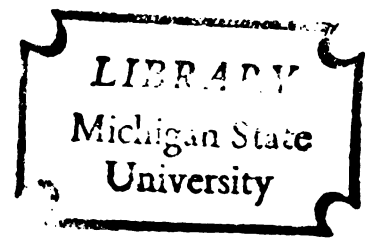


RADIATION IN A COMPRESSIVE PLASMA BY A
SMALL SOURCE SURROUNDED BY A PLASMA SHEATH

Thesis for the Degree of Ph. D.
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DAVID F. STRAWE

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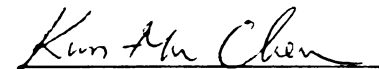
RADIATION IN A COMPRESSIVE PLASMA BY A SMALL
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presented by

David F. Strawe

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AN ABSTRACT

RADIATION IN A COMPRESSIVE PLASMA BY A SMALL SOURCE SURROUNDED BY A PLASMA SHEATH

by David F. Strawe

The purpose of this investigation is to determine the effect of the nonuniform plasma sheath region surrounding a small dipole antenna, immersed in an otherwise uniform compressive plasma, upon the driving point admittance and the radiated electroacoustic and electromagnetic energy. Most previous analyses have ignored the sheath assuming an entirely uniform surrounding plasma. These analyses generally predict relatively large amounts of power radiated in the form of longitudinal plasma waves or electroacoustic waves as they are called; because of this effect the driving point admittance of the antenna is modified greatly from the value it would assume in a free space environment. It is then a goal of this study to check the validity of these predictions by including in the analysis the effect of a plasma sheath using a geometry previously analyzed assuming a completely uniform surrounding plasma.

A simple geometry has been chosen consisting of two perfectly conducting spheres connected by an extremely thin straight feed wire; the system is driven symmetrically at the center of the feed wire with a sufficiently small excitation voltage that the perturbations in electron plasma density are linearly related to the excitation. Around each sphere spherically symmetric sheath regions are assumed to form

whose thicknesses are sufficiently small that they do not overlap. Also, the electron density perturbations are assumed spherically symmetric. These symmetry conditions can be obtained approximately if the spheres are sufficiently separated. Further, the dimensions are limited so that the fields surrounding the antenna are quasi-static. The input admittance is determined approximately from the quasi-static solution and corrected by a radiation conductance determined by the usual Poynting vector approach. The plasma is considered collisionless, and Landau damping is neglected; as a result, the plasma is lossless and the quasi-static input conductance can be related to the radiated electroacoustic power.

The symmetry assumed allows the perturbation in electron density around each of the spheres to be described by spherically symmetric wave equations which are determined from the first and second moment equations derived from Boltzmann's equation. Solution in the uniform plasma is particularly simple, but within the sheath regions a pair of wave equations must be solved simultaneously; each equation involves the electron density perturbation and the perturbation in electron drift velocity. These wave equations are extremely complicated and must be solved numerically. The method chosen here for solution is an iterative one involving solution of a simplified wave equation in the density perturbation only as a first step. This approximate solution can be substituted into the remaining wave equation allowing approximate solution for the drift velocity perturbation; this solution is reintroduced

into the first wave equation to obtain a first order iterative solution for the electron density perturbation. Once this solution is determined the input admittance is derived from it. When the EM radiation conductance correction is made the total input admittance is determined. The relative amount of power radiated in each radiation mode is determined by the ratio of the quasi-static input conductance to the EM radiation conductance.

Plots from which the quasi-static input admittance and the radiation conductance can be determined are given as are those of other parameters of interest. Comparison is made with the results obtained for the same geometry without the plasma sheath. Although there are many similarities in the two cases there are significant differences. The most salient of these is the general tendency for the magnitude of all solutions to be reduced when the sheath is considered. Although the trends of the solution for the quasi-static input conductance are the same in each case, with the sheath it is greatly reduced indicating greatly reduced radiated electroacoustic power. With the sheath there is a very significant increase in input susceptance (i. e. , toward the value corresponding to a free space environment). The radiation conductance is zero for each case at and below the plasma frequency, and each conductance increases monotonically with frequency to attain the same limiting forms at high frequencies although the uniform plasma case shows somewhat greater conductance, and hence, greater EM power in the intermediate

DAVID F. STRAWE

frequency range. It is generally concluded that previous analyses have unjustifiably neglected the sheath and have thereby predicted excessively large effects due to electroacoustic radiation.

RADIATION IN A COMPRESSIVE PLASMA BY A SMALL SOURCE
SURROUNDED BY A PLASMA SHEATH

By

David F.^{Francis} Strawe

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LIST OF SYMBOLS

I. Subscripts

Except where noted by the inclusion of exceptional terms in the nonstandard term section the following subscripts indicate:

e	Electron related quantities
i	Ion related quantities
o (or eo)	Unperturbed quantities
l	Perturbation quantities
le	EM mode perturbation quantities
lp	EA mode perturbation quantities
∞	Limiting values at infinity
s	Sheath edge values
r	Radial components
θ	Angular components
ϕ	Angular components
w	Values at the sphere's surface

II. Vector Quantities

Quantities with bars are vector quantities while the same symbols without bars are vector magnitudes of the same quantities, i.e., $\bar{E}_1$ versus E_1 for example.

III. Standard Symbolism

$\bar{E}$	Electric field
$\bar{B}$	Magnetic flux density
$\bar{H}$	Magnetic field intensity

$\overline{S}_{av}$	Time averaged Poynting vector
P_{av}	Time averaged radiated EM power
I	Feed current
σ	Surface charge density
V	Applied voltage
ϵ	Dielectric permittivity
ϵ_o	Permittivity of free space
$\overline{J}$	Volume current density
μ_o	Permeability of free space
e	Electronic charge
K	Boltzmann's constant
ϕ	Scalar potential
η	Normalized potential $(\frac{-e\phi}{KT_e})$
T	Gas temperature
r	Radial coordinate
λ_D	Debye shielding length $(\lambda_D^2 = \frac{KT_e \epsilon_o}{n_\infty e^2})$
ω_p	Radian plasma frequency $(\omega_p^2 = \frac{n_\infty e^2}{m_e \epsilon_o})$
$\overline{v}$	Velocity
$\overline{a}$	Acceleration
$\overline{u}$	Average drift velocity
v_o	RMS electron thermal velocity
n	Gas number density
ω	Radian frequency
t	Time
$d\overline{r}^3$	Differential volume in configuration space

$d\bar{v}^{-3}$	Differential volume in velocity space
$f(\bar{\mathbf{r}}, \bar{\mathbf{v}}, t)$	Probability distribution function in phase-time space
$\hat{\mathbf{n}}$	Unit normal in the direction of increasing $\mathbf{r}$
∇	Del operator
$\nabla_{\mathbf{v}}$	Del operator in velocity space

IV. Nonstandard Symbolism

P	Gas pressure
$P(\mathbf{r})$	Potential component
$d\mathbf{p}$	Differential linear momentum
$(\frac{\partial f}{\partial t})_{\text{coll}}$	Collision term in Boltzmann's equation
v_c	Critical electron velocity below which electrons have insufficient kinetic energy to reach the sphere
$\mathbf{v}(\mathbf{r}, \mathbf{r}')$	Velocity at $\mathbf{r}$ of an ion generated at $\mathbf{r}'$
$\bar{\omega}$	$\bar{\mathbf{v}} - \bar{\mathbf{u}}$
ω_i	The i-th component of $\bar{\omega}$
$d\bar{\omega}^{-3}$	The counterpart of $d\bar{v}^{-3}$ in $\bar{\omega}$ space
a	Sphere radius
d	Sheath edge radius in sheath model
$d\bar{x}_i$	Differential of i-th component of $\bar{\mathbf{r}}$
$d\omega_i$	Differential of ω_i
dA	Differential of $A(\mathbf{r})$
$A(\mathbf{r})$	Area
$\bar{a}$	Average of $\bar{a}$ over $\bar{\mathbf{v}}$ space
d_1	Boundary between regions II and III
R	The largest plasma dimension

ℓ_f	Mean free path -- ions through neutrals
s	Normalized distance variable in Self's analysis ($s = r/L$)
L	$\frac{n_\omega}{G(0)} \sqrt{\frac{2 KT_e}{M_i}}$
λ	Wave length of Bohm's plasma standing wave pattern
x_1	$\frac{r - a}{\lambda_D}$
x_2	$\frac{r - d}{\lambda_D}$
x	$- x_2$
$\overline{P}_c$	Collision term in second moment equation
$\overline{\Psi}$	Plasma stress tensor
f_c	Collision frequency
$\overline{\Phi}$	$- \phi$
$\overline{\Phi}_1(r)$	$r \phi_1$
m_e	Electron mass
M_i	Ionic mass
$G(r)$	Ion generation function (ions/vol/time)
δN	Ensemble of generated ions
$g(s)$	Normalized generation function $(\frac{G[r(s)]}{G[0]})$
α_{KB}	Balmain's absorption coefficient
u_{KB}	Velocity of sound in plasma
$I(\eta)$	Normalized ion current density $\sqrt{\omega^2 - \omega_p^2}$
β	EA propagation constant $(\beta = \frac{\omega_p}{v_o})$
β_p	β
b	$ \beta $

$R_1(E)$	E field contribution to complete plasma wave equation
$R_2(u)$	Drift velocity contribution to complete plasma wave equation
f_r	Effective normalized local plasma frequency
Y_a, Y_b	Cohen's surface admittance coefficients
D_1, B_1	Coefficients for hybrid boundary condition
$L_{1, 2, 3, 4}$	Linear ordinary second order differential operators
$N_1(r)$	$r n_1$
$N(r)$	Normalized spherical density solution $A N_1(r)$
C_0, C_1, C_2, A, B	Constants determining solutions for n_1
a_4, a_5, a_6	Constants determining solutions for ϕ_1
E, F, H	Plasma parameters involved in solutions for major plasma variables
K_c	Normalized version of C_2
K_y	Normalized quasi-static input admittance
K_a	Normalized solution from which n_1 is determined
K_e	Normalized version of G_e
K_p	Normalized EA - EM radiated power ratio
$\beta_1(r)$	Propagation factor analog in complete density wave equation
$\beta(r)$	$\frac{\sqrt{\omega^2 - \omega_p^2 f_r}}{v_o}$
G_e	Radiation conductance
Y_p	Quasi-static input admittance
G_p	$R_e[Y_p]$

P_p	Radiated EA power
P_e	Radiated EM power
$\bar{J}^s$	Source volume current density
ρ^s	Source volume charge density
J_p	Plasma volume current density at $r = a$
p	$\frac{\partial \eta}{\partial x_1}$
I_o	Dipole model feed current
Q	Dipole model charge
Ω_p	Singular point in Fejer's solution for $n_1(a)$
y	Dummy variable
$y_{1,2}$	Linearly independent solutions to density wave equation
$a_a, a_b,$ γ_a, γ_b	Integration constants
l	Geometry index
γ_1	Generation factor
a_e	$\frac{m_e}{2KT_e}$
α	Specific heat ratio
γ	Number of degrees of freedom in perturbation system
$N_H(r)$	Homogeneous solution for $N(r)$
$N_p(r)$	Particular solution for $N(r)$
u	$r u_1$
Ω	ω/ω_p

$$U = - \frac{2 n_{\infty} u_{\infty}}{v_o} r u_1 A$$

$$k_{pe} = \omega_p / c$$

$$\sigma_1, \sigma_2 \quad \text{Coefficients in boundary conditions}$$

$$\sigma_3 = d/a$$

$$\delta = \frac{\lambda_D}{a}$$

CHAPTER I

INTRODUCTION

1.1. The Problem and Its Motivation

The purpose of this investigation is to determine the effect of the nonuniform plasma sheath region surrounding a small dipole antenna, immersed in an otherwise uniform plasma, upon the driving point admittance and the radiated electroacoustic and electromagnetic waves. It is hoped that certain generalizations can be made as a result of this study which will be of general applicability to antenna problems where the effect of the plasma sheath is important.

Generally, when antennas in plasmas are analyzed the plasma sheath is ignored; this usually leads to the prediction of a relatively large amount of power radiated from the antenna in the form of electroacoustic waves and of a considerable modification of the driving point admittance over its free space value; many analyses of this form have appeared in the literature.

Wait¹ has analyzed the spherical aperture antenna in a uniform compressive plasma. He assumes the applicability of the linearized Maxwell and moment equations neglecting all loss mechanisms and replacing the stress tensor by the gradient of a scalar pressure as is, it seems, conventional in these analyses (this idealization neglects the existence of Landau damping² which is very significant for frequencies appreciably above the plasma frequency); his solution is a modal analysis exact within the framework of the basic equations which he uses. Wait's theory predicts a rather large effect due to

the radiation of EA waves. On the other hand, Larson³, in his doctoral dissertation, analyzed the symmetric version of the same model with essentially the same basic equations and approach but retained the plasma sheath with its nonuniform electron density and static potential distribution (he neglected the electron drift current); this theory predicts a much smaller effect due to EA propagation.

In the class of analyses in which the existence of the plasma sheath is ignored the common conclusion drawn is that the radiation of EA waves has a relatively great effect upon the antenna driving point admittance. Those analyses including the sheath show, as a general rule, a much reduced effect.

Chen⁴ has analyzed the hertzian dipole as well as the thin cylindrical dipole assuming a uniform plasma and no plasma sheath; he obtains relatively large effects in each case. Chen⁵ also has analyzed a conducting cylinder in a uniform plasma with no sheath; he has found that the radar cross section due to the excitation of EA waves is orders of magnitude greater than that due to EM wave excitation, i. e. again a large effect due to EA wave propagation.

Fejer⁶ has analyzed a short dipole antenna consisting of two metal spheres connected by a thin feed wire driven at its center and assuming a uniform plasma with no plasma sheath. He determined the quasi-static fields surrounding the dipole using Poisson's equation and the linearized moment equations derived from the collisionless Boltzmann equation; he calculated the driving point impedance based on this quasi-static solution and proceeded to correct this expression with a radiation resistance using the usual

Poynting vector method in conjunction with the radiation zone fields which were matched to those in the quasi-static zone of the dipole. The model was simple, if somewhat unrealistic, but yields some particularly simple and interesting analytical results; his results also predict a substantial effect due to the excitation of EA waves.

There are many criticisms which might be made of these various antenna-plasma analyses, but the most seemingly questionable assumption is that of a completely uniform surrounding plasma (i. e., neglecting the plasma sheath); the vast difference in the results of Wait and of Larson, which represents essentially a comparison of the same geometry with and without the sheath, dramatizes this contention. The exact mechanism accounting for this difference is in question. This mechanism may be simply a matter of the depleted plasma sheath region, which forms a relative void near the antenna's surface, effectively isolating the antenna from the plasma for EA excitation. It is well known that there is a coupling between EA and EM waves in the presence of a density nonuniformity or discontinuity; this latter explanation seems more plausible since the sheath seems too thin for a void of that thickness to be an effective decoupling agent.

The most rigorous of the analyses referenced here is that of Larson, whose approach does not depend upon a dipole approximation involving solution for only the quasi-static solution in the vicinity of the antenna but solves for the total fields and is, therefore, not limited in antenna dimension. Prior to this analysis, Schmitt⁷ analyzed acoustic plasma resonances in a cylindrical plasma column

solving a modified form of the homogeneous Klein-Gordon equation with the plasma frequency written as a function of position to account for the nonuniformity in the electron density; the electric field of charge separation and the electron drift current were neglected. As mentioned before, Larson included the electric field but neglected the electron drift current. Inspection of the electron density wave equation developed in chapter IV indicates that the effect of the drift current is not entirely negligible, if small, so that it should be included in a rigorous solution of antenna-plasma problems.

1.2. The Geometry

The geometry of the problem considered here consists of two perfectly conducting spheres, each of radius a , and separated by a distance D in a compressive plasma of infinite extent and of uniform density n_∞ except in the vicinity of the spheres' surfaces. The spheres are connected by and symmetrically driven in antiphase from thin feed wires lying along the line between the sphere centers as indicated in Figure 1.1. The main body of the plasma consists of electrons and positive ions in equal numbers per unit volume assuring plasma neutrality; each constituent gas has the constant density n_∞ there, and the electron gas is assumed to have a Maxwellian velocity distribution with a constant temperature T_e so that the gas obeys the ideal gas laws. Due to the thermal motion of the electrons and ions a "plasma sheath" of nonuniform plasma is formed around each sphere. Because of their small mass relative to the ions, the electrons accumulate on the spheres in sufficient

numbers to create a negative static potential on each of the spheres which accelerates ions to them and repels further ingress of electrons; this leads to an equilibrium situation in which electron and ion densities decrease to relatively small values and in which charge separation, with an accompanying electric field, occurs as the spheres are approached; in addition, there is a steady drift of electrons and ions toward the spheres in this sheath region. The densities and drift velocities as well as the electric field due to charge separation return to their uniform plasma values asymptotically with distance from the spheres; drift velocities and the electric field vanish in the uniform body of the plasma. Although the sheath variables return asymptotically to their uniform values, uniformity has been essentially reestablished within a distance of approximately ten debye shielding lengths so that, in order to facilitate analysis, the plasma may be considered uniform for distances from the spheres of the order of, or greater than, this figure. The plasma surrounding each sphere is to be modeled, then, by assuming a certain plasma density profile in a sheath region which returns to uniformity smoothly in a finite and arbitrarily specified distance from the sphere; the rest of the plasma is considered uniform so that analysis of the propagation of electroacoustic and electromagnetic waves is greatly simplified, in the absence of a constant magnetic field, by the mode separation technique developed by Cohen.⁸ The feed wires are considered to be so thin that there is no appreciable charge accumulation upon them and hence, negligible sheath formation around them. The assumption is made that the

sheath thickness is much smaller than the center to center sphere spacing D so that the sheath formation is essentially spherically symmetric about the spheres' centers; this requirement is not unreasonable since in most plasmas the sheath thickness will be quite small. Further, the sphere spacing D is assumed sufficiently greater than the sphere radius a so that when the system is driven the perturbation in the spheres surface charge density, as well as all perturbations in plasma variables within the sheath, have spherical symmetry about the center of the sphere in question. The dimension D is assumed to be much smaller than the electromagnetic wave length in the uniform portion of the plasma so that the radiated electromagnetic fields may be determined considering the antenna to be an electric dipole in the uniform plasma consisting of two point charges (located at the sphere centers with charge equal to the sum of that on the sphere's surface and that in the sheath) separated by a distance D ; for this assumption to be valid there must be no charge accumulation on the feed wires, a condition which requires that D not be extremely large relative to the sphere radius a (the antenna is an end loaded dipole and requires coupling between the spheres to guarantee uniform current in the feed wires and, hence, no charge accumulation upon them). There is obviously a conflict here with the symmetry requirement that D be much larger than a ; even though there may be a compromise range of values for D/a in which each requirement is adequately met it seems that this range would be small at best. Despite this conflict, the assumptions made are necessary to make the problem tractable,

and herein lies the justification for their use.

This geometry has precedent for its use; with the exception of deletion of the sheath region the same geometry has been analyzed by Fejer⁶ who obtained some very interesting results in surprisingly simple analytical form; a major goal of the present study is to check out Fejer's results by inclusion of the effect of the sheath region. This geometry, while not representing a practical antenna structure, lends itself nicely to solution, and the results obtained should shed some light on the effect of the plasma sheath on antenna performance in a compressive plasma. The spherical symmetry of the perturbed quantities in the sheath allows the analysis to center around the solution for electromagnetic and hydrodynamic (plasma density, drift velocity, etc.) quantities radiated from a single sphere; the problem then reduces to a one dimensional one greatly simplifying the analysis.

The plasma surrounding each sphere will be arbitrarily subdivided into three regions. First Region I, the sheath region, extending from $r = a$ to $r = d$ is considered to contain all of the nonuniform plasma. Second Region II, extending from $r = d$ to $r = d_1$, is considered to have uniform plasma everywhere, joining that of Region I smoothly at $r = d$, and to be entirely within the quasi-static zone of the antenna so that the EM fields can be approximated nicely by solutions to Poisson's equation. Region III extends from $r = d_1$ to infinity, has uniform plasma everywhere, and encompasses the radiation zone as well as the transition zone between the radiation zone and the quasi-static zone. The profiles

of pertinent plasma variables in the three regions are plotted in Figure 1.2.

1.3. The General Approach

The driving voltage applied to the antenna just described is limited to magnitudes much less than the static potential built up on each sphere by the thermal motion of electrons and ions in the plasma in order that the perturbations in the electromagnetic and hydrodynamic quantities in the plasma be linearly related to the applied voltage. This allows linearization of Maxwell's equations and the moment equations which represents a major simplification of the analysis; the development of the linearized equations is carried out in chapter II. This study ignores the possible existence of a static magnetic field; therefore, in the uniform plasma region (Regions II and III) Cohen's mode separation approach can be used to advantage. This approach separates the solutions for the electromagnetic and hydrodynamic perturbation terms from the linearized Maxwell and electron moment equations, developed in chapter II, into an EM mode (electromagnetic) and an EA (electro-acoustic) mode for a uniform plasma in the absence of a static magnetic field and loss mechanisms. The equations show that the modes are decoupled, with the EM mode containing all of the perturbation magnetic field and no charge accumulation, behaving as if in a lossless dielectric medium of relative permittivity $1 - \frac{\omega_p^2}{\omega^2}$ (where ω_p is the plasma frequency). No loss mechanism

is considered here for simplicity although there exist, in reality, two major sources of damping, namely, the electron-neutral particle collision and Landau damping. Although not always negligible, collision losses above the plasma frequency, where EA propagation is possible, should not be great in most laboratory plasmas and in the ionosphere. Landau damping results from the trapping of electrons of high thermal velocity in the potential troughs of the EA wave resulting in a net transfer of energy from the wave into the plasma in the form of heat; this effect greatly damps EA waves for frequencies higher than about twice the plasma frequency so that the results of this analysis should be applied with care beyond this point.

Regions I and II lie entirely in the quasi-static zone of the antenna so that the electromagnetic fields can be approximated by solutions to Poisson's equation. A one dimensional Poisson's equation will be solved in conjunction with a one dimensional wave equation in the electron density perturbation analogous to the inhomogeneous Klein-Gordon equation; in Region II, this equation is, in fact, the Klein-Gordon equation. In each of these regions the Poisson equation is solved in terms of the solution to the density wave equation. The density wave equation is independently solved--exactly analytically in Region II and approximately by numerical means in Region I. The solutions in Regions I and II are matched at the interregional boundary ($r=d$) assuming the continuous differentiability of the potential perturbation and the electron density perturbation. In Region I, at the surface of the

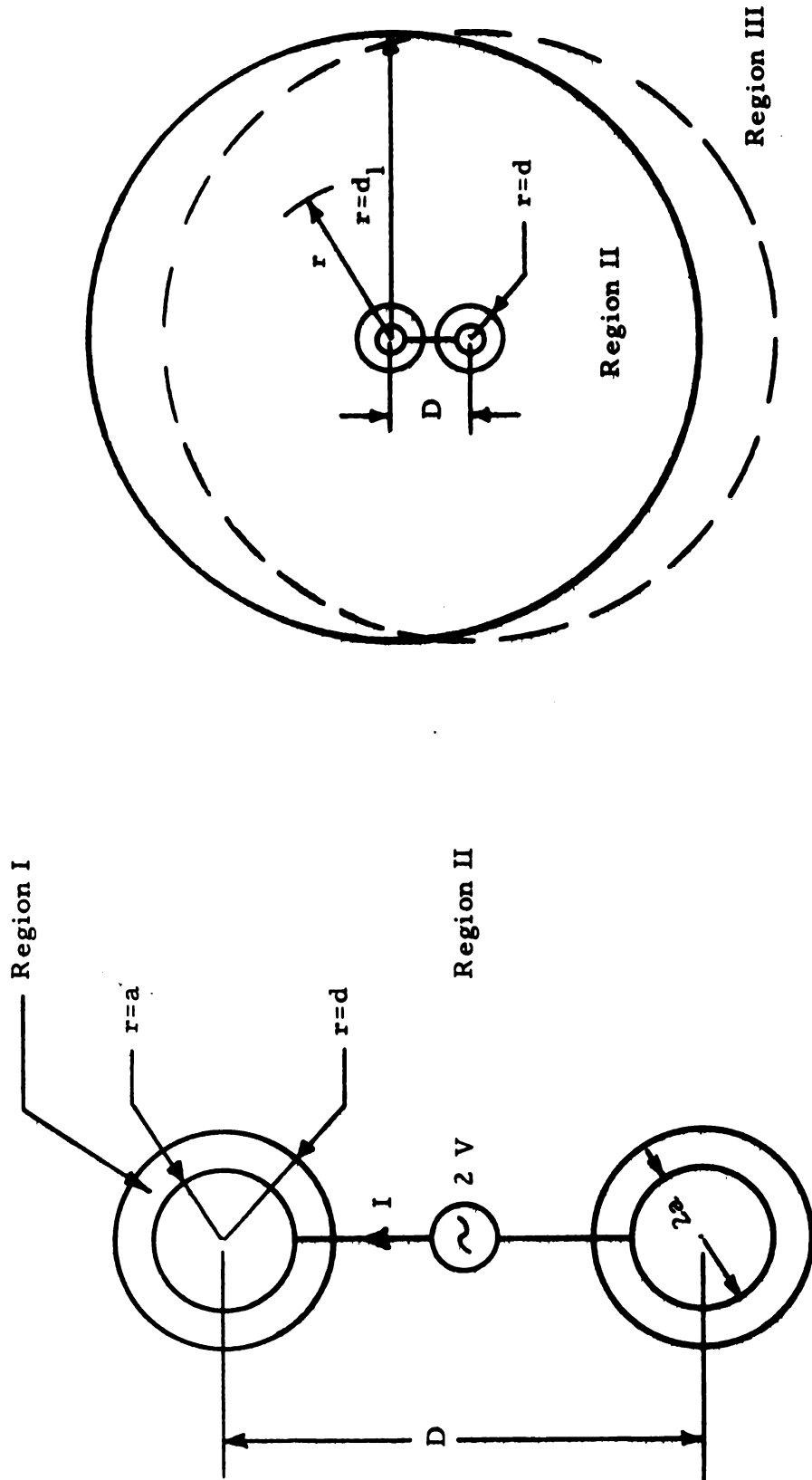
sphere, boundary conditions developed in chapter V are used to tie together the solutions for the density and potential perturbations as well as to tie the solutions to the driving potential V . In Region III, the mode separation technique of Cohen is employed to obtain solutions for the complete EM and EA terms. The EA terms are directly tied to those of Region II since the solutions are identical in the two regions. At the interregional boundary ($r=d_1$) the scalar potential term valid in the radiation zone as well as in Region II (taken from modal solutions) is matched to the scalar potential derived in Region II from Poisson's equation. This matching process is surprisingly simple here; the scalar potential in each region consists of a term due to the electron density perturbation and an "EM" term; the EM term from Region III matches directly the degenerate EM term from Region II, their forms being essentially the same while the EA terms are identical.

The antenna current I can be determined from the surface charge density perturbation on the sphere and the plasma current perturbation at $r = a$; each is easily related to the driving potential V so that a driving point admittance (denoted Y_p) can be calculated. Since no loss mechanism was included in the analysis and since no EM radiation effect is included in this admittance, because only the quasi-static solution was obtained in Region I, the real part of this admittance accounts for the existence of electroacoustic waves. The effect upon the driving point admittance of the EM radiation can be included, because of the absence of all loss mechanisms, by the usual Poynting vector method; a radiation conductance term

will be calculated by this method and displayed, in addition to Y_P , in chapter VIII. The ratio of the power radiated in the form of electroacoustic waves to that radiated in electromagnetic form is also given versus frequency.

Fejer describes the application of his theory to resonance probes and the modifications to the usual theory brought about by its application. He indicates that the usual theory can be grossly in error; this contention makes comparison of his theory with the present one on this particular point of interest. He argues that resonance indications take place at frequencies representing peaks (poles in his theory) of the electron density perturbation at the probes surface; the peaks in his theory occur appreciably below the plasma frequency whereas the conventional theory takes the resonance indications to occur at the plasma frequency. Plots of the electron density perturbation at the sphere's surface versus frequency from the present theory are presented in chapter VIII for comparison with Fejer's results.

Admittedly, the model analyzed here is highly idealized as well as being quite specialized, and the boundary conditions to be used (developed in chapter V) are somewhat arbitrary and perhaps not entirely realistic; hopefully, however, the results of the analysis will yield some insight into the extremely difficult problem which antenna-plasma problems represent.



A. Basic Antenna Geometry

B. Plasma Segmentation

Figure 1.1. Antenna-Plasma Geometry.

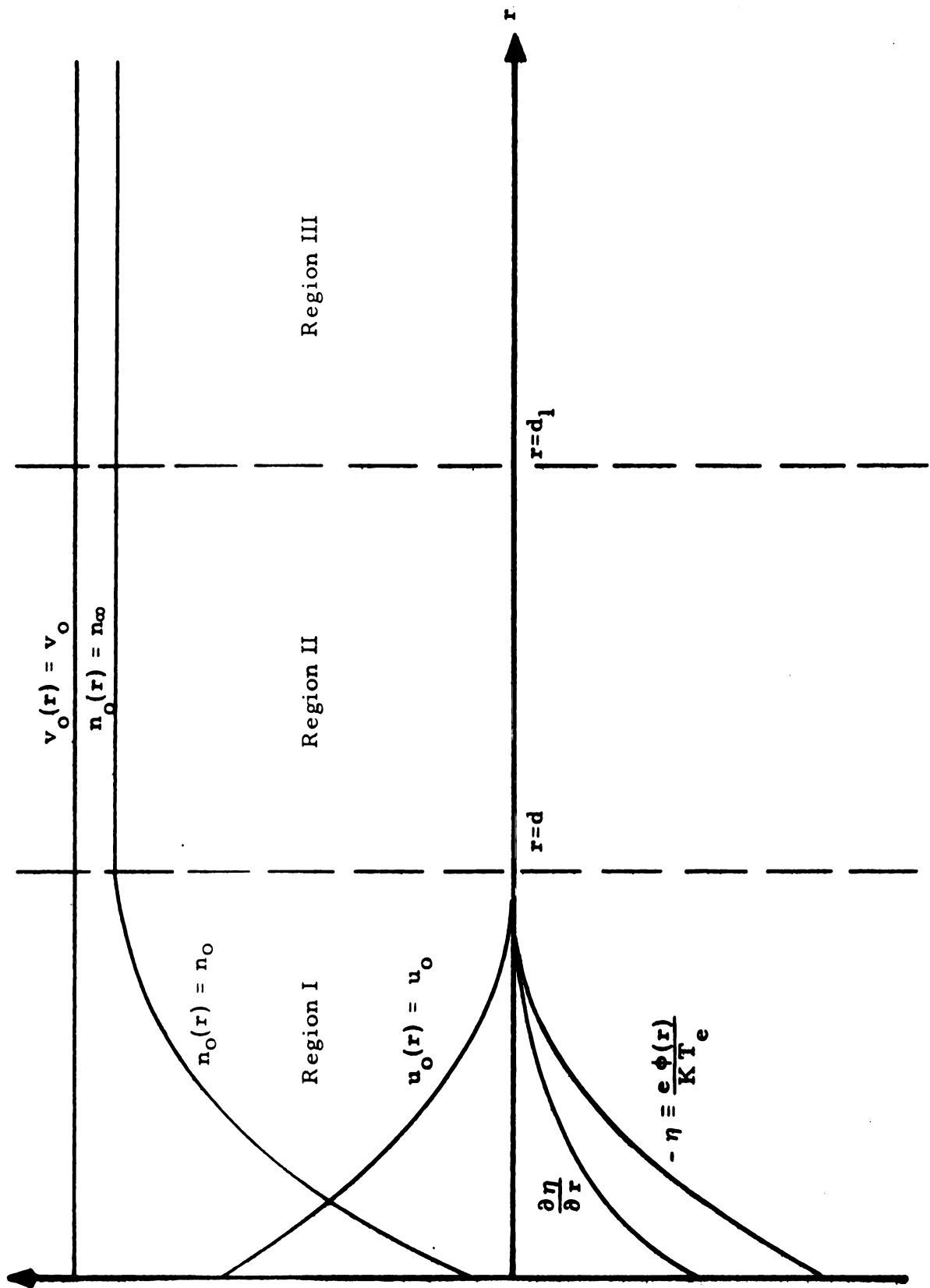


Figure 1.2. The Plasma Model.

CHAPTER II

BASIC EQUATIONS

2.1. Basic Equations

The most basic description of the dynamical behavior of a plasma would be a description of the position as a function of time of each constituent particle of that plasma. Obviously this kinetic or microscopic approach is impossible in general and recourse to the statistical hydrodynamic or macroscopic approach is required. A plasma being merely an ionized gas it is natural to apply the basic equations of gas dynamics to the description of the plasma. A basic set of equations applicable to gas dynamics consists of the Boltzmann equation

$$\frac{\partial f}{\partial t} + \bar{\mathbf{v}} \cdot \nabla f + \bar{\mathbf{a}} \cdot \nabla_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} \quad (2.1)$$

and its first and second moment equations

$$\nabla \cdot (n \bar{\mathbf{u}}) = - \frac{\partial n}{\partial t} \quad (2.2)$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} = \bar{\mathbf{a}} - \frac{1}{nm} \nabla \cdot \psi + \frac{1}{nm} \bar{\mathbf{P}}_c \quad (2.3)$$

where $f = f(\bar{\mathbf{r}}, \bar{\mathbf{v}}, t)$ is the statistical distribution function for the gas in question in phase space,

$n = \int_{\mathbf{v}s} f d\bar{\mathbf{v}}^3$ is the number density of the gas equal to the integral of f over velocity space,

$\bar{\mathbf{u}} = \frac{1}{n} \int_{\mathbf{v}s} f \bar{\mathbf{v}} d\bar{\mathbf{v}}^3$ is the mean velocity of the gas particles,

$\bar{\mathbf{a}} = \frac{1}{n} \int_{\mathbf{v}s} f \bar{\mathbf{a}} d\bar{\mathbf{v}}^3$ is the mean acceleration of the gas particles,

ψ is the stress tensor related to the compressibility of the gas and defined by $(\nabla \cdot \psi)_i = m \frac{\partial}{\partial X_i} \int_{vs} f \omega_i \bar{\omega} d\bar{\omega}^3$ where $\bar{\omega} = \bar{v} - \bar{u}$,

P_c is the mean momentum gain of the gas particle defined as

$$m \int_{vs} \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} \bar{v} d\bar{v}^3,$$

m is the particle mass.

The only other term requiring description is $\left(\frac{\partial f}{\partial t} \right)_{\text{coll}}$ which can be considered as the total derivative in time of the distribution function $f(\bar{r}, \bar{v}, t)$ due to collisions. Also $\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} d\bar{r}^3 d\bar{v}^3$ is the net time rate of increase of particles in a differential volume $d\bar{r}^3 d\bar{v}^3$ in phase space moving at velocity $\bar{v}$ and is due to collisions; this seems reasonable when it is recognized that the Boltzmann equation is just a continuity equation in phase space.

In addition when the particles of a gas are ionized the dynamics of the plasma is also determined by Maxwell's equations. The plasma consists of electrons, positive ions, and neutral atoms so that Maxwell's equations take the form

$$\nabla \cdot \bar{E} = \frac{e}{\epsilon_o} (n_i - n_e) \quad (2.4)$$

$$\nabla \times \bar{E} = - \frac{\partial \bar{B}}{\partial t} \quad (2.5)$$

$$\nabla \times \bar{B} = \mu_o e (n_i \bar{u}_i - n_e \bar{u}_e) + \mu_o \epsilon_o \frac{\partial \bar{E}}{\partial t} \quad (2.6)$$

$$\nabla \cdot \bar{B} = 0 \quad (2.7)$$

where $\bar{E}$ and $\bar{B}$ are the total electric field and the total magnetic flux density, respectively; e is the electronic charge; μ_o and ϵ_o are the permeability and the permittivity of free space, respectively.

The subscripts i and e on n and $\bar{u}$ denote positive ion and electron quantities, respectively.

Each of the three constituent gases in the plasma (electrons, positive ions, and neutrals) should be described in part by separate systems of equations of the form (2.1) through (2.7); the neutrals, of course, are not affected by the $\bar{E}$ and $\bar{B}$ fields so that only equations (2.1), (2.2), and (2.3) apply. The neutrals do not enter into the wave propagation picture and so are of little interest except in so far as they collide with electrons and ions thereby affecting their distributions. The ions are much too heavy relative to the electron mass to contribute to wave propagation directly although they do affect plasma sheath variations of electron density and the sheath potential distribution, and thus in this manner make their presence felt. Characterization of the positive ion gas by use of equations (2.1), (2.2), and (2.3) is found to be difficult so that different means must be found. The electron gas lends itself fairly well to description by means of these equations however, and use will be made of them as the basis of the following work.

