FREQUENCY DOMAIN METHODS FOR SYSTEMS WITH SLOW AND FAST DYNAMICS

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ABSTRACT

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The mathematical treatment of systems with slow and fast dynamics has traditionally involved singular perturbation theory for differential equations. This thesis suggests a set of conditions to be placed on a frequency domain description of a system to guarantee two time scale behavior. Some basic stability and approximation results are presented.

To my parents,

Herman and Catherine Luse

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

I. INTRODUCTION

The automatic control literature contains a wide variety of analysis and design methods for linear multivariable systems. Most of these, however, suffer from the so-called "curse of dimensionality." That is, the amount of computation required increases dramatically with the dimension of the system under consideration. This situation creates a need for efficient model reduction and decomposition methods. Some model simplification schemes [e.g., 13] assume no particular structural properties for the system being treated. Others rely on natural subsystem decomposition caused by spatial [e.g., 14] or temporal separation [1]. The subject of this thesis is the latter.

There is an extensive literature on time scale separation through the use of singular perturbation methods for differential equations. This thesis considers the same problem from a frequency domain approach. A discussion of some basic time domain results for linear systems is included here for later comparison.

An autonomous linear singularly perturbed system is shown in (1.1). The blocks A_{11}, A_{12}, A_{21}, and A_{22} are analytic at $\varepsilon = 0$. It can be shown [2] that the state matrix $A(\varepsilon)$ defined in (1.2) can be brought to block diagonal form

(1.3) by a similarity transformation $T(\varepsilon)$ where the following statements hold:

1. T, A_s, and A_f are analytic at $\varepsilon = 0$.
2. $A_s(0) = A_{11}(0) - A_{12}(0)\, A_{22}^{-1}(0)\, A_{21}(0) \triangleq A_0$
3. $A_f(0) = A_{22}(0)$
4. $T(0) = \begin{bmatrix} I & 0 \\ A_{22}^{-1} A_{21} & I \end{bmatrix}$.

$$\dot{x}_1 = A_{11}(\varepsilon)\, x_1 + A_{12}(\varepsilon)\, x_2 \tag{1.1}$$

$$\varepsilon \dot{x}_2 = A_{21}(\varepsilon)\, x_1 + A_{22}(\varepsilon)\, x_2, \quad \det A_{22}(0) \neq 0$$

$$A(\varepsilon) \triangleq \begin{bmatrix} A_{11}(\varepsilon) & A_{12}(\varepsilon) \\ \frac{A_{21}(\varepsilon)}{\varepsilon} & \frac{A_{22}(\varepsilon)}{\varepsilon} \end{bmatrix} \tag{1.2}$$

$$T(\varepsilon)\, A(\varepsilon)\, T^{-1}(\varepsilon) = \begin{bmatrix} A_s(\varepsilon) & 0 \\ 0 & \frac{1}{\varepsilon} A_f(\varepsilon) \end{bmatrix} \tag{1.3}$$

It is easily seen that the eigenvalues of $A_s(\varepsilon)$ approach finite limits as $\varepsilon \to 0$ and the eigenvalues of $A_f(\varepsilon)$ of all go to infinity as $0(1/\varepsilon)$ as $\varepsilon \to 0$. Thus, $A(\varepsilon)$ has been split into slow and fast parts. The limiting effect of this transformation can be interpreted: $A_s(0)$ is the state matrix of a system obtained by setting $\varepsilon = 0$ in (1.1), solving for x_2 in terms of x_1, and substituting in the first equation; $A_f(0)$ is the state matrix for the x_2 vector assuming that the x_1 vector is constant. The exponential matrix, which contains all information about the state space trajectories, can now be decomposed as shown in (1.4).

$$e^{A(\varepsilon)t} = T^{-1}(\varepsilon) \begin{bmatrix} e^{A_s(\varepsilon)t} & 0 \\ 0 & 0 \end{bmatrix} T(\varepsilon) + T^{-1}(\varepsilon) \begin{bmatrix} 0 & 0 \\ 0 & e^{\frac{1}{\varepsilon}A_f(\varepsilon)t} \end{bmatrix} T(\varepsilon) \qquad (1.4)$$

The transformation $T(\varepsilon)$ employed here is not unique. In fact, the transformation to Jordan form (varying with ε) would also suffice. The advantages here are that T, A_s, and A_f are analytic at $\varepsilon = 0$, and that limiting values for these matrices are easily computed.

Associated with the decomposition (1.4) is an approximation of the exponential matrix (1.5). It is formed by setting $\varepsilon = 0$ everywhere in (1.4) except where it divides $A_f(\varepsilon)$. If the matrices A_0 and $A_{22}(0)$ both have all eigenvalues in the open left half plane, then the relation (1.6) holds. Note that neither $\phi(t,\varepsilon)$ nor $\exp A(\varepsilon)t$ has a uniform limit function as $\varepsilon \to 0$. Pointwise limit functions exist but are discontinuous at the origin.

$$e^{A(\varepsilon)t} \approx \phi(t, \varepsilon) \triangleq T^{-1}(0) \begin{bmatrix} e^{A_0 t} & 0 \\ 0 & 0 \end{bmatrix} T(0) + T^{-1}(0) \begin{bmatrix} 0 & 0 \\ 0 & e^{A_{22}(0) \cdot \frac{t}{\varepsilon}} \end{bmatrix} T(0) \qquad (1.5)$$

$$\sup_{t \geq 0} \| e^{A(\varepsilon)t} - \phi(t,\varepsilon) \| = 0(\varepsilon) \text{ as } \varepsilon \to 0 \qquad (1.6)$$

$\|\cdot\|$ is any matrix norm.

The decomposition (1.4) and approximation (1.5) have been generalized for certain ε-dependent systems which do not have the form (1.1) [6]. There are methods of computing higher order approximations of the matrices T, A_s, and A_f [8]. In [9], multiple (more than two) time scales are allowed and $A(\varepsilon)$ is generalized to be a linear mapping on a Banach space.

The approximate time scale decomposition above has a number of other applications. As a representative example, its application to the pole assignment problem with state feedback is included here. The system (1.7) is split into slow and fast subsystems (1.8) and (1.9) respectively. The original problem is stated: Assign, by state feedback, the poles $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}, \frac{\lambda_{n_1+1}}{\varepsilon}, \frac{\lambda_{n_1+2}}{\varepsilon}, \ldots, \frac{\lambda_{n_1+n_2}}{\varepsilon}$. It is required that $\lambda_{n_1+1}, \ldots, \lambda_{n_1+n_2}$ be non-zero. The slow subproblem is: assign the poles $\lambda_1, \ldots, \lambda_{n_1}$ to the system (1.8) by state feedback G_s. Similarly, the fast subproblem is: assign the poles $\lambda_{n_1+1}, \ldots, \lambda_{n_1+n_2}$ to the system (1.9) with state feedback G_f.

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 u \tag{1.7}$$

$$\varepsilon \dot{x}_2 = A_{21} x_1 + A_{22} x_2 + B_2 u, \ \det A_{22} \neq 0.$$

$$A_{11} \text{ is } n_1 \times n_1, \ A_{22} \text{ is } n_2 \times n_2$$

$$\dot{x}_s = A_0 x_s + B_0 u_s \tag{1.8}$$

$$\dot{x}_f = A_{22} x_f + B_2 u_f \tag{1.9}$$

$$\text{where } A_0 = A_{11} - A_{12} A_{22}^{-1} A_{21}$$

$$B_0 = B_1 - A_{12} A_{22}^{-1} B_2$$

Assuming that the slow and fast subproblems have been solved, their solutions are combined in (1.10) and (1.11) to form the composite control law (1.12) for the system (1.7).

$$G_1 = (I + G_f A_{22}^{-1} B_2) G_s + G_f A_{22}^{-1} A_{21} \tag{1.10}$$

$$G_2 = G_f \tag{1.11}$$

$$u = G_1 x_1 + G_2 x_2 \tag{1.12}$$

If the feedback (1.12) is applied to (1.7), then the following result holds: The closed loop poles can be written (as functions of ε).

$$\nu_1(\varepsilon), \; . \; . \; . \; , \quad \nu_n(\varepsilon), \; \frac{\nu_{n_1+1}(\varepsilon)}{\varepsilon}, \; . \; . \; . \; , \; \frac{\nu_{n_1+n_2}(\varepsilon)}{\varepsilon}$$

$$\text{where } |\nu_j - \lambda_j| \to 0 \text{ as } \varepsilon \to 0 \text{ for } 1 \le j \le n_1 + n_2$$

At this time, there are very few applications of frequency domain methods for slow-fast subsystem decomposition in the literature. In [17], the "asymptotic forms" (1.14) for the transfer matrix of the system (1.13) are derived. However, no precise theoretical meaning is given to the terminology "asymptotic forms."

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3/\varepsilon & A_4/\varepsilon \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2/\varepsilon \end{bmatrix} u \tag{1.13}$$

$$y = C_1 x + C_2 z$$

where A_1 is $n \times n$, A_4 is $m \times m$

$$G_1(s) = C_0(sI_n - A_0)^{-1} B_0 - C_2 A_4^{-1} B_2 \tag{1.14}$$

$$G_h(s) = C_2(\varepsilon sI_n - A_4)^{-1} B_2$$

where $A_0 = A_1 - A_2 A_4^{-1} A_3$

$B_0 = B_1 - A_2 A_4^{-1} B_2$

$C_0 = C_1 - C_2 A_4^{-1} A_3$

The purpose of this thesis is to show frequency domain analogs for the fundamental time domain results. One of the achievements of this work is to give theoretical meaning to the frequency scale decomposition (1.14), in much the same way in which time domain results give meaning to the approximation (1.5). Throughout this work, however, the basic system description used is the transfer matrix, with internal descriptions used only for proofs and examples.

There are a number of reasons for investigating multiple time scale systems from a frequency domain viewpoint. Frequency domain methods have been revived in recent years with generalizations of classical methods to multivariable systems. This work should open the way for investigation of regularities which may occur in the generalized nyquist and root locus plots of multiple time scale systems. It may lead to convenient application of multivariable robustness [e.g., 15] and sensitivity [e.g., 16] methods, which could lead to design schemes which are fundamentally different from time domain methods.

Through this approach, multiple time scale methods may be extended to systems described by convolution operators, such as those involving time delays. For theoretical purposes, the approach is useful for clarifying some time domain results. This is because no particular internal structure for systems is assumed--only input-output relationships.

In summary, this thesis should be regarded as the theoretical basis for future application of frequency response methods to multiple time scale systems.

II. SYSTEM MATRIX THEORY FOR AN ARBITRARY FIELD

Rosenbrock [3] develops the theory of time invariant linear system through the use of system matrices. Two classes of system matrices are considered: rational and polynomial. Rational system matrices have elements which are rational over some field and may be regarded as a generalization of the concept of transfer matrix. The class of polynomial system matrices, that is, the class of system matrices with polynomial elements only, includes both the class of state space descriptions and the class of matrix fraction descriptions as special cases. Polynomial system matrices are useful because they provide very general internal descriptions of linear systems. Operations which preserve the relation of strict system equivalence are analogous to similarity and unimodular transformations for state space and matrix fraction descriptions respectively. Strict system equivalence preserves all external descriptions of a system as well as maintaining the values of all (possibly internal) poles.

In [3], the underlying number field is the field of complex numbers. An extension is needed, for the present work, to allow the parameter ε to appear in coefficients of polynomials and rational fractions of the frequency variables. Whatever manner ε is allowed to appear, the field properties of the coefficients must be preserved; otherwise most of the theory of system matrices would be lost. Also, all concern will be with small values of the parameter ε. With these observations in mind, we make the following definition:

Definition 2.0.1:

$$\mathcal{F}_\varepsilon \triangleq \left\{ \frac{f(\varepsilon)}{\varepsilon^r} : f(\varepsilon) \text{ is analytic at } \varepsilon = 0;\ r \text{ is an integer} \right\}$$

This variation of coefficients with ε is general enough for most physical purposes. A smaller field, such as the field of functions rational in ε could have been chosen, but $\mathcal{F}_\varepsilon$ is needed for a polynomial factorization later on.

As mentioned before, most of the theory of system matrices carries over when the field $\mathcal{F}_\varepsilon$ is used. All processes involving only field operations, such as block Gaussian elimination, are performed in an identical manner. The important Euclid's algorithm for computation of the greatest common factor of two polynomials is still available. There is one major difference, however: there exist nonlinear prime polynomials. Stated differently, there are polynomials over $\mathcal{F}_\varepsilon$ whose roots are not in $\mathcal{F}_\varepsilon$. An example is $s^2 - \varepsilon = 0$. The roots are $s = \pm\sqrt{\varepsilon}$ which are not analytic at $\varepsilon = 0$. Note that as a consequence, the Jordan form of a matrix cannot be generated without leaving the underlying field.

A number of Rosenbrock's proofs assume that any polynomial can be factorized into linear factors, but all essential results can be derived without using this property. The remainder of this chapter consists of two parts. First, a statement of the key results from [3] which are needed for this work. Second, the modifications of proofs necessary to alleviate the difficulty discussed above.

II.1. Basic Theorems from System Matrix Theory

This section contains results from system matrix theory which are needed later. All theorems in this section assume that the underlying field is arbitrary, and the field will be denoted by F. The development roughly follows [3]. A reader familiar with the system matrix approach can follow this section if he assumes that the mechanics of "extraction of decoupling zeros" has been extended to system matrices over a field F.

Definition 2.1.1: A rational matrix P(s) (over a field F) is a system matrix with m outputs and ℓ inputs if

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \tag{2.1.1}$$

where U is $r \times \ell$, V is $m \times r$, and $\det T(s) \neq 0$.

The associated transfer matrix H(s) is given by

$$H = VT^{-1}U + W \tag{2.1.2}$$

Also, P is called a system matrix representation of H. Relations between the matrices in (2.1.1) will be indicated by subscripts. For instance, P_1 will consist of the blocks T_1, U_1, V_1 and W_1.

<u>Definition 2.1.2</u>: P(s) is a polynomial system matrix of order n if:

1. P(s) is a system matrix with only polynomial elements;
2. deg det T(s) = n;
3. $r \geq \max(n, \ell, m)$.

<u>Definition 2.1.3</u>: A polynomial system matrix P(s) of order n with associated transfer matrix H(s) has least order if every other polynomial system matrix representation of H has order n or greater.

<u>Definition 2.1.4</u>: Polynomial system matrices P_1 and P_2 are strictly system equivalent if there exist polynomial matrices X and Y, and unimodular matrices M and N such that

$$P_2 = \begin{bmatrix} M & O \\ X & I \end{bmatrix} \cdot P_1 \cdot \begin{bmatrix} N & Y \\ O & I \end{bmatrix} \qquad (2.1.3)$$

<u>Definition 2.1.5</u>: System matrices P_1 and P_2 are system equivalent if P_2 can be obtained from P_1 by one or more of the following two types of transformations:

1. Transformations of the form of (2.1.3) with X,Y, M,N all rational and M,N non-singular;
2. Trivial expansions and contractions as shown in (2.1.4).

$$P \longleftrightarrow \begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix} \tag{2.1.4}$$

It can be shown that two equivalent system matrices have the same associated transfer matrix.

<u>Theorem 2.2.1</u>: Let H(s) be a rational matrix over a field F. Then there exist polynomial matrices T(s) and V(s) over F with T(s) non-singular such that H(s) has a least order polynomial system matrix representation of the form (2.1.5). (The form of system matrix (2.1.5) will be referred to as Matrix Fraction Description or MFD form).

$$\left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & T(s) & I \\ \hline 0 & -V(s) & 0 \end{array}\right] \tag{2.1.5}$$

<u>Proof</u>: Let d(s) be a least common denominator for the elements of H(s) so that H(s) = N(s)/d(s) with N(s) polynomial. Then (2.1.6) is a system matrix representation for H(s).

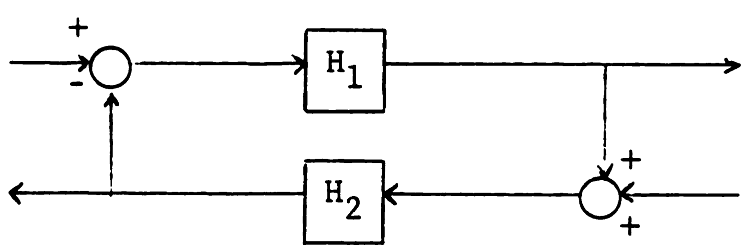
$$\begin{bmatrix} d(s)I & I \\ -N(s) & 0 \end{bmatrix} \tag{2.1.6}$$

All non-unimodular common right divisors of d(s)I and N(s) can be extracted using system equivalence operations followed by an expansion to meet condition 3 of Definition (2.1.2). (2.1.5) is least order by an extension of Theorem 3.2 of [3]. □

Definition 2.1.6: The characteristic polynomial of a rational matrix H(s) over a field F is the least common multiple of the denominators of all non-zero minors of all orders of H(s). The characteristic polynomial is assumed to be normalized in the case $F = \mathcal{F}_\varepsilon$ so that ε can be non-trivially set to zero.

Definition 2.1.7: Let P(s) be a polynomial system matrix over a field F. Then the input decoupling polynomial of P is the product of the diagonal elements of the Smith form of [T(s) U(s)]. The output decoupling polynomial of P is the product of the diagonal elements of the Smith form of $[T(s)^T \ -V(s)^T]^T$. Let $P_1(s)$ be a system matrix obtained by extracting the input decoupling polynomial from P(s), i.e., by extracting all non-unimodular common left divisions from [T(s) U(s)] in P(s). Then the input-output decoupling polynomial of P(s) is the quotient $\gamma(s)/\gamma_1(s)$ where $\gamma(s)$ is the output decoupling polynomial of P(s) and $\gamma_1(s)$ is the output decoupling polynomial of $P_1(s)$.

Figure 2.1.1



Theorem 2.1.2: Let H(s) be rational over a field F. Let P be a least order polynomial system matrix realization of

H(s). Denote the blocks of P by T,U,V, and W. Then

$$CP[H(s)] = f \cdot \det T \quad \text{where } f \in F \tag{2.1.7}$$

Proof: Follows almost exactly as in [3]. □

Theorem 2.1.3: Suppose two systems with transfer matrices H_1 and H_2 over a field F are connected in series, and put in the unity feedback configuration of Figure 2.1.1. Let H_1 and H_2 be described by system matrices P_1 and P_2, respectively. Then the closed loop system matrix can be represented by the system matrix P_{CL} as shown in (2.1.8) if $\det(I + H_1H_2) \neq 0$.

$$P_{CL} = \left[\begin{array}{cccccc|cc} T_1 & U_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_2 & U_2 & 0 & 0 & 0 & 0 \\ -V_1 & W_1 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & -V_2 & W_2 & 0 & -I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & I & -I & 0 \\ 0 & 0 & 0 & I & -I & 0 & 0 & -I \\ \hline -V_1 & W_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -V_2 & W_2 & 0 & 0 & 0 & 0 \end{array}\right] \tag{2.1.8}$$

Furthermore $\det T_{CL} = \pm\det T_1 \det T_2 \det(I + H_1H_2)$. (The sign depends upon the sizes of the blocks.)

Proof: Follows from a trivial extension of the derivation in Chapter 5, Section 1, of [3]. □

Theorem 2.1.4: Let H_1, H_2, P_1, P_2, etc. be as in Theorem 2.1.3. Suppose, furthermore, that P_1 and P_2 are polynomial system matrices. Let β, γ, and δ be generic input decoupling polynomial, output decoupling polynomial, and input-output decoupling polynomial, respectively. They will be subscripted according to the system matrices to which they refer (1,2,CL). Then

$$\beta_{CL} = \beta_1 \beta_2$$

$$\gamma_{CL} = \gamma_1 \gamma_2$$

$$\delta_{CL} = \delta_1 \delta_2$$

Proof: It is evident from (2.1.8) that $\beta_{CL} = \beta_1 \beta_2$. (2.1.8) can be transformed by operations of strict system equivalence to (2.1.9).

$$\left[\begin{array}{cccccc:cc} T_1 & 0 & U_1 & 0 & 0 & 0 & 0 & 0 \\ -V_1 & 0 & W_1 & 0 & -I & 0 & 0 & 0 \\ 0 & T_2 & 0 & U_2 & 0 & 0 & 0 & 0 \\ 0 & -V_2 & 0 & W_2 & 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 & 0 & I & -I & 0 \\ 0 & 0 & 0 & I & -I & 0 & 0 & -I \\ \hdashline 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{array}\right] \quad (2.1.9)$$

It can now be seen that $\gamma_{CL} = \gamma_1 \gamma_2$. To show that $\delta_{CL} = \delta_1 \delta_2$, first extract the input decoupling polynomial from (2.1.8) and perform the same operations of strict system equivalence that were used to arrive at (2.1.9). This new system matrix is identical to (2.1.9) except that the pairs (T_1U_1) and (T_2U_2) are replaced by reduced versions. □

The next theorem involves a commonly used regularity condition which guarantees that both open loop and closed loop systems have the same order.

Theorem 2.1.5: Suppose two systems with proper transfer matrices H_1 and H_2 over a field F are connected as in Figure 2.1.1. If $\det(I + H_1(\infty)\, H_2(\infty)) \neq 0$, then the matrix (2.1.8) is a system matrix. Furthermore, the number of closed loop poles of Figure 2.1.1 is equal to the sum of the number of poles of the two open loop systems.

Proof: Clearly, $\det(I + H_1(s)H_2(s)) \neq 0$ if $\det\,(I + H_1(\infty)\, H_2(\infty)) \neq 0$. Hence, (2.1.8) is a system matrix. Referring to (2.1.8),

$$\frac{\pm\det T_{CL}}{\det T_1 \det T_2} = \det\,(I + H_1H_2) \qquad (2.1.10)$$

If s is allowed to go to ∞, it is apparent that $\det T_{CL}$ and $\det T_1 \cdot \det T_2$ must have the same degree. □

II.2. Appendix: Extensions of Rosenbrock's Results

This section is not self-contained: it assumes the reader is familiar with the reference [3] and the theory of polynomial matrices [21]. In this section, an R preceeding a theorem number means that it corresponds to that theorem in [3].

