A.

It is feasible to employ a method of

individualized instruction to a study of fractions. The use of concrete aids manipulated by students appears to promote better understanding of the process of addition of fractions and more enthusiasm on the part of the students.

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CHAPTER I

General Problem

The general question "What mathematics do children need to learn?" has no commonly accepted answer. There is almost unanimous agreement that every child should be taught to add common fractions. The purpose of this paper is to investigate children's learning of some concepts and principles which enable them to perform examples of addition of common fractions.

A search of the literature concerned with fractions reveals a conspicuous absence of detailed studies that report the actual learning history of pupils during the period of time in which learning is taking place. Studies tend to base their conclusions on a pair of tests, a pretest and a posttest, (Howard, 1950; Aftreth, 1958; Fincher, 1963; Pigge, 1964)[1], or on posttests alone (Morton, 1924;, Hayes, 1927; Brueckner, 1928; Polkinghorn, 1935; Guiler, 1945). A history of learning would be expected to show variations in the rate at which pupils learn, variations in the accomplishments of learners at different stages during the learning process, variation

[1]Names and dates in parenthesis refer to listings in the bibliography.

in the total amount of learning, variation in the manner in which learning takes place, and variation in the nature of the difficulties which pupils encounter during learning (Edwards, 1932). Further, such a study should provide a variety of learning experiences so that any invariances displayed are not solely the result of the use of a single set of materials. In studies of this type investigators have generally ignored the field of fractions.

Historical Bases for the Study

Early educational research on the addition of fractions seems to reflect the prevailing psychology of the time in which it was conducted. With the emphasis on "drill" during the mental discipline era of the early twentieth century a profusion of error-analysis studies were made. If the causes of the errors were detected, it was thought, drill of the proper type could be provided and the errors could be eliminated (Hayes, 1927, p. 130). The results of these studies, however, indicated that a great many of the errors in addition were due to a lack of understanding of processes with fractions (Brueckner, 1928; Searle 1927; Morton, 1924). In a later study Sebold (1947, p. 71) reports that

> Although approximately two thirds of the pupils in grades five to seven who were interviewed could add simple similar and unlike fractions, most of them relied on

> mechanical procedures which they had acquired. Very few could tell why unlike fractions had to be changed to similar ones before adding. "That is how I learned it," was the common response to the question, "Why do you change these fractions to fractions having a common denominator?"

One consistent conclusion from error studies is that the subjects tested showed "incompetency" in addition of fractions (Guiler, 1945a; Guiler, 1945b; Sebold, 1947). None of the error studies, however, yielded information concerning those students who do become competent.

In the transition to "social" arithmetic in the late 1920's, 1930's, and the early 1940's, emphasis turned to instruction with only those common fractions which were socially useful (Wilson and Dalrymple, 1937). Social utility was again argued by Johnson (1956) who indicated that since adult usage favors decimals, only the most common of the common fractions should be taught and that the place value principle should be extended at an earlier age to include decimal fractions which are inherently easier.

During this same period of time leaders in mathematics education such as William A. Brownell and C. L. Thiele promoted the idea that more "meaning" must occur in the teaching of arithmetic (Brownell, 1935; Thiele, 1941). Investigations have shown that students performed arithmetic computations significantly better when specific efforts were made to assist the pupil in understanding (Steele, 1940; Howard, 1947; Pigge, 1964). Howard showed that the use of concrete materials in developing meaning significantly improved the performance level. Pigge's study showed that a combination devoting 50% or 75% of the time to developmental-meaningful activities enabled the pupils to perform significantly better than did pupils who had been exposed only 25% of the class time to developmental-meaningful activities with the remainder in each case being devoted to a drill.

Textbook changes seemed to reflect both the social utility arguments and the plea for the incorporation of more "meaning". Dooley (1950) reports that research resulted in the elimination of awkward, unrealistic fractions as well as the increased utilization of illustrations as visual aids.

Just as the aforementioned studies of the first half of this century reflect the psychology and philosophy of that time, so must a study done today reflect those of the present time. Some of todays thinking is indicated by the following:

> One thing we can all be quite certain of: Wherever in the vast realm of human learning we wish to look for individual differences, we surely will find them.
>
> ...Arthur Jensen (1967, p. 117)

> At the present time it seems fair to say that we know considerably more about learning, its varieties and conditions, than we did ten years ago. But we do not know much more about individual differences in learning than we did thirty years ago.
>
> ...Robert M. Gagne (1967, p. xi)

> Approaches to teaching can be better designed only when we better understand how people learn mathematics.
>
> ...E. Glenadine Gibb (1968, p. 434)

> At the turn of the century, the treatments became less axiomatic, and presentations were geared to what was believed to be a child's level of understanding. This theory caused a "low" as far as axiomatic insights into arithmetic were concerned, and the situation existed for a period of at least fifty years. Since the mid-fifties, however, textbook presentations have been based on the axiomatic understanding of the structure of the number system.
>
> ...Sister A. M. Sibilia (1959, p. 207)

Instead of reporting the problems of the groups of students who have not learned to add fractions, this study will report on the successes of those individuals who are learning to add fractions. Instead of emphasizing drill as a technique for promoting learning, it will emphasize pattern and definition as a technique for discovering the processes involved in addition of fractions. Instead of considering the social utility of the material being learned, it will consider the overall structure of number systems. Instead of looking only at grouped data on a pretest and/or posttests, it will look at the progress made by individuals at regular intervals during the learning process.

Fractions and Rational Numbers

Although some texts define a fraction to be a name for a rational number, there is a growing tendency to allow a fraction to be a number as well (Hill, 1967). In this study the latter definition will be used.

(1) A fraction is an ordered pair of natural numbers a and b which is usually named by the symbol "a/b". Two fractions a/b and c/d are said to be equivalent if $a \times d = c \times b$.

(2) A rational number is a class of ordered pairs of integers. The ordered pairs are written in the form m/n, with the restriction that "n" is never 0.[2] When an ordered pair for which m and n are both positive integers is chosen from the set to represent the rational number, we call this ordered pair a fraction. When two fractions are equivalent, they represent the same rational number.

A more complete development of the concept of fractions is given in Chapter II.

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[2]Peterson and Hashisaki, Theory of Arithmetic Second Edition, John Wiley & Sons, 1965, p. 172.

Design of the Study

By analyzing the processes involved in the addition of fractions, a list of concepts and principles deemed necessary for the performance of an example was established. These principles were arranged in a logical hierarchy with the goal represented by the example 3 3/10 + 2 5/6. The individual elements of the hierarchy were likewise represented in mathematical terms so that a measure of pupil understanding could be made. This was the first component of the study.

The second component was the development of achievement and diagnostic instruments to measure the student's achievement of the concepts and principles of the hierarchy. One test form was designated to be used as a pretest and a posttest. Five other forms were designated to be used at regular intervals between the pre and posttests. A seventh form was designated to be used as a retention test two months after the learning period.

The third component of this investigation was the materials. One set of materials A, consisted of the last two chapters of the Addison-Wesley Fourth Grade textboooks[3], pencils and paper. The

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[3]Eicholz, et al., Elementary School Mathematics, Book 4, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, 1964.

second set of materials B, were units specially prepared for this experiment. These materials began witha number line approach to rational numbers and permitted students to find the sums of fractions using concrete aids after two days of instruction.

The last component of this investigation was two classes of fourth graders; the classes being chosen at random from all fourth grades at 7 elementary schools in the East Lansing Public Schools. One of these classes, A, was assigned the commercial materials and instructed as a total group with everyone in the class working on the same material at the same time. The other class, B, which was not as far along in the textbook material as the first class, was assigned the second set of materials which maximized individual work and also allowed some free choice of the order in which certain of the principles were studied. Both of the classes were conducted by the writer for the 20 day period of the investigation.

Purposes of this Study

The purposes of this study were (1) to develop an instructional unit on the addition of fractions which is designed for individual instruction, (2) to compare the effectiveness of this unit with that of a standard textbook unit on the same material presented on a class basis, and (3) to determine invariances in the order in which students develop an understanding of the principles involved in adding fractions.

The primary hypothesis related to these purposes was:

(1) The order in which items from the hierarchy are mastered does not differ from one class to the other.

Three secondary hypothesis were also considered:

(2) The order in which items from the hierarchy are mastered supports the logical order as indicated on the hierarchy.

(3) Students using the experimental material will make no greater change in performance than students who used the textbook material.

(4) Students who already understand some basic concepts of fractions can progress further in the hierarchy than those who do not.

CHAPTER II

BACKGROUND FOR FRACTIONS

The Concept of Fraction

Contemporary literature exhibits a variety of interpretations of fraction and related concepts.

Peterson and Hashisaki (1965) describe four interpretations of fraction with Hill (1967) and Fehr (1968) offering a fifth. Botts (1968) explains three uses of the word "fraction" extending the list of two offered by SMSG (1962). Still others (Brumfiel, et. al, 1961) equate "fraction" to "rational number" in certain circumstances. To appreciate the full scope of the concept of fraction, each of these points of view needs to be considered.

Landau (1960) defined a fraction, developed the fraction as an element of a mathematical system, and then defined a rational number in terms of fractions. Key steps in his exposition are

Definition 7: By a fraction $\frac{x_1}{x_2}$ (read "x_1 over x_2") is meant the pair of natural numbers x_1, x_2 (in this order).

Definition 8:

$$\frac{x_1}{x_2} \sim \frac{y_1}{y_2}$$

($\sim$ to be read "equivalent") if

$$x_1 y_2 = y_1 x_2 \quad .$$

Definition 13: By $\frac{x_1}{x_2} + \frac{y_1}{y_2}$ (+ to be read "plus")

is meant the fraction $\frac{x_1 y_2 + y_1 x_2}{x_2 y_2}$.

It is called the sum of $\frac{x_1}{x_2}$ and $\frac{y_1}{y_2}$,

or the fraction obtained by the addition of

$$\frac{y_1}{y_2} \text{ to } \frac{x_1}{x_2}.$$

Definition 16: By a rational number, we mean the set of fractions which are equivalent to some fixed fraction.

Definition 17: $X = Y$
(= to be read "equals") if the two sets consist of the same fractions. Otherwise,

$$X \neq Y$$

($\neq$ to be read "is not equal to").
Let X and Y be integers, say $X = x$ and $Y = y$.

Then by Theorem 114, the rational number $\frac{x}{y}$ determined by Definitions 26 and 27 stands for the class to which the fraction $\frac{x}{y}$ (in the earlier sense) belongs.

Along with the above key definitions Landau proves theorems displaying the rules that apply to fractions under the operations of addition and multiplication. Fractions, the operations defined on fractions, and the rules governing these operations form a mathematical system. Are fractions in this context numbers?

Botts (1968) makes a distinction between fraction as a number and fraction as a pair of numbers. In regards to the latter he writes (p. 218)

> Now here we may be sure that we are _not_ speaking of fractions as numbers, for the numbers 3/2 and 6/4 are the same, and that is what we assert when we write 3/2 = 6/4. In this usage the term "fraction" applies to something whose essential feature is a _pair of numbers_, a numerator and a denominator.... Such a numerator-denominator pair does, to be sure, _define_ or _determine_ a number in a conventional way, namely the number that results from dividing the numerator by the denominator. But the fraction, in this usage, is really the numerator-denominator pair, not the number we get by dividing.

Hence, two interpretations of fraction are suggested: fraction as an element of a mathematical system vs. a fraction as a quotient. However, what is the nature of the number that is obtained by dividing?

Some authors would call this number a fraction as well. Gibb, et. al, (1959, p. 29) wrote

> Fractions were invented to deal with parts of things and to make division always possible.

The authors of SMSG (1962) chose to use _fractional number_ when they were talking about the number, although later in the unit the term _fraction_ was used, relying on the context to make clear what was meant.

Fouch and Nichols (1959, p.334) explain

> Thus, a fractional numeral is a symbol naming a fractional number. We use the phrase "fractional number" to be synonymous with

> "rational number". Since in common usage the word "fraction" is used to refer to a number, we may abbreviate and also use "fraction" to be synonymous with "fractional number" or "rational number".

Other authors use the desirability of a system to be closed under division or to have a root of the equation nx = m to motivate the axiomatic development of the rational numbers.

Sibilia (1959, p. 161) summarizes

> A modern approach to the study of fractions is based on the axiomatic construction of the rational number system. This provides an interpretation of fractions as elements of the rational number system.

The definition of fraction as a name for a rational number is one which cannot go unnoticed. SMSG (1962) defined fraction to be a numeral. As such it had a numerator and a denominator. Landau (1960, p. 42) used the same symbol x/y to refer both to the rational number and to the fraction which belongs to the rational number. In one sense, he was using a fraction from the set to represent the rational number.

The symbol for a fraction is a pair of numerals (Gibb, et. al, 1959, p. 29). Since the same symbol can be thought of as naming a rational number, the symbol and the fraction are often confused (cf. Mueller, 1961).

In summary the word "fraction" is used to denote many different ideas:

(1) An ordered pair of natural numbers as an element of a mathematical system. (Landau, 1960, p. 19).

(2) An ordered pair of numerals which name a rational number (Mueller, p. 196) or a fraction (in the sense of (1) above) (Gibb, 1959, p. 29).

(3) A rational number. (Fouch and Nichols, 1958, p. 334).

In this study, the context will dictate in which way the word "fraction" is used.

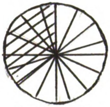
<u>Interpretations of Fractions</u>

Several interpretations of fractions are common in the elementary school. In contemporary textbooks (Cf. Eicholz, 1964; SMSG, 1962) fraction as a partition is generally developed first. Equivalent fractions are defined as those which name the same partition. Although some texts (Brueckner, et. al, 1963, p. 316) discuss addition in terms of partitions, others introduce rational numbers and develop addition in terms of the number line. (Eicholz, et. al, 1964, p. 222).

> For each set of equivalent fractions, think of one rational number and one point on the number line. Any fraction from a set of equivalent fractions can be used to name the rational number for that set.

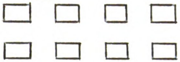
The Partition Interpretation

In a fraction a/b, the denominator, b, tells the number of equal parts an object or set is to be divided into and the numerator, a, tells the number of the parts which are to be considered. This interpretation is used in answering questions such as the following:



What part of this circle is shaded?

(Ans. 6/16 or 3/8).



Shade 3/4 of the squares.

What is another fraction which tells the part of the total number of squares you have shaded? (Ans. 6/8).

Historically, fractions owe their creation to the transition from counting to measuring. (Gibb, et. al., 1959, p. 29). As a measure "2/3" may be conceived of as (1) naming the number property of a set of 2 elements each of which is 1/3 of some unit.

Fractions as Operators

Although somewhat similar to the interpretation of fraction as a partition, the numerator and the denominator are considered in the opposite order. For the fraction a/b, the a is a stretcher (it magnifies a quantity a-times) and b is a shrinker which has the inverse effect to b. (Fehr, 1968). A fraction as an operator is used extensively in the UICSM Materials. (Braunfeld and Wolfe, 1966; Braunfeld, et. al., 1967).

Example: Find 2/3 of 12.

The 2 operates on 12, doubling its value.

Next the $\bar{3}$ operates on 24, shrinking it to 8.

A line segment 12 units long would be used to represent 12.

<u>The Division Interpretation</u>

A fraction a/b indicates the quotient a ÷ b. This interpretation is used in answering questions such as the following:

Find 15/3 (Ans. 5)

What is 3 ÷ 2 ? (Ans. 1 1/2 or 3/2).

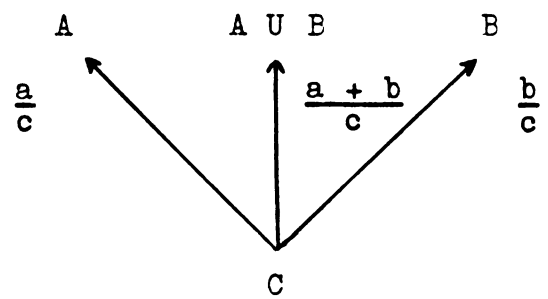
<u>The Ratio Interpretation</u>

Ratio denotes a relative comparison of quantities and as such is a pair of numbers. (Van Engen, 1960) calls such comparisons of quantities <u>rate pairs</u> and cautions that the addition of rate pairs does not follow the usual fraction rules. For this reason he does not call a rate pair a fraction. He writes

> It is now apparent why fractions and rate pairs (usually called ratios) are often confused. Both have in common the following properties:
> a. The test for equivalence;
> b. Membership in only one equivalent set.
> Here the similarity ends. Pairs of numbers used as fractions can be added according to the

> usual rules of arithmetic, but pairs of numbers used as a rate pair are not added according to the usual arithmetic rules.

Bidwell (1966), however, points out that if we consider fractions to be added as those which represent a ratio of disjoint sets to a given set C, then the sum of the fractions will be the ratio of the union of the disjoint sets to the given set.



Some examples that can be performed using the ratio interpretation are (Johnson, 1948)

> What part of 27 is 9?
>
> Ans. For each unit in 9, there are 3 units in 27, hence 9 is 1/3 of 27.
>
> Find 1/4 of 24.
>
> Ans. We need to find a cardinal number such that each element in its set can be made to correspond to 4 elements of a set of 24. Since 6 x 4 = 24, the answer is 6.

The Element of a Mathematical System Interpretation

By a number system is meant a set of numbers, operations defined on the numbers in the set, and rules governing these operations. (Peterson and Hashisaki, 1967, p. 74.) The system of fractions consists of the set of fractions with binary operations + and X possessing the properties

Closure for + and .
+ and X are commutative.
+ and X are associative.
There exist identities for + and for X.
For every element of the set there exists
an additive inverse and a multiplicative inverse (with the exception of 0).
X distributes over +.
The conditions for equivalence of fractions.
Order.

Several formal developments of **fraction** and of rational number are given by Hill (1967).

CHAPTER III

RELATED LITERATURE

The literature provides few results which have direct bearing on this study. Related investigations fall into three general categories: the comparison of methods by which students learn, the histories of class performance during the learning process, and the collection and analysis of data at specified times during the learning process.

Methods of Teaching

In an investigation comparing two methods of teaching Lankford and Pattishall (1956) found a significant difference in favor of an experimental method with two important features:

> (a) Ideas and rules of arithmetic are developed inductively through pupil participation.
>
> (b) Pupils are encouraged to learn arithmetic thoughtfully and independently. To this end we encourage mental arithmetic and varied approaches.

In an initial pilot study they had found (1956, p. 3) that many pupils had learned very little from the conventional teaching of arithmetic other than a list of processes which were apparently used in a highly mechanical manner with little thinking and often still getting incorrect answers. Another impression received during the pilot study was the unexpected indication that the bright pupils interviewed followed both literally and uniformly the conventional algorisms in arithmetic as did the pupils with lower ability. Regarding this impression Lankford and Pattishall wrote (p. 25)

> They get more correct answers because their memories were better and they were more careful workers. Their attention spans were also longer than those of the dull pupils. There was little indication in these interviews held prior to the experiment that bright pupils learn from conventional teaching to do arithmetic more independently, with more originality, or more thought than do duller pupils.

Lankford and Pattishall concluded (p. 67)

> Perhaps the most important fact demonstrated by this study is that it is a sound procedure to allow pupils to use as much freedom and exploration as they require to understand fully working with fractions.

Using a pretest, a posttest, and a retention test, Fincher (1963) found that the use of programmed materials is more effective than the use of conventional textbook approach in the teaching of addition and subtraction of fractions and no less effective than the conventional method for recall after a four-week interval.

In comparing learning by drill to learning with extensive use of audio-visual aids and considerable emphasis being placed on meaning (Howard, 1950, p. 29), no significant results were found in computation with fractions at the end of the initial learning period. However, when the same students were tested the following September, the results favored those who had made use of audio-visuals (either part or all of the time).

