GRAPHS AND THEIR SUBDIVISIONS

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
M. JAMES STEWART
1972

This is to certify that the

thesis entitled

Graphs and their Subdivisions

presented by

M. James Stewart

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

E. A. Nordhaus
Major professor

Date Nov. 10, 1972

O-7639

ABSTRACT

GRAPHS AND THEIR SUBDIVISIONS

By

M. James Stewart

The concept of a subdivision of a graph is an old concept which is widely used in various graph-theoretic constructions and underlies the notion of graph-homeomorphism. A particular type of subdivision of a graph G, denoted by $S_n(G)$ and referred to as the nth subdivision graph of G, is obtained by replacing every edge of G by a path of length n + 1. The nth subdivision graph of G is related to three other graph-valued functions: the nth power G^n, the total graph T(G), and the line graph L(G). Although the graphs G^n, T(G), and L(G) have received a great deal of attention in the literature, surprisingly little research has been devoted to the investigation of $S_n(G)$. The main purpose of this thesis is to add to the knowledge of subdivisions of graphs and, in particular, to study the properties of the nth subdivision graph and to obtain a characterization of it.

The definitions of terms referred to in this thesis are presented in Chapter 1, as well as much of the notation that will

be employed throughout. A few well known graph-theoretic results useful in this thesis are also stated here.

The first section of Chapter 2 traces the development of the graph-homeomorphism concept from its toplogical origins to the equivalence classes of the relation "is homeomorphic with," where particular attention is paid to the canonical representatives of these classes, the homeomorphically irreducible graphs. In addition to the multiple characterization of these homeomorphically irreducible graphs in terms of paths, triangles, and matrices, in sections 2.2 and 2.3 several graph-theoretic properties, including colorability, traversibility, and automorphism groups, and the way these relate to homeomorphically irreducible graphs, are discussed. In section 2.4 the hamiltonian index of graphs contained in certain homeomorphism classes of nonhamiltonian graphs is investigated.

A complete characterization of the nth subdivision graph involving lengths of trails is presented in the first section of Chapter 3, followed by some considerations of the structural relationship of a graph G with its nth subdivision graph $S_n(G)$. Except for section 3.4, which deals with the enumeration of trees having a 1-factor, the remainder of this chapter is devoted to the development of various salient characteristics of nth subdivision graphs. In particular, for any graph G, necessary and sufficient conditions for the existence of a 1-factor of $S_n(G)$, G^n $(n \geq 2)$, and T(G) are obtained in section 3.3 and generalizations of these results utilizing the edge independence number are discussed in the final section 3.5

GRAPHS AND THEIR SUBDIVISIONS

By

M. James Stewart

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1972

DEDICATION

To

Gary,

Shashi,

Tricia's Mother,

and the Local Leo.

ACKNOWLEDGMENTS

Words cannot suffice to thank my good friends Gary Chartrand and Shashi Kapoor, who are in large part responsible for giving me the motivation to initiate research of my own in graph theory.

A special debt of gratitude is owed to my "academic father," Professor E. A. Nordhaus, without whose inspiration, guidance, and psychological support this thesis could not exist. I thank him for his concern, his knowledge, and the time and effort he has cheerfully spent on my behalf.

My gratitude goes also to Professors E. M. Palmer and B. M. Stewart for the helpful suggestions and numerous kindnesses, often unsolicited, they have extended to me. I thank also Professors M. Fuchs and V. P. Sreedharan for serving on my advisory committee.

Most importantly, my family has been an unseen but indispensable source of strength to me, especially Joyce I. Stewart. I realize better in retrospect than I did at the time just how much her consideration, understanding, and support has contributed to the accomplishment this dissertation represents. I am very grateful.

Over and above all else, I thank my wife, Mary--without her there would be no point to this endeavor.

Finally, I am indebted to the people of the Lansing Community College district for their financial support through a sabbatical leave, and to the Department of Mathematics at Michigan State University for the gracious hospitality I enjoyed throughout the sabbatical year.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

Section 1.1
Introduction

One of the most celebrated theorems in graph theory is Kuratowski's theorem, which characterizes planar graphs as those graphs containing no subgraph isomorphic to a subdivision of the complete graph K_5 or to the complete bipartite graph K(3,3). In addition to its central role in Kuratowski's theorem and in numerous other graph-theoretic constructions, the concept of the subdivision of a graph underlies the notion of graph-homeomorphism. This relation is defined more precisely in Chapter 2 and there is shown to be an equivalence relation on the set of all graphs.

A graph S(G), referred to as the subdivision graph of G, is also studied and is an example of a graph-valued function which is intimately related to a well known trio of graph-valued functions: the total graph T(G), the line graph L(G), and the nth power G^n. In 1965, M. Behzad showed that the total graph T(G) of a graph G is isomorphic to the square of the subdivision graph S(G). The total graph T(G) of any graph G contains both S(G) and the line graph L(G) as subgraphs. The nth subdivision graph $S_n(G)$, a generalization of S(G), is a graph-valued function which is associated with a graph G and is obtained by replacing every edge of G by a path of length n +1.

Although the line graph, total graph, and, to a lesser extent, the nth power of a graph have received much attention in the literature, surprisingly little research has been devoted to the investigation of the nth subdivision graph function. One of the main purposes of this thesis is to study the properties of the nth subdivision graph $S_n(G)$ of a graph G from the viewpoint that the pair $[G, S_n(G)]$ is a specific instance of homeomorphic graphs.

Definitions of terms used in this thesis are presented in Chapter 1, along with much of the notation that will be employed. A few well known results are also stated here.

The first section of Chapter 2 traces the development of the graph-homeomorphism concept from its topological origins to the equivalence classes of the relation "is homeomorphic with," where particular attention is paid to the canonical representatives of these classes, the homeomorphically irreducible graphs. In addition to the multiple characterization of these homeomorphically irreducible graphs in terms of paths, triangles, and matrices, in sections 2.2 and 2.3 several graph-theoretic properties, including colorability, traversibility, and automorphism groups, and the way these relate to homeomorphically irreducible graphs, are discussed. In section 2.4 the hamiltonian index of those graphs in certain homeomorphism classes of nonhamiltonian graphs is investigated.

A complete characterization of the nth subdivision graph involving lengths of trails is presented in the first section of Chapter 3, followed by some considerations of the structural relationship of a graph G with its nth subdivision graph $S_n(G)$. Except for section 3.4, which deals with the enumeration of trees having a 1-factor, the remainder of this chapter is devoted to the development of various salient characteristics of nth subdivision graphs. In particular, for any graph G, necessary and sufficient conditions for the existence of a 1-factor of $S_n(G)$, G^n $(n \geq 2)$, and T(G) are obtained in section 3.3 and generalizations of these results utilizing the edge independence number are discussed in the final section 3.4.

Section 1.2
Basic Terminology

We set forth here some basic definitions and notation which will be used in the following chapters. For those terms not given here, the reader is directed to [2] or [9].

A graph G is a finite, nonempty set V together with a set E of two-element subsets of V. Each element of V is referred to as a vertex or a point and V itself as the vertex set or point set of G; the members of the edge set E are called edges. In general, the vertex set and edge set of a graph G will be denoted by V(G) and E(G), respectively.

There are variations of graphs to which we shall occasionally make reference. In a multigraph, we allow the presence of more than one edge joining a pair of distinct points; such edges are called multiple edges. If loops are also permitted, that is, edges which join a point to itself, we have a pseudograph.

The order of a graph G, denoted $|G|$, is the number of elements in V(G); if $|G| = p$ and E(G) has q elements, we say G is a (p,q)-graph. A graph G is called *empty* if E(G) is the empty set. The *degree* of a point v of G is the number of edges of G incident with v and is denoted $\deg_G v$, or simply deg v, if no confusion is likely to result by leaving the graph G unspecified. A point of degree 1 is called an *endpoint* of G. When every point of G has degree r, we say that G is *regular of degree r*, or *r-regular*. Two graphs G_1 and G_2 are called *isomorphic*, expressed by writing $G_1 = G_2$, if there exists a one-to-one correspondence between $V(G_1)$ and $V(G_2)$ which preserves adjacency and nonadjacency.

The subgraph *induced* by a set U of points of G, denoted <U>, is that subgraph having U as its point set and whose edge set consists of all edges of G incident with two points of U. In a completely analogous way, we speak of the subgraph <F> induced by a nonempty subset of E(G). If v is a point of G, then G-v denotes the graph <V(G) - {v}> and, in general, if S is a proper subset of V(G), then G-S represents the graph <V(G) - S>. Similarly, G-F represents <E(G) - F> for any proper subset F of E(G), and, in particular, G-x represents <G - {x}> for any edge x of G. A subset W of V(G) is *independent* if <W> is empty. Two graphs are said to be *disjoint* if their vertex sets are disjoint. The *union* of two graphs G_1 and G_2 is the graph G given by $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

For points u and v of a graph G, a *u-v trail* is a finite alternating sequence of points and edges of G, in which no edge is repeated, beginning with u and ending with v, such that every edge is immediately preceded and succeeded by the two points with which it is incident (if we allow an edge to be repeated, we have a walk from u to v). A u-v path P is a u-v trail in which no point is repeated and the points of P distinct from u and v are called interior points of P. Two u-v paths in a graph G are vertex-disjoint (or simply disjoint) if they have no vertices in common, other than u and v. Edge-disjoint paths have no common edge. We say a u-v trail is closed or open depending on whether u = v or u ≠ v, and a trivial trail is one containing no edges. A nontrivial closed trail of G in which no points are repeated (except the first and last) is called a cycle of G. The number of edges in a u-v trail is called its length, denoted as d(u,v), and a cycle of length n is an n-cycle. A graph of order n which is a path or a cycle is denoted by P_n or C_n, respectively.

A graph G is connected if there exists a u-v path in G for every two points u and v of G. The relation "is connected to" is an equivalence relation on the vertex set of a graph G and the subgraphs induced by the points in an equivalence class are called the components of G. The distance d(u,v) between points u and v of a connected graph is the length of the shortest u-v path. A point v of G is a cutpoint of G if G-v has more components than does G. A block of a graph G is a maximal nontrivial connected subgraph of G containing no cutpoint of itself. For a connected graph G with at least three points, the following are equivalent (see Chapter 3):

(1) G is a block.

(2) Every two points of G lie on a common cycle.

(3) For every three distinct points of G, there is a path joining any two of them which does not contain the third.

A block of order 3 or more is called a cyclic block while the block of order 2 is called the acyclic block. An endblock of a connected graph G having more than one block is a block containing exactly one cutpoint of G.

There are several special classes of graphs to which we will frequently make reference. An acyclic graph or forest is a graph with no cycles. A planar graph is one which can be embedded in the plane. A (planar) graph which can be embedded in the plane such that each of its points lies in the boundary of the exterior region is called outerplanar. The complete graph K_p has every pair of its p points adjacent. A triangle is a subgraph isomorphic to K_3. A bipartite graph G is a graph whose vertex set V(G) can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G is of the form v_1v_2 where $v_i \in V_i$, i = 1, 2. If V_1 and V_2 have m and n points and G has mn edges, we say that G is a complete bipartite graph and write G = K(m,n). G. Chartrand and F. Harary ([9], p. 107) have shown that a graph is outerplanar if and only if it has no subgraph homeomorphic with K_4 or K(2,3), except K_4-x (where x is any edge of K_4).

To any given nonempty graph G, there are associated with G several special graphs which we will encounter. The nth power G^n of G is that graph with $V(G^n) = V(G)$ such that an edge $uv \in E(G^n)$ if and only if $1 \leq d(u,v) \leq n$ in G. The line graph L(G) of G has vertex set V(L(G)) which can be placed in one-to-one correspondence with E(G) in such a way that two points of L(G) are adjacent if and only if the corresponding edges of G are adjacent. The _total graph_ T(G) has vertex set V(T(G)) which can be placed in one-to-one correspondence with the points and edges of G in such a way that two points of T(G) are adjacent if and only if the corresponding elements of G are adjacent or incident.

CHAPTER 2

Section 2.1
Homeomorphic Graphs

A collection of p points (0-simplexes) and q arcs (1-simplexes) joining certain pairs of points which is embedded in 3-space in such a way that every intersection or arcs occurs only at some of the p points is a finite geometric simplicial 1-complex, or simply a 1-complex. Two 1-complexes $\mathcal{C}_1$ and $\mathcal{C}_2$ are homeomorphic if there exists a one-to-one bicontinuous mapping from $\mathcal{C}_1$ onto $\mathcal{C}_2$. Since every graph is realizable as a 1-complex and every 1-complex can be embedded in 3-space, there is at least one associated geometric 1-complex for every graph. This observation serves as the motivation for defining two graphs to be <u>homeomorphic with</u> each other (or simply <u>homeomorphic</u>) if their associated 1-complexes are topologically

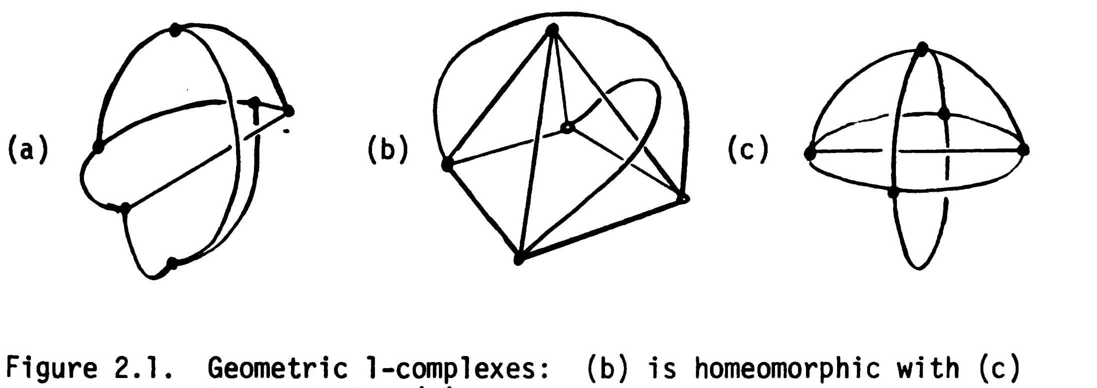


Figure 2.1. Geometric 1-complexes: (b) is homeomorphic with (c) but not with (a).

equivalent, that is, homeomorphic. Each graph is then said to be a homeomorph of the other. It is a consequence of this definition (see [17]) that two homeomorphic graphs can differ only by the number of points of degree 2 that they possess.

Another approach to the study of homeomorphic graphs involves the concept of a subdivision (see [2], Chapter 8). An elementary subdivision of a graph G is a graph obtained from G by the removal of a edge uv of G and the addition of a new point w together with the edges uw and vw. Occasionally we may describe an elementary subdivision of G by stating that a new vertex w is inserted on the edge uv of G, or that uv has been subdivided into the edges uw and vw. A subdivision of G is a graph obtained from G by a finite sequence of elementary subdivisions. Using this terminology, two graphs G_1 and G_2 are homeomorphic if either $G_1 = G_2$ or there exists a graph G_3 such that both G_1 and G_2 are subdivisions of G_3.

