THE THEORY OF CLUSTER SETS

Thesis for the Degree of Ph. D.
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Ruth Ann Su
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This is to certify that the

thesis entitled

THE THEORY OF CLUSTER SETS

presented by

Ruth Ann Su

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Peter A. Loeb
Major professor

Date November 14, 1974

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ABSTRACT

THE THEORY OF CLUSTER SETS

By

Ruth Ann Su

Since Painlevé founded the theory of cluster sets in 1895, mathematicians have discovered many significant properties pertaining to the set of limit points of a function at the boundary of its domain of definition. The functions studied may be divided into the following three major classes: Arbitrary Functions, Normal Functions and Class A Functions.

Arbitrary functions have limited patterns of behavior with respect to cluster sets because of the particular topologies of the plane and sphere. For example, according to the Bagemihl Ambiguous Point Theorem, any complex-valued function defined in the unit disk D has at most a countable number of boundary points $e^{i\theta}$ with the property that there exist two curves in D ending at $e^{i\theta}$ along which f has disjoint cluster sets. Also, globally, there are numerous relationships between the cluster set of a function relative to an angle at a point $e^{i\theta}$ and the cluster set of a function relative to a region between two circles each internally tangent to the unit circle at $e^{i\theta}$.

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A function is normal in a simply connected region if its family of arbitrary conformal mappings of the region onto itself has the property that every sequence of this family contains a subsequence which converges uniformly or tends uniformly to infinity on every compact subset of this region. A meromorphic function in D is normal if the function omits at least three points in D. In addition a complex function in D is normal if it is uniformly continuous from the disk with the hyperbolic metric to the sphere with the chordal metric. The sum of two analytic normal functions is not necessarily normal although the special type of normal functions called uniformly normal has the property that the sum of two uniformly normal functions is uniformly normal. The definition of a uniformly normal function is analogous to the definition of a normal function where the sphere with the chordal metric is replaced by the plane with the usual metric.

Suppose f is a holomorphic nonconstant function in D. Then f belongs to Class A if for each point in a dense set of C, f has a path in D ending at $e^{i\theta}$ along which f approaches a limit. f belongs to Class B if and only if the set of points $e^{i\theta}$, for which f has a path in D ending at $e^{i\theta}$ along which either f approaches infinity or the modulus of f is bounded by some finite number, is dense on C. For any constant λ greater than or equal to zero the level set consists of all points z in D for which the modulus of f is equal to λ. Then f belongs to Class L if the maximum diameter of the components of each level set intersected with the set of z having modulus greater than r approaches zero as r approaches one. A very important theorem in the study of Class A Functions states that a function is in Class A if and only if the function is in Class B if and only if th

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if and only if the function is in Class L. Class A Functions are not closed under the operations of addition and multiplication. In fact every nonconstant, holomorphic function in D can be written as the sum or as the product of pairs of functions in Class A.

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THE THEORY OF CLUSTER SETS

By

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I wish to express my sincere gratitude to Professor Peter Lappan for his continual assistance in the preparation of this thesis. I also wish to thank my husband Dr Lawrence Su for his helpful suggestions and for the typing of the thesis.

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PREFACE

The purpose of this thesis is to bring together recent important developments in the theory of cluster sets. We assume that the reader would have the mathematical training equivalent to that of a graduate course in the theory of complex variables and a cursory knowledge of the works by Collingwood and Lohwater (1), Noshiro (1) and MacLane(1).

Painleve (1) founded the theory of cluster sets in 1895 when he gave the name "domaine d'indetermination" to the set of limit points of a function at a boundary of its domain of definition. Today this set is called the cluster set of a function at a point. Although the theory was first considered for analytic functions, it is applicable to more general functions, and much of the present-day research is largely topological.

Early developments in the theory of cluster sets were mostly concerned with the behavior of an analytic function in the neighborhood of an isolated essential singularity or in a discontinuous set of singularities. The earliest result dealing with cluster sets was the theorem proved in a paper of Weierstrass (1) in 1876. It states that if z_o is an isolated point of a set E in the unit disk D and f(z) is meromorphic in D - E, then the set of limit points of f at z_o is either a single point or the entire Riemann sphere. In 1905 Painleve proved that this theorem is true for any z_o in a set of measure zero.