Souder (1943, p. 134) found that the use of the diagnostic readiness test for instructional purposes differentially affects the learning of pupils.

In a study to determine the extent to which the identification and correction of errors in sets of examples in addition and subtraction of fractions affected learning, no statistically significant differences were found to favor either the experimental or the control group. (Aftreth, 1958).

Anderson (1966) found no significant difference in teaching either of two procedures for the addition of fractions: that of setting up rows of equivalent fractions or that of factoring the denominators.

Histories of Class Performance

In 1932, Edwards investigated how students differ in learning about fractions. Using a plan of instruction which allowed each student to move step by step through the same materials at his own rate, he found large differences in the amount of progress made during identical periods of time. Pupils who ranked high in general arithmetical ability and mental ability required less attention from the teacher and attained a more complete mastery of the processes taught. In attempting to develop a regression equation from which to predict student success, he found the equations little better than a guess. Students in his study developed some ability to solve problems upon which no previous instruction or drill was given.

In another sequence of units designed for individual instruction, Brooks (1937) confirmed that there are very great individual differences in the time needed by pupils for the completion of a unit of learning. He studied the workbooks of the individual students as well as the pretest-posttest analysis. He found that units presented different degrees of difficulty to different students but that certain units of work were more difficult than others to the group as a whole. In the individual scores no pupil showed a steady and consistent gain from unit to unit of learning and none high in the early units dropped off greatly at the end. The data provided by Brooks is by units or by class. He did not study the question of sequencing problems for individual students. Each student follows the same pre-designed curriculum and no data is given indicating which of the concepts or principles of each unit are attained.

The Collection and Analysis of Data for the Early Elementary Grades

In investigations by Gunderson (1958) and by Gunderson and Gunderson (1957), it was found that seven year old children are interested in, like to, and are able to work with fractions when teaching is done without use of the fraction symbols and with the use of manipulative materials. The problems solved by seven-year-olds (1958, p. 237) involved addition of fractions, subtraction of fractions, and fractional equivalents. Both studies concluded that there is a need for a long acquaintance period between the child's first introduction to fractions and the time he is expected to work with fractions using algorithms and symbols.

By means of a single interview, each of 266 children from the kindergarten, first, second, and third grades were tested to discover when, what, and how concepts of fractions are acquired naturally by children (Polkinghorne, 1935). It was found that the acquisition of concepts without formal teaching is a continual process and a direct result of experience with fractions in daily living. Kindergarten and first grade children showed understanding of unit fractions only. In grades two and three other proper fractions, improper fractions, and the identification of fractions were known. No evidence was displayed for an understanding of equivalent fractions.

Preliminary to her investigation, Sebold (see next paragraph) found indications that concepts of fractions were known by some of the students prior to formal teaching. She also found several misconceptions (See Table I, page 26).

<u>The Collection and Analysis of Data for Later Grades</u>

Sebold (1946) used individual testing or interviews, group testing, and individual instruction in efforts to discover the mental processes through the development of which pupils arrive at an understanding of the basic concepts in fractions. She concluded that (p. 79)

> There is no uniformity in learning among the children. Not all children in a given grade are at the same level of learning in respect to all the concepts. Not every child is at the same stage in respect to all concepts, nor do all children traverse the same series of stages preliminary to final, meaningful understanding.
>
> In general, the learning of the basic concepts in fractions progresses through the following levels of understanding:
>
> (a) No knowledge of the concept.
> (b) Erroneous ideas of the concept.
> (c) Confusion of the concept with others, expecially with those which have been only partially learned. But in the confused ideas there is often an element of correctness.
> (d) Partially correct but vague ideas.
> (e) Knowledge that an incorrect presentation or illustration of the concept is not true.

TABLE I

CONCEPTS AND MISTAKEN CONCEPTS OF FRACTIONS DISPLAYED IN EARLY ELEMENTARY GRADES

	Concept Displayed	Mistaken Concepts Displayed
First and Second Grades	Name a few fractions Recognize the fractional parts of figures or objects which have been divided	
Third Grade	Unequal divisions of figures cannot be designated as fractional parts (1/4 of pupils) Name the fractional part of a figure that was equally divided Finds fractional parts of groups of figures with respect to unit and other proper fractions (1/3 of pupils) Understands the above (Far less than 1/3 of pupils) One half > one fourth (60% of pupils) Can add 1/2 + 1/2 Fraction is one or more of the equal parts of a group and of a number.	Any part of a figure is one half or some other fraction. Name the fractional part according to the number of parts into which the figure was divided, irrespective of equal or unequal divisions. 1/6 > 1/3, 1/6 > 1/2 The word fraction connotes "one-half" (50% of pupils in third and fourth grades)
Fourth Grade	Find fractional parts of groups (35% of pop.) 1/3 > 1/6, 1/2 > 1/6, 3/4 > 1/2, 2/3 > 1/3, Add a few fractions.	

* Based on data given by Sebold (1946, pp. 26-28).

(f) Recourse to the visualization of a previously acquired model with descriptive phrases explaining it.
(g) Memorized information on the concept, with or without understanding.
(h) Partial understanding.
(i) Full understanding.

In an analysis, by means of class tests, of children's mental processes in multiplying fractions in the fifth grade, Collier (1922) found that the child's mental processes related to whole numbers and not to fractions.

Knight and Setzafandt (1924) studied the problem of the extent to which training in the addition of fractions involving the denominators 2,3,6,8,10,12,16 and 24 transfers to the ability to handle the addition of fractions in which the numbers 3,5,7,9,14,15,16,21,28 and 30 are used as denominators. A substantial amount of transfer was shown to occur.

Hayes (1927) found that the same type of errors tend to appear with the same relative frequency throughout the grades. Gundlach (1936) observed that there is great variation in the ability of individuals within each grade for each of the four operations in fractions. He found a great variation in the ability in fractions between pupils within each group representing a different level of capacity but that the ability of those in a group of greatest capacity is less variable than those in the group of least capacity. In addition he computed that the curves of growth in ability in the operations with fractions for the three levels of capacity are similar to the curve of the entire group, the difference being in level of performance.

In summary: There appears to be little difference in the patterns of development followed by the more able student as compared to the less able. Meaningful materials on an individual basis and materials using concrete aids have met with relative success but other differential approaches have had little effect. Learning, as measured by existing testing devices and evidenced by grouped data, appears to progress smoothly rather than in an irregular fashion producing sharp changes when new concepts or principles are encountered.

CHAPTER IV

RESEARCH DESIGN AND DESCRIPTIVE DATA

Development of the Hierarchy

Lists of concepts and skills related to the addition of fractions (Cf. Becker, 1940; Sebold, 1947; Howard, 1948, p.24) have been compiled. These, however, were designed to include the entire spectrum of possibilities rather than simply those necessary for the attainment of a particular objective.

Gagne (1965) offers a totally different approach in developing a hierarchy:

> As described previously (...) the method employed was to ask the question of the final task, 'What would the learner have to know how to do in order to attain this final performance when given only instructions?' In this case, the question applied to the final task yielded the identification of five subordinate knowledges. When applied in turn to these subordinate classes of tasks, and then successively to the additional tasks so identified, the analysis yieldeda hierarchy of subordinate knowledges,..
>
> ..., each successive step in the analysis yielded one or more subordinate knowledge entities that are progressively simpler and more general as one proceeds downward in the hierarchy. The basic set of hypotheses generated by his 'knowledge structure' is the following: (1) the attainment of each entity of knowledge (measureable in each case as a particular performance) is dependent upon positive transfer of training from the next lower subordinate knowledge connected to it by an arrow; and (2) such transfer requires the high recallability of all the next lower subordinate knowledges (connected to it by arrows).

The ultimate goal set in this experiment was the principle of adding fractions, i.e. 2 5/6 + 3 3/10. The attainment of this goal is dependent upon first learning some other principles such as the associative and commutative properties of addition, the definition 3 3/10 = 3 + 3/10, the principle of adding two fractions when they have the same denominators, etc. These principles in turn depend upon knowing the concepts of 3/10, 1/10, 5/6, 1/6, 2, 3, etc. The logical organization of knowledge so developed was represented in a hierarchy of principles and concepts.

Since there are many interpretations of fraction, the three considered most appropriate for fourth grade were included, relying on later testing to determine which of these are prerequisites to addition. Fractions as ratios or operators were not included since interviews conducted by the writer with fifth graders who could perform the indicated final task have shown that the fifth graders did not understand these two interpretations. Hence, they could not be prerequisite knowledge.

Construction of the Tests and Revision of the Hierarchy

Using the hierarchy, several exercises were constructed to test each item. Six fourth graders from the Wardcliff School, Okemos Public

Schools were asked to work the exercises as best each could and the results were recorded. Trial materials were developed and the six fourth graders were given instruction with fractions using these materials and being tutored by the writer during five one hour periods. At the close of instruction, each was asked to again work the exercises. On the basis of the testing and instruction experience, two items were found to have been overlooked in the original development of the hierarchy and other relationships between items were discovered. The hierarchy was revised resulting in that shown in Table II, page 32. The revision included no deletions from the original but did include a reorganization of the order of items and the inclusion of the two additional items.

For each item on the revised hierarchy, one exercise was chosen (See Table III, page 33). These were composed to form the final survey test which was used for both pretest and posttest (See Appendix A). Exercises similar to those on the final test were written for each of the other six test required. The trial materials were redeveloped in their final form (See Appendix B).

The Materials, Treatment A

The materials for Group A consist of the Addison-Wesley Fourth Grade Text (1964), pencil and

TABLE II

HIERARCHY OF PRINCIPLES AND CONCEPTS NECESSARY FOR ADDITION OF FRACTIONS

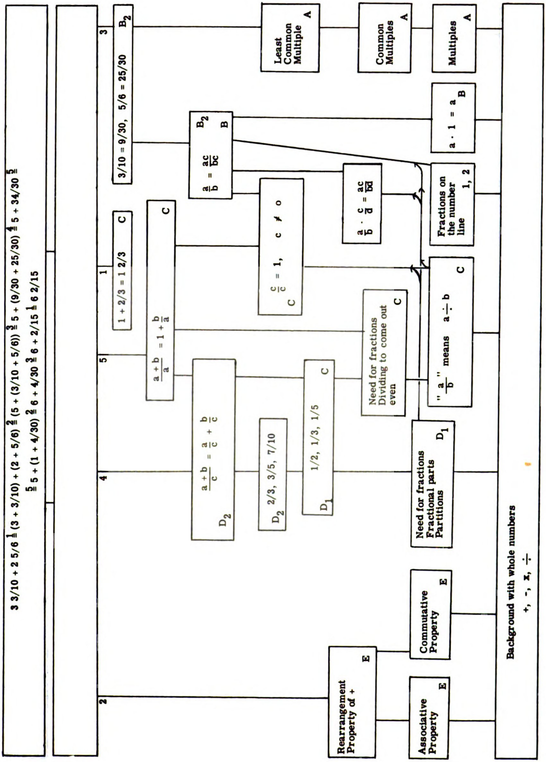
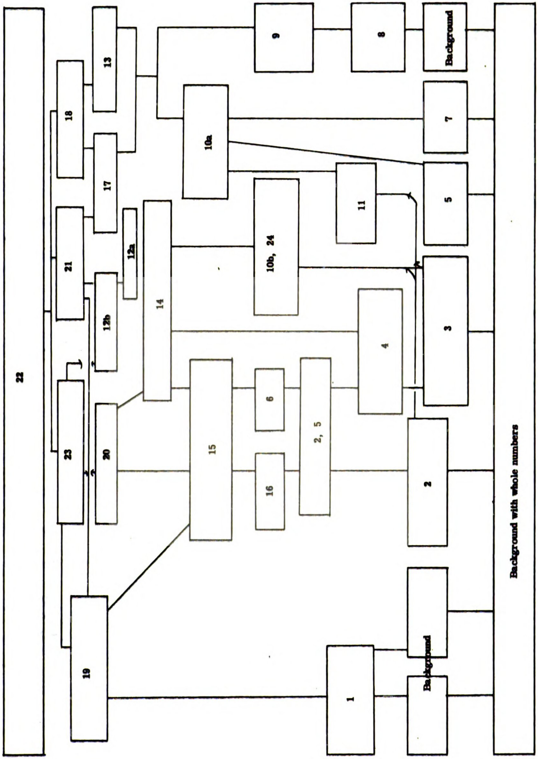


TABLE III

THE HIERARCHY, BY EXAMPLE FROM SURVEY TEST



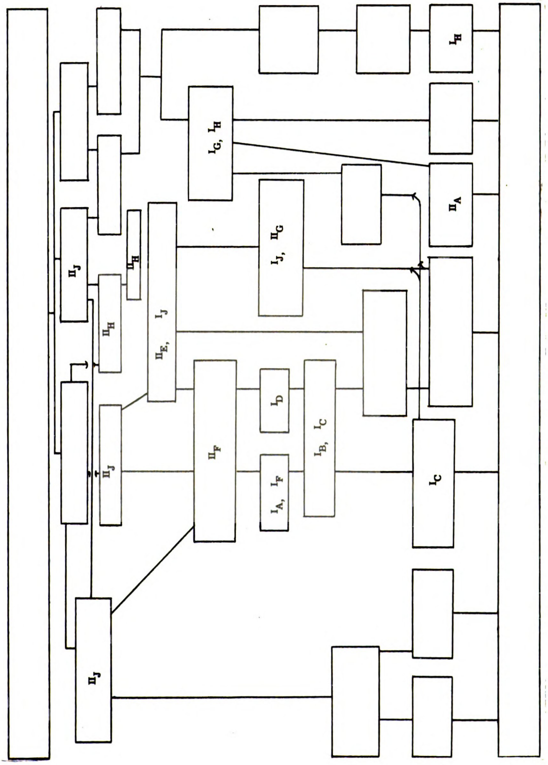
paper. Although some paper folding was used for demonstrations at two different occasions, no manipulative aids were employed by the students. The text made liberal use of diagrams and illustrations. The students were asked to respond orally to some pages and in written form to others. All pages which pertained to sections of the hierarchy were used. A few pages on the formal definition of equivalent fractions were omitted as well as pages unrelated to fractions.[4] An outline of the topics studied in the order in which they occured follows. Table IV illustrates how the materials related to the hierarchy, the labels correspond to the topics in the outline.

---

[4]Pages were assigned in this order: 240-255, 257, 258, 262, 263, 265, 269 (B parts), 282, 283, 285-289, 328 (Set 42), 295 - 296, 298, 299, 278, 300-304, 306, 310, 311. Extra 296, 292, 297, 328 (Set 43).

TABLE IV

CORRESPONDENCE OF MATERIALS TO THE HIERARCHY, TREATMENT A



Outline of Material, Treatment A

I. Fractions
   A. Partition interpretation of fractions (3/4, 2/7, 4/6, etc.)
   B. Fractions as pairs of numbers
   C. Fractions of rectangles
   D. A fraction a/b means a of b EQUAL parts
   E. Fractions of segments
   F. Fractions comparing a part of a set with the whole set
   G. Fractions and parts of an object
   H. Sets of equivalent fractions (partitions)
   I. Sets of equivalent fractions (patterns)
   J. Fractions with numerator > or = to the denominator
   K. Fractions with zero numerators
   L. Review

II. Rational Numbers
   A. Rational numbers as sets of equivalent fractions
      1. Think of one rational number
      2. One point on the number line
   B. Practice with the concept in A
   C. Equivalent fractions name the same rational number
   D. Inequalities
   E. Rational numbers greater than one
   F. Addition of fractions using the number line
   G. Fractions which name whole numbers
   H. Mixed numerals
   I. Addition of fractions using parts of wholes
   J. Review

The Materials, Treatment B

Before developing the materials for Group B, the current literature on teaching and learning was surveyed. Educational psychologists were consulted. A fourth grade class at the Wardcliff School in Okemos, Michigan was observed and tested to determine their understanding of fractions, as well as their general mathematics background. The units were tested in sequential order with a group of 6 children, revised, and put into final form for use in the experimental situation.

Jerome Bruner and Helen Kenney (1965, p. 51) described an experiment in representation and mathematics learning in which eight year old children were given instruction in various mathematical activities:

> Each child had available a series of graded problem cards to go through at his own pace. ...the problem sequences were designed to provide, first, an appreciation of mathematical ideas through concrete constructions using materials of various kinds for these constructions. From these, the child was encouraged to form perceptual images of the mathematical idea in terms of the forms that had been constructed. The child was then further encouraged to develop or adopt a notation to describe his construction...

A second experiment was also described in this paper (p. 57) in which a group of 10 nine-year-olds were instructed in the elements of group theory. Again the approach was one in which the children first worked with physical manuevers and later deVeloped notation and ability to work with the symbols.

In the concluding paragraph of this paper Bruner and Kenney (p. 59) wrote:

> We would suggest that learning mathematics may be viewed as a microcosm of intellectual development. It begins with instrumental activity, a kind of definition of things by doing. Such operations become represented and summarized in the form of particular images. Finally, and with the help of a symbolic notation that remains invariant across transformations in imagery, the learner comes to grasp the formal or abstract properties of the things he is dealing with. But while, once abstraction is achieved, the learner becomes free in a certain measure of the surface appearance of things, he nonetheless continues to rely upon the stock of imagery he has built en route to abstract mastery. It is this stock of imagery that permits him to work at the level of heuristic, through convenient and non-rigorous, means of exploring problems and relating them to problems already mastered.

It is in accord with the above lines of thinking that set B of materials on the addition of fractions developed. The instrument in this case is an expanded concept of the number line in which students in the early elementary grades are, in general, very familiar. A unit of measure was represented both by a rectangle (in the materials provided this is a rectangle $1\frac{1}{2}$" x $10\frac{1}{2}$") considered to be 1 unit in length and by a point on the number line that distance from zero. Rectangles 1 unit in length were divided into fractional parts of the unit. The children were to learn how the pieces could be constructed and used by constructing and using them. The standard fractional notation is used and the process of adding fractions is interpreted as the summing of lengths of rectangles. Students were to be encouraged to discover the algorithms for finding equivalent fractions and for adding fractions so that the concrete materials would be gradually neglected. This section of the materials was designed to be used for approximately five days of activity: two days for orientation to the materials and construction of them in groups of two and three days for manipulation with them on an individual basis. The instructions were provided on 4 groups of 5 X 8 cards (19 in all) and the exercises for practice on four 8 X 10 dittoed sheets upon which the answers could be written. The first five days of activity were to provide the active stage.

For the iconic state, 8 sequences leading to generalizations about the abstract processes with fractions were developed. Any of the eight could be chosen to be worked at any time during this stage and each was expected to require from 1 to 3 instructional periods for completion.

The regularly given tests provided an opportunity to develop at the symbolic level. The tests also served as a challenge to stimulate the discovery by the student of algorithms for solution;

the concrete materials provided verification of a correct solution. No algorithms were provided.