2.2. The Collision Term

The collision term $\bar{P}_c$ in equation (2.3) is, in general, an extremely difficult quantity to evaluate; it can be approximated only under special conditions. In the case of the weakly ionized gases which are being considered the dominant collision effect is that corresponding to the collision of electrons and neutral atoms. One commonly used model in this case where the plasma is assumed

to be nearly in equilibrium assumes a constant collision frequency;⁹ then the collision term in the Boltzmann equation can be shown to be approximately

$$\left(\frac{\partial f_e}{\partial t} \right)_{\text{coll}} \doteq - f_c (f_e - f_{e0}) \quad (2.8)$$

where f_c is the collision frequency between the electrons and the neutral atoms, f_e is the distribution function for the electrons, and f_{e0} is the equilibrium distribution function for the electrons. The acceleration term $\bar{a}$ for the electrons is related to the $\bar{E}$ and $\bar{B}$ fields by

$$\bar{a} = - \frac{e}{m_e} [\bar{E} + \bar{v} \times \bar{B}]$$

if gravity and other external potentials are assumed absent. In this case equation (2.1) becomes

$$\frac{\partial f_e}{\partial t} + \bar{v} \cdot \nabla f_e - \frac{e}{m_e} (\bar{E} + \bar{v} \times \bar{B}) \cdot \nabla_{\bar{v}} f_e = - f_c (f_e - f_{e0}) \quad (2.9)$$

The collision term $\bar{P}_c$ then becomes

$$\bar{P}_c = m_e \int_{\text{vs}} - f_c (f_e - f_{e0}) \bar{v} d\bar{v}^3 = - m_e f_c (n_e \bar{u}_e - n_e \bar{u}_{e0})$$

so that

$$\frac{\bar{P}_c}{n_e m_e} = - f_c (\bar{u}_e - \bar{u}_{e0}) \quad (2.10)$$

Other models for this weakly ionized case are possible, such as a constant mean free path, but equation (2.8) is likely to be as good of a model as any other yielding such a simple form for $\bar{P}_c$.

2.3. The Stress Tensor

The stress tensor term $\frac{1}{nm} \nabla \cdot \psi$ is another term which defies description. It seems impossible to evaluate for the ion gas, but it can be approximated for the electron gas by assuming that the ideal adiabatic gas law applies. For the electron gas the divergence of the stress tensor can be replaced by the gradient of a scalar pressure derived on the basis of an ideal gas formulation. This can be shown by direct calculation of the i -th component of the divergence of the stress tensor

$$(\nabla \cdot \psi)_i = m_e \frac{\partial}{\partial X_i} \int_{vs} f \omega_i \bar{\omega} d\bar{\omega}^3.$$

First it can be shown that by the definition of $\bar{\omega}$ the integral $\int_{vs} f \omega_i \omega_j d\bar{\omega}^3$ vanishes if $i \neq j$ so that

$$\begin{aligned} (\nabla \cdot \psi)_i &= m_e \frac{\partial}{\partial X_i} \int_{vs} f \omega_i^2 d\bar{\omega}^3 \\ &= \frac{\partial}{\partial X_i} [m_e n_e \langle \omega_i^2 \rangle] \end{aligned} \quad (2.11)$$

where $\langle \omega_i^2 \rangle$ is the mean square of the i -th component of particle velocity in the reference coordinate system moving with velocity $\bar{u}_e$.

Next a differential volume of cross section dA and length dX_i is considered. Within this volume all particles with $\hat{X}_i$ directed velocity components (in the moving system) equal to ω_i within $\pm \frac{d\omega_i}{2}$ are considered as a group to carry, in time dt , a momentum $d\mathbf{p}$ out of the volume in the $\hat{X}_i$ direction.

$$dp = m_e f(\omega_i) d\omega_i dA dX_i \omega_i$$

This corresponds to a pressure of

$$dP = \frac{1}{dA} \frac{dp}{dt} = \frac{m_e f(\omega_i) d\omega_i dA dX_i \omega_i}{dA dt}$$

But $\omega_i = \frac{dX_i}{dt}$ so that this ensemble of particles contributes a pressure

$$dP = m_e f(\omega_i) \omega_i^2 d\omega_i.$$

Integrating the contribution of all particles of all velocities in the X_i direction

$$P = \int dP = \int_{vs} m_e f(\omega) \omega_i^2 d\omega = m_e n_o <\omega_i^2>$$

so that

$$(\nabla \cdot \psi)_i = \frac{\partial}{\partial X_i} P$$

or

$$\nabla \cdot \psi = \nabla P \quad (2.12)$$

where P is the usual pressure in the moving system. It must be remembered that for this expression to be useful the gas considered must behave in a sufficiently ideal manner that the concept of a pressure is justified.

In the electron gas, collisions are assumed to be sufficiently frequent so that in the equilibrium state, i.e., when no external fields are present, the distribution function f_{e0} is nearly Maxwell-Boltzmann, and the gas obeys to a good approximation the ideal gas laws. If f_{e0} is Maxwell-Boltzmann the concept of a pressure and a temperature is valid; the electron temperature

T_e of the electron gas is assumed to be constant so that

$$\begin{aligned} P_e &= K T_e n_o \quad \text{and} \\ \nabla P_e &= K T_e \nabla n_o \end{aligned} \tag{2.13}$$

for the unperturbed gas (n_o will henceforth be used to denote the unperturbed value of n_e).

In all of the work to follow all of the electron plasma variables (n_e , $\bar{u}_e$, $\bar{E}$, etc.) are assumed to consist of their unperturbed values (those existing in the absence of external fields) plus a relatively small perturbation term due to external fields. All of the unperturbed terms will carry the subscript zero while the perturbation terms will carry the subscript one. The ion variables will be assumed to be unperturbed and will carry the usual symbolism.

In so far as perturbations are concerned it is assumed that collisions are sufficiently frequent to allow description in terms of a perturbation pressure term but that the effect of the external fields is so great as to reduce the number of degrees of freedom in the perturbation system to one. Perturbation density variations are assumed to form adiabatically so that the ideal adiabatic gas law applies. Just how well all of these conditions are satisfied is open to question, but these assumptions are common ones evidently representing the best approximations available. The ideal adiabatic gas law states that

$$P \sim n^{\alpha} \tag{2.14}$$

where α is the specific heat ratio $\frac{C_p}{C_v}$ which is equal to $\frac{\gamma + 2}{\gamma}$ from the kinetic theory, and γ is the number of degrees of freedom of the perturbation system. Here $\gamma = 1$ which implies that $\alpha = 3$. Equation (2.14) implies that

$$\frac{P_e}{P_o} = \left(\frac{n_e}{n_o} \right)^\alpha$$

or that

$$\nabla P_e = \nabla \left[P_o \left(\frac{n_e}{n_o} \right)^\alpha \right] = \nabla P_e (n_1, n_o) . \quad (2.15)$$

The chain rule can be used to write

$$\nabla P_e = \frac{\partial P_e}{\partial n_o} \nabla n_o + \frac{\partial P_e}{\partial n_1} \nabla n_1 . \quad (2.16)$$

Evaluating $\partial P_e / \partial n_o$ and $\partial P_e / \partial n_1$

$$\frac{\partial P_e}{\partial n_1} = K T_e \frac{\partial}{\partial n_1} \left[n_o \left(1 + \frac{n_1}{n_o} \right)^\alpha \right] = K T_e^\alpha \left(1 + \frac{n_1}{n_o} \right)^{\alpha-1}$$

or

$$\frac{\partial P_e}{\partial n_1} = \alpha \frac{P_e}{n_e} \quad (2.17)$$

and

$$\frac{\partial P_e}{\partial n_o} = K T_e \frac{\partial}{\partial n_o} \frac{(n_1 + n_o)^\alpha}{n_o^{\alpha-1}} = K T_e \frac{P_e}{P_o} \frac{n_o + (1-\alpha)n_1}{n_1 + n_o} . \quad (2.18)$$

Assuming the perturbation density term n_1 is much smaller than the unperturbed term n_o equations (2.17) and (2.18) become, respectively

$$\frac{\partial P_e}{\partial n_1} = \alpha \frac{P_o}{n_o} = \alpha K T_e \quad (2.19)$$

and

$$\frac{\partial P_e}{\partial n_o} = K T_e . \quad (2.20)$$

Substituting equations (2.19) and (2.20) into equation (2.16) ∇P_e takes the form

$$\nabla P_e = K T_e \nabla n_o + \alpha K T_e \nabla n_1 . \quad (2.21)$$

The assumption that the unperturbed electron gas obeys the ideal gas laws allows application of the equipartition principle. Since there are three degrees of freedom in the unperturbed gas the mean kinetic energy of the unperturbed electrons is

$$K. E. = \frac{m_e v_o^2}{2} = \frac{3}{2} K T_e \quad (2.22)$$

where v_o^2 is the mean square unperturbed electron velocity. The constant α has been given the value three; use of this value and equation (2.22) in equation (2.21) implies

$$\nabla P_e = K T_e \nabla n_o + m_e v_o^2 \nabla n_1 . \quad (2.23)$$

This is the common scalar pressure substitution for the stress tensor; it can be no better as an approximation than the set of assumptions leading to its development.

2.4. The Moment Equations

The mean electron acceleration term $\overline{\overline{a}}$ of equation (2.3) can easily be shown to be

$$\overline{\overline{a}} = - \frac{e}{m_e} (\overline{E} + \overline{u}_e \times \overline{B})$$

in the absence of gravity and other external potentials. If equations (2.23) and (2.10) are substituted in equation (2.3) the moment equations for the electron gas with small perturbations from equilibrium take the form

$$\nabla \cdot (n_e \overline{u}_e) = - \frac{\partial n_1}{\partial t} \quad (2.24)$$

and

$$\begin{aligned} \frac{\partial \overline{u}_1}{\partial t} + (\overline{u}_e \cdot \nabla) \overline{u}_e = & - \frac{e}{m_e} (\overline{E} + \overline{u}_e \times \overline{B}) \\ & - \frac{1}{n_e m_e} (K T_e \nabla n_o + m_e v_o^2 \nabla n_1) - f_c \overline{u}_1 . \end{aligned} \quad (2.25)$$

It is possible to separate equations for the unperturbed variables by dropping all perturbation terms, i. e.,

$$\nabla \cdot (n_o \overline{u}_o) = 0$$

and

$$(\overline{u}_o \cdot \nabla) \overline{u}_o = - \frac{e}{m_e} (\overline{E}_o + \overline{u}_o \times \overline{B}) - \frac{K T_e}{n_o m_e} \nabla n_o . \quad (2.26)$$

The moment equations for the perturbation terms can be simplified by retaining only first order terms in the perturbation terms, i. e., dropping all the unperturbed terms as well as all of those involving

the product of more than one perturbation term. Equation (2.24) is reduced by replacing $n_e \bar{u}_e$ by $n_o \bar{u}_1 + n_1 \bar{u}_o$ so that

$$\nabla \cdot [n_o \bar{u}_1 + n_1 \bar{u}_o] = - \frac{\partial n_1}{\partial t} . \quad (2.27)$$

The first order terms corresponding to $n_e (\bar{u}_e \cdot \nabla) \bar{u}_e$ are

$$[n_e (\bar{u}_e \cdot \nabla) \bar{u}_e]_1 = n_o (\bar{u}_o \cdot \nabla) \bar{u}_1 + n_o (\bar{u}_1 \cdot \nabla) \bar{u}_o + n_1 (\bar{u}_o \cdot \nabla) \bar{u}_o ; \quad (2.28)$$

those corresponding to $n_e [\bar{E} + \bar{u}_e \times \bar{B}]$ are

$$[n_e (\bar{E} + \bar{u}_e \times \bar{B})]_1 = n_1 \bar{E}_o + n_o \bar{E}_1 + n_o \bar{u}_o \times \bar{B}_1 + n_o \bar{u}_1 \times \bar{B}_o + n_1 \bar{u}_o \times \bar{B}_o . \quad (2.29)$$

It follows that equation (2.25) is reduced to

$$n_o \frac{\partial u_1}{\partial t} + [n_e (\bar{u}_e \cdot \nabla) \bar{u}_e]_1 + f_c \bar{u}_1 = - \frac{en_o}{m_e} [\bar{E}_1 + \bar{u}_e \times \bar{B}]_1 - v_o^2 \nabla n_1 . \quad (2.30)$$

Equations (2.26), (2.27), and (2.30) are general and complete for the case of small perturbation of the equilibrium plasma.

In the special case to be considered in the bulk of this work where collisions, the Lorentz force, and any constant magnetic field are neglected the moment equations for the perturbation terms become

$$\nabla \cdot [n_o \bar{u}_1 + n_1 \bar{u}_o] = - \frac{\partial n_1}{\partial t}$$

and

$$n_o \frac{\partial \bar{u}_1}{\partial t} + [n_e (\bar{u}_e \cdot \nabla) \bar{u}_e]_1 = - \frac{e}{m_e} [n_o \bar{E}_1 + n_1 \bar{E}_o] - v_o^2 \nabla n_1 . \quad (2.31)$$

In a region of uniform plasma where the densities of ions and electrons are constant and equal, $\bar{u}_o = 0$, collisions can be neglected, the perturbation Lorentz force term is neglected, and the unperturbed $\bar{E}$ field is zero equations (2.27) and (2.30) take the relatively simple form

$$n_o \nabla \cdot \bar{u}_1 = - \frac{\partial n_1}{\partial t}$$

and

$$n_o \frac{\partial \bar{u}_1}{\partial t} = - \frac{e n_o}{m_e} \bar{E}_1 - v_o^2 \nabla n_1 \quad (2.32)$$

Maxwell's equations can also be separated into equations in unperturbed and perturbation terms, respectively.

$$\nabla \cdot \bar{E}_o = \frac{e}{\epsilon_o} (n_i - n_o) \quad (2.33)$$

$$\nabla \cdot \bar{E}_1 = - \frac{e}{\epsilon_o} n_1 \quad (2.34)$$

$$\nabla \times \bar{E}_o = 0 \quad (2.35)$$

$$\nabla \times \bar{E}_1 = - \frac{\partial \bar{B}_1}{\partial t} \quad (2.36)$$

$$\nabla \times \bar{B}_o = 0 \quad (2.37)$$

$$\nabla \times \bar{B}_1 = - \mu_o e (n_o \bar{u}_1 + n_1 \bar{u}_o) + \mu_o \epsilon_o \frac{\partial \bar{E}_1}{\partial t} \quad (2.38)$$

$$\nabla \cdot \bar{B}_o = \nabla \cdot \bar{B}_1 = 0 \quad (2.39)$$

In the uniform plasma region just described equation (2.38) can be simplified to

$$\nabla \times \bar{B}_1 = - \mu_o e n_o \bar{u}_1 + \mu_o \epsilon_o \frac{\partial \bar{E}_1}{\partial t} \quad (2.40)$$

Thus equations (2.32), (2.34), (2.36), and (2.39), together with equation (2.40), completely specify the dynamics of the uniform plasma.

The solution of the perturbation equations for the uniform plasma can be simplified by separation of the plasma variables into two sets of variables corresponding to an electromagnetic and electroacoustic mode as described by Cohen.⁸ In this mode separation technique the four perturbation plasma variables

are considered each to be the sum of an "EM" and an "EA" component to be denoted with subscripts $1e$ and $1p$, respectively.

$$\begin{aligned}
 \overline{E}_1 &= \overline{E}_{1e} + \overline{E}_{1p} \\
 \overline{B}_1 &= \overline{B}_{1e} + \overline{B}_{1p} \\
 \overline{u}_1 &= \overline{u}_{1e} + \overline{u}_{1p} \\
 n_1 &= n_{1e} + n_{1p}
 \end{aligned} \tag{2.41}$$

It can be shown that such a mode separation is indeed valid in the frequency domain and that the new modal variables are determined by the following equations.

EM mode

$$\begin{aligned}
 \nabla \times \overline{E}_{1e} &= -j\omega \overline{B}_1 \\
 \nabla \times \overline{B}_1 &= \mu_o \overline{J}^s - \mu_o e n_o \overline{u}_{1e} + j\omega \mu_o \epsilon_o \overline{E}_{1e} \\
 \nabla \cdot \overline{E}_{1e} &= \frac{\rho^s}{\epsilon(\omega)} \\
 \epsilon(\omega) &= \epsilon_o \left(1 - \frac{\omega_p^2}{\omega^2} \right) \\
 \omega_p^2 &= \frac{n_o e^2}{m_e \epsilon_o}
 \end{aligned} \tag{2.42}$$

EA mode

$$\begin{aligned}
 \nabla \times \overline{E}_{1p} &= 0 \\
 j\omega \mu_o \epsilon_o \overline{E}_{1p} - \mu_o e n_o \overline{u}_{1p} &= 0 \\
 \nabla \cdot \overline{E}_{1p} &= \frac{\rho^s}{\epsilon_o} - \frac{e}{\epsilon_o} n_1 - \nabla \cdot \overline{E}_{1e} \\
 j\omega n_o \overline{u}_{1p} &= - \frac{e n_o \overline{E}_{1p}}{m_e} - v_o^2 \nabla n_1
 \end{aligned} \tag{2.43}$$

The terms $\bar{J}^s$ and ρ^s are source current density and source charge density terms which are independent of the plasma variables; ω_p is a plasma parameter called the plasma frequency. It can be seen that $\bar{B}_{1p} = \bar{n}_{1e} = 0$. Equations (2.42) and (2.43) indicate that the EM mode has an electric field, all of the magnetic field, but no charge accumulation. The EA mode consists of an electric field, no magnetic field, and all of the charge accumulation. It is a simple matter to manipulate equations (2.43) into a wave equation in n_1

$$\left[\nabla^2 + \frac{1}{v_o^2} (\omega^2 - \omega_p^2) \right] n_1 = - \frac{\omega_p^2 \rho^s}{e v_o^2} . \quad (2.44)$$

This is the well known inhomogeneous Klein-Gordon equation. This equation has the particular solution

$$n_1 = \frac{1}{4\pi} \frac{\omega_p^2}{e v_o^2} \int_{vol} \rho^s \frac{e^{-j\beta_p r}}{r} dv \quad (2.45)$$

where

$$\beta_p = \frac{\sqrt{\omega^2 - \omega_p^2}}{v_o} .$$

The EM equations can be solved considering that the sources exist in a region of constant permittivity

$$\epsilon(\omega) = \epsilon_o \left(1 - \frac{\omega_p^2}{\omega^2} \right) .$$

This mode separation technique is unfortunately not valid when external magnetic fields are present or when the plasma

density is not uniform, i. e., in the plasma sheath surrounding an antenna emersed in a compressive plasma, for example. For the case of the plasma sheath a wave equation for n_1 similar in form to equation (2.44) is developed in chapter IV; its extremely complicated nature requires that its solution be numerical.

CHAPTER III

THE PLASMA SHEATH

3.1. Sheath Formation

Whenever a conducting object is submerged in a hot plasma it assumes a negative potential relative to the surrounding plasma. If it is assumed that initially both electrons and ions are individually in thermal equilibrium at temperatures T_e and T_i , respectively, where $T_i < T_e$ (generally in arcs and ionospheric plasmas, where the plasma density is low, $T_i \ll T_e$) the electrons because of their small relative mass have much greater thermal velocities than do the ions. Therefore, upon introduction of the conductor into the previously uniform plasma, the electrons initially impact upon it in relatively large numbers before the ions can reach it in appreciable numbers. As a result a negative potential is produced at the conductor surface which accelerates the ions to the conductor and repels the electrons from it. Equilibrium is achieved when electrons and ions impact the conductor in equal numbers per unit time.

As a result of this process there exists in the plasma surrounding the conducting object a negative potential distribution which increases monotonically with distance from the conductor to the potential of the main body of the plasma. Also electron and ion densities as well as the corresponding drift velocities vary in the same manner in space as a function of the potential. If the conducting object is an antenna the nature of these spatial distributions of

of potential, densities, and drift velocities will influence the electromagnetic and electroacoustic waves radiated into the plasma.

This region of nonuniform plasma will henceforth be termed the "plasma sheath" or merely the "sheath". Specification of the plasma sheath parameters is a very difficult problem although approximate analyses of various geometries under restrictive assumptions have been carried out by numerous workers.

In general, the character of the plasma sheath is determined by the effects of particle collisions as well as the effect of the fields due to charge separation. The solution of the problem considering the effects of collisions, diffusion, and drift due to charge separation is a very elusive one, and no one to date has adequately attacked this general situation.

A large class of plasma problems involves the so-called "low density" or "low pressure" plasma where collision effects can be largely neglected relative to the dominant effect of drift due to the electric fields of charge separation. Once attention is limited to this class of plasmas a number of simplifying assumptions can be made allowing approximate solution for the major parameters describing the plasma sheath. Various sets of assumptions and corresponding analyses have been made by many authors.

3.2. Analyses of Low Density Plasmas

In the case of low density plasmas contained in a finite volume where the mean free path of the ions through the neutral particles (and therefore of the electrons through the neutrals) is much greater

than any linear dimension of the plasma a simplified analysis (such as that given by Tonks and Langmuir¹⁰ and extended by Self¹¹) for the sheath can be carried out. It is found that the magnitude of the conductor potential corresponds to several times the average electron thermal energy outside the sheath; therefore, nearly all electrons entering the sheath are reflected by the retarding sheath potential so that the distribution function for the electrons outside and inside (except perhaps very near the conductor surface) the sheath should be nearly Maxwell-Boltzmann. It is easily shown that the ion velocities cannot be Maxwellian distributed anywhere in the vicinity of the conductor since the conductor absorbs nearly every ion impacting it yielding a deficiency of ions traveling outward from the conductor. The ion density distribution can be related to the potential distribution by various orbital analyses; Poisson's equation can then be solved for the potential distribution.

To illustrate the mathematical difficulty incurred if a Maxwellian ion velocity distribution is used (whether totally Maxwellian or Maxwellian for incident ions only) to calculate the ion drift current density to the conductor a Maxwellian distribution is assumed valid outside the sheath at a distance far enough from the sheath to allow the ion velocity distribution to become essentially Maxwellian with a drift term yet close enough that collisions between this point and the conductor can be neglected. To simplify the description the conductor is assumed spherical and the plasma surrounding the sphere has spherical symmetry. The ion distribution function is

then

$$f(\bar{r}, \bar{v}) = n_{\infty} \left(\frac{M_i}{2\pi K T_i} \right)^{3/2} \epsilon^{-\frac{M_i}{2K T_i} [(v_r - u_{\infty})^2 + v_{\theta}^2 + v_{\phi}^2]} . \quad (3.1)$$

For the case of negligible collisions between $r = d$ and $r = a$ the continuity equation becomes

$$0 = \nabla \cdot (n \bar{u}) \quad \text{or} \quad 4\pi r^2 n \bar{u} = \text{constant} \quad (3.2)$$

so that the ion current directed to the sphere at $r = d$ equals that at $r = a$. The total ion current density at $r = d$ is then

$$n u \Big|_{r=d} = \int_{\text{vel. space}} f(\bar{r}, \bar{v}) \bar{v} d\bar{v}^3 = n_{\infty} \int_{\text{v. s.}} \left(\frac{M_i}{2\pi K T_i} \right)^{3/2} \epsilon^{-\frac{M_i}{2K T_i} [(v_r - u_{\infty})^2 + v_{\theta}^2 + v_{\phi}^2]} \bar{v} d\bar{v}^3$$

or

$$n u \Big|_{r=d} = n_{\infty} u_{\infty} . \quad (3.3)$$

This follows from the definition of the drift velocity and direct integration of $f(\bar{r}, \bar{v}) \bar{v}$ over the velocity space. No information on the value of u_{∞} is available from this integration process; also, the assumed distribution $f(\bar{r}, \bar{v})$ vanishes only at $v = \text{infinity}$ irrespective of the absorptive properties of the sphere. Generally, the sphere is assumed to absorb all ions impacting it, hence, in the absence of collisions in the sheath no "outward going" ions should exist at $r = a$ or $r = d$, i. e., $f(d, v)$ should vanish for $v_r > 0$ contradicting the Maxwellian form of the distribution for positive v_r . The standard Maxwellian distribution seems invalid and an alternative must be considered.

The next reasonable approach seems to be that of assuming that the incoming ions have a Maxwellian velocity distribution while the distribution function vanishes identically for $v_r > 0$. One might then attempt to calculate $n_\infty u_\infty$ as $\int_{v.s.} f(\bar{r}, \bar{v}) \bar{v} d\bar{v}^3$ by writing this integral in terms of u_∞ then solving the resulting equation in u_∞ for u_∞ . This, at first, might seem plausible since in this case

$$n_\infty u_\infty = \int_{v.s.} f(\bar{r}, \bar{v}) \bar{v} d\bar{v}^3$$

does not degenerate to

$$n_\infty u_\infty = n_\infty u_\infty$$

as in the case of the complete Maxwellian distribution.

This equation takes the form

$$n_\infty u_\infty = \int_{v.s.} f(\bar{r}, \bar{v}) \bar{v} d\bar{v}^3 = n_\infty \left(\frac{M_i}{2\pi K T_i} \right)^{3/2} \int_{v_r = -\infty}^0 \int_{v_\theta = -\infty}^{+\infty} \int_{v_\phi = -\infty}^{+\infty} e^{-\frac{M_i}{2 K T_i} [(v_r - u_\infty)^2 + v_\theta^2 + v_\phi^2]} v_r dv_r dv_\theta dv_\phi$$

or

$$n_\infty u_\infty = n_\infty \left(\frac{M_i}{2\pi K T_i} \right)^{1/2} \int_{v_r = -\infty}^0 e^{-\frac{M_i}{2 K T_i} (v_r - u_\infty)^2} v_r dv_r \quad (3.4)$$

This can be integrated to obtain

$$n_\infty u_\infty = n_\infty \sqrt{\frac{K T_i}{2\pi M_i}} e^{-\frac{M_i u_\infty^2}{2 K T_i}} + \frac{n_\infty u_\infty}{2} \left(1 + \operatorname{erf} \sqrt{\frac{M_i u_\infty^2}{2 K T_i}} \right)$$

or

$$x = \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{1 - \operatorname{erf} x} \quad \text{where} \quad x \equiv \sqrt{\frac{M_i u_\infty^2}{2KT_i}} \quad (3.5)$$

This equation, unfortunately, has only an infinite solution for u_∞ .

This can be seen from the fact that

$$f(x) \equiv 1 - \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x}$$

is an asymptotic approximation for $\operatorname{erf} x$ approaching it asymptotically "from below" and increasing monotonically as x approaches $+\infty$.

Therefore equation (3.5) is satisfied only for $u_\infty = -\infty$ which is non-physical, and therefore the assumed distribution is again invalid.

Although this technique is useless for determination of u_∞ , its failure does not indicate that the assumed distribution is grossly in error because $g(x)$ is a very good approximation for $\operatorname{erf} x$ if $x \geq \sqrt{2}$ so that the equation is very nearly balanced for u_∞^2 of the order of $\frac{4KT_i}{M_i}$ which is of the same order as that determined by various authors by different techniques.

The problem of including the temperature of the ions in the main body of the plasma in the sheath analysis is very difficult and has not been rigorously approached to date in the literature. The usual approach when including ion temperature is to assume that the ions are monoenergetic, i.e., all having the same average energy.

In contrast a Maxwellian distribution for the electrons at $r = d$ seems quite appropriate since most electrons are reflected back into the plasma maintaining the Maxwellian character of the outwardly

directed electrons at $r = d$. With the exception of the close vicinity of the sphere's surface, at every point in the sheath it is also true that most of the inwardly directed electrons are reflected by the sheath potential somewhere between that point and the sphere's surface. The velocity distribution in the sheath (except very near the sphere's surface) is essentially Maxwellian so that the electron density distribution can be assumed to be essentially Boltzmann. Corrections to the Boltzmann distribution close to the sphere have been given by various authors although most analyses merely assume that the electron density is perfectly Boltzmann everywhere.

Only the "tail" of the Maxwellian distribution contributes electrons to the drift current to the sphere. The drift velocity can be determined in terms of the sphere potential by assuming a strictly Maxwellian distribution of electron velocities at $r = d$ and integrating the contributions of the fast electrons in the Maxwellian "tail", i. e., those electrons with negative velocities at $r = d$ corresponding to energies sufficiently great to overcome the large negative sphere potential, i. e.,

$$v_r < v_c \equiv - \sqrt{\frac{2e\bar{\phi}_w}{m_e}}$$

where $\bar{\phi}_w \equiv -\phi_w$ is the potential of the sphere. The resulting integral equation can be solved exactly and reduced to the standard approximation by use of order of magnitude values of u_∞ and v_c from other analyses. Then

$$n_{\infty} u_{\infty} = \sqrt{\frac{a_e}{\pi}} \int_{-\infty}^{-v_c} e^{-a_e(v_r - u_{\infty})^2} v_r dv_r \quad \text{where } a_e = \frac{m_e}{2KT_e}. \quad (3.6)$$

This equation when integrated becomes

$$u_{\infty} = - \sqrt{\frac{KT_e}{2\pi m_e}} e^{-\eta_w + \frac{u_{\infty}}{2}} (1 - \operatorname{erf} \sqrt{a_e} (v_c - u_{\infty})) \quad (3.7)$$

where $\eta = -\frac{e\phi}{KT_e}$. Solving for u_{∞}

$$u_{\infty} = - \sqrt{\frac{KT_e}{2\pi m_e}} e^{-\eta_w} \frac{1}{\frac{1}{2} [1 + \operatorname{erf} \sqrt{a_e} (v_c - u_{\infty})]} \quad (3.8)$$

It is known from various sources that the order of magnitude of u_{∞} is $\sqrt{\frac{KT_e}{M_i}}$ and that of η_w is 4.50 so that

$$\sqrt{a_e} v_c = \sqrt{\frac{m_e}{2KT_e} \cdot \frac{2e\Phi_w}{M_e}} = \sqrt{\eta_w} = 2.1 \quad (3.9)$$

and

$$\sqrt{a_e} u_{\infty} \approx \sqrt{\frac{m_e}{2KT_e} \cdot \frac{KT_e}{M_i}} = \sqrt{\frac{m_e}{2M_i}} \ll 1. \quad (3.10)$$

This implies that $\sqrt{a_e} (v_c - u_{\infty})$ is of the order of 2 so that the $\frac{1}{2} [1 + \operatorname{erf} \sqrt{a_e} (v_c - u_{\infty})]$ is equal to unity within about 0.1%. The $\operatorname{erf} x$ "flattens" out to asymptotically approach 1.0 very closely for $x \geq 1$ so that variations in the values of η_w and u_{∞} from their assumed values have little effect upon the erf and hence upon the order of the approximation for $\frac{1}{2} [1 + \operatorname{erf} \sqrt{a_e} (v_c - u_{\infty})]$ and u_{∞} . The approximate relation for u_{∞} in terms of η_w which is normally used has been developed.

$$u_{\infty} \cong - \sqrt{\frac{KT_e}{2\pi m_e}} e^{-\eta_w} \quad (3.11)$$

It was assumed that collisions within the sheath could be neglected so that

$$\nabla \cdot (n u) = 0 \quad .$$

This implies that

$$n_{\infty} u_{\infty} \cong \left(\frac{r}{d}\right)^{\ell} n(r) u(r) \quad (3.12)$$

where $\ell = 0, 1, 2$ for planar, cylindrical, and spherical symmetry, respectively. In the spherical case if $\frac{d-a}{a} \ll 1$ (i.e., for a thin sheath) the factor $\left(\frac{r}{d}\right)^2$ is essentially unity so that

$$n_{\infty} u_{\infty} \cong n(r) u(r) \quad . \quad (3.13)$$

In accord with the assumption of a Boltzmann distribution of electron density

$$n(r) = n_{\infty} e^{-\eta(r)} \quad . \quad (3.14)$$

It follows directly from (3.11), (3.12), and (3.14) that

$$u(r) = - \left(\frac{d}{r}\right)^{\ell} \sqrt{\frac{KT_e}{2\pi m_e}} e^{\eta(r) - \eta_w} = \left(\frac{d}{r}\right)^{\ell} u_{\infty} e^{\eta(r)} \quad . \quad (3.15)$$

To complete the description of the sheath it is required that the ion density distribution and the ion current density be determined in terms of the potential distribution. Once this has been accomplished the Poisson equation can be solved for the potential distribution from which all other sheath variables can be determined.

$$\nabla^2 \phi(r) = -\frac{e}{\epsilon_0} (n_i(r) - n_e(r)) \quad (3.16)$$

Because of the difficulty in determining the ion density and drift current these quantities are normally determined by orbital analyses of some type. Normally the ions are considered to be generated cold so that the drift currents are the result of the potential distribution only;^{10, 11} when ion temperatures are considered the energy distribution is usually neglected and the ions are assumed to be monoenergetic^{12, 13} (i. e., all of equal energy). It might be questioned why a direct and rigorous approach such as solution of the Boltzmann equation is not employed. If such a solution were accomplished for both ions and electrons all drift velocities and densities can be determined hence the sheath would be completely described. If collisions are neglected and the previous symmetry (or any other symmetric situation which reduces the description to one space variable) is considered and if the only field present is that due to charge separation in the sheath then the Boltzmann equation becomes

$$v \frac{\partial f}{\partial r}(\bar{r}, \bar{v}) \mp \frac{e}{m} \frac{\partial \phi}{\partial r} \frac{\partial f}{\partial v}(\bar{r}, \bar{v}) = 0 \quad (3.17)$$

for ions and electrons, respectively, where r and v are the space and velocity coordinates, respectively. Since $\partial \phi / \partial r$ is continuous everywhere in the plasma it is easy to show that any differentiable function of $\frac{mv^2}{2} \mp e \phi(r)$ is a solution. The whole problem then lies in the selection of the correct function which satisfies a set of boundary conditions sufficient to uniquely describe the solution.

The fact of the matter is the specification of this basic set of sufficient boundary conditions is not easily accomplished.

Tonks and Langmuir¹⁰ first considered the ion drift into the sheath as due entirely to acceleration of ions generated in the plasma at zero velocity by the sheath potential which actually penetrates the plasma. They limited consideration to a finite plasma enclosed by a conductor with the ion-neutral mean free path $\ell_f \gg R$, the largest plasma dimension without symmetry, so that only drift due to charge separation is considered. They solved by a series technique a "plasma equation" which is really the Poisson equation (3.16) with the left hand side set equal to zero in the plasma since $\frac{n_i - n_e}{n_i}$ approximates zero everywhere except in the sheath. Their solution yields ion density and ion current density at the sheath edge which allows solution for the "wall" potential (conductor potential) ϕ_w by equating ion and electron currents at the conductor surface as demanded by continuity.

Self¹¹ following Langmuir's formulation obtains complete solutions for the sheath potential and density profiles. Since this formulation and the corresponding results are so commonly used a brief sketch seems in order here. Three symmetric geometries are considered: planar, cylindrical, and spherical with the spherical case pictured in Figure 3.2.