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at z_o may be a proper subset of the Riemann sphere. A lot of research has been done concerning the boundary behavior of functions defined in a simply connected domain whose boundary contains more than one point and which can therefore be mapped conformally onto the open unit disk.

The study of cluster sets at a continuous boundary begins with the Fatou paper (1) of 1906 on the radial limits of functions analytic in the unit disk. Caratheodory (4) studied the boundary correspondence between the unit disk and an arbitrary simply connected domain under a conformal mapping. This led to the notion of a prime end, the correspondence between the points of the unit circle and the prime ends of the domain whose impressions are the cluster sets of the mapping function at the corresponding points.

Since the 1930's, cluster sets have been widely studied. The books The Theory of Cluster Sets by Collingwood and Lohwater (1) and Cluster Sets by Noshiro (1) contain most of the important results before 1960. In this thesis we assume the above material to be background information and present some of the more significant developments since then.

We organize our material into three chapters which deal with the three major classes of functions: Arbitrary Functions, Normal Functions, and Class A Functions. We have selected results from The Theory of Cluster Sets by Collingwood and Lohwater (1) for preliminary work in the chapter on arbitrary functions and results from Cluster Sets by Noshiro (1) for introductory material in the chapter on normal functions. The last chapter uses as background the MacLane paper (1), Asymptotic Values of Holomorphic Functions, since this paper is the foundation for the study of Class A functions, which consist of non-constant holomorphic functions in the open unit disk which approach limits on a dense subset of the unit circle.

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In the first chapter we primarily consider functions in the unit disk without imposing any restrictions except that they be complex-valued. In spite of the lack of restrictions, these functions have limited patterns of behavior with respect to cluster sets. Much of this is the result of the particular topologies of the plane and the sphere.

Bagemihl's Ambiguous Point Theorem is an outstanding example of the limited patterns of behavior mentioned above. The theorem says that if f is a complex-valued function defined in D, then there are at most a countable number of boundary points ζ with the property that there exist two curves in D ending at ζ along which f has disjoint cluster sets. Even though this result is true in the plane, it does not apply if the domain of the function is the unit ball in three dimensions (Church, 1). Moreover the addition of some mild restrictions, such as requiring the function to be analytic, does not yield a stronger conclusion (Bagemihl and Seidel, 4). This theorem has been extended by the theory of prime ends to other domains, such as simply or multiply connected regions, with approximately the same result.

The above theorem has found wide application in the study of cluster sets. For example, let f have the n-separated-arc property at a point p if, for any integer $n > 1$, there exist n arcs in D ending at p which are mutually disjoint except for p where the intersection of the cluster sets of all n arcs is empty while that of any $n - 1$ of them is nonempty. Then if f is a homeomorphism of D onto itself, any point p satisfying the n-separated-arc property is an ambiguous point. Consequently these points are at most countable. However, this does not hold in general as there

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in general as there is a continuous function which has the 3-separated-arc property at all but at most a countable number of points (Piranian, 1).

An interesting relationship between different cluster sets is the relationship between the angular and the horocyclic cluster sets. An angular cluster set is the cluster set of a function relative to an angle at a point $e^{i\theta} \in C$ while a horocyclic cluster set is the cluster set of a function relative to a region between two circles each internally tangent to the unit circle at a common point. Since an angle is the region between two chords originating from $e^{i\theta}$, a horocyclic region and an angle have no points in common near $e^{i\theta}$. So a relationship between these two types of cluster sets should not be expected. However, there are numerous relationships between these two kinds of cluster sets on a global, not local, basis. For example, for any function the set of points $e^{i\theta}$ on the unit circle for which there exists an angle and a horocycle such that the angular cluster set is not contained in the horocyclic cluster set is a set of measure zero and of first category.