Outline of Materials, Treatment B

I. Construction and manipulation of concrete aids
   A. Construction of number lines with whole number designation
   B. Construction of number lines with fraction designations ($\frac{1}{2}$'s and $\frac{1}{4}$'s)
   C. Agreement to use 1 unit equal to the lengths provided ($10\frac{1}{2}$")
   D. Construction of a number line to the agreed upon scale (1/2's, 1/3's, 1/4's, 1/6's)
   E. Construction of rectangles of 1, 2, and 3 units of length
   F. Construction of rectangles of lengths a fraction of 1 unit (1/2's, 1/3's 1/4's, 1/6's, 1/7's, 1/14's)
   G. Finding different names for the same sized rectangles (Equivalent fractions)
   H. Finding a name for a rectangle equal to the sum of two rectangles (Addition of fractions)
   I. Names for lengths greater than 1 unit (Definition 1 1/6)
   J. Comparison of lengths of rectangles
   K. Practice materials

II. Sequences for development of generalizations
   A. Common multiples and least common multiples
   B. Equivalent fractions (By partitions)
   C. Names for
      1. Equivalent fractions (by multiplication)
      2. Equivalent fractions (by number line)
   D. Adding using equivalent fractions
   E. Interpretation of fraction as an indicated division
   F. Unit fractions using partitions
   G. Other fractions using partitions
   H. The associative and commutative properties

The Population Sample

The population sample was drawn from the East Lansing Public Schools, East Lansing, Michigan. It is a system of about 5048 students in grades kindergarten through twelve, contiguous to Lansing, and composed mostly of well-educated middle class residents. The two classes of fourth graders used in this study were chosen at random from seven elementary schools and 12 fourth grade classes.

One of the classes was assigned the commercial materials (Treatment A) and the other class was assigned the experimental materials (Treatment B). The choice was conditioned by the fact that one group (A) had been accustomed to traditional methods of teacher controlled instruction, whereas the second of the classes had been introduced to individual work two weeks prior to the initiation of the scheduled instruction period. Both classes were conducted by the writer during the period of the experiment.

In each of the groups there were a few students who had skipped either part or all of the third grade arithmetic and/or part of the fourth grade work. These students had not yet developed skill in the multiplication of whole numbers. In Group A, the majority of the class had completed the fourth grade work on multiplication and were working with a section on approximation which precedes work in long division. This group, with the exception of those four who had been moved ahead, displayed mastery of multiplication with whole numbers less than ten. In Group B, however, the majority of the students had not yet mastered the multiplication facts. This might have served to hinder their discovery of patterns in working with fraction which depended on this knowledge. Although these students had been working on the same textbook as Group A, they had not progressed as far.

Fourth graders were chosen for the experiment because it was felt that these students had limited instruction in fractions. In the planned curriculum for the East Lansing Schools, a unit of fractions first appears in the last quarter of the fourth grade.

Initial Characteristics

Initial characteristics of each student were obtained from three different sources. These measurements were his score on the mathematics section of the Sequential Tests of Educational Progress, a 26 item test on basic number facts of addition, subtraction, multiplication, and division, and his score on the pretest. The initial data is presented in Table V.

TABLE V

SUMMARY OF INITIAL TESTS

Treatment Received	Test	Number Tested	Mean	Standard Deviation
A	STEP 1/68	22	249.6	9.4
	Number Facts	22	21.1	5.4
	Pretest	22	3.6	2.4
B	STEP 5/68	22	249.8	11.4
	Number Facts	22	20.9	4.8
	Pretest	22	1.6	1.3

Teaching the Units

On February 1, 1968, the two regular teachers administered the pretest to their classes. Over the next five weeks, for 20 sessions, the writer conducted the two classes, administering the posttest during the twentieth session. The retention tests were given the last week in May, 1968 and each regular teacher said that she had not assigned any work with fractions in the interim. The results of the tests appear in Table VI, page 46.

The lessons for Group B were restricted to 50-55 minute periods daily; the lessons for Group A averaged 55 minutes in length. Both the regular teachers and the writer were available to answer any questions the students had during the mathematics period. Each day in Group A, the written work from the previous day was returned and discussed, some oral drill on material in the text was conducted, the new material was introduced and oral responses solicited for discussion questions, and a written assignment was made and supervised.

In Group B, two students worked together to construct the concrete materials, then each worked individually through the practice sheets using the concrete aids. The student was allowed to proceed to some other section of the hierarchy according to his choice and the availability of material. Each

TABLE VI

SUMMARY OF SURVEY TESTS

Treatment Received	Pretest Mean	Posttest Mean	Retention Mean
A	3.55	12.86	10.55
B	1.64	12.14	11.41
	Respective Standard Deviations		
A	2.42	4.23	4.88
B	1.26	6.56	5.92

	Total Number of Exercises Performed, Mean	Pretest-Posttest Gain, Mean	Posttest Retention Test Gain, Mean
A	15.41	9.32	-1.85
B	15.41	10.50	-0.73
	Respective Standard Deviations		
A	4.17	3.47	3.42
B	6.64	6.13	3.44

student had as his goals, the completion of the short units and trying to acquire enough concepts and principles to be able to perform the examples on the surveys when given. No direct teaching of how to do addition examples abstractly was provided.

Daily Reports

In Class A, daily reports of class procedure and material studied were prepared by the teacher. Time allowed for the lesson, times of testing, and other pertinent observations were recorded. Three sample reports for each group are included in Appendix C.

In Class B with each student operating individually daily reports of general procedures being followed, time allowed for the lessons, times of testing, and other pertinent observations were recorded. In addition a chart was kept on which a record was kept of the date on which each student finished a section of the work. A second blank was filled when the teacher felt that the work had been understood by the student, i.e. if the answers were 90% correct or if, upon questioning the student displayed understanding.

The Intermediate Testing

In addition to the pretest, posttest, retention tests, and the daily class records, alternate forms of the pretest were administered regularly beginning the sixth day. As each type of example was performed correctly it was eliminated from future tests for that individual.

Students in both classes were thus motivated to discover how those he had not yet solved could be achieved. Tables VII, VIIa, VIII and VIIIa show the order in which each individual student performed the exercises, the numbers used correspond to those numbers of boxes on the hierarchy.

<u>Some Individual Histories</u>

In this section two students from each class who performed few examples on the pretest are chosen for a more thorough analysis.

Of students receiving Treatment A, C_2 is an interesting example. C_2 was to be a third grader but was instead placed in the fourth grade class because she was exceptionally able. Her background in multiplication and division were particularly weak. On the pretest 1, 7 were accomplished. After the instruction on the partition interpretation of fraction 2 and 16 were mastered. Next 12b, then 11 because of their relationship to whole numbers. After discussions on equivalent fractions 10a was performed, then 12a. On the posttest she was able to project to 5, 6, 10b, 15 and 19. Other students weak in multiplication in the class performed the same examples, some adding 24 to the list or 8 (Cf. M_4, L_2,

TABLE VII

THE ORDER IN WHICH EXAMPLES WERE

PERFORMED, TREATMENT A

Student	Survey I Pretest	Survey II	Survey III	Survey IV	Survey V
A	1,7,12a,12b	16	10a	2	
C_1	1	16		7	5,6
C_2	1,7	2,16	12b	11	10a
D_1	7,10b	1,16		2	
D_2	1,7,8,10b	2,12a,12b,16	10a		11
D_3	1,7,10b,8	2,5,6,16	3	1,12a,12b	10a
H	1,5,6,7,11 12a,12b,16	2,8,10b	10a		3,4,9,14
J_1	1,7,8	2,5,6,16	3,10a,12a, 12b	11	
J_2	1,7,8	16		2	
K	1,7	2	10a,11,16, 8	3	5,6,10b
L_1	1,7,16			8,10a,10b	2,5,6,12a,12b
L_2	1,7,12a,12b	16	10a	11	2
M_1	7	16	1,8		5,6,10a,10b
M_2	1,7,8,9,12b, 16	2	6	5,10b	3,11
M_3	1,3,7	8,16		10a	2,11
M_4	8	1,7,16	5	2,11	6
N_1	1,5,8,10a 12a,12b,15	11,16,19	6,7,9	2,4	
N_2	1,6,7,10b,12a,16, 12b,15,19,21,23	2,8	5,9,10a,17, 24	3,4,14,20	
R	1,7	16	10a,12b	3,5,6,12a	2,8
S_1	1,7,8	12b,16		2	5,11,12a
S_2	1,7,8	2,5,12a,12b, 10a,16	10b		6,14
T	1,7	2,8,16	10a		11

TABLE VIIa

THE ORDER IN WHICH EXAMPLES WERE

PERFORMED, TREATMENT A

(continued)

Student	Survey VI	Survey VII	Survey VIII Retention
A	5,6	8,14,24,11	
C_1	15	12b,19	2,11
C_2	12a	5,6,10b,15,19	
D_1	5,11,12a,12b,15,19	8	6
D_2	6,15,19		5
D_3	14,15,19	24	
H	15,19	23,24	
J_1	15,19	9,10b	4
J_2	11	6,10a,15	
K	12a,12b,15,14	4,19,20,24	21
L_1	15,24	3,11,13,19	9
L_2	15,19,24		10b
M_1	12a,12b,15	2,3,19,11	
M_2	12a,14,15,19,20,24	10a,13,23	
M_3	12a,15	5,6,10b,14,19,24	
M_4	10a,12a,12b	15,19	
N_1		10b	3,14,20,23
N_2		11,13	
R	4,14	11,15,19	10b,20,23,24
S_1	15,19	10a	
S_2	4,15,24	3,9,11,17,19,20,21,23	
T	12a,12b,15,19,24	5,6,10b	

TABLE VIII

THE ORDER IN WHICH EXAMPLES WERE

PERFORMED, TREATMENT B

Student	Survey I Pretest	Survey II	Survey III	Survey IV
A	1,7			8,12a,12b,15
B_1	11	1,12a,12b	7	15
B_2	1	6,7	16	12a,12b,15
D_1	1,7,8	2,16,10a,10b	5,6,12a,19,24 12b,14,15	3,4,20
D_2	1,7	2,8,16	12a,12b,15	19
D_3	7,16	1,2,5,6	10a,10b	4,8,12a,12b,14, 15,17,19,21,23,24
D_4	1	8	7	10b,15
J_1		1,7,10a	15,16	
J_2	1	5,6,7,8,12a,12b	10b,15,19	
J_3		5,6,7,8,15,10b, 12a,12b,16	1,19	
J_4	1,3		7,12a,12b,15	
K	1,7	4,5,6,8,9,3,16	10a,10b,12a,12b, 14,19,20,21,23	2,18,15
L	1	7	8,10b,12a,12b	15,19
M	1,7,11,12a,12b, 16	2,8,10b,15,19,20	3,5,14,23,24	4,6,21
N	1	2,7,8		15,19
O	1,7		15	12a,12b,19
P_1	1,3,4	12a,12b	5,6,10b,14,15,19	2,7,16,21,24
R		7	3,4,6,8,10b	15,19
S	1,7	2,3,4,10a,14,15, 16,10b,12a	6,17	
T_1	1	2,4,8,10b		7
T_2	11	1,5,7,6,12b,16	10b,12a,15,19	3,4,14,24
M_2	1,7	2,3,4,5,6,10b	8,14,15,16,17, 23,24,19	9,12a,12b,20,21

TABLE VIIIa

THE ORDER IN WHICH EXAMPLES WERE

PERFORMED, TREATMENT B

(continued)

Student	Survey V	Survey VI	Survey VII	Survey VIII Retention
A	11,19			2,3,16
B_1	19,16			2
B_2	2,19	8,10b	5,14,24	
D_1			23	
D_2	5,9,14		11,20,24	10a
D_3		3,20	13	9
D_4	16	12a,12b,3	19	5
J_1			8,12a,12b	2,6
J_2		2,3,4,9,14, 17,20,24	11,21,23	16
J_3	2	3,4,24	14,20,23	
J_4	19	11		
K	13,17		11	
L		2,16		
M	9		10a,17	
N				5,6
O	2,10b	16	5	
P_1	20a,12a,12b	8,17		
R	10a,12a,12b,1,2,5	11		
S		5		
T_1	3,10a,12a,12b,14, 15,19,23,24			6,9,20
T_2	2	17	8,9,20,21,23	
M_2			10a	

C_1, etc.). Several of the students who were better prepared at the onset performed the same examples on the pretest (Cf. H, N_1, and N_2).

Another student who was able in multiplication made stronger gains under treatment A. S_2 began with the ability to do some basic whole number operations (1, 7, 8). After the introduction of and practice with the partition interpretation of fraction, 2, 5, 10a, 12a, 12b, and 16 were added to the list. As his understanding was strengthened with practice, 6 and 14, then 4, 15, and 24 were incorporated into his rapport. But it was not until after the addition of fractions was discussed that he was able to incorporate the firm conceptual background into the process of addition so that he performed 3, 9, 11, 17, 19, 20, 21, and 23 on the posttest which he had not performed previously. Fourteen other students in the class also performed their first addition of fractions after brief discussion prior to Survey VI with 5 more accomplishing these after classroom practice had been provided.

Of students receiving Treatment B patterns are not as easy to find. Surges are much more common as are lack of further progress. J_2 worked steadily and consistently throughout the period of experimentation. Performing only example 1 on the pretest, she showed greater strength with whole numbers on

Survey II (7, 8) and with some familiarity with fractions (5, 6) she was able to also do 12a, and 12b. By Survey III she was able to accomplish as much as many under Treatment A completed in the entire experimental period, adding 10b, 15 and 19. After 2 surveys showing no progress, she developed the partition interpretation of fractions (2) and the division interpretation (3, 4) and was able to again begin building (9,14,17, 20,24). J_2 was still growing in her understanding of fractions a week after the retention test when she joyously showed the teacher that she had figured out how to do examples which she had found in a more advanced arithmetic book and which were similar to example 22.

Other students also made steady gains and retained well on the retention test (M_1, P, T_1, T_2). Several made expecially rapid gains at the beginning but either lost interest or had reached a point that they needed actual instruction and practice in the algorithms for addition rather than in the concepts (Cf. S, K, M_2).

E had a very difficult time understanding the concept of fraction in any of its manifestations. Her development shows that each thing she did related the fraction work back to an algorithm regarding whole numbers. 1, 8, 7, her first successes were whole number operations. 3, 10b, 15, 12a, 12b, 19 were performed by relating to the whole numbers involved and to a pattern she discovered in them. It was not until Survey V that some fraction concept began to develop (16) and this was retained. 5 and 16 were both performed on the retention test.

Others, too, seemed to have trouble with the concept of fraction. B_2, N, J_3, R still had not developed them for the retention test. Some had so much trouble developing them that they retained little else. (B_1, J_1, L).

Some General Classroom Observations

The two classroom presentations place in direct confrontation two distinctly different methods: that of teaching the class as a whole and that of providing individual instruction. Of the two, teaching the class as a whole from a standard textbook is by far the less taxing on the teacher's time. A difficulty with discipline can occur when several students finish their written work ahead of the others and need to be directed to some individual project while the others finish. The faster students can be encouraged to look beyond the present work by setting a long range goal for them (as was done by giving the inventories fairly often so that they could be applying what they had learned to discovering how to solve future problems) or by providing additional problems of the challenge variety which help them to discover principles relating to what is likely to come up in the near future. On the whole, however, the faster students are bored with the in-class explanations and the classroom drill which is essential to push the slower student along with the rest. This time, for the more able student, might be better spent in some other way.

The individual instruction method demands extra time of the classroom teacher both in preparation of the materials (which are not available commercially) and in the burden of correcting many diversified types of papers each day. Without the convenience of oral drill, more work is done on paper, and the faster students turn in work at a much higher rate than they would under the classroom plan. However, after an initial organizational period, the students do keep busy during the entire time prescribed for arithmetic and often request extra time.

The individual plan attempts to provide the administrative machinery whereby the pupil is permitted to learn at his own rate, to receive help from the teacher only when he needs it, and only upon his own individual difficulties.

In this study an additional contrast was provided since extensive practice on the examples employing the concepts and principles studied was provided in the classroom group, but little practice was provided to those receiving Treatment B. If the unit itself were to be used as an instructional tool, many exercise sets should be developed to provide practice with each concept and skill which was developed in any particular unit. Developing the concept or principle does not of itself insure being able to use it again later. It is thought that using the concepts or principles after developing them will facilitate their later use and their application to the next level of difficulty.

Contrast in enthusiasm of the two groups was very noticeable. Those students in group A who were academically minded, i.e. who verbalized that they thought it was fun to learn, were mildly pleased to see the experimental teacher at the time she made a return visit. The girl who make the lowest scores, on the other hand, made a point to complain with some support from others who did not like being pushed to accomplish a great deal of work.

In group B, however, students jumped out of their seats and asked if the experimental teacher could come back and teach for a few days. Even when threatened with even harder work than ever before, they agreed that they would do it. This group had continued working as individuals using their textbooks after the experiment had been completed.

After the conclusion of the experimental period and after some time had passed, the teacher of those students receiving Treatment B reported that her students were solving their exercises in division by writing the remainders as fractions rather than simply R. __. This transfer indicates some degree of understanding of fraction concepts beyond that normally encountered in a fourth grade class.

Summary

In an effort to present the analysis of data in a comprehensive fashion this chapter first statistically compared the two treatments for resulting level of performance and statistically analyzed the order in which the concepts and principles were understood. Some individual histories which demonstrated certain patterns were described and some contrasts of the two methods of teaching were described. Some conclusions which may be drawn from this analysis are presented in Chapter V.

CHAPTER V

ANALYSIS OF DATA

Introduction

The data are analyzed both with regard to order and with regard to performance for each of the approaches: (A) use of commercial materials, and (B) use of experimental materials. In addition, individual progress and support for the hierarchy are considered.

To test the comparability of the two treatments, previous achievement (STEP) and level of performance (Pretest) are analyzed. Then, Statistical significance of the change in performance (Pretest-Posttest) is measured using the t-test.

The main criteria for the determining order were Surveys (I-VIII). The items from each of the Surveys were considered in pairs (a,b). For each class the number of students who performed a on an earlier survey than b, who performed b on an earlier survey than a, and who performed a and b simultaneously were tabulated. This tabulation was analyzed using the binomial test and where applicable an ordering $a < b$ or $b < a$ was established. The results for each class were compiled into a projected hierarchy.

At the end of this chapter observations are made, leading into the summary, conclusions, and

recommendations presented in Chapter VI.

Results of Initial Testing

Level of arithmetic achievement as measured by the Sequential Test of Educational Progress, Mathematics, was used as an independent variable to judge whether or not there were significant differences in the mathematical ability of the two groups. Approach A had a mean achievement of 249.60; approach B had a mean achievement of 249.77. The difference was less than 1 point and a t-test showed a nonsignificant t = .05 with 42 degrees of freedom. However, since the STEP was given to Group A in January and to Group B in May, the lack of significant difference indicates only that Group B did not achieve significantly better than Group A. Both of the means fall above the average school means quoted in the STEP Manual for Fall testing of the fifth grade.

On the test of 26 number facts in addition, subtraction, multiplication of single digit numbers and division by single digit numbers, Group A performed more accurately than Group B. With means of 21.10 (A) and 20.82 (B), this difference was not significant (t = .002).

The Survey of Skills in Fractions was used as a pretest and a posttest as the critical measure of the effect of a particular treatment on the two groups. The next section analyzes the two administrations of this test.

The Survey Tests

The main criteria for measuring performance was the Survey Test, administered as a pretest, a posttest, and a retention test. Table IX, page 62 reports the analysis of these tests.

Using the t-test with 42 degrees of freedom, no significant differences were found except on the pretest. The pretest indicates that the students receiving Treatment A had a lead on the students receiving Treatment B at the beginning of the study although this lead was completely eliminated by the time the retention test was administered. Pretest-Retention test results showed the mean of individual net gain to be 9.77 in Group B and 7.35 in Group A although 2 persons in Group A did not take the retention test. The data is not sufficient for conclusion that the net gain of students receiving Treatment B was significantly greater than that of Group A although data indicates that greater gains were made.

On all 3 tests, Pretest, Posttest, and Retention Test the variance of scores under Treatment B were 1/5-3/4 again as great as the variance of scores under Treatment A.