The concept of "extracting a decoupling zero" no longer makes sense, since this zero may not be in the field F. The next theorem has been reworded and reproved to reflect this.

Theorem R2.4.1: Let P be a polynomial system matrix over F. Let the blocks of P be labeled:

$$P = \begin{bmatrix} T & U \\ -V & W \end{bmatrix} \tag{2.2.1}$$

If either [T U] or $[T^T\ -V^T]^T$ is not Smith equivalent to [I 0] then there is a polynomial system matrix P_1 of lower order (i.e., deg det T_1 < deg det T) giving rise to the same transfer matrix.

Proof: Suppose [T U] ~ [S 0] where [S 0] is in Smith form with $S \neq I$. Then there exist unimodular R and Q such that

$$R\ [T\ U]\ Q = [S\ 0] \text{ or,}$$
$$R\ [T\ U] = [S\ 0]\ Q^{-1} \tag{2.2.2}$$

The rows of (2.2.2) are divisible by the corresponding diagonal elements of S. The system matrix (2.2.1) can be transformed by strict system equivalence:

$$\begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T & U \\ -V & W \end{bmatrix} = \begin{bmatrix} S & 0 \\ -VQ & W \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix}$$

This matrix has the first r rows divisible by the diagonal elements of S. S has no zero diagonal elements because T is nonsingular. Thus S^{-1} exists. Define P_1 as

$$P_1 \triangleq \begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T & U \\ -V & W \end{bmatrix}$$

Then P_1 is a polynomial system matrix with deg det T_1 = deg det T - deg det S < deg det T.

As mentioned in the last section, the concept of "set of decoupling zeros" must be replaced by "decoupling polynomial." We now introduce (and repeat some of) the following terminology for a polynomial system matrix P with associated transfer matrix G.

α = pole polynomial of G

β = input decoupling polynomial

γ = output decoupling polynomial

δ = input-output decoupling polynomial

η = pole polynomial of P

The definitions of the above are:

$\beta \triangleq$ det S where $[T \quad U] \sim [S \quad 0]$ where $[S \quad 0]$ is in Smith form.

$\gamma \triangleq$ det $\bar{S}$ where $[T^T \ - \ V^T] \sim [\bar{S} \quad 0]$ where $[\bar{S} \quad 0]$ is in Smith form.

$\Theta \triangleq$ the output decoupling polynomial of a system matrix obtained from P by "removing" the input decoupling polynomial as in Theorem R2.4.1.

$\delta \triangleq \frac{\gamma}{\Theta}$ (shown later to be a polynomial)

$\gamma \triangleq \frac{\eta\delta}{\beta\gamma}$ (shown later to be a polynomial)

$\eta \triangleq$ det T.

<u>Theorem R2.5.1</u>: Let $\alpha, \beta, \gamma, \delta, \eta$, and Θ be defined as above for a system matrix P over F. Then the polynomial division relations hold:

$$\Theta \mid \gamma, \quad \delta \mid \beta, \text{ and } \beta\gamma \mid \eta\delta.$$

Also, $\beta \mid \eta$ and $\gamma \mid \eta$.

<u>Proof</u>: Let P_1 be the system matrix resulting from removal of β from the input of P.

If [S 0] is the Smith form of [T U] then det S = the greatest common factor of all r x r minors of [T U].

Then det S | det T or $\beta \mid \eta$

By similar reasoning, $\gamma \mid \eta$.

Let R and Q be unimodular matrices which transfer [T U] to its Smith form:

$$R\,[T \quad U]\,Q = [S \quad 0]$$

Then P can be transformed through strict system equivalence:

$$\begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T & U \\ -V & W \end{bmatrix} = \begin{bmatrix} [S \quad 0] & Q^{-1} \\ [-V \quad W] & \end{bmatrix} \triangleq \hat{P} \qquad (2.2.3)$$

This transformation preserves the Smith form of the output pair:

$$\begin{bmatrix} T \\ V \end{bmatrix} \sim \begin{bmatrix} \bar{S} \\ 0 \end{bmatrix} \sim \begin{bmatrix} \hat{T} \\ -V \end{bmatrix}$$

We can left multiply (2.2.3) by the matrix (2.2.4). The result is still a polynomial system matrix and the transfer matrix is unchanged. However, the Smith form of the output pair may change.

$$\begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{2.2.4}$$

Consider the effect of left multiplying $\hat{P}$ in (2.2.3) by (2.2.4) on an r x r minor of $[\hat{T}^T, -V^T]^T$. The minors will be divided by the (possibly non-unity) diagonal elements of S which correspond to the positions of their rows from $\hat{T}$. P_1 can be written explicitly:

$$P_1 = \begin{bmatrix} S^{-1}\hat{T} & S^{-1}\hat{U} \\ -V & W \end{bmatrix}$$

Thus, if M is an r x r minor of $[\hat{T}^T, -V^T]^T$ and N is the corresponding r x r minor of $[T_1{}^T, -V^T]$, then $N \mid M$.

$$\text{r x r minors of} \begin{bmatrix} \hat{T} \\ -V \end{bmatrix} : \quad M_1, M_2, \ldots, M_p \quad \gamma = \underset{1 \le i \le p}{\text{GCF}}\, M_i$$

$$\text{r x r minors of} \begin{bmatrix} S^{-1}\hat{T} \\ -V \end{bmatrix} : \quad N_1, N_2, \ldots, N_p$$

$$\theta = \underset{1 \le i \le p}{\text{GCF}}\, N_i \tag{2.2.5}$$

Since $N_i \mid M_i$ for $1 \le i \le p$, then the minors of $[\hat{T}^T, -V^T]^T$ can be written

$$g_1 N_1,\ g_2 N_2,\ \ldots,\ g_p N_p; \text{ and } \gamma = \underset{1 \le i \le p}{\text{GCF}}\, g_i N_i \tag{2.2.6}$$

where each g_i is a polynomial. It is now clear that $\Theta \mid \gamma$.

Since $\beta = \det S$, $g_i \mid \beta$ for $1 \le i \le p$. Stated otherwise, β is a common multiple of the g_i's. We then have

$$\operatorname*{LCM}_{1 \le i \le p} g_i \mid \beta$$

since the least common multiple divides all other common multiples. Let h be a prime factor of δ of multiplicity K. That is, $h^K \mid \delta$ but $h^{K+1} \nmid \delta$. Viewing the list (2.2.6) as a modification of the list (2.2.5), it is seen that $h^K \mid g_j$ for some j. The same argument can be repeated for each prime factor of δ to show that

$$\delta \mid \operatorname*{LCM}_{1 \le i \le p} g_i$$

Thus, $\delta \mid \beta$.

Since $\det T_1$ is an r x r minor of $[T_1^T \ \ -V^T]^T$, $\Theta \mid \det T_1$. Also, $\det T_1 \cdot \det S = \det \hat{T}$. Therefore,

$$\Theta \cdot \beta \mid (\det T_1) \cdot \beta$$

$$\Theta \cdot \beta \mid \det T_1 \cdot \det S$$

$$\Theta \cdot \beta \mid \det \hat{T}$$

But $\det T \ \alpha \ \det \hat{T}$ and $\eta = \det T$. This gives

$$\Theta \cdot \beta \mid \eta$$

substituting $\Theta = \gamma/\delta$ yields

$$\frac{\gamma\beta}{\delta} \mid \eta \text{ or } \gamma\beta \mid \eta\delta. \qquad \square$$

Three standard conditions for two polynomial matrices to be coprime are generalized in the next theorem. Part (i) of this theorem in [3] is no longer applicable. That is, quantities may be involved which are not in $\mathcal{F}_\varepsilon$.

<u>Theorem R2.6.1</u>: Let T and U be polynomial matrices over a field F where [T U] has normal rank r. Then each of the following conditions are equivalent to T and U being left coprime.

(ii) [T U] is Smith equivalent to [I 0].

(iii) There exist polynomial matrices V and W respectively $\ell \times r$ and $\ell \times \ell$ such that

$$\begin{bmatrix} T & U \\ -V & W \end{bmatrix} \sim I_{r \times \ell}$$

(iv) There exist right coprime X and Y such that TX + UY = I.

<u>Proof</u>:

(ii) ($\rightarrow$) Suppose T and U are left coprime.

Suppose, to the contrary, that [T U] ~ [S 0] where [S 0] is in Smith form with det S $\neq$ 1. Then there exist unimodular R and Q such that

$$\begin{aligned} [T \quad U] &= R\,[S \quad 0]\,Q \\ &= RS\,[I \quad 0]\,Q \end{aligned}$$

RS is a left divisor of T and U, but det RS = det R det S which depends on s. This is a contradiction. Therefore, [T U] ~ [I 0].

($\leftarrow$) Suppose that $[T \;\; U] \sim [I \;\; 0]$. Let R be any common left divisor of T and U. Then there exist polynomial matrices T_1 and U_1 such that

$$[T \;\; U] = R\,[T_1 \;\; U_1].$$

Let M be an r x r minor of $[T \;\; U]$, and let N be the corresponding r x r minor of $[T_1 \;\; U_1]$. Then

$$M = (\det R) \cdot N$$

follows from the Cauchy - Binet formula. Thus, det R divides every r x r minor of $[T \;\; U]$. Det R must divide the greatest common factor of all r x r minors of $[T \;\; U]$. But the latter is equal to unity, because of a standard theorem on the Smith form. Since $\det R \mid 1$, det R is independent of s and thus R is unimodular. Coprimeness of T and U follows because R was an arbitrary common left divisor of T and U. It is also clear that the normal rank of $[T \;\; U]$ is r.

(iii) The proof in [3] holds as written.

(iv) ($\rightarrow$) Suppose T and U are left coprime. Then there exist M and N such that

$$\begin{aligned} [T \;\; U] &= M\,[I \;\; 0] \cdot N \\ &= [I \;\; 0] \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \cdot N \\ &= [I \;\; 0]\, Q^{-1} \end{aligned}$$

where $Q^{-1} = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \cdot N$

Clearly, Q is unimodular. Let blocks of Q be written as Q_{ij}. The last equation can be rewritten:

$$[T \quad U]\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = [I \quad 0]$$

Multiplying out yields:

$$T\,Q_{11} + U\,Q_{21} = I$$

We now set $X = Q_{11}$ and $Y = Q_{21}$. To prove that X and Y are right coprime, let $[X^T\ Y^T]^T$ have Smith form $[S \quad 0]^T$. Taking determinants of the above equation,

$$\det\,[T \quad U]\begin{bmatrix} X \\ Y \end{bmatrix} = \det I = 1 \tag{2.2.7}$$

The left hand side can be expanded using the Cauchy-Binet formula. Let the r x r minors of $[T \quad U]$ be listed: $M_1, \ldots, M_p$; let the corresponding (column for row) minors of $[X^T\ Y^T]^T$ be listed: $N_1, \ldots, N_p$. Then (2.2.7) becomes

$$\sum_{i=1}^{p} M_i\,N_i = 1 \tag{2.2.8}$$

We know that det S divides each minor N_i of $[X^T\ Y^T]^T$. Write $N_i = G_i \cdot \det S$ where each G_i is a polynomial. Then (2.2.8) becomes

$$\det S \cdot \sum_{i=1}^{p} M_i\,G_i = 1$$

Thus, det $S \mid 1$, and $S = I$. A transposed version of part (ii) of this theorem shows that X and Y are right coprime.

($\leftarrow$) Suppose there exist X and Y such that $T X + U Y = I$. Both sides can be transposed:

$$X^T T^T + Y^T U^T = I$$

This can be written in block form

$$[X^T \quad Y^T] \begin{bmatrix} T^T \\ U^T \end{bmatrix} = I$$

The argument starting just before (2.2.7) above can now be repeated to show that T^T and U^T are right coprime and therefore that T and U are left coprime. Note that the hypothesis of X and Y being right coprime was not needed. □

Theorem 2.6.2 of [3] gives several equivalent conditions for the matrices $sI - A$ and B to be left coprime. Only parts (iii), (iv), and (vi) are necessary for the essential theorems in the remainder of the book [3], so only these parts will be extended.

Lemma 2.2.1: The rank defects of the matrices (2.2.9) and (2.2.10), whose elements are in F, are equal.

$$\begin{bmatrix} I & 0 \ldots 0 & 0 & 0 \ldots 0 & 0 & B \\ -A & I \quad\; 0 & 0 & 0 \quad\; 0 & B & 0 \\ 0 & -A \quad 0 & 0 & 0 \quad\; B & 0 & 0 \\ \vdots & \vdots \quad\; \vdots & \vdots & \vdots \quad\; \vdots & \vdots & \vdots \\ 0 & 0 \quad\; I & 0 & B \quad\; 0 & 0 & 0 \\ 0 & 0 \ldots -A & B & 0 \ldots 0 & 0 & 0 \end{bmatrix} \tag{2.2.9}$$

$$[B \;\; AB \;\; A^2B \; \ldots \; A^{m-1} \, B] \tag{2.2.10}$$

where A is n x n, B is n x ℓ, and (2.2.9) has m block rows. (Thus, (2.2.9) is mn x [(m-1) n + mℓ] and (2.2.10) is n x mℓ.

<u>Proof</u>: Starting with the first block row of (2.2.9), left multiply each block row by A and add to the next row down. Repeating this m-1 times gives

$$\begin{bmatrix} I & 0 \ldots 0 & 0 & 0 \ldots 0 & 0 & B \\ 0 & I \quad\; 0 & 0 & 0 \quad\; 0 & B & AB \\ 0 & 0 \quad\; 0 & 0 & 0 \quad\; B & AB & A^2B \\ \vdots & \vdots \quad\; \vdots & \vdots & \vdots \quad\; \vdots & \vdots & \vdots \\ 0 & 0 \quad\; I & 0 & B \quad\; A^{m-4} & A^{m-3}B & A^{m-2}B \\ 0 & 0 \ldots 0 & B & AB \, .. \, A^{m-3}B & A^{m-2}B & A^{m-1}B \end{bmatrix} \tag{2.2.11}$$

The first n(m-1) rows of (2.2.11) are linearly independent from each other and from the last n rows. The last row obviously has the same rank as (2.2.10). Therefore, the rank defects are the same. □

<u>Lemma 2.2.2</u>: Let A and B be matrices whose elements are in F. Let q be the degree of the minimal polynomial of A. Then rank $[B\ AB\ \ldots\ A^{m-1}\ B] = \text{rank}\ [B\ AB\ \ldots\ A^{q-1}\ B]$ (2.2.12)

where $m \geq q$

<u>Proof</u>: Note--while some results such as the Jordan form are lost in the extension to an arbitrary field, others such as the Hamilton-Cayley theorem still hold. Thus, it is still true that $q \leq n$.

This lemma is proved by expanding the powers of A on the left hand side of (2.2.12) which are higher than q-1 in terms of lower powers of A. This is followed by zeroing out these terms by applying appropriate column operations. □

<u>Theorem R2.6.2</u>: Let A and B be matrices with elements in F. Then each of the following conditions is equivalent to sI - A and B being left coprime. A is n x n, B is n x ℓ, and the degree of the minimal polynomial of A is q.

(iii) The matrix (2.2.10) with m set equal to q has rank n.

(iv) The matrix (2.2.9) with m set equal to q has rank nq.

(vi) There exist X and Y such that

$$(sI - A)\ X + BY = I \qquad (2.2.13)$$

where $\deg X \leq q - 2$ and $\deg Y \leq q - 1$.

<u>Proof</u>: Lemma 2.2.1 shows that (iii) and (iv) are equivalent to each other. Figure 2.2.1 shows the circle of implications for the proof.

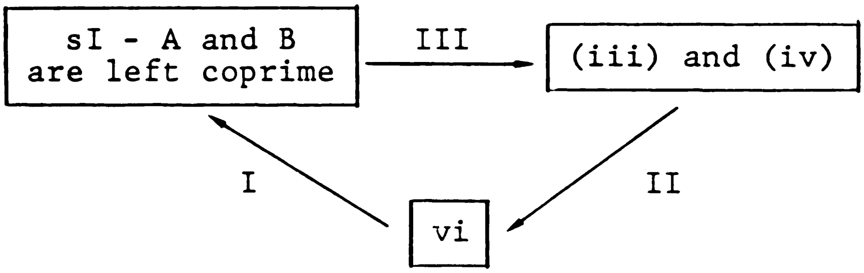


Figure 2.2.1

Implication I: This follows immediately from R2.6.1 (iv) (recall the last sentence of the proof of R2.6.1).

Implication II: Suppose that (iv) is true. To show that polynomial matrices with the specified properties exist, a system of equations, which the coefficients of X and Y must satisfy, is written. Define H_q as

$$H_q \triangleq \text{Matrix (2.2.9) with m set equal to q.}$$

Also define X and E (nq x n):

$$X = \begin{bmatrix} X_{q-2} \\ X_{q-1} \\ \vdots \\ X_0 \\ Y_0 \\ \vdots \\ Y_{q-1} \end{bmatrix} \qquad E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

where $X = X_0 + X_1 s + \ldots + X_{q-2}\, s^{q-2}$ and

$$Y = Y_0 + Y_1 s + \ldots + Y_{q-1}\, s^{q-1}.$$

Coefficients of s in (2.2.13) can be equated to yield

$$H_q X = E$$

This has a solution if rank H_q = rank $[H_q : E]$. By assumption, H_q has rank equal to its number of rows, so that adding more columns cannot change its rank. This proves Implication II.

Implication III: Suppose $sI - A$ and B are left coprime. Then R2.6.1 (iv) shows that there exist X and Y such that

$$(sI - A) X + BY = I \tag{2.2.14}$$

Since X and Y are polynomial matrices, they can be expanded in terms of their coefficients:

$$X = X_0 + X_1 s + \ldots + X_{m-2} s^{m-2}$$
$$Y = Y_0 + Y_1 s + \ldots + Y_{m-1} s^{m-1}$$

Proceeding in the same manner as above, define

$$X = \begin{bmatrix} X_{m-2} \\ \vdots \\ X_0 \\ Y_0 \\ \vdots \\ Y_{m-1} \end{bmatrix} \qquad E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \qquad H_m = \text{matrix (2.2.9)}$$

Coefficients of s in (2.2.14) can be equated:

$$H_m X = E \tag{2.2.15}$$

The row operations employed in the proof of Lemma 2.2.1 can be applied to (2.2.15). Let $\tilde{H}_m$ be matrix (2.2.11). Note that E does not change when these row operations are applied to it.

$$\tilde{H}_m X = E \tag{2.2.16}$$

The last block row of (2.2.16) is

$$[0\ 0\ \ldots\ 0\ B\ \ AB\ \ldots\ A^{m-1}B]\ X = I$$

This shows that rank $[B\ AB\ \ldots\ A^{m-1}B] \geq n$, so that $[B\ AB\ \ldots\ A^{m-1}B]$ has rank equal to n, its number of rows. There are now three possible cases:

1. If $m = q$, (iii) is true
2. If $m > q$, Lemma 2.2.2 implies (iii)
3. If $m < q$, addition of the columns $A^mB, \ldots, A^{q-1}B$ leaves the rank equal to n. Again, (iii) is true.

This proves Implication III. □

There is one point remaining which needs clarification. Theorem 3.2.2 of [3] requires the next lemma for extension. The notation is preserved from the proof in [3].

<u>Lemma 2.2.3</u>: Let $\hat{M} = \begin{bmatrix} M_{q-1} \\ t \end{bmatrix}$

where $M_{q-1}(s)$ is a polynomial matrix over F, with q-1 rows, such that $M_{q-1} \sim [I\ \ 0]$, $\hat{M}$ has normal rank q-1, and t is a polynomial row vector. Then there exists a polynomial row vector w such that

$$t = w \cdot M_{q-1}$$

<u>Proof</u>: There exist unimodular matrices R and Q such that

$$R\, M_{q-1}\, Q = [I \quad 0]$$

We now apply a unimodular transformation to $\hat{M}$

$$\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_{q-1} \\ t \end{bmatrix} Q = \begin{bmatrix} R\, M_{q-1}\, Q \\ tQ \end{bmatrix} = \begin{bmatrix} I & 0 \\ t_{11} & t_{12} \end{bmatrix} \quad (2.2.17)$$

where $t\, Q = [t_{11}\ t_{12}]$

The row t_{12} is zero in (2.2.17) since $\hat{M}$ has normal rank q-1. Then

$$\begin{aligned} t\, Q = [t_{11} \quad 0] &= t_{11} [I \quad 0] \\ &= t_{11}\, R\, M_{q-1}\, Q \end{aligned}$$

Therefore, $t = (t_{11}\ R)\ M_{q-1}$. The lemma is proved by setting $w = t_{11}\ R$. □

The following list of theorems from [3] consists of those which hold over a general field F. It is not exhaustive: others may have generalizations, especially when $F = \mathscr{F}_\varepsilon$. In this the theory of analytic functions of several complex variables may be of use. An asterisk indicates that the theorem needs superficial restatement.

<u>Chapter 1</u>: 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.10

The above theorems are developed for the general case in [21].

4.1

9.1*, 9.2* Some uniformity conditions in ε must be added to hypotheses here.

<u>Chapter 2</u>: 1.1, 1.2, 1.3, 1.4, 1.5

2.1

3.1, 3.2, 3.3, 3.4, 3.5

4.1*

5.1*, 5.2*

6.1* (except (i)), 6.2* (except (i) and (ii))

7.1

8.1*

<u>Chapter 3</u>: 1.1 (Theory is developed in [21])

2.1, 2.2

3.1, 3.2

4.1, 4.2, 4.3

6.1, 6.2, 6.3, 6.4

<u>Chapter 5</u>: 1.1, 1.2

III. TWO FREQUENCY SCALE RATIONAL MATRICES

The method of introducing ε into rational matrices which was presented in the previous chapter is very general. We now narrow this down in such a way that "two frequency scale" behavior, in analogy to "two time scale" behavior, is guaranteed. Before this is done, however, the definition is motivated by examining the variation of the roots of a polynomial in s whose coefficients are in the field $\mathcal{F}_\varepsilon$.