Γ is called a selector of arcs if it associates a nonempty collection of arcs with every point in C. The Γ-principal cluster set $\Pi_\Gamma(f,e^{i\theta_o})$ is the intersection of the cluster sets of all the arcs in Γ which end at the point $e^{i\theta_o}$. Let $\Pi_\Gamma^*(f,e^{i\theta_o},\mu)$ denote the closure of the union of the Γ-principal cluster sets for all points $e^{i\theta}$ for which $|e^{i\theta} - e^{i\theta_o}| < \mu$. Then the boundary Γ-principal cluster set at $e^{i\theta_o}$ is the intersection of the $\Pi_\Gamma(f,e^{i\theta_o},\mu)$'s for all positive μ. For any continuous function f in D and all points in C, the Γ-principal cluster set is equal to the boundary Γ-principal cluster set except for a set of first category on C if Γ is either the collection of all arcs or all chords.

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For continuous functions defined in D, the cluster sets along all Jordan arcs in D ending at a point on C form a topology where the distance between any two closed sets is defined as the greatest distance between a point in one set and a nearest point in the other set. This topology is called the M-topology. If $e^{i\theta}$ is not an ambiguous point, then the set $G(e^{i\theta})$ consisting of its Jordan-arc cluster sets is compact in the M-topology. So another consequence of the Ambiguous Point Theorem is the fact that the set of points for which $G(e^{i\theta})$ is not compact is at most countable.

The cluster sets of special classes of functions have been studied extensively. As might be expected the cluster sets of these functions possess properties which are not necessarily true for the cluster sets of arbitrary functions. Some of the functions investigated most frequently are normal functions and Class A functions.

Any normal meromorphic function $f(z)$ in D which approaches a limit α at a point z_o in C along a Jordan curve lying in D also has the angular limit α at z_o. Moreover, if $f(z)$ tends to a limit along a simple continuous curve $z(t)$ for which $|z(t)| \rightarrow 1$ as $t \rightarrow 1$ and its end contains more than one point, then it is a constant function. Another example where $f(z)$ must be a constant function occurs when it approaches a constant along a sequence of arcs in D which converge to a boundary arc in C.

Analytic normal functions are not closed under addition although the special type of functions called uniformly normal functions are closed under addition. These functions satisfy the condition $\sup_{z \in D} (1 - |z|^2) |f'(z)|$ is finite. If, in addition, a uniformly normal function f satisfies the condition $f(0) = 0$ then it is called a Bloch function. The collection of all Bloch functions form a Banach space. Each Bloch function, and thus

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function, and thus each uniformly normal function, has the property that it possesses angular limits on an uncountably dense subset of C.

Any normal holomorphic function in D belongs to the class I_p, which consists of those holomorphic functions f in D having the property that if there exists a pair of arcs t_1 and t_2 along which $f(z) \to \infty$ as $z \to p \in C$, then along any path between t_1 and t_2 the function f(z) is unbounded. If f is in class I_p, then the set $G(e^{i\theta})$ consisting of the set of all cluster sets of all Jordan arcs in D which end at p is compact in the M-topology. Consequently, for any normal holomorphic function the set $G(e^{i\theta})$ is compact in the M-topology.

In order to study the behavior of normal meromorphic functions near C, D has been compactified into a Hausdorff space M in such a way that any bounded holomorphic function f has a continuous extension $\hat{f}$ and D is dense in M. Two points $m_1, m_2 \in M$ are in the same Gleason part if for any bounded holomorphic function of modulus less than or equal to one, its extension $\hat{f}$ has the property that the difference in magnitude of $\hat{f}(m_1)$ and $\hat{f}(m_2)$ is strictly between 0 and 2. This determines an equivalence relation. Each Gleason part consists of either a single point or the image of a one-to-one analytic map of an open disk into M. The Gleason parts partition the boundary points of D in such a way that any bounded analytic function has a continuous extension onto the boundary of D. A function is normal in D if and only if it can be continued continuously to the set G consisting of the maximal ideal space M of H^∞ minus the trivial Gleason parts lying over the boundary of D. So, in this sense, normal functions are a generalization of bounded functions. If f is a normal meromorphic function, then it is so continuous that on every non-trivial Gleason part f is either meromorphic or identically infinite.