TABLE IX

ANALYSIS OF THE SURVEY OF SKILLS WITH FRACTIONS, PRETEST, POSTTEST, AND RETENTION TEST BY APPROACH TREATMENT

Analysis by t-test

Group		N	Mean	SD	t	p*
Pretest	A	22	3.55	2.4	3.03	.005
	B	22	1.64	1.3		
Posttest	A	22	12.86	4.2	.04	NS
	B	22	12.14	6.6		
Retention Test	A	22	10.55	4.9	.05	NS
	B	22	11.41	5.9		
Pre-Post Gain	A	22	9.32	3.5	.71	NS
	B	22	10.50	6.1		
Post-Retention	A	20	-1.85	3.4	1.06	NS
	B	22	-.73	3.4		
Total Problems Mastered by Each Individual	A	22	15.41	4.6	.00	NS
	B	22	15.41	6.6		

The Order of Performance

The hypothesis, H_1, to be tested is that the order predicted by the task analysis diagram is indeed the order in which each student performed each task; i.e. if a precedes b on the diagram, that the student would perform example a prior to performing example b. The null hypothesis, H_0 would indicate that there was no difference between the probability (p_1) of performing example a first and b second and the probability (p_2) of performing b first and a second. H_1 implies that $p_1 < p_2$.

The binomial test was chosen because the data was in two discrete categories and the design for each class was of the one-sample type. Since under the null hypothesis there was no reason to think that a should be learned prior to b, $P = Q = \frac{1}{2}$.

The significance level chosen was $\sigma = .001$. N, the number of cases, is the number of persons performing example a prior to example b, plus the number of persons performing example b prior to example a, plus the number of persons performing a and b both satisfactorily for the first time on the same test. The sampling distribution given by $\sum_{=0}^{x} \binom{N}{i} P^{i} Q^{N-i}$ was obtained from Siegel, Nonparametric Statistics for the Behavioral Sciences, Table D, p. 250. For probabilities that a would occur by chance in the tabulated relationship to b of less than .001, the conclusion is that the learning of a is prerequisite to the learning of b. The region of rejection of H_0 consisted of values of x (where x = the number of subjects who performed example b prior to example a) which are so small that the probability associated with their occurence under H_0 is equal to or less than = .001. Since the direction of the difference was predicted in advance, the region of rejection is one-tailed. Table X to XIII, pp. 65-68 give the tabulations of the number of students performing a and b on the same survey test and items and the tabulations of the number of students performing a on an earlier survey than b. These tabulations are given separately for students under Treatment A and under Treatment B. Tables XIV and XV, pp. 69-70, give the probabilities that the items would be performed as tabulated under the null hypothesis, H_0.

The Order of Performance Under Treatment B

Table XV, p. 70 gives the probabilities that a would occur before b in the quantities tabulated by Table XII, p. 67. In the event that the probability is less than or equal to .001 the conclusion is that a is prerequisite to b.

Since examples 1 and 7 are in general prerequisite to all other examples, and since neither

TABLE X

NUMBER OF STUDENTS PERFORMING a ON AN EARLIER SURVEY TEST THAN b, TREATMENT A

b	1	2	3	4	5	6	7	8	9	10a	10b	11	12a	12b	13	14	15	16	17	18	19	20	21	22	23	24
Total	22	22	12	8	20	21	22	19	7	20	16	21	20	20	4	10	21	21	2	0	20	6	3	0	6	12
a																										
1		22	21	22	20	20	2	12	21	21	18	21	17	16	22	22	20	15	22	22	21	22	21	22	21	22
2			19	21	12	13		6	20	13	15	15	12	9	22	22	18	1	22	22	18	22	21	22	21	22
3		2		8	2	2		2	7	3	4	5	4	3	9	8	7	1	10	12	7	9	10	12	9	9
4			2						3		2	3			7	2	2		7	8	3	5	7	8	6	4
5		6	17	20		5	1	6	18	8	9	12	7	7	20	19	13	1	19	20	14	20	19	20	19	16
6		6	18	21	4			6	19	8	9	10	8	8	21	19	12		21	21	13	21	20	21	20	18
7	2	22	21	22	20			13	20	20	18	20	16	16	22	22	20	14	22	22	20	22	21	22	21	22
8	1	12	17	19	14	15	2		18	14	13	15	13	12	19	18	16	8	19	19	17	19	17	19	18	18
9		2	3	5	1	1				1	2	2	1		6	5	2		5	7	2	6	5	7	5	4
10a		7	16	19	10	12	1	4	18		13	12	8	5	19	19	15	1	19	20	17	19	19	20	18	18
10b	1	5	12	14	5	6		3	13	6		7	7	7	16	14	9	3	16	16	9	15	15	16	14	12
11		3	12	18	7	11	1	5	18	9	13		8	5	19	18	13		19	21	13	19	19	21	19	18
12a		7	14	20	6	10	1	7	19	9	12	10		1	20	18	12	3	20	20	16	19	19	20	19	18
12b		9	16	20	10	11	1	8	19	11	13	14	5		20	19	13	3	20	20	16	20	19	20	19	19
13			0	1					1							1			3	4		2	3	4	1	
14			2	6					6	1	2	4		1	10		4		9	10	5	7	9	10	8	5
15		4	14	18	6	6	1	5	19	5	10	6	2	3	21	15		1	21	21	8	20	20	21	20	16
16	1	15	20	21	17	18	2	10	20	18	18	19	15	13	21	21	20		21	21	19	21	20	21	20	21
17			1	1					0			1			2	1				2		1		2		
18																										
19		2	9	16	4	5	1	5	17	5	8	2	1	1	19	12			19	20		17	18	20	18	13
20			0	1					3			1			6	0			4	6			4	6	2	
21		1	1	1	1			1	1	1		1			3	1			2	3		3		3	1	1
22			0	0	0			0		0		0														
23		1	1	1	1			1	2	1		1			4	2			5	6		2	4	6		1
24			6	8	2	2		1	9	1	3	4		1	12	3	1		11	12	3	8	11	12	8	

TABLE XI

NUMBER OF STUDENTS PERFORMING a ON THE

SAME SURVEY AS b, TREATMENT A

a \ b	1	2	3	4	5	6	7	8	9	10a	10b	11	12a	12b	13	14	15	16	17	19	20	21	23
2																							
3	1	1																					
4		1	2																				
5	2	4	1																				
6	2	3	1		12																		
7	18		1		1	3																	
8	9	4			1		7																
9	1		2	1	1	1	2	1															
10a	1	1	1	1	4	2		3	1														
10b	3	1			6	6	4	4	1	2													
11	1	4	4		3	1	1	2	1	1													
12a	5	3	2		6	4	5	1		4	1	4											
12b	7	3	1		5	3	5	2	1	5	1	2	15										
13			1	1						1		2											
14			3	3	1	2		1	1		1	0	2	1									
15	2			1	3	4	1	1		2	2	2	8	6			3						
16	6	6			4	4	5	4	1	2	1	3	4	6				1					
17			1		1				2	1		1											
18																							
19	1	1	3	1	3	4	1		1		3	5	4	4		1	3	13	2	1			
20			3	2					1		1	1	1				3	1		1	3		
21	1		1			1	1		1		1	1	1	1				1	1	1	2	1	
22																							
23	1		2	1		1	1		1	1	2	1	1	1		1	1	1	1	1	2	3	
24				2	2	1		1	1	1	2		2	1			3	5		1	5	3	2

TABLE XII

NUMBER OF STUDENTS PERFORMING a ON AN EARLIER SURVEY TEST THAN b, TREATMENT B

a \ b	TP*	1	2	3	4	5	6	7	8	9	10a	10b	11	12a	12b	13	14	15	16	17	18	19	20	21	22	23	24
1	22		19	19	20	18	18	9	18	22	20	20	19	18	17	22	22	20	18	22	22	20	22	22	22	22	22
2	20			12	12	12	10	1	8	18	15	7	16	6	6	20	15	6	5	19	19	7	16	17	20	18	15
3	15	1	6		4	7	7	2	5	13	11	4	12	5	5	15	8	5	5	13	15	6	5	13	15	11	7
4	12	1	4	2		4	3	2	4	10	9	2	10	5	5	12	6	5	4	10	12	5	8	9	12	9	5
5	16	1	5	9	11		4	1	6	11	13	6	14	5	4	16	10	5	5	15	16	6	14	15	16	14	12
6	15	2	6	8	9	5		1	6	13	10	7	13	7	6	15	10	7	7	14	15	9	12	13	15	12	11
7	22	3	18	20	20	18	17		16	22	21	19	19	16	15	22	21	18	17	22	22	21	22	21	22	22	21
8	18	2	8	13	11	11	9	2		16	16	8	16	8	8	18	14	9	8	14	18	12	15	15	18	14	14
9	9		1	1	2		1				6	2	6	1	1	8	1	1		4	8	1	2	4	9	3	3
10a	11		3	4	4	4	5		3	6		2	8	3	3	11	4	4	3	8	10	3	5	6	11	8	2
10b	16	2	6	11	10	9	7	2	6	15	12		14	6	6	16	13	7	7	16	16	8	13	14	16	14	14
11	10	2	5	5	6	5	6	2	4	6	7	6		2	2	9	6	3	5	7	9	3	6	7	10	8	6
12a	21	1	13	13	14	13	12	2	8	19	14	10	18			21	15	6	9	19	21	12	18	17	21	16	15
12b	21	1	13	13	14	13	12	2	8	18	14	11	18	1		21	15	7	9	19	21	13	18	17	21	16	14
13	2									1							0			0	1		0	0	2	0	1
14	13		3	4	5	2	4	1	3	10	8	1	11	1	1	13		1	2	10	13		8	10	13	6	12
15	22	2	12	15	15	14	12	1	8	21	16	8	19	6	6	22	15		10	20	21	9	19	19	22	17	16
16	18	2	9	12	13	12	11		9	16	16	9	13	8	8	18	14	8		16	18	10	16	15	18	16	14
17	8			1	1	1			2	5	4		6	1	1	7	0				7		3	3	8	2	0
18	2												1			2	0			1			0	0	2	0	0
19	21	1	11	13	14	13	11	1	7	20	15	7	17	3	3	21	14	2	9	19	21		18	18	21	16	15
20	12		1	2	2	3	3		2	6	7	1	9			12	1	1	1	8	12			6	12	4	1
21	8		1	1	0	1	1		2	3	5		5			8	0	1	0	4	8		2		8	2	0
22	0		0			0	0		0	0			0			0	0	0	0	0	0		0	0		0	0
23	11		1	1	1	2	3		2	7	6		8	1	1	11	0	1	1	7	11		3	5	11		0
24	13		2	4	5	2	4		3	10	8	1	10	1	1	13	1	1	1	10	13		8	9	13	7	

*TP indicates the total

TABLE XIII

NUMBER OF STUDENTS PERFORMING a ON THE SAME SURVEY AS b, TREATMENT B

a	b	1	2	3	4	5	6	7	8	9	10a	10b	11	12a	12b	13	14	15	16	17	19	20	21	23
2		2																						
3		2	4																					
4		1	4	9																				
5		3	3	3	2																			
6		2	3	3	4	9																		
7		10	2			3	4																	
8		1	4	2	4	3	4	4																
9			1	2	2	2	2		2															
10a		2	3	2	1	1		1		1														
10b			6	3	4	3	4	1	6		4													
11		1						1		1	1													
12a		3	2	3	2	4	3	4	6	1	4	5	1											
12b		4	2	3	2	5	4	5	6	1	4	4	1	20										
13																								
14			2	5	4	5	2		2	2	3	3		5	5									
15			3	2	2	3	3	2	5		2	7		10	9		6							
16		2	6	3	2	3	3	5	4	1	2	3	1	3	4		2	4						
17			1	1	2		1		3	1	1			1	1	1	2	2	1					
18			1							1			1					1						
19		1	3	2	2	2	2		3		3	6	1	7	6		7	11	3	2				
20			3	4	3		2		2	4	2	3	1	3	3		4	1	1	1	4			
21			2	1	3		1	1	2	2	2	2	1	4	4		2	2	2	1	3	4		
23			1	3	2	1			3	1	3	2	1	4	4		6	4	2	2	5	5	5	
24			3	6	5	3	1	1	2	1	3	2	1	5	5		9	5	3	3	6	4	4	6

TABLE XV

PROBABILITY THAT a WOULD OCCUR BEFORE b BY CHANCE IN THE TABULATED QUANTITIES OF TABLE XII

a \ b	1	2	3	4	5	6	7	8	9	10a	10b	11	12a	12b	13	14	15	16	17	18	19	20	21	22	23	24
1		01¯	01¯	01¯	02	02		01	00	01¯	01¯	01	02	08	00	00	01¯	02	00	00	01¯	00	00	00	00	00
2									01¯	39		13			00	21			01¯	01¯		06	01	00	01¯	21
3									11			72			00				04	00			04	00	59	
4															00				46	00			73	00	73	
5									11	48		32			00				01	00		06	01	00	06	72
6									11			84			00				01	00		72	04	00	18	
7		01	01¯	01¯	02	72		26	00	01¯	01¯	01¯	26	67	00	01¯	01	08	00	00	01¯	00	01¯	00	00	01¯
8									01	02		06			00	32			32	00		10	10	00	32	32
9															20					02				02		
10a															00					00				00		
10b									01			58			00				00	00		25	02	00	02	06
11															02				08	33				01		
12a				95					01¯	95		01			00	39			01¯	00		01	04	00	13	39
12b	01¯			95					01¯	95		01			00	39			01¯	00		01	04	00	39	58
13																										
14									46						00				19	00			19	00		
15			67	67					01¯	26		01¯			00	67			01¯	00		01¯	01¯	00	08	26
16				84					01¯	13		84			00	15			01¯	00		01	01¯	00	02	15
17															35					35				04		
18															35											
19				95					01¯	39		04			00	95			01¯	00		01	01	00	13	39
20															00					00				00		
21															04					04				04		
22																										
23															01					00				00		
24															01				46	00				00		

.0 precedes each entry
Blanks indicate nonsignificant entries

TABLE XIV

PROBABILITIES THAT a WILL OCCUR BEFORE b BY CHANCE IN THE QUANTITIES TABULATED BY TABLE X

a \ b	1	2	3	4	5	6	7	8	9	10a	10b	11	12a	12b	13	14	15	16	17	18	19	20	21	22	23	24
1		00	01-	00	01-	01-			01-	01-	02	01-	08	26	00	00	00	67	00	01-	01-	00	01-	00	01-	00
2			01-	01-					01-		39				00	00	02		00	00	01	00	01-	00	01-	00
3															11				19	00		73	19	00	73	
4															90				35	00			35	04		
5			01	00					01-						00	01-			01-	00	95	00	01-	00	01-	06
6			01	00					01-						00	01-			00	00		00	01-	00	01-	01
7		00	01-	00	01-	01-			01-	01-	02	01-	26	26	00	00	01-	95	00	00	01-	00	01-	00	01-	00
8			01-	00	95	39			01-	95		67			00	01-	26		00	00	08	00	01-	00	01-	01-
9															62					08				08		
10a			06	01-					01-						01-	01-	67		01-	00	08	00	01-	00	01-	01-
10b			38	02					11						00	06			00	00		01-	01-	00	02	72
11				01					01						01-	02			01-	00		01-	01-	00	01-	02
12a			58	00					01-						00	01-			00	00	13	01-	01-	00	01-	01-
12b			06	00					01-			95			00	01-			00	00	13	00	01-	00	01-	01-
13																										
14															06				11	01			11	01		
15			95	01					01-						00	67			00	00		01-	01-	00	01-	26
16		67	01-	00	08	02			01-	01	02	01-	67		00	00	01-		00	00	01-	00	01-	00	01-	00
17																										
18																										
19				06					01						01-				01-	00		00	01-	00	01	
20																								16		
21																										
22																										
23																								16		
24															01-				03	72		04	03	01-		

.0 precedes each entry
Blanks indicate nonsignificant entries

example involves fractions per se, it is indicated that these two examples be included among the basic arithmetic of whole numbers studied before fractions. Examples 13, 18, and 22 are in general those requiring all others for mastery. Hence, it is indicated that these three types be the last types studied. The other prerequisites under Treatment B are

$2 < 9, 17, 21, 23$

$5 < 17, 21$

$6 < 17$

$8 < 9$

$10b < 9, 17$

$12a < 11, 17, 20$

$12b < 9, 11, 17, 20$

$15 < 9, 11, 17, 20, 21$

$16 < 9, 17, 20, 21$

$19 < 9, 17, 20, 21$

The resulting suggested hierarchy appears in Table XVII, p.73 . The Table has been adjusted to reflect the number of students who performed each example correctly at any time during the experiment as given in Table XVI, p.72 .

Some relations of the hierarchy that are not supported by the data collected from the group receiving Treatment B are those involving 2, 5, 6, 16, 15, and 19. One would anticipate that unit fractions need to be understood before those with

TABLE XVI

PERCENT OF STUDENTS PERFORMING EACH EXAMPLE

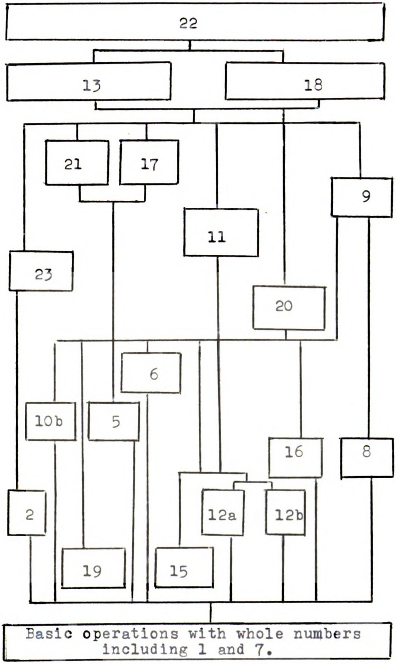
BY PERCENT AND NUMBER

Percent and Number	Examples Performed by the Percent Treatment A	Examples Performed by the Percent Treatment B
100%-22	1,2,7	1,7,15
95%-21	6,11,15,16	12a,12b,19
91%-20	5,10a,12a,12b,19	2
86%-19	8	
82%-18		8,16
77%-17	10b	
73%-16		5,10b
68%-15		3,6
64%-14		
59%-13	24	14,24
55%-12	3	4,20
50%-11		10a,23
45%-10	14	11
41%-9		9
36%-8	4	17,21
32%-7	9	
27%-6	20,23	
23%-5		
18%-4	13	
14%-3	21	
9%-2	17	13,18
5%-1		
0%-0	18,22	22

TABLE XVII

PROJECTED HIERARCHY UNDER

TREATMENT B, BY EXAMPLE



*3,4,10a,14,24 do not occur in this diagram. It is understood that they require 1 & 7 and are prerequisite to 13,18, and 22. But no other relationships were well defined.

non-unit numerators and also before one learns to add fractions. However, there was no strong indication that this was true. Students may have developed a pattern for addition which depended on the concrete aids without understanding fully how the concrete aids were developed.

The number of simultaneous solutions of 15 and 19 indicate that the commutative and associative laws are used by the students to the point that they do not interfere with the performance on example 19 once example 15 is understood. The early solutions of 12a and 12b indicate an easy association of fractions with fractions and whole numbers with whole numbers. There appears to be much more difficulty in relating whole numbers to fractions as is indicated by the lower number of students performing examples 14, 20, and 23 compared to the number performing 19.

Examples 4 and 14 and 15 are closely interrelated. The tendency was for 15 to be solved before either 4 or 14 but this might be accounted for because of the number of students who did not receive instruction in the definition of a fraction as an indicated division whereas all students had experience with material similar to example 15 either through the use of fractions on the number line or fractions as partitions.

Example 11 was included in the diagram to provide one route to the understanding of equivalent fractions. There is no indication that being able to perform 11 in any way aided further performance of examples.