Sometimes it is useful to be able to determine when a graph G_2 which is homeomorphic with a graph G_1 is a subdivision of G_1. We say that G_2 is homeomorphic from G_1 if either $G_2 = G_1$ or G_2 is a subdivision of G_1. In Figure 2.2, G_1 and G_3 are homeomorphic graphs, neither of which is homeomorphic from the other, but both are homeomorphic from G_2.

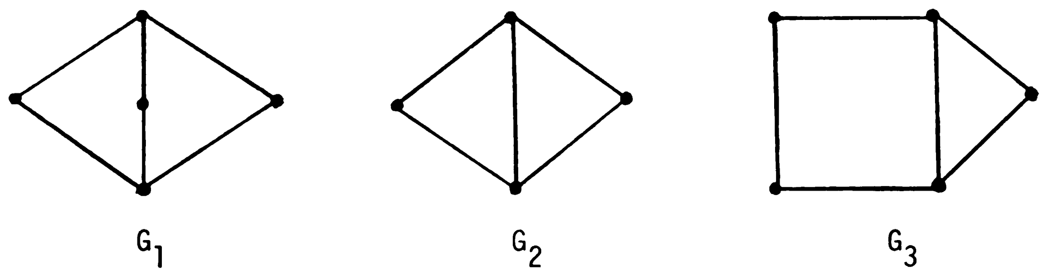


Figure 2.2. Homeomorphic graphs.

Although the relation "homeomorphic from" is not in general symmetric, the slightly more general relation "homeomorphic with" is clearly both symmetric and reflexive. We next verify that it is also transitive, showing that this relation is an equivalence relation. Before proceeding, however, we find it helpful to introduce some additional notation. Whenever a pair of points u and v of a given graph, neither deg u nor deg v being 2, are joined by r distinct u-v paths having interior points only of degree 2, let these paths be denoted by $Q_{uv}^{(i)}$, for $1 \leq i \leq r$.

To check the transitivity, suppose that the graph G_1 is homeomorphic with G_2 and also that G_2 is homeomorphic with G_3. This means, by definition, that there exist graphs G_4 and G_5 such that G_1 and G_2 are homeomorphic from G_4 and that G_2 and G_3 are homeomorphic from G_5. Now construct G_6 from G_2 by (1) replacing each $Q_{uv}^{(1)}$ by the edge uv if u and v are not adjacent, otherwise by a u-v path of length 2, and (2) replacing each $Q_{uv}^{(i)}$ by a u-v path of length 2, for $i \geq 2$. Since applying a sequence of elementary subdivisions to a graph only increases the length of paths $Q_{uv}^{(i)}$, both G_4 and G_5 are homeomorphic from G_6. This implies that both G_1 and G_3 are also homeomorphic from G_6, hence G_1 is homeomorphic with G_3, and we have established the fact that the relation "homeomorphic with" is an equivalence relation on the set of all graphs.

Section 2.2
Homeomorphically Irreducible Graphs

We have seen in the preceding section that under the equivalence relation "homeomorphic with" all graphs are partitioned into classes, with two graphs belonging to the same class if and only if they are homeomorphic with each other. In [2], it is shown that in each such equivalence class $\mathcal{C}$ there exists a unique graph H with the minimum number of points of all graphs in this class. For convenience, we include here a reproduction of that result and its proof.

Theorem 2.1. In every class $\mathcal{C}$ of homeomorphic graphs, there exists a unique graph H such that if $G \in \mathcal{C}$, then G is homeomorphic from H.

Proof. Let $\mathcal{C}$ be a class of homeomorphic graphs and let H be a member of $\mathcal{C}$ having the minimum number of points. For every graph $G \in \mathcal{C}$, G and H are homeomorphic, so that there is a graph H_1 such that both H and G are homeomorphic from H_1. Since H is homeomorphic from H_1, either $H = H_1$ or H is a subdivision of H_1, the latter implying that H_1 has fewer points than H (a contradiction because $H_1 \in \mathcal{C}$). Hence $H = H_1$, so that G is homeomorphic from H. It follows immediately that H is unique.

In light of Theorem 2.1, we define a graph H to be homeomorphically irreducible, or simply irreducible, if whenever a graph G is homeomorphic with H, then G is homeomorphic from H. By Theorem 2.1, each irreducible graph serves as a canonical representative of the homeomorphism class to which it belongs.

There are several ways in which irreducible graphs can be characterized; before presenting these, we introduce some additional terms.

Two graphs G_1 and G_2 are considered equal (written $G_1 = G_2$) if they are isomorphic. A graph G_1 is said to be less than G_2, denoted $G_1 < G_2$, if G_1 has fewer points than G_2, or if they have the same number of points and G_1 has fewer edges than G_2. If G_1 is less than or equal to G_2 according to the above definitions, we write $G_1 \leq G_2$. We observe in passing that if neither $G_1 < G_2$ nor $G_2 < G_1$ holds, it does not necessarily follow that $G_1 = G_2$.

The adjacency matrix of a graph G with p points v_i, $1 \leq i \leq p$, is the p x p matrix $A = [a_{ij}]$, where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise.

Theorem 2.2. For any graph G, the following statements are equivalent:

(1) G is irreducible.

(2) H a homeomorph of G implies $G \leq H$.

(3) No shortest path in G has an interior point of degree two.

(4) Every point of degree 2 in G lies on a triangle.

(5) If A denotes the adjacency matrix of G, then the matrix $A^2 + A^3$ has no diagonal entry equal to 2.

Proof. (1) implies (2). Let G be irreducible, and suppose H is homeomorphic with G. Then H is homeomorphic from G, and either G = H or H can be obtained from G by a sequence of elementary subdivisions. In the latter case, H contains more points than G, so that G < H. Hence $G \leq H$ in all cases.

(2) implies (3). Let G be a graph having the property that whenever a graph H is homeomorphic with G, then $G \leq H$. Suppose G has a shortest path P containing an interior point w of degree 2, where, say, w is adjacent to points u and v. Then by replacing the point w together with the edges uw, wv by the single edge uv, a homeomorph H of G is produced which has one less point than G, a contradiction. Therefore, no such shortest path P can exist.

(3) implies (4). Let G be a graph in which every shortest path contains no interior point of degree 2. Let w be a point of degree 2, adjacent, say, to points u and v. If u and v are not adjacent, there exists a shortest u-v path containing w as an interior point of degree 2. Hence the three points u, v, and w determine a triangle.

(4) implies (5). Let G be a graph in which every point of degree 2 lies on a triangle, and suppose the p points of G are labeled v_i, $1 \leq i \leq p$. If A is the adjacency matrix of G, then it is well known [9, p. 151] that the i,j entry of A^n is the number of walks of length n between v_i and v_j. In particular, the i,i entry

in A^2 is simply the degree of v_i in G, while the i,i entry in A^3 is twice the number of 3-cycles containing v_i. Hence the i,i entry in $A^2 + A^3$ is the sum of the degree of v_i and twice the number of 3-cycles containing v_i. Clearly this entry has the value 2 if and only if v_i has degree 2 but lies on no 3-cycle. Thus no diagonal entry in $A^2 + A^3$ can have the value 2.

(5) implies (1). Let G be a graph with adjacency matrix A, and assume the matrix $A^2 + A^3$ has no diagonal entries with the value 2. As we have just seen in the preceding paragraph, this is equivalent to saying every point of degree 2 lies on a 3-cycle. If G is not irreducible, then there exists a graph H different from G such that G is homeomorphic from H. Hence at least one line of H has been subdivided (one or more times) in the process of producing G. This implies that G contains a point which is adjacent to exactly two points which are not themselves adjacent, a contradiction. Thus G is irreducible.

A point of degree 2 not lying on a triangle is called suppressible. Thus Theorem 2.1 states that a graph G is irreducible if and only if it possesses no suppressible point. As a simple application of this, in Figure 2.2 only the graph G_2 is irreducible.

Section 2.3
Some Properties of Irreducible Graphs

In this section we investigate several graph-theoretic concepts as they relate to irreducible graphs and their homeomorphs.

The connectivity $\kappa(G)$ of a graph G is the minimum number of points whose removal either disconnects G or results in the trivial graph consisting of a single point. We say G is n-connected if $\kappa(G) \geq n$. If G is irreducible and H is homeomorphic with G, and if the removal of a certain set of points disconnects G, then the removal of this same set of points disconnects H. Furthermore, if H is not isomorphic to G, then H contains a point v of degree 2 so that $\kappa(H) \leq 2$, as seen by removing the points of H adjacent to v. These observations give our next result and its corollary.

Proposition 2.1. If G is irreducible and H is homeomorphic with G, then $\kappa(G) \geq \kappa(H)$. Indeed, if G is not isomorphic to H, then $\kappa(H) \leq 2$.

Corollary 2.1. If G is 3-connected, then G is irreducible.

A graph is said to be n-colorable if it is possible to assign to each point of G one color of a set of n colors so that every two adjacent points are assigned different colors. The chromatic number $\chi(G)$ of G is the smallest value of n for which G is n-colorable.

If G is a graph whose chromatic number $\chi(G)$ exceeds 2 and H is a subdivision of G, then H differs from G only by the presence of one suppressible point w for each elementary subdivision that was performed upon G to yield H; a k-coloring of H is then produced by assigning to each such suppressible point w of H any color which is different from the colors of the two points adjacent to w. Such a statement cannot be made, of course, if $\chi(G) = 2$ (for example, consider a cycle of length 4). This can now be summarized as:

Proposition 2.2. If G is irreducible with $\chi(G) \geq 3$ and if H is homeomorphic with G, then $\chi(H) \leq \chi(G)$.

Proposition 2.2 asserts that an irreducible graph G with $\chi(G) \geq 3$ has the largest chromatic number of all graphs in its homeomorphism class $\mathcal{C}$. Sharing the class $\mathcal{C}$ with G is the bipartite graph S(G) derived from G by performing one elementary subdivision on every edge of G, so that $\chi(S(G)) = 2$. Thus each graph $H \in \mathcal{C}$ has chromatic number $\chi(H)$ such that $2 \leq \chi(H) \leq \chi(G)$. From this arises the question of the existence of a graph $H \in \mathcal{C}$ with $\chi(H) = k$ for every value of k where $2 < k < \chi(G)$.

Theorem 2.3. Let G be any (not necessarily irreducible) graph with $\chi(G) \geq 3$. Then there exists a graph H homeographic from G with $\chi(H) = k$ for each value of k such that $2 < k < \chi(G)$.

Proof. Put $n = \chi(G)$ and let any n-coloring of G be given, $n \geq 4$ (if $n = 3$, there is nothing to prove). The set of all points with any one color is called a color class; thus the points of G are partitioned into n color classes $C_1, C_2, \ldots, C_n$, any two of which have at least one edge joining them. Let m be the minimum number of edges joining any pair of color classes; denote any two color classes joined by m edges as C and C'. Now obtain the graph G' from G by performing one elementary subdivision on each of the m edges joining C and C'. Then a minimal (n-1)-coloring is achieved for G' by placing the m new subdivision points in any class distinct from C and C', and coalescing C and C' into one single class. Thus $\chi(G') = n-1$.

Now if $\chi(G') \geq 4$, repeat this procedure with G', obtaining G'' with $\chi(G'') = n-2$. In this way we obtain a subdivision H of G with chromatic number k for every value of k such that $2 < k < n$.

Proposition 2.2 and Theorem 2.3 describe in general the possible alteration in chromatic number of a graph produced by a sequence of subdivisions. More specifically we may examine the possible change in the chromatic number produced by a single elementary subdivision. It is not difficult to see that such a subdivision H of G may result only in $\chi(H) = \chi(G)$, $\chi(H) = \chi(G) - 1$, or $\chi(H) = \chi(G) + 1$, as illustrated by the elementary subdivision of a path, 3-cycle, or 4-cycle, respectively. Of course if $\chi(G) \geq 3$, then the discussion leading to Proposition 2.2 also shows that $\chi(H) \leq \chi(G)$. In the case where $\chi(H) < \chi(G)$, an equivalent formulation is given by

Theorem 2.4. Let $\chi(G) \geq 3$. There exists an elementary subdivision G' of G with $\chi(G') = \chi(G) - 1$ if and only if there exists a $\chi(G)$-coloring such that for some two colors, precisely one edge of G joins points having these colors.

Proof. Sufficiency. Assume that there exists a $\chi(G)$-coloring in which for some two colors α and β, there is precisely one edge uv joining points u and v having these colors. Letting G' be the graph obtained from G by the insertion of a new point w on the edge uv, w can be assigned any color γ distinct from α and β, and now all α-colored and β-colored points can be colored alike, since in G' no two such points are adjacent. Thus $\chi(G') = \chi(G) - 1$.

<u>Necessity</u>. Let $\chi(G) = k \geq 3$, let G' be any elementary subdivision of G with $\chi(G') = \chi(G) - 1$, and assume that for every k-coloring and every pair of colors α, β, there exists at least two edges joining points colored α and β (no $\chi(G)$-coloring can exist in which there are two colors for which no edge of G joins points of these colors). The set of all points of G are partitioned into k color classes by every k-coloring. Let uv be the edge of G on which w is inserted to produce G', and let μ be any (k-1)-coloring of G'.

CASE 1. Assume u and v belong to distinct classes of μ. Then w is in a class distinct from those of u and v. But G can be reconstructed from G' by replacing uw, w, and wv by the edge uv, so that μ serves as a (k-1)-coloring of G--a contradiction.

CASE 2. Assume u and v are in the same class of μ. Then we may introduce a new color class C_u containing only u; call this new coloring μ^*. Now replace uw, w, and wv in G' by uv, reconstructing G. Then μ^* is a k-coloring of G, but C_u is a color class joined to only one other class by a single edge--again a contradiction.

Hence some k-coloring exists such that for some pair of colors, exactly one edge joins points having these colors.

Corollary 2.4. Let $\chi(G) = 3$. If G' is an elementary subdivision of G with $\chi(G') = 2$, then all odd cycles of G share a common edge.

Proof. By Theorem 2.4, there is a $\chi(G)$-coloring in which some pair of color classes have exactly one edge uv joining them. Since all edges of G join points in distinct color classes, all odd cycles (particularly 3-cycles) must pass through uv (see Figure 2.3).