<u>Theorem 3.0.1</u>: Suppose that in Equation (3.0.1), $a_j(\varepsilon) \in \mathcal{F}_\varepsilon$ for $0 \leq j \leq n$. Then each of the n roots of (3.0.1) can be expanded about $\varepsilon = 0$ as in Equation (3.0.2) where the b_{kj}'s are complex constants, N is an integer, and $\varepsilon^{1/q}$ is a branch of $z^q = \varepsilon$.

$$a_n(\varepsilon)s^n + \ldots + a_1(\varepsilon)s + a_o(\varepsilon) = 0 \qquad (3.0.1)$$

$$s_k(\varepsilon) = \sum_{j=N}^{\infty} b_{kj}\varepsilon^{j/q}, \quad 1 \leq k \leq n \qquad (3.0.2)$$

Furthermore, if one root has an expansion (3.0.2) with $q > 1$, then there will be $q - 1$ other roots having expansions with the same coefficients but using different branches of $z^q = \varepsilon$.

<u>Proof</u>: First, (3.0.1) is multiplied by a suitable power of ε so that the result is a polynomial in s with coefficients

analytic at $\varepsilon = 0$. Furthermore, setting ε to zero will not leave the left hand side identically zero. To do this, express the coefficients:

$$a_j(\varepsilon) = \varepsilon^{r_j}\, c_j(\varepsilon),\ 0 \le j \le n$$

where each r_j is an integer and each c_j is analytic and non-zero at $\varepsilon = 0$. Define

$$r = \min_{0 \le j \le n} r_j$$

When (3.0.1) is divided by $\varepsilon^r c_n(\varepsilon)$, the resulting polynomial has the above properties. (3.0.1) becomes

$$\varepsilon^t s^n + e_{n-1}(\varepsilon)\, s^{n-1} + \ldots + e_0(\varepsilon) = 0 \qquad (3.0.3)$$

where $e_j(\varepsilon) = \dfrac{a_j(\varepsilon)}{\varepsilon^r c_n(\varepsilon)}$ and

t is a non-negative integer

Up to this point, the roots are not changed. Now the substitution $p = \varepsilon^t s$ is made in (3.0.3) and the result is multiplied by $\varepsilon^{(n-1)t}$:

$$p^n + e_{n-1}(\varepsilon)\, p^{n-1} + \varepsilon^t e_{n-2}(\varepsilon)\, p^{n-2} + \ldots +$$
$$\varepsilon^{(n-1)t} e_0(\varepsilon) = 0 \qquad (3.0.4)$$

Theorem 4.12 of [20] shows that the roots of (3.0.4) can be expanded:

$$p_k(\varepsilon) = \sum_{i=0}^{\infty} \alpha_{ki} \cdot \varepsilon^{i/q},\ 1 \le k \le n$$

The roots of (3.0.1) can now be written

$$s_k(\varepsilon) = \frac{1}{\varepsilon^t} \sum_{i=0}^{\infty} \alpha_{ki} \, \varepsilon^{i/q} = \sum_{i=0}^{\infty} \alpha_{ki} \, \varepsilon^{(i-tq)/q}$$

Changing the index of summation gives (3.0.2). The last remark in the theorem is merely a statement that all branches of the q^{th} root appear. □

The case when the coefficients in (3.0.1) are rational is treated in most standard texts on complex variables under the topic of algebraic functions [e.g.,19, or the treatise 22]. (3.0.1) as stated defines s as an algebroidal function of s, which behaves locally as an algebraic function.

It is assumed that the slow (or low frequency) and the fast (or high frequency) behaviors are each described by transfer matrices which are independent of ε. Following the usual time domain treatment, a scaling ratio of $p = \varepsilon s$ is assumed. The next definition is made in view of these observations.

<u>Definition 3.0.1</u>: A matrix $H(s,\varepsilon)$ rational in s over the field $\mathscr{F}_\varepsilon$ is two frequency scale if:

1. $H(s,\varepsilon)$ is proper in s;
2. $H(s,0)$ is defined and proper; (3.0.5)
3. $H(\frac{p}{\varepsilon}, \varepsilon) \Big|_{\varepsilon = 0}$ is defined and proper; (3.0.6)
4. The expansions (3.0.1) of each of the poles of $H(s,\varepsilon)$ about $\varepsilon = 0$ take one of the following special forms:

$$\text{A.} \quad s_p(\varepsilon) = \sum_{j=0}^{\infty} b_j \, \varepsilon^{j/q}; \tag{3.0.7}$$

$$\text{B.} \quad s_p(\varepsilon) = \frac{1}{\varepsilon} \sum_{j=0}^{\infty} b_j \, \varepsilon^{j/q}, \ b_o \neq 0. \tag{3.0.8}$$

Note that each term of $H(s,\varepsilon)$ or $H(\frac{p}{\varepsilon}, \varepsilon)$ can be expressed as the ratio of two polynomials in s whose coefficients are analytic at $\varepsilon = 0$ by multiplying numerator and denominator by a suitable power of ε. When expressed in this way, the numerator and denominator are defined at $\varepsilon = 0$. Now, only indeterminate forms of the 0/0 type can occur when the evaluations (3.0.5) and (3.0.6) are made. These can always be resolved, however, by dividing both numerator and denominator by a suitable power of ε while maintaining analytic coefficients.

Although part 4 of Definition 3.0.1 may seem complicated, it is quite easy to verify once the characteristic polynomial $q(s,\varepsilon)$ of $H(s,\varepsilon)$ is known. By definition, ε can be set to zero in $q(s,\varepsilon)$ so that $q(s,0)$ is defined and $q(s,0) \not\equiv 0$. Let r be the smallest integer for which all coefficients of the polynomial (in p) $\varepsilon^r q(\frac{p}{\varepsilon},\varepsilon)$ are analytic at $\varepsilon = 0$. Then part 4 of Definition 3.1 is equivalent to (3.0.9) where L is the number of non-zero roots of $\varepsilon^r q(\frac{p}{\varepsilon}, \varepsilon)\Big|_{\varepsilon = 0}$

$$\deg q(s,\varepsilon) = \deg q(s,0) + L \tag{3.0.9}$$

The notation $H_S(s) = H(s,0)$ and $H_F(p) = H(\frac{p}{\varepsilon}, \varepsilon)\Big|_{\varepsilon = 0}$ is introduced for convenience. $H_S(s)$ and $H_F(p)$ can be interpreted as descriptions of the system at low and high frequencies respectively.

The following simple examples illustrate these concepts.

Example 3.0.1: Let $h(s,\varepsilon) = \frac{s+1}{(s+2)(\varepsilon s+1)}$.

$$\text{Then } h_S(s) = h(s,0) = \frac{s+1}{s+2}, \text{ and}$$

$$h_F(p) = h(\tfrac{p}{\varepsilon}, \varepsilon)\Big|_{\varepsilon=0} = \frac{\frac{p}{\varepsilon}+1}{(\frac{p}{\varepsilon}+2)(p+1)}\Bigg|_{\varepsilon=0}$$

$$= \frac{1}{p+1}$$

Example 3.0.2: Let $h(s,\varepsilon) = \frac{1}{(s+1)(\varepsilon s+1)}$.

Then $h_S(s) = \frac{1}{s+1}$, and

$$h_F(p) = \frac{\varepsilon}{(p+\varepsilon)(p+1)}\Bigg|_{\varepsilon=0} = 0$$

Example 3.0.3: Suppose a transfer matrix has characteristic polynomial

$$\varepsilon s^2 + s + 1 = 0 \qquad (3.0.9)$$

Letting $\varepsilon = 0$, we get

$$s + 1 = 0 \qquad (3.0.10)$$

Substituting $p = \varepsilon s$ and clearing,

$$p^2 + p + \varepsilon = 0.$$

Again, setting $\varepsilon = 0$,

$$p(p + 1) = 0 \qquad (3.0.11)$$

Thus, there is a pole which is $O(1)$ as $\varepsilon \to 0$ and a pole $O(1/\varepsilon)$ as $\varepsilon \to 0$ from (3.0.10) and (3.0.11) respectively. This shows that the polynomial (3.0.9) satisfies part 4 of Definition 3.0.1.

III.1. Algebraic Form of a Two Frequency Scale Rational Matrix

In this section, it is shown that a necessary and sufficient condition for a rational matrix $H(s,\varepsilon)$ to be two frequency scale is that its elements satisfy a certain algebraic form. This is accomplished by first finding a form for polynomials whose roots obey property 4 of Definition 3.0.1.

Lemma 3.1.1: Suppose that $q(s,\varepsilon)$ is a polynomial in s with coefficients analytic at $\varepsilon = 0$. Then q has roots obeying property 4 of Definition 3.0.1 if and only if

$$q(s,\varepsilon) = \varepsilon^{\alpha}[d_1(s,\varepsilon) + s^K d_2(\varepsilon s,\varepsilon)]$$

where

1. $K = \deg d_1(s,\varepsilon) = \deg d_1(s,0)$;
2. $\deg d_2(p,0) = \deg d_2(p,\varepsilon)$;
3. the constant term of d_2 is zero;
4. d_1 and d_2 have coefficients analytic at $\varepsilon = 0$;
5. α is an integer ≥ 0.

Proof: If $q(s,0) \equiv 0$, then each coefficient of $q(s,\varepsilon)$ has zero as the constant term in its power series in ε, and each coefficient can be divided by some power of ε while remaining analytic. This shows that

$q(s,\varepsilon) = \varepsilon^{\alpha} q_1(s,\varepsilon)$ where $q_1(s,0) \not\equiv 0$ and α is some integer ≥ 0. Let $q(s,\varepsilon)$ have roots obeying property 4 of Definition 3.0.1. Since $q_1(s,\varepsilon)$ has the same roots as $q(s,\varepsilon)$,

$$q_1(s,\varepsilon) = f(\varepsilon) \prod_{i=1}^{K} (s - a_i(\varepsilon)) \prod_{j=1}^{L} (\varepsilon s - b_j(\varepsilon)) \tag{3.1.1}$$

where the $a_i(\varepsilon)$ and $b_j(\varepsilon)$ may have algebraic singularities at $\varepsilon = 0$, but approach finite limits as $\varepsilon \to 0$. Matching of leading coefficients in Equation (3.1.1) shows that $f(\varepsilon) \in \mathscr{F}_{\varepsilon}$. Since $q_1(s,0) \not\equiv 0$, $f(0) \neq 0$. Matching coefficients of s^K and noting that all $b_j(0)$ are non-zero shows that $f(0) \neq \infty$. Therefore, $f(\varepsilon)$ is analytic and non-zero at $\varepsilon = 0$.

$$\text{Let } q_1(s,\varepsilon) = \sum_{i=0}^{K+L} C_i(\varepsilon) s^i \tag{3.1.2}$$

Setting $\varepsilon = 0$ in (3.1.1) yields

$$q_1(s,0) = f(0) \prod_{j=1}^{L} (-b_j(0)) \prod_{i=1}^{K} (s - a_i(0))$$

This polynomial has degree K and it is evident that $C_K(0) = f(0) \prod_{j=1}^{L} (-b_j(0)) \neq 0$.

Define $d_1(s,\varepsilon) = \sum_{i=0}^{K} C_i(\varepsilon) s^i$

Using (3.1.1)

$$\varepsilon^K q_1(\frac{p}{\varepsilon}, \varepsilon) = f(\varepsilon) \prod_{i=1}^{K} (p - \varepsilon a_i(\varepsilon)) .$$

$$\prod_{j=1}^{L} (p - b_j(\varepsilon)) \tag{3.1.3}$$

From (3.1.2),

$$\varepsilon^K q_1(\frac{p}{\varepsilon}, \varepsilon) = \sum_{i=0}^{K} \varepsilon^{K-1} C_k(\varepsilon) p^i +$$

$$\sum_{j=K+1}^{K+L} \frac{C_j(\varepsilon)}{\varepsilon^{j-K}} p^j \tag{3.1.4}$$

Evaluting (3.1.3) and (3.1.4) at $\varepsilon = 0$ gives

$$f(0)p^K \cdot \prod_{j=1}^{L} (p - b_j(0)) = C_K(0)p^K +$$

$$p^K \sum_{i=1}^{L} \left. \frac{C_{i+K}(\varepsilon)}{\varepsilon^i} \right|_{\varepsilon = 0} p^i$$

This shows that $\left. \frac{C_{i+K}(\varepsilon)}{\varepsilon^i} \right|_{\varepsilon = 0}$ is finite for $1 \leq i \leq L$, and $\left. \frac{C_{K+L}(\varepsilon)}{\varepsilon^L} \right|_{\varepsilon = 0} \neq 0$.

Then $C_{i+k}(\varepsilon) = \varepsilon^i e_i(\varepsilon)$ for $1 \leq i \leq L$ with $e_i(\varepsilon)$ analytic at $\varepsilon = 0$, and $e_L(0) \neq 0$.

$$\begin{aligned} q_i(s, \varepsilon) &= \sum_{i=0}^{K} C_i(\varepsilon) s^i + \sum_{j=1+K}^{L+K} C_j(\varepsilon) s^j \\ &= d_1(s, \varepsilon) + \sum_{j=1}^{L} e_j(\varepsilon)(\varepsilon s)^j s^K \\ &= d_1(s, \varepsilon) + s^K d_2(\varepsilon s, \varepsilon) \end{aligned}$$

On the other hand, if $q_1(s,\varepsilon) = d_1(s,\varepsilon) + s^K d_2(\varepsilon s,\varepsilon)$ then $q(s,0) = d_1(s,0)$ has K roots, so that K of the expansions will have no negative powers of ε. Therefore, K of the expansions will be of form A of Definition 3.0.1 part 4.

If $p = \varepsilon s$ is substituted into q,

$$\varepsilon^K \cdot q_1(\tfrac{p}{\varepsilon},\varepsilon) = \varepsilon^K d_1(\tfrac{p}{\varepsilon},\varepsilon) + p^K d_2(p,\varepsilon) \tag{3.1.5}$$

Evaluating at $\varepsilon = 0$,

$$\varepsilon^K q_1(\tfrac{p}{\varepsilon},\varepsilon)\Big|_{\varepsilon=0} = [C_K(0) + d_2(p,0)]p^K \tag{3.1.6}$$

The K zero roots of (3.1.6) correspond to the finite roots of $d_1(s,0)$. The L non-zero roots of (3.1.6) must have expansions of type A of Definition 3.0.1 part 4, with the additional constraint that they approach non-zero limits as $\varepsilon \to 0$. But the roots of (3.1.5) are scaled by ε, so the roots of $q_1(s,\varepsilon)$ have expansions of type B of Definition 3.0.1 part 4.

In the following discussions, it will be assumed that in each term $h_{ij}(s,\varepsilon)$ of $H(s,\varepsilon)$, the numerator and denominator are coprime in s.

<u>Theorem 3.1.1</u>: $H(s,\varepsilon)$ is two frequency scale if and only if each term can be expanded as

$$h_{ij}(s,\varepsilon) = \frac{n_{1ij}(s,\varepsilon) + s^K n_{2ij}(\varepsilon s,\varepsilon)}{d_{1ij}(s,\varepsilon) + s^K d_{2ij}(\varepsilon s,\varepsilon)}$$

where

1. n_{1ij}, n_{2ij}, d_{1ij}, and d_{2ij} are polynomial with coefficients analytic in ε at $\varepsilon = 0$;
2. $\deg d_{1ij}(s,\varepsilon) = \deg d_{1ij}(s,0) = K$;
3. $\deg d_{2ij}(p,\varepsilon) = \deg d_{2ij}(p,0)$;
4. $\deg n_{1ij} \le K$;
5. $\deg n_{2ij} \le \deg d_{2ij}$;
6. The constant terms of n_{2ij} and d_{2ij} are both zero.

Proof: Let each term $h_{ij}(s,\varepsilon)$ of $H(s,\varepsilon)$ be expressed as $n_{ij}(s,\varepsilon)/d_{ij}(s,\varepsilon)$. Then each denominator d_{ij} divides the characteristic polynomial. Thus, if the characteristic polynomial of $H(s,\varepsilon)$ satisfies the conditions of Lemma 3.1.1, then so must each d_{ij}. Conversely, suppose that each d_{ij} satisfies the conditions of Lemma 3.1.1. Let $M(s,\varepsilon) = n_M(s,\varepsilon)/d_M(s,\varepsilon)$ be a minor of $H(s,\varepsilon)$ computed in the following manner: all products are performed without cancellation and sums are computed by cross-multiplication. Thus, $d_M(s,\varepsilon)$ is the product of some of the $d_{ij}(s,\varepsilon)$. Clearly, $d_M(s,\varepsilon)$ satisfies the conditions of Lemma 3.1.1. Let $M(s,\varepsilon) = n_M'(s,\varepsilon)/d_M'(s,\varepsilon)$ be the result of performing all possible numerator and denominator cancellations. It can be shown from the last statement of Theorem 3.0.1 that $d_M'(s,\varepsilon)$ can be chosen to have coefficients analytic in ε at $\varepsilon = 0$. This shows that $d_M'(s,\varepsilon)$ satisfies the conditions of Lemma 3.1.1. Since the characteristic polynomial is found by taking the least common multiple of all such polynomials, it follows (by similar arguments) that the characteristic polynomial also satisfies the conditions of Lemma 3.1.1. Thus, $H(s,\varepsilon)$ is two frequency scale if and only if each $h_{ij}(s,\varepsilon)$ is so, and it is enough to prove the theorem for the scalar case.

$$(\rightarrow) \text{ Let } h_{ij}(s,\varepsilon) = \frac{n(s,\varepsilon)}{d(s,\varepsilon)}$$

$$= \frac{\varepsilon^{\beta}\hat{n}(s,\varepsilon)}{\varepsilon^{\alpha}\hat{d}(s,\varepsilon)},\quad \hat{n}(s,0) \not\equiv 0 \not\equiv \hat{d}(s,0)$$

$$= \frac{\varepsilon^{\beta-\alpha}\hat{n}(s,\varepsilon)}{\hat{d}(s,\varepsilon)}$$

$\beta - \alpha \geq 0$ since $h_{ij}(s,0)$ is defined.

Therefore, $h_{ij}(s,\varepsilon) = \dfrac{m(s,\varepsilon)}{d_1(s,\varepsilon) + s^K d_2(\varepsilon s,\varepsilon)}$

where $m(s,\varepsilon)$ has coefficients analytic in ε.

Properness of $H(s,\varepsilon)$ implies

$$m(s,\varepsilon) = \sum_{i=0}^{K+L} l_i(\varepsilon)s^i$$

$$h_{ij}\left(\frac{p}{\varepsilon},\varepsilon\right) = \frac{\sum_{i=0}^{K+L} l_i(\varepsilon)\varepsilon^{K-i}p^i}{\varepsilon^K d_1\left(\frac{p}{\varepsilon},\varepsilon\right) + p^K d_2(p,\varepsilon)}$$

Setting $\varepsilon = 0$, and using the notation $C_i(\varepsilon)$ from the previous lemma,

$$h_{ij}\left(\frac{p}{\varepsilon},\varepsilon\right)\bigg|_{\varepsilon=0} = \frac{\sum_{i=0}^{K+L} l_i(\varepsilon)\varepsilon^{K-i}\Big|_{\varepsilon=0} p^i}{p^K[C_K(0) + d_2(p,0)]}$$

$$= \frac{\displaystyle\sum_{i=K+1}^{K+L} \left.\frac{l_i(\varepsilon)}{\varepsilon^{i-K}}\right|_{\varepsilon=0} p^i + l_K(0)p^K}{p^K[C_K(0) + d_2(p,0)]}$$

$$= \frac{\displaystyle\sum_{j=1}^{L} \left.\frac{l_{j+K}(\varepsilon)}{\varepsilon^j}\right|_{\varepsilon=0} p^j + l_K(0)}{C_K(0) + d_2(p,0)}$$

For this to be defined,

$$l_{j+K}(\varepsilon) = \varepsilon^j f_j(\varepsilon), \quad 1 \le j \le L$$

with f_j analytic at $\varepsilon = 0$.

The proof is completed by setting

$$n_{1ij}(s,\varepsilon) = \sum_{i=0}^{K} l_i(\varepsilon)s^i, \qquad d_{1ij} = d_1$$

$$n_{2ij}(\varepsilon s,\varepsilon) = \sum_{j=1}^{L} f_j(\varepsilon)(\varepsilon s)^j, \qquad d_{2ij} = d_2$$

($\leftarrow$) Inspection shows that $h_{ij}(s,\varepsilon)$ is proper.

$$h_{ij}(s,0) = \frac{n_{1ij}(s,0)}{d_{1ij}(s,0)} \tag{3.1.7}$$

This is defined and proper.

$$h_{ij}(\tfrac{p}{\varepsilon},\varepsilon)\Big|_{\varepsilon=0} = \left.\frac{\varepsilon^K n_{1ij}(\frac{p}{\varepsilon},\varepsilon) + p^K n_{2ij}(p,\varepsilon)}{\varepsilon^K d_{1ij}(\frac{p}{\varepsilon},\varepsilon) + p^K d_{2ij}(p,\varepsilon)}\right|_{\varepsilon=0}$$

$$= \frac{\chi + n_{2ij}(p.0)}{C_K(0) + d_{2ij}(p,0)} \tag{3.1.8}$$

where χ is the coefficient of degree K in $n_{1ij}(s,0)$. This is also defined and proper.

Examination of (3.1.7) and (3.1.8) shows the following corollary.

Corollary 3.1.1: If $H(s,\varepsilon)$ is two frequency scale, then

$H_S(\infty) = H_F(0)$ and

$$[H(\tfrac{p}{\varepsilon},\varepsilon)\,\big|_{p=\infty}]\big|_{\varepsilon=0} = H_F(p)\,\big|_{p=\infty}$$

III.2. Exact Frequency Scale Decomposition

This section presents a frequency scale decomposition analogous to (1.4). It can be viewed as a partial Laplace expansion of a transfer matrix $H(s,\varepsilon)$. Most of the complications associated with the complete Laplace expansion are avoided here because the denominators in the decomposition are coprime. The first step is to show that the denominators of the terms of a two frequency scale transfer matrix can be factorized into slow and fast parts, with each of the parts having coefficients in $\mathcal{F}_\varepsilon$.