One would not anticipate that 2, 10b, 12b, 15, 16, and 19 would be prerequisite to 9 since the concepts involved in 9 relate only to whole numbers. Perhaps, since the processes of multiplication were not fully developed by the students receiving Treatment B, 9 proved so much more difficult than the others as to influence the authenticity of the statistical test. Similarly, the difficulty with which students relate whole numbers to fractions as was apparent in the relationships of 19 to 14, 20, and 21, may have caused the unexpected result of showing some items prerequisite to 17 and 20.

The types of errors made by students under Treatment B seem to progress in a set pattern. At first students seem to add all four numbers together and arrive at a single whole number answer. Next the numerators were added together and the denominators were added together giving a fraction for an answer (similar to the addition of ratios). Following this the students became able to correctly perform those having common denominators but merely substituted a convenient number in the denominator of the other sums such number being the sum of the denominators or the product of the denominators or one of the denominators. The fourth stage was one in which a common denominator was determined but it was not understood how to deal with the numerators. Correct addition of easy combinations which could be analyzed though diagrams occured next with successful performance coming last.

Order of Performance Under Treatment A

Table **XIV** gives the probabilities that a will occur before b according to the tabulations in Tables X and XI.

It is apparent that examples 1 and 7 were performed satisfactorily prior to the satisfactory performance of most other examples. Since neither of these involve knowledge of fractions they could be considered part of the basic arithmetic prerequisite to understanding fractions.

Of the other examples, 16 shows the highest number of examples to which it can be considered prerequisite. On Survey II all except one of the students receiving Treatment A performed satisfactorily on this item. This predominance of success may result from the fact that the first instruction received introduced the concept necessary for the performance of this item and provided practice with similar examples. Only 2 students failed to retain the concept well enough to perform item 16 on the posttest or retention test.

The next most required item was number 8 which may indicate that the students receiving Treatment A had developed their concept of multiplication to a higher degree than that of the students receiving Treatment B. This example required no understanding of fractions.

At the other end of the scale, items 13, 17, 18, and 22 require the greater number of prerequisite understandings with 20, 21, 23, and 24 followed by 4 and 9 requiring the next most. Since 18 and 22 were performed by no student receiving Treatment A and 13, 21, and 17 by only 4, 3, and 2 respectively, we chart these at the top of the hierarchy. Other prerequisites are listed here:

2	3,4,9,14,19,20,23,24
5	3,4,9,14,20,23
6	3,4,9,14,20,23,24
8	3,4,9,20,23,24
10a	4,9,14,20,23,24
10b	20,23,24
11	4,9,20,23
12a	4,9,14,20,23,24
12b	4,9,14,20,23,24
15	4,9,20,23
16	3,4,9,10a,11,14,15,19,20,23,24
19	9,20,23

Table XVIII, page 79 gives a projected hierarchy for these relationships. One striking feature of the projected hierarchy is the fact that those which were performed by the most students were also performed on earlier tests and were prerequisite to most of the others. Of items performed by all of the group 1 & 7 are prerequisite to at least 19 items, 2 to 14 items. Of items performed by all but 1 student in the group, 16 was prerequisite to 16 items, 6 to 12 items, and 11 and 15 each to 9 items. Of items performed by less than 17 persons, none is prerequisite to more than 2 others at the .001 significance.

Another phenomenon is the performance of an item for the first time on approximately the same test for all students. Examples of this are 16 and 2 (on Survey II), 10a (on Survey III), 1 and 7 (on Survey I), 19 and 24 (on Surveys VI and VII).

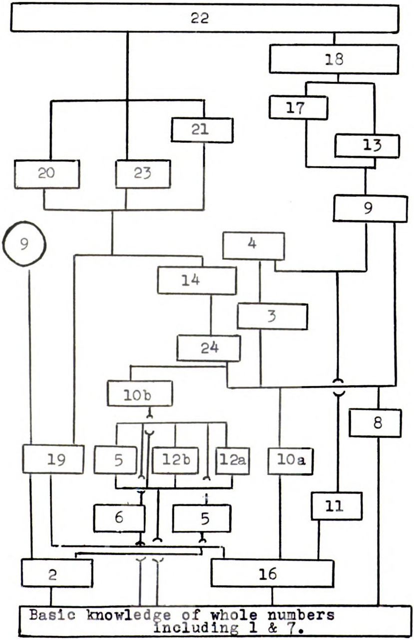
The processes of multiplication may not have been developed by fourth graders sufficiently to enable 9 to have been performed in a natural order. However, in the teaching of material similar to example 10a, students showed relative ease in assimilating this concept which also relied on multiplication. No teaching was directed toward examples similar to 9.

Another unexpected order was the position which 3 and 4 take in the hierarchy. Instead of the

TABLE XVIII

PROJECTED HIERARCHY UNDER TREATMENT A,

BY EXAMPLE



concept of fraction as indicated division being a building block, it is an application of other concepts. Perhaps this was the result of no formal instruction in the use of the bar to indicate division. Those who performed 3 and 4 may have done so through considerations of equivalent fractions.

Seemingly unrelated on the hierarchy are 10a and 10b. 10a asks for an equivalent name for a rational number less than one and 10b asks for an equivalent fraction name for one. Only 2 persons performed this for the first time on the same test; others were randomly split, some performing 10a first and some performing 10b first.

Exercise 2 required the student to divide a figure into equal parts and to name a unit fraction for the shaded part. Exercise 16 was already marked in equal parts, the student was required to write a fraction for the shaded part which involved the use of a non-unit numerator. No relation between the two appeared in the data. 16 was performed on the average on Survey II, 2 on the average on Survey IV.

CHAPTER VI

SUMMARY AND CONCLUSIONS

Summary of the Investigation

This study investigated the order in which children learned some concepts and principles which enabled them to perform examples of addition of common fractions. To delineate the concepts and principles necessary to the addition of common fractions a hierarchy was developed which had as its base the understanding of operations with whole numbers and as its apex the performance of the example 3 3/10 + 2 5/6. Two classes were instructed in the concepts of fractions and tested at regular intervals. One of the classes used commercial materials and was instructed as a total group with everyone working on the same material at the same time (Treatment A). The other class used a specially developed set of materials which maximized individual work and allowed some free choice of the order in which certain of the principles were studied (Treatment B). The test results were analyzed to determine invariances in the order in which students develop an understanding of the concepts and principles of the hierarchy. The data was examined for patterns of learning. The relative performances of two classes were compared. Contrasts in the two methods were reported.

Consideration of the hypotheses, the relevant findings of each, and the conclusions follow. Suggestions for further research in teaching fractions conclude the chapter and the dissertation.

Findings and Conclusions

Hypothesis 1

The order in which items from the hierarchy are mastered does not differ from one class to the other.

Findings

The main criteria for determining order were the Surveys (I-VIII). The items from the surveys were considered in pairs (a,b). For each class the number of students who performed a on an earlier survey than b, who performed b on an earlier survey a, and who performed a and b simultaneously were tabulated. This tabulation was analyzed and where applicable an order $a < b$ or $b < a$ was established. The results for each class were compiled into a projected hierarchy.

Comparisons of the two hierarchies indicates that:

(i) There were more significant pairings under Treatment A than under Treatment B.

(ii) In general all of the prerequisites under Treatment B are also prerequisites under Treatment A. Two exceptions are those items prerequisite to understanding the concept of least common multiple (9) and the principle of multiplication of fractions (11). One explanation may be that since the students receiving Treatment B had not received as much practice in whole number multiplication as those receiving Treatment A, these two items proved more difficult to them and were delayed by reason of their difficulty.

(iii) Students receiving either treatment seem to find the same items easy and the same items difficult. The items performed by the most students in each group are considered the easy items and are 1 and 7, then 2, 12a, 12b, 15, 19 and 16. The items considered difficult are those which few performed: 13, 17, 18 or 22. There were noticeable differences, however, in the number of students of each class performing 10a and 11. More of the students receiving Treatment A could name rational numbers on the number line even though direct instruction was provided during the first week of Treatment B.

(iv) Even though most all students under either treatment performed the addition examples involving fractions with the same denominators, there was a noticeable difference in the survey on which these items were first performed. Under

Treatment A, 2 performed both on the pretest, 15 did not perform either of the examples until Survey VI and 5 not until Survey VII. Only 4 (of 20) retained both principles and 2 others retained one. At the time Survey V was given, all except one receiving Treatment B had performed both of the items, that one had performed only one. Only 5 of the students failed to perform one of the items on the retention test.

Conclusion

Evidence indicates that with the exception of the two items cited above (9 and 11), those orders established under Treatment B are invariant. It also appears that prescribing the order of instructions has a direct effect upon the order of learning. More students seem to have an understanding of the partition and the rational interpretation is taught first and drill provided. More students seem to have an understanding of the addition of fractions having the same denominators if they approach addition through the rational number interpretation of fractions using concrete aids than if they approach addition through the partition interpretation.

Hypothesis 2

The order in which items from the hierarchy are mastered supports the logical order as indicated on the hierarchy.

Findings

The hierarchy indicated that n/n should be recognized as a name for 1 (item 24) before p/q can be renamed by np/nq (item 10a). The basis for this inclusion was that students could see the pattern in equivalent fractions through a multiplication by 1 = n/n. (Item 11 tests understanding of the principles of multiplication of fractions.) Students receiving Treatment A, however, learned principles for renaming fractions, p/q = np/nq prior to learning that n/n = 1. This is the only contradiction to the logical order as indicated on the hierarchy.

There are some other relationships, however, that are indicated by the data which are not indicated on the hierarchy. One of these is the paralleling development of the concept of unit fraction as compared to non-unit fractions as embodied in examples 2 and 16. Results indicated that 16 would be more likely to precede 2 rather that the other way around as would be logically expected.

Under both treatments 9 and 11 required many elements not indicated as prerequisite by the hierarchy. This may be because they are more difficult concepts or because no direct teaching of either item was provided. Items 14 (a + b)/a = 1 + (b/a) and 24 also seemed to be more difficult than anticipated and required 12a and 12b for performance under

Treatment A. Items 3 and 4, which can be performed using the division interpretation of fractions, did not prove to be as basic as indicated on the hierarchy.

Conclusions

Understanding equivalent fractions can be acquired without understanding of the multiplication of fractions or of fractional names for one. An understanding of mixed numerals may aid in the understanding of fractional names for one or in renaming an improper fraction with a mixed numeral. The division interpretation of fraction was not utilized by either group to aid in the development of the principles of addition of fractions. Direct teaching of least common multiple may be necessary for the understanding of this item.

Hypothesis 3

Students using the experimental material will make gains in performance no greater than students who used the textbook material.

Findings

Students receiving Treatment B did make greater gains in performance than students who used the textbook materials and retained them better. However, these gains were not statistically significant.

Although the mean number of problems performed by each individual did not differ under the two treatments, the standard deviation under Treatment B was $1\frac{1}{2}$ times that under Treatment A. There was greater diversity in the order in which problems were performed under Treatment B than under Treatment A. Informal observation in later mathematics lessons seemed to indicate that the students who had received Treatment B were more enthusiastic than students who had received Treatment A.

Conclusions

It is feasible to employ a method of individualized instruction to a study of fractions. The use of concrete aids manipulated by students appears to promote better understanding of the process of addition of fractions and more enthusiasm on the part of the students.

Hypothesis 4

Students who already understand some basic concepts of fractions can progress further in the hierarchy than those who do not.

Findings

For those 6 students who understood the partition interpretation of fraction at the time of the pretest (Item 16), the average pretest-posttest gain was 12.7 as compared to the average class pretest-posttest gains of 9.32 and 10,50. One phenomenon which might account for this difference would be the translation of operations with fractions into algorithms with whole numbers by those recently introduced to fraction concepts. Several students individual histories showed that they performed only examples which had easy algorithms involving whole numbers.

Conclusion

Students who have some previous understanding of the concept of fraction seem to make greater gains than those who do not.

Limitations of This Study

It is recognized that most students in this study had average or above average mathematics backgrounds and came from a suburban middle class neighborhood. By using materials related to the last two chapters of a textbook in the middle of the year, some of the whole number backgrounds necessary for the study of fractions may have been missing. In other situations different results may have been obtained. It is this writers belief, however, that in situations where the mathematics background is not as strong, students will show even greater gains with the concrete materials than with regular textbook materials and that their use need not be limited to the fourth grade. It is the individual classroom teacher, in her individual situation, with each individual child, who will determine whether the results of this study have implications for her. It is hoped that the study will provide some basic ideas from which she can draw.

Suggestions for Further Research

This study has investigated the order in which some of the concepts and principles related to the addition of common fractions are learned. Introducing the partition interpretation of fraction first followed by appropriate drill seemed to promote greater understanding than introducing fractions as represented by points on the number line. Would the results have differed significantly if the division interpretation of fraction had been introduced first?

Another of the findings was that the earlier introduction to fractions on the number line seemed to facilitate understanding and retention of addition of fractions. If the concept of fraction were introduced using the partition or division interpretation followed by the use of fractions to represent points on the number line before any other concepts or principles, would the result prove significantly different?

Do plateaus of learning exist? Knowledge of the associative and commutative properties and of the multiplicative identity seemed to precede knowledge of other concepts. Another level appeared to exist which included the partition and rational number interpretations of fraction, the addition of fractions having common denominators and the understanding of common multiples along with mixed numerals. This was approximately the average performance level. This level seemed to be accompanied by an understanding of fraction in terms of its number components and often is accompanied with $1/6 > 1/4$ or similar misconceptions. What is the nature of the transition from thinking with whole numbers to thinking with fractions? Is the transition enhanced by the introduction of the concept of fraction as early as the first grade?

The present investigation involved fourth graders. Discussions pertaining to the proper placement of fractions range from beginning in the first grade without using symbols to waiting until the latest possible time. At what states of child development should each of the concepts and principles be introduced? Does the answer to this question depend upon mathematical, cultural, or economic backgrounds? Does it depend on the child's level of ability or upon something else? Should the concept of fraction initially be introduced without symbolism?

The present study placed in direct confrontation two distinctly different methods. Will a mixture of teaching the class as a whole with short sequences of individual instruction promote better understanding with fractions than having the entire unit taught by one method or the other?

Do students in the individual situation develop attitudes, habits, ability to discover or other abilities that are not developed by those in the class dominated by the teacher?

Whereas the two methods were equally successful with the two groups, there is no guarantee that the same results would have been achieved had the methods been reversed. What past learning experiences of the students make one method more efficient for them than another? Does having participated in individualized instruction affect the rate or kind of learning that takes place under other treatments?

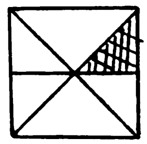
APPENDIX A

Survey I

1) Find

2+9+3+8+1+7

Show your work.

2) 

What part of the figure is shaded?

3) Find

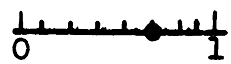
$\frac{15}{3}$

4) Find

$\frac{15}{2}$

5) Name the point indicated by a dot on this line.

6) Name the point indicated by a dot on this line.



7) What is

3,679,215 x 1?

8) Find a number which can be found by multiplying a number by 10 or by multiplying another number by 4.

9) Find the least number which is a multiple of both 4 and 10.

10)

$\frac{3}{5} = \frac{\square}{15}$

$1 = \frac{\square}{13}$

11)

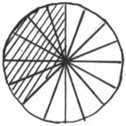
$\frac{2}{3} \times \frac{4}{5} =$

12)

$3 + \frac{3}{10} =$

$3\frac{2}{5} + 2 =$

Survey I

13) $\frac{1}{2} + \frac{1}{5} =$	14) $\frac{12}{11} =$	15) $\frac{2}{7} + \frac{3}{7} =$
16) What part of this figure is shaded? 	17) $\frac{1}{2} + \frac{1}{6} =$	18) $\frac{3}{4} + \frac{1}{10} =$
19) $3\,\frac{1}{7} + 2\,\frac{3}{7} =$	20) $\frac{3}{8} + \frac{7}{8} =$	21) $7\,\frac{1}{2} + 9\,\frac{3}{4} =$
22) $3\,\frac{3}{10} + 2\,\frac{5}{6} =$	23) $3\,\frac{3}{7} + 6\,\frac{5}{7} =$	24) $\frac{8}{8} =$

Basic Number Facts

9 + 4 =	8 x 4 =
6 + 7 =	7 x 6 =
8 + 5 =	5 x 9 =
3 + 6 =	4 x 7 =
7 + 8 =	8 x 6 =
9 + 7 =	7 x 7 =
8 + 6 =	8 x 8 =
72 ÷ 8 =	9 x 4 =
36 ÷ 6 =	8 x 7 =
24 ÷ 4 =	9 x 9 =
35 ÷ 7 =	13 - 9 =
54 ÷ 9 =	15 - 6 =
63 ÷ 7 =	14 - 5 =

APPENDIX B

Pack 1 Card 1

1. Take a strip of cardboard which has a line drawn on it. Choose a point on the line and label it "0" (zero). Choose a point 1 unit from 0 and label it "1" (one).

2. What point should you label 2? Label it.

3. What point should you label 3? Label it.

4. The line you are constructing is called a <u>number line</u>. Label two more points on this line.

Materials needed: Several strips of heavy paper, 1½" wide and approximately 9" long. Several short strips of heavy paper 1½" wide and approximately 2" long labeled "1 unit".

Pack 1 Card 2

1. Take a short strip which is labeled "1 unit". Place this piece so that it fits between 0 and 1 on your number line. Did you put 1 in the right place?

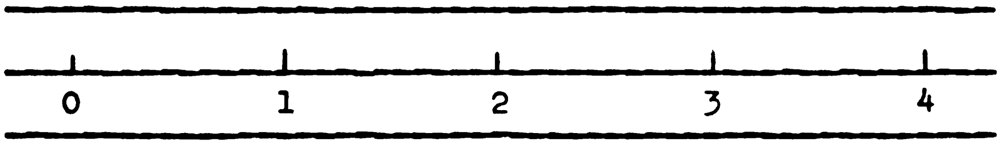
2. Place another short strip, "1 unit", between 1 and 2 on your number line. It's left edge should be at 1. Where is it's right edge? What is another name for 1+1?

3. Place a third piece to the right of the first two. Where should its right edge lie? What is another name for 1+1+1?

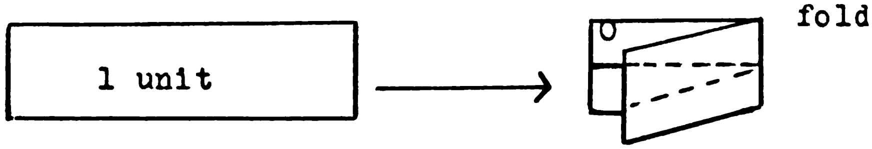
Pack 1 Card 3

Make another number line. This time place the point 0 near the left edge of your strip. Be accurate.

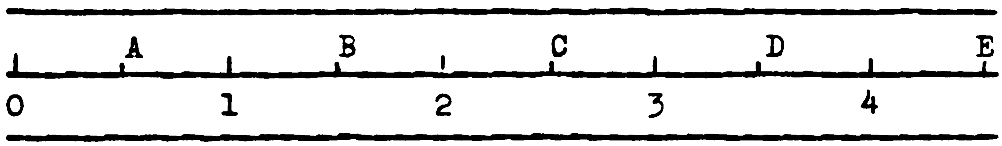
Example:



Take a one unit piece and fold it in the middle; fold it in 2 pieces each the same length. Cut the piece on this fold. Mark a point on your number line which is the distance from 0.