Ions are assumed generated at rest everywhere in the main plasma body at a rate $G(\bar{r})$ ions/cm³/sec. $\eta(\bar{r})$ is taken as zero at the plasma center. The drift of ions to the sheath is considered

as a result of the presence of $\eta(\bar{r})$ only, i.e., thermal effects are neglected. Further all collisions of the ions once formed are neglected so that the ion drift and density can be determined from simple orbital analysis. To determine $n(r) u(r)$ we consider the number δN of ions generated in the differential volume pictured in Figure 3.2 of volume

$$dv' = A(r') dr' = 4\pi r'^2 dr'$$

at $r = r'$ in time dt with zero velocity and accelerated to

$$v(r, r') = \sqrt{\frac{2e}{M_i} (\Phi(r) - \Phi(r'))}$$

at $r = r$. Now

$$\delta N = G(r') dv' dt \quad (3.18)$$

and the contribution to $n(r) u(r)$ in time dt when they arrive there (at $r = r$) is

$$d(nu) = \frac{\delta N}{A(r) dt} = \frac{G(r') dv' dt}{A(r) dt} = G(r') \left(\frac{r'}{r}\right)^\ell dr' \quad (3.19)$$

where $\ell = 0, 1, 2$ for the planar, cylindrical, and spherical cases, respectively. $n(r) u(r)$ is obtained by integrating the contributions of each differential volume over $0 \leq r' \leq r$. Then

$$n(r) u(r) = \int_0^r G(r') \left(\frac{r'}{r}\right)^\ell dr' \quad (3.20)$$

This integral is generally transformed to an integral over $\eta(r)$ and the variables are transformed to dimensionless ones to facilitate numerical integration, i.e., the following definitions are applied:

$$g(s') = \frac{G(s')}{G(0)} ; \quad s = \frac{r}{L} ; \quad L = \frac{n_{\infty}}{G(0)} \sqrt{\frac{2KT_e}{M_i}} . \quad (3.21)$$

Then (3.20) becomes

$$n(r) u(r) = n_{\infty} \sqrt{\frac{2KT_e}{M_i}} I(\eta(r)) \quad (3.22)$$

where

$$I(\eta) = \int_0^{\eta} g(s') \left(\frac{s'}{s}\right)^{\ell} \frac{\partial s'}{\partial \eta} d\eta . \quad (3.23)$$

$I(\eta)$ is obtained numerically and tabulated for $\ell = 0, 1, 2$. Except for the planar case $I(\eta)$ is a weak function of the form of the generation function. Two cases are considered: (1) ionization rate constant in space and (2) ionization rate proportional to electron density. The electrons are assumed Boltzmann distributed so that $g(s)$ can be written

$$g(s) = e^{\gamma_1 \eta}$$

where $\gamma_1 = 0$ for case (1) and $\gamma_1 = 1$ for case (2). Self indicates the following values are valid for $I(\eta_w)$:

TABLE 3.1. Self's Values for $I(\eta_w)$

$\ell = 0$	$\gamma_1 = 0$	$I(\eta_w) = 0.3444$
0	1	0.3444
1	0	0.2914
1	1	0.2703
2	0	0.2571
2	1	0.2136

The contribution to $n(r)$ due to δN is

$$\begin{aligned} d n(r) &= \frac{d(nu)_r}{v(r, r')} = G(r') \left(\frac{r'}{r}\right)^\ell \frac{dr'}{\sqrt{\frac{2e}{M_i} (\Phi(r) - \Phi(r'))}} \\ &= \sqrt{\frac{M_i}{2KT_e}} G(r') \left(\frac{r'}{r}\right)^\ell \frac{dr'}{\sqrt{\eta - \eta'}} . \end{aligned} \quad (3.24)$$

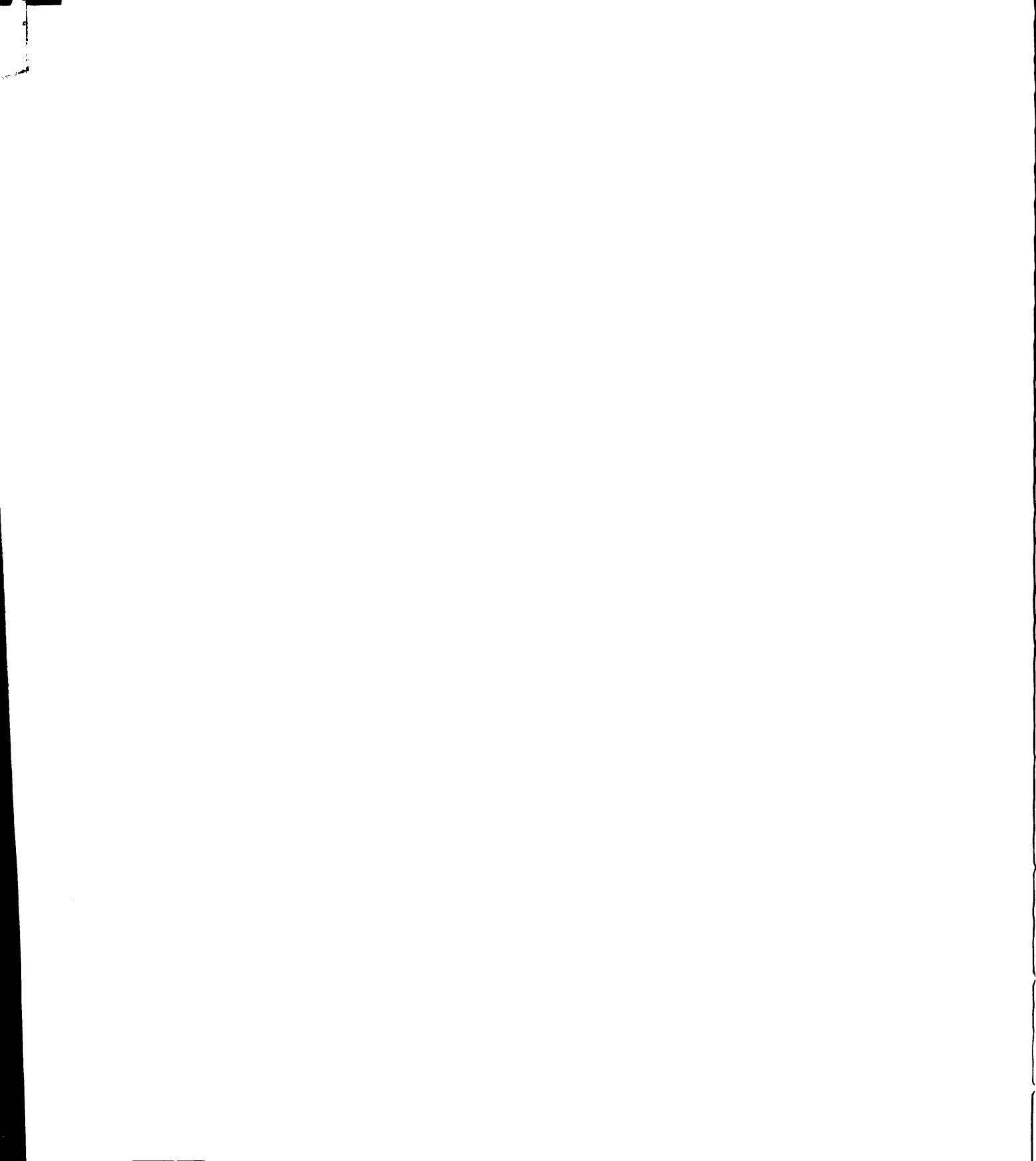
Then integrating over r'

$$n(r) = \int_0^r \sqrt{\frac{M_i}{2KT_e}} G(r') \left(\frac{r'}{r}\right)^\ell \frac{dr'}{\sqrt{\eta - \eta'}} \quad (3.25)$$

or in terms of the dimensionless variables

$$n(r) = \sqrt{\frac{M_i}{2KT_e}} \int_0^\eta g(s') \left(\frac{s'}{s}\right)^\ell \frac{1}{\sqrt{\eta - \eta'}} \frac{\partial s'}{\partial \eta} d\eta . \quad (3.26)$$

This latter equation has been solved in conjunction with Poisson's equation for the density and potential distributions. The geometry analyzed considered a finite plasma (at least the range of the space variable without symmetry is finite). If the external symmetric cases are considered the formulation of equations for $n(r)$ $u(r)$ and $n(r)$ is identical and yields equations of identical form so that one might be tempted to use Self's result for the external infinite plasma cases. No doubt the results apply well enough for sheath thicknesses much less than the radius for cylindrical and spherical cases, but these integrals involve integration over r from infinity to $r = a$. It is a simple matter to show that each diverges. For thin sheaths the sheath geometry is nearly planar;



further, it is difficult to imagine much difference between external and internal sheath formation. This contradiction of physical intuition with mathematical results can be rationalized by realizing that in the infinite plasma region surrounding the conductor the collisionless assumption of Self's internal plasma analysis cannot be justified as $r \rightarrow \infty$. Eventually, as r increases, collisions become more important, the analysis breaks down, and drift currents and densities are maintained finite.

Bernstein and Rabinowitz¹² analyzed the external infinite plasma for the spherical and cylindrical cases. Electrons are assumed to be Boltzmann distributed while an attempt to solve the Boltzmann equation for the ion distribution function is made. Then the Poisson equation is solved for the potential distribution. The ion current density is formulated as a complicated integral. This can be solved in conjunction with Poisson's equation only by tedious numerical computation. To simplify the problem ions are assumed to be monoenergetic. Solution for the potential and density distributions still involves numerical calculation but Bernstein and Rabinowitz present the results of this computation. Although more dependence of the results on ion temperature was found than previously predicted by a majority of analyses indicating essentially no dependence of sheath parameters on ion temperature, the dependence is still not great (approximately 20% over the entire range of possible ion temperatures).

In 1949 Bohm¹⁴ gave a simplified treatment ignoring the distribution of energy of the positive ions entering the sheath. Bohm

showed that the positive ions have an energy given by

$$\frac{1}{2} M_i v_r^2 = \frac{1}{2} K T_e$$

when they arrive at the sheath edge. This result is obtained by considering the planar case with ions assumed entering the sheath region all at the same velocity u_0 , electrons are Boltzmann distributed, and collisions of ions in the sheath are neglected. The density distribution of the ions is computed assuming

$$n(r) u(r) = n_\infty u_\infty = \text{constant}$$

where the drift velocity is computed from the potential distribution

$$u(r) = \sqrt{u_\infty^2 + \frac{2e\Phi(r)}{M_i}} \quad (3.27)$$

so that the density distribution becomes

$$n_i = \frac{n_\infty}{\sqrt{1 + \sigma \eta(r)}} \quad (3.28)$$

where

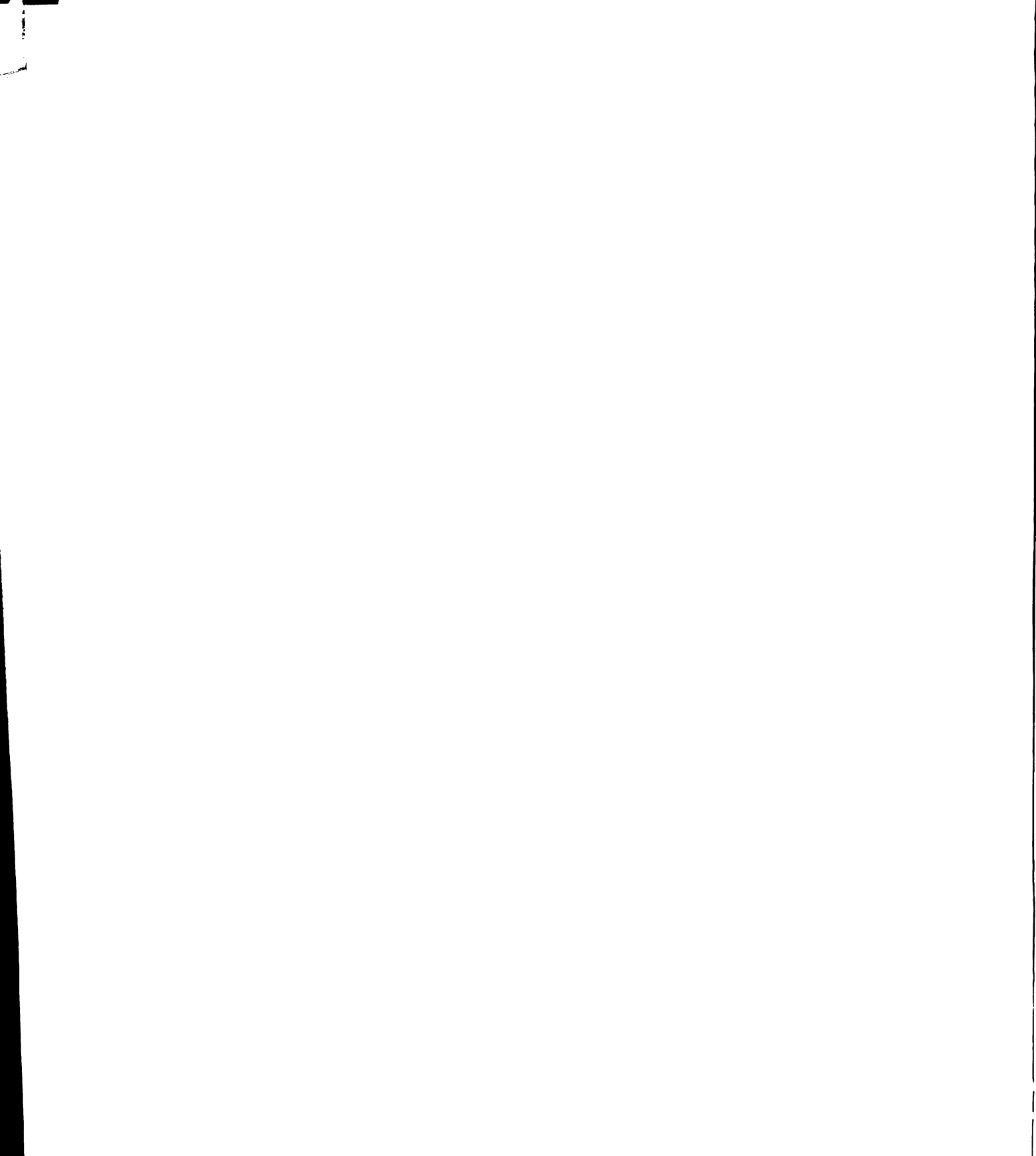
$$\sigma \equiv \frac{2KT_e}{M_i u_\infty^2} .$$

Poisson's equation then becomes

$$\frac{d^2 \eta}{dx^2} = \frac{1}{\sqrt{1 + \sigma \eta}} - e^{-\eta} \quad (3.29)$$

where $x \equiv r/\lambda_D$ and λ_D is the debye shielding length for the main plasma defined by

$$\lambda_D^2 = \frac{K T_e \epsilon_0}{n_\infty e^2} .$$



If the R. H. S. of equation (3.29) is expanded in a power series in η for small η , i. e., in the plasma near the sheath edge, equation (3.29), becomes

$$\frac{d^2\eta}{dx^2} = 1 - \frac{1}{2} \sigma \eta - (1 - \eta) + \dots \doteq \eta \left(1 - \frac{\sigma}{2}\right). \quad (3.30)$$

If $\sigma > 2$ the solutions of equation (3.30) in the plasma region will oscillate with wavelength

$$\lambda = \frac{\lambda_o}{2\pi \sqrt{\frac{\sigma}{2} - 1}}$$

which obviously is not allowable. Therefore $\sigma \leq 2$ and

$$u_\infty^2 \geq \frac{K T_e}{M_i}$$

from a physical argument; unfortunately nothing can be said at this point about an upper bound for u_∞^2 although $\frac{K T_e}{M_i}$ is found to be of the right order of magnitude from other analyses.

Allen and Thonemann¹⁵ also considered the planar case and arrived at the same density expressions starting with the same physical model. They note, however, that at the sheath edge

$$n_i = n_e \equiv n_s$$

and

$$\frac{dn_i}{dx} = \frac{dn_e}{dx} \quad (3.31)$$

to a high degree of accuracy. It is found as a result that

$$\frac{M_i u_\infty^2}{2} = \frac{1}{2} K T_e$$



from these conditions; this agrees with Bohm's result.

Later Allen, Boyd, and Reynolds¹⁶ analyzed the spherical case and published potential and density profile data as well as ion drift current. The drift current density was approximated by the equation valid at the sheath edge

$$n_s u_s = n_\infty 0.61 \sqrt{\frac{KT_e}{M_i}}. \quad (3.32)$$

It is interesting to note that Bohm, Burhop, and Massey obtain the same result for $n_s u_s$ considering monoenergetic ions except that 0.61 is replaced by 0.57 and 0.54 for ion energies of 0.01 and 0.5 times KT_e , respectively, indicating a very weak dependence on ion temperature. The apparent contradiction with Bohm's result

$$u_\infty \geq \sqrt{\frac{KT_e}{M_i}}$$

can be understood if it is noted that the more rigorous analyses of Langmuir and of Self, for example, indicate that η is not zero at the sheath edge and that consequently the density is less than half that in the uniform body of the plasma, n_∞ . It follows then that since

$$n_s u_s = n_\infty (0.61) \sqrt{\frac{KT_e}{M_i}}$$

that $u_\infty > \sqrt{\frac{KT_e}{M_i}}$ at the sheath edge and the conflict is resolved.

Additionally, Laframboise¹⁷ has performed an analysis and numerical integration similar to the approach of Bernstein and Rabinowitz but retaining a Boltzmann distribution for the ion density.

Other authors have attacked the problem with various approaches largely similar to those previously described.

A general result of all these analyses is that the ion drift velocity at the sheath edge is approximately $\sqrt{\frac{KT_e}{M_i}}$. The ion and electron densities at the sheath edge are roughly half their values in the main plasma. The sheath potential and density profiles are nearly independent of ion temperature being dependent almost entirely on the electron density and temperature in the main plasma body.

3.3. Conductor Potential

The potential of the conducting object must be determined by equating the electron and ion currents to the conductor in accordance with continuity. It has been previously show that the ion current density is given by

$$\left(\frac{2KT_e}{M_i}\right)^{1/2} n_{\infty} e I(\eta_w)$$

while the electron current density is

$$n_{\infty} e \sqrt{\frac{KT_e}{2\pi m_e}} e^{-\eta_w} .$$

$I(\eta_w)$ is determined by geometry and generation function form as is shown in Table 3.1. Equating the current densities yields the potential at the conductor surface as

$$\eta_w = \ln \frac{1}{I(\eta_w)} \sqrt{\frac{M_i}{4\pi m_e}} . \quad (3.33)$$

In the planar case Self¹¹ obtains

$$I(\eta_w) = 0.3444$$

so that

$$\eta_w = 3.56 + \frac{1}{2} \ln \frac{M}{M_p} \quad (3.34)$$

where M/M_p is the molecular weight of the gas in question.

Larson³ has collected typical profiles of potential and electron density from the analyses of various authors for the spherical case and for conductor potentials corresponding to $\eta_w = 3.5$. These data are presented in Figure 3.3. In each case Larson has plotted the curves to correspond to $\eta_w = 3.5$ and $a/\lambda_D = 19.6$.

3.4. Selection of a Sheath Model

The value $\eta_w = 3.5$ was arbitrarily chosen by Larson because most theories predict η_w near 3.5 for hydrogen and because several of the investigators presented profile data for hydrogen. Since heavier gases are normally encountered a value of $\eta_w = 4.5$ was arbitrarily chosen for the following work as it better approximates the values expected for most gases.

Larson's analysis indicates that EA wave propagation is not greatly affected by changing the sheath profiles, hence the arbitrary adoption of a set of profiles from any one of the sheath analyses reviewed is justified. As stated in the introduction the sheath thickness will commonly be many times less than the sphere radius; in this "thin sheath" case the geometry within the sheath and that of the plasma adjacent to the sheath becomes nearly planar. The profiles

to be used correspond to the solution of equation (3.29) developed by Bohm for the planar case with u_∞ taken as $\sqrt{\frac{KT_e}{M_i}}$, i.e.,

$$\frac{d^2 \eta}{dx^2} = \frac{1}{\sqrt{1+2\eta}} - \epsilon^{-\eta} \quad (3.35)$$

where $x = r/\lambda_D$ with the boundary conditions

$$\eta \Big|_{r=a} \equiv \eta_w = 4.5$$

and

$$\eta \rightarrow 0 \quad \text{as} \quad x \rightarrow +\infty. \quad (3.36)$$

Although the sheath potential profile is determined from the planar case for simplicity (because the potential profile description is not a function of the sphere radius a) the other sheath parameters are determined on the basis of spherically symmetric geometry. The equations governing the sheath parameters in terms of η are repeated here for ease of reference.

$$n(r) \equiv n_0 = n_\infty \epsilon^{-\eta} \quad (3.37)$$

$$u(r) \equiv u_0 = \left(\frac{d}{r}\right)^2 u_\infty \epsilon^{\eta} \quad (3.38)$$

$$u_r = - \sqrt{\frac{KT_e}{2\pi m_e}} \epsilon^{-\eta_w} \quad (3.39)$$

$$n_0 u_0 = \frac{d^2}{r^2} n_\infty u_\infty \quad (3.40)$$

$$n_i = n_\infty \frac{1}{\sqrt{1+2\eta}} \quad (3.41)$$

The solution of equation (3.35) is given in Figure 3.4. Also given in Figure 3.4 are the slope of $\eta(x)$ from equation (3.35) and the corresponding electron density profile.

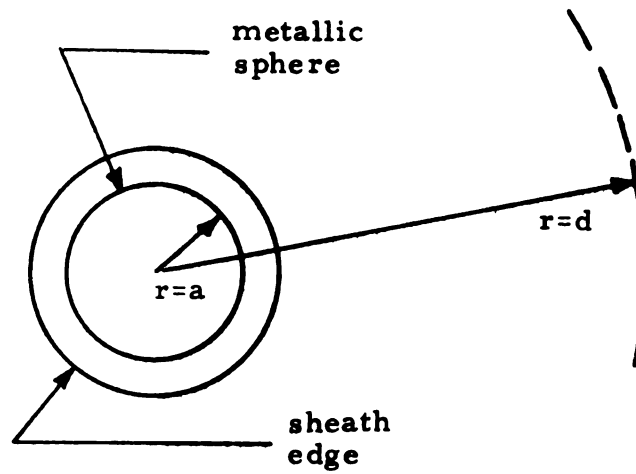


Figure 3.1. Maxwellian Distribution at $r=d$
 $a < d \ll l_f$

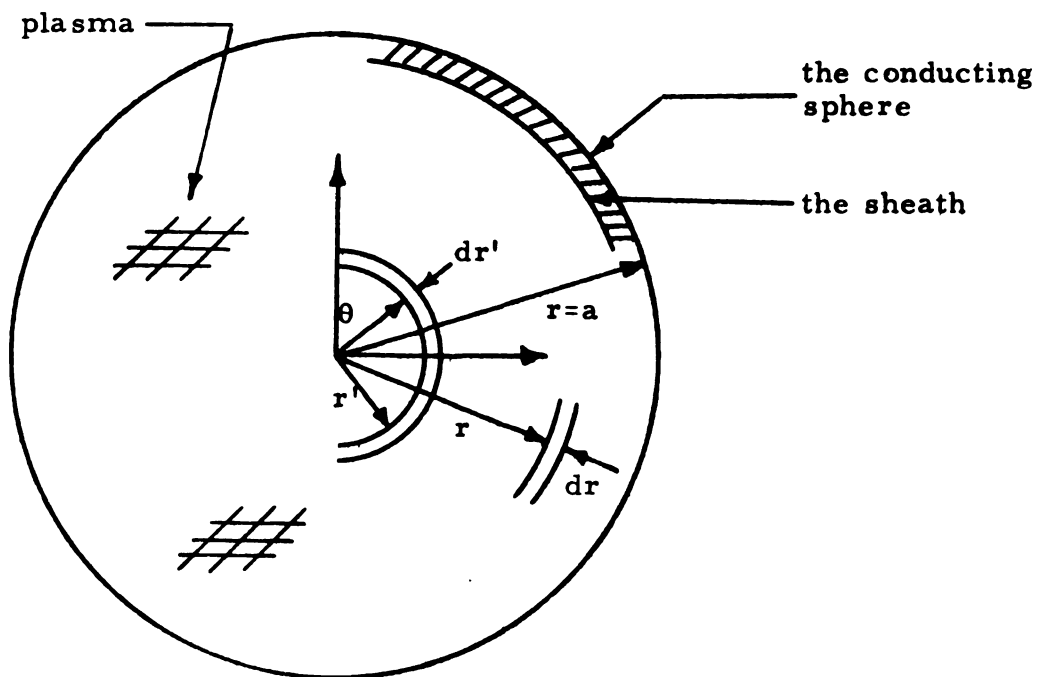


Figure 3.2. Self's Plasma Geometry.

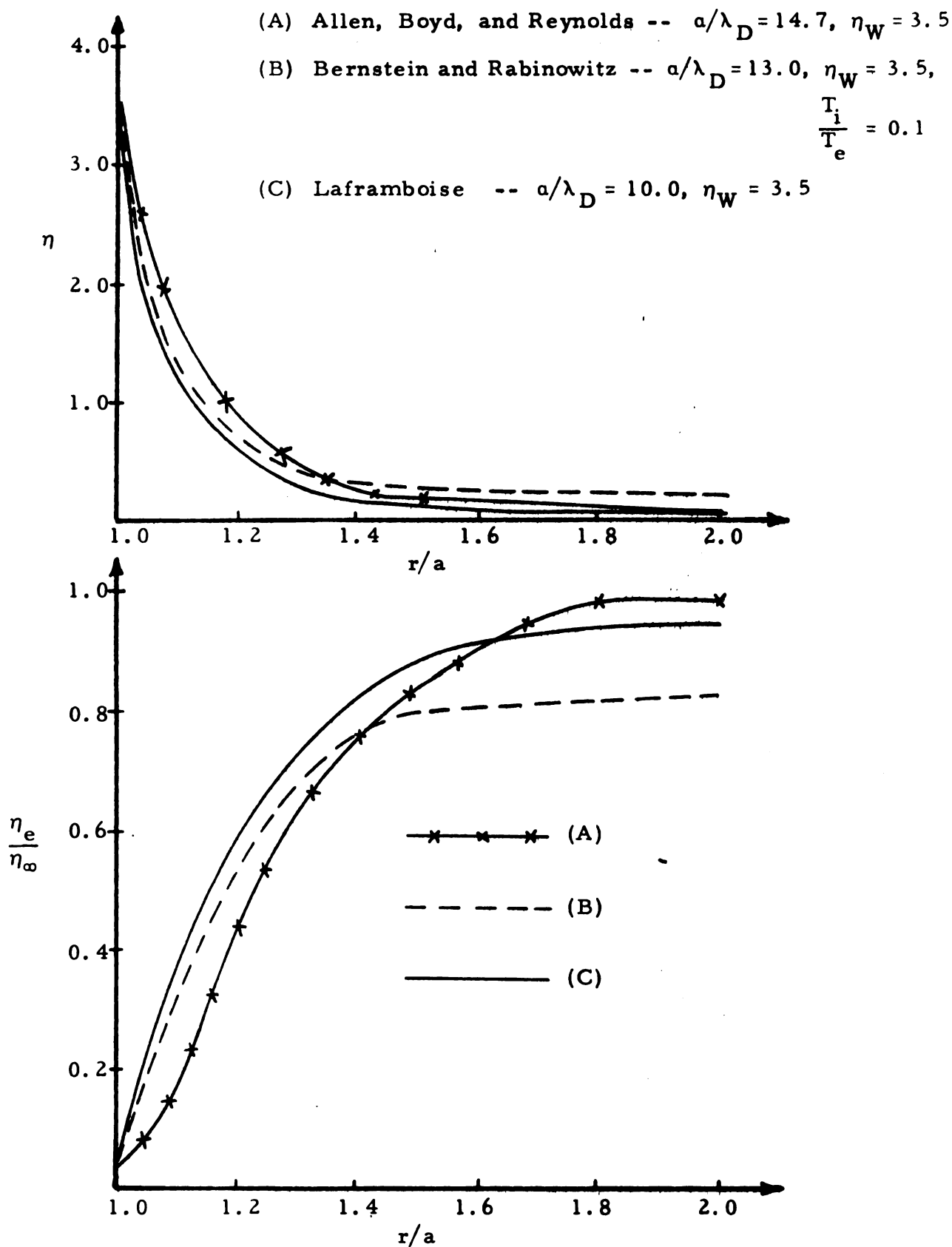


Figure 3.3. Comparison of Sheath Potential and Density Profiles.

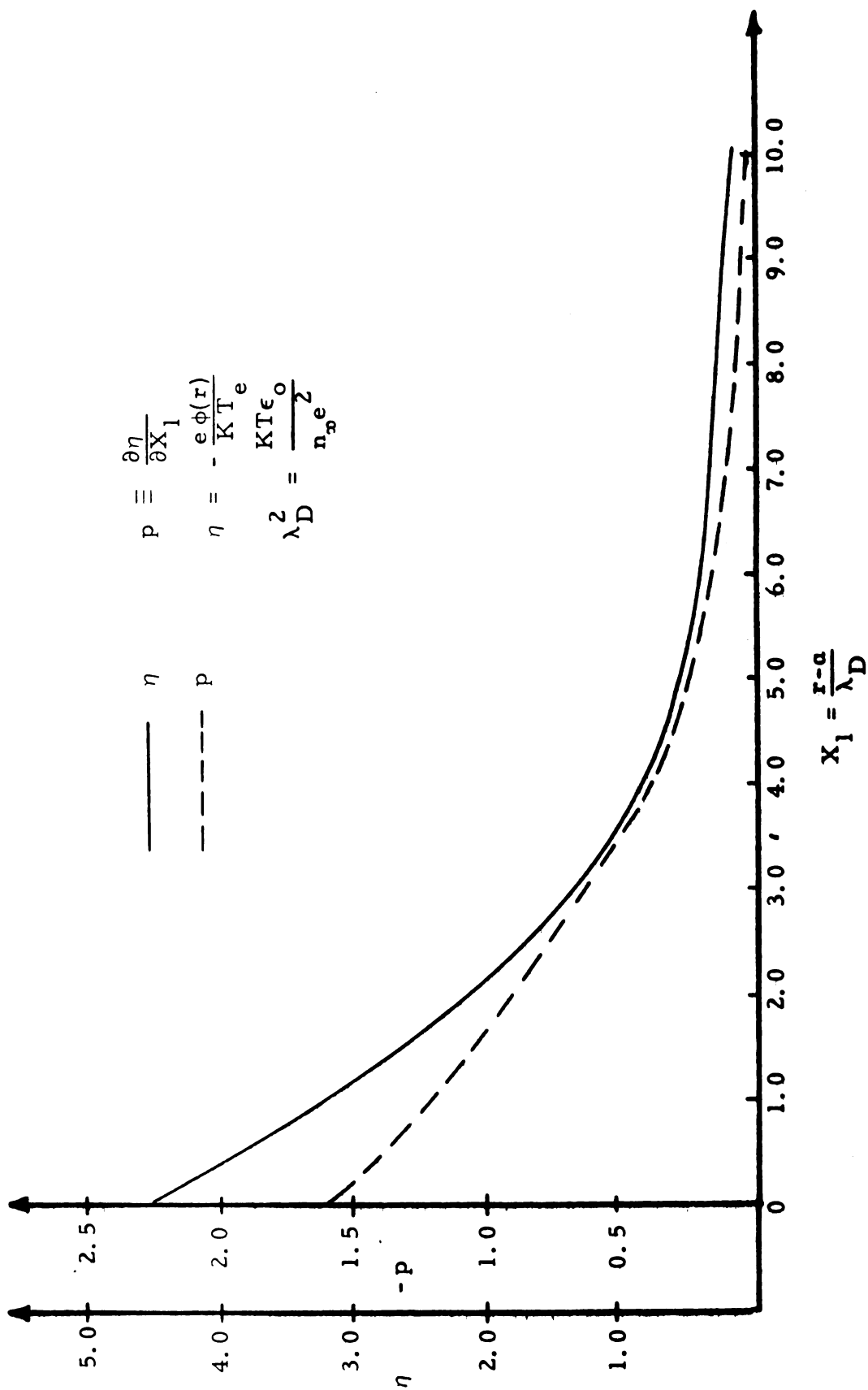


Figure 3.4. Sheath Potential and Electric Field.

CHAPTER IV

DEVELOPMENT OF THE WAVE EQUATION

4.1. Wave Equation Form

In this section a wave equation for the density perturbation n_1 will be developed from the collisionless first-order moment equations developed in chapter II; this equation will be of general validity inside as well as outside of the sheath. No consideration is given to any possible static magnetic field in the interest of maintaining symmetry in order to simplify the problem. When the moment equations are combined to form a Klein-Gordon type equation for n_1 it is found that the equation contains terms in electron drift velocity perturbation u_1 as well as a term in the electric field perturbation E_1 . It can be shown that the E_1 term contributes negligibly to the solution for n_1 once the drift velocity terms are eliminated. Elimination of the E_1 term leaves one equation in n_1 and u_1 which cannot be solved since u_1 cannot be independently related to n_1 . It is possible to rewrite this equation in conjunction with the moment equations to form a system of two coupled second order linear partial differential equations in n_1 and u_1 . In the symmetrical geometry considered in this work this system can be simplified to a system of two coupled second order linear ordinary differential equations in n_1 and u_1 each of the basic form

$$\frac{d^2 y}{dx^2} + a(x) \frac{dy}{dx} + b(x) y = F(x, n_1, u_1) \quad (4.1)$$

where y represents n_1 or u_1 and x represents a space coordinate. The coefficients, though rather complicated in form, are readily determined so that the system could be, in principle, solved if a complete set of boundary conditions could be specified. The exact form of this system will be given subsequently, and the required boundary conditions are given in chapter V.

The complicated nature of the coefficients of the system makes analytical solution impossible, and recourse to numerical methods must be made; even this approach is not a simple one. Most workers who have considered the effect of the plasma sheath on the wave equation have completely sidestepped this mathematical difficulty by neglecting all drift velocity terms. This obviously greatly simplifies the problem reducing it to the solution of a single linear second order ordinary differential equation in n_1 for which, depending on the extent of the simplification of the remaining terms, there may even be an analytical solution. The arbitrary exclusion of the drift velocity is not generally justified, and an attempt will be made in the following work to determine, at least approximately, the effect of the drift terms.

In chapter V it will be shown that the appropriate boundary condition for u_1 at the metallic surface is $u_1 = 0$. This will be used to determine the boundary conditions required for the solution of the approximate wave equation for n_1 (i. e., neglecting drift terms); these boundary conditions can be reapplied, along with the approximate solution for n_1 , to the solution of the equation in u_1 .

The solution to the latter equation can be substituted into the complete equation for n_1 so that it can be solved for a more accurate value of n_1 including drift velocity effects. This iterative approach could be repeated for greater accuracy if desired. This iterative approach is an alternative to the solution by the simultaneous system approach to the inclusion of the drift effect in the analysis; it is justified heuristically on the assumption that the approximate wave equation for n_1 (i.e., neglecting drift terms) yields a good approximation to the true value so that the calculated value of u_1 is a good approximation to its true value, and the value of n_1 calculated in the final calculation differs from the initial solution by only a small perturbation. The technique is justified, then, on the basis of being a perturbation calculation.

In many cases the plasma sheath thickness is small compared with antenna geometry; this case corresponds to $\lambda_D \ll d$ in the spherically symmetric problem considered in this work. Great simplification of the equations for n_1 and u_1 is possible here because the sheath geometry is essentially planar, and the equations may be formulated on this basis eliminating many complex terms in the coefficients of the differential equations. The form of the major equations will be indicated for this "thin sheath" case.

4.2. Solution from the Moment Equations

The first order moment equations for the perturbation terms neglecting collisions, any static magnetic field, and the perturbation Lorentz force as derived in chapter II are equations (2.31)

$$\nabla \cdot (n_0 \bar{u}_1 + n_1 \bar{u}_0) = - \frac{\partial n_1}{\partial t}$$

$$n_0 \frac{\partial \bar{u}_1}{\partial t} + [n_e (\bar{u}_e \cdot \nabla) \bar{u}_e]_1 = - \frac{e}{m_e} (n_0 \bar{E}_1 + n_1 \bar{E}_0) - v_0^2 \nabla n_1$$

where

$$[n_e (\bar{u}_e \cdot \nabla) \bar{u}_e]_1 \equiv n_0 (\bar{u}_1 \cdot \nabla) \bar{u}_0 + n_0 (\bar{u}_0 \cdot \nabla) \bar{u}_1 + n_1 (\bar{u}_0 \cdot \nabla) \bar{u}_0 .$$

Since complete spherical symmetry has been assumed in the fields and density distributions around each sphere these vector partial differential equations can be reduced, ultimately, to a set of scalar ordinary differential equations with a single independent variable r . For the present the vector form will be maintained in order to develop, within the limits imposed by the assumptions under which the moment equations were developed, a generally applicable wave equation. Time domain description will also be maintained at first for generality although ultimately the description will be transformed to the frequency domain for solution.