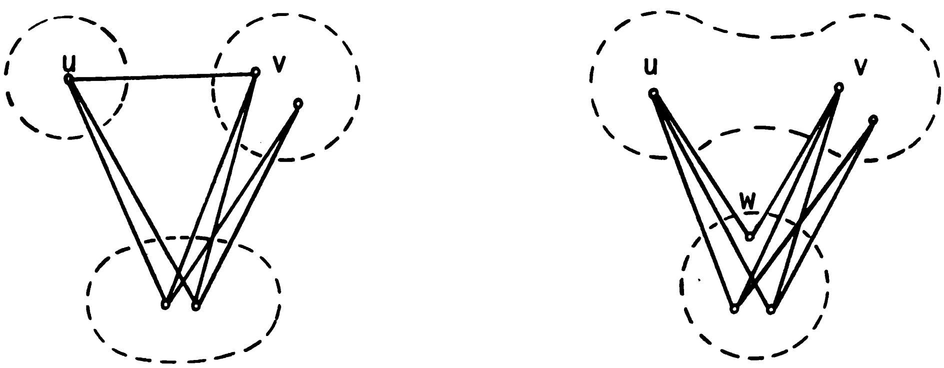


Figure 2.3. An example illustrating Corollary 2.4.

Another parameter which involves partitioning the point set of a graph G is the point-outerthickness $\pi(G)$ of G (denoted $\pi_3(G)$ in [4]). For any graph G, $\pi(G)$ is the minimum number of subsets contained in every partition Π of the point set of G, where the subgraph induced by each subset is outerplanar. Such a partition Π for G consisting of exactly $\pi(G)$ subsets will be henceforth referred to as a minimal partition. Clearly if G' is any elementary subdivision of G, then $\pi(G')$ can exceed $\pi(G)$ by at most 1 (a new partition Π' can be formed from an old minimal partition Π by adding to Π the singleton subset consisting of the new suppressible point).

However, if $\pi(G') = \pi(G) + 1$, and uv is the edge of G subdivided by w to produce G', then in each minimal partition Π for G there must be some subset $S \in \Pi$ to which both u and v belong (if u and v belong to distinct subsets S, S' in Π, then $S \cup \{w\}$ is outerplanar and $\Pi \cup [S \cup \{w\}]$ is a minimal partition for G' composed of only $\pi(G)$ subsets). Since the insertion of w makes the induced subgraph $\langle S \rangle$ nonouterplanar, then u and v must belong to some cycle in $\langle S \rangle$ not containing uv. Then the insertion of w produces a subgraph of $\langle S \cup \{w\} \rangle$ which is homeomorphic from $K(2,3)$ (see [9], p. 107). Conversely, if u and v are situated as prescribed for every partition of G, then $\pi(G') = \pi(G) + 1$. We summarize this as:

Proposition 2.3. Let G' be the elementary subdivision of G produced by subdividing the edge uv. Then $\pi(G') = \pi(G) + 1$ if and only if each minimal partition contains a subset in which u and v belong to a cycle not containing uv.

The group $\Gamma(G)$ of a graph G consists of the set of all isomorphisms of G under the operation of composition. Many of the basic properties possessed by these groups can be found in [9, Chapter 14]. Here we discuss the relationship of the automorphism group of an irreducible graph G with the automorphism group of a homeomorph of G. We note that each point of a graph must map into a point of like degree under isomorphism.

Many possibilities arise when one considers the orders of the groups of an irreducible graph and one of its homeomorphs. For example, if G is a path P_2 with two points and if H is homeomorphic with G, then $|\Gamma(G)| = |\Gamma(H)| = 2$. If G is a cycle of length 3 and H is homeomorphic with G, then H is a cycle of length n for some $n \geq 3$. Hence, $|\Gamma(G)| = 6$, while $|\Gamma(H)| = 2n$, so that $|\Gamma(H)| \geq |\Gamma(G)|$. For the irreducible graph G of Figure 2.3 and the graphs H_1 and H_2 homeomorphic with G, we have $|\Gamma(G)| = 2$, $|\Gamma(H_1)| = 1$, and $|\Gamma(H_2)| = 8$, so that $|\Gamma(G)| > |\Gamma(H_1)|$ but $|\Gamma(G)| < |\Gamma(H_2)|$.

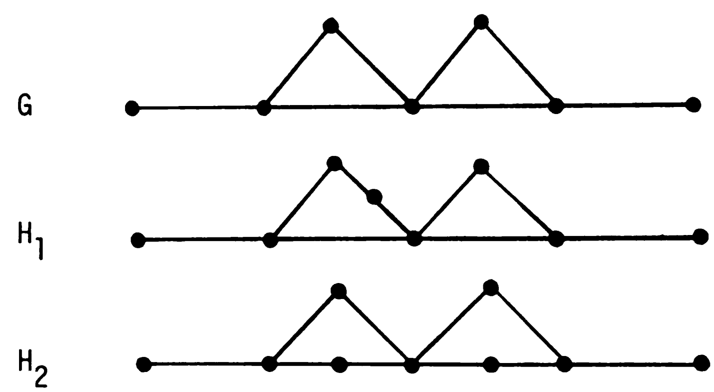


Figure 2.4. Homeomorphs H_1 and H_2 of an irreducible graph G having $|\Gamma(H_1)| < |\Gamma(G)| < |\Gamma(H_2)|$.

This uncertain situation can be remedied, however, by requiring that G have no points of degree 2.

<u>Theorem 2.5</u>. If a graph G has no points of degree 2 (and so is irreducible) and if H is homeomorphic with G, then $\Gamma(H)$ is isomorphic to a subgroup of $\Gamma(G)$.

Proof. Since H is homeomorphic from G, the points of H consist of the points of G together with the points of degree 2 resulting from however many elementary subdivisions are required to transform G into H. If α denotes any isomorphism of H, then α must map the points of G onto themselves, since the points of G have degrees other than 2. Now let α' be the restriction of α to the set of points of G (so α' is one-to-one since α is). To show that α' is an isomorphism of G, consider any two adjacent points u and v of G. If u and v are also adjacent in H, then $\alpha(u)$ is adjacent to $\alpha(v)$ so that $\alpha'(u)$ is adjacent to $\alpha'(v)$. If u and v are not adjacent in H, then the edge uv of G has been subdivided, say $u, u_1, u_2, \ldots, u_n, v$, in the process of producing H. However, $\alpha(u), \alpha(u_1), \alpha(u_2), \ldots, \alpha(u_n), \alpha(v)$ is a path in H in which $\alpha(u_i)$, $i = 1, 2, \ldots, n$, is a point of degree 2; thus this path is a subdivision of the edge $\alpha(u)\alpha(v)$. Hence, in this case, $\alpha'(u)$ is also adjacent to $\alpha'(v)$ so that α' is an isomorphism of G.

Under any isomorphism of H, a path whose interior points have degree 2 can only be mapped into a like path and the endpoints of such a path must be mapped into the endpoints of the image path. Therefore, each element $\alpha \in \Gamma(H)$ uniquely determines an element $\alpha' \in \Gamma(G)$, and two distinct elements of $\Gamma(H)$ produce two distinct elements of $\Gamma(G)$. From these observations and the manner in which the mappings α' were defined, it follows that the set Γ' of all such mappings is a subgroup of $\Gamma(G)$ isomorphic to $\Gamma(H)$.

Corollary 2.5. If a graph G has no point of degree 2, and if H is homeomorphic with G, then $|\Gamma(H)| \leq |\Gamma(G)|$.

Pertaining to the area of traversibility, we say that a graph G is eulerian if it contains a closed trail, called an eulerian trail, which includes every edge of G exactly once and every point at least once. A graph is hamiltonian if it has a cycle, which we refer to as a hamiltonian cycle, which encounters every point of the graph exactly once. A well-known theorem of Euler (see [9], p. 64) states that a graph is eulerian if and only if it is connected and every point has even degree. Thus for an irreducible graph H and any homeomorph G of H, we have

Proposition 2.4. If H is irreducible and G is homeomorphic from H, then G is eulerian if and only if H is eulerian.

The statement obtained through the substitution of "hamiltonian" for "eulerian" in Proposition 2.4 is false, as can be seen by taking H to be the graph made up of two 3-cycles sharing a common edge x and G obtained from H by inserting a new point on x.

Two rather more stringent traversibility requirements are for a graph G to be randomly eulerian from a point $v \in G$ and to be randomly hamiltonian from a point $v \in G$. The first of these two requirements means that G contains a point v such that the following procedure always results in an eulerian trail: Begin at v and traverse any incident edge. On arriving at a point, choose any incident edge which has not yet been traversed. When no unused edges are available, the procedure terminates. Analogously, G is randomly hamiltonian from one of its points v if a hamiltonian cycle always results from this procedure: Begin at v, proceed to any adjacent point. On arriving at a point, select any adjacent point not previously encountered. When no unused points remain, and an edge exists between the final point chosen and v, the process terminates.

In 1951 Ore [12] characterized graphs which are randomly eluerian from a point v as those graphs in which every cycle contains v. The following result is an immediate corollary.

Proposition 2.5. If H is irreducible and G is homeomorphic from H, then G is randomly eulerian from one of its points if and only if H is randomly eulerian from one of its points.

As was the case with Proposition 2.4, the substitution of "hamiltonian" for "eulerian" in Proposition 2.5 produces a false statement, as illustrated by the example in Figure 2.5.

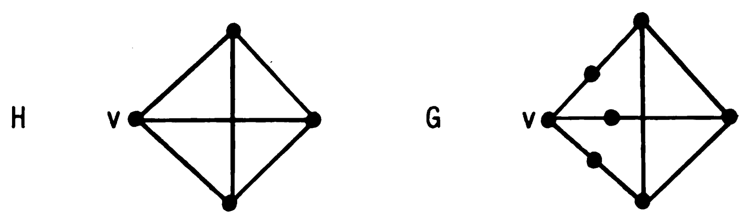


Figure 2.5. H is randomly hamiltonian from v, but possesses a subdivision G which is not hamiltonian.

The example of Figure 2.5 prompts an additional remark about homeomorphism equivalence classes. Although the irreducible graph of a class containing infinitely many nonhamiltonian graphs may be hamiltonian, the existence of one hamiltonian graph in a class guarantees that the irreducible graph for that class is also hamiltonian (and similarly for the property of being randomly hamiltonian from a point).

Section 2.4
Certain Homeomorphism Classes and the Hamiltonian Index

A related aspect of the hamiltonian property is the <u>hamiltonian index</u> h(G) of a graph G. For any connected graph G, which is not a simple path, h(G) is defined as the smallest nonnegative integer n such that the iterated line graph $L^n(G)$ is hamiltonian.

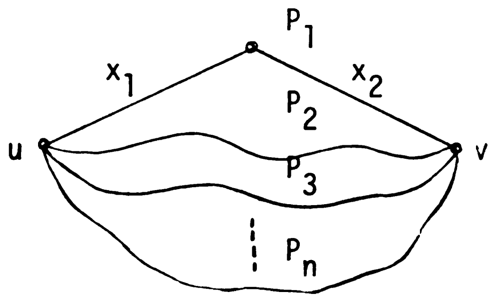
It is well known (see [9], p. 66) that every hamiltonian graph is 2-connected and that every 2-connected nonhamiltonian graph contains a subgraph homeomorphic from K(2,3). Thus the graphs homeomorphic from K(2,3), and, more generally, graphs homeomorphic from K(2,n) (for $n \geq 3$), provide an especially notorious class of nonhamiltonian graphs. Consequently, for such graphs G, the value of h(G) is positive. A natural question, then, to ask of a homeomorph of K(2,n) is this: Is 1 the best possible lower bound, in general, for h(G), and can the precise value of h(G) be determined? This question is answered in the affirmative by our next result.

<u>Theorem 2.6</u>. Let G be any graph homeomorphic from K(2,n), $n \geq 3$, and let u and v denote the points of G having degree n. Then

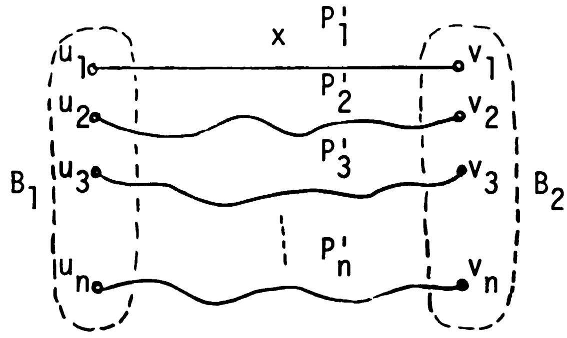
$$h(G) = \begin{cases} 1 & \text{if } n \text{ is even.} \\ d(u,v) - 1 & \text{if } n \text{ is odd.} \end{cases}$$

<u>Proof</u>. If G is homeomorphic from K(2,n) with n even, then u and v are joined by an even number of paths, say $P_1, P_2, \ldots, P_n$, so that every edge of G appears in the trail constructed by travelling P_1 from u to v, then P_2 from v back to u, and so forth, exhausting all n paths. In [3], a graph with the property that every edge of a graph can be ordered in a sequence such that consecutive edges are adjacent is called <u>sequential</u> and it is shown there that the line graph of a graph is hamiltonian if and only if the original graph is sequential. Thus h(G) = 1.

For G homeomorphic from K(2,n) with n odd, we proceed via induction on d(u,v), where u and v denote the points of G with degree n. If d(u,v) = 2, then at least one of the n paths P_i (i = 1, 2, ..., n) joining u to v in G, say P_1, consists of only two edges x_1 and x_2. The line graph L(G) consists of two copies B_1, B_2 of K_n whose points are pairwise joined by n disjoint paths P_i' corresponding to the n paths P_i in G such that each path P_i' in L(G) has length one less than its counterpart P_i in G (see the following diagram). Thus the path P_1' in L(G) consists of a single edge x. A hamiltonian cycle is then obtained for L(G) as follows: $u_1, u_1u_2, u_2, P_2', v_2, v_2v_1, v_1, v_1v_3, P_3', u_3, u_3u_4, u_4, P_4', v_4, \ldots, v_n, P_n', u_n, u_nu_1, u_1$. Hence h(G) = 1 = d(u,v) - 1.

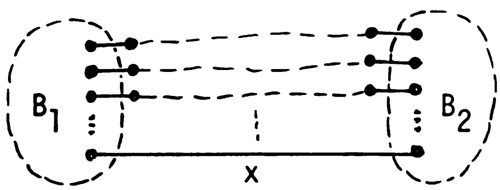


G

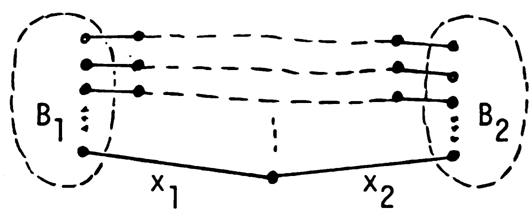


L(G)

Assuming that h(G) = d(u,v) - 1 for all homeomorphs G with d(u,v) < k, we seek to verify that h(G) = d(u,v) - 1 when d(u,v) = k. From G, construct G* by removing one edge from exactly one shortest path of G; this yields d(u,v) = k - 1 in G*, so that G* falls under the induction hypothesis, so that h(G*) = k - 2, implying that $L^{k-2}(G^*)$ is hamiltonian. Now consider the k - 2nd line graphs of G and G*. Since the shortest u-v paths present in G and G* have lengths k and k-1, respectively, we are assured that $L^{k-2}(G^*)$ and $L^{k-2}(G)$ each consist of identical subgraphs B_1 and B_2 joined by n disjoint paths, the only distinction between them being that precisely one path of $L^{k-2}(G^*)$ is a single edge x while the corresponding shortest path in $L^{k-2}(G)$ consists of two edges x_1, x_2 (see the figure below).