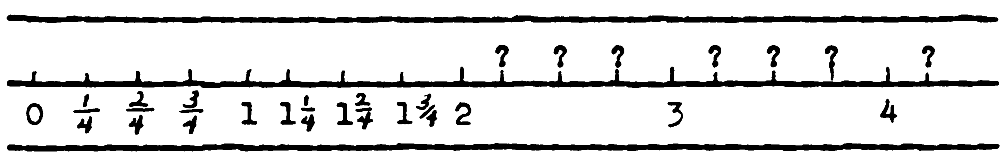
Pack 1 Card 4



1. If we label point A, "½", and point B, "1½", how should we label point C? ______

2. How should we label points D and E? ______

3. Fold a piece 1 unit long into 4 equal pieces. Cut on the folds. Mark the corresponding points on your number line. Label these points. Mark and label all of your line.



Pack 1 Card 5

Materials needed: A long strip of paper approximately 1 yard long. 2 or 3 strips of cardboard $10\frac{1}{2}$" long by $1\frac{1}{2}$" wide, one of them marked off into thirds.

1. Take a long strip of paper and a shorter strip of cardboard marked 1 unit. Construct a number line with 0, 1, 2 and 3 labeled on it.

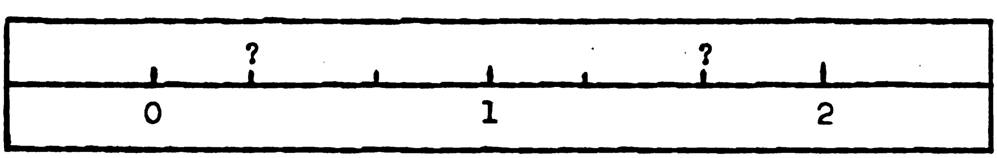
2. Fold the cardboard "1 unit" into 2 equal parts and cut it. Mark the points "$\frac{1}{2}$", "$1\frac{1}{2}$", "$2\frac{1}{2}$" on your number line.

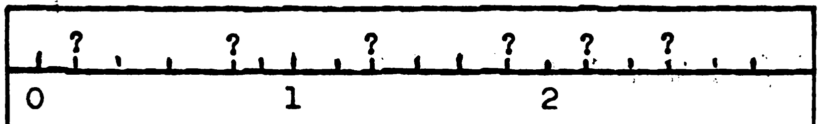
3. Fold each piece of the cardboard into 2 equal parts and cut it. Mark the points "$\frac{1}{4}$", "$\frac{2}{4}$", "$\frac{3}{4}$", "$1\frac{1}{4}$", "$1\frac{2}{4}$", "$1\frac{3}{4}$", "$2\frac{1}{4}$", "$2\frac{2}{4}$", "$2\frac{3}{4}$", on your number line.

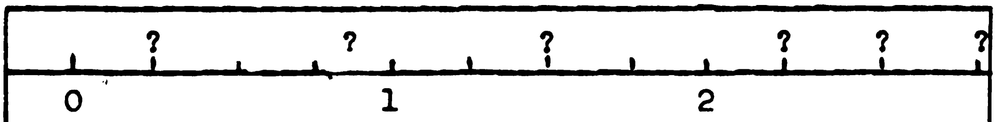
4. Take another strip of cardboard 1 unit long. Cut it into 3 equal parts. Mark and label the points "$\frac{1}{3}$", "$\frac{2}{3}$", "$1\frac{1}{3}$", "$1\frac{2}{3}$", "$2\frac{1}{3}$", "$2\frac{2}{3}$" on your number line. Save your number line.

Pack 1 Card 6

How can we label the points marked on each of these number lines? Label them. Then take this card to your teacher to be checked.







Pack 2 Card 1

1. Find a 1 unit strip which is marked off in 3 equal parts. Cut on the marks. Label each piece "$\frac{1}{3}$".

2. Find a 1 unit strip which is marked off in 2 equal parts. Cut on the marks. Label each piece "$\frac{1}{2}$".

3. Find a 1 unit strip which is marked off in 7 equal parts. Cut on the marks. Label each piece "$\frac{1}{7}$".

4. Make some "$\frac{1}{6}$"'s and some "$\frac{1}{4}$"'s. Label them.

Materials needed: Number line made on Card 5 of Pack 1. Several strips of cardboard 1 unit long marked off in 2,3,4,6 and 7 equal parts. (Strips should measure 10$\frac{1}{2}$" long by 1$\frac{1}{2}$" wide). Box to keep pieces in.

Pack 2 Card 2

1. Find a 1 unit strip which is marked off in 3 equal parts. Cut off 1 of these parts. Label it "$\frac{1}{3}$". Label the other piece "$\frac{2}{3}$".

2. Find a 1 unit strip which is marked off in 4 equal parts. Cut off 1 of these parts. Label it "$\frac{1}{4}$". Label the other piece "$\frac{3}{4}$".

3. Find a 1 unit strip which is marked off in 4 equal parts. Cut off 2 of these parts. Label each of these "$\frac{1}{4}$". Label the other piece "$\frac{2}{4}$".

4. Sort out all of the pieces you have made by looking at the number under the bar. Put the "$\frac{1}{3}$"'s in one pile, the "$\frac{1}{4}$"'s in another and so on.

Pack 2 Card 3

1. Find a 1 unit strip which is marked off in 7 equal parts. Cut 1 piece. Label it "1/7". Label the other piece "6/7".

2. Find a 1 unit strip which is marked off in 7 equal parts. Cut a piece 2 parts long. Label it "2/7". Label the other piece "5/7".

3. Find a 1 unit strip which is marked off in 7 equal parts. Cut a piece 3 parts long. Label it "3/7". Label the other piece "4/7".

4. Put all the pieces which have 7 under the bar together in one pile.

Pack 2 Card 4

1. Make pieces the correct lengths for $\frac{1}{6}$, $\frac{5}{6}$, $\frac{2}{6}$, $\frac{4}{6}$, $\frac{3}{6}$, $\frac{3}{6}$, and label them.

2. Make pieces the correct lengths $\frac{1}{14}$, $\frac{13}{14}$, $\frac{2}{14}$, $\frac{12}{14}$, $\frac{3}{14}$, $\frac{11}{14}$, $\frac{4}{14}$, $\frac{10}{14}$, $\frac{5}{14}$, $\frac{9}{14}$, $\frac{6}{14}$, $\frac{8}{14}$, $\frac{7}{14}$, $\frac{7}{14}$.

3. The numeral under the bar is called the denominator. Put all the pieces which have a 6 in the denominator together. Place all the pieces which have a 14 in the denominator together. You should have 6 piles.

Pack 3 Card 1

1. Find a piece which is labeled 1 whole. The length of this piece is chosen to be 1 unit.

2. Find a piece which is 2 units long.

3. Find a piece which is 3 units long.

4. Can you show how long a piece would be that was 5 units long?

> Materials needed: A strip $10\frac{1}{2}$" long and $1\frac{1}{2}$" wide of heavy paper labeled "1 whole". Other pieces $1\frac{1}{2}$" wide of approximate lengths labeled: $\frac{1}{2}$, $\frac{1}{2}$, 1, 2, 3, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$, $\frac{6}{6}$, $\frac{1}{7}$, $\frac{1}{7}$, $\frac{1}{7}$, $\frac{1}{7}$, $\frac{1}{7}$, $\frac{1}{7}$, $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$.

Pack 3 Card 2

1. Find two pieces of the same length which together make a piece 1 unit long. What is the name on each of these two pieces? $\square + \square = \frac{2}{2} = 1$.

2. Find three pieces the same size which together make a piece 1 unit long. What is the name on these 3 pieces? $\square + \square + \square = \frac{3}{3} = 1$.

3. Find 6 pieces the same size which together make a piece 1 unit long. What is the name on each of these pieces?

4. How would pieces be labeled if 4 of them made a length 1 unit long? $\square + \square + \square + \square = \frac{4}{4} = 1$.

5. $\frac{\triangle}{7} = 1$; $\frac{\bigcirc}{6} = 1$.

Have the teacher check your answers.

Pack 3 Card 3

1. Take two pieces each labeled "1/7" and place them end to end. Can you find one piece the same length as the two of these together? How is it labeled?

2. Take three pieces each labeled "1/7" and place them end to end. Can you find one piece equal in length to the sum of these three? How is it labeled?

3. Can you find one piece equal in length to 4 pieces labeled "1/7" ? How is it labeled?

4. How would you label a piece the same length as 6 pieces together, each labeled "1/7" ?

5. What is another name for 1/7 + 1/7 ? 1/7 + 1/7 + 1/7 + 1/7 ? 1/7 + 1/7 + 1/7 + 1/7 + 1/7 + 1/7 ?

Pack 3 Card 4

1. Take 2 pieces each labeled "1/3" and place them end to end. Can you find a piece equal in length to both of these together? How is it labeled?

2. How would you label a piece the length of 3 pieces labeled "1/3" ? Can you think of two names for this piece?

3. How would you label a piece the length of 5 pieces labeled "1/6" ? How would you label a piece the same length as 6 pieces labeled "1/6" ? Can you think of 2 names for this piece?

4. Find two names for 1/3 + 1/3 + 1/3, 1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6, 1/2 + 1/2, 1/7 + 1/7 + 1/7 + 1/7 + 1/7 + 1/7 + 1/7 .

Pack 3 Card 5

1. Find several pieces labeled "1/6". Find a piece the same length as 5 of these pieces. How is it labeled?

2. How would you label a piece equal in length to 4 pieces each labeled "¼" ? Find this piece. Can you find another piece the same length? How is it labeled?

3. How would you label a piece equal in length to three pieces labeled "1/6" ? Find this piece. Can you find another piece this same length?

4. Write two names for 1/6 + 1/6 + 1/6 ______, ______ .

5. Write two names for 1/6 + 1/6 + 1/6 + 1/6 ______, ______ .

6. Write a name for 1/6 + 1/6 + 1/6 + 1/6 + 1/6 ______, ______ .

Pack 3 Card 6

1. Find a piece labeled "1". Find two pieces that can be placed end to end to equal "1". (These two pieces do not have to be equal to one another.)

2. List all the sets you can find which equal 1 unit when placed end to end.

   Example: $\frac{1}{6} + \frac{5}{6} = 1$

Have your teacher check Cards 5 and 6.

Pack 4 Card 1

1. Find a piece labeled "2/6". Find another piece the same length as "2/6". Find two pieces which together make "2/6". Complete the following:

$$\frac{2}{6} = \frac{\square}{3} \quad ; \quad \triangle + \triangle = \frac{2}{6}$$

2. Find a piece labeled "½". Find another piece the same length as "½". Find two pieces which together make "½". Find three pieces which together make "½". Complete the following:

$$\frac{1}{2} = \frac{\square}{6} \quad ; \quad \bigcirc + \bigcirc = \frac{1}{2}$$

$$\frac{1}{2} = \frac{\triangle}{4} \quad ; \quad ▱ + ◠ = \frac{1}{2}$$

3. Can you write another equation?

Pack 4 Card 2

1. Place a "1" and a "1/6" end to end. A piece equal in length to "1" + "1/6" is "1 + 1/6" or "1 1/6". Find some pairs of pieces which make "1 1/6" when placed end to end:

1 + 1/6 = 1 1/6 ______________________

5/6 + 2/6 = 1 1/6 ______________________

2. Complete the following:

1 + 1/4 = ______ 3/4 + 2/4 = ______

1 + 1/3 = ______ 2/3 + 2/3 = ______

Check cards 1 and 2 with your teacher.

Practice Sheet 1

Place the sign > , = , or < in the space provided so that the following statements will be true

$\frac{1}{7}$ ____ $\frac{1}{6}$ $\frac{3}{7}$ ____ $\frac{4}{7}$ $\frac{6}{6}$ ____ $\frac{7}{7}$

$\frac{3}{6}$ ____ $\frac{2}{3}$ $\frac{6}{7}$ ____ $\frac{5}{7}$ $\frac{5}{7}$ ____ $\frac{5}{6}$

$\frac{1}{2}$ ____ $\frac{3}{6}$ $\frac{1}{3}$ ____ $\frac{2}{6}$ $\frac{2}{3}$ ____ $\frac{2}{7}$

* * *

Find a fraction which will name the same number as

$\frac{1}{3} + \frac{1}{3}$ $\frac{1}{6} + \frac{4}{6}$

$\frac{1}{7} + \frac{2}{7}$ $\frac{3}{6} + \frac{2}{6}$

$\frac{4}{7} + \frac{1}{7}$ $\frac{3}{7} + \frac{2}{7}$

* * *

$\frac{1}{3} + \frac{2}{3}$ $\frac{5}{6} + \frac{1}{6}$

$\frac{1}{7} + \frac{6}{7}$ $\frac{3}{7} + \frac{4}{7}$

$\frac{1}{2} + \frac{1}{2}$ $\frac{2}{6} + \frac{4}{6}$

* * *

$\frac{2}{3} + \frac{2}{3}$ $\frac{5}{6} + \frac{2}{6}$

$\frac{5}{7} + \frac{4}{7}$ $\frac{6}{7} + \frac{6}{7}$

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$ $\frac{1}{3} + \frac{1}{2} + \frac{1}{2}$

Practice Sheet 2

Place the sign > , = , or < in the space provided, so that the following statements will be true.

$\frac{1}{2}$ ____ $\frac{2}{4}$	$\frac{1}{3}$ ____ $\frac{2}{6}$	$\frac{1}{2}$ ____ $\frac{6}{14}$
$\frac{1}{2}$ ____ $\frac{3}{6}$	$\frac{1}{3}$ ____ $\frac{7}{21}$	$\frac{2}{3}$ ____ $\frac{3}{6}$
$\frac{1}{2}$ ____ $\frac{7}{14}$	$\frac{1}{2}$ ____ $\frac{2}{6}$	$\frac{2}{3}$ ____ $\frac{13}{21}$

Find a fraction which names the same number as

$\frac{3}{6}$ ________	$\frac{2}{4}$ ________
$\frac{1}{2} + \frac{3}{6}$ ________	$\frac{1}{2} + \frac{2}{4}$ ________
$\frac{2}{6}$ ________	$\frac{1}{2} + \frac{1}{4}$ ________
$\frac{1}{3} + \frac{2}{6}$ ________	$\frac{1}{2} + \frac{1}{6}$ ________

* * *

$\frac{1}{3} + \frac{1}{6}$ ______	$\frac{3}{7} + \frac{1}{14}$ ______
$\frac{2}{14}$ ______	$\frac{6}{14}$ ______
$\frac{1}{7} + \frac{2}{14}$ ______	$\frac{7}{14}$ ______
$\frac{1}{7} + \frac{1}{14}$ ______	$\frac{1}{2} + \frac{3}{7}$ ______

Practice Sheet 3

Place the sign , = or in the space provided so that the following statements will be true.

$\frac{3}{3}$ ____ 1 $\qquad$ $\frac{4}{3}$ ____ $1\frac{1}{3}$ $\qquad$ $\frac{8}{7}$ ____ 1

$\frac{6}{6}$ ____ $\frac{3}{3}$ $\qquad$ $\frac{5}{3}$ ____ 1 $\qquad$ $\frac{7}{7}$ ____ 1

$\frac{7}{6}$ ____ $1\frac{1}{6}$ $\qquad$ $\frac{9}{6}$ ____ 1 $\qquad$ $\frac{3}{4}$ ____ 1

* * *

$\frac{3}{7} + \frac{2}{7} =$ ______ $\qquad$ $1 + \frac{1}{6} =$ ______

$\frac{3}{7} + \frac{5}{7} =$ ______ $\qquad$ $1 + \frac{1}{5} =$ ______

$\frac{2}{3} + \frac{1}{3} =$ ______ $\qquad$ $2 + \frac{3}{4} =$ ______

$\frac{2}{3} + \frac{2}{3} =$ ______ $\qquad$ $2\frac{1}{3} + \frac{1}{3} =$ ______

* * *

$1\frac{3}{7} + 2\frac{1}{7} =$ ______ $\qquad$ $3\frac{2}{3} + \frac{1}{3} =$ ______

$3\frac{1}{6} + 4\frac{2}{6} =$ ______ $\qquad$ $3\frac{2}{3} + \frac{1}{3} =$ ______

$11\frac{1}{3} + 7\frac{1}{3} =$ ______ $\qquad$ $5\frac{3}{4} + \frac{2}{4} =$ ______

$\frac{5}{6} + \frac{1}{3} =$ ______ $\qquad$ $\frac{3}{4} + \frac{1}{2} =$ ______

Practice Sheet 4

Find another name for

$\frac{2}{4} =$ ___ $\qquad \frac{1}{2} + \frac{1}{4} =$ ___ $\qquad 1\frac{1}{2} + 2\frac{1}{4} =$ ___

$\frac{2}{6} =$ ___ $\qquad \frac{1}{3} + \frac{1}{6} =$ ___ $\qquad 1\frac{1}{3} + \frac{1}{6} =$ ___

$\frac{7}{14} =$ ___ $\qquad \frac{1}{2} + \frac{1}{14} =$ ___ $=$ ___

$\frac{1}{7} = \frac{\quad}{14} \qquad 3\frac{1}{7} + 2\frac{1}{14} =$ ___

$\frac{3}{6} =$ ___ $\qquad \frac{1}{2} + \frac{1}{3} =$ ___

$\frac{2}{14} =$ ___ $\qquad \frac{7}{14} =$ ___ $\qquad \frac{1}{7} + \frac{1}{2} =$ ___

* * *

$1\frac{1}{2} + \frac{2}{3} = \qquad 3\frac{1}{2} + 2\frac{1}{2} =$

$1\frac{1}{3} + \frac{5}{6} = \qquad 6\frac{2}{3} + 3\frac{5}{6} =$

$2\frac{2}{3} + \frac{3}{7} = \qquad 1\frac{6}{7} + 2\frac{11}{14} =$

* * *

$\frac{1}{4} + \frac{1}{3} =$

$\frac{1}{6} + \frac{1}{4} =$

$$\begin{array}{r} 3\frac{5}{6} \\ +\ 2\frac{8}{21} \\ \hline \end{array}$$

Box A Card 1

Do the work from this card on a separate sheet of paper.

1. Multiply 10 by 1. Multiply 10 by 2. Multiply 10 by 3, by 4. Your answers are called multiples of 10.

2. List seven multiples of 10.

3. Do you notice anything special about all multiples of 10?

4. Multiply 4 by 1. Multiply 4 by 2. Multiply 4 by 3, by 4. Your answers are multiples of 4.

5. List seven multiples of 4.

6. Do you notice anything special about all multiples of 4?

7. Look at your answers to exercises 2 and 5 above. Are there any answers in 2 that are also in 5? Are there any numbers that are multiples of 10 that are also multiples of 4? List two.

8. Multiply 6 by 1. Multiply 6 by 2, by 3, by 4. Your answers are called multiples of 6.

9. List seven multiples of 6.

10. Are there any multiples of 6 that are also multiples of 4? List 3 of them.

11. List seven multiples of 3.

12. Are there any multiples of 3 that are also multiples of 6? List three of them.

13. Are there any multiples of 3 that are also multiples of 4? List three of them.

14. Find a number which is a multiple of 3, 4, and 6.

Box A Card 2

Do the work from this card on a separate sheet of paper.

1. List 6 multiples of 8.
   (8 x 1 = , 8 x 2 = , 8 x 3 = , 8 x 4 = , 8 x 5 = , 8 x 6 =).

2. List 8 multiples of 6.
   (6 x 1 = , 6 x 2 = , 6 x 3 = , 6 x 4 = , etc.)

3. Can you find two numbers that are multiples of both 6 and 8?

4. What is the least number that is a multiple of both 6 and 8? This is called the least common multiple of 6 and 8.

5. List 8 multiples of 2.

6. List 8 multiples of 3.

7. What numbers are multiples of both 2 and 3?

8. What is the least common multiple of 2 and 3?

9. List 6 multiples of 7.

10. What numbers are multiples of both 2 and 7?

11. What is the least common multiple of 2 and 7?

12. What is the least common multiple of 3 and 7?

Complete the chart showing the least common multiple of each of the pairs of numbers.

l.c.m.	2	3	7	4	6	8
2	2					
3	6	3			6	
7		21	7			56
4			28	4		
6				12	6	
8	8				24	8

Box A Card 3

1. What is the least common multiple of 2 and 3?

2. What is the least common multiple of 2 and 7?

3. What is the least common multiple of 3 and 7?

4. Do you see a pattern?

5. What is the least common multiple of 4 and 6?

6. What is the least common multiple of 6 and 8?

7. What is the least common multiple of 4 and 8?

8. Does the pattern noticed in 4 still hold?

9. Can you describe how you would find the least common multiple of three numbers? Use 4, 6, and 9 as an example.

Have the teacher check your chart from card 2 and problem 9 on this page.