Differentiating the first equation with respect to time and taking the divergence of each term in the second, one obtains

$$- \frac{1}{v_0^2} \frac{\partial^2 n_1}{\partial t^2} = \frac{1}{v_0^2} \frac{\partial}{\partial t} \nabla \cdot [n_0 \bar{u}_1 + n_1 \bar{u}_0] = \frac{1}{v_0^2} \nabla \cdot \frac{\partial}{\partial t} [n_0 \bar{u}_1 + n_1 \bar{u}_0] \quad (4.2)$$

and

$$\nabla^2 n_1 = - \frac{e}{m_e v_0^2} \nabla \cdot [n_0 \bar{E}_1 + n_1 \bar{E}_0] - \frac{1}{v_0^2} \nabla \cdot \left\{ n_0 \frac{\partial \bar{u}_1}{\partial t} + [n_e (\bar{u}_e \cdot \nabla) \bar{u}_e]_1 \right\} \quad (4.3)$$

Adding equations (4.2) and (4.3)

$$\begin{aligned} \nabla^2 n_1 - \frac{1}{v_o^2} \frac{\partial^2 n_1}{\partial t^2} = & - \frac{e}{m_e v_o^2} [n_o \nabla \cdot \bar{E}_1 + n_1 \nabla \cdot \bar{E}_o + \nabla n_o \cdot \bar{E}_1 + \nabla n_1 \cdot \bar{E}_o] + \\ & \frac{1}{v_o^2} \nabla \cdot \left\{ \frac{\partial}{\partial t} [n_o \bar{u}_1 + n_1 \bar{u}_o] - n_o \frac{\partial \bar{u}_1}{\partial t} - [n_e (\bar{u}_e \cdot \nabla) \bar{u}_e]_1 \right\}. \end{aligned} \quad (4.4)$$

The last collection of terms on the RHS of equation (4.4) accounts for the effect of the drift velocity on the solution for n_1 . This term can be expanded into the form

$$\begin{aligned} R_2(u) \equiv & \frac{1}{v_o^2} \nabla \cdot \left\{ \frac{\partial}{\partial t} [n_o \bar{u}_1 + n_1 \bar{u}_o] - n_o \frac{\partial \bar{u}_1}{\partial t} - [n_e (\bar{u}_e \cdot \nabla) \bar{u}_e]_1 \right\} \\ = & \frac{1}{v_o^2} \nabla \cdot \left[\bar{u}_o \frac{\partial n_1}{\partial t} - n_o (\bar{u}_1 \cdot \nabla) \bar{u}_o - n_o (\bar{u}_o \cdot \nabla) \bar{u}_1 - n_1 (\bar{u}_o \cdot \nabla) \bar{u}_o \right]. \end{aligned} \quad (4.5)$$

The first collection of terms on the RHS of equation (4.4) can be simplified by taking note of a few relations developed in chapters II and III. From chapter II, equations (2.34) and (2.33) are useful.

$$\nabla \cdot \bar{E}_1 = - \frac{e}{\epsilon_o} n_1$$

$$\nabla \cdot \bar{E}_o = \frac{e}{\epsilon_o} (n_i - n_o)$$

From chapter III, equations (3.37), (3.7), and (3.41), respectively, are:

$$n_o = n_\infty e^{-\eta}$$

$$\eta = - \frac{e \phi_o(r)}{K T_e}$$

$$n_i = \frac{n_\infty}{\sqrt{1 + 2\eta}}.$$

These equations allow reduction of ∇n_o , $\nabla \cdot \bar{E}_o$, and $\bar{E}_o$ to more useful forms.

$$\nabla n_o = -n_o \nabla \eta \quad (4.6)$$

$$\nabla \cdot \bar{E}_o = \frac{en_o}{\epsilon_o} \left[\frac{1}{\sqrt{1+2\eta}} - \epsilon^{-\eta} \right] \quad (4.7)$$

$$\bar{E}_o = \frac{KT}{e} \nabla \eta \quad (4.8)$$

The function $\eta = \eta(r)$ was completely specified in chapter III so that all of these terms are likewise completely specified. With the substitution of these terms into the first collection of terms, it becomes

$$\begin{aligned} R_1(E) &\equiv -\frac{e}{m_e v_o^2} [n_o \nabla \cdot \bar{E}_1 + n_1 \nabla \cdot \bar{E}_o + \nabla n_o \cdot \bar{E}_1 + \nabla n_1 \cdot \bar{E}_o] \\ &= -\frac{e}{m_e v_o^2} \left[-n_o \epsilon^{-\eta} \frac{e}{\epsilon_o} n_1 + n_o \frac{e}{\epsilon_o} \left(\frac{1}{\sqrt{1+2\eta}} - \epsilon^{-\eta} \right) n_1 - n_o \nabla \eta \cdot \bar{E}_1 + \right. \\ &\quad \left. \frac{KT}{e} \nabla \eta \cdot \nabla n_1 \right] \\ &= \frac{n_o e^2}{m_e \epsilon_o v_o^2} \left(2 \epsilon^{-\eta} - \frac{1}{\sqrt{1+2\eta}} \right) n_1 + \frac{en_o}{m_e v_o^2} \nabla \eta \cdot \bar{E}_1 - \frac{KT}{m_e v_o^2} \nabla \eta \cdot \nabla n_1. \end{aligned}$$

Noting that $n_o e^2 / m_e \epsilon_o \equiv \omega_p^2$ and $3 KT_e = m_e v_o^2$, $R_1(E)$ becomes

$$R_1(E) = \frac{\omega_p^2}{v_o^2} \left(2 \epsilon^{-\eta} - \frac{1}{\sqrt{1+2\eta}} \right) n_1 + \frac{en_o}{m_e v_o^2} \nabla \eta \cdot \bar{E}_1 - \frac{1}{3} \nabla \eta \cdot \nabla n_1. \quad (4.9)$$

Using this expression to rewrite equation (4.4) the wave equation takes the form

$$\nabla^2 n_1 - \frac{1}{v_o^2} \frac{\partial^2 n_1}{\partial t^2} - \frac{\omega_p^2}{v_o^2} f_r n_1 = \frac{en_o \nabla \eta \cdot \bar{E}_1}{m_e v_o^2} - \frac{1}{3} \nabla \eta \cdot \nabla n_1 + R_2(u)$$

These equations allow reduction of ∇n_o , $\nabla \cdot \bar{E}_o$, and $\bar{E}_o$ to more useful forms.

$$\nabla n_o = -n_o \nabla \eta \quad (4.6)$$

$$\nabla \cdot \bar{E}_o = \frac{en_o}{\epsilon_o} \left[\frac{1}{\sqrt{1+2\eta}} - \epsilon^{-\eta} \right] \quad (4.7)$$

$$\bar{E}_o = \frac{KT}{e} \nabla \eta \quad (4.8)$$

The function $\eta = \eta(r)$ was completely specified in chapter III so that all of these terms are likewise completely specified. With the substitution of these terms into the first collection of terms, it becomes

$$\begin{aligned} R_1(E) &\equiv -\frac{e}{m_e v_o^2} [n_o \nabla \cdot \bar{E}_1 + n_1 \nabla \cdot \bar{E}_o + \nabla n_o \cdot \bar{E}_1 + \nabla n_1 \cdot \bar{E}_o] \\ &= -\frac{e}{m_e v_o^2} \left[-n_o \epsilon^{-\eta} \frac{e}{\epsilon_o} n_1 + n_o \frac{e}{\epsilon_o} \left(\frac{1}{\sqrt{1+2\eta}} - \epsilon^{-\eta} \right) n_1 - n_o \nabla \eta \cdot \bar{E}_1 + \right. \\ &\quad \left. \frac{KT}{e} \nabla \eta \cdot \nabla n_1 \right] \\ &= \frac{n_o e^2}{m_e \epsilon_o v_o^2} \left(2 \epsilon^{-\eta} - \frac{1}{\sqrt{1+2\eta}} \right) n_1 + \frac{en_o}{m_e v_o^2} \nabla \eta \cdot \bar{E}_1 - \frac{KT}{m_e v_o^2} \nabla \eta \cdot \nabla n_1 . \end{aligned}$$

Noting that $n_o e^2 / m_e \epsilon_o \equiv \omega_p^2$ and $3 KT_e = m_e v_o^2$, $R_1(E)$ becomes

$$R_1(E) = \frac{\omega_p^2}{v_o^2} \left(2 \epsilon^{-\eta} - \frac{1}{\sqrt{1+2\eta}} \right) n_1 + \frac{en_o}{m_e v_o^2} \nabla \eta \cdot \bar{E}_1 - \frac{1}{3} \nabla \eta \cdot \nabla n_1 . \quad (4.9)$$

Using this expression to rewrite equation (4.4) the wave equation takes the form

$$\nabla^2 n_1 - \frac{1}{v_o^2} \frac{\partial^2 n_1}{\partial t^2} - \frac{\omega_p^2}{v_o^2} f_r n_1 = \frac{en_o \nabla \eta \cdot \bar{E}_1}{m_e v_o^2} - \frac{1}{3} \nabla \eta \cdot \nabla n_1 + R_2(u)$$

where

$$f_r \equiv \frac{n_o}{n_\infty} - \frac{\partial^2 \eta}{\partial r^2} = 2 \epsilon^{-\eta} - \frac{1}{\sqrt{1+2\eta}} \quad (4.10)$$

Equation (4.10) bears a strong resemblance to the Klein-Gordon equation previously described; it is easily shown that the two become identical in a region of uniform plasma where $\nabla \eta$ and $R_2(u)$ are identically zero and $f_r = 2 \epsilon^{-\eta} - \frac{1}{\sqrt{1+2\eta}} = 1$ identically. The basic equation from which n_1 can be determined initially is obtained from equation (4.10) by dropping $R_2(u)$ and the term involving $\bar{E}_1$; in the time domain it takes the form

$$\nabla^2 n_1 + \frac{1}{3} \nabla \eta \cdot \nabla n_1 - \frac{1}{2} \frac{\partial^2 n_1}{\partial t^2} - \frac{\omega_p^2}{v_o^2} f_r n_1 = 0 \quad (4.11)$$

where

$$f_r = 2 \epsilon^{-\eta} - \frac{1}{\sqrt{1+2\eta}} \quad .$$

Equation (4.11) can be written more compactly in the frequency domain assuming that n_1 has solutions of form $n_1(r) \epsilon^{j\omega t}$. If the spherical symmetry of the problem considered in this work is used to simplify equation (4.11), it becomes

$$\nabla_r^2 n_1 + \frac{1}{3} \frac{\partial \eta}{\partial r} \frac{\partial n_1}{\partial r} + \beta^2(r) n_1 = 0 \quad (4.12)$$

where

$$\beta^2(r) \equiv \frac{\omega^2 - \omega_p^2 f_r}{v_o^2} \quad \text{and} \quad \nabla_r^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \quad .$$

It might be noted that most people^{7,18} attempting solution for n_1 in the sheath start with

$$\nabla_r^2 n_1 + \beta^2(r) n_1 = 0$$

solving this equation by either the WKB method (a rather inaccurate method for most sheath problems) or some series technique (slow convergence usually limits the usefulness of this method for analytical purposes, although for numerical purposes it is usually more economical in terms of computer time than is the standard integration procedure involving, say, a Runge-Kutta routine).

4.3. Elimination of the $\bar{E}_1$ Term

Elimination of the $\bar{E}_1$ term has not been justified as yet. It is quite possible to include this term in the analysis using Poisson's equation to relate n_1 to $\bar{E}_1$. Upon dropping the drift term $R_2(u)$ equation (4.10) becomes a linear third order D. E. in $\bar{E}_1$. In the frequency domain with spherical symmetry in space this D. E. becomes a linear third order ordinary D. E. which can be easily solved by numerical means for three linearly independent solutions. There would be no problem in matching these three linearly independent solutions in the sheath to the corresponding solutions in the uniform plasma region. The problem lies in matching these solutions to each other (by means of boundary conditions) at the metallic surface as the third order system requires one more boundary condition than was required for the solution of the second order system in n_1 . This additional boundary condition is simply not available at this state of the art; boundary conditions for the hydrodynamic quantities in general represent a major weakness in the theory. Hence, inclusion of the $\bar{E}_1$ term is not possible since no direct way of relating $\bar{E}_1$ to n_1 in linear algebraic form is possible; its

exclusion is necessary, and hopefully, its contribution can be shown to be small anyway. The fact is that this can be done in a rather crude fashion as follows.

The sheath thickness can generally be assumed to be small in comparison to the radius of the metallic sphere hence the geometry in the sheath is essentially planar, and reduction of equation (4.10) to a planar equation in E_1 is justified. Using

$$\nabla \cdot \bar{E}_1 = - \frac{en_1}{\epsilon_0} ,$$

which in planar symmetric form is

$$\frac{\partial E_1}{\partial r} = - \frac{en_1}{\epsilon_0} ,$$

to replace n_1 in equation (4.10) while dropping $R_2(u)$ and transforming description to the frequency domain, one obtains

$$\frac{\partial^3 E_1}{\partial r^3} + \frac{1}{3} \frac{\partial \eta}{\partial r} \frac{\partial^2 E_1}{\partial r^2} + \beta^2(r) \frac{\partial E_1}{\partial r} + \frac{\omega_p^2}{v_o^2} \epsilon^{-\eta} \frac{\partial \eta}{\partial r} E_1 = 0 . \quad (4.13)$$

It is assumed that a WKB type solution is valid, at least roughly, in this case, i. e.,

$$E_1 = E_{10}(r) \epsilon^{-j \int^r \beta(r) dr} .$$

Further, the amplitude $E_{10}(r)$ can change in the sheath only if there is appreciable charge accumulation there by Gauss's law. The magnitude of this charge accumulation appears not to be large compared with that on the sphere's surface so that $E_{10}(r)$ is not a strong function

of r . As a rough but very convenient simplifying assumption $E_{10}(r)$ is assumed constant so that

$$E_1 = E_{10} e^{-j \int^r \beta(r) dr}.$$

This expression is substituted into equation (4.13), and the magnitude of each of the resulting terms is compared to that of the last one which it is hoped contributes negligibly. The resulting expression is a very complicated one and will not be reproduced here; it is sufficient to say that at every point within the sheath the last term is the smallest and is normally much less than the largest term. The last term vanishes at the sheath edge, is very small at the sphere's surface, and has a single peak near the sphere's surface at which point it approximates one fifth of the amplitude of the largest term in the equation. The contribution of the $\overline{E}_1$ term is then rather small which is fortunate since it cannot be handled owing to a shortage of boundary conditions.

4.4. The Drift Velocity Wave Equation

In a manner similar to that in which the modified Klein-Gordon equation for n_1 was derived in terms of the unperturbed electric field, electron density, and drift terms a wave equation for u_1 can be derived in terms of the unperturbed parameters and n_1 . This equation is a second order linear partial differential equation in vector form of rather complicated form; it will be developed here only for the case of spherical symmetry.

Consider the moment equation (2.27). Also, note that in the case of spherical symmetry the divergence operation reduces to

$$\nabla \cdot \bar{A} = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 A_r .$$

The moment equation becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 [n_o u_1 + n_1 u_o] = -j\omega n_1$$

or

$$n_o \frac{\partial u_1}{\partial r} + \frac{\partial n_o}{\partial r} u_1 + \frac{\partial u_o}{\partial r} n_1 + u_o \frac{\partial n_1}{\partial r} + \frac{2}{r} [n_o u_1 + n_1 u_o] = -j\omega n_1 . \quad (4.14)$$

Differentiating each term with respect to r and collecting terms

yields the equation

$$\begin{aligned} n_o \frac{\partial^2 u_1}{\partial r^2} + \left(2 \frac{\partial n_o}{\partial r} + \frac{2n_o}{r} \right) \frac{\partial u_1}{\partial r} + \left(\frac{\partial^2 n_o}{\partial r^2} + \frac{2}{r} \frac{\partial n_o}{\partial r} - \frac{2n_o}{r^2} \right) u_1 = \\ - \left[u_o \frac{\partial^2 n_1}{\partial r^2} + \left(2 \frac{\partial u_o}{\partial r} + \frac{2u_o}{r} + j\omega \right) \frac{\partial n_1}{\partial r} + \left(\frac{\partial^2 u_o}{\partial r^2} + \frac{2}{r} \frac{\partial u_o}{\partial r} - \frac{2u_o}{r^2} \right) n_1 \right] , \end{aligned} \quad (4.15)$$

or compressing this equation by noting that in the spherically symmetric case

$$\nabla^2 A = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial A}{\partial r} = \frac{\partial^2 A}{\partial r^2} + \frac{2}{r} \frac{\partial A}{\partial r}$$

equation (4.15) becomes

$$\begin{aligned} n_o \nabla^2 u_1 + 2 \nabla n_o \cdot \nabla u_1 + \left(\nabla^2 n_o - \frac{2n_o}{r^2} \right) u_1 = - \left[u_o \frac{\partial^2 n_1}{\partial r^2} + \right. \\ \left. \left(\frac{2}{r} \frac{\partial}{\partial r} r u_o + j\omega \right) \frac{\partial n_1}{\partial r} + \left(\nabla^2 u_o - \frac{2u_o}{r^2} \right) n_1 \right] . \end{aligned} \quad (4.16)$$

It is easily shown that in the thin sheath case equation (4.15) reduces

to

$$n_o \frac{\partial^2 u_1}{\partial r^2} + 2 \frac{\partial n_o}{\partial r} \frac{\partial u_1}{\partial r} + \frac{\partial^2 n_o}{\partial r^2} u_1 = - \left[u_o \frac{\partial^2 n_1}{\partial r^2} + (2 \frac{\partial u_o}{\partial r} + j\omega) \frac{\partial n_1}{\partial r} + \frac{\partial^2 u_o}{\partial r^2} n_1 \right] . \quad (4.17)$$

4.5. The Drift Correction to the Density Wave Equation

Before equation (4.10) can be written in its complete form the terms $R_2(u)$ as defined in equation (4.5) must be evaluated. Making use of the moment equation (2.27) $R_2(u)$ becomes

$$-v_o^2 R_2(u) = \nabla \cdot [\bar{u}_o \nabla \cdot (n_o \bar{u}_1 + n_1 \bar{u}_o) + n_o (\bar{u}_o \cdot \nabla) \bar{u}_1 + n_o (\bar{u}_1 \cdot \nabla) \bar{u}_o + n_1 (\bar{u}_o \cdot \nabla) \bar{u}_o] . \quad (4.18)$$

Recalling the vector identity

$$(\bar{A} \cdot \nabla) \bar{B} = \sum_{k=1}^3 A_k \frac{\partial \bar{B}}{\partial x_k} ,$$

which for spherical symmetry becomes

$$(\bar{A} \cdot \nabla) \bar{B} = A_r \frac{\partial \bar{B}_r}{\partial r} ,$$

equation (4.18) becomes

$$-v_o^2 R_2(u) = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \left[u_o \frac{1}{r^2} \frac{\partial}{\partial r} r^2 (n_o u_1 + n_1 u_o) + n_o u_o \frac{\partial u_1}{\partial r} + n_o \frac{\partial u_o}{\partial r} u_1 + u_o \frac{\partial u_o}{\partial r} n_1 \right] \quad (4.19)$$

or

$$\begin{aligned}
 -v_o^2 R_2(u) = & \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \left[u_o \left(n_o \frac{\partial u_1}{\partial r} + \frac{\partial n_o}{\partial r} u_1 + u_o \frac{\partial n_1}{\partial r} + \frac{\partial u_o}{\partial r} n_1 \right) + \right. \\
 & \left. \frac{2u_o}{r} (n_o u_1 + n_1 u_o) + n_o u_o \frac{\partial u_1}{\partial r} + n_o \frac{\partial u_o}{\partial r} u_1 + u_o \frac{\partial u_o}{\partial r} n_1 \right] .
 \end{aligned} \tag{4.20}$$

Collecting similar terms this becomes

$$\begin{aligned}
 -v_o^2 R_2(u) = & \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \left[2 n_o u_o \frac{\partial u_1}{\partial r} + \left(u_o \frac{\partial n_o}{\partial r} + n_o \frac{\partial u_o}{\partial r} + \frac{2}{r} n_o u_o \right) u_1 + \right. \\
 & \left. u_o^2 \frac{\partial n_1}{\partial r} + \left(2 u_o \frac{\partial u_o}{\partial r} + \frac{2 u_o^2}{r} \right) n_1 \right] .
 \end{aligned} \tag{4.21}$$

Some additional relations taken from chapter III are useful, i. e., equations (3.38), (3.39), and (3.40).

$$\begin{aligned}
 u_o &= - \sqrt{\frac{KT_e}{2\pi m_e}} \frac{d^2}{r^2} \epsilon^{\eta - \eta_w} \\
 n_o u_o &= \frac{d^2}{r^2} n_\infty u_\infty
 \end{aligned}$$

These relations lead to the following equations:

$$\begin{aligned}
 \frac{\partial^2 n_o}{\partial r^2} &= n_o \left[\left(\frac{\partial \eta}{\partial r} \right)^2 - \frac{\partial^2 \eta}{\partial r^2} \right] \\
 \frac{\partial u_o}{\partial r} &= u_o \left[\frac{\partial \eta}{\partial r} - \frac{2}{r} \right] \\
 \frac{\partial^2 u_o}{\partial r^2} &= u_o \left[\frac{\partial^2 \eta}{\partial r^2} + \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{6}{r^2} - \frac{4}{r} \frac{\partial \eta}{\partial r} \right]
 \end{aligned} \tag{4.22}$$

Substitution of these relations into equation (4.21) reduces it to

$$-v_o^2 R_2(u) = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \left[2 n_o u_o \frac{\partial u_1}{\partial r} + u_o^2 \frac{\partial n_1}{\partial r} + 2 u_o^2 \left(\frac{\partial \eta}{\partial r} - \frac{1}{r} \right) n_1 \right] \quad (4.23)$$

because the coefficient of u_1 in equation (4.21) conveniently vanishes.

Carrying out the indicated differentiation and simplifying the results

equation (4.23) becomes

$$\begin{aligned} -v_o^2 R_2(u) = & 2 n_o u_o \frac{\partial^2 u_1}{\partial r^2} + u_o^2 \frac{\partial^2 n_1}{\partial r^2} + 4 u_o^2 \left(\frac{\partial \eta}{\partial r} - \frac{1}{r} \right) \frac{\partial n_1}{\partial r} + 2 u_o^2 \left[\frac{\partial^2 \eta}{\partial r^2} + \right. \\ & \left. 2 \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{3}{r^2} - \frac{4}{r} \frac{\partial \eta}{\partial r} \right] n_1 \end{aligned} \quad (4.24)$$

or

$$\begin{aligned} -R_2(u) = & \frac{2 n_o u_o}{v_o^2} \frac{\partial^2 u_1}{\partial r^2} + \frac{u_o^2}{v_o^2} \nabla^2 n_1 + \frac{2 u_o^2}{v_o^2} \left[2 \frac{\partial \eta}{\partial r} - \frac{3}{r} \right] \frac{\partial n_1}{\partial r} + \\ & \frac{2 u_o^2}{v_o^2} \left[\frac{\partial^2 \eta}{\partial r^2} + 2 \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{3}{r^2} - \frac{4}{r} \frac{\partial \eta}{\partial r} \right] n_1 . \end{aligned} \quad (4.25)$$

The density wave equation (4.10) now becomes

$$\begin{aligned} \left[1 + \frac{u_o^2}{v_o^2} \right] \nabla^2 n_1 - \frac{1}{v_o^2} \frac{\partial^2 n_1}{\partial t^2} + \left[\frac{1}{3} \frac{\partial \eta}{\partial r} + \frac{2 u_o^2}{v_o^2} \left(2 \frac{\partial \eta}{\partial r} - \frac{3}{r} \right) \right] \frac{\partial n_1}{\partial r} \\ - \frac{\omega_p^2}{v_o^2} f_r n_1 + \frac{2 u_o^2}{v_o^2} \left[\frac{\partial^2 \eta}{\partial r^2} + 2 \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{3}{r^2} - \frac{4}{r} \frac{\partial \eta}{\partial r} \right] n_1 = \\ - \frac{2 n_o u_o}{v_o^2} \frac{\partial^2 u_1}{\partial r^2} \end{aligned} \quad (4.26)$$

where

$$f_r = \frac{n_o}{n_\infty} - \lambda_D^2 \frac{\partial^2 \eta}{\partial r^2} = \frac{2 n_o}{n_\infty} - \frac{n_i}{n_\infty} .$$

Equation (4.26) is complete and general within the framework of the development of the moment equations (2.31). With the sheath model chosen in chapter III

$$f_r = 2 \epsilon^{-\eta} - \frac{1}{\sqrt{1+2\eta}} ,$$

but equation (4.26) is independent of the sheath model chosen. If

u_o is replaced by its equivalent function of η equation (4.26) becomes

$$\begin{aligned} & \left[1 + \frac{1}{6\pi} \frac{d^4}{r^4} \epsilon^{2(\eta-\eta_w)} \right] \nabla^2 n_1 - \frac{1}{v_o^2} \frac{\partial^2 n_1}{\partial t^2} + \frac{1}{3\lambda_D} \left[p + \frac{d^4}{r^4} \frac{\epsilon^{2(\eta-\eta_w)}}{\pi} (2p - \frac{3\lambda_D}{r}) \right] \frac{\partial n_1}{\partial r} \\ & - \frac{\omega_p^2}{v_o^2} \left[\left(2 \epsilon^{-\eta} - \frac{1}{\sqrt{1+2\eta}} \right) - \frac{d^4}{r^4} \frac{\epsilon^{2(\eta-\eta_w)}}{\pi} \left(\frac{1}{\sqrt{1+2\eta}} - \epsilon^{-\eta} + 2p^2 - \frac{4\lambda_D}{r} p \right. \right. \\ & \left. \left. + \frac{3\lambda_D^2}{r^2} \right) \right] n_1 = - \frac{2 n_o u_o}{v_o^2} \frac{\partial^2 u_1}{\partial r^2} = \frac{+ 2 n_\infty \sqrt{\frac{KT_e}{2\pi m_e}} \epsilon^{\eta-\eta_w}}{v_o^2} \\ & \cdot \frac{d^2}{r^2} \frac{\partial^2 u_1}{\partial r^2} \end{aligned} \quad (4.27)$$

where $p = \lambda_D \frac{\partial \eta}{\partial r}$. For the thin sheath case equation (4.27) reduces to

$$\begin{aligned} & \left[1 + \frac{1}{6\pi} \epsilon^{2(\eta-\eta_w)} \right] \nabla^2 n_1 - \frac{1}{v_o^2} \frac{\partial^2 n_1}{\partial t^2} + \frac{p}{3\lambda_D} \left[1 + \frac{2 \epsilon^{2(\eta-\eta_w)}}{\pi} \right] \frac{\partial n_1}{\partial r} \\ & - \frac{\omega_p^2}{v_o^2} \left[\left(2 \epsilon^{-\eta} - \frac{1}{\sqrt{1+2\eta}} \right) - \frac{\epsilon^{2(\eta-\eta_w)}}{\pi} \left(\frac{1}{\sqrt{1+2\eta}} - \epsilon^{-\eta} + 2 p^2 \right) \right] n_1 = \\ & - \frac{2 n_\infty u_\infty}{v_o^2} \frac{\partial^2 u_1}{\partial r^2} . \end{aligned} \quad (4.28)$$

4.6. Final Forms

The drift wave equation (4.15) can be expressed in a more useful form by substituting functions of η for the coefficients.

Performing this substitution, equation (4.15) becomes

$$\begin{aligned} \frac{\partial^2 u_1}{\partial r^2} + 2 \left[\frac{1}{r} - \frac{\partial \eta}{\partial r} \right] \frac{\partial u_1}{\partial r} + \left[\left(\frac{\partial \eta}{\partial r} \right)^2 - \frac{\partial^2 \eta}{\partial r^2} - \frac{2}{r} \frac{\partial \eta}{\partial r} - \frac{2}{r^2} \right] u_1 = \\ - \frac{u_0}{n_0} \left\{ \frac{\partial^2 n_1}{\partial r^2} + \left[2 \frac{\partial \eta}{\partial r} - \frac{2}{r} + j \frac{\omega}{u_0} \right] \frac{\partial n_1}{\partial r} + \left[\frac{\partial^2 \eta}{\partial r^2} + 2 \left(\frac{\partial \eta}{\partial r} \right)^2 - \frac{2}{r} \frac{\partial \eta}{\partial r} \right] n_1 \right\} \quad (4.29) \end{aligned}$$

or

$$\begin{aligned} \frac{\partial^2 u_1}{\partial r^2} + \frac{2}{\lambda_D} \left[\frac{\lambda_D}{r} - p \right] \frac{\partial u_1}{\partial r} + \frac{1}{\lambda_D^2} \left[p^2 - \frac{1}{\sqrt{1+2\eta}} + \epsilon^{-\eta} - 2 \frac{\lambda_D}{r} p - \right. \\ \left. 2 \left(\frac{\lambda_D}{r} \right)^2 \right] u_1 = \left[\frac{1}{n_\infty} \sqrt{\frac{KT_e}{2\pi m_e}} \frac{d^2}{r^2} \epsilon^{\eta-\eta_w} \right] \left[\frac{\partial^2 n_1}{\partial r^2} + \frac{1}{\lambda_D} (2p - \right. \\ \left. 2 \frac{\lambda_D}{r} - j \sqrt{2\pi} \Omega \frac{r^2}{d^2} \epsilon^{\eta_w-\eta}) \frac{\partial n_1}{\partial r} + \frac{1}{\lambda_D^2} \left(\frac{1}{\sqrt{1+2\eta}} - \epsilon^{-\eta} + p^2 \right. \right. \\ \left. \left. - 2 \frac{\lambda_D}{r} p \right) n_1 \right] \quad (4.30) \end{aligned}$$

where Ω is the normalized frequency variable ω/ω_p . In the case of the thin sheath equation (4.30) reduces to

$$\begin{aligned} \frac{\partial^2 u_1}{\partial r^2} - \frac{2p}{\lambda_D} \frac{\partial u_1}{\partial r} + \frac{1}{\lambda_D^2} \left[p^2 - \frac{1}{\sqrt{1+2\eta}} + \epsilon^{-\eta} \right] u_1 = \\ - \frac{u_0}{n_0} \left[\frac{\partial^2 n_1}{\partial r^2} + \frac{1}{\lambda_D} (2p - j \sqrt{2\pi} \Omega \epsilon^{\eta_w-\eta}) \frac{\partial n_1}{\partial r} + \frac{1}{\lambda_D^2} \left(\frac{1}{\sqrt{1+2\eta}} - \epsilon^{-\eta} + p^2 \right) n_1 \right]. \quad (4.31) \end{aligned}$$

The final form of the density wave equation in the frequency domain is obtained from equation (4.27) as

$$\left[1 + \frac{1}{6\pi} \frac{d^4}{r^4} \epsilon^{2(\eta-\eta_w)} \right] \nabla^2 n_1 + \frac{1}{3\lambda_D} \left[p + \frac{d^4}{r^4} \frac{\epsilon^{2(\eta-\eta_w)}}{\pi} \left(2p - \frac{3\lambda_D}{r} \right) \right] \frac{\partial n_1}{\partial r} + \beta_1^2(r) n_1 = - \frac{2 n_\infty \sqrt{\frac{KT}{2\pi m}} e}{v_o^2} \frac{d^2}{r^2} \epsilon^{\eta-\eta_w} \frac{\partial^2 u_1}{\partial r^2} \quad (4.32)$$

where

$$\beta_1^2(r) = \frac{\omega^2}{v_o^2} - \frac{p^2}{v_o^2} \left[\left(2\epsilon^{-\eta} - \frac{1}{\sqrt{1+2\eta}} \right) - \frac{d^4}{r^4} \frac{\epsilon^{2(\eta-\eta_w)}}{\pi} \cdot \left(\frac{1}{\sqrt{1+2\eta}} - \epsilon^{-\eta} + 2p^2 + \frac{3\lambda_D^2}{r^2} - \frac{4\lambda_D^2}{r} p \right) \right].$$

Inspection of equations (4.30) and (4.32) make it evident that the wave equations in n_1 and u_1 are coupled as indicated previously. They form a system of two linear ordinary differential equations requiring simultaneous solution; the system may be written in the form

$$\begin{aligned} L_1 n_1 + L_2 u_1 &= 0 \\ L_3 n_1 + L_4 u_1 &= 0 \end{aligned} \quad (4.33)$$

where the L 's are second order linear operators of the general form

$$L = a(r) \frac{\partial^2}{\partial r^2} + b(r) \frac{\partial}{\partial r} + c(r) = a(r) \frac{d^2}{dr^2} + b(r) \frac{d}{dr} + c(r)$$

and the values corresponding to the set $[a_i(r), b_i(r), c_i(r)]_{i=1,2,3,4}$ are obvious from equations (4.30) and (4.32).

Some insight into the extent of the modification of the solution for n_1 to be expected upon inclusion of the drift velocity effects can be gained by consideration of the amplitude of the drift terms added to equation (4.12) to form equation (4.32). For example, in the first term of equation (4.32) $\frac{1}{6\pi} \epsilon^{2(\eta-\eta_w)}$ is extremely small compared to unity in almost all of the sheath; it increases rapidly near the metallic surface to $\frac{1}{6\pi}$ or approximately 0.05. This term should have very little effect on the solution for n_1 . In the second term $\frac{2}{\pi} \epsilon^{2(\eta-\eta_w)}$ increases rapidly to $2/\pi$ as the metallic surface is approached; although appreciable at this surface its overall effect should be small. The same general argument can be given for the third term. Unfortunately, there is no accurate means of comparing the RHS to the other terms in the equation; only solution for u_1 can yield this information.

CHAPTER V

BOUNDARY CONDITIONS

5.1. Introduction

Before any of the equations for n_1 and u_1 developed in chapter IV can be solved sufficient boundary conditions must be described. This can be done only approximately and will be based on the assumption that no electrons take part in the r.f. (perturbation) motion at the surface of the metallic sphere, i.e.,

$$\bar{u}_1 \cdot \hat{n} \Big|_{r=a} = 0,$$

and upon the continuity of the potential perturbation at $r = a$.

Boundary conditions must also be specified at the sheath-plasma boundary where, as will be detailed in chapter VI, solutions for both the electromagnetic fields and the hydrodynamic variables will be matched; this sheath-plasma boundary is assumed to be just inside of the region of uniform plasma so that all electromagnetic and hydrodynamic variables are infinitely differentiable, and the matching process is trivial. The matching of the far zone electromagnetic and hydrodynamic variables to their corresponding forms in the "quasi-static" zone (Region II) is simple and straightforward; it is covered in chapter VI and need not be further mentioned here. The basic boundary conditions will be used to derive the boundary conditions required for the solution of the specific equations or systems of equations solved in chapters VI and VII.

5.2. Sheath-Plasma Boundary Conditions

The sheath-plasma boundary is a surface chosen, at $r = d$, sufficiently far away from the metallic sphere's surface that the electron density has returned essentially to its uniform value n_∞ , the unperturbed potential is essentially zero, and the unperturbed electron drift velocity is negligible, while still close enough to the sphere to be in the quasi-static region surrounding it. In chapter VI solutions for the electromagnetic and hydrodynamic variables valid in the sheath are matched to those valid in the uniform "quasi-static" region at the sheath-plasma boundary. As stated before at the sheath-plasma boundary all electromagnetic and hydrodynamic variables are infinitely differentiable so that matching is easily accomplished. The continuity relations for the variables of interest are given for completeness.

$$\lim_{r \rightarrow d^-} n_1 = \lim_{r \rightarrow d^+} n_1 \quad (5.1)$$

$$\lim_{r \rightarrow d^-} \frac{\partial n_1}{\partial r} = \lim_{r \rightarrow d^+} \frac{\partial n_1}{\partial r} \quad (5.2)$$

$$\lim_{r \rightarrow d^-} \phi_1 = \lim_{r \rightarrow d^+} \phi_1 \quad (5.3)$$

$$\lim_{r \rightarrow d^-} \frac{\partial \phi_1}{\partial r} = \lim_{r \rightarrow d^+} \frac{\partial \phi_1}{\partial r} \quad (5.4)$$

$$\lim_{r \rightarrow d^-} u_1 = \lim_{r \rightarrow d^+} u_1 \quad (5.5)$$

$$\lim_{r \rightarrow d^-} \frac{\partial u_1}{\partial r} = \lim_{r \rightarrow d^+} \frac{\partial u_1}{\partial r} \quad (5.6)$$

where ϕ_1 is the scalar potential associated with the perturbation E field, i.e., since E_1 is assumed quasi-static

$$\overline{E}_1 = -\nabla \phi_1 . \quad (5.7)$$

5.3. Boundary Conditions at the Surface of the Metal Sphere

At the sphere's surface the following field relations are valid

$$\hat{n} \times \overline{E}_1 = 0 \quad (5.8)$$

$$\epsilon_o E_1 = \sigma_s \quad (5.9)$$

$$\phi_1 = V \quad (5.10)$$

where V is the perturbation potential applied to the sphere from an external source, and σ_s is the perturbation surface charge density on the sphere. Equation (5.9) will be used in the determination of the current to the sphere, while equation (5.10) will be used directly as a boundary condition on ϕ_1 .

The boundary conditions for the hydrodynamic variables are not so easily specified, in fact, this difficulty represents one of the major weaknesses of the theory. If only the second order equation for n_1 is to be solved it can be shown that only one boundary condition for the hydrodynamic variables is required; in fact, it can be shown that only one relation, independent of the moment equations, in the variables n_1 , u_1 , $\frac{\partial n_1}{\partial r}$, $\frac{\partial u_1}{\partial r}$, and $\frac{\partial \phi_1}{\partial r}$ is required to specify the problem. Obviously the independent specification of any one of these variables would suffice. Various authors have used various relations here. Cohen⁸ suggests the following linear relation

$$\hat{n} \cdot \bar{u}_1 = Y_a \hat{n} \cdot \bar{E}_1 + Y_b n_1 \quad (5.11)$$

in analogy with acoustics describing the boundary in terms of a surface admittance. He argues that the acoustic analogy to this reflection problem would lead to a linear relation between $\hat{n} \cdot u_1$ and n_1 at the boundary surface, since the admittance is the ratio of the perturbation in velocity to the perturbation in pressure (the so-called "excess pressure"). "The excess pressure is the body pressure in acoustics; but in the plasma the electric field also contributes to the body force, and we should include it in the admittance." Linear inclusion of the electric field leads directly to equation (5.11). The admittance coefficients Y_a and Y_b may be functions of the configuration of the fields at the metallic surface and are, in general, functions of frequency. Cohen indicates that, "in principle, these admittance coefficients can be measured, since for each incident wave (impinging on the metal) there are two reflected waves (one EM and one EA wave). A calculation of the coefficients themselves, however, would have to start from the opposite point of view. One would have to solve in detail the plasma-metal boundary problem, including the fields and electron motions inside the metal." No one has, as yet, devised a method of evaluating these coefficients so that the general form of equation (5.11), while of theoretical interest, is not useful. There are two assumptions commonly used here which are justified on the basis of analogy with acoustics. The first is the analog of the so-called acoustic "soft" boundary condition in which the metallic surface is assumed to be perfectly absorptive so that

$$n_1 \Big|_{\text{surface}} \equiv 0 . \quad (5.12)$$

This model cannot be justified rigorously, but it has the advantages of great simplicity and of representing one extreme of surface absorptivity. Another more commonly used surface model is the analog of the so-called acoustic "hard" boundary condition in which the metallic surface is assumed to be perfectly reflective for electrons so that there is no net perturbation in drift velocity at the surface, i. e., microscopically every electron involved in perturbation motion which strikes the surface is elastically reflected. This leads to

$$\hat{n} \cdot \bar{u}_1 \Big|_{\text{surface}} = 0 . \quad (5.13)$$

A criticism of this model is that, while in the case of the acoustically hard surface the neutral molecules of the fluid cannot penetrate the surface and $\hat{n} \cdot \bar{u}_1$ is indeed zero, in the plasma case the fluid consists of electrons which can, by nature of their charge, penetrate the surface. The advantages of this model lie in its simplicity, its ease of application, and in the fact that it represents the opposite conceptual extreme of the soft model. Larson³ performing an analysis similar to the present one uses both hard and soft boundary conditions and achieves similar results in each case. This might be quite surprising considering the extreme difference in the surface models if it were not recognized that the unperturbed density drops to very small values at the metallic surface so that the perturbation density should be small also regardless of the surface model used.

In any case this result is a consoling one since the proper model is in doubt. It can be noted that the hard and soft boundary conditions correspond to the following admittance coefficients, respectively:

$$\frac{\text{Soft B.C.}}{Y_b = \infty} \qquad \frac{\text{Hard B.C.}}{Y_a = Y_b = 0}$$

Balmain²⁰ suggested the following absorptive boundary condition

$$u_1 = - \frac{a_{KB}}{u_{KB} n_o m_e} P_1 \quad \text{at } r = a \quad (5.14)$$

where u_{KB} is the velocity of sound in the electron gas, n_o is the unperturbed electron density in the uniform plasma, and a_{KB} is a dimensionless constant depending on the nature of the surface. For a "completely collapsed sheath" (totally absorptive surface) a_{KB} approximates $\sqrt{2/\pi}$, whereas $a_{KB} = 0$ for the "perfectly rigid" (perfectly reflective) boundary. While this absorptive boundary condition obviously represents an oversimplified picture of the actual behavior at the metallic surface, it would seem to be an improvement over the hard or soft boundary conditions; the problem with its use is the determination of a_{KB} .

All of the boundary conditions discussed have been used by various authors in various applications; each has its advantages. For the purposes of the present work the more conventional hard boundary condition is arbitrarily adopted, primarily for reasons of simplicity, however use of a conventional boundary condition facilitates comparison of results with those of others.

5.4. Hybrid Boundary Condition for the Density Solution

The solution of the density wave equation in chapter VI requires a linear relation in $\frac{\partial \phi_1}{\partial r}$, n_1 , and $\frac{\partial n_1}{\partial r}$. The perturbation moment equations contain these variables plus u_1 and $\frac{\partial u_1}{\partial r}$ in addition. Use of the hard boundary condition $u_1 = 0$ eliminates u_1 , and $\frac{\partial u_1}{\partial r}$ can be eliminated from the moment equations by subtraction yielding a relation of the proper type.