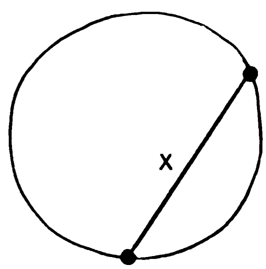


$L^{k-2}(G^*)$

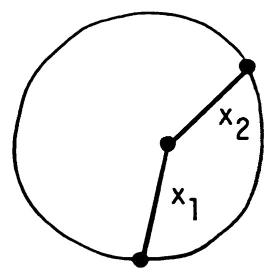


$L^{k-2}(G)$

Although $L^{k-2}(G^*)$ is hamiltonian, no hamiltonian cycle of it contains the edge x since such a cycle must also contain the other n-1 paths (possessing interior points) connecting B_1 and B_2, making an odd number of disjoint paths that such a cycle must contain, which is impossible. Similar reasoning precludes $L^{k-2}(G)$ being hamiltonian, as it contains an odd number of paths with interior points connecting B_1 to B_2. However, $L^{k-2}(G^*)$ and $L^{k-2}(G)$ can now be redrawn as shown in the following figure.



$L^{k-2}(G^*)$



$L^{k-2}(G)$

Another result of G. Chartrand ([3] p. 562) states that if a graph H contains a cycle C with the property that every edge of H is incident with at least one point of C, then L(H) is hamiltonian. The graph $L^{k-2}(G)$ satisfies this condition, thus insuring that $L(L^{k-2}(G)) = L^{k-1}(G)$ is hamiltonian. Hence $h(G) = k - 1 = d(u,v) - 1$. This completes the proof.

To illustrate Theorem 2.6, consider the graph G_1 composed of 2 points joined by 3 disjoint paths of length 3 and the graph G_2

consisting of 2 points joined by 4 disjoint paths of length 3. Then $h(G_1) = 2$ while $h(G_2) = 1$. If G_1' and G_2' represent the graphs obtained from G_1 and G_2 by the insertion of an additional point into each edge, then $h(G_1') = 5$ while $h(G_2') = h(G_2) = 1$.

CHAPTER 3

Section 3.1
The Nth Subdivision Graph and Its Structure

In 1965 Harary and Nash-Williams [10] introduced the concept of the nth subdivision graph $S_n(M)$ of a multigraph M. By definition, this is a graph obtained from M by replacing every edge uv of M by a u-v path of length n + 1, where $n \geq 1$. The graph $S_1(M)$ is denoted simply S(M) and is called the subdivision graph of M. Behzad [1], also in 1965, utilized the subdivision graph S(G) of a graph G in order to characterize the total graph T(G) as isomorphic to the square of S(G). While M ordinarily is not a graph, $S_n(M)$ is always a graph; and if M has p points and q edges, then $S_n(M)$ is a (p + nq, (n + 1)q)-graph. The above definition for the nth subdivision graph could be extended to the case where G is a pseudograph, but we shall not consider this possibility since when n = 1, S(G) may be a multigraph and not a graph.

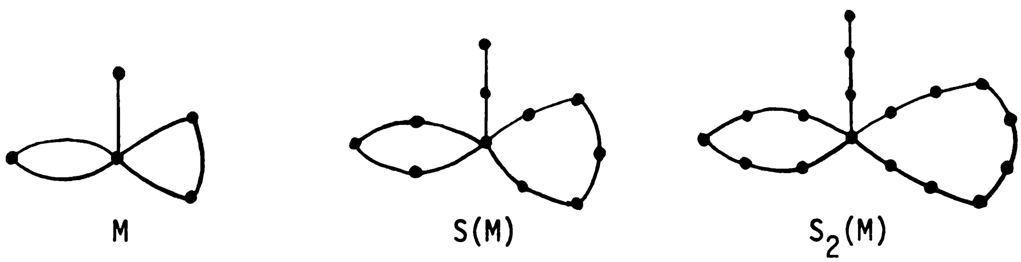


Figure 3.1. A multigraph M together with S(M) and $S_2(M)$.

We next consider some elementary properties of the subdivision graph $S(M)$ of a multigraph M. If we regard $S(M)$ as obtained from M by the insertion of one new point on every edge of M, then the original points of M are nonadjacent in $S(M)$ and form an independent set in $S(M)$. Since each new point is adjacent to exactly two old points, the set of all the new points is also independent in $S(M)$ and contains only points of degree 2. It is clearly necessary, then, that for a graph G to be a subdivision graph of a multigraph M, G must be bipartite, and one of the two independent sets which partition the point set of G contains only points of degree 2. We prove later that this restriction is also a sufficient condition for a graph to be a subdivision graph. Thus the only regular graphs which are subdivision graphs are those that are 2-regular, namely those graphs which are unions of disjoint cycles of even length. Of course, for $n \geq 1$, $S_n(M)$ may contain odd cycles.

We next consider the question of deciding when a given graph G is the nth subdivision graph of some multigraph M, and examine a few simple choices for G. Certainly, (a) no nonempty complete graph can be an nth subdivision graph, (b) every cycle $C_{k(n + 1)}$, for $k \geq 3$, is isomorphic to the nth subdivision graph of C_k (for $k = 2$, $C_{2(n + 1)} = S_n(\Theta_0)$ where Θ_0 is the multigraph consisting of two points joined by a pair of edges), (c) every path $P_{(k - 1)(n + 1) + 1}$ is isomorphic to $S_n(P_k)$, and (d) the complete bipartite graph $K(m,n)$ is a subdivision graph if and only if at least one of m and n is 2.

Since we have already characterized 2-regular subdivison graphs as unions of disjoint cycles, we now examine graphs containing at least one point whose degree is different from 2. An uncomplicated criterion for identifying those graphs which are nth subdivision graphs is presented as the following theorem.

Theorem 3.1. Let G be a graph which is not 2-regular. Then G is the n^{th} subdivision graph of a multigraph if and only if no block of G is a cycle of length $n + 1$ and every u_1-u_2 trail in G has a length which is a multiple of $n + 1$ for all points u_1 and u_2 with deg $u_i \neq 2$, for $i = 1, 2$.

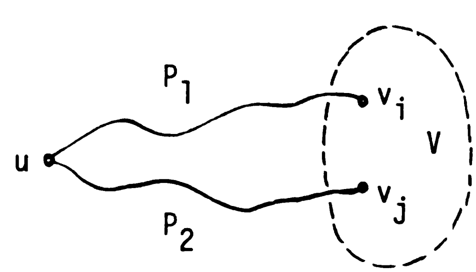
Proof. Necessity. Let M be any multigraph and consider $G = S_n(M)$. In obtaining G from M, every edge uv of M is replaced by a path of length $n + 1$. Thus every u_1-u_2 trail in G joining points u_1 and u_2 in M has length $(k - 1)(n + 1)$, where k is the number of points of M that appear in this trail. Since the only points of G which are not of degree 2 are also points of M, then every u_1-u_2 trail in G joining points u_1 and u_2 which are not of degree 2 has length that is a multiple of $n + 1$. However, no block composed entirely by a u_1-u_1 cycle of length $n + 1$, deg $u_1 \neq 2$, can occur in G, since such a cycle corresponds to a loop in M.

Sufficiency. Now let G be any graph, not 2-regular, with the property that every u_1-u_2 trail joining points u_1 and u_2, which are not of degree 2, has length a multiple of $n + 1$. We must show the existence of a multigraph M such that $S_n(M) = G$. We may assume G is connected, for otherwise the ensuing argument can be applied to each component of G in order to construct the corresponding component for M.

Select any point v_1 in G which does not have degree 2. At least one such point exists, since G is not 2-regular. Let V be the set of all points in G whose distance from v_1 is a multiple of n + 1 and let U be the set of all other points. Thus, by hypothesis, all points of G which are not of degree 2 belong to V so that U contains only points of degree 2.

Every pair of points v_i, v_j of V are joined only by trails whose length is a multiple of n + 1, for if v_i were joined to v_j by a trail T_0 of length d which is not a multiple of n + 1, then T_0, together with trails T_1, T_2 from v_1 to v_i and from v_1 to v_j, forms a v_1-v_1 trail whose length is not a multiple of n + 1.

Now let P_1 and P_2 be shortest paths from any point $u \in U$ to the set V, say to points v_i and v_j in V. Inasmuch as the v_i-v_j trail T formed by P_1 and P_2 has no interior point that is a point of V, the length of T must be exactly n + 1. Furthermore, $v_i \neq v_j$ because if $v_i = v_j$, then all points of T except v_i have degree 2, making T a block of G, a contradiction.



Thus every point u of U lies on some v_i-v_j path $P_{i,j}$ of length n + 1, all of whose interior points are in U, joining two points of V. Each point of U has degree 2, making it impossible for any point of U to simultaneously lie on two such paths; thus each point $u \in U$ uniquely determines one of these v_i-v_j paths $P_{i,j}$.

If we label all paths of length n + 1 joining each pair of points v_i, v_j of V by $P_{i,j}^{(1)}$, $P_{i,j}^{(2)}$, ..., $P_{i,j}^{(t)}$, the desired multigraph M can now easily be described. The point set of M is the set V and two points v_i and v_j of M are joined by one edge $x_{i,j}^{(k)}$ for each path $P_{i,j}^{(k)}$, k = 1, 2, ..., t.

All that remains is to verify that $S_n(M) = G$, that is, to exhibit an isomorphism ϕ between $S_n(M)$ and G. The point set of $S_n(M)$ is $V \cup V_1$, where V_1 consists of n points of degree two $w_{i,j,1}^{(k)}$, $w_{i,j,2}^{(k)}$, ..., $w_{i,j,n}^{(k)}$ for each edge $x_{i,j}^{(k)}$ of M, with $w_{i,j,r}^{(k)}$ adjacent to $w_{i,j,r+1}^{(k)}$ for r = 1, 2, ..., n - 1. In G each path $P_{i,j}^{(k)}$ of length n + 1 from v_i to v_j contains precisely n points of U, say $u_{i,j,1}^{(k)}$, $u_{i,j,2}^{(k)}$, ..., $u_{i,j,n}^{(k)}$, where $u_{i,j,r}^{(k)}$ is adjacent to $u_{i,j,r+1}^{(k)}$ r = 1, 2, ..., n - 1. Thus the mapping ϕ of $S_n(M)$ onto G defined by the equations $\phi(v_i) = v_i$ for $v_i \in V$, $\phi(w_{i,j,r}^{(k)}) = u_{i,j,r}^{(k)}$ for $w_{i,j,r}^{(k)} \in V_1$, r = 1, 2, ..., n, is one-to-one and adjacency-preserving. This completes the proof of Theorem 3.1.

<u>Corollary 3.1</u>. Let G be a graph which is not 2-regular. Then G is the subdivision graph of a multigraph if and only if every u_1-u_2 trail in G has even length for all points u_1 and u_2 with deg $u_i \neq 2$, i = 1, 2.

Proof. This corollary follows immediately from Theorem 3.1 with the observation that when $n = 1$, the hypothesis forbidding the presence of a cycle of length $n + 1$ as a block of G is superfluous.

When a graph G is known to be an nth subdivision graph of some multigraph M, one might wonder if M is unique up to isomorphism; that is, can G be the nth subdivision graph of two or more nonisomorphic multigraphs? This question is settled in the negative by the following:

Theorem 3.2. If M_1 and M_2 are connected multigraphs whose nth subdivision graphs are isomorphic, then M_1 and M_2 are isomorphic.

Proof. It is convenient to deal first with connected multigraphs M_1 and M_2 having 3 or fewer points and which have isomorphic nth subdivision graphs. If we temporarily agree to call the multigraph Θ_0 a cycle, then whenever $S_n(M_1) = S_n(M_2)$ is 2-regular, namely a cycle, then M_1 and M_2 must also both be cycles, necessarily with the same number of points, and hence isomorphic. If $S_n(M_1) = S_n(M_2)$ is not 2-regular, then it contains a point of degree $k \neq 2$. Since the only points of $S_n(M_1)$ and $S_n(M_2)$ not of degree two are those points which are in the original multigraph, consequently M_1 and M_2 must both consist of one point v joined by k edges to precisely those points connected to v in the nth subdivision graph by a path of length $n + 1$.

Next we consider connected multigraphs M_1 and M_2 having more than 3 points with $S_n(M_1) = S_n(M_2)$. The remainder of this proof was suggested by an elegant analogous proof for line graphs given by Jung [12]. For any isomorphism $\phi_1 : S_n(M_1) \to S_n(M_2)$, we shall exhibit an induced isomorphism $\phi : M_1 \to M_2$.

Suppose v is any point of M_1, and let E(v) be the set of all edges incident to v. We claim that to every $v \in M_1$ there is exactly one point v' in M_2 such that $E(v) \to E(v')$ under ϕ_1. If deg $v \geq 2$, let x_1 and x_2 be edges at v and let v' be the common point of edges $\phi_1(x_1)$ and $\phi_1(x_2)$. Then for each edge x at v, v' is incident with $\phi_1(x)$; and for each edge x' at v', v is incident with $\phi_1^{-1}(x')$. If deg v = 1, let x = uv be the edge at v. Then deg $u \geq 2$, hence E(u) corresponds to E(u') and $\phi_1(x) = u'v'$, where $v' = \phi_1(v)$. Since for every edge x' at v', the edges $\phi_1^{-1}(x')$ and x have u as a common point and u' is on x', therefore $x' = \phi_1(x)$ and deg v' = 1. Thus in all cases there is precisely one v' in M_2 for each v in M_1 such that $E(v) \to E(v')$ under ϕ_1; call this induced mapping ϕ. Then ϕ is evidently one-to-one on the point sets of M_1 and M_2 since E(u) = E(v) only if u = v. Also ϕ is onto, since to any point v' in M_2 there exists an incident edge x' which determines an edge $\phi_1^{-1}(x')$ in M_1, one of whose endpoints maps into v'. The adjacency-preserving property of ϕ is inherited from ϕ_1, and the proof is complete.