Lemma 3.2.1: Let $d(s,\varepsilon)$ be a polynomial whose coefficients are analytic in ε at $\varepsilon = 0$. Suppose that the roots of $d(s,\varepsilon)$ obey property 4 of Definition 3.0.1. Then $d(s,\varepsilon)$ has a unique factorization.

$$d(s,\varepsilon) = f(\varepsilon)\cdot d_S(s,\varepsilon)\cdot d_F(\varepsilon s,\varepsilon)$$

where 1. $\deg d(s,\varepsilon) = \deg d_S(s,0) + \deg d_F(p,0)$

2. $d_F(0,0) \neq 0$

3. all of the following are analytic at $\varepsilon = 0$:

f, the coefficients of $d_S(s,\varepsilon)$, and the coefficients of $d_F(p,\varepsilon)$

4. d_S and d_F are monic

Proof: $d(s,\varepsilon)$ can be factorized as in the proof of Lemma 3.1.1:

$$d(s,\varepsilon) = f(\varepsilon) \cdot \prod_{i=1}^{K} (s - \alpha_i(\varepsilon)) \cdot \prod_{j=1}^{L} (\varepsilon s - \beta_j(\varepsilon)) \qquad (3.2.1)$$

where each $\alpha_i(\varepsilon)$ and $\beta_j(\varepsilon)$ has an expansion of the type (3.0.7), with each $\beta_j(0) \neq 0$. If we set

$$d_S(s,\varepsilon) \triangleq \prod_{i=1}^{K} (s - \alpha_i(\varepsilon)) \qquad (3.2.2)$$

$$d_F(p,\varepsilon) \triangleq \prod_{j=1}^{L} (p - \beta_j(\varepsilon)) \qquad (3.2.3)$$

then all points of the lemma are easily varified except the statement concerning $d_S(s,\varepsilon)$ and $d_F(p,\varepsilon)$ in part 3. Note that uniqueness follows because no other grouping of the linear terms (in s) of (3.2.1) into two groups can satisfy all of the other three properties simultaneously. What remains to be proved is that the products (3.2.2) and (3.2.3) have coefficients analytic at $\varepsilon = 0$. To show this, we first note that the product of two or more polynomials with analytic coefficients is a polynomial with analytic coefficients. Then we show that the product of all the linear terms corresponding to each "q^{th} root group" mentioned in Theorem 3.0.1 yields a polynomial with coefficients analytic at $\varepsilon = 0$.

The proof of the above statement, although simple in principle, is somewhat tedious. We show that only integral powers of s remain after each "q^{th} root group" is multiplied out. Let q of the roots of $d_S(s,\varepsilon)$ or $d_F(p,\varepsilon)$ have expansions (taking here the d_S case):

$$s_i(\varepsilon) = \sum_{j=0}^{\infty} b_j \, \varepsilon_{(i)}^{j/q}, \quad 1 \le i \le q$$
$$= \emptyset(\varepsilon_{(i)}^{1/q})$$

where $\emptyset\,(\cdot)$ is analytic at 0.

Here, there are q different functions of ε, each defined by taking a different branch of $\varepsilon^{1/q}$. This is indicated by the subscript (i) above. Define

$$W_i(\varepsilon) = \varepsilon^{1/q} e^{\frac{j2\pi i}{q}}, \quad 1 \le i \le q$$

The "q^{th} root group" product is:

$$\prod_{i=1}^{q} (s - \emptyset(W_i(\varepsilon))) = s^q + p_1 s^{q-1} + \dots + p_q$$

where $p_1 = \sum_{i=1}^{q} \emptyset(W_i(\varepsilon))$

$$p_2 = \sum_{i,\ j\neq} \emptyset(W_i(\varepsilon))\ \emptyset(W_j(\varepsilon))$$

$$p_3 = \sum_{i,j,k\neq} \emptyset(W_i(\varepsilon))\ \emptyset(W_j(\varepsilon))\ \emptyset(W_k(\varepsilon))$$

$$\vdots$$

$$p_q = \prod_{i=1}^{q} \emptyset(W_i(\varepsilon))$$

We first claim that

$$\sum_{i_1, \ldots, i_n \neq} e^{j \frac{2\pi(i_1 K_1 + \ldots + i_n K_n)}{q}} = 0, \quad 1 \le n \le q \tag{3.2.4}$$

when $q \nmid K_1 + K_2 + \ldots + K_q$

The summation of (3.2.4) is taken over all possible combinations of the indices i_j, $1 \le j \le n$ with $1 \le i_j \le q$ for which no indices are repeated. Thus, (3.2.4) has $\binom{q}{n}$ terms. Proof of this claim is by induction on n with q fixed.

n = 1 case: $\sum_{i=1}^{q} e^{j \frac{2\pi i K}{q}}$ is seen to be the discrete Fourier transform of the constant function 1, evaluated at the frequency variable K. It is well known that this is zero if $q \nmid K$.

Suppose now that the claim is true for n = 1, 2, ..., n-1. Assume that $q \nmid K_1 + K_2 + \ldots + K_n$. The "missing" terms can be added to (3.2.4):

$$\sum_{i_1=1}^{q} \cdots \sum_{i_n=1}^{q} e^{j \frac{2\pi(i_1 K_1 + \ldots + i_n K_n)}{q}} \tag{3.2.5}$$

The difference between (3.2.5) and (3.2.4) consists of the terms:

$\binom{n}{2}$ summations with 2 of $i_1, \ldots, i_n$ set equal

$\binom{n}{3}$ summations with 3 of $i_1, \ldots, i_n$ set equal

$\vdots$

$\binom{n}{n}$ summations with all of $i_1, \ldots, i_n$ set equal

All of these summations are zero by the previous cases. For instance, the last sum is zero by the $n = 1$ case. (3.2.5) is the n-dimensional discrete Fourier transform of the constant function 1 evaluated at the frequency variables K_1, ..., K_n. As in the $n = 1$ case, this is zero. This proves the claim (3.2.4). The coefficient p_n can be written

$$p_n(\varepsilon) = \sum_{i_1, \ldots, i_n \neq} \emptyset(W_{i_1}(\varepsilon)) \ldots \emptyset(W_{i_n}(\varepsilon))$$

$$= \sum_{K_1=0}^{\infty} \ldots \sum_{K_n=0}^{\infty} \left\{ b_{K_1} \ldots b_{K_n} \varepsilon^{\frac{K_1 + K_2 + \ldots K_{qn}}{q}} \cdot \sum_{i_1, \ldots, i_n \neq} e^{j\frac{2\pi(i_1K_1 + \ldots + i_nK_n)}{q}} \right\}$$

(3.2.4) shows that all fractional powers of ε have zero coefficients. Therefore, each p_n is analytic in ε. □

The next series of lemmas provides a way of finding numerators for the previously mentioned partial Laplace expansion, once the denominator has been factorized. All polynomials are over F in Lemmas 3.2.2, 3.2.3, and 3.2.4.

Lemma 3.2.2: Suppose a, b, x and y are non-zero polynomials such that

$$ax + by = 0 \qquad (3.2.6)$$

with deg x < deg b and deg y < deg a.

Then a and b are not coprime.

Proof: If (3.2.6) is written $ax = -by$, then the lemma easily follows by cancellation of prime factors on both sides. □

<u>Lemma 3.2.3</u>: Let $h = \frac{n}{d_1 \cdot d_2}$ be strictly proper with d_1, d_2, and n polynomials, and d_1 and d_2 are coprime. Suppose h has a strictly proper decomposition:

$$h = \frac{a}{d_1} + \frac{b}{d_2}$$

That is, deg a < deg d_1 and deg b < deg d_2. Then a and b are unique.

<u>Proof</u>: Suppose $h = \frac{\hat{a}}{d_1} + \frac{\hat{b}}{d_2}$ where $\hat{a} \neq a$ or $\hat{b} \neq b$. Then $\frac{a-\hat{a}}{d_1} + \frac{b-\hat{b}}{d_2} = 0$; and this gives $(a-\hat{a})d_2 + (b-\hat{b})d_1 = 0$ If $a = \hat{a}$, then $b = \hat{b}$ and the lemma follows; and similarly in the case that $b = \hat{b}$. If $a \neq \hat{a}$ and $b \neq \hat{b}$, Lemma 3.2.2 shows that d_1 and d_2 are not coprime. The lemma follows from this contradiction. □

<u>Lemma 3.2.4</u>: Let $h = \frac{n}{d_1 d_2}$ be strictly proper with d_1, d_2, and n polynomials, and d_1 and d_2 coprime. Then there exist r_1 and r_2 such that

$$h = \frac{r_1}{d_1} + \frac{r_2}{d_2}$$

with deg r_1 < deg d_1 and deg r_2 < deg d_2.

<u>Proof</u>: By applying Euclid's algorithm, or as a special case of Theorem R2.6.1(iv) in Section 2.2, there exist x and y such that

$$d_1 x + d_2 y = 1$$

Then $$nd_1x + nd_2y = n$$

$$h = \frac{nd_1x + nd_2y}{d_1d_2}$$

$$= \frac{nx}{d_2} + \frac{ny}{d_1}$$

$$= \frac{q_1d_1 + r_1}{d_1} + \frac{q_2d_2 + r_2}{d_2}$$

where $ny = q_1d_1 + r_1$ and

$nx = q_2d_2 + r_2$

with $\deg r_1 < \deg d_1$ and $\deg r_2 < \deg d_2$.

h can now be written

$$h = \frac{r_1}{d_1} + \frac{r_2}{d_2} + q_1 + q_2$$

$$= \frac{r_1}{d_1} + \frac{r_2}{d_2}$$

since h is strictly proper q_1 must equal $-q_2$. □

Lemma 3.2.5: Let $h(s,\varepsilon)$ be a two frequency scale scalar. Then $h(s,\varepsilon)$ can be expressed

$$h(s,\varepsilon) = h_1(s,\varepsilon) + h_2(\varepsilon s,\varepsilon) + A(\varepsilon)$$

where 1. $h_1(s,\varepsilon)$ and $h_2(\varepsilon s,\varepsilon)$ are both strictly proper and two frequency scale;

2. The poles of $h_1(s,\varepsilon)$ and $h_2(p,\varepsilon)$ approach finite limits as $\varepsilon \to 0$ and all poles of $h_2(p,\varepsilon)$ approach non-zero limits;

3. $A(\varepsilon)$ is analytic at $\varepsilon = 0$.

Proof: By division, $h(s,\varepsilon)$ can be written

$$h(s,\varepsilon) = g(s,\varepsilon) + A(\varepsilon)$$

where g is strictly proper.

That $A(\varepsilon)$ is analytic follows from Theorem 3.1.1. Clearly, $g(s,\varepsilon)$ is two frequency scale.

Let g be expressed

$$g(s,\varepsilon) = \frac{n(s,\varepsilon)}{d(s,\varepsilon)}$$

where $0 \not\equiv d(s,0) \not\equiv \infty$

Let $\alpha_i(\varepsilon)$, $1 \le i \le K$ be the slow poles of g and let $\beta_j(\varepsilon)/\varepsilon$, $1 \le j \le L$ be the fast poles of g.

Then
$$d(s,\varepsilon) = f(\varepsilon) \cdot \prod_{i=1}^{K} (s - \alpha_i(\varepsilon)) \cdot \prod_{j=1}^{L} (\varepsilon s - \beta_j(\varepsilon)) \tag{3.2.7}$$

where f is analytic at $\varepsilon = 0$ and $f(0) \neq 0$.

Lemma 3.2.1 shows that the two products in (3.2.7) are polynomials with analytic coefficients.

Define:
$$d_1(s,\varepsilon) = f(\varepsilon) \cdot \prod_{i=1}^{K} (s - \alpha_i(\varepsilon))$$
$$d_2(p,\varepsilon) = \prod_{j=1}^{L} (p - \beta_j(\varepsilon))$$

Properties of d_1 and d_2 include: $d_1(s,0) \not\equiv 0$, $d_2(0,0) \neq 0$; $d_1(s,\varepsilon)$ and $d_2(\varepsilon s,\varepsilon)$ are coprime; and $\deg d_1(s,0) = \deg d_1(s,\varepsilon)$. These observations show property 2 of the lemma. d_1, d_2, and n satisfy the hypothesis of Lemma 3.2.4 so that

$$g(s,\varepsilon) = \frac{r_1(s,\varepsilon)}{d_1(s,\varepsilon)} + \frac{r_2(s,\varepsilon)}{d_2(\varepsilon s,\varepsilon)}$$

The next step is to show that the coefficients of r_1 and r_2 are analytic--we know only that they are in $\mathcal{F}_\varepsilon$ at this point. Expand r_1 and r_2 in powers of ε:

$$r_1(s,\varepsilon) = \sum_{i=i_o}^{\infty} a_i(s)\varepsilon^i$$

$$r_2(s,\varepsilon) = \sum_{j=j_o}^{\infty} b_j(s)\varepsilon^j$$

The a_i's and b_j's are polynomials in s with

$$\deg a_i(s) < \deg d_1(s,\varepsilon),\ i = i_o,\ i_o + 1,\ \ldots$$

$$\deg b_j(s) < \deg d_2(p,\varepsilon),\ j = j_o,\ j_o + 1,\ \ldots$$

Let $K_o = \min(i_o, j_o)$ so that K_o is the smallest integer for which $a_{K_o}(s) \neq 0$ or $b_{K_o}(s) \neq 0$. g can now be expressed

$$g(s,\varepsilon) = \sum_{i=K_o}^{\infty} \left\{\frac{a_i(s)}{d_1(s,\varepsilon)} + \frac{b_i(s)}{d_2(\varepsilon s,\varepsilon)}\right\} \varepsilon^i$$

The quantity in braces is analytic at $\varepsilon = 0$ and has a power series expansion. Thus, $g(s,\varepsilon)$ has a Laurent series expansion

$$g(s,\varepsilon) = \left\{\frac{a_{K_o}(s)}{d_1(s,0)} + \frac{b_{K_o}(s)}{d_2(0,0)}\right\} \varepsilon^{K_o} + \sum_{i=K_o+1}^{\infty} g_i(s)\varepsilon^i$$

Suppose that $K_o < 0$. Since $g(s,0)$ is defined,

$$\frac{a_{K_o}(s)}{d_1(s,0)} + \frac{b_{K_o}(s)}{d_2(0,0)} \equiv 0$$

By letting $s \to \infty$ and observing that a_{K_o}/d_1 is strictly proper, we must have

$$b_{K_o}(s) \equiv 0$$

It then follows that $a_{K_o}(s) \equiv 0$. But this is a contradiction since either a_{K_o} or b_{K_o} was assumed to be non-zero. Therefore, $K_o \geq 0$ and it has been shown that r_1 and r_2 have coefficients which are analytic at $\varepsilon = 0$.

Only the dependence of $h_2(p,\varepsilon)$ on ε remains to be shown. Substituting $p = \varepsilon s$,

$$g(\tfrac{p}{\varepsilon},\varepsilon) = \frac{r_1(\frac{p}{\varepsilon},\varepsilon)}{d_1(\frac{p}{\varepsilon},\varepsilon)} + \frac{r_2(\frac{p}{\varepsilon},\varepsilon)}{d_2(p,\varepsilon)}$$

$$= \frac{\varepsilon^K r_1(\frac{p}{\varepsilon},\varepsilon)}{\varepsilon^K d_1(\frac{p}{\varepsilon},\varepsilon)} + \frac{r_2(\frac{p}{\varepsilon},\varepsilon)}{d_2(p,\varepsilon)}$$

Setting ε to zero,

$$g(\tfrac{p}{\varepsilon},\varepsilon)\Big|_{\varepsilon=0} = \frac{r_2(\frac{p}{\varepsilon},\varepsilon)}{d_2(p,\varepsilon)}\Bigg|_{\varepsilon=0} \qquad (3.2.8)$$

Since (3.2.8) must be defined, $r_2(p/\varepsilon,\varepsilon)$ must be a polynomial in p with analytic coefficients.

Let $r_2(\frac{p}{\varepsilon},\varepsilon) = \hat{r}_2(p,\varepsilon)$.

The lemma follows when we set

$$h_1(s,\varepsilon) \triangleq \frac{r_1(s,\varepsilon)}{d_1(s,\varepsilon)}$$

$$h_2(p,\varepsilon) \triangleq \frac{\hat{r}_2(p,\varepsilon)}{d_2(p,\varepsilon)} \qquad \square$$

Theorem 3.2.1: Let $H(s,\varepsilon)$ be a two frequency scale transfer matrix. Then $H(s,\varepsilon)$ can be expressed

$$H(s,\varepsilon) = H_1(s,\varepsilon) + H_2(\varepsilon s,\varepsilon) + A(\varepsilon)$$

where 1. $H_1(s,\varepsilon)$ and $H_2(\varepsilon s,\varepsilon)$ are both strictly proper and two frequency scale;

2. The poles of $H_1(s,\varepsilon)$ and $H_2(p,\varepsilon)$ approach finite limits as $\varepsilon \to 0$ and all poles of $H_2(p,0)$ approach non-zero limits;

3. $A(\varepsilon)$ is analytic at $\varepsilon = 0$.

Proof: Apply Lemma 3.2.5 term by term. We have used the fact a pole of a rational matrix must be a pole of at least one of its terms. $\square$

Two simple examples are now given to illustrate this theorem. They demonstrate that the major step is factorization of the denominator. The computation of the numerators in Lemma 3.2.4 is mainly for theoretical purposes. In practice it is more efficient to set up equations for the coefficients of r_1 and r_2 directly.

Example 3.2.1:

$$
\begin{aligned}
h(s,\varepsilon) &= \frac{1}{(s+1)(\varepsilon s+1)} \\
&= \frac{\frac{1}{1-\varepsilon}}{s+1} + \frac{\frac{\varepsilon}{\varepsilon-1}}{\varepsilon s+1}
\end{aligned}
$$

Example 3.2.2:

$$
\begin{aligned}
h(s,\varepsilon) &= \frac{1}{\varepsilon s^2+s+1} \\
&= \frac{\frac{1}{\sqrt{1-4\varepsilon}}}{s+\frac{1-\sqrt{1-4\varepsilon}}{2\varepsilon}} + \frac{\frac{-\varepsilon}{\sqrt{1-4\varepsilon}}}{\varepsilon s+\frac{1+\sqrt{1-4\varepsilon}}{2}}
\end{aligned}
$$

III.3. System Matrices for Two Frequency Scale Transfer Matrices

This section specializes the system matrix approach of Chapter II to the case of two frequency scale transfer matrices. This will aid in the study of closed loop systems. It will also be seen that system matrices can be used as a convenient tool for evaluating the slow and fast descriptions $H_S(s)$ and $H_F(p)$.

Theorem 2.1 shows that a rational matrix $H(s,\varepsilon)$ over $\mathscr{F}_\varepsilon$ can be represented by a least order polynomial system matrix $P(s,\varepsilon)$ in MFD form. If any coefficients of s in $P(s,\varepsilon)$ have a pole at $\varepsilon = 0$, the columns can be multiplied by suitable powers of ε so that all coefficients are analytic at $\varepsilon = 0$. Let the system matrix with cleared columns be $P_1(s,\varepsilon)$. Even if $H(s,0)$ is defined, the matrix $P_1(s,0)$ may not be a system matrix because the upper left block of $P_1(s,0)$ may be singular. It will be convenient later to work with system matrices over $\mathscr{F}_\varepsilon$ for which ε can be set to zero. The next theorem shows how this difficulty can be alleviated.

<u>Lemma 3.3.1</u>: Let $H(s,\varepsilon)$ be a rational matrix over $\mathscr{F}_\varepsilon$ for which $H(s,0)$ is defined. Let $P(s,\varepsilon)$ be a polynomial system matrix representation for $H(s,\varepsilon)$ in MFD form with coefficients analytic at $\varepsilon = 0$. Then $P(s,\varepsilon)$ is strictly system equivalent to a system matrix $P_1(s,\varepsilon)$ in MFD form where $P_1(s,0)$ is a polynomial matrix over $\mathbb{C}$.

<u>Proof</u>:

$$\text{Let } P(s,\varepsilon) = \begin{bmatrix} I & 0 & 0 \\ 0 & T(s,\varepsilon) & I \\ 0 & -V(s,\varepsilon) & 0 \end{bmatrix} \tag{3.3.1}$$

If $\det T(s,0) = 0$, then $\det T(s,\varepsilon) = \varepsilon^r q(s,\varepsilon)$ where $q(s,\varepsilon)$ is a polynomial in s with $q(s,0) \not\equiv 0$ and $r \geq 1$. Also, if $\det T(s,0) = 0$, then the Smith form of $T(s,0)$ must have some zero diagonal element. Let $R(s)$ and $Q(s)$ be unimodular matrices over $\mathbb{C}$ which transform $T(s,0)$ to its Smith form as in (3.3.2).

$$R(s)T(s,0)Q(s) = \begin{bmatrix} S(s) & 0 \\ 0 & 0_{txt} \end{bmatrix} \tag{3.3.2}$$

Since premultiplication by $R^{-1}(s)$ can be interpreted as a sequence of row operations,

$$T(s,0)Q(s) = R^{-1}(s) \begin{bmatrix} S(s) & 0 \\ 0 & 0_{txt} \end{bmatrix} = \begin{bmatrix} * & 0 \\ * & 0_{txt} \end{bmatrix} \quad (3.3.3)$$

The *'s represent possibly non-zero polynomial entries. Consider the effect of performing the column operations of post-multiplication by Q(s) on the system matrix (3.3.1).

$$\begin{bmatrix} I & 0 & 0 \\ 0 & T(s,\varepsilon)Q(s) & I \\ 0 & -V(s,\varepsilon)Q(s) & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & \hat{T}(s,\varepsilon) & I \\ 0 & -\hat{V}(s,\varepsilon) & 0 \end{bmatrix} = \hat{P}(s,\varepsilon) \quad (3.3.4)$$

All coefficients in the last t columns of $\hat{T}(s,\varepsilon)$ are zero when $\varepsilon = 0$. The rule for computing the transfer matrix yields (3.3.5). This is expressed column by column in (3.3.6).

$$H(s,\varepsilon) = \hat{V}(s,\varepsilon)[\hat{T}(s,\varepsilon)]^{-1} \quad (3.3.5)$$

$$H(s,\varepsilon) \cdot \hat{T}_{\cdot j}(s,\varepsilon) = \hat{V}_{\cdot j}(s,\varepsilon) \quad (3.3.6)$$

Suppose that $\hat{T}_{\cdot j}(s,\varepsilon)$ above is one of the last t columns of $T(s,\varepsilon)$. Then (3.3.7) holds.

$$\hat{V}_{\cdot j}(s,0) = H(s,0) \cdot 0 = 0 \quad (3.3.7)$$

This shows that the entire j^{th} column of $\hat{P}(s,\varepsilon)$ is zero when $\varepsilon = 0$. This column can now be divided by ε while leaving all of its coefficients analytic at $\varepsilon = 0$. The operation of dividing this column by ε produces a new system matrix for which the exponent r in det $T(s,\varepsilon) = \varepsilon^r q(s,\varepsilon)$ is reduced by one.