Consider the perturbation moment equations (2.31) reduced to the frequency domain.

$$j\omega n_o \bar{u}_1 + [n_e (\bar{u}_e \cdot \nabla) \bar{u}_e]_1 = -\frac{e}{m_e} [n_o \bar{E}_1 + n_1 \bar{E}_o] - v_o^2 \nabla n_1 \quad (5.15)$$

and

$$\nabla \cdot [n_o \bar{u}_1 + n_1 \bar{u}_o] = -j\omega n_1. \quad (5.16)$$

In the case of spherical symmetry these equations become

$$j n_o \omega u_1 + n_o u_1 \frac{\partial u_o}{\partial r} + n_o u_o \frac{\partial u_1}{\partial r} + n_1 u_o \frac{\partial u_o}{\partial r} = -\frac{e}{m_e} [n_1 E_o + n_o E_1] - v_o^2 \frac{\partial n_1}{\partial r}$$

and

$$n_1 \frac{\partial u_o}{\partial r} + u_o \frac{\partial n_1}{\partial r} + n_o \frac{\partial u_1}{\partial r} + \frac{\partial n_o}{\partial r} u_1 + \frac{2}{r} (n_1 u_o + n_o u_1) = -j\omega n_1.$$

Applying the conditions $u_1 = 0$ and $r = a$ while multiplying the second equation through by u_o leads to

$$n_o u_o \frac{\partial u_1}{\partial r} + [u_o \frac{\partial u_o}{\partial r} + \frac{e}{m_e} E_o] n_1 + v_o^2 \frac{\partial n_1}{\partial r} = \frac{e}{m_e} n_o \frac{\partial \phi_1}{\partial r} \quad (5.17)$$

and

$$n_o u_o \frac{\partial u_1}{\partial r} + \left[u_o \frac{\partial u_o}{\partial r} + \frac{2 u_o^2}{a} + j \omega u_o \right] n_1 + u_o^2 \frac{\partial n_1}{\partial r} = 0. \quad (5.18)$$

Subtracting equation (5.18) from equation (5.17) a relation of the desired form is obtained as

$$\left(\frac{e}{m} E_o - \frac{2 u_o^2}{a} - j \omega u_o \right) n_1 + (v_o^2 - u_o^2) \frac{\partial n_1}{\partial r} = \frac{e n_o}{m_e} \frac{\partial \phi_1}{\partial r}. \quad (5.19)$$

Dividing through by v_o^2 and substituting for u_o , E_o , v_o^2 , and n_o equation (5.19) becomes

$$\begin{aligned} \frac{\omega_p^2 \epsilon_o}{v_o^2 e} \epsilon^{-\eta_w} \frac{\partial \phi_1}{\partial r} &= \left[1 - \frac{1}{6\pi} \left(\frac{d}{a} \right)^4 \right] \frac{\partial n_1}{\partial r} + \frac{1}{3\lambda_D} \\ &\cdot \left[p - \frac{\delta}{\pi} \left(\frac{d}{a} \right)^4 + j \frac{\Omega}{\sqrt{2\pi}} \left(\frac{d}{a} \right)^2 \right] n_1 \end{aligned} \quad (5.20)$$

where the parameters ω_p , λ_D , η_w , v_o^2 , p , Ω , d , and a have been previously defined, and δ is defined as λ_D/a . Equation (5.20) can be written in a more compact form for use in chapter VI.

$$\left. \frac{\partial \phi_1}{\partial r} \right|_{r=a} = D_1 \left. \frac{\partial n_1}{\partial r} \right|_{r=a} + B_1 n_1 \Big|_{r=a} \quad (5.21)$$

where

$$D_1 = \frac{v_o^2 e}{\omega_p^2 \epsilon_o} \epsilon^{\eta_w} \left[1 - \frac{1}{6\pi} \left(\frac{d}{a} \right)^4 \right]$$

and

$$B_1 = \frac{v_o^2 e}{\omega_p^2 \epsilon_o} \epsilon^{\eta_w} \frac{1}{3\lambda_D} \left[p \Big|_a - \frac{\delta}{\pi} \left(\frac{d}{a} \right)^4 + j \frac{\Omega}{\sqrt{2\pi}} \left(\frac{d}{a} \right)^2 \right].$$

For the thin sheath case these coefficients become

$$D_1 = \frac{v_o^2 e}{\omega_p^2 \epsilon_o} \epsilon^{\eta_w} \left[1 - \frac{1}{6\pi} \right]$$

$$B_1 = \frac{v_o^2 e}{\omega_p^2 \epsilon_o} \epsilon^{\eta_w} \frac{1}{3\lambda_D} \left[p \Big|_a + j \frac{\Omega}{\sqrt{2\pi}} \right] . \quad (5.22)$$

5.5. Boundary Conditions for the Drift Equation

If the wave equations for n_1 and u_1 are solved by the iterative method discussed in chapter IV the solution of the drift wave equation (4.30) requires specification of the values of u_1 and $\frac{\partial u_1}{\partial r}$ at the metal boundary. The specification of $u_1 = 0$ at the boundary fulfills the first requirement. It is not difficult to apply this condition to the first moment equation thus determining $\frac{\partial u_1}{\partial r}$ at the boundary in terms of the values of n_1 and $\frac{\partial n_1}{\partial r}$.

The first perturbation moment equation in the frequency domain for spherical symmetry is

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 (n_o u_1 + n_1 u_o) = -j\omega n_1$$

which can be written in the form

$$n_o \frac{\partial u_1}{\partial r} + \left(\frac{\partial n_o}{\partial r} + \frac{2 n_o}{r} \right) u_1 + u_o \frac{\partial n_1}{\partial r} + \left(\frac{\partial u_o}{\partial r} + \frac{2 u_o}{r} + j\omega \right) n_1 = 0 . \quad (5.23)$$

Letting $u_1 = 0$ and $r = a$ equation (5.23) becomes

$$- n_o \frac{\partial u_1}{\partial r} = u_o \frac{\partial n_1}{\partial r} + \left(\frac{\partial u_o}{\partial r} + \frac{2}{a} u_o + j\omega \right) n_1 \quad (5.24)$$

or multiplying by $\frac{u_o}{v_o}$ this becomes

$$- \frac{n_o u_o}{v_o^2} \frac{\partial u_1}{\partial r} = \frac{u_o^2}{v_o^2} \frac{\partial n_1}{\partial r} + \frac{1}{v_o^2} (u_o \frac{\partial u_o}{\partial r} + \frac{2}{a} u_o^2 + j\omega u_o \omega) n_1 .$$

Remembering that

$$u_o = - \sqrt{\frac{KT_e}{2\pi m_e}} \frac{d^2}{r^2} e^{\eta - \eta_w}$$

so that

$$\frac{\partial u_o}{\partial r} = u_o \left[\frac{\partial \eta}{\partial r} - \frac{2}{r} \right]$$

then

$$- \frac{n_o u_o}{v_o^2} \frac{\partial u_1}{\partial r} = \frac{u_o^2}{v_o^2} \frac{\partial n_1}{\partial r} + \left[\frac{u_o^2}{v_o^2} \frac{\partial \eta}{\partial r} + j \frac{\omega u_o}{v_o^2} \right] n_1 . \quad (5.25)$$

Substituting for u_o equation (5.25) can be written as

$$- \frac{n_o u_o}{v_o^2} \frac{\partial u_1}{\partial r} \Big|_a = \frac{1}{6\pi} \left(\frac{d}{a} \right)^2 \frac{\partial n_1}{\partial r} \Big|_a + \frac{1}{3\lambda_D} \left[\frac{p|_a}{2\pi} \left(\frac{d}{a} \right)^2 - j \frac{\Omega}{\sqrt{2\pi}} \right] n_1 \Big|_a . \quad (5.26)$$

For the thin sheath case this degenerates to

$$- \frac{n_o u_o}{v_o^2} \frac{\partial u_1}{\partial r} \Big|_a = \frac{1}{6\pi} \frac{\partial n_1}{\partial r} \Big|_a + \frac{1}{3\lambda_D} \left[\frac{p|_a}{2\pi} - j \frac{\Omega}{\sqrt{2\pi}} \right] n_1 \Big|_a . \quad (5.27)$$

Equation (5.26) is the desired form of the boundary condition for $\frac{\partial u_1}{\partial r}$.

The possibility of solving the wave equations for n_1 and u_1 simultaneously has been discussed in chapter IV. Without going into the details of such a solution, it can be observed that specification

of a set of boundary conditions for n_1 , u_1 , $\frac{\partial n_1}{\partial r}$, and $\frac{\partial u_1}{\partial r}$ will be required at some common point r on the interval $[a, d]$. If the boundary conditions for n_1 and $\frac{\partial n_1}{\partial r}$ can be determined at $r = d$, then it can be shown that u_1 and $\partial u_1 / \partial r$ can be specified in terms of the values of n_1 and $\partial n_1 / \partial r$. This relationship can be developed as follows.

Since the plasma is assumed uniform at $r = d$ with negligible unperturbed electron drift velocity or electric field, the uniform plasma relations described in section 2.3 are applicable. If the $\bar{B}$ field is neglected the following relations can be developed.

$$-e n_o \bar{u}_1 + j \omega \epsilon_o \bar{E}_1 = 0 \quad (5.28)$$

$$\nabla \cdot \bar{E}_1 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 E_1 = \frac{\partial E_1}{\partial r} + \frac{2}{r} E_1 = -\frac{e}{\epsilon_o} n_1 \quad (5.29)$$

$$j \omega n_o m_e \bar{u}_1 = -n_o e \bar{E}_1 - m_e v_o^2 \nabla n_1 \quad (5.30)$$

Solving for E_1 from equation (5.28)

$$E_1 = \frac{e n_o}{j \omega \epsilon_o} u_1 \quad (5.31)$$

Differentiating this expression with respect to r

$$\frac{\partial \bar{E}_1}{\partial r} = \frac{e n_o}{j \omega \epsilon_o} \frac{\partial u_1}{\partial r} \quad (5.32)$$

Substituting equation (5.31) into equation (5.30) and simplifying yields

$$\begin{aligned} u_1 &= j \frac{\omega}{n_o} \frac{v_o^2}{\omega^2 - \omega_p^2} \frac{\partial n_1}{\partial r} \\ &= j \frac{\omega}{n_o} \frac{v_o^2}{\omega_p^2 (\Omega^2 - 1)} \frac{\partial n_1}{\partial r} \end{aligned} \quad (5.33)$$

Combining equations (5.29), (5.31), and (5.32)

$$\frac{\partial u_1}{\partial r} = -j \frac{\omega}{n_o} \left[n_1 + \frac{2}{a} \frac{v_o^2}{\omega_p^2 (\Omega^2 - 1)} \frac{\partial n_1}{\partial r} \right]. \quad (5.34)$$

Equations (5.33) and (5.34) specify u_1 and $\partial u_1 / \partial r$ in terms of n_1 and $\partial n_1 / \partial r$ in the uniform plasma region which includes $r = d$. These equations then specify boundary conditions for u_1 and $\partial u_1 / \partial r$ in terms of those for n_1 and $\partial n_1 / \partial r$.

CHAPTER VI

ANALYTICAL FORMULATION

6.1. Introduction

As indicated in chapter I, the plasma surrounding each sphere of the antenna can be divided into three separate concentric spherical regions. The first of these is the sheath region, designated Region I, extending from the sphere's surface, $r = a$, to an arbitrarily chosen radius $r = d$ sufficiently far out into the plasma so that the unperturbed electron and ion densities have returned very nearly to their values in the uniform body of the plasma; the unperturbed electron drift velocity and the unperturbed potential have very nearly vanished at $r = d$ so that for $r \geq d$ they can be taken as identically zero with little error. Region II extends from $r = d$ to $r = d_1$ where d_1 is chosen to be within the "quasi-static" zone surrounding the sphere; since Regions I and II are within the quasi-static zone the solution for the "electromagnetic" fields can be simplified accordingly there. Region III includes the radiation zone and the transition zone between the radiation zone and the quasi-static zone; it extends from $r = d_1$ to $+\infty$. Solutions for the electromagnetic and hydrodynamic quantities in regions II and III are easily obtained since the mode separation discussed in chapter II is valid. The solution for these quantities in Region I is greatly complicated by the effects of the nonuniformity in the plasma density; the only practical approach seems to involve numerical methods; a numerical approach to this solution is outlined in chapter VII.

The goals of this chapter include indicating the basic forms of the solutions for the pertinent electromagnetic and hydrodynamic quantities in each of the three regions and appropriately matching the corresponding solutions at the interregional boundaries in order to obtain the complete solutions. This matching process makes use of the boundary conditions developed in chapter V. The input admittance will be approximated making use of the quasi-static solutions for the electron density and electric field perturbations. The effect of electromagnetic radiation will be included by computing an effective radiation conductance term calculated, as usual, by use of Poynting's theorem; the electromagnetic energy radiated is determined by comparing the far field scalar potential to that of an electric dipole and writing the classical electric dipole expressions in terms of the appropriate electric dipole moment. The density perturbation at the sphere's surface ($r=a$) is of interest for resonance probe applications; this is derived in a form useful for comparison with other theories.

6.2. The Hydrodynamical Wave Equations

The basic equations describing the propagation of the hydrodynamic quantities n_1 and u_1 were derived in chapter IV, i. e., equations (4.32) and (4.30), respectively. In Region I, where these equations assume their most complicated form, both must be solved. In Regions II and III, they assume relatively simple forms, are decoupled, and only the density wave equation need be solved; the drift velocity term u_1 can be determined from the density term n_1 by use of equation (5.33). In chapter VII, equations (4.32) and

(4.30) for Region I will be transformed to forms suitable for solution by iteration. Only equation (4.32) in its simplified form is solved in Regions II and III; exact analytical solution is accomplished in these regions.

The form of equations (4.32) and (4.30) is expressed in operational form in equation (4.33), i. e.,

$$L_1 n_1 + L_2 u_1 = 0$$

$$L_3 n_1 + L_4 u_1 = 0$$

where the L's are second order linear ordinary differential operators of the general form

$$L = a(r) \frac{\partial^2}{\partial r^2} + b(r) \frac{\partial}{\partial r} + c(r) \quad .$$

In the iterative approach to the solution of this system, L_2 is first assumed identically zero, the coefficients of L_1 are simplified by retaining only major terms, and the resultant zeroth order decoupled homogeneous equation is solved for two linearly independent solutions. These solutions are appropriately joined to the solutions for n_1 in Region II; the boundary conditions at $r = a$, developed in chapter V, for n_1 in terms of $\partial n_1 / \partial r$ and $\partial \phi_1 / \partial r$ are applied completely determining the zeroth order solution for n_1 . Next this zeroth order solution is used in conjunction with $L_3 n_1$ to determine a decoupled inhomogeneous equation in u_1 ; this equation is solved in conjunction with boundary conditions developed previously to obtain the first order iterated solution for u_1 . This solution is then substituted into the first equation of (4.33) (i. e., as $L_2 u_1$), and the resultant inhomogeneous equation in n_1 is solved as before to obtain the first order iterated solution for n_1 . The exact form of the zeroth and first order operators are determined from equations

(4.12), (4.30), and (4.32).

In any of the three regions the basic equation to be solved, then, is a second order linear differential equation of the form

$$\nabla^2 n_1 + G_1(r) \frac{\partial n_1}{\partial r} + \beta_1^2(r) n_1 = G_2(r) \quad (6.1)$$

where the coefficients are determined from equation (4.32). In Regions II and III, where u_0 and η are taken as identically zero, it is evident that equation (6.1) reduces to the particularly simple form

$$\nabla^2 n_1 + \beta^2 n_1 = 0 \quad (6.2)$$

since $G_1(r)$ and $G_2(r)$ vanish and $\beta_1^2(r)$ goes over to

$$\beta_1^2(r) \equiv \beta^2 = \frac{\omega^2 - \omega_p^2}{v_o^2} .$$

In Region I, for the zeroth order solution for n_1 , $G_2(r)$, which corresponds to the inclusion of the drift terms, is dropped and $G_1(r)$ is reduced to

$$G_1(r) = \frac{1}{3} \frac{\partial \eta}{\partial r} = \frac{1}{3\lambda_D} p , \quad (6.3)$$

and only the dominant terms of $\beta_1^2(r)$ are retained so that

$$\beta_1^2(r) = \frac{\omega^2 - \omega_p^2 f_r}{v_o^2} = \beta^2(r) \quad (6.4)$$

where

$$f_r = 2 \epsilon^{-\eta} - \frac{1}{\sqrt{1 + 2\eta}} ,$$

and equation (6.1) becomes

$$\nabla^2 n_1 + \frac{1}{3\lambda_D} p \frac{\partial n_1}{\partial r} + \beta^2(r) n_1 = 0. \quad (6.5)$$

It is evident that equation (6.5) goes over to equation (6.2) in accord with the assumption previously made regarding plasma uniformity at $r = d$. The solutions to either should apply equally well at $r = d$; this concept will be used to match the solutions at $r = d$.

One further comment on the solution of equation (6.1) is required so that no further reference to the form of the equation need be made in this chapter. It will be useful to consider that equation (6.1) has two linearly independent solutions so that independent boundary conditions can be specified for each solution; i.e., if $G_2(r) = 0$, equation (6.1) is homogeneous and has two linearly independent solutions which can easily be shown to be of the form

$$\begin{aligned} y_1 &= 1 + a_1 x_2^2 + a_3 x_2^3 + \dots \\ y_2 &= x_2 + b_3 x_2^3 + b_4 x_2^4 + \dots \end{aligned} \quad (6.6)$$

where $x_2 \equiv r - d/\lambda_D$, so that the following relation is true at $x_2 = 0$ ($r = d$):

$$\begin{bmatrix} y_1(0) & y_2(0) \\ \frac{\partial y_1(0)}{\partial x_2} & \frac{\partial y_2(0)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (6.7)$$

The form of equations (6.6) follows from the fact that equation (6.1) is homogeneous with coefficients regular everywhere in the region of interest which guarantees the existence of two linearly independent solutions in the form of Frobenius series (convergence is open to

question of course). Equation (6.7) will prove very useful in numerical computations as well as in setting up analytical solutions. The problem arises when the first order inhomogeneous equation for n_1 is solved (i.e., $G_2(r) \neq 0$ in equation (6.1)); in addition to the two linearly independent solutions, there is a particular solution determined entirely by the zeroth order solution for u_1 . The question arises as to the usefulness of a relation such as equation (6.7). This problem can, fortunately, be dodged in the following manner. The drift terms are considered zero at $r = d$, and they remain relatively insignificant except for the portion of the sheath nearest the spheres surface. Since the effect of the drift terms is negligible or zero at and about $r = d$, so also should be the particular solution of equation (6.1) in the vicinity of $r = d$. Thus the homogeneous solutions represent the complete solution at $r = d$, and the situation is the same as for the zeroth order density solution, i.e., equation (6.7) holds for the complete solution; adding the particular solution does not remove the possibility of forming two linearly independent solutions. The complete solution then is determined by superimposing the particular and homogeneous solutions. The point of this argument, the justification of the general use of equation (6.7), has been made.

6.3. Transformation of the Density Wave Equation

Equation (6.2) can be written in a more useful form by noting that for spherical symmetry

$$\nabla^2 f = \frac{1}{r} \frac{\partial^2}{\partial r^2} (rf) \quad (\text{rf})$$

so that

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r n_1) + \beta^2 n_1 = 0$$

or

$$\frac{\partial^2 N_1}{\partial r^2} + \beta^2 N_1 = 0 \quad (6.8)$$

where

$$N_1 \equiv r n_1$$

and

$$\beta^2 \equiv \frac{\omega^2 - \omega_p^2}{v_o^2}.$$

This equation, valid in Regions II and III, has the general solution

$$N_1 = C_o e^{j\beta r} + C_1 e^{-j\beta r}$$

where

$$\beta = \frac{1}{v_o} \sqrt{\omega^2 - \omega_p^2}$$

if $\omega > \omega_p$, and if $\omega < \omega_p$

$$N_1 = C_o e^{br} + C_1 e^{-br}$$

where

$$b = |\beta|. \quad (6.9)$$

In this geometry, only the forward propagating term need be retained since there is nothing to justify the assumption of the existence of a reflected wave; since these expressions are valid on the infinite interval only the decaying term can be used for $\omega < \omega_p$ because N_1 must remain finite as $r \rightarrow +\infty$ (n_1 must decay at least as fast as $1/r$ so that N_1 must remain finite). The general solution to

equation (6.8) is then

$$N_1 = C_1 \epsilon^{-j\beta r} \quad (6.10)$$

for $\omega > \omega_p$ and

$$N_1 = C_1 \epsilon^{-br} \quad (6.11)$$

for $\omega < \omega_p$. It might be questioned at this point whether or not it is advisable to denote the density coefficients in equations (6.10) and (6.11) by the same symbol C_1 ; no trouble arises from its use however since these equations apply to different frequency ranges, and the results are analogous and nearly identical, $j\beta$ being merely replaced by $b = |\beta|$ in extending the results for $\omega > \omega_p$ to the range $\omega < \omega_p$. It is noted that equation (6.8) has solutions of the form of equation (6.6) satisfying equation (6.7); these are given by

$$\epsilon^{-j\beta r} = \cos \beta r - j \sin \beta r$$

and

$$\epsilon^{-br} = \cosh \beta r - \sinh \beta r$$

where the sets

$$y_1 = \cos \beta r$$

$$y_2 = \frac{1}{\beta} \sin \beta r$$

and

$$y_1 = \cosh \beta r$$

$$y_2 = \frac{1}{\beta} \sinh \beta r$$

satisfy equations (6.6) and (6.7).

Equation (6.1), valid in Region I, can be similarly transformed using the same variable changes and noting that

$$\frac{1}{r} \frac{\partial}{\partial r} (r n_1) = \frac{1}{r} \frac{\partial N_1}{\partial r} = \frac{\partial n_1}{\partial r} + \frac{n_1}{r} .$$

Equation (6.1) becomes

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r n_1) + G_1(r) \left(\frac{1}{r} \frac{\partial N_1}{\partial r} - \frac{N_1}{r^2} \right) + \beta^2(r) n_1 = G_2(r) \quad (6.13)$$

which is easily rewritten in the form

$$\frac{\partial^2 N_1}{\partial r^2} + g_1(r) \frac{\partial N_1}{\partial r} + g_3(r) N_1 = g_2(r) \quad (6.14)$$

where the coefficients are obvious from equations (6.1) and (6.13) although not of any particular interest here except in so far as their regularity is concerned; the transformation to equation (6.14) involves multiplication of certain coefficients by r and $\frac{1}{r}$ which, as can easily be seen, cannot effect their regularity on the interval $r = a$ to $r = d$. On this interval equation (6.14) has two linearly independent solutions of the form of equation (6.6) which satisfy the relation in equation (6.7).

6.4 Matching Solutions at $r = d$

In Region I, the general solution for $N_1(r)$ is a linear combination of the linearly independent solutions justified.

$$N_1(r) = A y_1 + B y_2 \quad (6.15)$$

where A and B are constants to be determined. The continuity of N_1 and $\partial n_1 / \partial r$ at $r = d$ guarantees the continuity of N_1 and N_1' there (primes will henceforth be used to designate derivatives with respect to r , except where noted) so that

$$\begin{aligned} N_1(d^-) &= N_1(d^+) \\ N_1'(d^-) &= N_1'(d^+) . \end{aligned} \quad (6.16)$$

In Region II, from equation (6.10),

$$\begin{aligned} N_1(d^+) &= C_1 e^{-j\beta d} \\ N_1'(d^+) &= -j\beta C_1 e^{-j\beta d} \end{aligned} \quad (6.17)$$

where the notation $N(d^+)$ is shorthand for $\lim_{r \rightarrow d^+} N(r)$. In section 6.3, it was indicated that the development for $\omega < \omega_p$ is completely analogous to the $\omega > \omega_p$ case presented, the steps being identical if $j\beta$ is replaced by $|\beta|$. That this is true can easily be seen from the similarity of equations (6.10) and (6.11); if use of equation (6.11) is visualized in lieu of equation (6.10) in the development of this section the analogy is obvious.

From equations (6.15), (6.6), and (6.7)

$$\begin{aligned} N_1(d^-) &= A \\ N_1'(d^-) &= \frac{1}{\lambda_D} B \end{aligned} \quad (6.18)$$

so that the following relations are developed

$$\begin{aligned} A &= C_1 e^{-j\beta d} \\ B &= -j\beta \lambda_D A \end{aligned} \quad (6.19)$$

Now if the following definition is made

$$N(r) \equiv y_1 - j\beta \lambda_D y_2, \quad (6.20)$$

$N_1(r)$ can then be written as

$$\begin{aligned} \text{Region I: } N_1(r) &= A(y_1 - j\beta \lambda_D y_2) = A N(r) \\ \text{Region II: } N_1(r) &= C_1 e^{-j\beta r} = A e^{-j\beta(r-d)} \end{aligned} \quad (6.21)$$

The perturbation potential ϕ_1 must be determined everywhere and matched to the driving potential perturbation V at $r = a$. The potential perturbation can be determined in general form by integrating Poisson's equation after transforming it in the manner of equation (6.8).

Poisson's equation in the perturbation terms as derived in chapter II, equation (2.34), for the present symmetric geometry takes the form

$$\nabla^2 \phi_1 = \frac{e n_1}{\epsilon_o} = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\phi)$$

or

$$\frac{\partial^2}{\partial r^2} \bar{\Phi}_1(r) = \frac{e}{\epsilon_o} N_1(r) \quad (6.22)$$

where $\bar{\Phi}_1(r)$ is defined as $r\phi_1$. If equation (6.22) is integrated twice with respect to r in indefinite form $\phi_1(r)$ is determined as

$$\bar{\Phi}_1(r) = C_2 + C_3 r + P(r)$$

where

$$P(r) \equiv \frac{e}{\epsilon_o} \iint N_1(r) dr dr . \quad (6.23)$$

In Region II

$$N_1(r) = C_1 e^{-j\beta r}$$

which implies

$$P(r) = - \frac{e}{\epsilon_o \beta^2} C_1 e^{-j\beta r} . \quad (6.24)$$

Since ϕ_1 must go to zero at least as fast as $\frac{1}{r}$ as r approaches infinity, $\bar{\Phi}_1(r) = r\phi_1$ should remain finite as r approaches infinity; $C_3 r$ is unbounded so that C_3 must be taken as zero. In Region II, $\bar{\Phi}_1(r)$ is then given by

$$\bar{\Phi}_1 = C_2 + P(r) . \quad (6.25)$$

In Region I the situation is more complex. Evaluating $P(r)$, the first integral of $N_1(r)$ is

$$\begin{aligned} \int N_1(r) dr &= A \int N(r) dr \\ &= A \left[\int_d^r N(r) dr + a_a - j\beta \lambda_D \gamma_a \right] \end{aligned}$$

and the second integral becomes

$$\begin{aligned} \int_0^r \int_0^r N_1(r) dr &= A \int_0^r \int_0^r N(r) dr dr \\ &= A \left[\int_d^r \int_d^r N(r) dr dr + (a_a - j\beta\lambda_D \gamma_a)(r-d) + a_b - j\beta\lambda_D \gamma_b \right]. \end{aligned} \quad (6.26)$$

The continuity of ϕ_1 and ϕ_1' at $r = d$ guarantees that of $\bar{\Phi}_1(r)$ and $\bar{\Phi}_1'$ there; this, in conjunction with equation (6.25), guarantees the continuity of $P(r)$ and $P'(r)$ there also. Potential matching at $r = d$ will be carried out on the basis of the matching $P(r)$ and $P'(r)$; this process leads to a specialization of the integration constants a_a, a_b, γ_a , and γ_b . At $r = d$, the following relations are true:

$$P(d^+) = -\frac{e}{\epsilon_o \beta^2} C_1 \epsilon^{-j\beta d} = -\frac{e}{\epsilon_o \beta^2} A \quad (6.27)$$

$$P'(d^+) = j \frac{e}{\epsilon_o \beta} C_1 \epsilon^{-j\beta d} = j \frac{e}{\epsilon_o \beta} A \quad (6.28)$$

$$P(d^-) = \frac{e}{\epsilon_o} A [a_b - j\beta\lambda_D \gamma_b] \quad (6.29)$$

$$P'(d^-) = \frac{e}{\epsilon_o} A [a_a - j\beta\lambda_D \gamma_a] \quad (6.30)$$

Applying the conditions

$$\begin{aligned} P(d^+) &= P(d^-) \\ P'(d^+) &= P'(d^-) \end{aligned} \quad (6.31)$$

the following relationships between the integration constants are obtained.

$$\begin{aligned} a_b - j\beta\lambda_D \gamma_b &= -\frac{1}{\beta^2} \\ a_a - j\beta\lambda_D \gamma_a &= j \frac{1}{\beta} \end{aligned} \quad (6.32)$$

The set of integration constants satisfying equations (6.32) is not unique, but if that set is considered real, then the set is uniquely determined and potential matching is accomplished if

$$\begin{aligned} a_a &= 0 \\ a_b &= -\frac{1}{\beta^2} \\ \gamma_a &= -\frac{1}{\beta^2 \lambda_D} \\ \gamma_b &= 0 \end{aligned} \quad (6.33)$$

Then, in Region I, $P(r)$ becomes

$$P(r) = \frac{e}{\epsilon_o} A \left[\int_d^r \int_d^r N(r) dr dr + j \frac{1}{\beta} (r-d) - \frac{1}{\beta^2} \right] \quad (6.34)$$

so that the potential perturbation is given by:

$$\text{Region I:} \quad \phi_1 = \frac{C_2}{r} - \frac{e}{\epsilon_o \beta^2} \frac{A}{r} \left[1 - j\beta(r-d) - \beta^2 \int_d^r \int_d^r N(r) dr dr \right] \quad (6.35)$$

$$\text{Region II:} \quad \phi_1 = \frac{C_2}{r} - \frac{e}{\epsilon_o \beta^2} \frac{A}{r} e^{-j\beta(r-d)} \quad (6.36)$$

6.5. Matching at $r = a$

In chapter V, a hybrid boundary condition was developed from the hard boundary condition. This condition, (5.21), and the condition that $\phi_1 = V$ at $r = a$ (where V is the applied potential perturbation) can be used to determine the constants A and C_2 in terms of V ; this process completely specifies the solutions for ϕ_1 and n_1 , as well as the radiation fields, in terms of V as will be shown subsequently. The hybrid boundary condition to be applied has the form

$$\left. \frac{\partial \phi_1}{\partial r} \right|_{r=a} = D_1 \left. \frac{\partial n_1}{\partial r} \right|_{r=a} + B_1 n_1 \Big|_{r=a}$$

where the coefficients are given in section 5.4. At $r = a$, $\partial\phi_1/\partial r$ determined from equation (6.35) is

$$\left. \frac{\partial\phi_1}{\partial r} \right|_{r=a} = -\frac{C_2}{a^2} - \frac{eA}{\epsilon_0\beta^2} \left\{ -\frac{1}{a^2} \left[1 - j\beta(a-d) - \beta^2 \int_d^a \int_d^r N(r) dr dr \right] + \frac{1}{a} \left[-j\beta - \beta^2 \int_d^a N(r) dr \right] \right\}$$

from which

$$\left. \frac{\partial\phi_1}{\partial r} \right|_{r=a} = -\frac{C_2}{a^2} + \frac{eE}{\epsilon_0\beta^2 a^2} A \quad (6.37)$$

where

$$E = 1 + j\beta d - \beta^2 \int_d^a \int_d^r N(r) dr dr + \beta^2 a \int_d^a N(r) dr.$$

From the definition

$$n_1 = \frac{N_1}{r} = \frac{A N(r)}{r}$$

so that

$$\frac{\partial n_1}{\partial r} = A \left[-\frac{1}{r^2} N(r) + \frac{1}{r} N'(r) \right]$$

and

$$\left. \frac{\partial n_1}{\partial r} \right|_{r=a} = -\frac{A F}{a^2} \quad (6.38)$$

where

$$F \equiv N(a) - a N'(a) .$$

If equations (6.37) and (6.38) are used to substitute for $\partial n_1/\partial r$ and $\partial\phi_1/\partial r$ in the hybrid boundary condition the following relation between C_2 and A is obtained

$$-\frac{C_2}{a^2} + \frac{eE}{\epsilon_0\beta^2 a^2} A = -\frac{D_1 F}{a^2} A + \frac{B_1 N(a) A}{a}$$

or

$$C_2 = A \left[D_1 F + \frac{e E}{\epsilon_o \beta^2} - B_1 N(a) a \right] . \quad (6.39)$$

Matching ϕ_1 at $r = a$ to the driving potential perturbation V allows the determination of A in terms of V . From equation (6.35)

$$V = \phi_1 \Big|_{r=a} = \frac{C_2}{a} - \frac{e}{\epsilon_o \beta^2} \frac{A}{a} \left[1 - j\beta(a-d) - \beta^2 \int_d^a \int_d^r N(r) dr dr \right] . \quad (6.40)$$

Substituting for C_2 from equation (6.39) equation (6.40) can be written

$$a V = A \left\{ D_1 F + \frac{e E}{\epsilon_o \beta^2} - B_1 N(a) a - \frac{e}{\epsilon_o \beta^2} \left[1 - j\beta(a-d) - \beta^2 \int_d^a \int_d^r N(r) dr dr \right] \right\}$$

or

$$a V = A \left[D_1 F + \frac{e H}{\epsilon_o \beta^2} - B_1 N(a) a \right] \quad (6.41)$$

where

$$H \equiv j\beta a + \beta^2 a \int_d^a N(r) dr .$$

A and C_2 are completely determined in terms of V as

$$A = a V \left[D_1 F + \frac{e H}{\epsilon_o \beta^2} - B_1 N(a) a \right]^{-1} \quad (6.42)$$

and

$$C_2 = a V \frac{D_1 F + \frac{e E}{\epsilon_o \beta^2} - B_1 N(a) a}{D_1 F + \frac{e H}{\epsilon_o \beta^2} - B_1 N(a) a} . \quad (6.43)$$

6.6. Matching at $r = d_1$

Equations (2.42) describe the EM mode in the uniform plasma without a constant magnetic field. They can be written in the same form as Maxwell's equations in a linear isotropic lossless medium of

permittivity

$$\epsilon = \epsilon_o \left(1 - \frac{\omega_p^2}{\omega^2} \right) .$$

This can be seen if equation (2.43) is subtracted from the frequency domain version of equation (2.32) yielding

$$j\omega \bar{u}_{1e} = -\frac{e}{m_e} \bar{E}_{1e} . \quad (6.44)$$

If equation (6.44) is substituted into the second equation of (2.42) then

$$\nabla \times \bar{B}_1 = \mu_o \bar{J}^s + \frac{\mu_o e^2 n_o \bar{E}_{1e}}{j m_e \omega} + j\omega \mu_o \epsilon_o \bar{E}_{1e}$$

or

$$\nabla \times \bar{B} = \mu_o \bar{J}^s + j\omega \mu_o \epsilon_o \left(1 - \frac{\omega_p^2}{\omega^2} \right) \bar{E}_{1e} . \quad (6.45)$$

This being the case, it is a simple matter to derive wave equations for the EM field quantities. For the purposes of this analysis the usual wave equation for the scalar potential in a homogeneous medium is sufficient, i.e.,

$$\nabla^2 \phi_{1e} + \beta_e^2 \phi_{1e} = 0 \quad (6.46)$$

where

$$\beta_e^2 = \omega^2 \mu_o \epsilon = \frac{\omega^2 - \omega_p^2}{C^2}$$

and

$$c^2 = \frac{1}{\mu_o \epsilon_o} .$$

Equation (6.46) is applicable everywhere in Regions II and III and has there the general solution

$$\phi_{1e} = \frac{\epsilon^{\pm j\beta_e r}}{r} ;$$

only the forward propagating term is retained since the geometry does not justify a reflected term so that

$$\phi_{1e} = a_4 \frac{\epsilon^{-j\beta_e r}}{r} . \quad (6.47)$$

In the quasi-static region (I and II) the term $\epsilon^{-j\beta_e r}$ is nearly constant so that ϕ_{1e} can be considered as being included in the C_2/r term; that this is true is the heart of the "quasi-static" approximation.

The EA mode has a conservative field ($\nabla \times \bar{E}_{1p} = 0$ from equation (2.43)) so that

$$\bar{E}_{1p} = -\nabla \phi_{1p} . \quad (6.48)$$

There being no independent source charge in regions II and III equation (2.43) yields

$$\nabla \cdot \bar{E}_{1p} = -\frac{e n_1}{\epsilon_o} , \quad (6.49)$$

and

$$\nabla^2 \phi_{1p} = \frac{e n_1}{\epsilon_o} . \quad (6.50)$$

Equation (6.2) is easily shown to apply here for the EA mode since this equation involves the total density perturbation n_1 (the equation's derivation did not involve the concept of mode separation), and the EA mode contains all the charge accumulation (i.e., $n_{1p} = n_1$).