One useful benefit which we can derive from Theorem 3.2 is that since each nth subdivision graph G is obtained from only one multigraph M (up to isomorphism), then we can always regard M as obtainable from G by the procedure used in the proof of Theorem 3.1. Furthermore, if G is an nth subdivision graph of a graph M, we can regard G as composed of an independent subset V of points v_1, v_2, ..., v_t together with certain v_i-v_j paths P_{ij} of length n + 1, where each edge in G belongs to some P_{ij} and no more than one path P_{ij} joins v_i and v_j.

A related question is whether there exist nonisomorphic multigraphs M_1 and M_2 with $S_m(M_1) = S_n(M_2)$ for $m \neq n$. This can occur as is seen by taking m = 2, n = 3, $M_1 = P_5$, and $M_2 = P_4$ (or $M_1 = C_4$, and $M_2 = C_3$). A more general example of this is obtained by taking $M_1 = S_n(M)$ and $M_2 = S_m(M)$ for any multigraph M and distinct integers m and n; then $S_m(M_1) = S_m(S_n(M)) = S_{mn+m+n}(M) = S_n(S_m(M)) = S_n(M_2)$ and M_1 and M_2 are nonisomorphic graphs.

Although each acyclic block of a graph G gives rise to n + 1 acyclic blocks in $S_n(G)$, the next proposition shows that G and $S_n(G)$ have the same number of cyclic blocks.

<u>Proposition 3.1</u>. The set of cyclic blocks of $S_n(G)$ is $\{S_n(B_i)\}$ where $\{B_i\}$, i = i, 2, ..., k, is the set of cyclic blocks of G.

<u>Proof</u>. Suppose that B' is any cyclic block of $S_n(G)$ and denote those points of B' which are also points of G by v_1, v_2, ..., v_r. We claim that the induced subgraph $\langle v_1, v_2, \ldots, v_r\rangle = B$ of G is a cyclic block whose nth subdivision $S_n(B)$ equals B'. One way to show that B is a cyclic block is, for every three distinct points of B, to exhibit a path joining any two of them which does not contain the third (B surely contains three or more points since B' cyclic implies that B' contains a cycle which is an nth subdivision of a corresponding

cycle in G). For each trio of points u, v, w of B, there is a path P' in B' joining u to v which does not contain w, by virtue of B' being a cyclic block. Since B' is a subgraph of an nth subdivision graph, P' has length a multiple of n + 1 and is of the form u, $u_{0,1}$, $u_{0,2}$, ..., $u_{0,n}$, u_1, $u_{1,1}$, $u_{1,2}$, ..., $u_{1,n}$, u_2, ..., $u_{t,n}$, $u_{t+1} = v$. Thus in B, the path P : u, u_1, u_2, ..., u_t, $u_{t+1} = v$ joins u and v, omitting w.

Now let B be any cyclic block of G and consider the subgraph $B' = S_n(B)$ of $S_n(G)$. To demonstrate that B' is a cyclic block of $S_n(G)$, it suffices to show that each pair of points of B' lie on a common cycle. Letting u and v be any two points of B', we distinguish three cases concerning them.

CASE 1. If both u and v are points of B, then since B is a cyclic block, there exists a cycle C containing both u and v; thus $S_n(C)$ is a cycle in B' containing both u and v.

CASE 2. If exactly one of u and v is not a point of B, say u, then u is an interior point of a u_1-u_2 path P' of length n + 1 in B' where u_1 and u_2 are points of B. Since u_1, v, and u_2 all belong to the block B, there exists a u_1-v path P_1 not containing u_2 and a v-u_2 path P_2 not containing u_1. Thus in B', the paths P', $S_n(P_1)$, and $S_n(P_2)$ form a cycle containing u and v.

CASE 3. If neither u nor v is a point of B, then in a manner almost identical to the approach employed in CASE 2, a cycle can be constructed in B' which is common to u and v.

Thus every cyclic block B_i of G gives rise to a cyclic block $S_n(B_i)$ of $S_n(G)$ and, conversely, each cyclic block $S_n(B_i)$ of $S_n(G)$ corresponds to precisely one cyclic block B_i of G.

If we restrict our attention to those graphs G for which G = S(H) for some graph H, we obtain several additional equivalent conditions.

Theorem 3.3. Let G be a graph which is not a union of disjoint cycles. Then the following statements are equivalent:

(1) G is the subdivision graph of a graph.

(2) G does not contain K(2,2) as a subgraph and every u_1-u_2 trail in G has even length for all points u_1 and u_2 with deg $u_i \neq 2$, i = 1, 2.

(3) G is bipartite having a point set which is a disjoint union of independent sets U and V such that U contains only points of degree 2 and no two points in U are mutually adjacent to the same two points of V.

Proof. Throughout the proof we shall assume G is connected since if the theorem holds for every component of G, it holds for G itself.

(2)$\Longrightarrow$(1). Since a connected graph which is not a cycle cannot be 2-regular, G satisfies the hypothesis of Corollary 3.1. It remains to be checked that by forbidding the presence of K(2,2) as a subgraph of G we guarantee that the multigraph M constructed in the proof of Theorem 3.2 is actually a graph. Referring to the construction of M in the sufficiency portion of the proof of Theorem 3.1, we find that the only way that one pair of points v_i, v_j can be joined by two lines $x^1_{i,j}$, $x^2_{i,j}$ in M is by the occurrence of at least two paths $P^{(1)}_{i,j}$, $P^{(2)}_{i,j}$ of length 2 from v_i to v_j in G. But these paths $P^{(1)}_{i,j}$, $P^{(2)}_{i,j}$ together form the subgraph K(2,2) in G. Thus every pair of edges in M with common endpoints corresponds to a K(2,2) subgraph of G.

(1)$\Longrightarrow$(3). For any point v_1 of G which is not of degree 2, we define V as the set of all points of G having even distance from v_1 and U as the set of all other points of G. Then, as in the proof of Theorem 3.1, all points of U have degree 2 and both U and V are independent sets, implying that G is bipartite. Again referring to the construction of M such that S(M) = G detailed in the proof of Theorem 3.1, no two points of U can be interior points of two paths $P^{(1)}_{i,j}$, $P^{(2)}_{i,j}$ joining the same pair of points v_i, v_j of V since this would mean that in the graph M v_i and v_j are joined by 2 edges. Hence no two points of U are mutually adjacent to the same pair of points of V.

(3)$\Longrightarrow$(2). If G is a bigraph whose point set is partitioned into subsets U and V such that U contains only points of degree 2, then every v_1-v_2 trail, where v_1 and v_2 are not 2-valent, joins points in V and hence has even length. Thus by Corollary 3.1, G is the subdivision graph of a multigraph M. A repetition of the same argument given in the preceding paragraph suffices to show that no multiple edges can occur in M; that is, M is a graph.

Corollary 3.3a. A graph G is the subdivision graph of K_n, $n \geq 2$, if and only if G is a bipartite graph, containing no 4-cycle, and whose point set can be partitioned into two independent subsets composed of n points of degree n - 1 and $\binom{n}{2}$ points of degree 2.

Proof. First we show that the above condition is sufficient. By Theorem 3.3(3), G = S(H) for some graph H. Assuming that the above condition holds, it remains to be shown that $H = K_t$ for some integer $t \geq 2$. If H is a (p, q)-graph, then G has p + q points and 2q edges. Since each edge of G is incident with exactly one of the n points of the independent set V, we have that 2q = n(n - 1), or that $q = \frac{n(n-1)}{2} = \binom{n}{2}$. All of the p + q points of G lie in U V, thus $p + q = \binom{n}{2} + n$ implies that p = n. Hence H is the complete graph on n points. The necessity follows immediately.

Corollary 3.3b. A graph G is the subdivision graph of the complete bipartite graph K(m,n) if and only if G contains no 4-cycle and the point set of G is the disjoint union of independent sets U, V, and W, composed respectively of mn points of degree 2, m points of degree n, and n points of degree m.

Proof. The necessity is immediate. To show that the above condition is sufficient, we note that Theorem 3.3(3) guarantees that G = S(H) for some graph H; it remains to be seen that H = K(s,t) for some pair of positive integers s and t. If H is a (p, q)-graph, then G has p + q points and 2 q edges. Summing the degrees of all the points of G, we find that $2q = 1/2(2mn + mn + nm)$, or that $q = mn$. Then $p + q = mn + m + n$ implies that $p = m + n$. Hence $H = K(m,n)$.

Section 3.2
Properties of Nth Subdivision Graphs

Since a graph G and its nth subdivision graph $S_n(G)$ belong to the same homeomorphism equivalence class, they possess many common characteristics. Although such concepts as order or the number of edges are obviously not preserved, planarity and the invariance of the number of connected components is maintained. Outerplanarity, on the other hand, is not in general inherited by $S_n(G)$ from G, as shown by K_4-x and it subdivision graph.

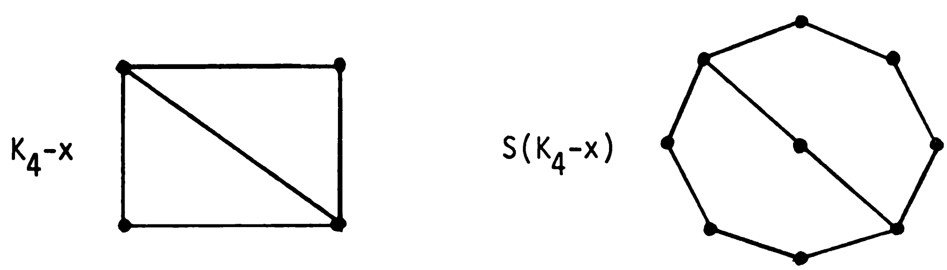


Figure 3.2. An outerplanar graph K_4-x and its nonouterplanar subdivision graph.

It is possible to completely delineate those nth subdivision graphs which are outerplanar, as shown by our next theorem; before proceeding, however, we present a definition.

A <u>cactus</u> is a connected graph every block of which is either a cycle or K_2. Within the context of nth subdivision graphs, the concepts of cacti and outerplanar graphs coincide, as we now see.

Theorem 3.4. Let G be an nth subdivision graph. Then G is outerplanar if and only if every component of G is a cactus.

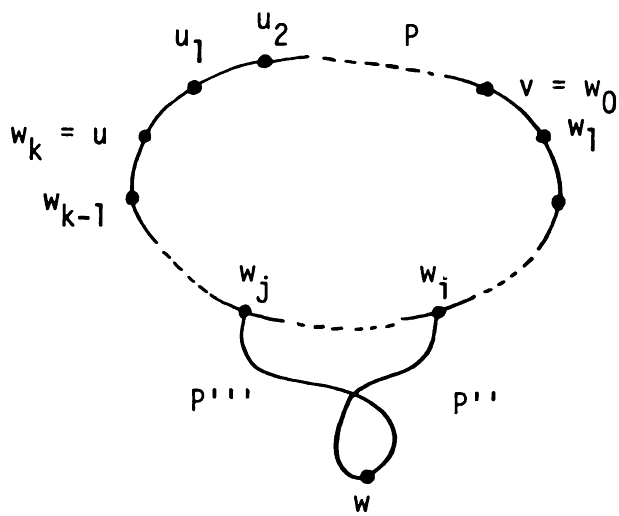
Proof. Since the sufficiency is evident, we proceed directly to the necessity portion of the proof. We may assume that G is a connected outerplanar nth subdivision graph, for otherwise we may consider the individual components of G. Moreover, we may also assume that G contains at least one point not of degree two since the theorem holds for 2-regular connected outerplanar nth subdivision graphs, i.e., disjoint unions of cycles. To verify that G is a cactus, we proceed via induction on the order of G. The theorem is immediate for connected nth subdivision graphs on three or fewer points. Assuming that every outerplanar connected nth subdivision graph with fewer than p points is a cactus, we consider such a graph G on p points.

The blocks of G having 3 points or less are either single edges or 3-cycles, and both types are cacti. Accordingly, we let B be any block of G having 4 or more points. We may regard the nth subdivision graph G, by means of the construction given in the proof of Theorem 3.1, as having a point set made up of two subsets U and V, where all noncarriers are contained in V and all points of U have degree 2, and where all the edges of G occur in paths $P_{i,j}^{(k)}$ of length n + 1 joining distinct points v_i, v_j in V. Since B is a cyclic block, it contains a point of U which is an interior point of some such path P of length n + 1 joining two points u and v of V; u and v are also points of B. Denote the subgraph obtained from G by deleting the n interior points of degree 2 of the path P as G*. Note that G* is an nth subdivision graph since all u_1-u_2 trails in G* joining points u_1 and u_2 not of degree 2 (which are elements of the set V) have length a multiple of n + 1. To be sure, G outerplanar implies the subgraph G* is outerplanar; thus by our induction hypothesis, G* is a cactus.

In the block B, the points u and v belong to a common cycle C. If we can demonstrate that C comprises all of B, then we will have shown that every block of the graph G is either a cycle or a single edge, so that G is indeed a cactus.

For every three distinct points of B, there exists a path joining any two of them which excludes the third. Thus B contains a u-v path P' which is disjoint from the path P. No two points of P' lie on a cycle in G* because this would generate a subgraph homeomorphic from K(2,3) in B, contradicting the outerplanarity of G. Thus the cycle C is composed of the paths P and P', so that we can label C by u, u_1, u_2, ..., u_n, v = w_0, w_1, w_2, ..., w_{k-1}, w_k = u, where the u_i, $1 \leq i \leq n$, are carriers in P and the w_i, $0 \leq i \leq k$, are the points of P'.

If B contains a point w not belonging to C, then again by utilizing the criterion of the first sentence in the preceding paragraph, we can find paths P'', P''' joining w to distinct points w_i, w_j of C (if every path from w to C joins w to the same point w_i, then w_i is a cutpoint of B).



But this implies the existence of a cycle in G* containing the two points w_i and w_j of P', which we've already seen is impossible. Therefore C passes through every point of B. This completes the proof.

Section 2.3 of Chapter 2 contained a discussion of various characteristics of a graph G and any subdivision H of G, in particular their chromatic numbers and their automorphism groups. The remainder of this section is devoted to obtaining more precise results in these two areas when $H = S_n(G)$.

Proposition 2.2 of Chapter 2 states that if G is irreducible, with chromatic number $\chi(G) \geq 3$, then for any graph H in the same homeomorphism class with G, we have $2 \leq \chi(H) \leq \chi(G)$. In particular, for all positive integers n, we have $2 \leq \chi(S_n(G)) \leq \chi(G)$. This inequality can in general be greatly improved, as our next result shows.

<u>Theorem 3.5</u>. Let $\chi(G) \geq 3$. Then $\chi(S_n(G)) = \begin{cases} 2 \text{ if } n \text{ is odd.} \\ 3 \text{ if } n \text{ is even.} \end{cases}$

<u>Proof</u>. Suppose the point set of G is partitioned into $\chi(G) = k$ color classes $C_1, C_2, \ldots, C_k$, where $k \geq 3$.