In summary, if $P(s,\varepsilon)$ is a polynomial system matrix representation for $H(s,\varepsilon)$ in MFD form with det $T(s,\varepsilon) = \varepsilon^r q(s,\varepsilon)$, a new system matrix $P_2(s,\varepsilon)$ strictly system equivalent to $P(s,\varepsilon)$ can be found for which det $T_2(s,\varepsilon) = K\,\varepsilon^{r-1} q(s,\varepsilon)$ where K is a complex constant. This process can be repeated until a system matrix $P_1(s,\varepsilon)$ is formed for which det $T_1(s,0) \neq 0$.

If $P(s,\varepsilon)$ is a system matrix for a two frequency scale transfer matrix $H(s,\varepsilon)$, then Lemma 3.3.1 can be applied to the system matrix $P(\frac{p}{\varepsilon},\varepsilon)$ after all negative powers of are eliminated from the coefficients. The various results can now be combined.

<u>Theorem 3.3.1</u>: Let $H(s,\varepsilon)$ be a two frequency scale transfer matrix. Then $H(s,\varepsilon)$ has a least order polynomial system matrix representation $P(s,\varepsilon)$ in MFD form for which $P(s,0)$ is a polynomial system matrix over ¢. Furthermore $H(\frac{p}{\varepsilon},\varepsilon)$ also has a least order polynomial system matrix representation $\tilde{P}(p,\varepsilon)$ in MFD form for which $\tilde{P}(p,0)$ is a polynomial system matrix over ¢. □

In Theorem 3.3.1, $P(s,0)$ and $\tilde{P}(p,0)$ may not be of least order even though $P(s,\varepsilon)$ and $\tilde{P}(p,\varepsilon)$ are. It is evident that some poles of $H(s,\varepsilon)$ may be undergoing numerator-denominator or other types of cancellations as $\varepsilon \to 0$. These will be called "lost poles" of $H(s,\varepsilon)$ in accordance with the next definition. This behavior is discussed in [4] for singularly perturbed systems.

Definition 3.3.1: Let $P(s,\varepsilon)$ and $\tilde{P}(p,\varepsilon)$ be least order polynomial system matrices for a two frequency scale rational matrix $H(s,\varepsilon)$ and its scaled version $H(\frac{p}{\varepsilon},\varepsilon)$ respectively. Suppose that $P(s,0)$ and $\tilde{P}(p,0)$ are polynomial system matrices. Let $P_1(s)$ and $\tilde{P}_1(p)$ be least order polynomial system matrices equivalent to $P(s,0)$ and $\tilde{P}(p,0)$ respectively. The "lost slow poles" of $H(s,\varepsilon)$ are the roots of the "lost slow polynomial" (3.3.8).

$$q_{LS}(s) \;\alpha\; \frac{\det T(s,0)}{\det T_1(s)} \tag{3.3.8}$$

The "lost fast poles" of $H(s,\varepsilon)$ are the non-zero roots of (3.3.9).

$$p^M q_{LF}(p) \;\alpha\; \frac{\det \tilde{T}(p,0)}{\det \tilde{T}_1(p)} \tag{3.3.9}$$

The "lost fast polynomial" is a polynomial having the lost fast poles as roots. q_{LS} and q_{LF} are chosen to be monic.

Since the polynomial $\det \tilde{T}(p,\varepsilon)$ is simply a "scaled" version of $\det T(s,\varepsilon)$ and since the zeros of $\det T(s,\varepsilon) = 0$ obey property 4 of Definition 3.0.1,

$$\det \tilde{T}(p,0) \;=\; p^K \cdot \phi(p) \tag{3.3.10}$$

where $\phi(0) \neq 0$ and $K = \deg \det T(s,0)$. It can be seen from Section 2 of this chapter that $H_F(p)$ has no poles at the origin. Thus, $\det \tilde{T}_1(0) \neq 0$. This shows that $M = K$ in (3.3.11).

Alternatively, the lost slow and fast poles are the roots of (3.3.11) and the non-zero roots of (3.3.12) respectively.

$$\frac{CP[H(s,\varepsilon)]|_{\varepsilon=0}}{CP[H(s,0)]} \tag{3.3.11}$$

$$\frac{CP[H(\frac{p}{\varepsilon},\varepsilon)]|_{\varepsilon=0}}{p^K\, CP[H(\frac{p}{\varepsilon},\varepsilon)\,|_{\varepsilon=0}]} \tag{3.3.12}$$

For system matrices in MFD form, the lost slow poles are roots of the output decoupling polynomial of $P(s,0)$ and, similarly, the lost fast poles are non-zero roots of the output decoupling polynomial of $\tilde{P}(p,0)$.

The next three examples demonstrate the decomposition method on non-trivial systems. Although the use of system matrices is not necessary here, their use makes the computations simple and obvious.

<u>Example 3.3.1</u>: (Singularly Perturbed System)

The system of equations (3.3.13) determine a transfer matrix for the output y in terms of the input u, so that $Y(s) = H(s,\varepsilon)u(s)$. The matrices on the right hand side of (3.3.13) are analytic at $\varepsilon = 0$, and $\det A_{22}(0) \neq 0$.

$$
\begin{aligned}
\dot{x}_1 &= A_{11}(\varepsilon)x_1 + A_{12}(\varepsilon)x_2 + B_1(\varepsilon)u \\
\varepsilon\dot{x}_2 &= A_{21}(\varepsilon)x_1 + A_{22}(\varepsilon)x_2 + B_2(\varepsilon)u \qquad (3.3.13) \\
y &= C_1(\varepsilon)x_1 + C_2(\varepsilon)x_2 + D(\varepsilon)u
\end{aligned}
$$

A system matrix representation is

$$
\begin{bmatrix}
sI - A_{11}(\varepsilon) & -A(\varepsilon) & B_1(\varepsilon) \\
\dfrac{-A_{21}(\varepsilon)}{\varepsilon} & sI - \dfrac{A_{22}(\varepsilon)}{\varepsilon} & \dfrac{B_2(\varepsilon)}{\varepsilon} \\
-C_1(\varepsilon) & -C_2(\varepsilon) & D(\varepsilon)
\end{bmatrix}
$$

The second row can be multiplied by ε to give a system matrix for which ε can be set to zero. This yields a system matrix for the slow subsystem.

$$
\begin{bmatrix}
sI - A_{11}(0) & -A_{12}(0) & B_1(0) \\
-A_{21}(0) & -A_{22}(0) & B_2(0) \\
-C_1(0) & -C_2(0) & D(0)
\end{bmatrix}
$$

This is easily shown to give the transfer matrix (3.3.14a) with A_o, B_o, C_o and D_o defined by (3.3.15). After substituting $p = \varepsilon s$, clearing ε's from denominators, and setting $\varepsilon = 0$, a system matrix for the fast subsystem is obtained.

$$
\begin{bmatrix}
pI & 0 & 0 \\
-A_{21} & pI - A_{22} & B_2 \\
-C_1 & -C_2 & D
\end{bmatrix}
$$

This gives $H_F(p)$ as in (3.3.14b). Since $A_{22}(0)$ is non-singular, all poles are either $O(1)$ or $O(\frac{1}{\varepsilon})$ as $\varepsilon \to 0$. Thus, $H(s,\varepsilon)$ is two frequency scale. The expressions (3.3.14) are similar to those in [17]. The matrices (3.3.15) for the slow sybsystem are the same as those for the reduced subsystem in [2].

$$H_S(s) = C_o(sI - A_o)^{-1}B_o + D_o \tag{3.3.14a}$$

$$H_F(p) = C_2(0)[pI - A_{22}(0)]^{-1}B_2(0) + D(0) \tag{3.3.14b}$$

$$A_o = A_{11} - A_{12}A_{22}^{-1}A_{21}\big|_{\varepsilon=0} \tag{3.3.15}$$

$$B_o = B_1 - A_{12}A_{22}^{-1}B_2\big|_{\varepsilon=0}$$

$$C_o = C_1 - C_2A_{22}^{-1}A_{21}\big|_{\varepsilon=0}$$

$$D_o = D - C_2A_{22}^{-1}B_2\big|_{\varepsilon=0}$$

<u>Example 3.3.2</u>: (Implicit Singularly Perturbed System)

Chow [5] considers the nxn system (3.3.16) where $A_1(\varepsilon)$ remains bounded at $\varepsilon = 0$. In [5] it is shown that if A_o satisfies condition (3.3.17), then (3.3.16) can be brought to explicit singularly perturbed form (3.3.13) by a similarity transformation.

Here it will be shown that condition (3.3.17) is sufficient for the transfer matrix defined by (3.3.16) (with appropriate input and output matrices added) to satisfy condition 4 of Definition 3.0.1.

$$\varepsilon \dot{x} = (A_o + \varepsilon A_1(\varepsilon))x \qquad (3.3.16)$$

$$R(A_o) \oplus N(A_o) = R^n \qquad (3.3.17)$$

where $R(\cdot)$ and $N(\cdot)$ indicate range and null spaces respectively, and $\oplus$ indicates the direct sum of subspaces.

It is easily shown that (3.3.17) is equivalent to: There exists a non-singular matrix T such that (3.3.19) holds, where J_1 is an rxr, non-singular Jordan matrix. For simplicity, the input and output matrices will be suppressed in the discussion that follows.

$$TA_oT^{-1} = J = \mathrm{diag}(J_1, 0) \qquad (3.3.18)$$

The system matrix in the unscaled frequency variables is given by (3.3.19). It is assumed here that $A_1(\varepsilon)$ is analytic in ε at $\varepsilon = 0$.

$$P(s,\varepsilon) = sI - \frac{1}{\varepsilon} A_o - A_1(\varepsilon) \qquad (3.3.19)$$

Now the above similarity transformation is applied and the denominators are cleared of ε.

$$P_1(s,\varepsilon) = TP(s,\varepsilon)T^{-1}\,\mathrm{diag}(\underbrace{\varepsilon, \varepsilon, \ldots, \varepsilon}_{r\ \varepsilon\text{'s}}, 0, 0, \ldots, 0)$$

$$= \begin{bmatrix} \varepsilon sI - J_1 - \varepsilon B_{11}(\varepsilon) & -\varepsilon B_{12}(\varepsilon) \\ -B_{21}(\varepsilon) & sI - B_{22}(\varepsilon) \end{bmatrix}.$$

where $$B(\varepsilon) = \begin{bmatrix} B_{11}(\varepsilon) & B_{12}(\varepsilon) \\ B_{21}(\varepsilon) & B_{22}(\varepsilon) \end{bmatrix} = TA_1(\varepsilon)T^{-1}$$

The matrix $P_1(s,0)$ is a system matrix, and det $P(s,0)$ = det$(-J_1)$. det$(sI - B_{22}(0))$. Clearly, det $P(s,0)$ has degree $n - r$.

The system matrix for the scaled frequency variable is

$$\tilde{P}(p,\varepsilon) \quad = \quad pI - A_o - \varepsilon A_1$$

$\tilde{P}(p,0)$ is a system matrix and det $\tilde{P}(p,0)$ = det$(pI - A_o)$ has r non-zero roots since A_o has rank r. This shows that condition 4 of Definition 3.0.1 holds. Condition 1 holds for any choice of input and output matrices independent of s. Conditions 2 and 3 are satisfied only for certain choices of input and output matrices.

If (3.3.18) is not satisfied, then $A_1(\varepsilon)$ is important, as shown by (3.3.20) and (3.3.21).

$$\varepsilon \dot{x}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\varepsilon & 1 \\ 0 & 0 & -\varepsilon \end{bmatrix} x_1 \tag{3.3.20}$$

$$\varepsilon \dot{x}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\varepsilon & 0 \end{bmatrix} x_2 \tag{3.3.21}$$

(3.3.20) has eigenvalues -1, -1, and $-\frac{1}{\varepsilon}$, satisfying condition 4 of Definition 3.0.1. (3.3.21), however, has eigenvalues $-\frac{1}{\varepsilon}$ and $\pm j/\sqrt{\varepsilon}$.

<u>Example 3.3.3</u>: (High Frequency Oscillatory Modes)

Given in [7] (without inputs and outputs) is the second order system (3.3.22) where Q_4 has simple, positive eigenvalues.

$$\ddot{x} + B\dot{x} + Qx = Lu \tag{3.3.22}$$

$$y = Cx$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$ $Q = \begin{bmatrix} Q_1 & \frac{Q_2}{\mu^2} \\ Q_3 & \frac{Q_4}{\mu^2} \end{bmatrix}$

$C = [C_1 \quad \frac{1}{\mu^2} C_2]$ $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ $\dim x_1 = n_1$, $\dim x_2 = n_2$

The system matrix (3.3.23) can be written.

$$P(s,\mu) = \begin{bmatrix} s^2I + sB_1 + Q_1 & \mu^2 sB_2 + Q_2 & L_1 \\ sB_3 + Q_3 & \mu^2 s^2 I + \mu^2 sB_4 + Q_4 & L_2 \\ -C_1 & -C_2 & 0 \end{bmatrix} \tag{3.3.23}$$

The slow and fast characteristic polynomials are shown in (3.3.24) and (3.3.25).

$$\det T(s,0) = \det \begin{bmatrix} s^2I + sB_1 + Q_1 & Q_2 \\ sB_3 + Q_3 & Q_4 \end{bmatrix}$$

$$= \det Q_4 \cdot \det[s^2I + sB_1 + Q_1 - Q_2Q_4^{-1}(sB_3 + Q_3)] \tag{3.3.24}$$

$$\det \tilde{T}(p,0) = \det \begin{bmatrix} p^2 I & Q_2 \\ 0 & p^2 I + Q_4 \end{bmatrix}$$

$$= p^{2n_1} \cdot \det(p^2 I + Q_4) \qquad (3.3.25)$$

Again, the high frequency oscillations are obvious from (3.3.25).

IV. APPROXIMATION OF TWO FREQUENCY SCALE RATIONAL MATRICES

The main result of this chapter may be regarded as a frequency domain version of (1.6). Even though this result holds for more general cases, most practical applications will require stability conditions on the slow and fast subsystems. This parallels the stability requirements for (1.6) to hold.

Robustness and sensitivity results for linear feedback systems typically involve properties of stable rational matrices along the imaginary axis, e.g. [15], [16], [18]. The next theorem shows that under certain conditions, the values of $H_S(s)$ and $H_F(p)$ along the imaginary axis determine a uniform $O(\varepsilon)$ approximation of $H(s,\varepsilon)$ along the imaginary axis. If such a rational matrix represents a signal gain, then $H_S(j\omega)$ and $H_F(j\varepsilon\omega)$ are approximate signal gains for low and high frequency sinusoidal inputs. The reciprocal of singular value graphs used for robustness evaluation can be approximated from $H_S(s)$ and $H_F(p)$. Note, however, that the $O(\varepsilon)$ approximation may be lost here. If $\underline{\sigma}[H_S(j\omega)]$ goes to zero at some specific value ω_1 of ω, it cannot be concluded that $1/\underline{\sigma}[H(j\omega_1,\varepsilon)]$ is infinity. It can only be concluded that $1/\underline{\sigma}[H(j\omega_1,\varepsilon)]$ is large when ε is small.

<u>Lemma 4.1.1</u>: Let $h(s,\varepsilon)$ be a two frequency scale scalar rational function. Let $h(s,\varepsilon)$ be expressed as the ratio of two polynomials with analytic coefficients:

$$h(s,\varepsilon) = \frac{n(s,\varepsilon)}{d(s,\varepsilon)}$$

Let the denominator have the expansion

$$d(s,\varepsilon) = f(\varepsilon) \prod_{i=1}^{K} (s - a_i(\varepsilon)) \prod_{j=1}^{L} (\varepsilon s - b_j(\varepsilon))$$

Let D be the imaginary axis. Suppose that for all i and j, $a_i(0)$ and $b_j(0)$ are not on D. Then

$$\sup_{s\in D} \left| h(s,\varepsilon) - h_S(s) - h_F(\varepsilon s) + w \right| = 0(\varepsilon) \qquad (4.1.1)$$

where $w = h_S(\infty) = h_F(0)$ and ε is restricted to be real.

<u>Proof</u>: Let $h(s,\varepsilon)$ be expressed as guaranteed by Lemma 3.2.5.

$$h(s,\varepsilon) = h_1(s,\varepsilon) + h_2(\varepsilon s,\varepsilon) + A(\varepsilon)$$

Then $h_S(s) = h_1(s,0) + h_2(0,0) + A(0)$

$$h_F(p) = h_1(\tfrac{p}{\varepsilon},\varepsilon)\Big|_{\varepsilon=0} + h_2(p,0) + A(0)$$

$$= h_2(p,0) + A(0)$$

(4.1.1) can be written:

$$\sup_{s \in D} \left| h(s,\varepsilon) - h_S(s) - h_F(\varepsilon s) + w \right|$$

$$= \sup_{s \in D} \left| h_1(s,\varepsilon) - h_1(s,0) + h_2(\varepsilon s,\varepsilon) - h_2(\varepsilon s,0) + A(\varepsilon) - A(0) \right|$$

$$\le \sup_{s \in D} \left| h_1(s,\varepsilon) - h_1(s,0) \right| + \sup_{s \in D} \left| h_2(\varepsilon s,\varepsilon) - h_2(\varepsilon s,0) \right| + \left| A(\varepsilon) - A(0) \right|$$

Since $A(\varepsilon)$ is analytic, $A(\varepsilon) - A(0) = 0(\varepsilon)$.

Let $h_1(s,\varepsilon)$ be expressed as

$$h_1(s,\varepsilon) = \frac{n_1(s,\varepsilon)}{d_1(s,\varepsilon)}$$

where n_1 and d_1 are polynomials in s with analytic coefficients. Then

$$\sup_{s \in D} \left| h_1(s,\varepsilon) - h_1(s,0) \right| = \sup_{s \in D} \left| \frac{\alpha(s,\varepsilon)}{d_1(s,\varepsilon)\; d_1(s,0)} \right|$$

where $\alpha(s,\varepsilon) = n_1(s,\varepsilon)\; d_1(s,0) - n_1(s,0)\; d_1(s,\varepsilon)$.

Let $\deg d_1(s,\varepsilon) = K$. Then $\deg d_1(s,0) = K$ and $\deg n_1(s,\varepsilon) \le K - 1$. Thus, $\deg \alpha(s,\varepsilon) \le 2K - 1$. Since $\alpha(s,0) \equiv 0$, $\alpha(s,\varepsilon) = \varepsilon \cdot \beta(s,\varepsilon)$ where β is a polynomial in s with analytic coefficients, and having degree $\le 2K - 1$. Combining,

$$\sup_{s \in D} \left| h_1(s,\varepsilon) - h_1(s,0) \right| = |\varepsilon| \sup_{s \in D} \left| \frac{\beta(s,\varepsilon)}{d_1(s,\varepsilon) \cdot d_1(s,0)} \right|$$

The sup on the right hand side is uniformly bounded for sufficiently small ε by Corollary 4.2.2.

Proceeding in a like manner, $h_2(\varepsilon s, \varepsilon)$ can be expressed:

$$h_2(\varepsilon s, \varepsilon) = \frac{n_2(\varepsilon s, \varepsilon)}{d_2(\varepsilon s, \varepsilon)}$$

We then have

$$\sup_{s \in D} \left| h_2(\varepsilon s, \varepsilon) - h_2(\varepsilon s, 0) \right| = |\varepsilon| \cdot \sup_{s \in D} \left| \frac{\gamma(\varepsilon s, \varepsilon)}{d_2(\varepsilon s, \varepsilon) d_2(\varepsilon s, 0)} \right|$$

The rational function on the right hand side is two frequency scale and has no slow or fast poles on the imaginary axis. Theorem 4.2.5 shows that the sup on the right hand side is uniformly bounded for sufficiently small real ε. (Note that the requirement of real ε appears only at this point.) This proves the lemma. □

Lemma 4.1.2: Suppose that all of the elements of a rational matrix $H(s, \varepsilon)$ satisfy the conditions of Lemma 4.1.1. Then

$$\sup_{s \in D} \| H(s, \varepsilon) - H_S(s) - H_F(\varepsilon s) + W \| = O(\varepsilon)$$

where $W = H_S(\infty) = H_F(0)$, $\| \cdot \|$ is some matrix norm, and D is the imaginary axis.