Pack A_2 Card 1

1. Take a large sheet of paper. (Notebook size is O.K.) Fold it in half. Fold it in half again. Open the sheet. The folds should divide it into 4 equally sized parts. Label each part $\frac{1}{4}$. Shade 3 of the parts grey with your pencil. What fraction of the whole sheet is shaded?

2. Fold the sheet back into the fourths. Fold it once more. How many pieces is the sheet folded into now? How many of these are shaded? What fraction of the whole sheet is shaded? (Open up the sheet and check.)

3. Was the same part of the sheet shaded for both questions 1 and 2? We say 3/4 and 6/8 are equivalent fractions because they name the same part of the whole sheet.

$$3 \times \quad = 6$$

$$4 \times \quad = 8$$

4. How can we obtain 6/8 as an equivalent fraction to 3/4? When 3 parts of the 4 parts are shaded, we make each part into ? pieces.

Pack A_2 Card 2

1. Take a large sheet of paper folded into 4 parts with 3 of the 4 parts shaded. What fraction of the whole sheet is shaded?

2. With a dark pencil or ballpoint, draw lines dividing each of the four parts into 3 pieces. How many pieces do you have all together? How many of them are shaded? What fraction of the whole sheet is shaded?

3. On card 1 we found that 3/4 and 6/8 were equivalent fractions. What is another fraction equivalent to 3/4?

4. 3 x   = 9

   4 x   = 12

5. How can we find 9/12 as an equivalent fraction to 3/4?

Pack A_2 Card 3

1. On cards 1 and 2 we found that the fractions 3/4, 6/8, and 9/12 were equivalent.

$$\frac{3}{4} = \frac{3 \times 1}{4 \times 1}, \quad \frac{6}{8} = \frac{3 \times 2}{4 \times 2}, \quad \frac{9}{12} = \frac{3 \times 3}{4 \times 3}$$

$$\frac{3 \times 4}{4 \times 4} = \frac{\square}{\bigcirc}.$$

2. Is 12/16 equivalent to 3/4?

3. Name some other fractions equivalent to 3/4.
______, ______, and ______ .

4. Find 3 more fractions for this set.
{3/4,6/8,9/12,12/16,15/20,18/21,__,__,__,...}.

5. Find 3 fractions equivalent to $\frac{1}{2}$.

6. Find 3 fractions equivalent to 1/3.

7. Find 3 fractions equivalent to 2/3.

Box B Card 1

Materials: Cuisenaire rods: 6 white (W), 3 red (R), 2 lightgreen (G), 2 purple (P), 2 yellow (Y) and 1 darkgreen (D).

1. Take 5 whites (W) and place them in a row. Which one rod is the same length as these 5 whites? Complete the following: W x 5 = ____ .

2. How many whites (W) end to end is the same length as 1 lightgreen (G)? Fill in W x _ = G.

3. Fill in the blanks: R x 3 = __, R x __ = P, G x __ = D, G x 3 = G + G + G = __ + G.

Box B Card 2

1. Take 1 white (W). Complete the following: W x 1 = __ .

2. Take 1 purple (P). Complete the following: P x 1 = __ .

3. If any color could be filled into the box, what is $\square$ x 1 ?

4. Let the darkgreen (D) be 2. How long then is the lightgreen (G)? How long then is the W? How long then is the R?

5. Complete the following: 2 x 1 = __, 1 x 1 = __, 1/3 x 1 = __, 2/3 x 1 = __ .

Box B Card 3

1. If the white is 1/3, how long is the red (R)? How long is the lightgreen?

2. In problem 1 you probably said red was 1/3 + 1/3 or 2/3. G = W + W + W. So G = 1/3 + 1/3 + 1/3. Did you call lightgreen 3/3 or did you call it 1? Is 1 = 3/3 ?

3. Complete the following: $1 = \frac{\square}{4}$; $1 = \frac{\square}{5}$; $1 = \frac{\square}{6}$, $1 = \frac{\square}{10}$.

4. Fill in the blanks: $1 \times 1 = \frac{\square}{3} \times 1 = \frac{\square}{3} \times \frac{1}{1} = \frac{3}{3}$.
$1 \times 1 = \frac{2}{2} \times \frac{\square}{3} = \frac{2 \times \square}{2 \times 3} = \frac{6}{6} = 1$.

Box B Card 4

1. How many 1/2 does it take to make 1? How many 1/3 does it take to make 1?

   $1 = \frac{\square}{2}$; $1 = \frac{\bigcirc}{3}$; $1 \times 1 = \frac{\square}{2} \times \frac{\bigcirc}{3} = \frac{6}{6}$

2. Fill in the blanks:

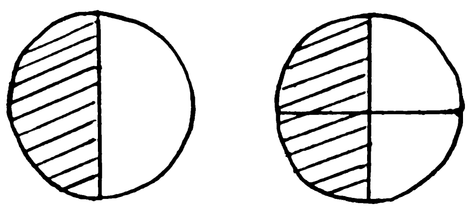
   $1 \times 1 = \frac{\square}{2} \times \frac{\Delta}{4} = \frac{\square \times \Delta}{2 \times 4} = \frac{8}{8} = \bigcirc$

   Is 1 x 1 = 1 ?

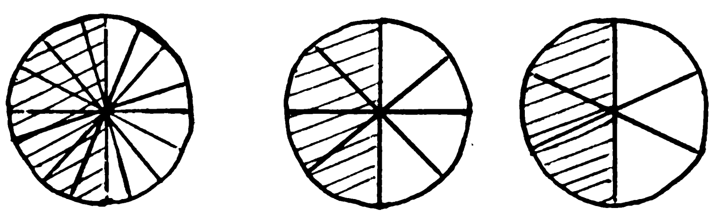
3. Find 1/2 of 1. $\frac{1}{2} \times 1 = \square$

   or $\frac{1}{2} \times 1 = \frac{1}{2} \times \frac{2}{2} = \frac{\bigcirc}{4}$. Does $\frac{1}{2} = \frac{2}{4}$?

Box B Card 5



(a) Into how many parts is each figure divided?
(b) How many of these parts are shaded?
(c) Write a fraction to represent the shaded area of each.
(d) Does 1/2 = 2/4?
(e) Find some other names for 1/2.



Box B Card 6

1. On card 5 we found that:

$$\frac{1}{2} = \frac{2}{4} = \frac{6}{12} = \frac{4}{8} = \frac{3}{6}.$$

Complete the following:

$$\frac{1}{2} = \frac{1}{2} \times 1 = \frac{1}{2} \times \frac{\square}{2} = \frac{\nabla}{4}$$

$$\frac{1}{2} = \frac{1}{2} \times \bigcirc = \frac{1}{2} \times \frac{3}{3} = \frac{\quad}{6}$$

$$\frac{1}{2} = \frac{1}{2} \times 1 = \frac{1}{2} \times \frac{\quad}{6} = \text{—}$$

$$\frac{1}{2} = \frac{1}{2} \times 1 = \frac{1}{2} \times \frac{4}{\triangle} = \frac{\quad}{8}$$

Notice: $\frac{2}{3} = \frac{2}{3} \times 1 = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$

Find: $\frac{2}{3} = \frac{2}{3} \times \bigcirc = \frac{2}{3} \times \frac{\square}{\square} = \frac{6}{9}$

Pack B_2 Card 1

Materials: A numberline having 1/2's, 1/3's, 1/4's, 1/6's, 1/7's, 1/14's, 1/21's, 1/12's.

1. Look at the number line. What are some other fractions that name the same number as 1/2? List these.

2. What are some other fractions that name the same number as 1/3? List these.

3. What are some other fractions that name the same number as 2/3? List these.

4. Can you find a pattern?

Pack B_2 Card 2

1. Try your pattern on these fractions. $\frac{1}{4} = \frac{}{8}$;

   $\frac{3}{4} = \text{—}$; $\frac{5}{6} = \frac{}{12}$

   Check your answers with the number line. Does your pattern work?

2. If your pattern did not work, see if you can find another pattern. Try your pattern on these fractions:

   $\frac{1}{7} = \frac{}{14}$; $\frac{1}{7} = \frac{}{21}$; $\frac{3}{7} = \frac{}{14}$; $\frac{4}{7} = \frac{}{21}$

   Check your answers with the number line. Does your pattern work?

Pack B_2 Card 3

1. Add: $\frac{3}{7} + \frac{2}{7}$

2. Did you find $\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$? What do the fractions

   $\frac{3}{7}$ and $\frac{2}{7}$ have in common?

3. Complete the statement: When two fractions have a common __________, you can add the numbers by ______________________________ .

   $$\frac{4}{7} + \frac{2}{7} = \frac{4+2}{7} = \frac{}{7} .$$

4. Can you add 1/2 + 1/3 by adding the numerators? Why not?

Pack B_2 Card 4

1. $\frac{1}{2} = \frac{\square}{6}$; $\frac{1}{3} = \frac{\bigcirc}{6}$; $\frac{1}{2} + \frac{1}{3} = \frac{\square}{6} + \frac{\bigcirc}{6} = \frac{\square + \bigcirc}{6} = \frac{\triangle}{6}$.

2. (1) above shows how 1/2 and 1/3 might be added. Try this same method with 1/2 + 1/7.

   $\left(\frac{1}{2} = \frac{\square}{14}\right) \quad \left(\frac{1}{7} = \frac{}{14}\right)$

3. Add $\frac{1}{3} + \frac{1}{6}$; Add $\frac{1}{4} + \frac{1}{6}$

4. Do problem 3 again using 6 and 12 as denominators.

Pack B_2 Card 5

1. Find at least 6 other names for 1/2, 1/3, 1/7.
   1/2 = ____ = ____ = ____ = ____ = ____ = ____ .
   1/3 = ____ = ____ = ____ = ____ = ____ = ____ .
   1/7 = ____ = ____ = ____ = ____ = ____ = ____ .

2. If you were going to add 1/3 and 1/2, what denominators could you choose?

3. If you were going to add 1/3 and 1/7, what denominators could you use? Add these.

4. If you were going to add 2/3 and 1/7, what denominators would you use? What is 2/3 + 1/7?

Pack B_2 Card 6

Going backwards:

1. Fill in the blanks: $\frac{1}{2} + \frac{1}{6} = \frac{}{6} + \frac{1}{6} = \frac{}{6}$. Can you find another name for 4/6 which has a smaller' denominator?

2. Find other names for each of these with smaller denominators: (Use the number line.)

   $\frac{4}{8}$, $\frac{9}{12}$, $\frac{12}{14}$, $\frac{14}{21}$

3. Check your pattern on these exercises:

   $\frac{10}{14} = \frac{}{7}$; $\frac{12}{21} = \frac{}{7}$; $\frac{8}{12} = \frac{}{3}$; $\frac{9}{6} = \frac{}{2}$.

Box C Card 1

Definition: When we write $\frac{\square}{\triangle}$ we are indicating the quotient when $\square$ is divided evenly by $\triangle$.
Example: $\frac{27}{3}$ is another name for 9, and $\frac{6}{2}$ is another name for 3.

1. Tell what whole number is named by the following:

   $\frac{15}{5}$, $\frac{56}{7}$, $\frac{63}{9}$, $\frac{9}{9}$, $\frac{48}{6}$, $\frac{8}{8}$, $\frac{7}{7}$.

2. List two or three fractions which name the same number as: 4, 7, 9, 11, 1 .

Box C Card 2

1. Divide 9 sticks between 2 people in your group. How many whole sticks may each have? How many are left over? If we wanted to divide this between the 2 people how might we do it?

2. $9 \div 2 =$ ___ R 1 or $\frac{9}{2} = 4\frac{1}{2}$. Why do you think we write $\frac{1}{2}$ to mean 1 stick divided between 2 people?

Box C Card 3

1. Divide 19 sticks between 3 people. How many whole sticks will each one receive? How many sticks must be broken to make this come out even? Into how many pieces should this be broken?

2. Fill in the blanks:

   $19 \div 3 =$ ____ R ____

   $\frac{19}{3} =$ ____ + ____ = ____ ____

Box C Card 4

1. Divide 17 sticks between 2 people. Write the problem mathematically.

2. Repeat the same process for 22 sticks divided between 2 people. Divide 17 sticks between 4 people. Divide 26 sticks between 5 people. Divide 37 sticks between 6 people. (It is difficult to break sticks into these small parts. Maybe you can write the answer without breaking them.)

Box C Card 5

1. Divide 14 sticks between 4 people. Can you do this by only breaking **two sticks?**

2. Divide 21 sticks between 6 people. Can you do this by only breaking 3 sticks? Can you do this by making only 2 breaks?

Box C Card 6

Answers to cards 1-5

Card 1: $\frac{15}{5} = 3$; $\frac{56}{7} = 8$; $\frac{63}{9} = 7$; $\frac{9}{9} = 1$

$4 = \frac{4}{1} = \frac{8}{2} = \frac{12}{3} = \frac{16}{4} = \frac{20}{5} = \frac{24}{6}$

$\frac{48}{6} = 8$, $\frac{8}{8} = 1$, $\frac{7}{7} = 1$

Card 3: $\frac{7}{2} = 3+\frac{1}{2} = 3\frac{1}{2}$; $19 \div 3 = 6$ R 1; $\frac{19}{3} = 6 + 1/3 = 6\ 1/3$

Card 4: $\frac{17}{2} = 8\frac{1}{2}$; $\frac{22}{2} = 11$; $\frac{22}{3} = 7\ \frac{1}{3}$; $\frac{17}{4} = 4\frac{1}{4}$; $\frac{26}{5} = 5\ \frac{1}{5}$;

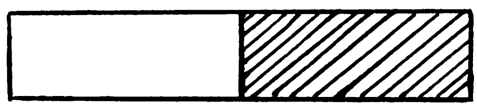
$\frac{37}{6} = 6\ \frac{1}{6}$.

Card 5: $\frac{14}{4} = 3\frac{1}{2}$ (Put 3 sticks on each pile. There are 2 left over. Break each of these in $\frac{1}{2}$.)
$21 \div 6 = 3$ R3. $\frac{21}{6} = 3 + \frac{1}{2} = 3\frac{1}{2}$.

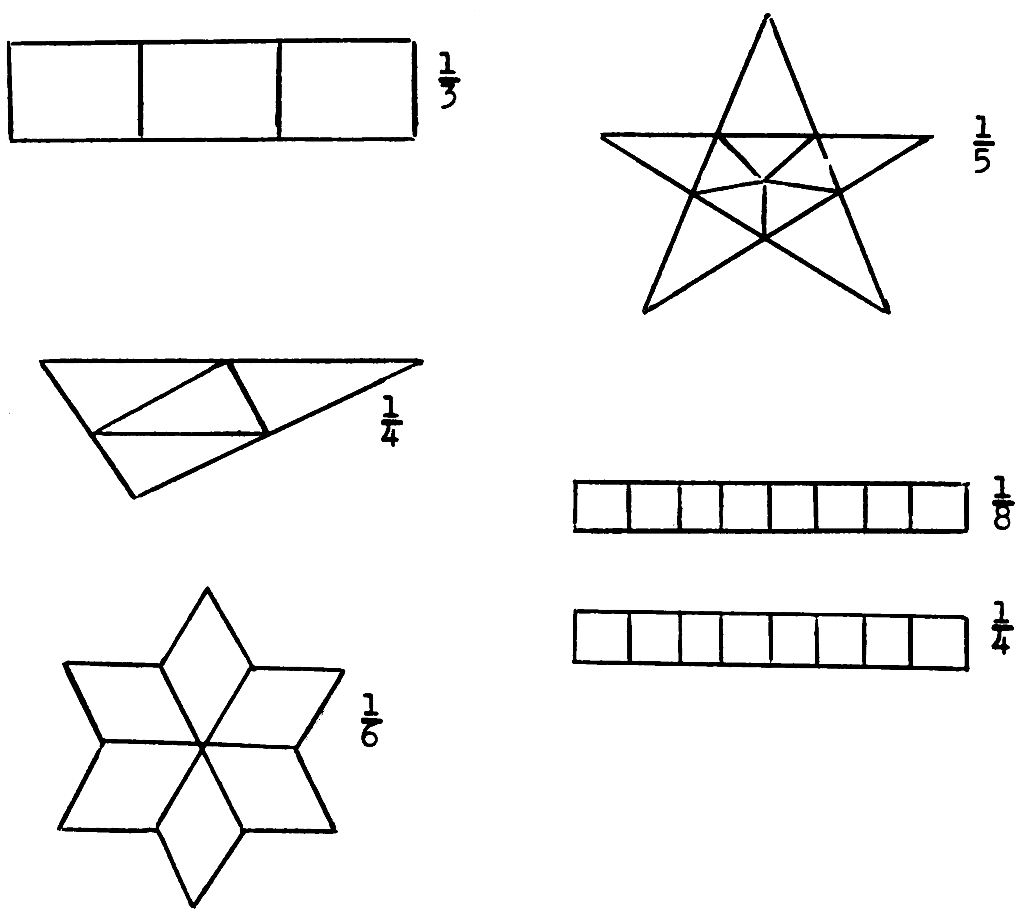
Worksheet D_1

John has a candy bar. He wants to divide it with Bill so that each will have an equal amount. How big a piece should each one have?

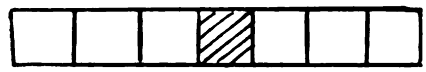
You can probably answer this question easily. (The answer is $\frac{1}{2}$, one half.) But have you thought about what the fraction $\frac{1}{2}$ means? In the example of this problem $\frac{1}{2}$ is the name of each part when one thing is divided into two equal parts. If the rectangle shown below represents one whole, what part represents $\frac{1}{2}$? You should answer, "The shaded part".

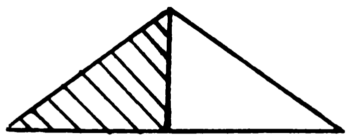


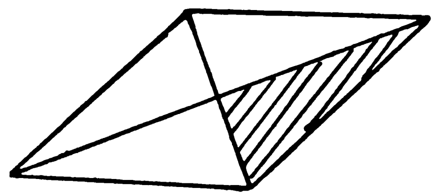
Beside each of the figures below is a fraction. Shade the part of the whole figure which represents that fractional part of the figure.



Beside each of the figures below write the fraction which represents which fractional part of the figure is shaded.



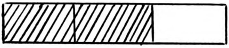




Worksheet D_2

John has a candy bar. He wants to divide it with Bill so that he will have twice as much as Bill. He decided that if he divided it into three pieces and gives Bill only only one piece, he will then have two pieces or twice as much as Bill. How big a piece of the whole candy bar does each one have?

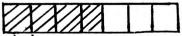
Since the candy bar was diveded into three pieces, each piece is 1/3 (one-third) of the entire candy bar. Bill has one of these pieces so Bill has 1/3 (one-third) of the bar. John has two of these pieces so John has 1/3 + 1/3 or two-thirds of the candy bar. "Two thirds" can be written "2/3".