For odd values of n, say $n = 2r - 1$ with $r = 1, 2, 3, \ldots$, $S_n(G)$ is obtained from G by inserting new points $u_1, u_2, \ldots, u_{2r-1}$ into each edge uv of G. A minimal 2-coloring of $S_n(G)$ is then achieved by assigning color α to the points of the type $u_2, u_4, \ldots, u_{2r-2}$ and also to the points of $S_n(G)$ which are in G, and color β to points of type $u_1, u_3, \ldots, u_{2r-1}$.

When n is even, say $n = 2r$ with $r = 1, 2, 3, \ldots$, $\chi(G) > 2$ implies the existence of at least one cycle of odd length s in G (since every pair of color classes are joined by at least one edge). Thus in $S_n(G)$ there exists at least one cycle of odd length $s(2r + 1)$, so that $\chi(S_n(G)) > 2$. Now $S_n(G)$ is obtained from G by inserting 2r points $u_1, u_2, \ldots, u_{2r}$ into each edge uv of G; if the points of $S_n(G)$ which are in G are assigned color α, points of the type $u_1, u_3, \ldots, u_{2r-1}$ are colored β, points of the type $u_2, u_4, \ldots, u_{2r-2}$ are colored α, and u_{2r} is colored γ, then we have produced a 3-coloring of $S_n(G)$. Hence $\chi(S_n(G)) = 3$.

For any irreducible graph G, Theorem 2.5 guarantees that $\Gamma(S_n(G))$ is isomorphic to a subgroup of $\Gamma(G)$. In the specific case concerning nth subdivision graphs, this result can be strengthened considerably.

Theorem 3.6. For any graph G, no component of which is a cycle, $\Gamma(S_n(G))$ is isomorphic to $\Gamma(G)$ for every positive integer n.

<u>Proof</u>. As in the proof of Theorem 3.1 and the remarks following Theorem 3.2, both G and $S_n(G)$ can be regarded as consisting of an independent set V of points $v_1, v_2, \ldots, v_t$ with one path P_{ij} of length n + 1 joining v_i to v_j in $S_n(G)$ whenever the edge v_iv_j is present in G. Each path P_{ij} has interior points of degree 2 only, so that under any automorphism α of $S_n(G)$, each path P_{ij} must map into a like path and the endpoints of P_{ij} must be mapped into the endpoints of the image path. (If any interior point u_i of some path P_{ij} maps into a point $v_j \in V$, then deg $v_j = 2$ and since the component of G containing v_j is not a cycle, there exists a point $v_k \in V$ with deg $v_k \neq 2$. Then $d_{S_n(G)}(v_j, v_k) \equiv 0 \bmod(n + 1)$ and $d_{S_n(G)}(u_i, v_j) \not\equiv 0 \bmod(n + 1)$. Hence $d_{S_n(G)}(u_i, v_k) \not\equiv 0 \bmod (n + 1)$, but $d_{S_n(G)}(\alpha(u_i), \alpha(v_k)) = d_{S_n(G)}(v_j, \alpha(v_k)) \equiv 0 \bmod(n + 1)$, inasmuch as $\alpha(v_k) \in V$, contradicting the assumption that α is an automorphism.) Thus each element $\alpha \in \Gamma(S_n(G))$ uniquely determines an element $\alpha' \in \Gamma(G)$, and conversely. From this observation and the way in which these automorphisms are defined, it follows that $\Gamma(G)$ and $\Gamma(S_n(G))$ are isomorphic.

Section 3.3
Factorization in $S_n(G)$

An <u>r-factor</u> of a graph G is a nonempty spanning subgraph of G which is regular of degree $r \geq 1$. If a graph G can be expressed as an edge-disjoint union of r-factors, we say that G is r-factorable. Clearly a graph G may have an r-factor without being r-factorable. Inasmuch as $S_n(G)$ always contains points of degree two, we shall deal only with the question of the existence of a 1-factor, or a 2-factor, for $S_n(G)$ and certain other related graphs.

It is immediate that a necessary condition for a connected graph G to contain a 1-factor is that G have even order. It is easily seen that this simple condition in general is not sufficient; for example, the complete bipartite graph K(1, n-1), for n an even integer (n > 2), does not contain two nonadjacent edges, much less n/2 such edges.

We continue our discussion of factorization with an elementary observation which will be used in the sequel. A path or cycle on n points is denoted by P_n or C_n, respectively.

Proposition 3.2. For k = 1, 2, 3, ..., the path P_{2k} has a unique 1-factor while the cycle C_{2k} has exactly two 1-factors.

Proof. For any path $P_{2k} : v_1, v_2, \ldots, v_{2k}$, the unique 1-factor must consist of the edges $v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}$. In the case of the cycle $C_{2k} : v_1, v_2, \ldots, v_{2k}, v_1$, in addition to the aforementioned set of edges, the edges $v_{2k}v_1, v_2v_3, \ldots, v_{2k-2}v_{2k-1}$ also form a 1-factor.

Obviously P_{2k} is not 1-factorable, while C_{2k} is 1-factorable.

Whenever a graph G possesses a 1-factor F, it follows that $S_n(G)$ also has a 1-factor provided that n is an even positive integer.

To verify this assertion, it is helpful to observe that if the point set of G is $V = \{v_1, v_2, \ldots, v_p\}$, then $S_n(G)$ consists of the set V together with q disjoint v_iv_j paths P_{ij} of length n + 1, where each path P_{ij} in $S_n(G)$ corresponds to the edge v_iv_j in G. In $S_{2k}(G)$, each path which corresponds to an edge x_i of G in F has odd length, hence possesses a unique 1-factor F_i, for i = 1, 2, 3, ..., p/2. Letting F_o denote the set of all edges of the form $u_{2i-1}u_{2i}$, for i = 1, 2, 3, ..., 2k, in all paths $P_{ij} : v_i, u_1, u_2, \ldots, u_{2k}, v_j$ which correspond to an edge of G not in F, $\bigcup_{i=o}^{p/2} F_i$ is the desired 1-factor of $S_{2k}(G)$.

Conversely, if $S_{2k}(G)$ possesses a 1-factor F, a 1-factor for G is formed by all edges v_iv_j corresponding to those v_i-v_j paths P_{ij} in $S_{2k}(G)$ which contain an edge $x_i \in F$ incident with v_i (notice that two paths P_{ij}, P_{ik} cannot both contain an edge from F which is incident with v_i). Thus we have shown

<u>Theorem 3.7</u>. A graph G has a 1-factor if and only if $S_{2k}(G)$ has a 1-factor, where k = 1, 2, 3, ...

We next consider the question of deciding when an odd subdivision graph $S_{2k-1}(G)$ has a 1-factor, for k = 1, 2, 3, ... The following lemma reduces this question to a simpler one.

<u>Lemma 3.1</u>. For any graph G and positive integer k, $S_{2k-1}(G)$ has a 1-factor if and only if S(G) has a 1-factor.

Proof. Let $V = \{v_1, v_2, \ldots, v_p\}$ be the point set of G. Then S(G) and $S_{2k-1}(G)$ each consist of the set V together with exactly one v_i-v_j path of length 2 or 2k, respectively, whenever v_iv_j is an edge of G. To any 1-factor of S(G), there is an associated 1-factor F' of $S_{2k-1}(G)$ obtained in the following way. In S(G), each point v_i is incident with exactly one edge $x_i \in F$ and this edge lies in exactly one v_i-v_j path P_{ij}. Labelling the corresponding v_i-v_j path P_{ij}' in $S_{2k-1}(G)$ as $v_i, u_1, u_2, \ldots, u_{2k-1}, v_j$, let F' be the set $\{v_iu_1, u_2u_3, \ldots, u_{2k-2}u_{2k-1}\}$. Then $F' = \bigcup_{i=1}^{p} F_i'$ is a 1-factor of $S_{2k-1}(G)$. Of course, if $S_{2k-1}(G)$ has a 1-factor for each positive integer k, then by taking k = 1, we have that S(G) has a 1-factor.

One additional definition is needed in dealing with the question of the existence of a 1-factor for $S_{2k-1}(G)$. A connected graph is called <u>unicyclic</u> if it contains precisely one cycle.

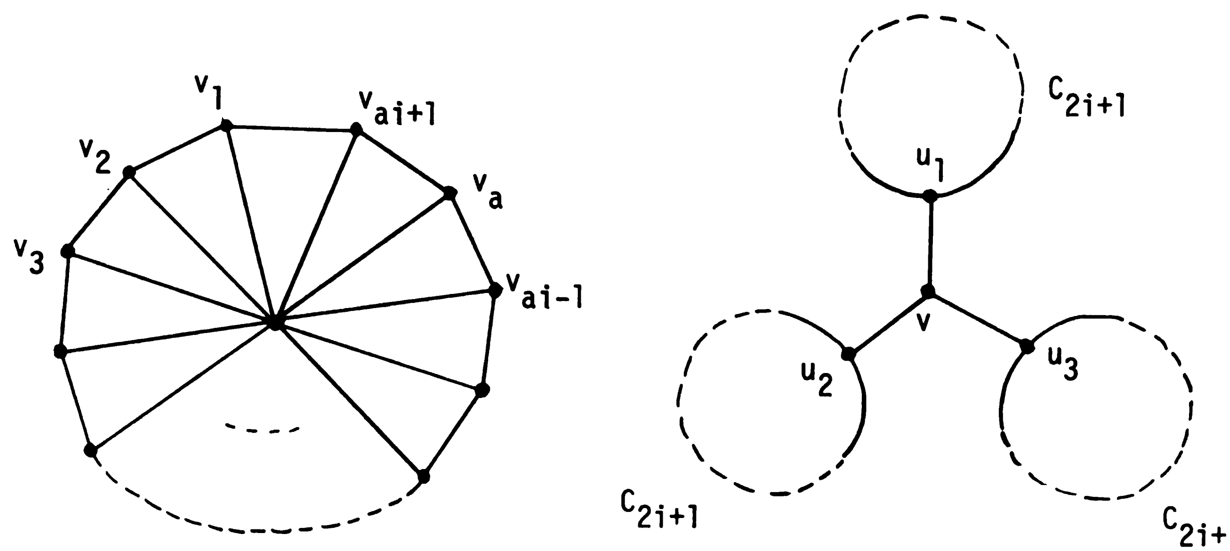
Theorem 3.8. For any graph G and positive integer k, $S_{2k-1}(G)$ has a 1-factor if and only if each component of G is unicyclic.

Proof. By Lemma 3.1, it suffices to show that S(G) has a 1-factor if and only if each component of G is unicyclic. In showing the necessity of this condition, we follow a suggestion of J. Zaks. If G has a component C which is not unicyclic, then the number of points p of C is not equal to the number of edges q of C (a connected (p, q)-graph is unicyclic if and only if p = q). All edges of the bipartite component S(C) of S(G) are of the form u_iv_j, where $u_1, u_2, \ldots, u_p$ and $v_1, v_2, \ldots, v_q$ correspond to the points and edges, respectively, of C. Thus any 1-factor of S(C) is composed of nonadjacent edges u_iv_j, implying that there is a one-to-one correspondence between the u_i and the v_j. But this means that p = q, a contradiction. Hence each component of G is unicyclic.

Conversely, assume that each component of G is unicyclic and consider any component C' of S(G), where C' = S(C) for some component C of G. Then C unicyclic implies that C' is unicyclic; denote the unique cycle in C' by $Z : v_1, u_1, v_2, u_2, \ldots, v_t, u_t, v_1$, where the points v_i correspond to points in C and deg $u_i = 2$, for i = 1, 2, ..., t. As in Proposition 3.2, $Z = C_{2t}$ has a 1-factor F_o. C' - Z is a union of trees T_k, for k = 1, 2, ..., r, where each tree T_k contains a point w_k adjacent to some point v_j of Z with deg $v_j \neq 2$. The distance d(u,v) in C' = S(C) between points u and v, neither of whose degrees is 2, is always even; thus the distance $d(w_k,v)$ in T_k, where deg $v \neq 2$, is always odd. This implies that T_k has a unique 1-factor F_k composed of all edges uv in T_k, where $d(w_k,u)$ is odd and $d(w_k,v)$ is even. Then $F = \bigcup_{k=0}^{r} F_k$ is a 1-factor for S(G).

<u>Corollary 3.8</u>. Let G be a connected (p, q)-graph and k a positive integer. Then $S_{2k-1}(G)$ has a 1-factor if and only if p = q.

In general, the existence of a 1-factor for a (p - q)-graph G is independent of the existence of a 1-factor for S(G), even if we restrict our attention to connected graphs G having both p and q even. This lack of relationship is illustrated by the following four infinite classes of connected graphs. For each positive integer i, if G_i is the cycle C_{2i+2}, then G_i has a 1-factor, as does $S(G_i)$, since G_i is unicyclic. Taking G_i as the cycle C_{2i+2} together with edges wv_1 and wv_2 joining any point w of C_{2i+2} to two additional new points v_1 and v_2, we find that G_i has no 1-factor, although $S(G_i)$ does. In Figure 3.3, the remaining two situations are illustrated.



(a) Graphs $G_i = W_{2i+2}$ having a 1-factor while $S(G_i)$ has no 1-factor.

(b) Graphs G_i having no 1-factor and such that $S(G_i)$ also has no 1-factor.

Figure 3.3. Graphs G such that S(G) has no 1-factor.

Choosing G_i as the "wheel" W_{2i+2} of Figure 3.5(a), it is easy to find a 1-factor of G_i, but $S(G_i)$ has no 1-factor. If G_i consists of three copies of C_{2i+1} joined by single edges with a new point v, as in Figure 3.5(b), then neither G_i nor $S(G_i)$ possess a 1-factor.

A criterion for $S_n(G)$ to be 1-factorable is much easier to establish than a test for the existence of a 1-factor. For $S_n(G)$ to be 1-factorable, it must be an edge-disjoint union of 1-factors F_1, F_2, ..., F_t, so that $S_n(G)$ is regular of degree t. Thus t = 2, implying that $S_n(G)$, and G as well, is a disjoint union of cycles. We summarize this as

Proposition 3.3. For any graph G, $S_n(G)$ is 1-factorable, for $n \geq 1$, if and only if each component of G is a cycle.

For any graph G, the total graph T(G) is related to the subdivision graph S(G) by the equation $T(G) = S^2(G)$. We shall make use of this to develop necessary and sufficient conditions for T(G) to possess a 1-factor.

In order for the nth power G^n ($n \geq 2$) of any graph G to have a 1-factor, G^n must possess an even number of points. The partnership of G. Chartrand, A. Polimeni, and the author was able to prove that this simple necessary condition is also sufficient for G^n to have a 1-factor.

Theorem 3.9. Let G be an arbitrary graph. Then G^n ($n \geq 2$) has a 1-factor if and only if each component of G has even order.