Proof: By the norm equivalence theorem, there is a constant B such that

$$\| A \| < B \| A \|_M$$

$$\text{where } \| A \|_M = \max_{i,j} | A_{ij} |$$

For brevity, let $H(s,\varepsilon) - H_S(s) - H_F(\varepsilon s) + W = \Delta(s,\varepsilon)$

$$\sup_{s\in D} \left\| \Delta(s,\varepsilon) \right\| \leq \sup_{s\in D} B \left\| \Delta(s,\varepsilon) \right\|_M$$

$$= \sup_{s\in D} B \max_{i,j} \left| \Delta_{ij}(s,\varepsilon) \right|$$

$$= B \max_{i,j} \sup_{s\in D} \left| \Delta_{ij}(s,\varepsilon) \right|$$

By Lemma 4.1.1, there exists ε^* and constants C_{ij} such that for $\varepsilon \in [-\varepsilon^*, \varepsilon^*]$,

$$B \max_{i,j} \sup_{s\in D} \left| \Delta_{ij}(s,\varepsilon) \right| \leq B \max_{i,j} C_{ij} \left| \varepsilon \right| = \left| \varepsilon \right| B \max_{i,j} C_{ij}$$

□

<u>Theorem 4.1.1</u>: Let $H(s,\varepsilon)$ be a two frequency scale rational matrix. Suppose that $H_S(s)$ and $H_F(p)$ have no pure imaginary poles and that $H(s,\varepsilon)$ has no pure imaginary lost poles. Then

$$\sup_{s\in D} \left\| H(s,\varepsilon) - H_S(s) - H_F(\varepsilon s) + W \right\| = O(\varepsilon)$$

where $W = H_S(\infty) = H_F(0)$, $\|\cdot\|$ is some matrix norm, and D is the imaginary axis.

<u>Proof</u>: Let $h_{ij}(s,\varepsilon) = n_{ij}(s,\varepsilon)/d_{ij}(s,\varepsilon)$ be any term of $H(s,\varepsilon)$. Then h_{ij} clearly satisfies the first three conditions of Definition 3.0.1. The characteristic polynomial $q(s,\varepsilon)$ of $H(s,\varepsilon)$ can be expressed as (4.1.4).

$$q(s,\varepsilon) = f(\varepsilon) \prod_{n=1}^{K} (s - a_n(\varepsilon)) \prod_{m=1}^{L} (\varepsilon s - b_m(\varepsilon)) \quad (4.1.4)$$

where $f(0) \neq 0$ and for all $1 \leq n \leq K$ and $1 \leq m \leq L$, $a_n(0)$ and $b_m(0)$ do not lie on the imaginary axis. Since each $d_{ij}(s,\varepsilon)$ divides $q(s,\varepsilon)$, it is seen that all hypothesis of Lemma 4.1.2 are met. □

IV.2. Appendix--Bounds on Rational Functions

This appendix shows in detail the boundedness of the rational functions as needed in the proof of Lemma 4.1.1.

Theorem 4.2.1: Let $g(s) = n(s)/d(s)$ be a proper rational function over $\mathbb{C}$ with $n(s)$ and $d(s)$ polynomials. Suppose that none of the zeros of $d(s)$ lie in the closed set D. Then $d(s)$ is uniformly bounded on D.

Proof: $g(s)$ can be expressed

$$g(s) = \frac{n_K s^K + n_{K-1} s^{K-1} + \ldots + n_o}{d_K s^K + d_{K-1} s^{K-1} + \ldots + d_o}$$

where the n_j's and d_i's are complex.

Rewrite $g(s)$

$$g(s) = \frac{n_K + n_{K-1} s^{-1} + \ldots + n_o s^{-K}}{d_K + d_{K-1} s^{-1} + \ldots + d_o s^{-K}}$$

Choose R large enough so that the values $n_{K-1} R^{-1}$, ..., $n_o R^{-K}$, $d_{K-1} R^{-1}$, ..., $d_o R^{-K}$ are all less than $\frac{1}{2K}$ · min $(G, |d_K|)$, where $G = \max(|n_K|, 1)$.

then for $|s| > R$, $|g(s)| < \dfrac{\frac{3}{2}G}{\frac{1}{2}|d_K|}$

The set $D \cap \{s | s \leq R\}$ is closed and bounded and therefore compact. g is analytic and therefore continuous on this set. Since the continuous image of a compact set is compact, g is uniformly bounded on $D \cap \{s | s \leq R\}$. Thus, g is bounded on all of D. □

<u>Theorem 4.2.2</u>: Let $g(s,\varepsilon) = n(s,\varepsilon)/d(s)$ where g is proper, d is a polynomial with complex coefficients, and $n(s,\varepsilon)$ is a polynomial with coefficients analytic at $\varepsilon = 0$. Suppose that none of the zeros of d lie in a closed set D. Then there exists $\varepsilon^* > 0$ such that $g(s,\varepsilon)$ is uniformly bounded for all $(s,\varepsilon) \in D \times \{\varepsilon | \ |\varepsilon| < \varepsilon^*\}$.

<u>Proof</u>: Choose ε^* so that all coefficients of $n(s,\varepsilon)$ are analytic in the disc $E = \{\varepsilon | \ |\varepsilon| \leq \varepsilon^*\}$. Let $n(s,\varepsilon) = n_K(\varepsilon)s^K + \ldots + n_o(\varepsilon)$ where the n_j's are analytic at $\varepsilon = 0$. Let $B_j = \max_{\varepsilon \in E} |n_j(\varepsilon)|$.

$$|g(s,\varepsilon)| = \left|\frac{n_K(\varepsilon)s^K + \ldots + n_o(\varepsilon)}{d(s)}\right|$$

$$\leq B_K \left|\frac{s^K}{d(s)}\right| + \ldots + B_1 \left|\frac{s}{d(s)}\right| + B_o \left|\frac{1}{d(s)}\right|$$

Each of the functions $s^j/d(s)$ satisfies the conditions of Theorem 4.2.1 on D. This shows that $g(s,\varepsilon)$ is uniformly bounded on DxE. □

<u>Theorem 4.2.3</u>: Let D be the imaginary axis and restrict ε to be real. Suppose $g(s,\varepsilon) = n(\varepsilon s,\varepsilon)/d(\varepsilon s)$ where g is proper, $n(p,\varepsilon)$ is a polynomial with coefficients analytic

at $\varepsilon = 0$, and $d(p)$ is a polynomial with complex coefficients with no zeros on D. Then there exists $\varepsilon^* > 0$ such that $g(s,\varepsilon)$ is uniformly bounded for all $(s,\varepsilon) \in D \times [-\varepsilon^*, \varepsilon^*]$.

<u>Proof</u>: Choose ε^* such that all coefficients of n are analytic on $\{\varepsilon \mid |\varepsilon| \leq \varepsilon^*\}$. By changing variables,

$$\sup_{s\in D} |g(s,\varepsilon)| = \sup_{s\in D} \left|\frac{n(\varepsilon s,\varepsilon)}{d(\varepsilon s)}\right| = \sup_{p\in D} \left|\frac{n(p,\varepsilon)}{d(p)}\right|$$

Boundedness follows from Theorem 4.2.2.

<u>Theorem 4.2.4</u>: Let D be the imaginary axis and let $d(s,\varepsilon) = d_1(s,\varepsilon) + s^K d_2(\varepsilon s,\varepsilon)$ where

1. $d_1(s,\varepsilon)$ and $d_2(p,\varepsilon)$ have coefficients analytic in ε at $\varepsilon = 0$;
2. $\deg d_1(s,\varepsilon) = \deg d_1(s,0) = K$;
3. $d_2(0,\varepsilon) = 0$

Define $d_S(s) = d_1(s,0)$

$$d_F(p) = \frac{1}{\chi}\,[\chi + d_2(p,0)]$$

where χ is the leading coefficient of $d_1(s,0)$. Let d_S and d_F have no roots in D. Let $r(s,\varepsilon) = d_S(s)d_F(\varepsilon s)/d(s,\varepsilon)$. Then

$$\lim_{\substack{\varepsilon\to 0\\ \text{real}}} r(s,\varepsilon) = 1 \text{ uniformly for } s \in D.$$

In particular, there exists $\varepsilon^* > 0$ such that $r(s,\varepsilon)$ is uniformly bounded for $(s,\varepsilon) \in D\times[-\varepsilon^*, \varepsilon^*]$.

Proof: Using previously derived expressions,

$$d(s,\varepsilon) = f(\varepsilon) \cdot \prod_{i=1}^{K} (s-a_i(\varepsilon)) \cdot \prod_{j=1}^{L} (\varepsilon s - b_j(\varepsilon))$$

where f is analytic at $\varepsilon = 0$, all $a_i(\varepsilon)$ and $b_j(\varepsilon)$ approach finite limits as $\varepsilon \to 0$, and all $b_j(0)$ are non-zero. Also,

$$d_S(s)\ d_F(\varepsilon s) = f(0) \cdot \prod_{i=1}^{K} (s - a_i(0)) \prod_{j=1}^{L} (\varepsilon s - b_j(0))$$

$$r(s,\varepsilon) = \frac{f(0)}{f(\varepsilon)} \cdot \prod_{i=1}^{K} \frac{s - a_i(0)}{s - a_i(\varepsilon)} \cdot \prod_{j=1}^{L} \frac{\varepsilon s - b_j(0)}{\varepsilon s - b_j(\varepsilon)}$$

If $s \in D$, then

$$\left| \frac{s - a_i(\varepsilon)}{s - a_i(0)} - 1 \right| = \left| \frac{a_i(0) - a_i(\varepsilon)}{s - a_i(0)} \right|$$

$$\leq \left| \frac{a_i(0) - a_i(\varepsilon)}{\mathrm{Re}\ a_i(0)} \right| \to 0 \text{ as } \varepsilon \to 0.$$

This limit is uniform in s.

Therefore, $\dfrac{s - a_i(0)}{s - a_i(\varepsilon)} \to 1$ as $\varepsilon \to 0$ uniformly in s.

Similarly, $\dfrac{\varepsilon s - b_j(0)}{\varepsilon s - b_j(\varepsilon)} \to 1$ as $\varepsilon \to 0$ uniformly in s.

The theorem follows using standard limit theorems. □

Corollary 4.2.1: Let D be the imaginary axis and let $d(s,\varepsilon)$ be a polynomial with coefficients analytic in ε at $\varepsilon = 0$. Suppose that the leading coefficient of $d(s,\varepsilon)$ does not vanish when ε is set to zero, and that $d(s,0)$ has no roots in D. Define $r(s,\varepsilon) = d(s,0)/d(s,\varepsilon)$. Then

$$\lim_{\varepsilon \to 0} r(s,\varepsilon) = 1 \quad \text{uniformly for } s \in D.$$

Proof: Easily extracted from proof of Theorem 4.2.4. □

Theorem 4.2.5: Let $h(s,\varepsilon)$ be two frequency scale. Suppose that the limits of the roots of the characteristic polynomial do not lie on the imaginary axis D (as in Lemma 4.1.1). Then there exists $\varepsilon^* > 0$ such that $h(s,\varepsilon)$ is uniformly bounded for $(s,\varepsilon) \in D \times [-\varepsilon^*, \varepsilon^*]$.

Proof: $|h(s,\varepsilon)|$ can be written

$$|h(s,\varepsilon)| = \left| \frac{n_1(s,\varepsilon) + s^K n_2(\varepsilon s,\varepsilon)}{d_1(s,\varepsilon) + s^K d_2(\varepsilon s,\varepsilon)} \right|$$

where the right hand side is the form guaranteed by Theorem 3.1.1.

$$|h(s,\varepsilon)| = \left| \frac{n_1(s,\varepsilon) + s^K n_2(\varepsilon s,\varepsilon)}{d_S(s)\; d_F(\varepsilon s)} \right| \cdot \left| \frac{d_S(s) \cdot d_F(\varepsilon s)}{d_1(s,\varepsilon) + s^K d_2(\varepsilon s,\varepsilon)} \right|$$

The right hand factor is bounded by virtue of Theorem 4.2.4. The left hand factor can be rewritten

$$\left| \frac{n_1(s,\varepsilon) + s^K n_2(\varepsilon s,\varepsilon)}{d_S(s)\; d_F(\varepsilon s)} \right|$$

$$= \left| \frac{n_1(s,\varepsilon)}{d_S(s)} \cdot \frac{1}{d_F(\varepsilon s)} + \frac{s^K}{d_S(s)} \cdot \frac{n_2(\varepsilon s,\varepsilon)}{d_F(\varepsilon s)} \right|$$

The theorem follows from application of Theorems 4.2.2 and 4.2.3 to the individual functions on the right hand side. □

<u>Corollary 4.2.2</u>: Let $h(s,\varepsilon)$ be a rational, proper function of s with coefficients analytic in ε at $\varepsilon = 0$. Let h be expressed

$$h(s,\varepsilon) = \frac{n(s,\varepsilon)}{d(s,\varepsilon)}$$

where $d(s,\varepsilon)$ satisfies the hypothesis of Corollary 4.2.1. Then there exists $\varepsilon^* > 0$ such that $h(s,\varepsilon)$ is uniformly bounded for all imaginary s and $|\varepsilon| < \varepsilon^*$.

<u>Proof</u>: Similar to proof of Theorem 4.2.5. □

V. CLOSED LOOP SYSTEMS

This chapter investigates what can be said about a two frequency scale system when a feedback loop is closed around it. A cascade of two two frequency scale systems with feedback applied will be considered. It is shown that under certain conditions, knowledge of the open loop lost poles and the poles of the closed loop slow and fast subsystems is sufficient to approximate all of the closed loop poles.

Figure 5.1 shows the inputs and outputs and the manner in which the loop is closed. Figures 5.2 and 5.3 show the closed loop slow and fast subsystems, respectively.

Figure 5.1

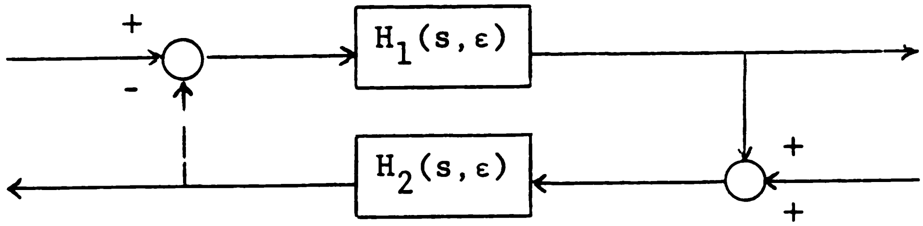


Figure 5.2

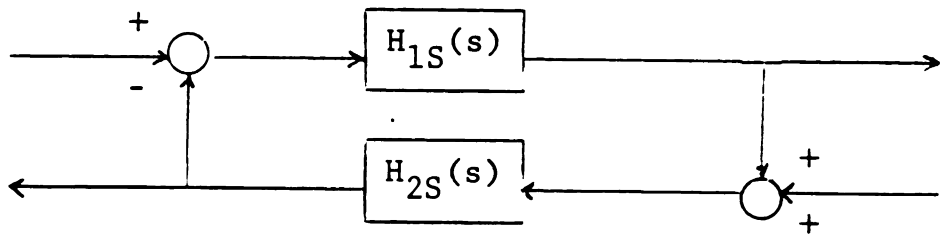
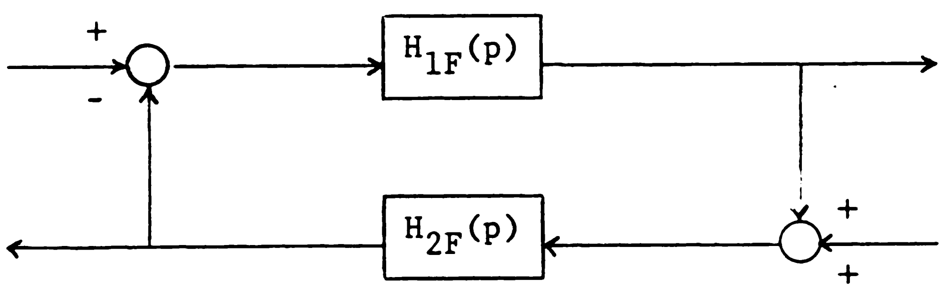


Figure 5.3



The following notation is introduced for convenience. Let $i = 1,2$.

1. $H_i(s,\varepsilon)$ has a least order polynomial system matrix representation $P_i(s,\varepsilon)$ in MFD form such that $P_i(s,0)$ is a polynomial system matrix.
2. H_{iS} has a least order polynomial system matrix P_{iS} derived from $P_i(s,0)$ by extracting all output decoupling zeros.
3. $H_i(\frac{p}{\varepsilon},\varepsilon)$ has a least order polynomial system matrix representation $\tilde{P}_i(p,\varepsilon)$ in MFD form such that $\tilde{P}_i(p,0)$ is a polynomial system matrix.

4. H_{iF} has a least order polynomial system matrix P_{iF} derived from $\tilde{P}_i(p,0)$ by extracting all output decoupling zeros.
5. $P_{CL}(s,\varepsilon)$ is formed by inserting the blocks of $P_i(s,\varepsilon)$ $(i = 1, 2)$ into (2.1.8). This is a candidate polynomial system matrix for Figure 5.1.
6. $P_{CLS}(s)$ is formed by inserting the blocks of $P_{iS}(s)$ $(i = 1, 2)$ into (2.1.8). This is a candidate polynomial system matrix for Figure 5.2.
7. $\tilde{P}_{CL}(p,\varepsilon)$ is formed by inserting the blocks of $\tilde{P}_i(p,\varepsilon)$ into (2.1.8). This is a candidate frequency scaled polynomial system matrix for Figure 5.1.
8. $P_{CLF}(p)$ is formed by inserting the blocks of P_{iF} into (2.1.8). This is a candidate polynomial system matrix for Figure 5.3.
9. γ is a generic output decoupling polynomial.
10. q is a generic lost polynomial.

<u>Theorem 5.1</u>: Let $H_1(s,\varepsilon)$ and $H_2(s,\varepsilon)$ be two frequency scale rational matrices. Suppose that (5.1) and (5.2) hold.

$$\det(I + H_{1S}(\infty)H_{2S}(\infty)) \neq 0 \tag{5.1}$$

$$\det(I + H_{1F}(\infty)H_{2F}(\infty)) \neq 0 \tag{5.2}$$

Then (with the above and previous notation), $P_{CL}(s,\varepsilon)$ and $\tilde{P}_{CL}(p,\varepsilon)$ are polynomial system matrices for Figure 5.1, and $P_{CLS}(s)$ and $P_{CLF}(p)$ are polynomial system matrices for Figures 5.2 and 5.3, respectively. The relations (5.3) and (5.4) for limiting forms of the closed loop characteristic polynomial hold.

$$\det T_{CL}(s,0) \alpha q_{1LS}(s) \cdot q_{2LS}(s) \cdot \det T_{CLS}(s) \qquad (5.3)$$

$$\det \tilde{T}_{CL}(p,0) \alpha p^{K_1+K_2} \cdot q_{1LF}(p) q_{2LF}(p) \cdot \det T_{CLF}(p) \qquad (5.4)$$

where $K_i = \deg \det T_i(s,0)$, $i = 1,2$.

Furthermore, the lost poles of Figure 5.1 are the open loop lost poles of $H_1(s,\varepsilon)$ and $H_2(s,\varepsilon)$.

Proof: By Theorem 2.1.5, (5.1) and (5.2) guarantee that $P_{CLS}(s)$ and $P_{CLF}(p)$ are polynomial system matrices for the closed-loop slow and fast subsystems, respectively. From Corollary 3.1.1,

$$\det(I + H_1(\infty,\varepsilon)H_2(\infty,\varepsilon))\Big|_{\varepsilon=0} = \det(I + H_{1F}(\infty)H_{2F}(\infty)) \neq 0$$

Thus, $\det(I + H_1(\infty,\varepsilon)H_2(\infty,\varepsilon)) \neq 0$. Applying Theorem 2.1.5 shows that $P_{CL}(s,\varepsilon)$ and $\tilde{P}_{CL}(p,\varepsilon)$ are polynomial system matrices for the closed-loop system of Figure 5.1.

Since $P_i(s,0)$ is in MFD form, it follows that

$$\det T_i(s,0) \alpha \gamma_{iS}(s) \cdot \det T_{iS}(s)$$

where $\gamma_{iS}(s)$ is the output decoupling polynomial of $P_i(s,0)$.

Moreover, $\gamma_{iS} = q_{iLS}$. Theorem 2.1.3 shows that

$$\det T_{CL}(s,0) \alpha \det T_1(s,0) \det T_2(s,0) \det(I + H_{1S}(s)H_{2S}(s))$$

$$\alpha q_{1LS}(s) q_{2LS}(s) \det T_{1S}(s) \det T_{2S}$$

$$\det(I + H_{1S}(s)H_{2S}(s))$$

$$\alpha q_{1LS}(s) q_{2LS}(s) \det T_{CLS}(s)$$

which proves (5.3). (5.4) is proved similarly.