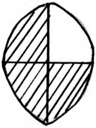
The above rectangle is separated into three parts of the same size and two of these parts are shaded. In terms of the size of the whole figure as a unit, what number tells the size of the shaded part of this figure? (Answer 2/3 or two-thirds). How are the 2 and 3 in the fraction 2/3 related to the rectangle shown? (The 3 tells the number of parts of the same size into which the figure is separated, and the 2 tells the number of parts of this size in the shaded part of the figure.)

Answer each of these three questions about the following figures:

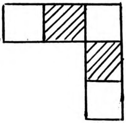
(a) Into how many parts of the same size is the figure separated?
(b) How many parts of this size are shaded?
(c) What fractional part of the whole is shaded?



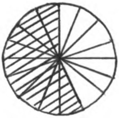
(a)
(b)
(c)



(a)
(b)
(c)



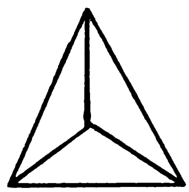
(a)
(b)
(c)



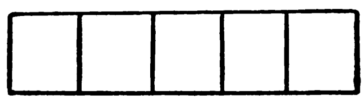
(a)
(b)
(c)

Worksheet D_2

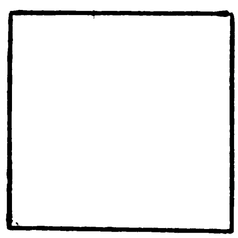
Beside each of the figures below is a fraction. Shade the part of the whole figure which represents that fractional part of the figure.



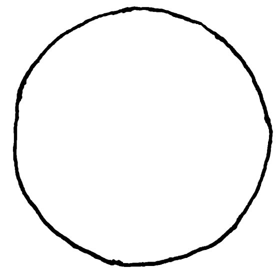
$\frac{2}{3}$



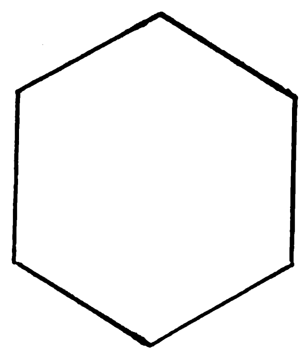
$\frac{3}{5}$



$\frac{3}{4}$



$\frac{5}{8}$



$\frac{5}{6}$

Write the fraction:

two thirds

one tenth

three fourths

seven eighths

Challenge questions:

1. Is 1/2 of two different things the same?

2. In figures of the same size and shape, which fractional part is bigger? (e.g. 1/2, 1/3, 1/4, 1/5, or 1/2, 2/3, 3/4, 5/8).

3. How does three fourths differ from three fours?

4. In how many ways can a figure be divided into fractional parts, e.g. 1/4's ?

Box E Card 1

1. In the kit you are provided (cuisenaire rods, 2 of each color) let W = white, R = red, G = light green, P = purple, Y = yellow, D = dark green, E = blue.

2. Make a train by putting end to end a lightgreen (G) for the engine and a red (R) for the caboose. We can name this train G + R (with G first).

3. Make a second train by putting end to end a red for the engine and a lightgreen for the caboose. We can name this train R + G (with R first).

4. Are these two trains the same length? We can write this by saying G + R = R + G.

Box E Card 2

1. Make a train by putting first the yellow (Y) and then the purple (P). Make a second train by putting first a purple (P) and then a yellow (Y). Is Y + P = P + Y?

2. Is G + D = D + G?

3. Can you find two rods for which the length is different when the order is changed?

4. If we say the lightgreen (G) is 1 unit long, how long is the dark green (D) rod? Is 1 + 2 = 2 + 1? How long is W + R? Is W + R = R + W?

Box E Card 3

1. The switchman is hooking up some longer trains. First he hooks the lightgreen (G) engine onto the yellow (Y) passenger car. Then he hooks on the red (R) caboose. He has the train (G+Y)+R. The "( )" show that these two were hooked together first. Make the train (G+Y)+R.

2. Make the train G+(Y+R). In this train you should hook the yellow and red together first with the red at the end. Then the lightgreen engine should be hooked on afterwards.

3. Are the two trains different because they were hooked together in a different order? Is (G+Y)+R = G+(Y+R) ?

Box E Card 4

1. Make the train (G+D)+E. Then make the train D+(E+G). Are these trains the same length?

2. If light green (G) is 1 unit long, how long is the dark green? How long is the blue?

3. Find (1+2)+3. Then find 2+(3+1). Does (1+2)+3 = 2+(3+1) ?

4. Find 4+(7+11)+2. Then find (4+11)+(7+2). Did you get 2 different answers? Why or why not?

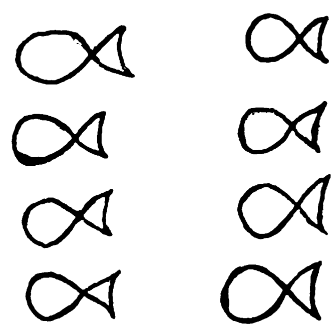
APPENDIX C

Group A
Lesson 3
February 8, 1968
Time - 55 minutes

The papers were returned at the beginning of the period. Exercises 2, 3 and 4 were discussed, p. 243 with the correct answers written on the board. The words numerator and denominator were introduced. The discussion exercises at the bottom of the page were discussed using this vocabulary. The discussion exercises on page 244 were answered in class. The top of page 246 was discussed in class.

The new assignment was given and instructions were given as to how to set up the homework paper. The assignment was page 245 (1-10) and page 246 (1-3).

The challenge problem was solved on the board by dividing the 6 circles into 3 groups and shading those circles in 2 of the groups. A new challenge problem was given.



Shade 3/4 of the fish.

Students worked at their seats with very few questions. Exercise 2 page 246 needed to have the directions explained. It was suggested that students who finished early read page 247.

For drill on multiplication tables, a math down was held, giving everyone at least 2 questions.

Group A
Lesson 9
February 19, 1968
Time - 60 minutes

Those papers that were turned in on Thursday were returned. The answers for page 257 were given by volunteers so that those who did not turn theirs in on Thursday could correct theirs.

The top of page 258 was read by various students in the class. We had previously discussed numerator and denominator so this was not new. All of page 262 and the top of page 263 was discussed by having a student read the explanation, calling on students to respond to the questions asked in the book, asking additional questions where it seemed necessary. Exercises 1 and 2 on page 265 were discussed. Each student in the room had at least two chances to recite.

The written assignment was made: page 258 (1,2), page 263 (1-6) and page 265 (3-8). Since this work could be accomplished by most in 15 minutes and 30 minutes remained in the hour, page 269, any (B) part was assigned as a challenge.

The last 5 minutes of the period were devoted to drill in multiplication by 3 and 4 via the "math-down" technique.

Group A
Lesson 14
February 26, 1968
Time - 60 minutes

Papers from Lesson 12 were returned on Set 42. Each student in the class was given a chance to work at the board, either to write and complete an exercise or to find a point named by a given fraction on the number line. Those at their seats were also involved in the process by describing what it was necessary for each to do to locate a point or to decide which rational number is greater or to decide which sign might be used.

Pages 298-299 were assigned to be written beginning with 2 (B). Exercise 1 was done orally. This exercise was somewhat confusing because the capital letters (A), (B) etc. were used for two different purposes.

The surveys were passed out and it was requested that these be completed before the assignment. Each student finished this and exercise 2 and part of exercise 3. However, very few finished the entire assignment.

Group B
Lesson 3
February 7, 1968

Boxes were distributed and students began where they had left off the day before. After answering initial questions, the teacher worked with 3 girls reviewing the construction of the number line and the concept that when 1 thing is divided into 6 parts, each part is called 1/6, 2 of them are called 2/6, etc. Also that if, e.g., 2 is a whole number point on the number line, 2 plus 1/6 is written 2 1/6.

Next 20 minutes of the period were spent answering miscellaneous questions and passing out new materials as students became ready for them.

When 40 minutes of the period were up, students were instructed to write requests for help or supplies on the inside of the top of their boxes, put everything in their boxes, and be ready for a game.

The function game was played using the sum of the number plus the next larger, the square of a number, and multiplication by 7.

Group B
Lesson 9
February 15, 1968
Time - 55 minutes

Progress of each individual was recorded on a newly posted chart by shading in those packs, etc., that had been completed by each. Separate shadings were made for marking when the teacher had checked a written exercise and for when the students had made the appropriate corrections.

A tag system was instigated allowing each student who needed help to take a number and then see the teacher in that order. This worked better than the system of having the teacher circulate, since fewer students ask questions - merely because it is easier to stop the teacher and ask her than to figure it out themselves.

Students worked on materials individually (or in twos for Packs 1-4). The materials labeled Box A, B, B2, C, D1, D2, E are available for choosing whenever the practice sheets (to be used with concrete materials) are completed. Each student makes a written response to each of these and turns it in to be graded by the teacher.

During the last 10 minutes of the period the "math-down" of drill with multiplication was

continued. About 2/3 of the students give good responses. The others just do not seem to even be able to figure the products when given sufficient time.

Group B
Lesson 14
February 22, 1968
Time - 55 minutes

In an effort to motivate children somewhat to making an effort to accomplish a little more and to also provide the recognition they each seem to need, the following was done: each child's name was called and he was asked what he was working on, how much he has done, and what he was doing this period. If he needed help he was told to take a number. If he hadn't turned in any work lately, he was encouraged to complete something today.

This same process was used at the close of the period to provide direction for the following day's work.

Seven children of the 16 who took number tags (to be helped in the order of the numbers) were helped. Seven others were helped in two groups, one on Pack A, and one on Pack D2.

BIBLIOGRAPHY

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Books and Monographs

Brooks, W. G., The Relation Between Quantity and Quality of Learning and Quantity and Quality of Stimulus in Addition of Fractions, Doctoral Dissertation, State University of Iowa, 1937.

Brownell, William A, "Psychological Considerations in the Learning and the Teaching of Arithmetic", Tenth Yearbook of the National Councel of Teachers of Mathematics, Teachers College, Columbia University, New York, 1935, pp. 1-31.

Brueckner, Leo J. et. al, Moving Ahead in Arithmetic, Book Four, Holt, Rinehart & Winston, Inc., New York, 1963, pp. 308-321.

Brumfiel, Charles F. and others, Introduction to Mathematics, Addison-Wesley Publishing Company, Inc., 1961, Reading, Massachusetts, p. 70.

Bruner, Jerome S., and Kenney, Helen J., "Representation and Mathematics Learning", Mathematical Learning, Society for Research in Child Development, Monograph Volume 30, No. 1, University of Chicago Press, 1965, pp. 50-59.

Edwards, A. U., Internal Evidence Concerning the Nature of Learning in Beginning Fractions and Mixed Numbers, Doctoral Dissertation, University of Iowa, 1932.

Eicholz, Robert, et. al, Elementary School Mathematics, Book 4, Addison-Wesley Publishing Co., Palo Alto, California, 1964.

Eicholz, Robert E. and O'Daffer, Phares G., Elementary School Mathematics, Book 5, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, 1964, p. 222.

Fehr, Howard F., and Hill, Thomas J., Contemporary Mathematics for Elementary Teachers, D. C. Heath and Company, Boston, 1966.

Fouch, Robert S. and Nichols, Eugene D., "Language and Symbolism in Mathematics", The Growth of Mathematical Ideas, Grades K-12, 24th Yearbook, National Council of Teachers of Mathematics, Washington, D. C., 1959, pp. 327-369.

Gagne, Robert M., "Learning and Individual Differences, Introduction to the Conference", Learning and Individual Differences, Charles E. Merrill Books, Inc., Columbus, Ohio, 1967, p. xi.

Gagne, Robert M., and Staff, "Some Factors in Learning Non-Metric Geometry", Mathematical Learning, Society for Research in Child Development, Monograph Volume 30, No. 1, University of Chicago Press, 1965, pp. 42-49.

Gibb, E. Glenadine; Jones, Phillip S. and Junge, Charlotte, "Number and Operation", The Growth of Mathematical Ideas, Grades K-12, National Council of Teachers of Mathematics, 24th Yearbook, Washington, D. C., 1959, pp. 29-30.

Hayes, Marion C., An Analysis of Responses Made in Four Narrow Mental Functions Involved in the Computation of Fractions, State University of Iowa, Doctoral Dissertations, 1927.

Hill, Thomas J., The Concept of Rational Number and Its Role in the Secondary School Curriculum, Doctoral Dissertation, Columbia University, 1967.

Howard, C. F., Three Methods of Developing Children's Concept of Fractions in Grades Five and Six of the Elementary School, Doctoral Dissertation, University of California, Berkeley, 1948.

Jensen, Arthur R., "Varieties of Individual Differences in Learning", Learning and Individual Differences, Charles E. Merrill Books, Inc., Columbus, Ohio, 1967, p. 117.

Landau, Edmund, Foundations of Analysis, Chelsea Publishing Company, New York, 1960, pp. 19-42. Translated by F. Steinhardt, **pp. 19-42.**

Lankford, F. G., Jr. and Pattishall, E. G., Jr., Development of Independence in Adding and Subtracting Fractions, School of Education, Division of Educational Research, University of Virginia, Charlottesville, 1956.

Peterson, John A., and Hashisaki, Joseph, Theory of Arithmetic, Second Edition, John Wiley & Sons, New York, 1967, pp. 169-204.

School Mathematics Study Group, Mathematics for the Elementary School, Volume 5, Volume 2, 1962, p. 2.

Searle, Albert Henry, An Analysis of the Causes of Pupils Errors in Common Fractions, Doctoral Dissertation, The State University of Iowa, Iowa City, Iowa, 1927.

Sebold, M. T. Sister, Learning the Basic Concepts in Fractions and Their Application in the Addition and Subtraction of Simple Fractions, Catholic University, Doctors Dissertations, 1947.

Sibilia, Sister A. M., M.P.F., An Axiomatic Analysis of the Fundamental Concepts of the Arithmetic of Natural and Rational Numbers as Presented in Textbooks from the Colonial Period to the Present Day, Pennsylvania State University, Doctoral Dissertation, 1959.

Steele, Don Carlisle, Teaching and Testing the Understanding of Common Fractions, University of Pittsburgh, Doctoral Dissertation, Pittsburgh, Pennsylvania, 1940.

Thiele, C. L., "Arithmetic in the Early Grade", Sixteenth Yearbook of the National Council of Teachers of Mathematics, 1941, Bureau of Publications, Teachers College, Columbia University, New York, pp. 45-79.

Articles and Periodicals

Aftreth, Orville B., "The Effect of the Systematic Analysis of Errors on Achievement in the Study of Fractions at Sixth Grade Level", Dissertation Abstracts, 14(1); 1954, p. 501.

Aftreth, Orville B., "The Effect of the Systematic Analysis of Errors in the Study of Fractions at the Sixth Grade Level", Journal of Educational Research, 52; September, 1958, pp. 31-4.

Anderson, Rosemary C., "A Comparison of Two Procedures For Finding the Least Common Denominator in the Addition of Unlike, Unrelated Fractions", Dissertation Abstracts, 26; 1966, p. 5901.

Becker, Angeline L., "Remedial Work in the Addition of Common Fractions", California Journal of Elementary Education, 9; August, 1940, pp. 43-7.

Bidwell, James King, "From Ratio to Rational Number", School Science and Mathematics, October, 1966, pp. 661-9.

Botts, Truman, "Fractions in the New Elementary Curricula", The Arithmetic Teacher, March, 1968, pp. 216-220.

Braunfeld, Peter; Dilley, Clyde, and Rucker, Walter, "A New UICSM Approach to Fractions for the Junior High School", The Mathematics Teacher March, 1967, pp. 215-221.

Braunfeld, Peter and Wolfe, Martin, "Fractions for Low Achievers", Arithmetic Teacher, December, 1966, pp. 647-655.

Brueckner, Lee J., "Analysis of Errors in Fractions", Elementary School Journal, XXVIII, 1928, pp. 760-70.

Collier, Myrtie, "Learning to Multiply Fractions", School Science and Mathematics, April, 1922, pp. 324-29.

Dooley, Sister Marie Constance, "The Relation Between Arithmetic Research and the Content of Elementary Arithmetic Textbooks,1900-1957", Dissertation Abstracts, 20(1); 1959, p. 562.

Fehr, Howard F., "Fractions as Operators", The Arithmetic Teacher, March, 1968, pp. 228-232.

Fincher, Glen Eugene, "The Construction and Experimental Application of a Programmed Course of the Addition and Subtraction of Fractions for Grade Five", Dissertation Abstracts, 24(2); 1963, p. 650.

Gibb, E. Glenadine, "The Years Ahead", The Arithmetic Teacher, May, 1968, p. 434.

Guiler, Walter S., "Difficulties in Fractions Encountered by Ninth Grade Pupils", Elementary School Journal, 46, November, 1945a, pp. 146-56.

Guiler, Walter S., "Difficulties Encountered by College Freshmen in Fractions", Journal of Educational Research, 39: October, 1945b, pp. 102-115.

Gunderson, Agnes and Gunderson, Ethel, "Fraction Concepts Held by Young Children", The Arithmetic Teacher, October, 1957, pp. 168-73.

Gunderson, Ethel, "Fractions--Seven Year Olds Use Them", The Arithmetic Teacher, November, 1958, pp. 233-238.

Gundlach, W. B. "The Ability of High School Pupils in Common Fractions", Journal of Educational Research, 29, 1936, pp. 347-354.

Howard, Charles F., "Three Methods of Teaching Arithmetic", California Journal of Educational Research 1, January, 1950, pp. 25-29.

Johnson, J. T., "Are Ratios Fractions?", Elementary School Journal, 48: March, 1948, pp. 374-78.

Knight, R. B., and Setzafandt, A. O. H., "Transfer Within a Narrow Mental Function", Elementary School Journal, June, 1924, pp. 780-87.

Morton, R. L., "An Analysis of Pupils Errors in Fractions", Journal of Educational Research, IX (1924), pp. 117-25.

Mueller, Francis J., "On the Fraction as a Numeral", The Arithmetic Teacher, 8, 1961, pp. 234-38.

Pigge, F. L. "An Experimental Comparison of Three Methods of Teaching Addition and Subtraction of Fractions in Grade Five", Dissertation Abstracts, 25, p. 1789.

Polkinghorne, A. R., "Young Children and Fractions", Childhood Education II, May, 1935, pp. 354-58.

Souder, H. C., "Construction and Evaluation of Certain Readiness Tests in Common Fractions", Journal of Educational Research, October, 1943, pp. 127-234.

Van Engen, Henry, "Rate Pairs, Fractions, and Rational Numbers", The Arithmetic Teacher, December, 1960, pp. 389-403.

Wilson, G. M. and Dalrymple, C. O., "Useful Fractions", Journal of Educational Research, Volume 30, January, 1937, pp. 341-347.

References Used in the Development of the Teaching Material, Treatment B

Bruner, Jerome S., The Process of Education, New York, Random House, Inc., 1960.

Flavell, J. H., The Developmental Psychology of Jean Piaget, D. Can Nostrand Company, Princeton, New Jersey, 1963.

Gagne, Robert M., The Conditions of Learning, Holt, Rinehart and Winston, Inc., New York, 1965.

Gagne, Robert M., Learning and Individual Differences, Charles E. Merrill, Columbus, Ohio, 1967.

Land, F. W., New Approaches to Mathematics Teaching, Macmillan and Company Ltd., London, 1963.

Lankford, Francis G., "Implications of the Psychology of Learning for the Teaching of Mathematics", The Growth of Mathematical Ideas, K-12, The Twenty Fourth Yearbook of the National Council of Teachers of Mathematics, Washington, D. C., 1959, pp. 405-430.

National Council of Teachers of Mathematics, Research in Mathematics Education, Washington, D. C., 1967.

Shulman, Lee S., and Keisler, Evan R., Learning By Discovery, Rand McNally & Company, 1966, Chicago.