Equations (6.2) and (6.50) can be combined to form

$$\nabla^2 \phi_{1p} = -\frac{e}{\epsilon_o \beta^2} \nabla^2 n_1 . \quad (6.51)$$

Equation (6.51) can be transformed to

$$\frac{\partial^2}{\partial r^2} (r \phi_{1p}) = -\frac{e}{\epsilon_o \beta^2} \frac{\partial^2}{\partial r^2} (r n_1) ; \quad (6.52)$$

integrating twice with respect to r yields

$$\phi_{1p} = \frac{-en_1}{\epsilon_o \beta^2} + a_5 + \frac{a_6}{r} \quad (6.53)$$

so that

$$\phi_1 = -\frac{e}{\epsilon_o \beta^2} n_1 + a_4 \frac{\epsilon^{-j\beta_e r}}{r} + a_5 + \frac{a_6}{r} . \quad (6.54)$$

Demanding that ϕ_1 tend to zero as r approaches infinity and dropping the nonphysical nonpropagating terms a_6/r equation (6.54) becomes

$$\phi_1 = -\frac{en_1}{\epsilon_o \beta^2} + a_4 \frac{\epsilon^{-j\beta_e r}}{r} . \quad (6.55)$$

The potential expression (6.55) can now be used to match the far field scalar potential perturbation to the quasi-static potential perturbation given in equation (6.36). Demanding that both equations (6.55) and (6.36) hold at $r = d_1$ leads to

$$\phi_1 \Big|_{r=d_1} = -\frac{en_1}{\epsilon_o \beta^2} \Big|_{r=d_1} + a_4 \frac{\epsilon^{-j\beta_e d_1}}{d_1} = \frac{C_2}{d_1} - \frac{e}{\epsilon_o \beta^2} n_1 \Big|_{r=d_1} \quad (6.56)$$

which implies immediately that

$$a_4 = C_2 \epsilon^{j\beta_e d_1} . \quad (6.57)$$

6.7. The Far Zone Field

As an immediate consequence of this matching process the radiation zone scalar potential is determined in terms of C_2 , which has been determined in terms of the driving potential perturbation V , as

$$\phi_{1e} = C_2 \frac{\epsilon^{-j\beta_e(r-d_1)}}{r} . \quad (6.58)$$

Consider an oscillating point charge Q at $r = 0$ in a region of uniform permittivity

$$\epsilon = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right) ;$$

the general particular solution to equation (6.46) in terms of the source charge density ρ_s is

$$\phi_{1e} = \frac{1}{4\pi\epsilon} \int_{\text{vol}} \frac{\rho_s}{r} e^{-j\beta_e r} dv ,$$

and for the point charge

$$\rho_s = Q \delta(r)$$

so that

$$\phi_{1e} = \frac{Q e^{-j\beta_e r}}{4\pi\epsilon r} . \quad (6.59)$$

Comparing equations (6.58) and (6.59) it is seen that a single sphere of the antenna looks like a point charge Q located at the sphere's center where Q is related to C_2 (and hence to V) by the relation

$$Q = 4\pi\epsilon C_2 e^{j\beta_e d_1} . \quad (6.60)$$

Since d_1 is arbitrary the phase relationship between the source and the far field is lost, but the amplitude and relative phase variations are maintained. By symmetry this description is valid for each sphere of the antenna and the resultant field is the superposition of those due the individual spheres; in performing this superposition the antenna-space geometry must be considered, i. e., the relative phasing of the fields at any given point in space will depend upon the orientation of that point relative to the antenna as well as upon the separation between the spheres.

The geometry of the antenna, as detailed in the introduction,

was chosen so as to meet the requirements of an electric dipole; it is electrically small, its magnetic dipole moment is zero, and it is modeled as far as the radiation field is considered as two point charges Q oscillating at frequency ω in antiphase and separated by a distance D .

Since the phase relation between the source and a radius r in the radiation zone is lost in the expression for Q and only the relative phase of the fields from each sphere is important, Q can be taken as

$$Q = 4\pi\epsilon |C_2| \quad . \quad (6.61)$$

Neither EA or EM propagation is possible when $\omega < \omega_p$ since both β^2 and β_e^2 are negative indicating attenuation so that this radiation model breaks down in this case.

The radiation zone EM fields can be determined from the scalar potential given in equation (6.58) and the vector potential due to the current in the antenna feed wires. A more simple method consists of merely writing down the radiation zone field expressions for an electric dipole in terms of the dipole moment QD ; these classic dipole field expressions are given in many texts.^{21, 22} Ramo²¹ gives the following expressions in terms of the feed current assuming the dipole to be oriented with both sphere centers on the vertical axis in spherical coordinates.

$$\begin{aligned} H_\phi &= - \frac{j\beta_e I_o D}{4\pi r} \sin\theta \epsilon^{-j\beta_e r} \\ E_\theta &= - \frac{j\omega\mu_o I_o D}{4\pi r} \sin\theta \epsilon^{-j\beta_e r} \quad . \end{aligned} \quad (6.62)$$

The feed current is related to the dipole charge by

$$I_o = j\omega Q \quad (6.63)$$

so that equations (6.62) become

$$\begin{aligned} H_\phi &= - \frac{\beta_e \omega Q D}{4 \pi r} \sin \theta e^{-j\beta_e r} \\ E_\theta &= - \frac{\omega^2 \mu_o Q D}{4 \pi r} \sin \theta e^{-j\beta_e r} . \end{aligned} \quad (6.64)$$

The field expressions can be linearly related to V through equations (6.61) and (6.43) replacing QD by

$$QD = 4\pi\epsilon |C_2| D = 4\pi\epsilon a DV \left| \frac{D_1 F + \frac{e E}{\epsilon_o \beta^2} - B_1 N(a) a}{D_1 F + \frac{e H}{\epsilon_o \beta^2} - B_1 N(a) a} \right| . \quad (6.65)$$

The radiated power is computed by integrating the time averaged outwardly directed Poynting vector over a large sphere of radius R concentric with the antenna center. The time averaged Poynting vector is given by

$$\begin{aligned} S_{av} &= \frac{\bar{E} \times \bar{H}^*}{2} = \frac{E_\theta H_\phi^*}{2} \\ &= \sqrt{\frac{\mu_o}{\epsilon}} \frac{\beta_e^2 \omega^2 Q^2 D^2}{32 \pi^2 R^2} \sin^2 \theta . \end{aligned} \quad (6.66)$$

The average EM power radiated is then

$$P_{av} = \sqrt{\frac{\mu_o}{\epsilon}} \frac{\beta_e^2 \omega^2 Q^2 D^2}{12 \pi} \quad (6.67)$$

or

$$P_{av} = \frac{1}{90} \left(\frac{\omega}{c}\right)^4 \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{5/2} |C_2|^2 D^2 . \quad (6.68)$$

6.8. Radiation Conductance

Any solution for the driving point admittance (or impedance) using the quasi-static solutions cannot include the effect of electromagnetic radiation. Because of the short electrical length of the antenna, its driving point admittance is largely capacitive so that a radiation conductance computed from the radiated EM power can very nicely be added to the quasi-static input admittance to yield a good approximation to the total input admittance. The total potential applied to the antenna is $2V$ so that the effective radiation conductance is related to the EM power by

$$G_e = \frac{P_{av}}{2V^2} \quad (6.69)$$

so that G_e can be written in terms of C_2 in the form

$$G_e = \frac{1}{180} \left(\frac{\omega}{c}\right)^4 \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{5/2} \left|\frac{C_2}{V}\right|^2 D^2. \quad (6.70)$$

This expression can be written in a form more useful for numerical calculation by recalling the definition of the normalized frequency variable $\Omega \equiv \frac{\omega}{\omega_p}$ and by noting that equation (6.43) for C_2 can be written in terms of a dimensionless quantity K_c , i.e.,

$$K_c \equiv \frac{D_1 F + \frac{eE}{\epsilon_o \beta^2} - B_1 a N(a)}{D_1 F + \frac{eH}{\epsilon_o \beta^2} - B_1 a N(a)}, \quad (6.71)$$

so that

$$C_2 = a V K_c. \quad (6.72)$$

With these definitions equation (6.70) can be written as

$$G_e = \frac{1}{180} \left(\frac{\omega_p}{c}\right)^4 \frac{1}{\Omega} (\Omega^2 - 1)^{5/2} D^2 a^2 |K_c|^2$$

or if ω_p/c is defined as k_{pe} this equation takes the form

$$\begin{aligned} G_e &= \frac{1}{180} (k_{pe} D)^2 (k_{pe} a)^2 \left[\frac{1}{\Omega} (\Omega^2 - 1)^{5/2} |K_c|^2 \right] \\ &= \frac{1}{180} (k_{pe} D)^2 (k_{pe} a)^2 K_e \end{aligned} \quad (6.73)$$

where

$$K_e = \frac{1}{\Omega} (\Omega^2 - 1)^{5/2} |K_c|^2.$$

6.9. Quasi-Static Input Admittance

The perturbation current to each sphere can be calculated in terms of the quasi-static solutions indicated previously. The current supplied to the antenna is related to the sphere's surface charge and the plasma current as follows:

$$\begin{aligned} I &= J 4 \pi a^2 \\ J &= \frac{\partial \sigma_s}{\partial t} + J_p \end{aligned}$$

and

$$J_p = -e(nu)_1 \quad (6.74)$$

where σ_s is the surface charge density on the upper sphere, J_p is the magnitude of the plasma current density perturbation directed away from it, and $(nu)_1$ is the magnitude of the perturbation component of the electron drift current there. The $(nu)_1$ term can be reduced to

$$(nu)_1 = n_o u_1 + n_1 u_o$$

or

$$(nu)_1 = n_1 u_o \quad (6.75)$$

since u_1 is taken as being identically zero at the sphere's surface.

The surface charge density σ_s is related to the field parameters

by the relation

$$\begin{aligned} \sigma_s &= \epsilon_o E_1 \Big|_{r=a} \\ &= -\epsilon_o \frac{\partial \phi_1}{\partial r} \Big|_{r=a} . \end{aligned} \quad (6.76)$$

Combining equations (6.74), (6.75), and (6.76) while transforming the

first to frequency domain description, the current I can be written in

the following terms

$$I = 4\pi a^2 \left[-j\omega\epsilon_o \frac{\partial \phi_1}{\partial r} \Big|_{r=a} - e(u_o n_1) \Big|_{r=a} \right] . \quad (6.77)$$

Use can once again be made of the hybrid boundary condition given

in equation (5.21) to reduce this current expression to one in n_1

and $\partial n_1 / \partial r$ only.

$$I = 4\pi a^2 \left[-j\omega\epsilon_o \left(D_1 \frac{\partial n_1}{\partial r} \Big|_{r=a} + B_1 n_1 \Big|_{r=a} \right) - e(u_o n_1) \Big|_{r=a} \right] . \quad (6.78)$$

Replacing $\frac{\partial n_1}{\partial r} \Big|_{r=a}$ using equation (6.38) and noting that

$$n_1 \Big|_{r=a} = \frac{A N(a)}{a}$$

and

$$u_o \Big|_{r=a} = - \sqrt{\frac{KT_e}{2\pi m_e}} \left(\frac{d}{a}\right)^2$$

from equation (3.38), equation (6.78) becomes

$$I = 4\pi \left[e \sqrt{\frac{KT_e}{2\pi m_e}} \left(\frac{d}{a}\right)^2 N(a) a + j\omega \epsilon_o (D_1 F - B_1 N(a) a) \right] A. \quad (6.79)$$

Replacing A , using equation (6.42), a complete expression for I in terms of V is obtained as

$$I = 4\pi a V \frac{e \sqrt{\frac{KT_e}{2\pi m_e}} \left(\frac{d}{a}\right)^2 N(a) a + j\omega \epsilon_o (D_1 F - B_1 N(a) a)}{D_1 F + \frac{eH}{\epsilon_o \beta^2} - B_1 N(a) a}. \quad (6.80)$$

The quasi-static input admittance to the antenna is given by

$$Y_p = \frac{I}{2V} \quad (6.81)$$

since the potential applied to the antenna is twice the applied potential perturbation of the individual spheres. From equation (6.80), Y_p is

$$Y_p = 2\pi a \frac{e \sqrt{\frac{KT_e}{2\pi m_e}} \left(\frac{d}{a}\right)^2 N(a) a + j\omega \epsilon_o (D_1 F - B_1 N(a) a)}{D_1 F + \frac{eH}{\epsilon_o \beta^2} - B_1 N(a) a}. \quad (6.82)$$

It will be shown in chapter VII that Y_p can be written in the form

$$Y_p = 2\pi a \epsilon_o \omega_p K_y \quad (6.83)$$

where the dimensionless quantity K_y is a function of frequency and plasma parameters; the definition of K_y is obvious from equations (6.82) and (6.83).

6.10. EA-EM Power Ratio

It is of interest to compare the ratio of the power radiated in the form of electroacoustic waves to that radiated in the form of electromagnetic waves. No loss mechanism has been included in the quasi-static analysis so that

$$G_p \equiv R_e [Y_p] = 2 \pi a \epsilon_o \omega_p R_e [K_y] \quad (6.84)$$

accounts for the power radiated in the form of electroacoustic waves. Similarly, no loss mechanism is included in the EM mode assumed here, and the radiation conductance computed in section 6.8 from the power radiated in electromagnetic form represents a valid radiation correction to the quasi-static admittance. It follows that the ratio of the radiated electroacoustic power to the radiated electromagnetic power is given by the ratio of the conductances G_p and G_e , i. e.,

$$\frac{P_p}{P_e} = \frac{G_p}{G_e} \quad (6.85)$$

From equations (6.84) and (6.73) this becomes

$$\frac{P_p}{P_e} = \frac{360 \pi a \epsilon_o \omega_p R_e [K_y]}{k_{pe}^4 D^2 a^2 \frac{1}{\Omega} (\Omega^2 - 1)^{5/2} |K_c|^2} \quad (6.86)$$

Applying the definition of k_{pe} , equation (6.86) can be written as

$$\frac{P_p}{P_e} = \frac{3 \Omega R_e [K_y]}{(k_{pe} D)^2 k_{pe} a (\Omega^2 - 1)^{5/2} |K_c|^2} \quad (6.87)$$

In chapter VII, K_y and $|K_c|^2$ are reduced to a form suitable for numerical evaluation. These dimensionless quantities, as well as others, are plotted in chapter VIII. Equation (6.87) can be written

more compactly in terms of the dimensionless quantity K_p defined as

$$K_p \equiv \frac{3 \Omega}{(\Omega^2 - 1)^{5/2}} \frac{R_e [K_y]}{|K_c|^2}$$

so that

$$\frac{P_p}{P_e} = \frac{1}{(k_{pe} D)^2 (k_{pe} a)} K_p . \quad (6.88)$$

6.11. Application to Resonance Probes

In as much as one of the goals of this study is to check out the relatively simple analytical results of Fejer⁶ who studied essentially the same geometry but neglected the sheath, the application of his theory to resonance probes will be checked with the present theory. Without going into the theory of resonance probes in detail, it can be said that resonance probes operate on the principle that when a radio frequency voltage is applied to a Langmuir probe the perturbation in the current as well as the direct current collected is a function of the applied frequency. It is normally accepted as an experimental fact that the collected DC current shows a sharp increase at the plasma frequency. Fejer disputes this claiming that (1) the change in the DC current is due to rectification caused by the nonlinear characteristic of the Langmuir probe, (2) the amplitude of the radio frequency perturbation in the collected current will be proportional to the density perturbation n_1 at the probes surface, and (3) the peak in $n_1 \Big|_{r=a}$ in his theory does not occur at ω_p but at a frequency appreciably lower than ω_p . To the extent that his arguments are valid, the

conventional theory of resonance probes leads to very serious errors.

Fejer's expression for n_1 in the notation used here is

$$n_1 \Big|_{r=a} = V \frac{\epsilon_o \omega_p^2}{e v_o^2} \left[1 - \frac{a \frac{\omega_p}{v_o} \Omega^2}{(1 - \Omega^2)^{1/2}} \right]^{-1} ; \quad (6.89)$$

this expression is relatively small for $\Omega > 1$, has a zero at $\Omega = 1$, and is entirely real for $\Omega < 1$ with a pole at

$$\Omega_p = \left(\frac{v_o^2}{2 a^2 \omega_p^2} \left[\left(1 + 4 \frac{a^2 \omega_p^2}{v_o^2} \right)^{1/2} - 1 \right] \right)^{1/2} . \quad (6.90)$$

It might be noted that as the factor $a \omega_p / v_o$, which can be written $a / \sqrt{3} \lambda_D$, approaches infinity the singular point Ω_p approaches zero so that for the case of the thin sheath very appreciable errors could be made assuming that $\Omega_p = 1$. The present study indicates a somewhat different density-frequency profile; this is plotted in chapter VIII. The density perturbation can be written in terms of V and the plasma parameters in the form

$$n_1 \Big|_{r=a} = \frac{\omega_p^2 \epsilon_o}{v_o^2 e} V K_a N(a) \quad (6.91)$$

where the dimensionless factor K_a is a function of frequency and plasma parameters; the form of K_a is determined from equation (6.42) and is developed in chapter VII. The quantity A is related to K_a by the relation

$$\frac{A}{a V} = \frac{\omega_p^2 \epsilon_o}{v_o^2 e} K_a . \quad (6.92)$$

CHAPTER VII

FORMULATION FOR NUMERICAL SOLUTION

7.1. The Mathematical Sheath Model

The purpose of this chapter is to manipulate the results of chapters IV, V, and VI into a form amenable to numerical solution. It will be noted that the separation of the plasma into the sheath region and the uniform region (Regions II and III) as described in chapter VI is an artificial idealization of the actual situation described in chapter III. If the point $r = d$ is chosen sufficiently far from $r = a$ the conditions of uniformity at $r = d$ assumed in the model will be very closely achieved; however, if d is chosen too far from $r = a$ practical difficulties arise in the numerical solution of the wave equations on the interval $[a, d]$. The second order differential equations involved are solved by numerical integration on this interval, and since the solutions are either oscillatory, at times involving many cycles of sinusoidal like functions on the interval, or exponential like functions with large growth rates, the stability of the method over the entire interval may be a problem. Thus, if d is chosen too far from $r = a$ stability requirements may force the use of extremely fine subdivisions of the interval for calculation, and uneconomical amounts of machine time may be required. The choice of the arbitrary value of d represents a compromise between the extremes indicated. The sheath model chosen in chapter III for this study is not dependent upon the sphere radius and is shown in Figure 3.4; the normalized potential η drops to less than 2.5% of its value at the spheres

surface in 10 debye lengths from the sphere, and the electron density returns to within 10% of its uniform value. The sheath thickness for the mathematical model is arbitrarily chosen as 10 debye lengths regardless of sphere radius; a better approximation could be had if this thickness were extended to 20 debye lengths at the cost of appreciably more machine time. The various coefficients in the differential equations will be evaluated from an arbitrarily chosen mathematical model used in lieu of the physical model of Figure 3.4; this mathematical model is chosen so that the normalized potential η (1) is described by a low order polynomial in r for simplicity of evaluation, (2) assumes the same value as the physical model at the spheres surface, chosen in chapter III to be 4.50, and (3) goes to zero with zero slope at $r = d$ maintaining uniformity (i. e., continuity of derivatives of plasma variables) there. Plots of the physical model potential determined in chapter III, as well as the approximate mathematical model for η , and the coefficients of the zeroth order wave equation for n_1 will be given in Figures 7.1 and 7.2. The analytical forms of the polynomials used are indicated in the figures.

7.2. Transformation of the Wave Equations

First consider the complete wave equation for n_1 derived in chapter IV, namely equation (4.32). If the variable substitution

$$N_1(r) \equiv r n_1$$

is applied, and if it is noted that

$$\frac{1}{r} \frac{\partial}{\partial r} (r n_1) = \frac{\partial n_1}{\partial r} + \frac{1}{r} n_1$$

equation (4.32) becomes

$$\left[1 + \frac{1}{6\pi} \left(\frac{d}{r} \right)^4 \epsilon^{2(\eta - \eta_w)} \right] \frac{1}{r} \frac{\partial^2 N_1}{\partial r^2} + \frac{1}{3\lambda_D} \left[p + \left(\frac{d}{r} \right)^4 \frac{\epsilon^{2(\eta - \eta_w)}}{\pi} (2p - 3 \frac{\lambda_D}{r}) \right] \cdot \left[\frac{1}{r} \frac{\partial N_1}{\partial r} - \frac{N_1}{r^2} \right] + \beta_1^2(r) \frac{N_1}{r} = - \frac{2 n_o u_o}{v_o^2} \frac{\partial^2 u_1}{\partial r^2} \quad (7.1)$$

If this equation is multiplied through by $r \lambda_D^2 A$ and the common terms are collected it becomes

$$\left[1 + \frac{1}{6\pi} \left(\frac{d}{r} \right)^4 \epsilon^{2(\eta - \eta_w)} \right] \frac{\partial^2 N(r)}{\partial x_1^2} + \frac{1}{3} \left[p + \frac{\epsilon^{2(\eta - \eta_w)}}{\pi} \left(\frac{d}{r} \right)^4 (2p - 3 \frac{\lambda_D}{r}) \right] \frac{\partial N(r)}{\partial x_1} + \left\{ \beta^2(r) \lambda_D^2 - \frac{p}{3} \frac{\lambda_D}{r} + \frac{\epsilon^{2(\eta - \eta_w)}}{3\pi} \left(\frac{d}{r} \right)^4 \left[\frac{\partial^2 \eta}{\partial x_1^2} + 2p^2 + 6 \left(\frac{\lambda_D}{r} \right)^2 - 6p \frac{\lambda_D}{r} \right] \right\} N(r) = - \frac{2 n_o u_o A r}{v_o^2} \frac{\partial^2 u_1}{\partial x_1^2} \quad (7.2)$$

where $\beta^2(r)$ is defined in equation (4.12) and

$$x_1 \equiv \frac{r - a}{\lambda_D} \quad (7.3)$$

Once η is defined all the terms in the coefficients of equation (7.2) are easily determined; this equation is in the proper dimensionless form for computation except for the right hand side which will be modified by the definition of the dimensionless counterpart to $N(r)$ for u_1 .

Since boundary conditions for the linearly independent solutions for n_1 are given at $r = d$, a different normalized independent variable

$$x \equiv \frac{d - r}{\lambda_D} \equiv -x_2$$

will be used for the density wave equations of zeroth and first order for convenience; x_1 will be retained for the drift wave equation since its boundary conditions have been given at $r = a$ ($x_1 = 0$).

In the development of the normalized drift wave equation it is convenient to start with equation (4.15) transforming it in terms of N_1 as in equation (7.1).

$$n_o \frac{\partial^2 u}{\partial r^2} + 2 \frac{\partial n_o}{\partial r} \frac{\partial u}{\partial r} + \left(\frac{\partial^2 n_o}{\partial r^2} - \frac{2}{r} n_o \right) u = - \left[u_o \frac{\partial^2 N_1}{\partial r^2} + \left(2 \frac{\partial u_o}{\partial r} + j\omega \right) \frac{\partial N_1}{\partial r} + \left(\frac{\partial^2 u_o}{\partial r^2} - \frac{2}{r} u_o - j \frac{\omega}{r} \right) N_1 \right] \quad (7.4)$$

where

$$u \equiv r u_1 \quad (7.5)$$

If equation (7.4) is multiplied through by $2 u_o / v_o^2$, and if the coefficients are written as explicit functions of η the following equation is obtained.

$$\begin{aligned} & \frac{2 n_o u_o}{v_o^2} \left(\frac{d}{r} \right)^2 \frac{\partial^2 u}{\partial r^2} - \frac{4 p}{\lambda_D} \frac{n_o u_o}{v_o^2} \left(\frac{d}{r} \right)^2 \frac{\partial u}{\partial r} + \frac{2 n_o u_o}{v_o^2} \left(\frac{d}{r} \right)^2 \frac{1}{\lambda_D^2} \left[p^2 - \right. \\ & \left. \frac{\partial^2 \eta}{\partial x_1^2} - 2 \left(\frac{\lambda_D}{r} \right)^2 \right] u = - \left\{ \frac{\epsilon^{2(\eta - \eta_w)}}{3\pi} \left(\frac{d}{r} \right)^4 \frac{\partial^2 N_1}{\partial r^2} + \right. \\ & \left. \frac{2}{3} \left[\frac{\epsilon^{2(\eta - \eta_w)}}{\pi} \left(\frac{d}{r} \right)^4 \left(p - 2 \frac{\lambda_D}{r} \right) - j \frac{\Omega}{\sqrt{2\pi}} \left(\frac{d}{r} \right)^2 \epsilon^{\eta - \eta_w} \right] \frac{\partial N_1}{\partial x_1} + \right. \\ & \left. \left[\frac{\epsilon^{2(\eta - \eta_w)}}{3\pi} \left(\frac{d}{r} \right)^4 \left(\frac{\partial^2 \eta}{\partial x_1^2} + p^2 + 4 \frac{\lambda_D^2}{r^2} - 4 \frac{\lambda_D}{r} p \right) + \right. \right. \\ & \left. \left. j \frac{2}{3} \frac{\Omega}{\sqrt{2\pi}} \epsilon^{\eta - \eta_w} \left(\frac{d}{r} \right)^2 \left(\frac{\lambda_D}{r} \right) \right] N_1 \right\} \quad (7.6) \end{aligned}$$

Further, if the following definition is made

$$U \equiv - \frac{2 n_{\infty} u_{\infty} A r u_1}{v_o^2}, \quad (7.7)$$

and if equation (7.6) is multiplied through by $\lambda_D^2 r^2 / d^2$ then the complete normalized drift wave equation is determined as

$$\begin{aligned} \frac{\partial^2 U}{\partial x_1^2} - 2p \frac{\partial U}{\partial x_1} + \left[p^2 - \frac{\partial^2 \eta}{\partial x_1^2} - 2 \left(\frac{\lambda_D}{r} \right)^2 \right] U &= \frac{\epsilon^{2(\eta - \eta_w)}}{3\pi} \left(\frac{d}{r} \right)^2 \frac{\partial^2 N(r)}{\partial x_1^2} + \\ \frac{2}{3} \left[\frac{\epsilon^{2(\eta - \eta_w)}}{\pi} \left(\frac{d}{r} \right)^2 \left(p - 2 \frac{\lambda_D}{r} \right) - j \frac{\Omega}{\sqrt{2\pi}} \epsilon^{\eta - \eta_w} \right] \frac{\partial N(r)}{\partial x_1} &+ \\ \left[\frac{\epsilon^{2(\eta - \eta_w)}}{3\pi} \left(\frac{d}{r} \right)^2 \left(\frac{\partial^2 \eta}{\partial r^2} + p^2 + 4 \frac{\lambda_D}{r^2} - 4 \frac{\lambda_D}{r} p \right) + j \frac{2\Omega}{3\sqrt{2\pi}} \epsilon^{\eta - \eta_w} \frac{\lambda_D}{r} \right] N(r). \end{aligned} \quad (7.8)$$

The definition equation (7.7) suggests the way to proceed in reducing the right hand side of equation (7.2). From the definition of u it is easily shown that

$$\frac{\partial u}{\partial r} = r \frac{\partial u_1}{\partial r} + u_1$$

and

$$\frac{\partial^2 u}{\partial r^2} = r \frac{\partial^2 u_1}{\partial r^2} + 2 \frac{\partial u_1}{\partial r}$$

from which

$$r \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial r^2} - \frac{2}{r} \frac{\partial u}{\partial r} + \frac{2u}{r^2}. \quad (7.9)$$

Multiplying equation (7.9) through by

$$- \frac{2 n_o u_o A}{v_o^2} = - \frac{2 n_{\infty} u_{\infty} A}{v_o^2} \left(\frac{d}{r} \right)^2$$

it becomes

$$- \frac{2n_o u_o Ar}{v_o^2} \frac{\partial^2 u_1}{\partial r^2} = \left(\frac{d}{r}\right)^2 \left(\frac{\partial^2 U}{\partial r^2} - \frac{2}{r} \frac{\partial U}{\partial r} + \frac{2}{r^2} U \right). \quad (7.10)$$

Substituting from equation (7.10) in equation (7.2) the complete normalized density wave equation is obtained after transformation of the independent variable x_1 to x .

$$\begin{aligned} & \left[1 + \frac{1}{6\pi} \left(\frac{d}{r}\right)^4 \epsilon^{2(\eta-\eta_w)} \right] \frac{\partial^2 N(r)}{\partial x^2} - \left[\frac{p}{3} + \frac{\epsilon^{2(\eta-\eta_w)}}{3\pi} \left(\frac{d}{r}\right)^4 (2p - 3 \frac{\lambda_D}{r}) \right] \frac{\partial N(r)}{\partial x} + \\ & \left[\frac{1}{3} (\Omega^2 - f_r) - \frac{p}{3} \frac{\lambda_D}{r} + \frac{\epsilon^{2(\eta-\eta_w)}}{3\pi} \left(\frac{d}{r}\right)^4 \left(\frac{\partial^2 \eta}{\partial x^2} + 2p^2 + 6 \frac{\lambda_D^2}{r^2} - 6 \frac{\lambda_D}{r} p \right) \right] N(r) \\ & = \left(\frac{d}{r}\right)^2 \left[\frac{\partial^2 U}{\partial x_1^2} - 2 \frac{\lambda_D}{r} \frac{\partial U}{\partial x_1} + 2 U \left(\frac{\lambda_D}{r}\right)^2 \right] \end{aligned} \quad (7.11)$$

where f_r is defined in equation (4.11). This equation will be used as the first order density wave equation in the iterative solution.

A word about notation seems in order here. The forms of the functional coefficients of these wave equations are not changed as the independent variables are changed, although, strictly speaking, they are different functions of the new variables, and this fact should be recognized, by using different symbols, for example. The approach used is simple and probably less confusing. Such terms as d/r , d/a , λ_D/r , and λ_D/a can be written in terms of a single parameter, the debye length-sphere radius ratio defined as δ ; this description will be given later.

The zeroth order density equation is obtained from equation (7.11) by dropping the drift terms on the right hand side and retaining

only the leading terms in the coefficients (these are considered dominant), i. e.,

$$\frac{\partial^2 N(r)}{\partial x^2} - \frac{p}{3} \frac{\partial N(r)}{\partial x} + \frac{1}{3} \left[\Omega^2 - f_r \right] N(r) = 0 ; \quad (7.12)$$

it is expected that equation (7.12) will yield a good approximate solution for $N(r)$ which can be used to start the iterative solution.

7.3. The Iterative Solution

The iterative approach to the solution for n_1 is simple in concept and has been previously described in general in section 6.2, however there are a few procedural details which should be mentioned. It is clear that equation (7.12) has two linearly independent solutions of the form given in equation (6.6) satisfying the boundary conditions given in equation (6.7); the boundary conditions will be used also for solution of the first order equation (7.11) although this must be justified since equation (7.11) is inhomogeneous and has a particular solution depending on the drift terms on the right hand side in addition to the two linearly independent solutions. It is obvious that the two linearly independent solutions can be made to satisfy equation (6.7); but what of the particular solution? The mathematical model calls for uniformity at $r = d$ ($x = 0$) so that the drift terms become inconsequential in the vicinity of $x = 0$ where the boundary conditions are specified. To the extent that the drift terms on the right hand side of equation (7.11) are negligible the equation becomes homogeneous, and the particular solution vanishes in the vicinity of $x = 0$. Boundary conditions suitable for particular solution of equation (7.11) are then given by

$$N_p(0) = N_p'(0) = 0 \quad (7.13)$$

where N_p represents the particular solution to equation (7.12), and the complete solutions (as well as the homogeneous solutions) satisfy

$$\begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (7.14)$$

the same set of boundary conditions satisfied by solutions to the zeroth order equation (7.12). (Note: primes here indicate derivatives with respect to x -- here only.)

The results of the solution of the zeroth order density wave equation are substituted into the drift wave equation (7.8), and this equation is solved for its particular solution using the boundary conditions developed, in the next section, from the basic form given in section 5.5. The results of this solution are then substituted into the right hand side of the first order density wave equation (7.11) which now becomes an inhomogeneous equation in $N(r)$. Although the boundary conditions given by equation (7.14) could be used to solve equation (7.11) for the complete solution for $N(r)$, it is more convenient to solve independently for the particular and homogeneous solutions. The homogeneous solution for $N(r)$ is then

$$N_H(r) = y_1 - j\beta \lambda_D y_2 \quad (7.15)$$

in accord with the definition in equation (6.20). The first order iterated solution for $N(r)$ is obtained by addition of $N_p(r)$ and $N_H(r)$.

7.4. Boundary Conditions for the Drift Equation

The boundary conditions for equation (7.8) are obtained starting from equation (5.26); making the usual variables substitutions and noting that $u_1 \Big|_{r=a} = 0$ it becomes

$$-\frac{n_\infty u_\infty}{v_o^2} \frac{\partial u_1}{\partial r} \Big|_{r=a} = \frac{1}{6\pi} \left(\frac{d}{a}\right)^2 \left[\frac{1}{r} \frac{\partial N_1}{\partial r} - \frac{n_1}{r} \right]_{r=a}$$

$$+ \frac{1}{3\lambda_D} \left[\frac{1}{2\pi} \left(\frac{d}{a}\right)^2 p \Big|_{r=a} - j \frac{\Omega}{\sqrt{2\pi}} \right] n_1 \Big|_{r=a}$$

or

$$-\frac{n_\infty u_\infty}{v_o^2} \frac{\partial u_1}{\partial r} \Big|_{r=a} = \frac{1}{6\pi} \left(\frac{d}{a}\right)^2 \frac{\partial N_1}{\partial r} \Big|_{r=a}$$

$$+ \frac{1}{3\lambda_D} \left[\frac{1}{2\pi} \left(\frac{d}{a}\right)^2 \left(p \Big|_{r=a} - \frac{\lambda_D}{a} \right) - j \frac{\Omega}{\sqrt{2\pi}} \right] N_1 \Big|_{r=a} \quad (7.16)$$

Multiplying equation (7.16) through by $2A\lambda_D$ while noting the definition of U yields

$$\frac{\partial U}{\partial x_1} \Big|_{x_1=0} = \frac{1}{3\pi} \left(\frac{d}{a}\right)^2 \frac{\partial N_1}{\partial x_1} \Big|_{x_1=0}$$

$$+ \left[\frac{1}{3\pi} \left(\frac{d}{a}\right)^2 \left(p \Big|_{r=a} - \frac{\lambda_D}{a} \right) - j \frac{2\Omega}{3\sqrt{2\pi}} \right] N_1 \Big|_{x_1=0} \quad (7.17)$$

Completing the picture, the fact that u_1 vanishes at $r = a$ implies that

$$U \Big|_{x_1=0} = 0 . \quad (7.18)$$

Equations (7.17) and (7.18) are the desired boundary conditions for the dimensionless drift wave equation.

7.5. Reduction of the K Forms

The evaluation of the major functions derived in chapter VI depend upon the calculation of the dimensionless forms K_a , K_c , K_y , K_e , and K_p since it has been indicated that

$$A = a V \frac{\omega_p^2 \epsilon_0}{v_o^2 e} K_a$$

$$C_2 = a V K_c$$

$$Y_p = 2\pi a \epsilon_0 \omega_p K_y$$

$$G_e = \frac{1}{180} (k_{pe} D)^2 (k_{pe} a)^2 K_e$$

$$\frac{P_p}{P_e} = \frac{1}{(k_{pe} D)^2 k_{pe} a} K_p . \quad (7.19)$$

First solving for K_a , from equation (6.42)

$$\frac{A}{aV} = \left[D_1 F - B_1 N(a)a + \frac{e}{\epsilon_0} \frac{H}{\beta^2} \right]^{-1}$$

where D_1 and B_1 are defined in equation (5.21) while F and H are given in equations (6.38) and (6.41). Substitution for these terms leads to

$$\frac{A}{aV} = \left[\frac{v_o^2 e}{\omega_p^2 \epsilon_o} \sigma_1 (N(a) - N'(a)a) + \frac{v_o^2 e}{\omega_p^2 \epsilon_o} \frac{1}{\lambda_D} \sigma_2 N(a)a + \frac{e}{\epsilon_o \beta^2} (j\beta a + \beta^2 a \int_d^a N(r) dr) \right]^{-1} \quad (7.20)$$

where

$$\sigma_1 \equiv \epsilon^{\eta_w} \left[1 - \frac{1}{6\pi} \left(\frac{d}{a} \right)^4 \right]$$

and

$$\sigma_2 = \frac{\epsilon^{\eta_w}}{3} \left[p \Big|_{r=a} - \frac{1}{\pi} \frac{\lambda_D}{a} \left(\frac{d}{a} \right)^4 + j \frac{\Omega}{\sqrt{2\pi}} \left(\frac{d}{a} \right)^2 \right].$$

Factoring common terms and writing the derivative and the integral in terms of x

$$\frac{A}{aV} = \frac{\omega_p^2 \epsilon_o}{v_o^2 e} \left[\sigma_1 \left(N(a) + \frac{a}{\lambda_D} \frac{\partial}{\partial x} N(a) \right) + \frac{a}{\lambda_D} \sigma_2 N(a) + \frac{a}{\lambda_D} \frac{1}{\Omega^2 - 1} (j\beta \lambda_D - \beta^2 \lambda_D^2) \int_0^{10} N(r) dx \right]^{-1} \quad (7.21)$$

If the δ parameter is defined as

$$\delta \equiv \frac{\lambda_D}{a} \quad (7.22)$$

then factoring δ from equation (7.21) K_a is determined as

$$K_a = \delta \left[\sigma_1 \left(\delta N(a) + \frac{\partial}{\partial X} N(a) \right) + \sigma_2 N(a) + j \frac{1}{\sqrt{3} \sqrt{\Omega^2 - 1}} - \frac{1}{3} \int_0^{10} N(r) dx \right]^{-1} \quad (7.23)$$

It should be noted that, as mentioned in chapter VI, all of the forms derived are valid for $\Omega > 1$; for $\Omega < 1$ the same forms are valid where applicable if $j\beta$ is replaced by $b \equiv |\beta|$ or equivalently if $j\sqrt{\Omega^2 - 1}$ is replaced by $\sqrt{1 - \Omega^2}$.