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Let $f(z)$ be a holomorphic nonconstant function in D. Then $f(z)$ belongs to Class A if and only if for each point in a dense set of C, it has a path in D ending at $e^{i\theta}$ along which it approaches a limit. $f(z)$ belongs to Class B if and only if the set of points $e^{i\theta}$ for which it has a path in D ending at $e^{i\theta}$ along which either $f \to \infty$ or $|f| < a$ where a is finite is dense on C. For any constant $\lambda \geq 0$, the level set $LS(\lambda)$ consists of all points z in D for which $|f(z)| = \lambda$. Then $f(z)$ belongs to Class L if and only if the maximum diameter of the components of each level set intersected with $\{z: |z| > r\} \to 0$ as $r \to 1$. A very important theorem in the study of Class A functions states that a function is in Class A if and only if the function is in Class B if and only if the function is in Class L.

In order to generalize Class A functions, Classes A_m, B_m and L_m are defined by replacing the word "holomorphic" with "meromorphic" in the appropriate definitions. Holomorphic normal functions are in Class A and meromorphic normal functions are in Class B_m. Class A_m is contained in Class B_m. Class L_m is contained in Class B_m. However, there are examples of functions in Class B_m that are not in Class A_m and examples of functions in Class L_m that are not contained in Class B_m.

A tract associated with a constant a is a collection of nonempty domains $D(\epsilon)$ such that each $D(\epsilon)$ is a component of the open set in D which is mapped by f into the open disk about a of radius ϵ and the intersection of all of the $D(\epsilon)$'s is called the end of the tract. It is a nonempty, connected closed subset of C. A tract is called global if and only if its end consists of the entire circumference C and for each arc γ contained in C there exists a sequence of arcs γ_n contained in $D(1/n)$ such that the γ_n's approach γ. If f is in Class A_m and $\{\gamma_n\}$ is a sequence of disjoi

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sequence of disjoint simple arcs in D which tend to the arc γ on C and $\sup(f(z) - a) \rightarrow 0$ where a is a complex number, then f has a tract with end K which contains γ. In addition for any interior point of K, the only asymptotic values come from the tract. If f is in A, then f has a global tract if and only if f is unbounded on every curve in D on which $|z| \rightarrow 1$.

If a is a finite asymptotic value along an arc that f maps one-to-one onto a linear segment, then this asymptotic value is called linearly accessible. If f is in Class A and omits some finite constant, then the set of linearly accessible points is dense on C.

Class A functions are not closed under the operations of addition and multiplication. In fact every nonconstant, holomorphic function in D can be written as the sum or as the product of pairs of functions in Class A. Furthermore, any nonconstant meromorphic function in D may be represented in each of the following ways: (i) the quotient of two holomorphic functions in Class A, (ii) the product of a function in Class A and a function in Class $A_m \cap$ Class L_m, (iii) the sum of two functions in Class $A_m \cap$ Class L_m.

Our bibliography consists primarily of the publications which contain the recent developments in the theory of cluster sets. Older references may be found in the bibliographies of the books of Collingwood and Lohwater, and Noshiro.

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CHAPTER I

ARBITRARY FUNCTIONS

INTRODUCTION

In this section we will first introduce some of the important definitions which will be used throughout the paper. Then we will summarize some of the major results in the theory of cluster sets which are included in the book by E.P. Collingwood and A.J. Lohwater (1).

We will consistently use the notation D for the open unit disk, C for the unit circle, and W for the Riemann sphere.

Some of the concepts which we will use repeatedly include those of cluster sets, asymptotic values, and range of values. If z_o is any point in $\overline{D}$ and f is an arbitrary function defined in D, then the <u>cluster set</u> $C(f,z_o)$ of $f(z)$ at z_o is defined in one of the following two equivalent ways:

(i) $C(f,z_o)$ is the set of points a on the Riemann sphere W for which there exists a sequence $\{z_n\}$ in $D - \{z_o\}$ such that, as $n \to \infty$, $\lim z_n = z_o$ and $\lim f(z_n) = a$ where $D - \{z_o\}$ is D with z_o removed.