The last statement in the theorem follows from Theorem 2.1.4. Consider the lost slow poles of Figure 5.1, i.e., the output decoupling polynomial Γ_S of $P_{CL}(s,0)$ (Note that the input decoupling polynomial is unity.) Then

$$\Gamma_S(s) = \gamma_{1S}(s)\gamma_{2S}(s)$$

$$= q_{1L}(s) q_{2L}(s). \qquad \square$$

<u>Theorem 5.2</u>: With the hypothesis and notation of Theorem 5.1, all closed loop poles in Figure 5.1 can be approximated by either (5.5) or (5.6).

$$s_i(\varepsilon) = s_{i0} + \Delta_i(\varepsilon), \quad 1 \le i \le K_1 + K_2 \tag{5.5}$$

$$s_j(\varepsilon) = \frac{p_{j0} + D_j(\varepsilon)}{\varepsilon}, \quad 1 \le j \le L_1 + L_2 \tag{5.6}$$

where $K_\alpha = \deg \det T_\alpha(s,0) \qquad \alpha = 1,2$

$L_\alpha = \deg \det T_\alpha(p,0) - \deg \det T_\alpha(s,0) \qquad \alpha = 1,2$

$\Delta_i(\varepsilon) \to 0$ as $\varepsilon \to 0$, $1 \le i \le K_1 + K_2$

$D_j(\varepsilon) \to 0$ as $\varepsilon \to 0$, $1 \le j \le L_1 + L_2$

s_{i0} is a root of $g_S(s)$, $1 \le i \le K_1 + K_2$

p_{j0} is a root of $g_F(p)$, $1 \le j \le L_1 + L_2$

$$g_S(s) = q_{1LS}(s)q_{2LS}(s)\det T_{CLS}(s) \tag{5.7}$$

$$g_F(p) = q_{1LF}(p)q_{2LF}(p)\det T_{CLF}(p) \tag{5.8}$$

Proof: Since the roots of a polynomial vary continuously with its coefficients, for every s_{i0} satisfying (5.7) there is a root $s_i(\varepsilon)$ of $\det T_{CL}(s,\varepsilon)$ satisfying (5.5). Similarly, for every $p_{j0} \neq 0$ satisfying (5.8), there is a root $p_j(\varepsilon)$ of $\det T_{CL}(p,\varepsilon)$ satisfying (5.9).

$$p_j(\varepsilon) = p_{j0} + D_j(\varepsilon) \tag{5.9}$$

If $s_j(\varepsilon) = p_j(\varepsilon)/\varepsilon$, $s_j(\varepsilon)$ satisfies (5.6). It remains to be shown that all roots of $\det T_{CL}(s,\varepsilon) = 0$ satisfy (5.5) or (5.6). This is done by showing that deg det $T_{CL}(s,\varepsilon)$ is the sum of the degrees of g_S and g_F. By Theorem 5.1,

$$\begin{aligned}\deg g_S(s) &= \deg\det T_{CL}(s,0)\\ &= \deg\det T_1(s,0) + \deg\det T_2(s,0)\\ &= K_1 + K_2\end{aligned}$$

$$\begin{aligned}\deg p^{K_1+K_2} g_F(p) &= \deg\det \tilde{T}_{CL}(p,0)\\ &= \deg\det \tilde{T}_1(p,0) + \deg\det \tilde{T}_2(p,0)\\ &= K_1 + K_2 + L_1 + L_2\end{aligned}$$

This shows that

$$\deg g_F(p) = L_1 + L_2$$

Together, (5.7) and (5.8) determine $K_1 + K_2 + L_1 + L_2$ roots of $\det T_{CL}(s,\varepsilon) = 0$. By Corollary 3.1.1, $\det(I + H_1(\infty,\varepsilon) \cdot H_2(\infty,\varepsilon)) \neq 0$ for sufficiently small ε, and Theorem 2.1.5 can be applied to the system of Figure 5.1.

$$\begin{aligned} \deg \det T_{CL}(s,\varepsilon) &= \deg \det T_1(s,\varepsilon) + \deg \det T_2(s,\varepsilon) \\ &= \deg \det T_1(p,0) + \deg \det T_2(p,0) \\ &= K_1 + K_2 + L_1 + L_2 \qquad \square \end{aligned}$$

In words, Theorem 5.2 shows that for sufficiently small ε the closed-loop poles of the system of Figure 5.1 can be approximated by the closed-loop poles of the slow and fast subsystems and the lost poles.

<u>Corollary 5.1</u>: Suppose $H_1(s,\varepsilon)$ and $H_2(s,\varepsilon)$ satisfy the hypothesis of Theorem 5.1. Let $G(s,\varepsilon)$ be any point to point transfer matrix in Figure 5.1. Then $G(s,\varepsilon)$ is two frequency scale.

<u>Proof</u>: The four conditions of Definition 3.0.1 are easily verified. $\square$

This corollary shows that Theorem 4.1.1 can be applied to the point to point transfer matrix $G(s,\varepsilon)$ when both slow and fast closed-loop subsystems as well as all lost poles are stable.

<u>Corollary 5.2</u>: Suppose $H_1(s,\varepsilon)$ and $H_2(s,\varepsilon)$ satisfy the hypothesis of Theorem 5.1, and both the slow and fast

closed-loop subsystems as well as all lost poles are stable. Then, for sufficiently small positive ε, the closed-loop system of Figure 5.1 is stable. □

The results of this section are illustrated with two simple examples.

Example 5.1: Consider the transfer function of Example 3.0.1 with unity feedback applied.

Entire closed-loop transfer function:

$$h_{CL}(s,\varepsilon) = \frac{s+1}{s+1+(s+2)(\varepsilon s+1)}$$

$$= \frac{s+1}{\varepsilon s^2 + (2+2\varepsilon)s + 3}$$

This has two poles: one which approaches $-\frac{3}{2}$, and one which asymptotically approaches $-\frac{2}{\varepsilon}$.

Slow closed-loop transfer function:

$$h_{CLS}(s) = \frac{s+1}{2s+3}$$

This has a pole at $-\frac{3}{2}$.

Fast closed-loop transfer function:

$$h_{CLF}(p) = \frac{1}{p+2}$$

This has a pole at -2.

<u>Example 5.2</u>: Consider the transfer function of Example 3.0.2 with unity feedback applied.

$$h_{CL}(s,\varepsilon) = \frac{1}{1 + (s + 1)(\varepsilon s + 1)}$$

$$= \frac{1}{\varepsilon s^2 + (1 + \varepsilon)s + 2} \qquad (5.10)$$

This has two poles: one which approaches -2, and one which asymptotically appraoches $-\frac{1}{\varepsilon}$.

Slow closed-loop transfer function:

$$h_{CLS}(s) = \frac{1}{s + 2}$$

This has a pole at -2.

The fast closed-loop transfer function (like the fast open-loop transfer function) is zero. However, there is a lost fast pole at p = -1. Note that -1 is also a lost fast pole of (5.10).

VI. APPLICATIONS

VI.1. Steady State LQG Controller for a Singularly Perturbed System

This section provides an interpretation of the time domain solution of an output feedback regulator problem for a singularly perturbed system [10]. The solution employs the usual division of the problem into slow and fast subproblems, followed by synthesis of the two subproblem solutions into a composite controller. A number of matrix manipulations are involved, making it not intuitively obvious why the solution works. The problem statement is: find a control law for the system.

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1u + G_1w \\ \varepsilon\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2u + G_2w \\ y &= H_1x_1 + H_2x_2 + v \end{aligned} \tag{6.1.1}$$

such that the performance criterion J in (6.1.2) is minimized

$$J = \lim_{\substack{t_o \to -\infty \\ t_f \to +\infty}} \frac{1}{t_f - t_o} \, E\left\{\int_{t_o}^{t_f} [z^Tz + u^TRu]\,dt\right\} \tag{6.1.2}$$

where $z = C_1x_1 + C_2x_2$

The following conditions hold:

1. All coefficient matrices (capital letters) in (6.1.1) are independent of t.
2. R, C_1, and C_2 are constant and R is positive definite.
3. w and v are independent, zero mean, stationary, white Gaussian noise processes with intensities I and V respectively. V is positive definite.
4. The input u and output y are available for feedback purposes.

For fixed values of ε, this is a standard linear-quadratic-Gaussian problem [e.g., 11]. It can be shown, however, that a slightly sub-optimal solution can be found by using a time scale decomposition.

The slow subproblem is: find a control law for the system

$$\dot{x}_S = A_o x_S + B_o u_S + G_o w$$

$$y_S = H_o x_S + S_o u_S + N_o w + u$$

where
$$A_o = A_{11} - A_{12}A_{22}^{-1} A_{21}$$
$$B_o = B_1 - A_{12}A_{22}^{-1} B_2$$
$$H_o = H_1 - H_2A_{22}^{-1} A_{21}$$
$$S_o = -H_2A_{22}^{-1} B_2$$
$$N_o = -H_2A_{22}^{-1} G_2$$
$$G_o = G_1 - A_{12}A_{22}^{-1} G_2$$

which minimizes the criterion

$$J_S = \lim_{\substack{t_o \to -\infty \\ t_f \to \infty}} \frac{1}{t_f - t_o} \cdot E \left\{ \int_{t_o}^{t_f} ([C_o x_S + D_o u_S]^T [C_o x_S + D_o u_S] + u_S^T R u_S)\, dt \right\}$$

where $C_o = C_1 - C_2 A_{22}^{-1} A_{21}$

$$C_o = -C_2 A_{22}^{-1} B_2$$

The fast subproblem is: find a control law for the system

$$\varepsilon \dot{x}_f = A_{22} x_f + B_2 u_f + G_2 w$$

$$y_f = H_2 x_f + v$$

which minimizes the criterion

$$J_f = \lim_{\substack{t_o \to -\infty \\ t_f \to \infty}} \frac{1}{t_f - t_o} E \left\{ \int_{t_o}^{t_f} (x_f^T C_2^T C_2 x_f + u_f^T R u_f)\, dt \right\}$$

It can be shown that the solution for the slow subproblem has the form

$$u_S = -F_S x_S$$

$$\dot{\hat{x}}_S = A_o \hat{x}_S + B_o u_S + Q_S [y_S - S_o u_S - H_o \hat{x}_S]$$

Similarly, it can be shown that the fast subproblem solution can be written

$$u_f = -F_f\hat{x}_f$$

$$\varepsilon\dot{\hat{x}}_f = A_{22}\hat{x}_f + B_2u_f + Q_f\,[y_f - H_2\hat{x}_f]$$

The matrices F_S, F_f, Q_S, and Q_f are found by least squares methods. We assume that stabilizing solutions to the two subproblems exist.

After the above solutions have been computed, a composite control is formed:

$$u_c = -F_S\hat{x}_S - F_f\hat{x}_f \qquad (6.1.3)$$

$$\dot{\hat{x}}_S = A_o\hat{x}_S + B_ou_c + Q_S\,[y - S_ou_c - H_o\hat{x}_S]$$

$$\varepsilon\dot{\hat{x}}_f = (A_{22} - B_2F_f - Q_fH_2)\,\hat{x}_f + Q_f\,[y + S_oF_S\hat{x}_S - H_o\hat{x}_S]$$

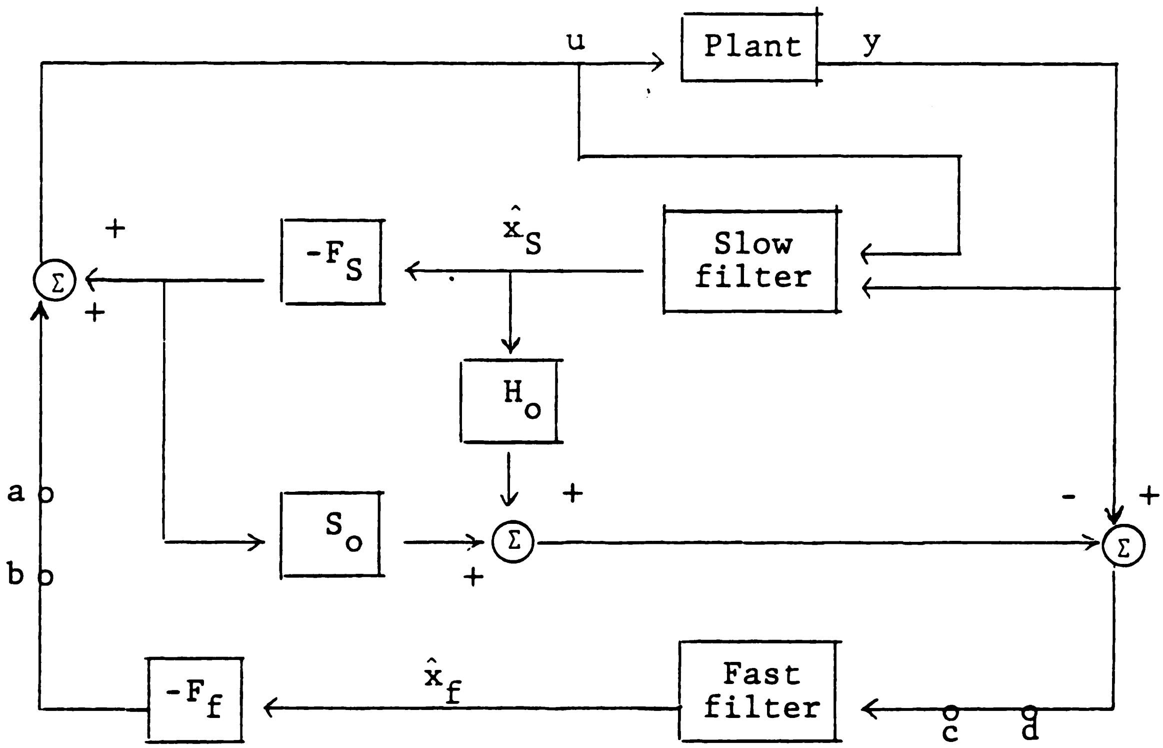
A block diagram for the entire closed-loop system is shown in Figure 6.1.1. It can be shown that the relative error in the criterion (6.1.2) for the system (6.1.1) with the controller (6.1.3) applied asymptotically approaches zero as $\varepsilon \to 0$.

We assume here that the fast filter has no poles at the origin. Let $H_1(s,\varepsilon)$ be the transfer matrix from a to d with the fast filter removed in Figure 6.1.1. It can be shown that

$$H_{1S}(s) = S_o$$

$$H_{1F}(p) = H_2(pI - A_{22})^{-1}\,B_2$$

Figure 6.1.1



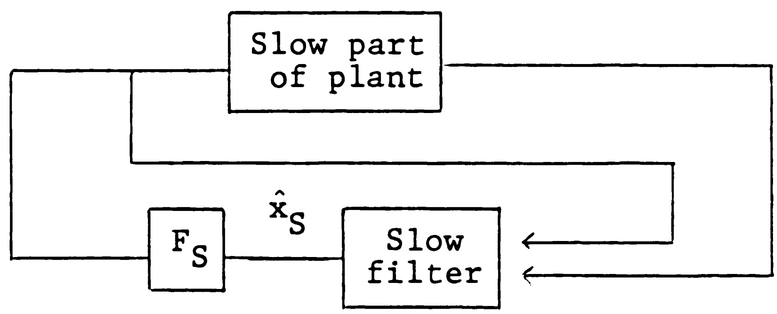
Let $H_2(s,\varepsilon)$ be the transfer matrix for the fast filter by itself, that is, from c to b in Figure 6.1.1.

Then

$$H_{2S}(s) = -F_f\ (-A_{22} + B_2F_f + Q_fH_2)^{-1}\ Q_f$$

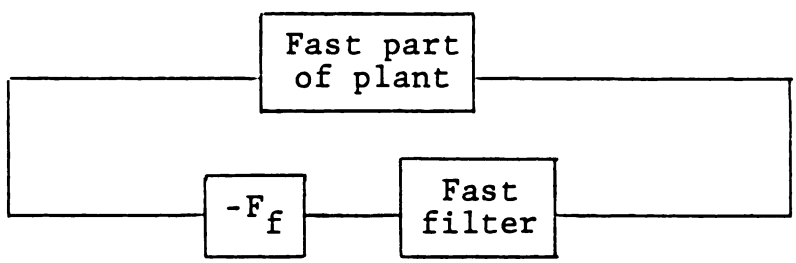
which is a constant independent of s. Theorem 5.2 shows that all of the closed loop slow poles are the lost slow poles of $H_1(s,\varepsilon)$ which are clearly the poles of the slow design subsystem shown in Figure 6.1.2 (in addition to the lost slow poles of the plant).

Figure 6.1.2



Also, the fast poles are the lost fast poles of the plant together with the poles of the system of Figure 6.1.3, which is the fast design subsystem.

Figure 6.1.3



The case when the fast filter has a pole at the origin can be treated in a similar way after $H_1(s,\varepsilon)$ above is split into exact slow and fast components using Theorem 3.2.1.

VI.2. Feedback Design Strategies

Theorems 5.2 and 4.1.1 can be used to derive various feedback design strategies. Although there are many possibilities, only two will be discussed here.

<u>Design Strategy #1</u>: Let $H(s,\varepsilon)$ be a two frequency scale transfer matrix which has no unstable lost poles.

Step 1: Design slow and fast cascade and feedback compensators so that the closed loop systems in Figure 6.2.1 and 6.2.2 have desirable characteristics including asymptotic stability.

Figure 6.2.1

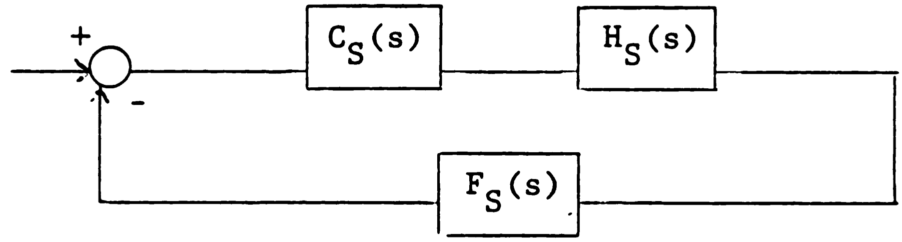
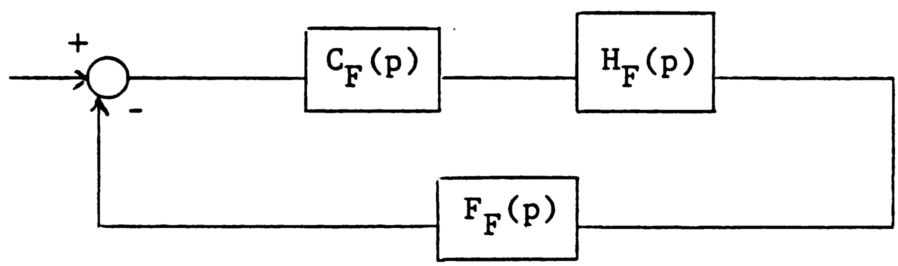


Figure 6.2.2



The designs are subject to the constraints

$$C_S(\infty) = C_F(0) \triangleq \hat{C} \tag{6.2.1}$$

$$F_S(\infty) = F_F(0) \triangleq \hat{F} \tag{6.2.2}$$

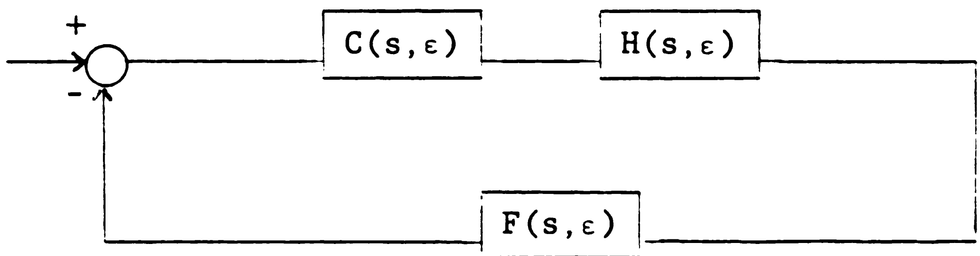
Step 2: Form composite cascade and feedback compensators in one of two different ways:

$$F(s,\varepsilon) \triangleq F_S(s) + F_F(s) - \hat{F} \quad \text{or}$$

$$F(s,\varepsilon) \triangleq F_S(s)\, \hat{F}^{-1}\, F_F(\varepsilon s) \quad \text{if } \hat{F} \text{ is invertable}$$

Similarly, $C(s,\varepsilon)$ is computed from $C_S(s)$ and $C_F(p)$. These are combined to form the composite control of Figure 6.2.3.

Figure 6.2.3



Result: The poles of the system in Figure 6.2.3 can be approximated from the lost poles of $H(s,\varepsilon)$ and from the poles of the systems in Figures 6.2.1 and 6.2.2. Furthermore, any point to point transfer matrix in Figure 6.2.3 can be approximated along the imaginary axis from the corresponding transfer matrices in Figures 6.2.1 and 6.2.2.

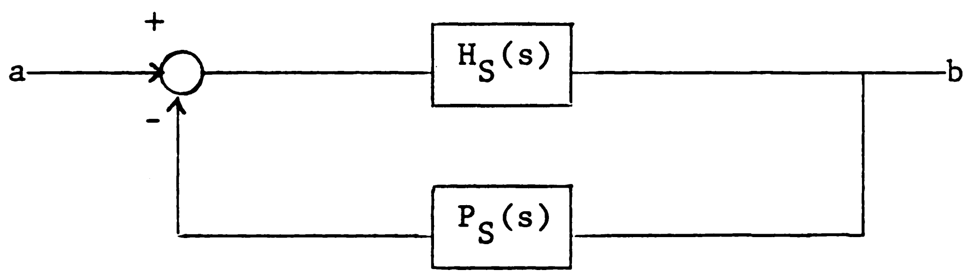
Proof: Follows by direct application of Theorems 5.2 and 4.1.1. Note that the compensators have no lost poles.

The main difficulty with the above strategy is the constraints (6.2.1) and (6.2.2) which make the subproblems nonstandard. Although not as satisfying theoretically, the next strategy circumvents this problem.

<u>Design Strategy #2</u>: Let $H(s,\varepsilon)$ be a two frequency scale transfer matrix with no unstable lost poles.

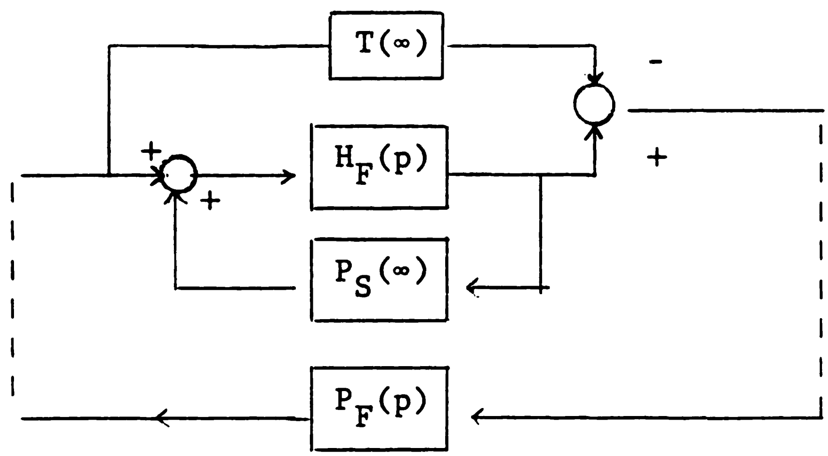
Step 1: Design a controller $P_S(s)$ to stabilize $H_S(s)$ and to shape the transfer matrix from a to b in a desirable way, as shown in Figure 6.2.4. Let T(s) be the transfer matrix from a to b.