In equation (6.43), C_2 is given as

$$C_2 = aV \frac{D_1 F + \frac{eE}{\epsilon_0 \beta^2} - B_1 N(a) a}{D_1 F + \frac{eH}{\epsilon_0 \beta^2} - B_1 N(a) a} \quad .$$

Substitution from equations (5.21), (6.37), (6.38), and (6.41) yields the following:

$$\begin{aligned} & \frac{C_2}{aV} \\ &= \frac{\sigma_1 (N(a) - aN'(a)) + \sigma_2 \frac{a}{\lambda_D} N(a) + \frac{1}{\Omega^2 - 1} [1 + j\beta d + \beta^2 a \int_d^a N(r) dr - \beta^2 \int_d^a \int_d^r N(r) dr dr]}{\sigma_1 (N(a) - aN'(a)) + \sigma_2 \frac{a}{\lambda_D} N(a) + \frac{1}{\Omega^2 - 1} \frac{a}{\lambda_D} [j\beta \lambda_D + \beta^2 \lambda_D \int_d^a N(r) dr]} \end{aligned} \quad (7.24)$$

$$\begin{aligned}
& K_c \\
& \sigma_1 \left(\delta N(a) + \frac{\partial N(a)}{\partial x} \right) + \sigma_2 N(a) + \frac{\delta}{\Omega^2 - 1} + j \frac{1}{\sqrt{3}\sqrt{\Omega^2 - 1}} - \frac{1}{3} \int_0^{10} N(r) dx - \frac{\delta}{3} \int_0^{10} \int_0^{10} N(r) dx dx \\
& = \frac{\sigma_1 \left(\delta N(a) + \frac{\partial N(a)}{\partial x} \right) + \sigma_2 N(a) + j \frac{1}{\sqrt{3}\sqrt{\Omega^2 - 1}} - \frac{1}{3} \int_0^{10} N(r) dx}{\sigma_1 \left(\delta N(a) + \frac{\partial N(a)}{\partial x} \right) + \sigma_2 N(a) + j \frac{1}{\sqrt{3}\sqrt{\Omega^2 - 1}} - \frac{1}{3} \int_0^{10} N(r) dx}
\end{aligned} \quad (7.25)$$

K_y can be determined from equation (6.82) which gives Y_p in the form

$$Y_p = 2\pi a \frac{e \sqrt{\frac{KT_e}{2\pi m_e}} \left(\frac{d}{a} \right)^2 N(a) a + j \omega \epsilon_0 (D_1 F - B_1 N(a) a)}{D_1 F + \frac{eH}{\epsilon_0 \beta^2} - B_1 N(a) a}$$

Substituting for D_1 , F , B_1 , and H

$$\begin{aligned}
Y_p &= 2\pi a \\
& e \sqrt{\frac{KT_e}{2\pi m_e}} \left(\frac{d}{a} \right)^2 N(a) a + j \omega \epsilon_0 \left[\frac{v_o^2 e}{\omega_p^2 \epsilon_0} \sigma_1 (N(a) - a N'(a)) + \frac{e v_o^2}{\epsilon_0 \omega_p^2} \frac{\sigma_2}{\lambda_D} N(a) a \right] \\
& \cdot \frac{\frac{v_o^2 e}{\omega_p^2 \epsilon_0} \sigma_1 (N(a) - a N'(a)) + \frac{e v_o^2}{\epsilon_0 \omega_p^2} \frac{\sigma_2}{\lambda_D} N(a) a + \frac{e}{\epsilon_0 \beta^2} (j \beta a + \beta^2 a) \int_d^a N(r) dr}{\frac{v_o^2 e}{\omega_p^2 \epsilon_0} \sigma_1 (N(a) - a N'(a)) + \frac{e v_o^2}{\epsilon_0 \omega_p^2} \frac{\sigma_2}{\lambda_D} N(a) a + \frac{e}{\epsilon_0 \beta^2} (j \beta a + \beta^2 a) \int_d^a N(r) dr}
\end{aligned} \quad (7.26)$$

Further reduction, proceeding as before, leads to

$$\begin{aligned}
Y_p &= 2\pi a \\
& \frac{\epsilon_0 \omega}{3\sqrt{2\pi}} \left(\frac{d}{a} \right)^2 N(a) + j \omega \epsilon_0 \left[\sigma_1 \left(\delta N(a) + \frac{\partial N(a)}{\partial x} \right) + \sigma_2 N(a) \right] \\
& \cdot \frac{\sigma_1 \left(\delta N(a) + \frac{\partial N(a)}{\partial x} \right) + \sigma_2 N(a) + j \frac{1}{\sqrt{3}\sqrt{\Omega^2 - 1}} - \frac{1}{3} \int_0^{10} N(r) dx}{\sigma_1 \left(\delta N(a) + \frac{\partial N(a)}{\partial x} \right) + \sigma_2 N(a) + j \frac{1}{\sqrt{3}\sqrt{\Omega^2 - 1}} - \frac{1}{3} \int_0^{10} N(r) dx}
\end{aligned} \quad (7.27)$$

from which K_y becomes

$$K_y = \frac{\frac{1}{3\sqrt{2\pi}} \left(\frac{d}{a}\right)^2 N(a) + j \Omega [\sigma_1 (\delta N(a) + \frac{\partial}{\partial x} N(a)) + \sigma_2 N(a)]}{\sigma_1 (\delta N(a) + \frac{\partial}{\partial x} N(a)) + \sigma_2 N(a) + j \frac{1}{\sqrt{3}\sqrt{\Omega^2 - 1}} - \frac{1}{3} \int_0^{10} N(r) dx} \quad (7.28)$$

Next K_e is determined from equation (6.73) where it was defined as

$$K_e = \frac{1}{\Omega} (\Omega^2 - 1)^{5/2} |K_c|^2$$

K_e is determined once K_c is obtained.

K_p is determined from K_y and K_c as indicated in equation (6.88),

i. e.,

$$K_p = \frac{3\Omega R_e [K_y]}{(\Omega^2 - 1)^{5/2} |K_c|^2}$$

Earlier reduction of $\frac{d}{r}$, $\frac{\lambda_D}{r}$, and $\frac{d}{a}$ was promised. These terms can be written in terms of the parameter δ defined in equation (7.22) noting that the sheath thickness has been taken as a constant 10 debye lengths, i. e.,

$$d - a = 10 \lambda_D$$

Dividing by a yields

$$\frac{d}{a} = 1 + 10 \delta \quad (7.29)$$

The factor $\frac{\lambda_D}{r}$ can be written

$$\frac{\lambda_D}{r} = \frac{1}{x_1 + \frac{1}{\delta}} \quad (7.30)$$

or

$$\frac{\lambda_D}{r} = \frac{1}{10 + \frac{1}{\delta} - x} \quad (7.31)$$

since, as it is easily shown,

$$x = 10 - x_1 = -x_2 \quad (7.32)$$

Finally $\frac{d}{r}$ can be reduced to

$$\frac{d}{r} = \frac{10 + \frac{1}{\delta}}{x_1 + \frac{1}{\delta}} \quad (7.33)$$

7.6. Comparison of Degenerate Forms with Fejer's Results

It might be noticed that the major differences between this analysis and that of Fejer stem from the inclusion of the plasma sheath. At first glance there appears to be little similarity in the results obtained, or in the analytical forms of Y_p , P_e , $n(a)$, etc. obtained in this chapter and those of Fejer. Fejer ignores the existence of an unperturbed electric field, an electron drift velocity, and of the static potential of the spheres; these terms effect the results of this theory to a great extent. If the analytical forms for Y_p , P_e , $n(a)$, etc. (i.e., equations (6.82), (6.68), (6.92), and (7.23)), are modified by the deletion of the terms which Fejer ignores it is easily shown that the forms become identical to those derived by Fejer. To accomplish this modification, first let the sheath parameters, which Fejer ignores, take on their uniform plasma values, i.e., let

$$u_0 = \eta \equiv 0$$

and

$$n_o = n_\infty \quad (7.34)$$

in the sheath region; also let the static potential of each sphere vanish.

Second let $d \rightarrow a$, i. e., set d equal to a so that

$$x_1(d) = \frac{d - a}{\lambda_D} = 0 \quad (7.35)$$

This is equivalent to shrinking the sheath to zero which is consistent with Fejer's uniform plasma assumption. As a result of this shrinking process all of the integrals in the equations mentioned vanish and the following degeneration occurs.

$$N(a) \rightarrow 1$$

$$\frac{\partial N(a)}{\partial x} \rightarrow j \beta \lambda_D = j \frac{1}{\sqrt{3}} \sqrt{\Omega^2 - 1}$$

$$\sigma_1 \rightarrow 1$$

$$\sigma_2 \rightarrow 0$$

$$\sigma_3 \rightarrow 1$$

$$\int_d^a \rightarrow 0$$

$$\int_d^a \int_d^x \rightarrow 0$$

$$\frac{\eta_w}{\epsilon} \rightarrow 1 \quad (7.36)$$

If a parameter δ_f is defined by

$$\delta_f \equiv \frac{1}{\sqrt{3} \delta} \quad (7.37)$$

and if substitutions are made using equations (7.36) and (7.37) into the expressions for Y_p , P_e , $n(a)$, etc., it is seen that all of these expressions go over exactly to those developed by Fejer if it is noted that δ_f has the same definition as Fejer's δ parameter.

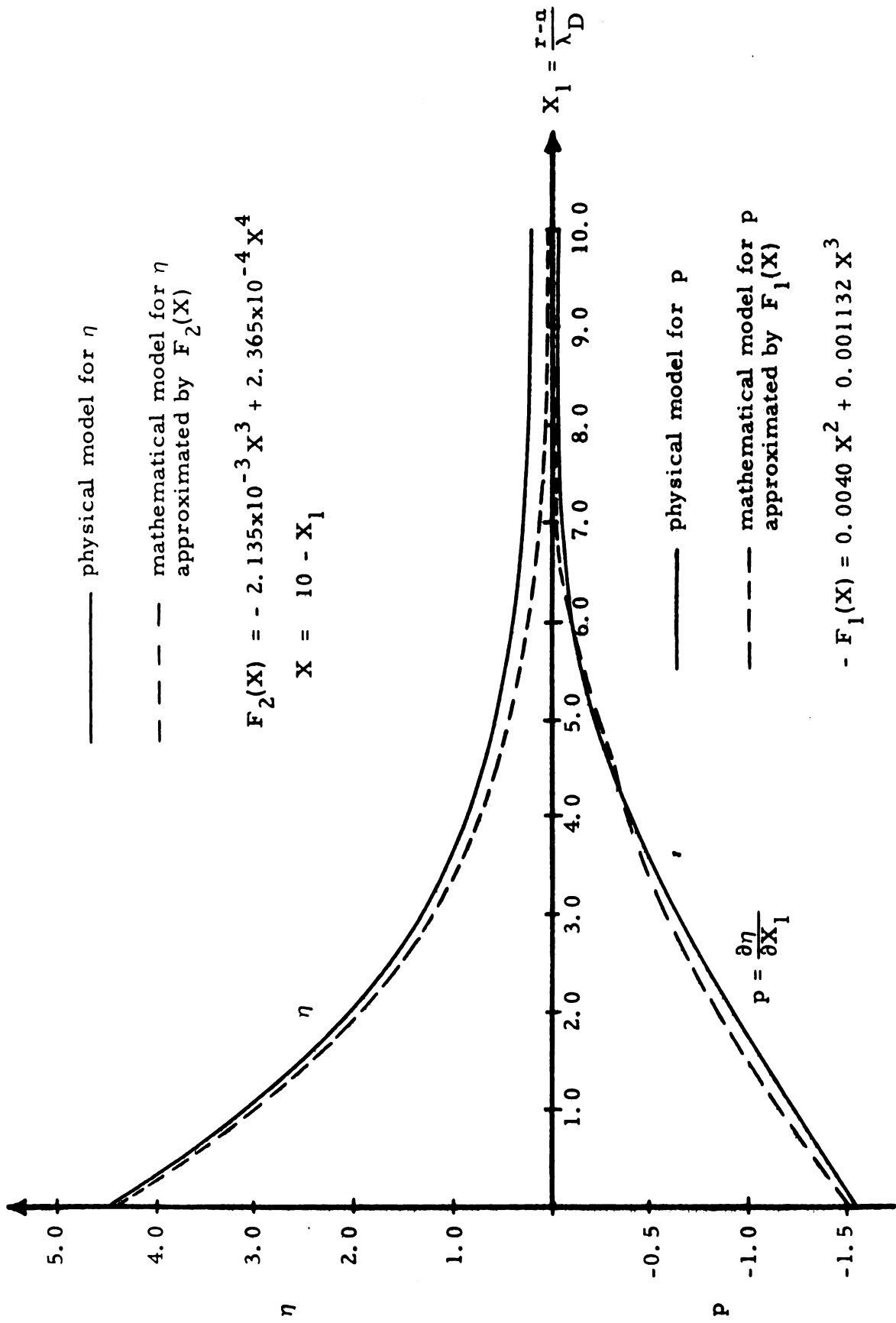
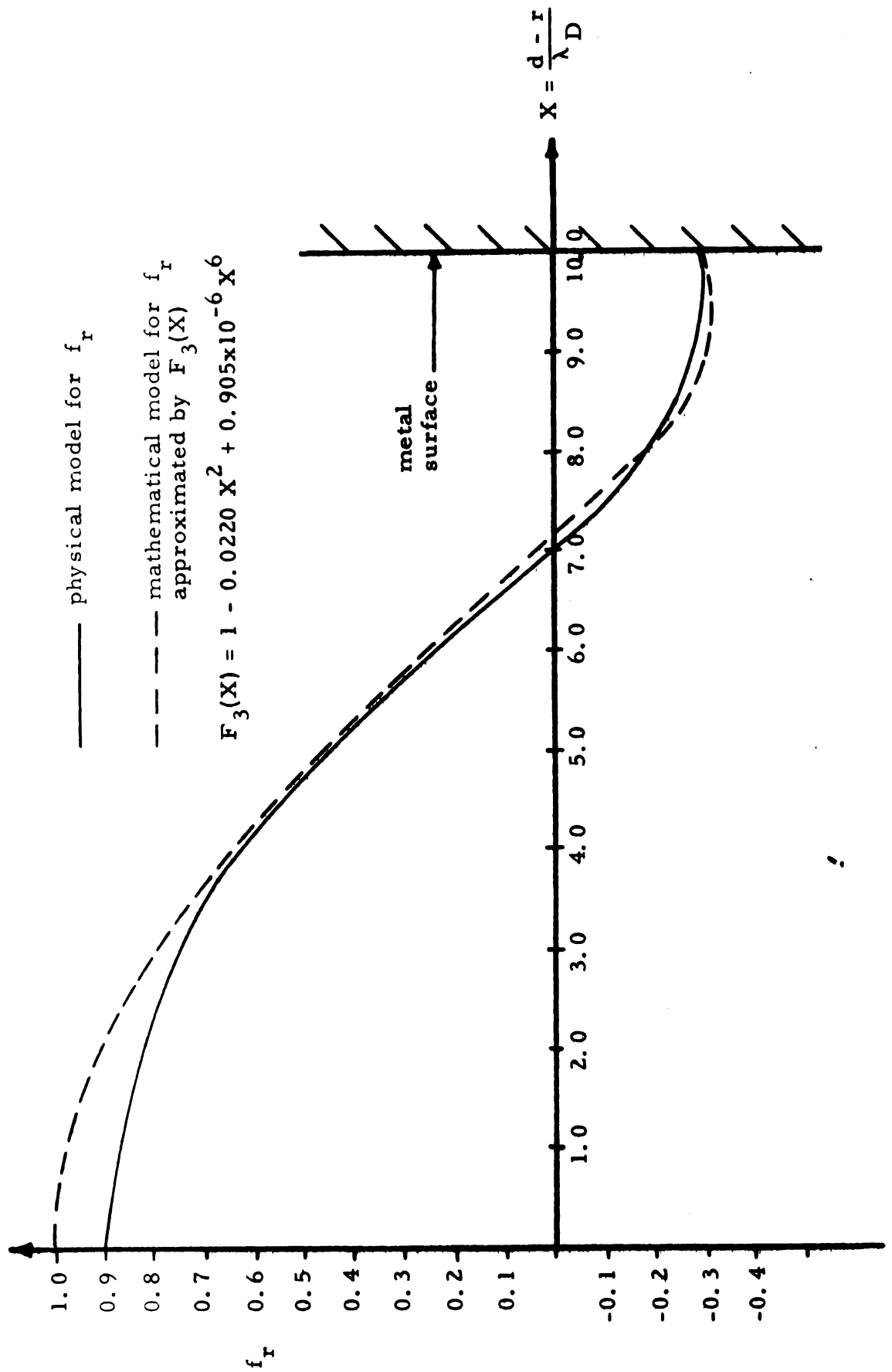


Figure 7.1 Physical and Mathematical Models for η and p

Figure 7.2. Physical and Mathematical Models for f_r .

CHAPTER VIII

NUMERICAL RESULTS

In sections 6.2 and 7.3 the iterative solution of equations (7.8) and (7.11) for the normalized versions of n_1 and u_1 has been described. The first step in the procedure is to solve the simplified homogeneous equation (7.12) for its two linearly independent solutions y_1 and y_2 . Examples of these solutions are plotted for the case of $\delta = 0$ in Figures 8.8 and 8.9. They exhibit the expected decaying oscillatory nature in that portion of the interval $[0 \leq x \leq 10]$ where ω exceeds the effective local plasma frequency $\omega_p \sqrt{f_r}$; on the rest of that interval they experience a growth with increasing X . This character is expected because if f_r and p are replaced by their "uniform plasma" values (i.e., 1.0 and 0, respectively) equation (7.12) becomes

$$\frac{\partial^2 N(r)}{\partial X^2} + \beta^2 N(r) = 0 \quad (8.1)$$

which has solutions of the form of sines and cosines if β^2 is positive and of the form of hyperbolic sines and cosines if β^2 is negative; the solutions to equation (7.12) bare a strong resemblance to these functions in the related regions. The increase of oscillation rate with X is due to the decrease of f_r with X ; the damping of the oscillatory solutions is due to the presence of the first derivative term involving p . If $\omega > \omega_p$ then y_1 and y_2 with the imposed boundary conditions given by equation (7.14) should approximate cosine and negative sine solutions, respectively,

which they appear to do.

The solutions for $N(r)$ and its appropriate derivatives and integrals are substituted into the K relations given in chapter VII; the results are plotted or tabulated in this section and represent the results of this study in that all of the quantities of interest are related by constants to the K 's. Y_p is related to K_y by equation (6.83), and K_y is defined by equation (7.28). Zeroth order solutions for K_y are plotted in Figure 8.1 for $\delta = 0, 0.01$, and 0.05 .

A geometrical limitation has been placed on this analysis and the results presented herein by the use of a fixed sheath configuration. Reference to the work of Bernstein and Rabinowitz¹² indicates that sheath thickness and potential profile change negligibly on the range $\delta = 0$ to $\delta = 0.05$ but for larger δ values the sheath shrinks. Extrapolation from their work indicates that the appropriate sheath thicknesses for this study were approximately $9 \lambda_D$ for $\delta = 0.10$ and $5 \lambda_D$ for $\delta = 0.50$. The sheath was shortened by changing the variable X to allow the mathematical sheath potential model to assume its terminal value at $X = 9$ and $X = 5$, respectively. Values of the zeroth order solution for K_z determined by this process are tabulated in Tables 8.7 and 8.8.

It is seen that the imaginary portion of K_y is a very weak function of δ (at least over the range of δ values considered); further, with the exception of a zero at $\Omega = 1$, it corresponds fairly closely to a capacitor of capacitance $2\pi\epsilon_0 a$ which would be the capacitance of the antenna if only the capacitance of each sphere

with respect to infinity were considered with the sphere to sphere capacitance ignored and with a free space medium. The real part of K_y corresponds to EA wave propagation since no loss mechanism was included in the "quasi-static" analysis from which it was derived; it is negligible with respect to the imaginary portion for Ω greater than about 1.5 regardless of δ but increases with decreasing Ω as $\Omega = 1$ is approached from above until it reaches a large sharp peak very near $\Omega = 1$ beyond which it drops rapidly to zero at $\Omega = 1$; it vanishes identically for Ω less than 1.0 since EA propagation in the surrounding plasma is not possible. Figure 8.1 would tend to indicate a general increase in $R_e K_y$ with δ for a given Ω although when the sheath model is appropriately shortened to accommodate larger δ 's it is found that somewhere between $\delta = 0.05$ and 0.10 a maximum is achieved and $R_e K_y$ decreases monotonically with δ thereafter.

Figure 8.2 is a plot of the zeroth order solution for K_z ; it is simply the reciprocal of K_y , but it is plotted to facilitate comparison with other analytical and experimental results. The results of Fejer⁶ for the case of $\delta = 0.01$ are plotted in the same figure for $\Omega > 1$ for comparison purposes. It might be noted that Fejer's result is generally of much greater magnitude although it has similar trends and has the same limiting values as $\Omega \rightarrow +\infty$ and as $\Omega \rightarrow 1^+$. As in the case of K_y , $R_e K_z$ increases with δ , for fixed Ω , until a peak is reached between $\delta = 0.05$ and 0.10 thereafter decreasing with δ .

The form $|N(a) K_a|$ is proportional to the density perturbation at the sphere's surface, i.e., $n_1(a)$; the specific relation to $n_1(a)$ is given in equation (6.91); the zeroth order solution for this quantity is plotted in Figure 8.3. It can be seen that there are three major peaks in this term. The first peak occurs between $\Omega = 0$ and $\Omega = 1$, its amplitude increasing and its position migrating upward with increasing δ ; the second peak occurs very near but below $\Omega = 1$; the third occurs very near but above $\Omega = 1$. The second and third peaks appear to be essentially part of the same local maximum with the very sharp notch caused by a zero at $\Omega = 1$ (it is noted that if a loss mechanism such as collision damping were considered this zero would not occur; there would most likely appear a more shallow but wider notch). These multiple peaks contrast with Fejer's results; his analysis yields a single pole between $\Omega = 0$ and $\Omega = 1$ although his solutions do increase in magnitude with δ and his pole does migrate upward with increasing δ , and he does show a zero at $\Omega = 1$; there are then several major similarities in the two sets of results. Between the first and second peaks there is a local minimum of very small magnitude; it appears not to migrate appreciably with δ . Finally it is noted that $|N(a) K_a| = 0$ for all Ω if $\delta = 0$; this is true also in Fejer's analysis.

K_a is defined by equation (7.23); zeroth order solutions for K_a are plotted in Figure 8.4. K_a is identically zero for all Ω if $\delta = 0$. It is seen that K_a increases in magnitude with increasing δ ; both the real and imaginary parts approximate slowly decaying sinusoids for $\Omega > 1$ except near $\Omega = 1$ where each assumes

relatively large and sharp peaks before going to zero at $\Omega = 1$.

The value of the far field static potential is related to the parameter C_2 by equation (6.58); C_2 , in turn, is related to K_c by equation (6.72); K_c is defined by equation (6.71) and more explicitly by equation (7.25). Plots of zeroth order K_c are given in Figure 8.5. It is seen that

$$K_c = 1 + j0 \quad (8.2)$$

for $\delta = 0$; it assumes this value very nearly for all δ except very near $\Omega = 1$ where a minor deviation can be noted.

The radiation conductance term G_e is related to K_e by equations (6.73); K_e is defined in terms of K_c also in equations (6.73) as

$$K_e = \frac{1}{\Omega} (\Omega^2 - 1)^{5/2} |K_c|^2.$$

Zeroth order solutions for K_e are given in Figure 8.6. Since K_c is very nearly equal to $1 + j0$ except near $\Omega = 1$ where its deviation is not great and since in this region of deviation $(\Omega^2 - 1)$ is very small, K_e can be represented to a good approximation by

$$K_e \doteq \frac{1}{\Omega} (\Omega^2 - 1)^{5/2} \quad (8.3)$$

for all δ .

The ratio of power radiated in electroacoustic form to that radiated in electromagnetic form P_r/P_e is related to K_p by equation (7.19); K_p is defined by equation (6.88). Zeroth order solutions for K_p are plotted in Figure 8.7. K_p appears to be a weak function

of δ with greatest dependence near $\Omega = 1$; it increases without bound as $\Omega = 1$ is approached from above and decreases monotonically with Ω for all $\Omega > 1$. At $\Omega = 1.1$, P_p is many times larger than P_e for all antenna geometries satisfying the "short antenna" requirement imposed by the adopted model, but for $\Omega > 3.0$ the situation is reversed for all antenna dimensions except the most minute; the point of power equality falls at the value of Ω where

$$K_p = (k_{pe} D)^2 k_{pe} a. \quad (8.4)$$

If, for example, the antenna geometry is such that

$$\begin{aligned} k_{pe} D &= 0.5 \\ k_{pe} a &= 0.1 \end{aligned} \quad (8.5)$$

then at the point of equality $K_p = 0.025$ which corresponds approximately to $\Omega = 1.35$ essentially independent of δ .

In order that greater accuracy may be conveyed than that possible with curves K_z , K_a , $|N(a) K_a|$, and K_p are tabulated for $\delta = 0.01$ in Tables 8.1 through 8.4.

In order that a quantitative picture of the effect of the plasma on the input admittance may be obtained values of Y_p and G_e were calculated for $\delta = 0.01$ and the dimensions given in equations (8.5); the results are tabulated in Table 8.5. In this case it is easily shown that

$$\begin{aligned} G_e &= 1.39 \times 10^{-5} K_e \\ P_p/P_e &= 40 K_p \\ Y_p &= 1.667 \times 10^{-3} K_y. \end{aligned} \quad (8.6)$$

For the same case P_p/P_e was calculated and tabulated in Table 8.6. The total input admittance is

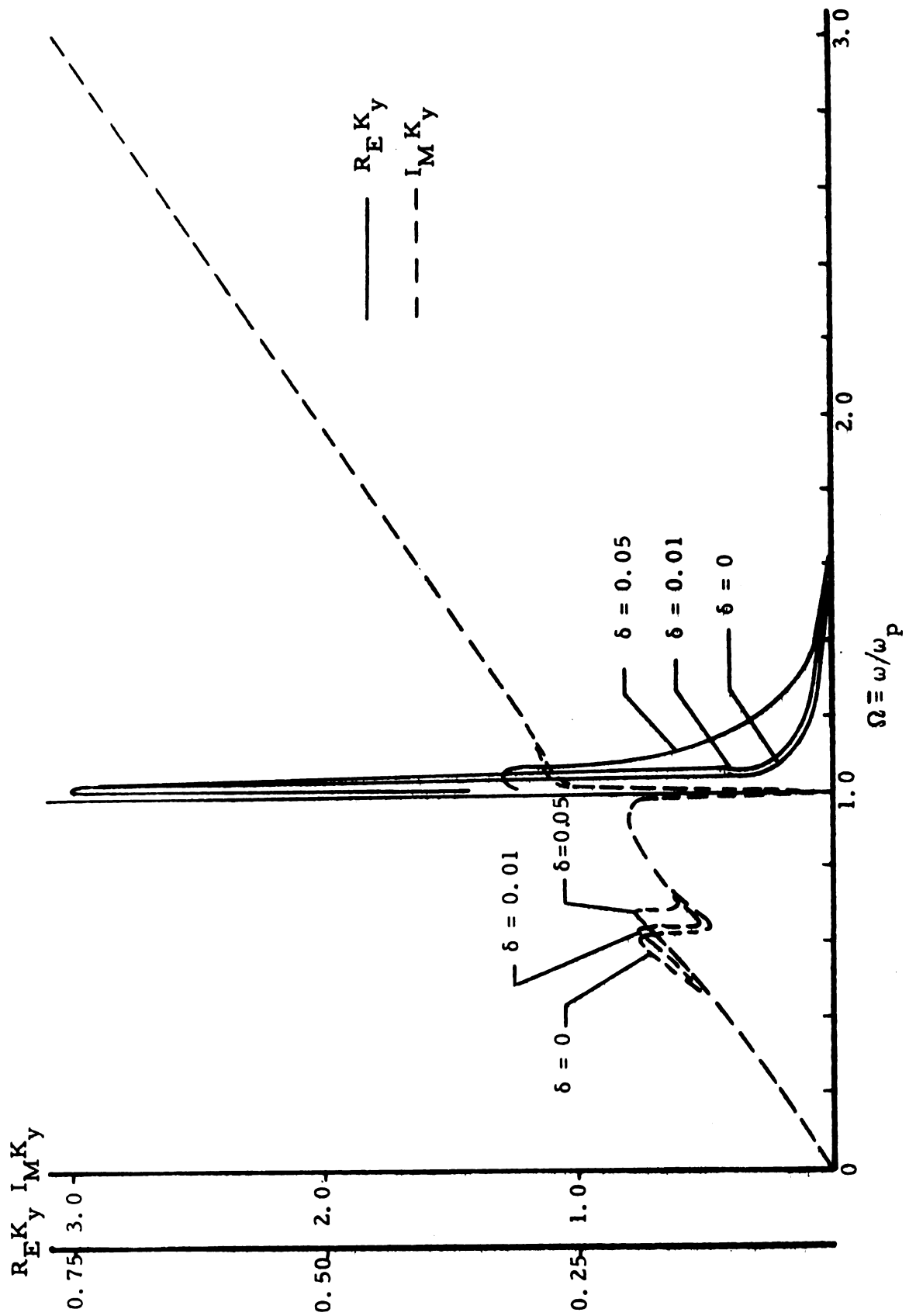
$$Y_T \equiv G_e + Y_p \quad (8.7)$$

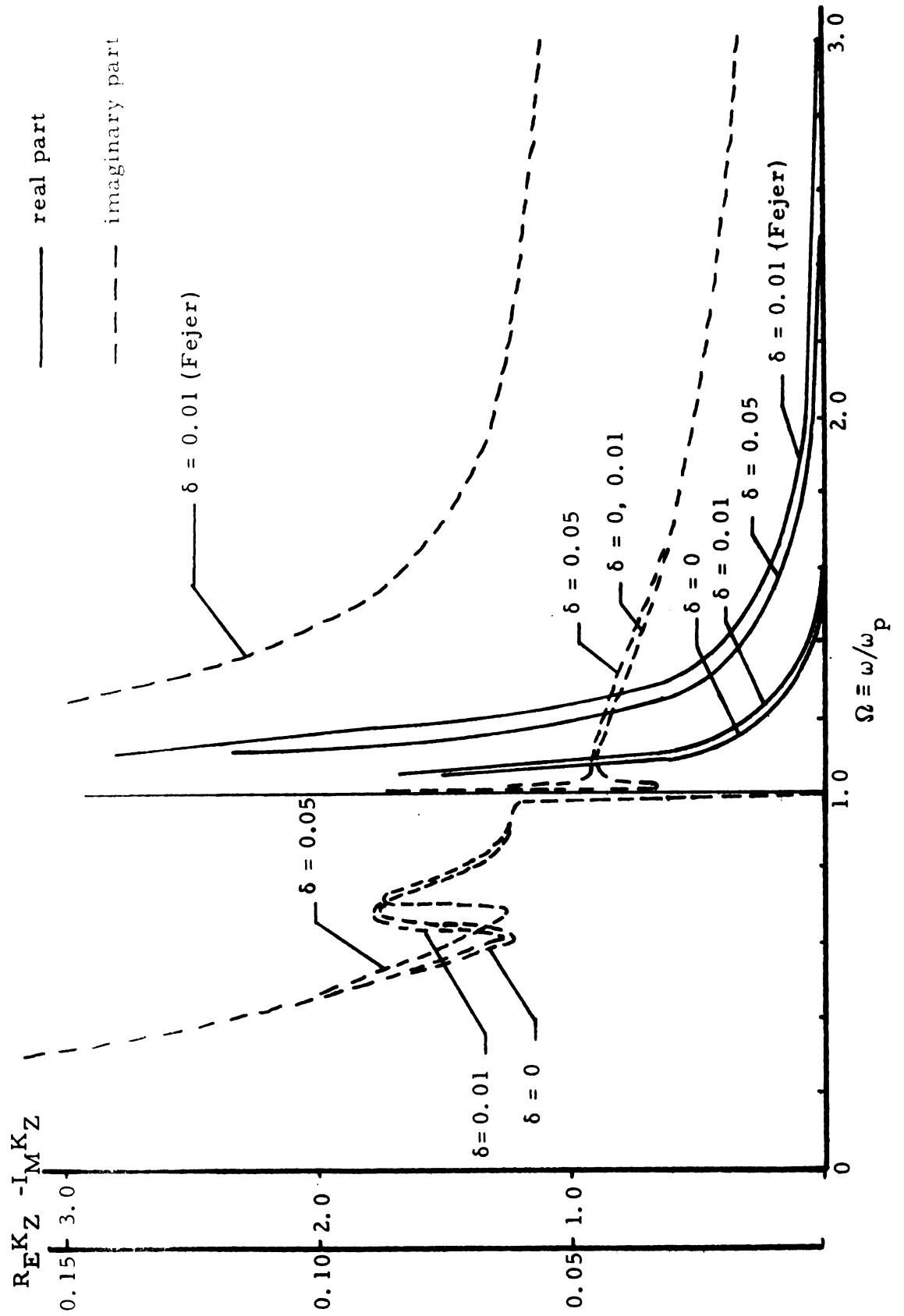
so that

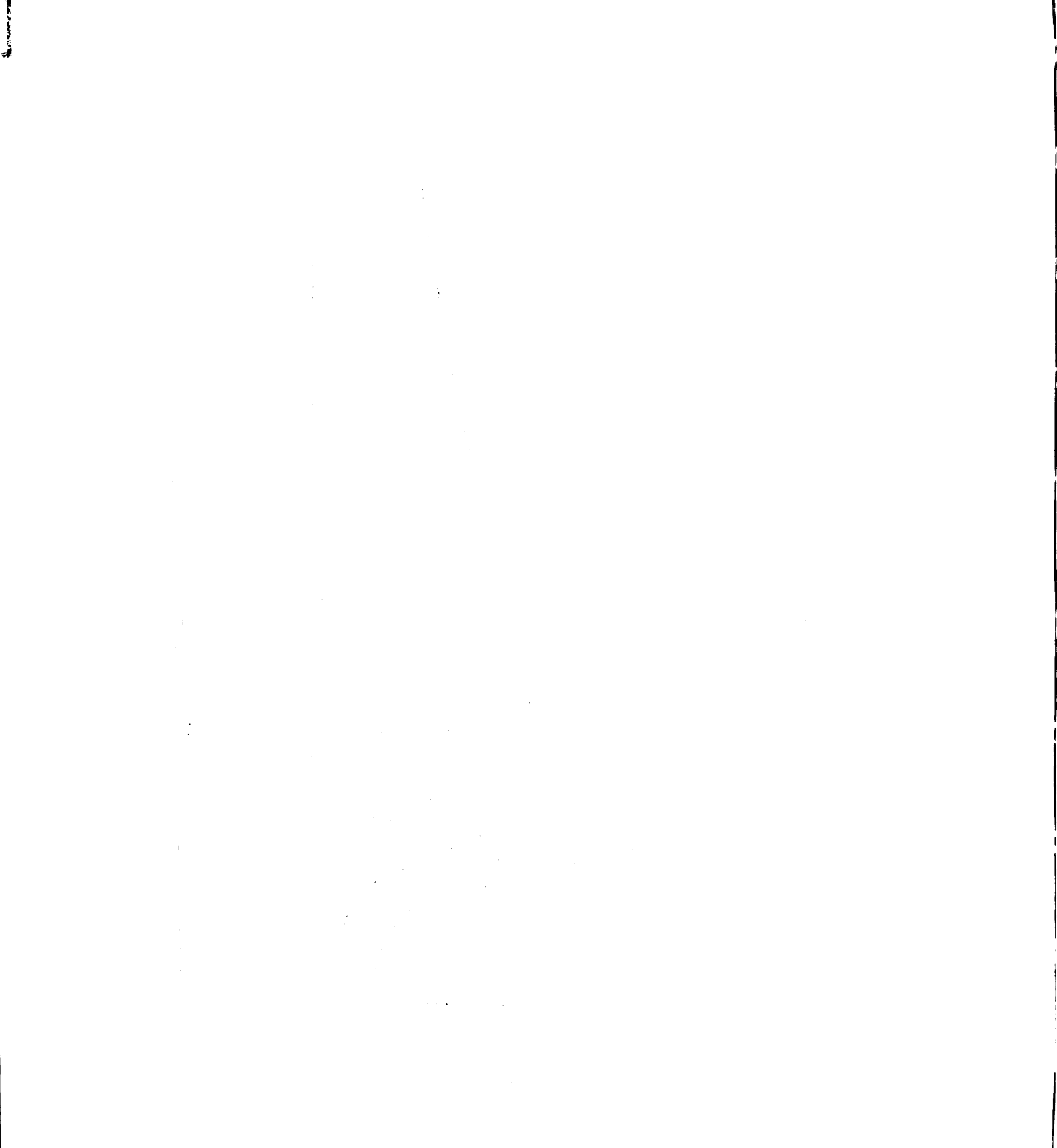
$$G_T = G_e + R_e Y_p. \quad (8.8)$$

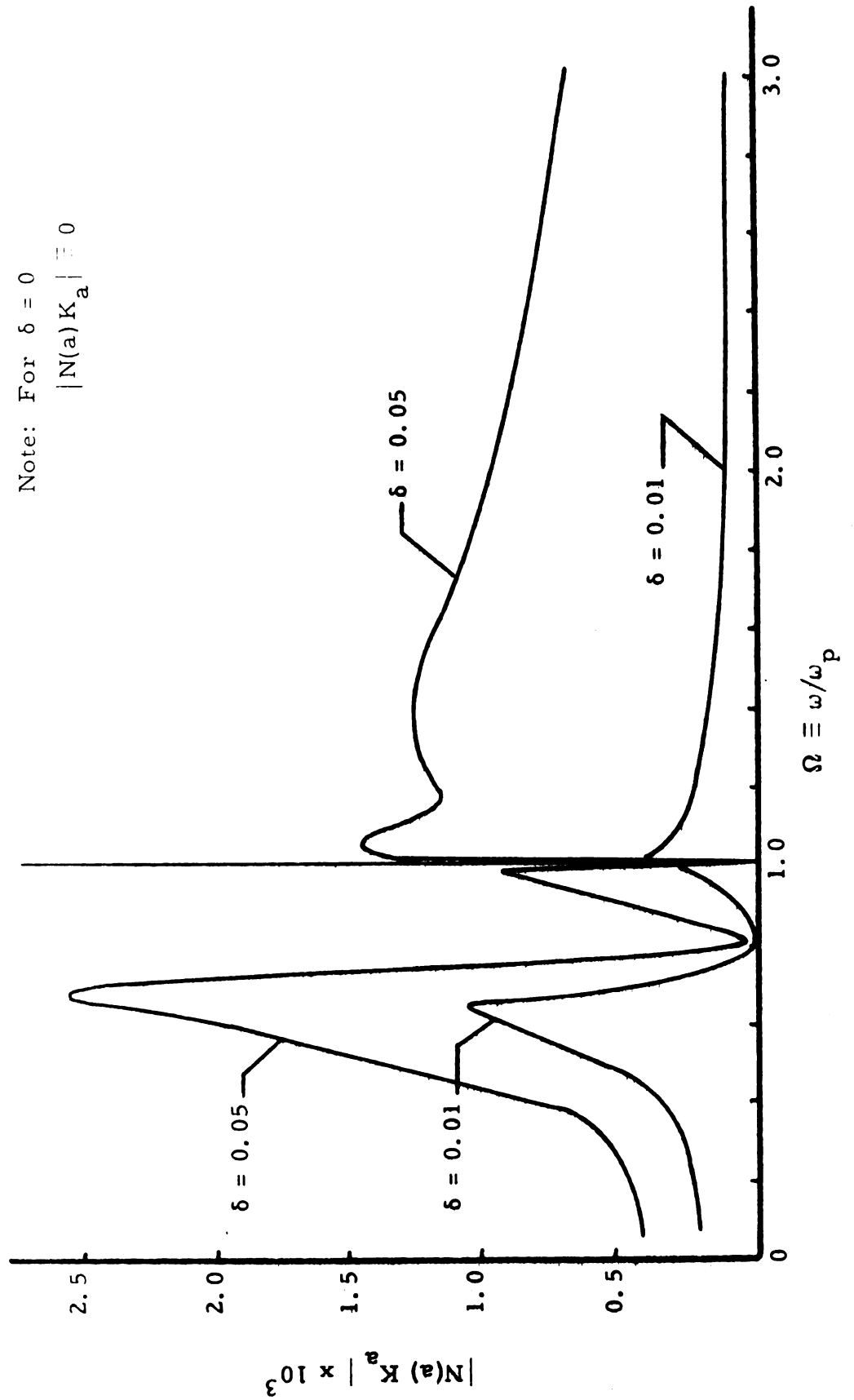
Zeroth order solutions for G_T are also tabulated in Table 8.6 for $\delta = 0.01$ and the dimensions of equations (8.5). It is seen that G_T takes on relatively large values near $\Omega = 1$ (in fact, G_T increases without bound as $\Omega = 1$ is approached from above) due to the excessive amount of EA power radiated there (EM power is negligible here). At first ($\Omega > 1$) G_T decreases with Ω as the dominant EA power decreases; as the increasing EA power becomes appreciable a minimum is approached near $\Omega = 1.4$, and beyond this point G_T increases along with the then dominant EM power.