Proof. It suffices to show that G^2 has a 1-factor if and only if G has even order, inasmuch as G^n ($n \geq 2$) and G share the same point set. We may further assume that G is a connected graph, for otherwise we may apply the following argument to each component of G.

The necessity of the given condition being obvious, we proceed to the sufficiency portion of the proof. Letting the order of G be 2n, the proof is by induction on n. For connected graphs on 2 points, the assertion is obvious. Assuming the assertion is true for all appropriate graphs on fewer than 2n points, we consider a connected graph G with order 2n. We may further assume that G consists of two or more blocks, for if G is a single cyclic block, then G^2 has a hamiltonian cycle (see [8]), alternate edges of which furnish a 1-factor. It is convenient to distinguish two cases.

CASE 1. G contains a cyclic endblock B. Suppose that v is the cutpoint of G in B. In [6], it is shown that for any cyclic block B with at least 4 points, $B^2 - u$ is hamiltonian for any u B. Thus if B contains an odd number p of points, $p \geq 5$, then $B^2 - v$ is hamiltonian and has an even number of points; hence there is a 1-factor F_0 for $B_2 - v$. If B has exactly three points, then $B^2 - v$ is simply a copy of K_2, and therefore is a 1-factor, say F_0. Also the connected graph $G' = G - (B - v)$ has an even number of points, so that by the induction assumption there is a 1-factor F_1 of $(G')^2$. Then $F_0 \cup F_1$ is a 1-factor of G^2.

CASE 2. Each endblock of G is acyclic. Let v denote a cutpoint of G such that all blocks containing v, except at most one, are endblocks. If there are at least two endblocks vv_1 and vv_2 containing v, then the connected graph $G' = G - \{v_1,v_2\}$ has an even number of points and by the induction assumption, $(G')^2$ has a 1-factor F. Since the edge v_1v_2

is present in G^2, $F \cup \{v_1 v_2\}$ is a 1-factor of G^2. In the event that there is just one acyclic endblock vv_1 containing v, let B be the other block containing v. Then $G' = G - \{v, v_1\}$ is connected and has 2n - 2 points, so that the induction hypothesis implies that there is a 1-factor F for $(G')^2$. Then $F \cup \{vv_1\}$ is a 1-factor of G^2. This completes the proof of the theorem.

<u>Corollary 3.9a</u>. Let G be a connected (p,q)-graph. Then T(G) has a 1-factor if and only if p + q is even.

<u>Corollary 3.9b</u>. Let G be a connected graph. Then $\begin{cases} T(G) \\ G^n \ (n \geq 2) \end{cases}$ has a 1-factor if and only if $\begin{cases} T(G) \\ G^n \ (n \geq 2) \end{cases}$ has even order.

An interesting special case of Corollary 3.9a occurs when G is the complete graph K_p. Here we have $q = \frac{p(p-1)}{2}$ and $p + q = \frac{p(p+1)}{2}$, so that we may conclude that $T(K_p)$ has a 1-factor if and only if $p \equiv 0$ or 3 (mod 4).

Whenever a graph H has a 2-factor, by definition it contains a collection of disjoint cycles $C_1, C_2, \ldots, C_k$ which span H. If H is an nth subdivision graph $S_n(G)$ of a graph G, then it turns out that $S_n(G)$ is composed of these cycles only.

<u>Theorem 3.10</u>. For any graph G, $S_n(G)$ has a 2-factor if and only if each component of G is a cycle.

<u>Proof</u>. We may assume G is connected, for otherwise we may work with each component separately. Suppose first that $S_n(G)$ has a 2-factor

F; then F is a collection of disjoint cycles C_1, C_2, ..., C_k which span $S_n(G)$. Now if any point $u \notin C_1$ is adjacent to a point $v \in C_1$, then deg $v \geq 3$ implies that v corresponds to a point of G, so that deg u = 2. But then the cycle C_i containing u must also contain v, which means that $C_i = C_1$, violating the choice of u. Thus k = 1, that is, C_1 is a spanning subgraph of $S_n(G)$. Furthermore, if two points v_1 and v_2 in C_1 are joined by a diagonal edge (an edge not belonging to C_1 which joins two points of C_1), then deg $v_i \neq 2$ for i = 1, 2; this is a contradiction since by Theorem 3.1, every v_1-v_2 path has length at least n + 1. Thus $C_1 = S_n(G)$ implies that G is also a cycle. The converse is immediate.

Corollary 3.10. The following statements are equivalent for any graph G:

(1) $S_n(G)$ is 1-factorable.

(2) $S_n(G)$ is 2-factorable.

(3) Each component of G is a cycle.

Section 3.4
The Enumeration of Trees Having a 1-Factor

In the course of this research a considerable number of enumeration problems have arisen. Some of these problems, for example the enumeration of homeomorphically irreducible trees, have been dealt with elsewhere (see [11]). Although extensive consideration of the area of graphical enumeration is beyond the focus of this dissertation, we do present one such problem as an example: the number of trees of given order which possess a 1-factor.

Before proceeding further, we require some additional definitions. A rooted graph is a graph in which one of its points, called the root, is distinguished from the others. Thus two rooted graphs are isomorphic if there exists a one-to-one correspondence between their vertex sets which preserves not only adjacency, but also the roots.

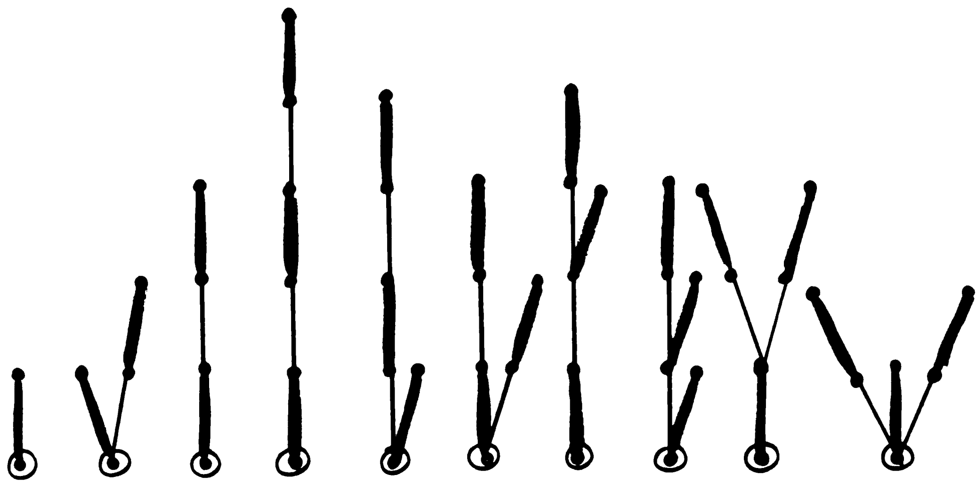


Figure 3.4. The 10 nonisomorphic rooted trees of order 6 or less having a 1-factor.

The generating function T(x) for rooted trees possessing a 1-factor is defined as $T(x) = \sum_{k=1}^{\infty} a_{2k}x^{2k}$, where a_{2k} is the number of nonisomorphic rooted trees of order 2k having a 1-factor. The generating function t(x) for (unrooted) trees possessing a 1-factor is similarly defined ad $t(x) = \sum_{k=1}^{\infty} b_{2k}x^{2k}$, where b_{2k} is the number of nonisomorphic trees of order 2k having a 1-factor. Figure 3.4 displays the nonisomorphic rooted trees of orders 2, 4, and 6 which have a 1-factor--each tree has the edges of its 1-factor darkened and its root encircled. Thus by direct observation we conclude that

$$T(x) = x^2 + 2x^4 + 7x^6 + \ldots.$$

E. M. Palmer has suggested an approach originated by Pólya [16] and refined by Otter [14], details of which can be found in Chapter 3 of [11], which shows that T(x) and t(x) satisfy the following functional equations:

$$T(x) = V^2(x), \text{ and}$$

$$t(x) = T(x) - 1/2(T(x) - V(x^2))$$

$$\text{where } V(x) = x \exp \sum_{k=1}^{\infty} \frac{T(x^k)}{k}.$$

From these equations we are able to obtain the following by direct computation.

$$V(x) = x + x^3 + 3x^5 + 10x^7 + 39x^9 + 160x^{11} + 702x^{13} + 3177x^{15} + \ldots$$
$$T(x) = x^2 + 2x^4 + 7x^6 + 26x^8 + 107x^{10} + 458x^{12} + 2058x^{14} + 9498x^{16} + \ldots$$
$$t(x) = x^2 + x^4 + 2x^6 + 5x^8 + 15x^{10} + 49x^{12} + 180x^{14} + 701x^{16} + \ldots$$

If desired, the correctness of the first 5 coefficients of t(x) can easily be checked firsthand by means of the tree diagrams found in [9], Appendix 3.

Section 3.5
The Edge Independence Number for $S_n(G)$

For any graph G, the <u>edge independence number</u> $\beta_1(G)$ of G is the maximum number of edges among all the 1-regular subgraphs of G, or equivalently, the maximum cardinality of any independent set of edges in G. For example, $\beta_1(K(m,n)) = \min(m,n)$. The observation that a graph G has a 1-factor if and only if $\beta_1(G) = |G|/2$ suggests that the parameter β_1 can be viewed as a generalization of the 1-factor concept. In general, $|G|/2$ is an upper bound for $\beta_1(G)$, but when G has no 1-factor, $\beta_1(G)$ may be much less than this. For example, K(1,p-1) is connected and has arbitrarily large order p, yet $\beta_1(K(1,p-1)) = 1$.

If, for any real number r, we denote by [r] the greatest integer not exceeding r, then without specializing the graph G under consideration, the best possible upper bound for $\beta_1(G)$ is $[|G|/2]$. Letting f(g) denote either of the graph-valued functions T(G) or G^n $(n \geq 2)$, we next show that $\beta_1(f(G))$ invariably attains this upper bound, thereby generalizing Corollary 3.9b.

Theorem 3.11. Let G be a connected graph and f(G) denote either one of the graph-valued functions G^n $(n \geq 2)$ or T(G). Then $\beta_1(f(G)) = [|f(G)|/2]$.

<u>Proof</u>. Suppose that G is a connected (p,q)-graph and f(G) is either G^n $(n \geq 2)$ or T(G). In the event that $|f(G)|$ is even, the assertion reduces to Corollary 3.9b. Assume, then, that $|f(G)|$ is odd.

For $f(G) = G^n$ $(n \geq 2)$, $|f(G)| = p$, and since G^2 is a subgraph of G^n, it suffices to show that $\beta_1(G^2) = \frac{p-1}{2} = [\frac{p}{2}]$. Let v be any point in G which is not a cutpoint of G (this is impossible only if $G = K_1$, in which case there is nothing to prove). Since G-v is connected and $(G-v)^2$ has even order, by Corollary 3.9b, $\beta_1((G-v)^2) = \frac{p-1}{2}$. Since $(G-v)^2$ is a subgraph of G^2, $\beta_1(G^2) = \frac{p-1}{2}$ also.

For $f(G) = T(G) = S^2(G)$, since S(G) is a connected graph of order p+q, $\beta_1(T(G)) = \beta_1(S^2(G)) = [\frac{p+q}{2}] = [\frac{|T(G)|}{2}]$ follows from the result shown in the preceding paragraph.

Corollary 3.11. Let G be a graph having components $C_1, C_2, \ldots, C_k$, and let f(G) denote either one of the graph-valued functions G^n $(n \geq 2)$ or T(G). Then $\beta_1(f(G)) = \sum_{i=1}^{k} \left[\frac{|f(C_i)|}{2}\right]$.

As regards the nth subdivision graph $S_n(G)$, Theorem 3.7 and Lemma 3.1 generalize to the following.

Theorem 3.12. Let G be a connected graph and k any positive integer. Then,

(1) $\beta_1(S_{2k}(G)) = [|S_{2k}(G)|/2]$ if and only if $\beta_1(G) = [|G|/2]$, and

(2) $\beta_1(S_{2k-1}(G)) = [|S_{2k-1}(G)|/2]$ if and only if $\beta_1(S(G)) = [|S(G)|/2]$.

Proof. Let G be a connected (p,q)-graph, so that $|S(G)| = p + q$, $|S_{2k-1}(G)| = p + (2k-1)q$, and $|S_{2k}(G)| = p + 2kq$. Each of the graphs $S_i(G)$, i = 1, 2k - 1, or 2k, is a union of q disjoint v_i-v_j

paths P_{ij} where each path P_{ij} corresponds to the edge v_iv_j in G and $V = \{v_1, v_2, \ldots, v_p\}$ is the set of points in $S_i(G)$ corresponding to the points of G.

For part (1). When p is even, (1) is a restatement of Theorem 3.7; we therefore assume that p is odd. For the sufficiency portion of (1), we assume $\beta_1(G) = (p - 1)/2$ and seek to prove that $\beta_1(S_{2k}(G)) = (p + 2kq - 1)/2$. Let F be a set of $(p - 1)/2$ independent edges in G. Each of the $(p - 1)/2$ v_i-v_j paths of length $2k + 1$ in $S_{2k}(G)$ corresponding to edges of F yield $k + 1$ independent edges; denote the set of these $(p - 1)(k + 1)/2$ independent edges by F'. Each of the remaining $q - (p - 1)/2$ v_i-v_j paths yield k additional independent edges which are mutually nonadjacent with the edges in F', so that altogether $S_{2k}(G)$ contains $(p - 1)(k + 1)/2 + (q - (p - 1)/2)k = (p + 2kq - 1)/2$ independent edges.

Conversely, if $\beta_1(S_{2k}(G)) = (p + 2kq - 1)/2$, to show that $\beta_1(G) = (p - 1)/2$, assume to the contrary that $\beta_1(G) \leq (p - 3)/2$. Then no more than $(p - 3)/2$ of the v_i-v_j paths of length $2k + 1$ in $S_{2k}(G)$ can fail to share a common point (otherwise the edges in G corresponding to these paths form a set of independent edges with cardinality greater than $(p - 3)/2$) and each such path yields at most $k + 1$ independent edges; denote the set of all such independent edges from these paths by F'. Each of the remaining v_i-v_j paths of length $2k + 1$ contains at least one point common to an edge of F', so that each such path yields at most k additional independent edges which are mutually nonadjacent with the edges in F'. Thus the number of independent edges in $S_{2k}(G)$ is no more than $(p - 3)(k + 1)/2 + (q - (p - 3)/2)k = (p + 2kq - 3)/2$, a contradiction. Hence $\beta_1(G) = (p - 1)/2$.