Figure 6.2.4



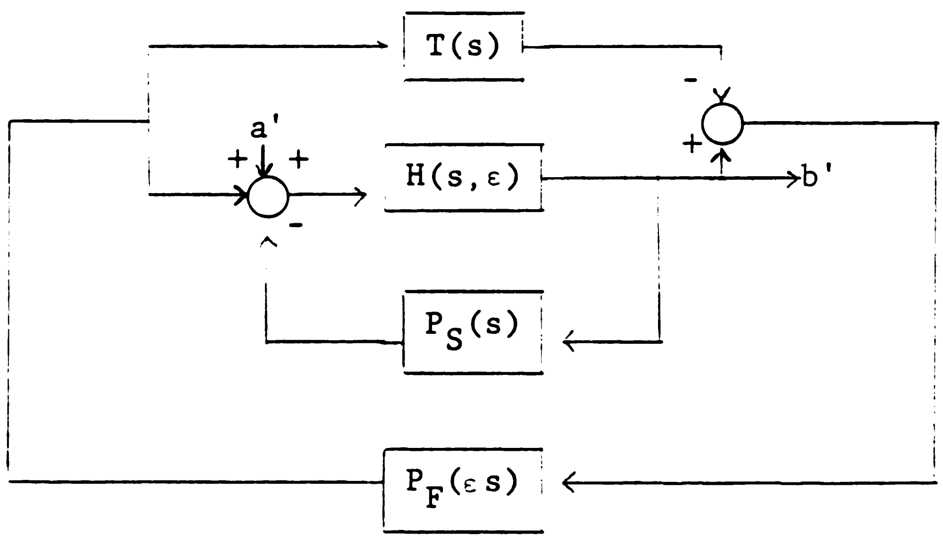
Step 2: Design $P_F(p)$ to stabilize the system of Figure 6.2.5.

Figure 6.2.5



Step 3: Form the composite control of Figure 6.2.6.

Figure 6.2.6



Result: Pole approximation holds as for Design Strategy #1. The transfer matrix from a' to b' is approximated at low frequencies by T(s).

Proof: Direct application of Theoream 5.2 and 4.1.1 suffices. Theorem 3.2.1 is required if $P_F(p)$ has poles at the origin.

This method is similar to the explanation of the LQG design in the previous section. In this case, however, the stable transfer matrix T(s) must be realized separately since the states of $P_S(s)$ do not replicate the states of $H_S(s)$ in general.

VI.3. Numerical Examples

If a numerically described transfer matrix is given, these methods cannot be applied unless ε is introduced in some manner. This can be done quite freely in the single variable case (that is, for 1xm or mx1 transfer matrices). The next example will show that the problem is nontrivial for the multivariable case.

Example 6.3.1: Suppose a system is described by the transfer matrix (6.3.1). Then the characteristic polynomial is (s-1). Suppose we perturb one of the numerators by ε as in (6.3.2). The characteristic polynomial becomes $(s-1)^2$.

$$\hat{H}(s) = \begin{bmatrix} \frac{2}{s-1} & \frac{2}{s-1} \\ \frac{2}{s-1} & \frac{2}{s-1} \end{bmatrix} \tag{6.3.1}$$

$$H(s,\varepsilon) = \begin{bmatrix} \frac{2}{s-1} & \frac{2}{s-1} \\ \frac{2}{s-1} & \frac{2+\varepsilon}{s-1} \end{bmatrix} \tag{6.3.2}$$

It might be argued that only existing poles have been doubled, but if unity feedback is placed around (6.3.1), then

$$\begin{aligned} \mathrm{CLCP}(s) &= \mathrm{OLCP}(s) \cdot \det\,(I + \hat{H}(s)) && (6.3.3) \\ &= (s-1) \cdot \frac{1}{(s-1)^2} \cdot [(s+1)^2 - 4] \\ &= s + 3 \end{aligned}$$

where CLCP and OLCP stand for "closed loop characteristic polynomial" and "open loop characteristic polynomial", respectively. Note that (6.3.3) is a restatement of (2.1.10). Unity feedback around (6.3.2) gives the closed loop characteristic polynomial

$$\begin{aligned} CLCP(s) &= (s-1)^2 \cdot \frac{1}{(s-1)^2} \cdot [(s+1)(s+1+\varepsilon) - 4] \\ &= s^2 + (2+\varepsilon)s + \varepsilon - 3 \end{aligned} \tag{6.3.4}$$

(6.3.4) has two roots: $-3 + 0(\varepsilon)$ and $1 + 0(\varepsilon)$. In other words, 1 is a lost (slow) pole of (6.3.2) and the system formed by placing unity feedback around (6.3.2). Thus, introducing ε has created new poles in this case.

Given a transfer function $\hat{h}(s)$, there are two obvious ways of introducing ε if the roots of the denominator seem to fall into fast and slow groups. Let $\hat{h}(s)$ be written

$$\hat{h}(s) = \frac{n(s)}{d_S(s)\, d_F(s)} \tag{6.3.5}$$

By equating coefficients and solving the resulting set of linear equations, (6.3.5) can be rewritten

$$\hat{h}(s) = \frac{n_S(s)}{d_S(s)} + \frac{n_F(s)}{d_F(s)} + a$$

We can now choose $\varepsilon_1 > 0$ arbitrarily and write

$$h(s,\varepsilon) = \frac{n_S(s)}{d_S(s)} + \frac{n_F(\frac{\varepsilon}{\varepsilon_1}s)}{d_F(\frac{\varepsilon}{\varepsilon_1}s)} + a$$

Now $h(s,\varepsilon_1) = \hat{h}(s)$. If ε_1 is chosen so that it represents an "average" of the ratio of magnitudes of slow and fast poles, then $d_S(s)$ and $d_F(\frac{p}{\varepsilon_1})$ will have zeros which are of approximately the same magnitude. For computational simplicity, however, it is best to choose $\varepsilon_1 = 1$. This method can be applied to a transfer matrix of arbitrary dimensions without encountering the problem of Example 6.3.1.

Suppose that the zeros of the numerator of $\hat{h}(s)$ also fall fast and slow groups. Let $\hat{h}(s)$ be written

$$\hat{h}(s) = \frac{n_S(s)\ n_F(s)}{d_S(s)\ d_F(s)}$$

Here we must have deg $n_S \le$ deg d_S. ε can be introduced:

$$h(s,\varepsilon) = \frac{n_S(s)\ n_F(\frac{\varepsilon s}{\varepsilon 1})}{d_S(s)\ d_F(\frac{\varepsilon s}{\varepsilon_1})}$$

Again, ε_1 is arbitrary and $\hat{h}(s) = h(s,\varepsilon_1)$. This cannot be applied to multivariable transfer matrices because of the pole duplication problem. Note that all fast poles will be lost poles if n_S/d_S is strictly proper so that this method tends to give more simple reduced models. Also, the computations are more simple than with the previous method.

The next example applies the sum method to a nontrivial numerical system.

<u>Example 6.3.2</u>: The transfer matrix $\hat{H}(s)$ below is for an open loop unstable chemical reactor and $\hat{C}(s)$ is the controller design derived in [12].

$$\hat{H}(s) = \frac{1}{\Delta(s)} \cdot \begin{bmatrix} \hat{N}_{11}(s) & \hat{N}_{12}(s) \\ \hat{N}_{21}(s) & \hat{N}_{22}(s) \end{bmatrix}$$

where

$$\hat{N}_{11}(s) = 29.24 + 263.3$$

$$\hat{N}_{12}(s) = -3.146s^3 - 32.62s^2 - 89.83s - 31.81$$

$$\hat{N}_{21}(s) = 5.679s^3 + 42.67s^2 - 68.84s - 106.8$$

$$\hat{N}_{22}(s) = 9.43s + 15.15$$

$$\Delta(s) = s^4 + 11.67s^3 + 15.75s^2 - 88.31s + 5.514$$

$$= (s - .06318)(s - 1.991)(s + 5.057)(s + 8.666)$$

The four roots of $\Delta(s)$ are the poles of this system. They are all simple because the determinant $\hat{N}_{11}\hat{N}_{22} - \hat{N}_{12}\hat{N}_{21}$ has $\Delta(s)$ as an exact division.

$$\hat{C}(s) = \frac{1}{s}\begin{bmatrix} 0 & 20s + 18 \\ -(20s + 42) & -16 \end{bmatrix}$$

Note that $\hat{C}(s)$ is second order.

The exact characteristic polynomial of the closed loop system is

$$s^2\,\Delta(s) \cdot \det\,[I + \hat{H}(s)\,\hat{C}(s)]$$

$$= \frac{1}{\Delta(s)} \cdot [447315 - 5991450s - 16389700s^2$$

$$- 12180500s^3 - 425309s^4 + 2731360s^5 + 1111830s^6$$

$$+ 175071s^7 + 11114.05s^8 + 199.84s^9 + s^{10}]$$

The roots of the polynomial in brackets are

$$\begin{array}{ll} .06321 & * \\ -8.666 & * \\ 1.991 & * \\ -4.9128 & * \\ -1.0098 \pm j.075002 & \\ -2.6739 & \\ -4.1631 & \\ -62.567 & \\ -116.89 & \end{array}$$

The four roots marked with asterisks are cancelled by $\Delta(s)$, leaving the remaining roots as the closed loop poles. The slight discrepancies are caused by round-off errors.

We now factorize $\Delta(s)$ as $\Delta_S(s)\,\Delta_F(s)$ where

$$\begin{aligned} \Delta_S(s) &= (s - .06318)(s - 1.991) \\ &= s^2 - 2.05418s + .12579 \end{aligned}$$

$$\begin{aligned} \Delta_F(s) &= (s + 5.057)(s + 8.666) \\ &= s^2 + 13.723s + 43.824 \end{aligned}$$

By equating coefficients of like powers of s and solving the resulting systems of linear equations,

$$\hat{H}_{11}(s) = \frac{5.98648 - 8.57339s}{\Delta_S(s)} + \frac{7.5399 + .857339s}{\Delta_F(s)} \tag{6.3.6a}$$

$$\hat{H}_{12}(s) = \frac{-.710036 - 2.08277s}{\Delta_S(s)} + \frac{-5.51215 - 1.06323s}{\Delta_F(s)} \tag{6.3.6b}$$

$$\hat{H}_{21}(s) = \frac{-2.55133 + 1.081547s}{\Delta_S(s)} + \frac{39.82326 + 4.59745s}{\Delta_F(s)} \tag{6.3.6c}$$

$$\hat{H}_{22}(s) = \frac{.349037 + .051548s}{\Delta_S(s)} + \frac{-1.16232 - .051548s}{\Delta_F(s)} \qquad (6.3.6d)$$

$H(s,\varepsilon)$ is formed as described above. For simplicity, ε_1 is taken as 1. $H_S(s)$ can now be computed ($H(s,\varepsilon)$ need not be witten out):

$$H_{S11}(s) = \frac{5.98648 - 8.57339s}{\Delta_S(s)} + \frac{7.5399}{43.824}$$
$$= \frac{.17205s^2 - 1.21076s + 6.11227}{\Delta_S(s)}$$

Similarly,

$$H_{S12}(s) = \frac{-.125779s^2 - 1.8244s - .725858}{\Delta_S(s)}$$
$$H_{S21}(s) = \frac{.90871s^2 - .785106s - 2.43702}{\Delta_S(s)}$$
$$H_{S22}(s) = \frac{-.0265225s^2 + .10603s + .345701}{\Delta_S(s)}$$

It is easily seen that $H_F(p)$ consists of the right hand terms of the sums (6.3.6) with s replaced by p. For example,

$$H_{F11}(p) = \frac{7.5399 + .857339p}{\Delta_F(p)}.$$

The compensator $\hat{C}(s)$ has no fast poles. Thus,

$$C_S(s) = \hat{C}(s)$$

and

$$C_F(p) = \begin{bmatrix} 0 & 20 \\ -20 & 0 \end{bmatrix}$$

The slow and fast closed loop characteristic polynomials are computed in the same manner as for the entire system. The slow closed loop characteristic polynomial is:

$$s^2\Delta_S(s) \cdot \det\,[I + H_S(s)\; C_S(s)]$$

$$= \frac{1}{\Delta_S(s)} \cdot [43.8934s^6 + 796.152s^5 + 1491.54s^4$$

$$- 2333.14s^3 - 6660.68s^2 - 3163.98s + 257.77]$$

The roots of the polynomial in brackets are:

.070699 *

1.98732 *

-.856861

-1.91192

-1.61333

-15.842

Again, the roots marked by asterisks are cancelled (with some round-off discrepancy) by $\Delta_S(s)$.

The fast closed loop characteristic polynomial is:

$$\Delta_F(p) \cdot \det\,[I + H_F(p)\; C_F(p)]$$

$$= \frac{1}{\Delta_F(p)} \cdot [p^4 + 140.66p^3 + 4673.89p^2$$

$$+ 45126.2p + 125955]$$

The roots of the polynomial in brackets are:

-5.02555 *

-8.66673 *

-29.7445

-97.2232

Table 6.3.1 gives a comparison of exact and approximate poles. Note that, as may be expected, the approximation is worst for poles of intermediate magnitude.

Table 6.3.1

Exact roots	Appoximate roots
$-1.0098 \pm j.075002$	-.856861, -1.61333
-2.6739	-1.91192
-4.1631	-15.842
-62.567	-29.7445
-116.89	-97.2232

An example of the product method is included for completeness.

Example 6.3.3: Let $\hat{h}(s) = \frac{s+1}{(s-1)(s+10)}$. Factorize $\hat{h}$ as

$$\hat{h}(s) = \frac{s+1}{s-1} \frac{1}{s+10}.$$ Then

$$h(s,\varepsilon) = \frac{s+1}{s-1} \cdot \frac{1}{\varepsilon s+10}$$

where $\varepsilon_1 = 1$.

$$h_S(s) = \frac{1}{10} \cdot \frac{s+1}{s-1}$$

$$h_F(p) = \frac{1}{p+10}$$

If static negative feedback of 20 is placed around $\hat{h}(s)$ then the characteristic polynomial is

$$(s-1)(s+10) + 20(s+1) = s^2 + 29s + 10,$$

yielding poles at -.34903 and -28.65.

The slow closed loop pole is computed:

$$s - 1 + 20 \cdot \frac{1}{10}(s+1) = 3s + 1,$$

which gives a slow pole at s = -.3333.

Similarly, the fast closed loop pole is given by:

$$p + 10 + 20 = p + 30 = 0.$$

The approximate poles are in good agreement with the exact poles.

VII. CONCLUSION

Some advantages of this approach are now apparent. The LQG regulator structure of Section VI.1 became intuitively obvious because internal structures for the plant and controller blocks were removed. Similarly for the design strategies of Section VI.2. The sum form decomposition of Section VI.3 corresponds to an exact block diagonalization of a system in state space form into slow and fast blocks. Again, no internal model is needed. There are some basic questions which still remain, however.

The first is one which should be easily answered. It was shown in Section III.3 that a transfer matrix described by a set of singularly perturbed state space equations is two frequency scale. The converse statement that any two frequency scale transfer matrix has a singularly perturbed state space description is most likely true. Approaching the problem through MFD methods runs into some technical difficulties. Theorem 3.2.1 reduces the question to that of realizing a "regularly perturbed" transfer matrix with a state space system whose matrices are analytic in ε. To see this, let a two frequency scale rational matrix $H(s,\varepsilon)$ be written as guaranteed by Theorem 3.2.1:

$$H(s,\varepsilon) = H_1(s,\varepsilon) + H_2(\varepsilon s,\varepsilon) + D(\varepsilon) \qquad (7.1)$$

Suppose $H_1(s,\varepsilon) = C_1(\varepsilon)(sI - A_1(\varepsilon))^{-1}B_1(\varepsilon)$ and $H_2(p,\varepsilon) = C_2(\varepsilon)(pI - A_2(\varepsilon))^{-1}B_2(\varepsilon)$ where the matrices A_i, B_i, and C_i are analytic at $\varepsilon = 0$. Then

$$H(s,\varepsilon) = C(\varepsilon)(sI - A(\varepsilon))^{-1}B(\varepsilon) + D(\varepsilon)$$

where

$$C(\varepsilon) = [C_1(\varepsilon) \ C_2(\varepsilon)]$$

$$A(\varepsilon) = \begin{bmatrix} A_1(\varepsilon) & 0 \\ 0 & \frac{1}{\varepsilon} A_2(\varepsilon) \end{bmatrix}$$

$$B(\varepsilon) = \begin{bmatrix} B_1(\varepsilon) \\ \frac{1}{\varepsilon}B_2(\varepsilon) \end{bmatrix}$$

$D(\varepsilon)$ as in (7.1).

Clearly, these are the matrices of a singularly perturbed system.

As stated in Section VI.3, the numerator denominator factorization method of introducing ε into a numerically described transfer matrix can, when applied term by term, cause duplication of poles in the multivariable case. Application of feedback can cause these multiple poles to separate and cause problems, especially in machine computation. The Smith-MacMillan form for a transfer matrix has suggestive possibilities for circumventing this difficulty. Let $H(s)$ be transformed to Smith-MacMillan form by the unimodular matrices P and Q.

$$H(s) = P(s) \ S^*(s) \ Q(s)$$

If all poles and zeros of the diagonal elements of S^* are clustered into slow and fast groups, then S^* can be factorized as $S^*(s) = S_S^*(s) \cdot S_F^*(s)$. For any unimodular matrix $R(s)$,

$$H(s) = [P(s)S_S^*(s)R(s)][R(s)^{-1}S_F^*(s)Q(s)] \qquad (7.2)$$

A number of questions arise now, the most important being under what conditions the two factors in (7.2) are proper. This is not guaranteed by having S_S^* and S_F^* proper. Given that the two factors are proper, one can proceed as in VI.3 to introduce ε. There will be no additional poles created in this case. Once the question of properness is answered, a "best" choice of the matrices P, Q, and R in (7.2) is needed. In general, P and Q are not unique even though S^* is.

One final remark should be made on maintaining real quantities in this work, as is done in [3]. In Definition 2.0.1, the field $\mathscr{F}_\varepsilon$ could have been defined with the added condition "$f(\varepsilon)$ has only real coefficients in its power series expansion about $\varepsilon = 0$." Then all results would follow as before, since $\mathscr{F}_\varepsilon$ is still a number field with this added restriction. Thus, we are assured that if a two frequency scale rational matrix $H(s,\varepsilon)$ has coefficients of s which are real for real values of ε, then $H_S(s)$ and $H_F(p)$ will have only real coefficients. Furthermore, the decomposition of Theorem 3.2.1 yields rational matrices with the same property.

BIBLIOGRAPHY

[1] P. V. Kokotovic, R. E. O'Malley, Jr., and P. Sannuti, (Singular Perturbation and Order Reduction in Control Theory-An Overview," Automatica, Vol. 12, pp. 123-132, 1976.

[2] J. H. Chow and P. V. Kokotovic, "A Decomposition of Near-Optimum Regulators for Systems with Slow and Fast Modes," IEEE Trans. Automat. Control, Vol. AC-21, pp. 701-705, October 1976.

[3] H. H. Rosenbrock, State and Space and Multivariable Theory, (New York: John Wiley & Sons, Inc.), 1970.

[4] J. H. Chow, "Preservation of Controllability in Linear Time Invariant Perturbed Systems," Int. J. Control, Vol. 25, No. 5, pp. 697-704, 1977.

[5] J. H. Chow, Ed., Time Scale Modeling of Dynamic Networks with Applications to Power Systems, (Berlin, Heidelberg, New York: Springer-Verlag), 1982.

[6] S. J. Campbell and N. J. Rose, "Singular Perturbation of Autonomous Linear Systems," Siam J. Math. Anal., Vol. 10, No. 3, pp. 542-551, May 1979.

[7] J. H. Chow, J. J. Allemong, and P. V. Kokotovic, "Singular Perturbation Analysis of Systems with Sustained High Frequency Oscillations," Automatica, Vol. 14, pp. 271-279, 1978.

[8] P. V. Kokotovic, J. J. Allemong, J. R. Winkelman, and J. H. Chow, "Singular Perturbation and Iterative Separation of Time Scales," Automatica, Vol. 16, pp. 22-23, January 1980.

[9] F. Hoppensteadt, "Analysis of Some Problems Having Matched Asymptotic Expansion Solutions," SIAM Review, Vol. 17, No. 1, January 1975.

[10] A. H. Haddad and P. V. Kokotovic, "Stochastic Control of Linear Singularly Perturbed Systems," IEEE Trans. Automat. Control, Vol. AC-22, pp. 815-821, October 1977.

[11] H. Kwakernaak and R. Sivan, Linear Optimal Control Systems, (Wiley-Interscience: New York), 1972.

[12] A. G. J. MacFarlane and B. Kouvaritakis, "A Design Technique for Linear Multivariable Feedback Systems,: Int. J. Control, Vol. 25, No. 6, pp. 837-874.

[13] B. C. Moore, "Principle Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction," IEEE Trans. Automat. Control, Vol. AC-26, pp. 17-31, February 1981.

[14] N. R. Sandell, P. Varaiya, M. Athans, and M. G. Safanov, "Survey of Decentralized Control Methods for Large Scale Systems," IEEE Trans. Automat. Control, Vol. AC-23, No. 2, pp. 108-128, April 1978.

[15] J. C. Doyle and G. Stein, "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," IEEE Trans. Automat. Control, Vol. AC-26, pp. 4-16, February 1981.

[16] J. B. Cruz, J. S. Freudenberg, and D. P. Looze, "A Relationship Between Sensitivity and Stability of Multivariable Feedback Systems," IEEE Trans. Automat. Control, Vol. AC-26, pp. 66-74, February 1981.

[17] B. Porter and A. F. Shenton, "Singular Perturbation Analysis of the Transfer Function Matrices of a Class of Multivariable Linear Systems," Int. J. Control, Vol. 21, No. 4, pp. 655-660., 1975.

[18] A. G. J. MacFarlane, Ed., Frequency-Response Methods in Control Systems, (New York: IEEE Press), 1979.

[19] K. Knopp, Theory of Functions--Part II, (New York: Dover Publications), 1947.

[20] B. A. Fuks, Analytic Functions of Several Complex Variables, (Providence, Rhode Island: American Mathematical Society), 1963.

[21] F. R. Gantmacher, The Theory of Matrices, (New York: Chelsea Publishing Company), 1977.

[22] G. A. Bliss, Algebraic Functions, AMS Colloquium Pub., Vol. VI, 1933.