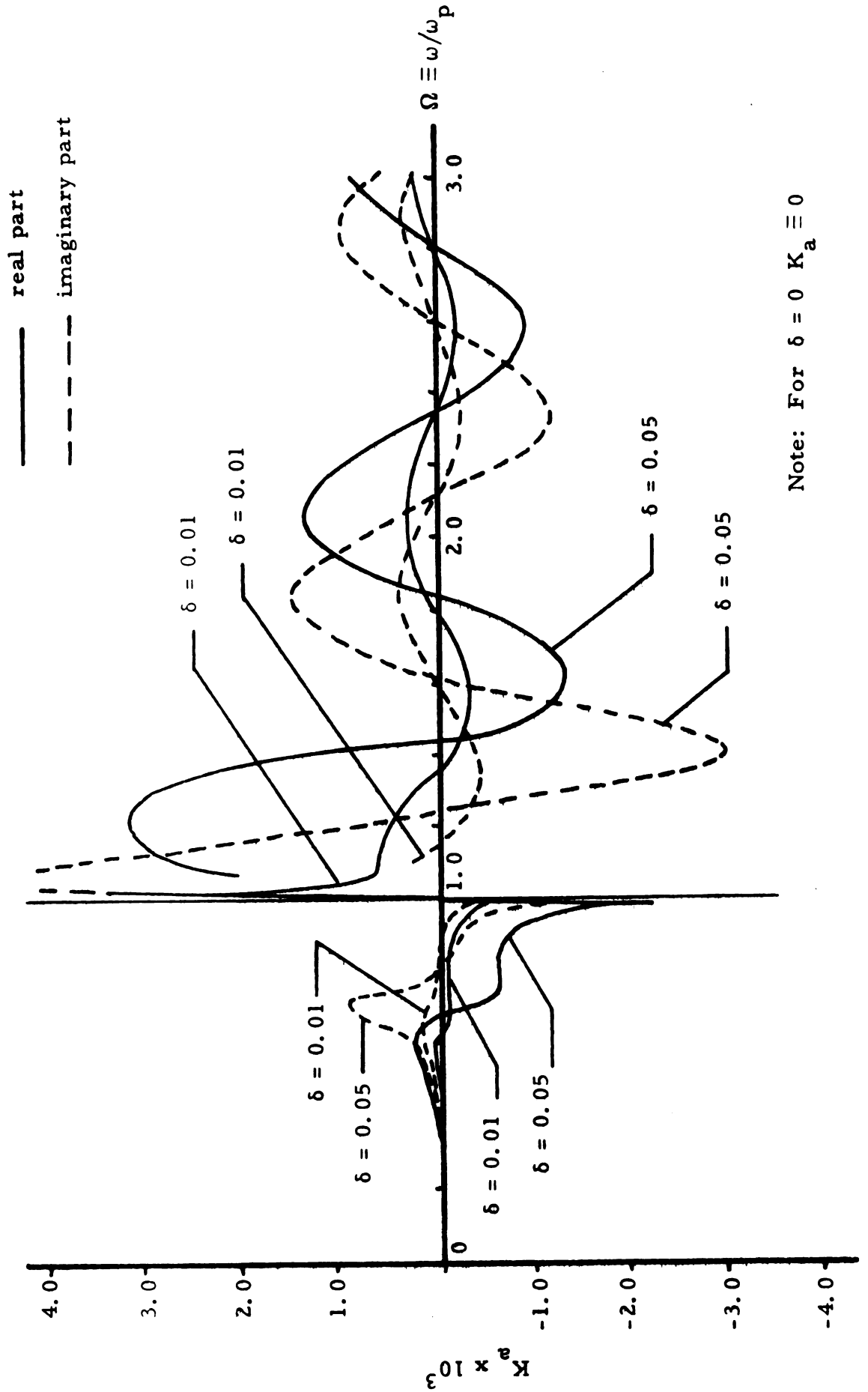
For reasons of economy in the use of computer time, first order iterative solutions were sparingly computed. First order solutions for K_z , K_a , $|N(a) K_a|$, and K_p for $\delta = 0.01$ are tabulated in Tables 8.9 through 8.12.

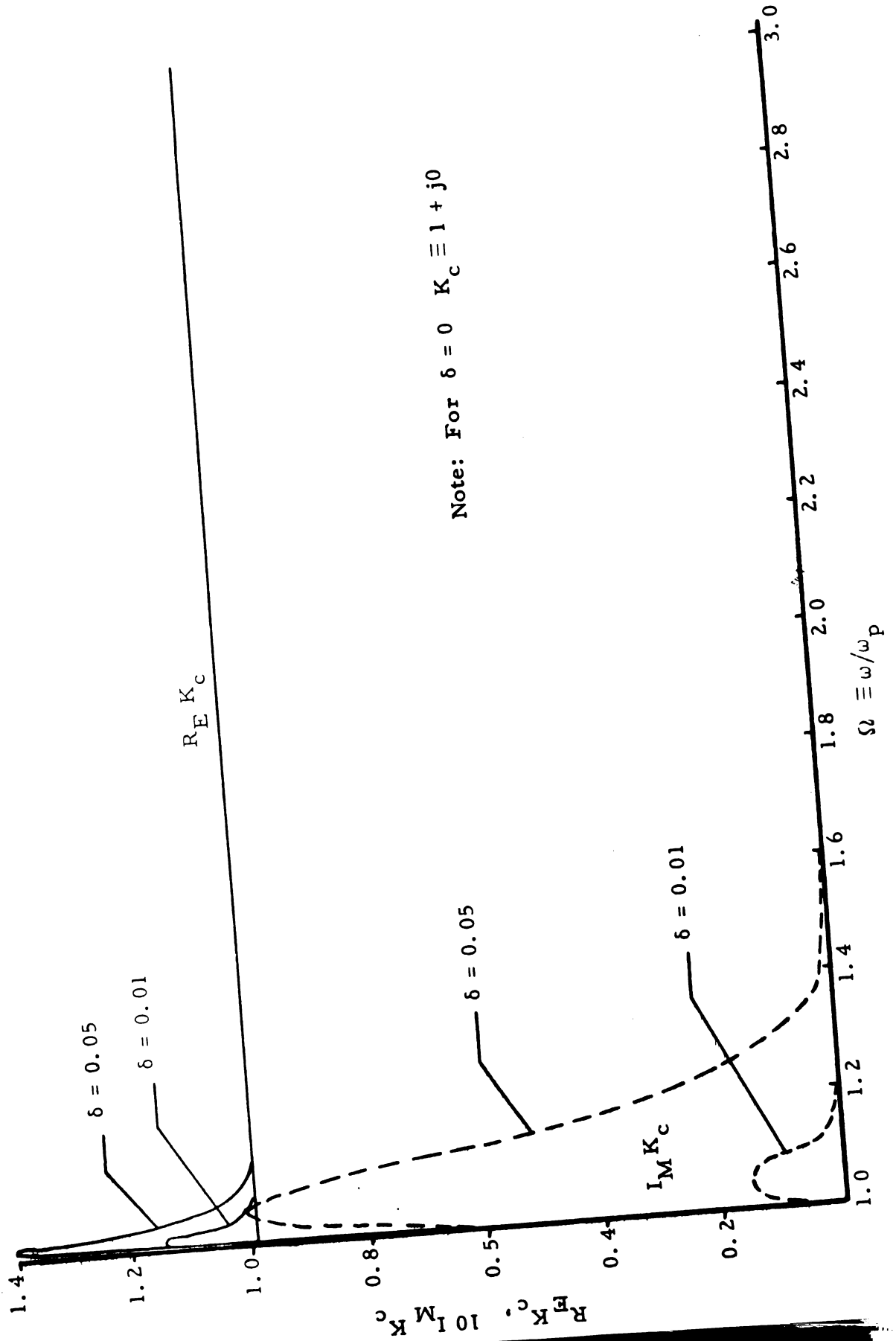
Figure 8.1. Zeroth Order K_y .

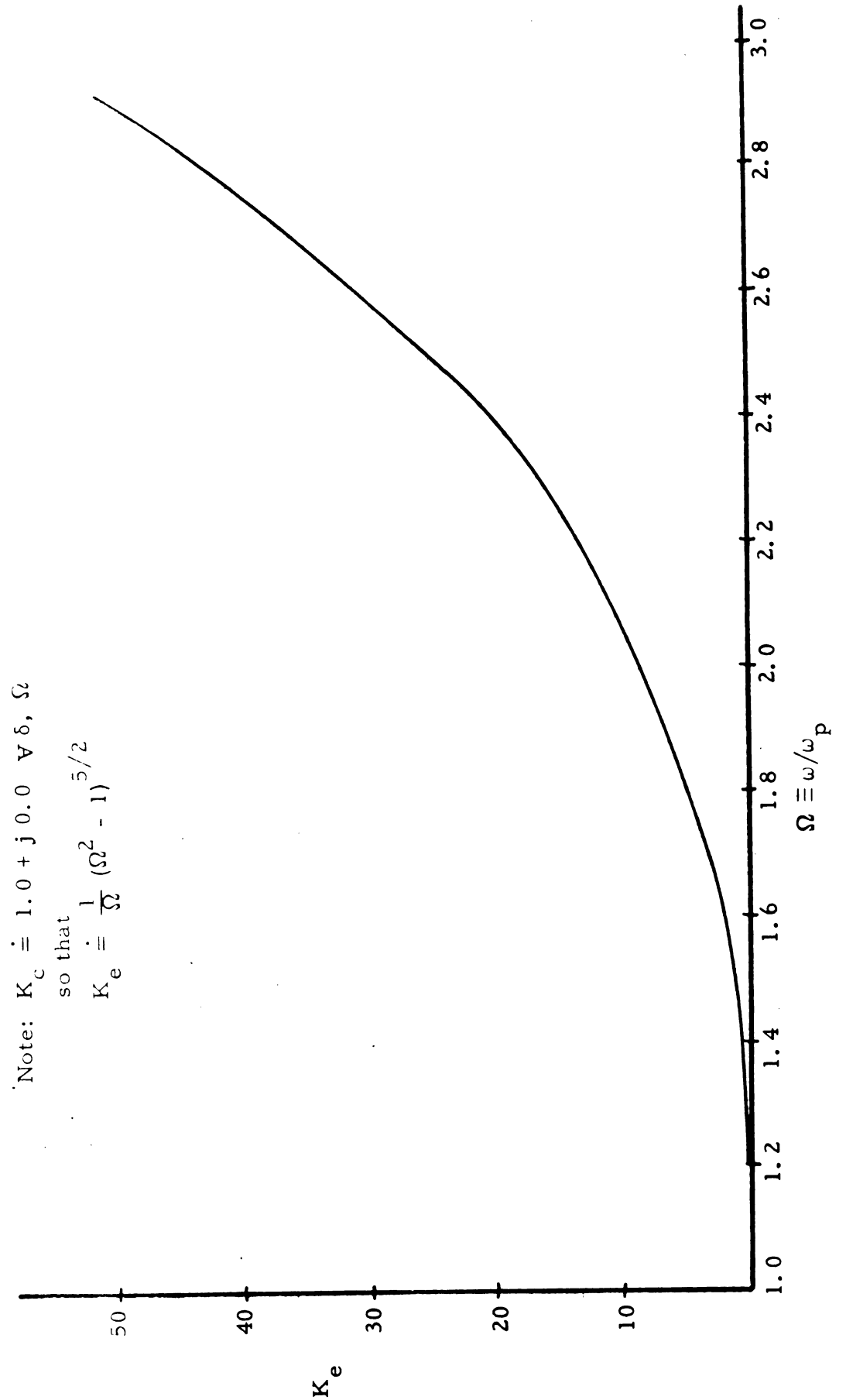
Figure 8.2. Zeroth Order K_z .

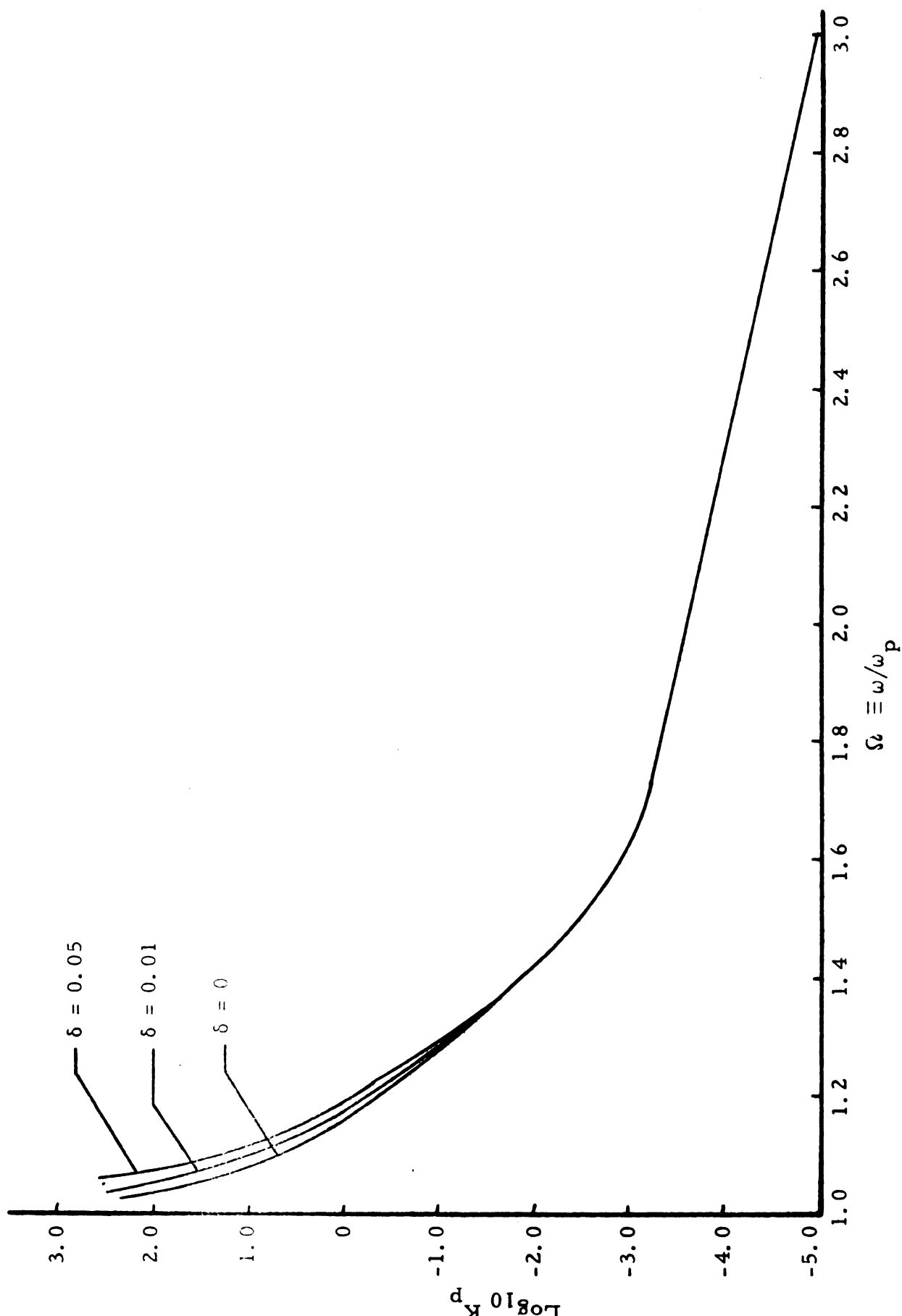


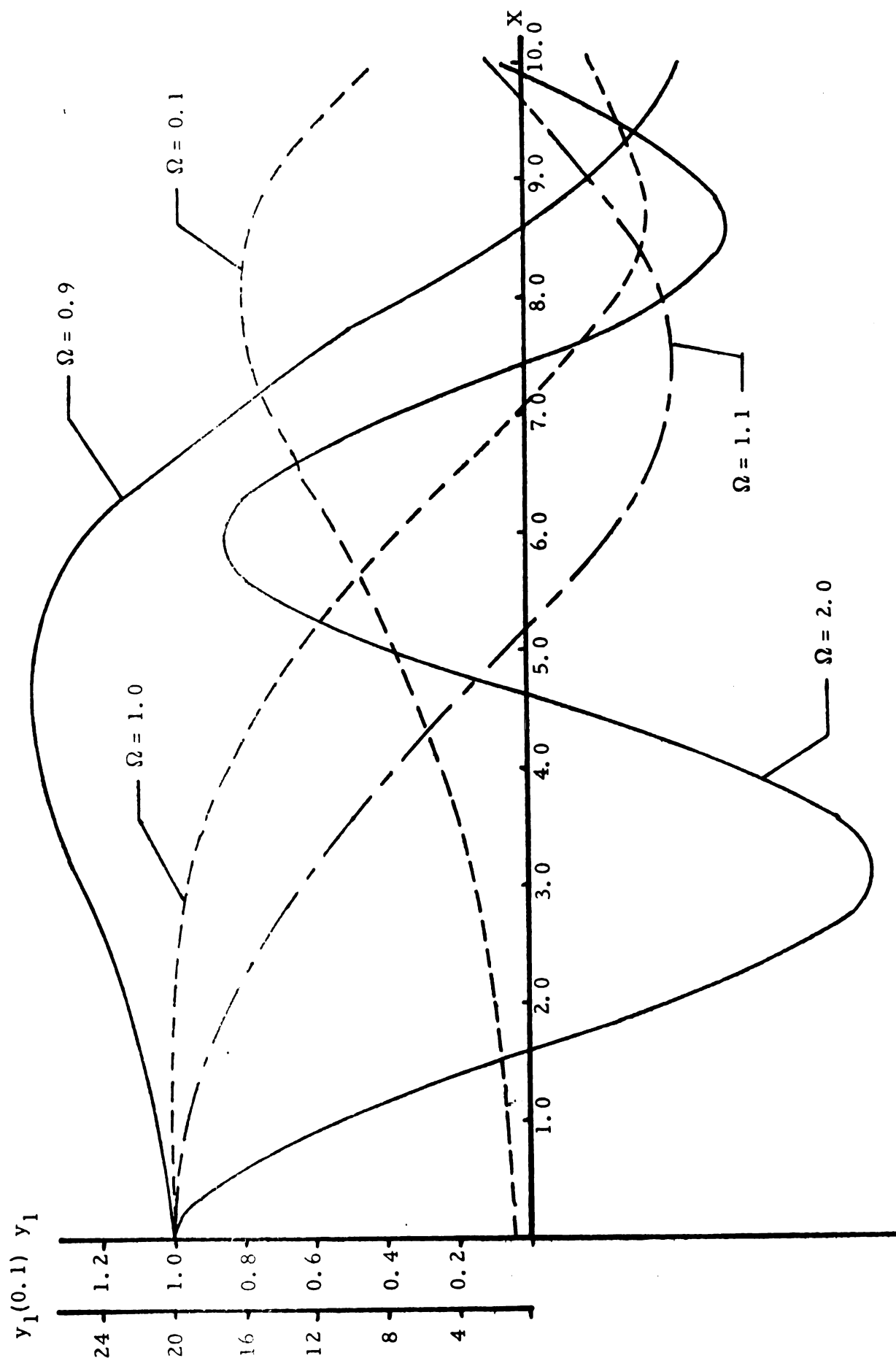
Figure 8.3. Zeroth Order $|N(a)K_a|$.

Figure 8.4. Zeroth Order K_a .

Figure 8.5. Zeroth Order K_c .

Figure 8.6. Zeroth Order K_e .

Figure 8.7. Zeroth Order K_p .

Figure 8.8. Preliminary Solutions for y_1 .

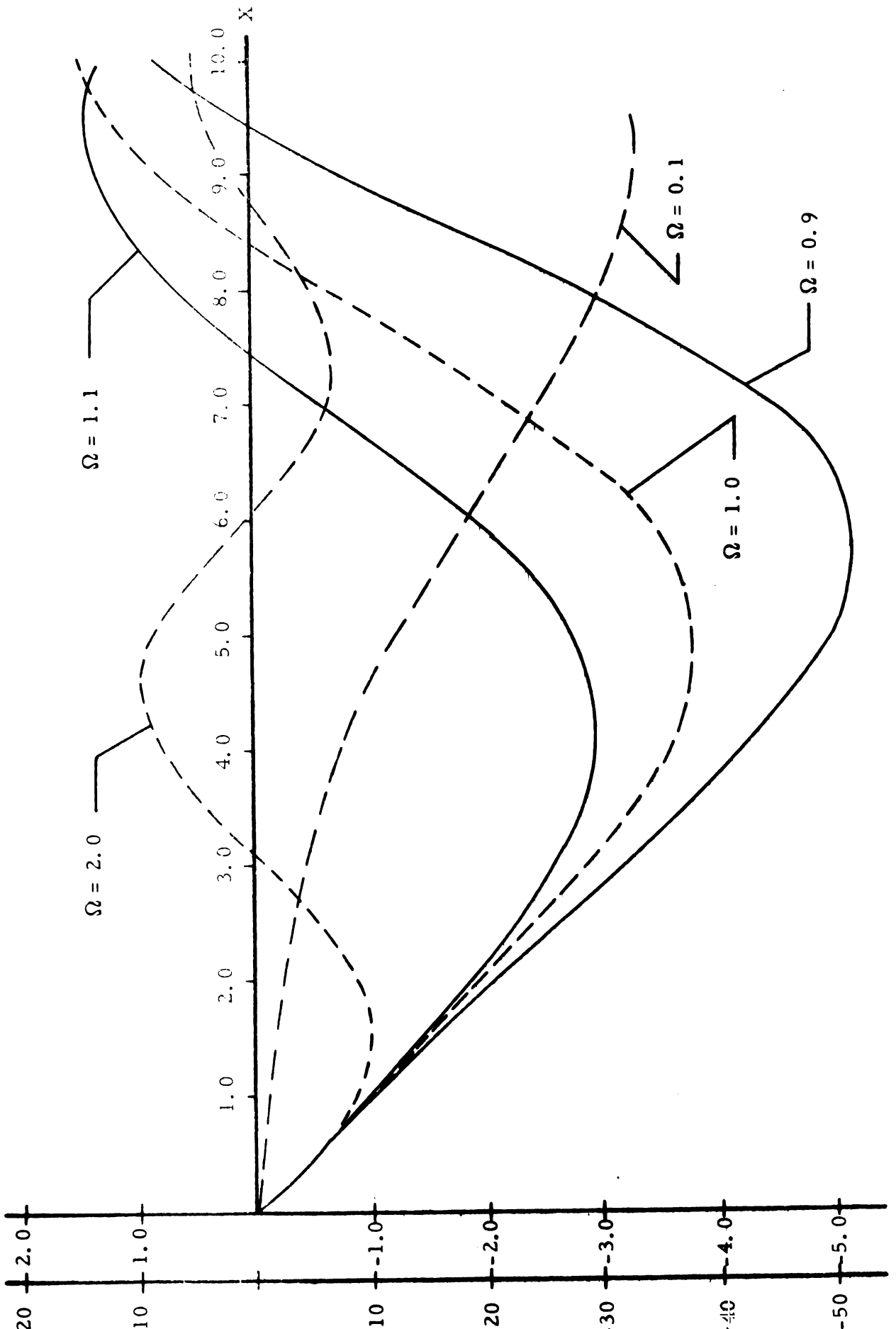
Figure 8.9. Preliminary Solutions for y_2 .

TABLE 8.1. Zeroth Order K_z for $\delta = 0.01$.

<u>Ω</u>	<u>$R_E K_z$</u>	<u>$I_M K_z$</u>
0.10	---	- 9.565
0.20	---	- 4.770
0.30	---	- 3.156
0.40	---	- 2.328
0.50	---	- 1.790
0.60	---	- 1.315
0.70	---	- 1.757
0.80	---	- 1.398
0.90	---	- 1.230
0.95	---	- 1.211
0.99	---	- 1.499
1.01	0.6503	- 0.7708
1.05	0.0836	- 0.8642
1.10	0.0314	- 0.8827
1.20	0.0128	- 0.8360
1.30	0.0057	- 0.7781
1.40	0.0018	- 0.7228
1.50	0.0006	- 0.6735
1.70	0.0002	- 0.5928
1.90	0.0002	- 0.5297
2.10	0.0001	- 0.4788
2.50	0.0000	- 0.4015
3.00	0.0000	- 0.3342

TABLE 8.2. Zeroth Order K_a for $\delta = 0.01$.

Ω	$10^2 R_E K_a$	$10^2 I_M K_a$
0.10	- 0.024	- 0.002
0.20	0.001	0.000
0.30	0.001	0.000
0.40	0.002	0.000
0.50	0.004	0.001
0.60	0.009	0.013
0.70	- 0.014	0.005
0.80	- 0.011	0.000
0.90	- 0.017	- 0.002
0.95	- 0.029	- 0.007
0.99	- 0.065	- 0.028
1.01	0.146	- 0.107
1.05	0.113	0.020
1.10	0.082	0.000
1.20	0.054	- 0.040
1.30	0.012	- 0.060
1.40	- 0.028	- 0.046
1.50	- 0.044	- 0.017
1.70	- 0.026	0.033
1.90	0.022	0.029
2.10	0.031	- 0.010
2.50	- 0.025	- 0.010
3.00	0.019	0.011

TABLE 8.3. Zeroth Order $|N(a) K_a|$ for $\delta = 0.01$

Ω	$10^2 N(a) K_a $
0.10	0.026
0.20	0.027
0.30	0.029
0.40	0.034
0.50	0.046
0.60	0.094
0.70	0.033
0.80	0.002
0.90	0.010
0.95	0.017
0.99	0.027
1.01	0.043
1.05	0.029
1.10	0.023
1.20	0.023
1.30	0.024
1.40	0.021
1.50	0.019
1.70	0.018
1.90	0.016
2.10	0.014
2.50	0.012
3.00	0.010

TABLE 8.4. Zeroth Order K_p for $\delta = 0.01$.

Ω	K_p
1.01	2.80×10^4
1.05	1.04×10^2
1.10	6.59×10^0
1.20	5.15×10^{-1}
1.30	9.28×10^{-2}
1.40	1.64×10^{-2}
1.50	3.11×10^{-3}
1.70	6.25×10^{-4}
1.90	3.45×10^{-4}
2.10	2.10×10^{-4}
2.50	5.10×10^{-5}
3.00	8.24×10^{-6}

TABLE 8.5. Zeroth Order Y_p and G_e for $\delta = 0.01$.

<u>Ω</u>	<u>$R_e Y_p$</u>	<u>$I_M Y_p$</u>	<u>G_e</u>
1.01	1.0657	1.263	0.0000
1.05	0.1848	1.910	0.0000
1.10	0.0671	1.885	0.0003
1.20	0.0306	1.993	0.0015
1.30	0.0157	2.142	0.0042
1.40	0.0059	2.305	0.0091
1.50	0.0020	2.473	0.0162
1.70	0.0010	2.812	0.0402
1.90	0.0008	3.147	0.0805
2.10	0.0007	3.482	0.1422
2.50	0.0005	4.150	0.3511
3.00	0.0003	4.987	0.8387

Note: All values expressed in millimhos. For the case presented $k_{pe} D = 0.50$ and $k_{pe} a = 0.10$.

TABLE 8.6.: Zeroth Order P_p/P_e and G_T for $\delta = 0.01$.

<u>Ω</u>	<u>P_p/P_e</u>	<u>G_T (millimhos)</u>
1.01	1.120×10^6	1.0657
1.05	0.416×10^4	0.1848
1.10	0.264×10^3	0.0674
1.20	0.206×10^2	0.0321
1.30	0.371×10^1	0.0199
1.40	0.683	0.0148
1.50	0.124	0.0182
1.70	0.250×10^{-1}	0.0412
1.90	0.138×10^{-1}	0.0813
2.10	0.840×10^{-2}	0.1429
2.50	0.204×10^{-2}	0.3516
3.00	0.330×10^{-3}	0.8390

TABLE 8.7. Zeroth Order K_z for $\delta = 0.5$.

Ω	$R_E K_z$	$I_M K_z$
0.10	---	- 9.989
0.30	---	- 3.330
0.50	---	- 1.997
0.60	---	- 1.664
0.70	---	- 1.426
0.80	---	- 1.247
0.90	---	- 1.107
0.95	---	- 1.046
0.99	---	- 0.984
1.01	0.01300	- 1.016
1.05	0.00245	- 0.958
1.10	0.00146	- 0.911
1.20	0.00095	- 0.834
1.30	0.00075	- 0.770
1.40	0.00063	- 0.774
1.50	0.00054	- 0.667
1.60	0.00047	- 0.625
1.70	0.00042	- 0.588
1.80	0.00038	- 0.556
1.90	0.00034	- 0.526
2.00	0.00031	- 0.500
2.50	0.00019	- 0.400
3.00	0.00013	- 0.333

TABLE 8.8. Zeroth Order K_z for $\delta = 0.1$.

Ω	$R_E K_z$	$-I_M K_z$
0.10	---	8.786
0.30	---	3.007
0.40	---	2.260
0.50	---	1.807
0.60	---	1.498
0.70	---	1.285
0.80	---	1.721
0.85	---	1.667
0.90	---	1.290
0.95	---	1.205
0.99	---	1.206
1.01	0.20374	1.296
1.05	0.05216	1.041
1.10	0.02649	0.933
1.20	0.01662	0.825
1.30	0.01127	0.759
1.40	0.00861	0.704
1.50	0.00595	0.658
1.70	0.00394	0.584
1.90	0.00257	0.523
2.10	0.00195	0.473
2.50	0.00111	0.398
3.00	0.00053	0.332

TABLE 8.9. First Order K_z for $\delta = 0.01$.

<u>Ω</u>	<u>$R_E K_z$</u>	<u>$I_M K_z$</u>
1.01	0.5217	- 1.2599
1.05	0.0792	- 1.0711
1.10	0.0301	- 0.9628
1.20	0.0119	- 0.8354
1.30	0.0055	- 0.7666
1.40	0.0017	- 0.7122
1.50	0.0005	- 0.6579
1.70	0.0003	- 0.5858
1.90	0.0002	- 0.5288
2.10	0.0001	- 0.4861
2.50	0.0000	- 0.3947
3.00	0.0000	- 0.3313

TABLE 8.10. First Order K_a for $\delta = 0.01$

Ω	$10^2 R_e K_a$	$10^2 I_m K_a$
1.01	0.011	- 0.059
1.05	- 0.006	- 0.072
1.10	- 0.007	- 0.065
1.20	0.008	- 0.045
1.30	0.010	- 0.035
1.40	0.009	- 0.039
1.50	0.018	- 0.090
1.70	0.006	0.037
1.90	- 0.007	0.045
2.10	- 0.017	- 0.029
2.50	0.018	- 0.028
3.00	0.005	0.037

TABLE 8.11. First Order $|N(a) K_a|$ for $\delta = 0.01$

Ω	$10^2 N(a) K_a $
1.01	0.017
1.05	0.024
1.10	0.024
1.20	0.014
1.30	0.007
1.40	0.013
1.50	0.043
1.70	0.012
1.90	0.018
2.10	0.014
2.50	0.018
3.00	0.017

TABLE 8.12. First Order K_p for $\delta = 0.01$

<u>Ω</u>	<u>K_p</u>
1.01	2.24×10^4
1.05	9.84×10^1
1.10	6.30×10^0
1.20	4.78×10^{-1}
1.30	8.95×10^{-2}
1.40	1.55×10^{-2}
1.50	2.60×10^{-3}
1.70	5.71×10^{-4}
1.90	3.23×10^{-4}
2.10	1.98×10^{-4}
2.50	4.51×10^{-5}
3.00	7.32×10^{-6}

CHAPTER IX

CONCLUSION

The major conclusion to be drawn from the results of this analysis is that Larson³ was seemingly quite justified in concluding, after his analysis of the spherical aperture antenna including sheath effects, that previous analyses ignoring the sheath had predicted excessively large effects due to EA radiation. His analysis involved the same geometry as and an approach similar to that of Wait¹ but included the plasma sheath which Wait did not do; this made it quite possible to compare directly his results with those of Wait to determine the effect of the sheath. The present geometry is the same as that of Fejer⁶ although he did not include the sheath; it is apparent here also that consideration of the sheath leads to the prediction of considerably reduced effects of EA radiation over those predicted by the corresponding "sheath-less" analysis. Larson noticed a considerably increased input susceptance (more nearly that for a free space environment) relative to Wait's result; the present study shows considerably increased input susceptance over Fejer's result. Larson noticed a considerably reduced amount of radiated EA power (hence a much reduced input conductance) in the region near $\Omega = 1$; in chapter VIII, the same trend is observed when the results there are compared with Fejer's results. Figure 8.2 shows Fejer's input resistance in the vicinity of $\Omega = 1$ to be greater by nearly an order of magnitude for $\delta = 0.01$; comparison for large δ values shows an even greater disparity.

It is easily seen that except for the very close proximity of $\Omega = 1$ the input reactance follows that of a capacitor with capacitance $2\pi\epsilon_0 a$ which would be the capacitance of the antenna if it were placed in a vacuum and the capacitance between the spheres were neglected (i. e., only the self capacitance or the capacitance to infinity of each sphere were considered). Since the antenna is small electrically the input admittance is essentially capacitive. The fact that the radiated EA power is greatly reduced (as evidenced by a greatly reduced input conductance) and the input susceptance is greatly increased by the inclusion of the sheath makes it apparent that the sheath greatly decouples the antenna from the plasma. This is not surprising because the sheath represents a near void of plasma at the surface of the antenna; that coupling to the plasma is reduced by such a void seems reasonable.

If a conventional radiation resistance term is derived from the total input admittance expression and compared with Fejer's result it is seen that each expression vanishes at $\Omega = 1$ and increases monotonically with Ω to achieve the same limiting values for large Ω ; it is seen also that Fejer's term increases much faster initially although the ratio of the two is always less than 2.0 if Ω is greater than about 1.10 so that the two analyses yield very similar results in the range where the radiation resistance term is appreciable.

Larson considered in his sheath model the unperturbed electron density and the unperturbed electric field due to charge separation; he did not consider the unperturbed drift of electrons

toward the spheres. This study has included in an approximate manner the effect of electron drift; it is found to be inconsequential as far as the input susceptance is concerned but to have considerable effect upon the magnitude of the input conductance and upon the electron density perturbation. Both analyses dropped the effect of the perturbation in the electric field; it was indicated in chapter IV that its effect was small although there is reason to believe that its effect might not be entirely negligible upon the relatively sensitive solutions for the input conductance and the electron density perturbation. The inclusion of this electric field term would require the solution of a system of third order linear ordinary differential equations if drift effects were simultaneously included; this could be done in theory but at present a lack of suitable and sufficient boundary conditions makes it impossible.

Boundary conditions represent a major weakness of this type of analysis and work should be done in the area of developing more reasonable boundary conditions. The "hard" boundary condition used here is arbitrary and somewhat unbelievable, but it is conventional. Other boundary conditions have been suggested by various people some of which offer greater flexibility although all are arbitrary and open to question.

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