For part (2). When $p + q$ is even, (2) becomes Lemma 3.1; we therefore assume $p + q$ is odd (so that $p + (2k - 1)q$ is also odd). Supposing that $\beta_1(S(G)) = (p + q - 1)/2$, we must check that $\beta_1(S_{2k-1}(G)) = (p + (2k - 1)q - 1)/2$. All edges in $S(G)$ have the form u_iv_j, where $v_j \in V$ and $u_i \notin V$, so that for any set F of $(p + q - 1)/2$ independent edges in $S(G)$, every point of $S(G)$, except one, belongs to an edge in F. Furthermore, the edges of F determine $(p + q - 1)/2$ v_i-v_j paths P_{ij} of length 2 in $S(G)$, each of which contributes one edge to F, and $q - (p + q - 1)/2$ other v_k-v_l paths P_{kl}, each of which contains no edge of F; thus the points v_k and v_l of such paths P_{kl} must belong to edges in F (otherwise $S(G)$ contains more than $(p + q - 1)/2$ independent edges). Consequently, in $S_{2k-1}(G)$ each path of length $2k$ corresponding to some P_{ij} yields k independent edges and each path of length $2k$ corresponding to some P_{kl} yields $(k - 1)$ additional independent edges, so that altogether we have a set of $k(p + q - 1)/2 + (k - 1)(q - (p + q - 1)/2) = 1/2(p + (2k - 1)q - 1)$ independent edges.

The necessity portion of (2) follows immediately by taking $k = 1$. This completes the proof of Theorem 3.12.

Theorem 3.12(2) is illustrated in Figure 3.5 with a graph G having order 6 and 7 edges for which $\beta_1(S(G)) = [13/2] = 6$ and $\beta_1(S_3(G)) = [1/2(6 + 3(7))] = [27/2] = 13$ (the edges in the independent

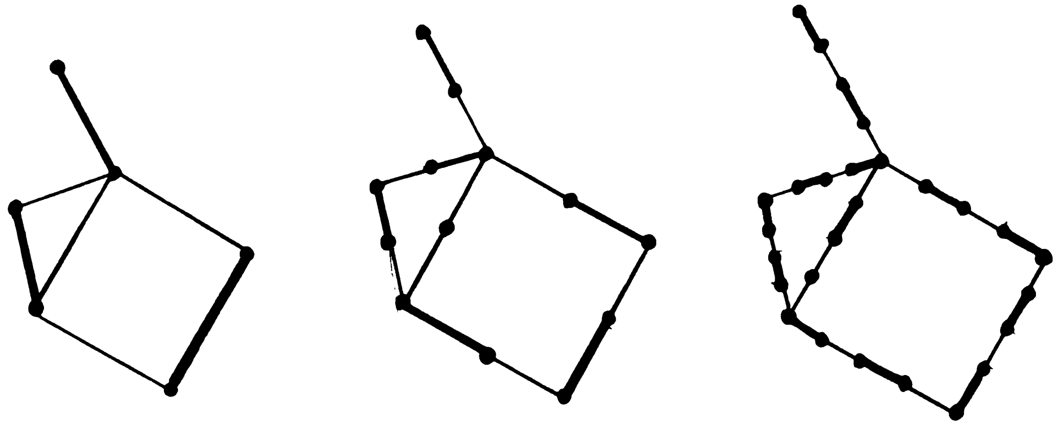


Figure 3.5. A (6,7)-graph G along with S(G) and S_3(G).

sets of maximum cardinality are darkened). Figure 3.5 also illustrates another result that applies to $S_n(G)$, for an an odd integer, which we state next.

Theorem 3.13. Let G be a connected (p,q)-graph. Then

$$\beta_1(S_{2k-1}(G)) = \begin{cases} kq & \text{if G is a tree.} \\ kq + p - q & \text{otherwise.} \end{cases}$$

Proof. If the connected (p,q)-graph G is a tree, then we shall prove that $\beta_1(S_{2k-1}(G)) = kq$ by showing that for every tree with p points, a maximal independent set F of edges in $S_{2k-1}(G)$ has cardinality kq and that for any endpoint w in $S_{2k-1}(G)$, the edges u_iv_i, for $1 \leq i \leq kq$, of F may be selected so that $d(w,u_i)$ is odd and $d(w,v_i) = d(w,u_i) + 1$. We proceed using induction on the order p of G.

The assertion is immediate for trees of order 1 or 2.

Now assume the assertion holds for trees on fewer than p points and consider a tree G with order p. For any endpoint v of G, let G' denote the tree on p - 1 points obtained from G by deleting v and its incident edge uv; by the induction assumption, an independent set F' of maximum cardinality of edges in $S_{2k-1}(G')$ has k(p - 2) members $u_i v_i$, for $1 \leq i \leq k(p - 2)$, which can be chosen in such a way that for any endpoint w in $S_{2k-1}(G')$ which is distinct from u, $d(w,u_i)$ is odd and $d(w, v_i) = d(w,u_i) + 1$. Since $S_{2k-1}(G)$ can be regarded as formed from $S_{2k-1}(G')$ by the addition of a u-v path P of even length 2k, it follows that the desired largest independent set F of edges for $S_{2k-1}(G)$ results by annexing the k alternate edges of P, beginning with the edge incident with v, to F'.

Next we consider connected graphs which contain at least one cycle. Proceeding by induction on the number of edges q, we shall prove that for any such (p,q)-graph G, each largest independent set F of edges of $S_{2k-1}(G)$ has cardinality p + (k - 1)q and contains an edge incident with each point of $S_{2k-1}(G)$ which corresponds to a point in G. In particular, we have that $\beta_1(S_{2k-1}(G)) = kq + p - q$.

If G is unicyclic (so that p = q), then Theorem 3.8 shows that $S_{2k-1}(G)$ has a 1-factor F, hence F is a largest independent set of edges of cardinality $1/2|S_{2k-1}(G)| = 1/2(p + (2k - 1)q) = kq = kq + p - q$.

The induction begins with q = 3; the only appropriate graph having 3 edges is K_3, a unicyclic graph, which therefore satisfies the induction assumption.

So now let G be a connected (p,q)-graph with two or more cycles and let G' denote the connected (p, q-1)-graph obtained from

G by deleting an edge uv not belonging to every cycle in G. Then G' contains at least one cycle and, by the induction assumption, every independent set F' of k(q - 1) + p - (q - 1) edges in $S_{2k-1}(G')$ contains an edge incident with each point of $S_{2k-1}(G')$ corresponding to a point in G'. Since $S_{2k-1}(G)$ differs from $S_{2k-1}(G')$ only by the presence of a u-v path P of length 2k and both u and v are incident with members of F', then an independent set of F edges with maximum cardinality for $S_{2k-1}(G)$ is obtained by adding to some set F' any collection of k - 1 mutually nonadjacent edges from P, none of which is incident with either u or v. Then $|F| = k(q - 1) + p - (q - 1) + (k - 1) = kq + p - q$ and, moreover, every point of $S_{2k-1}(G)$ corresponding to a point in G is incident with an element of F.

This completes the proof of Theorem 3.13.

In the case of a connected (p,q)-graph G. Theorem 3.8 is a corollary to Theorem 3.13. For if the graph $S_{2k-1}(G)$ has a 1-factor, then since its order is p + (2k - 1)q, we have $1/2(p + (2k - 1)q) = \beta_1(G) = kq + p - q$ (G is not a tree since that would imply that p + (2k - 1)q is odd). Simplifying, we obtain p = q, which means that G is unicyclic. Conversely, if G is unicyclic, then $\beta_1(S_{2k-1}(G)) = kq + p - q$. But since p = q, this value is equal to $1/2|S_{2k-1}(G)|$, implying that $S_{2k-1}(G)$ has a 1-factor.

By combining Theorem 3.13, Theorem 3.7, and the well known facts that the complete graph K_p has a 1-factor if and only if p is even and that the complete bipartite graph K(r,s) has a 1-factor if and only if r = s, we obtain the following corollaries.

<u>Corollary 3.13a</u>. $S_n(K_p)$ has a 1-factor if and only if either $p = 3$ and n is odd or both n and p are even.

<u>Corollary 3.13b</u>. $S_n(K(r,s))$ has a 1-factor if and only if either $r = s = 2$ and n is odd or $r = s$ and n is even.

The analogous result to Theorem 3.13 for $\beta_1(S_n(G))$, with n an even integer, is simpler to state but somewhat more difficult to apply, as the value of $\beta_1(G)$ is required.

<u>Theorem 3.14</u>. Let G be a connected (p,q)-graph. Then $\beta_1(S_{2k}(G)) = kq + \beta_1(G)$.

<u>Proof</u>. Suppose that F_0 is any independent set of edges of G having cardinality $|F_0| = \beta_1(G) = r$. Let both the points of G and the points of $S_{2k}(G)$ corresponding to the points of G be labeled as $v_1, v_2, \ldots, v_p$; to each edge v_iv_j in G, denote the corresponding v_i-v_j path of length $2k + 1$ in $S_{2k}(G)$ by P_{ij}. From each P_{ij} corresponding to an edge v_iv_j in F_0, we select the available $k + 1$ mutually nonadjacent edges; from each P_{ij} corresponding to an edge v_iv_j not in F_0, we select the k mutually nonadjacent edges, none of which is incident with v_i or v_j. Clearly this selection is always possible. Let F_1 denote this collection of $r(k + 1) + (q - r)k = kq + r$ independent edges in $S_{2k}(G)$. All that remains is to verify that F_1 is an independent set of edges in $S_{2k}(G)$ having maximum cardinality.

For any independent set F_2 of edges in $S_{2k}(G)$ with maximum cardinality $|F_2| \geq |F_1|$, we obtain another independent set of edges

F_3 from F_2 having the same cardinality in the following way. The edges of F_2 are distributed among the q paths P_{ij} of $S_{2k}(G)$, some of these paths--say s in number--containing k + 1 of these edges each, while the other q-s paths each contain only k edges of F_2. Whenever a v_iv_j path P_{ij} contains exactly k edges of F_2, at least one of v_i and v_j must be incident with an edge of F_2 not in P_{ij}, since otherwise k + 1 edges could be used in F_2 for that path P_{ij}, contradicting the maximality of $|F_2|$. Replacing this set of k edges by the set of k mutually nonadjacent edges of P_{ij}, no member of which is incident with v_i or v_j (if this latter set is distinct from the former) yields a maximal independent set of edges having the same cardinality as F_2. Repeating this process for every path P_{ij} containing only k edges of F_2, we obtain the promised independent set F_3 where $|F_3| = |F_2|$. Now choose a new independent set of edges in G consisting of the edges v_iv_j corresponding to the s v_i-v_j paths P_{ij} containing k + 1 edges of F_3. Since $|F_3| = s(k + 1) + (q - s)k \geq r(k + 1) + (q - r)k = |F_1|$ implies $s \geq r$, and since F_0 has maximum cardinality, we have s = r. Therefore, the cardinality of F_1 is equal to that of F_3, and hence to $|F_2|$ as well. Thus F_1 is a largest independent set of edges in $S_{2k}(G)$.

For connected (p,q)-graphs, Theorem 3.7 follows directly from Theorem 3.14. For if G has a 1-factor (so that $\beta_1(G) = p/2$), then $\beta_1(S_{2k}(G)) = kq + p/2 = 1/2|S_{2k}(G)|$ implies that $S_{2k}(G)$ has a 1-factor. Conversely, if $S_{2k}(G)$ has a 1-factor, then $\beta_1(S_{2k}(G)) =$

$1/2|S_{2k}(G)| = kq + p/2 = kq + \beta_1(G)$ implies that $\beta_1(G) = p/2$, so that G also has a 1-factor.

Another application of Theorem 3.14 is obtained by setting $G = K_p$ or $G = K(r,s)$. This procedure yields well known necessary and sufficient conditions that G have a 1-factor, as noted in the remarks preceding Corollary 3.13a.

The formulas presented in Theorems 3.13 and 3.14 are sufficiently similar so that one is tempted to try to combine them. One such formulation is the following.

Corollary 3.14. Let G be a connected (p,q)-graph. Then.

$$\beta_1(S_n(G)) = \begin{cases} \{\frac{n}{2}\}q + \frac{1}{2}[(-1)^n + 1]\beta_1(G), & \text{if } G \text{ is a tree,} \\ \{\frac{n}{2}\}q + \frac{(1 - (-1)^n)}{2}(p - q) + \frac{(1 + (-1)^n)}{2}\beta_1(G), & \text{otherwise,} \end{cases}$$

where curly brackets denote the greatest integer function.

BIBLIOGRAPHY

1. Mehdi Behzad. Graphs and Their Chromatic Numbers. Ph.D. Thesis, Michigan State University (1965).

2. Mehdi Behzad and Gary Chartrand. An Introduction to the Theory of Graphs, Allyn and Bacon, Inc., 1972.

3. Gary Chartrand. "On Hamiltonian Line Graphs." Transactions of the American Mathematical Society 3 (1968), 559-566.

4. Gary Chartrand; Dennis Geller; Stephen Hedetniemi. "Graphs with Forbidden Subgraphs." Journal of Combinatorial Theory 1 (1971), 12-41.

5. Gary Chartrand and Frank Harary. "Planar Permutation Graphs." Les Annales de l'Institut Henri Poincare (Section B), 3 (1967), 433-438.

6. Gary Chartrand; A. M. Hobbs; C. St. J. A. Nash-Williams; S. F. Kapoor. "The Square of a Block Is Hamiltonian Connected." (To appear.)

7. Gary Chartrand and H. V. Kronk. "Randomly Traceable Graphs." SIAM Journal of Applied Mathematics 4 (1968), 696-700.

8. H. Fleischner. "The Square of Every Non-separable Graph Is Hamiltonian Connected." (To appear.)

9. Frank Harary. Graph Theory. Addison-Wesley Co., 1969.

10. Frank Harary and C. St. J. A. Nash-Williams. "On Eulerian and Hamiltonian Graphs and Line Graphs." Canadian Mathematical Bulletin 8 (1965), 701-709.

11. F. Harary and E. M. Palmer. Graphical Enumeration. Academic Press, New York, 1973. (To appear.)

12. H. A. Jung. "Zu Einem Isomorphiesatz von Whitney für Graphen." Math. Ann. 164 (1966), 270-271.

13. O. Ore. "A Problem Regarding the Tracing of Graphs." Elem. Math. 6 (1951), 49-53.

14. R. Otter. "The Number of Trees." Ann. of Math. 49 (1948), 583-599.

15. Wayne S. Petroelje and Curtiss E. Wall. "Graph-Valued Functions and Hamiltonian Graphs." Recent Trends in Graph Theory (M. Capobianco, J. B. Frechen, and M. Krolik, editors). Springer-Verlag, Berlin (1971), 211-213.

16. G. Pólya. "Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und Chemische Verbindungen." Acta. Math. 68 (1937), 145-254.

17. M. James Stewart. "A Topological Influence: Homeomorphically Irreducible Graphs." The Many Facets of Graph Theory (G. Chartrand and S. F. Kapoor, editors). Springer-Verlag, Berlin (1969), 281-282.