(ii) For $r > 0$, $C(f,z_o) = \cap \overline{D}_r$ where $D_r = f(d_r \cap (D - \{z_o\}))$ and d_r is the disk $|z - z_o| < r$.

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If G is any infinite subset of D, then the cluster set $C_G(f,z_o)$ of f(z) relative to G is defined by

$$C_G(f,z_o) = \cap \overline{D_r(G)} \subset C(f,z_o)$$

where $D_r(G) = f(d_r \cap (D - \{z_o\}) \cap G)$.

The <u>range of values</u> $R(f,z_o)$ is defined to be the set of values a such that there exists a sequence $\{z_n\}$ in D such that as $n \longrightarrow \infty$ and $z_n \neq z_o$, $\lim z_n = z_o$ and $f(z_n) = a$. The <u>set of asymptotic values</u> $A(f,z_o)$ at z_o consists of those complex numbers a for which there exists a continuous curve $z = z(t)$, $0 < t < 1$, such that $z(t) \subset D - \{z_o\}$, $\lim z(t) = z_o$ and $\lim f(z(t)) = a$ as $t \longrightarrow 1$. We will use the symbol A(f) to denote the union of all of the $A(f,z_o)$'s for all z_o's in C and the symbol R(f) to denote the union of all of the $R(f,z_o)$'s.

If $\gamma = z(t)$, $0 \leq t \leq 1$, is a simple continuous arc lying in D except for $T = z(1) \in C$, then γ is called a <u>boundary arc</u> at T.

<u>Theorem 1</u>: If f(z) is an arbitrary function defined in D and if ζ is an arbitrary point of C, then there exists a simple arc γ, lying in D and terminating at ζ, such that $C_\gamma(f,\zeta) = C(f,\zeta)$. (Collingwood, 2)

A set E on C is of <u>first category</u> if E is the union of a countable set of nowhere dense sets; a set which is not of first category is said to be of <u>second category</u>. A set E on C is called <u>residual</u> on C if the complement of E on C is of first category.

For G any subset of $\overline{D}$, a <u>rotation</u> G_θ of G is obtained by mapping each point $z \in G$ to the point $ze^{i\theta}$.

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Theorem 2: If the real or complex function $f(z)$ is continuous in D and if $\{G_\theta\}$ is the family of rotations of a continuum G_o such that $G_o \cap C$ is the point $z = 1$, then $C_{G\theta}(f, e^{i\theta}) = C(f, e^{i\theta})$ on a residual set of points $e^{i\theta}$ on C. (Collingwood, 3)

Let $\Delta(1)$ be an open connected subset of D such that $\overline{\Delta(1)} \cap C$ is equal to $\{1\}$, and let $\Delta(e^{i\theta})$ denote the transform of $\Delta(1)$ under the rotation about the origin that sends 1 into $e^{i\theta}$. Dragosh (2, Lemma 1, p.58) proves that $C_{\Delta(e^{i\theta})}(f, e^{i\theta}) = C(f, e^{i\theta})$ for a residual G_δ subset of C.

In order to define boundary cluster sets, we use the notation

$$C(f,\ 0 < |\theta - \theta_o| < \eta) = \bigcup\ C(f, e^{i\theta}). \tag{1}$$

where the union is over $0 < |\theta - \theta_o| < \eta$. Then the boundary cluster set $C_{B_o}(f, e^{i\theta_o})$ may be expressed as

$$C_{B_o}(f, e^{i\theta_o}) = \bigcap_{\eta > 0} \overline{C(f,\ 0 < |\theta - \theta_o| < \eta)}. \tag{2}$$

The left-hand and right-hand boundary cluster sets $C_{Bl}(f, e^{i\theta})$ and $C_{Br}(f, e^{i\theta})$ are defined by (1) and (2) and the restrictions that $0 < \theta - \theta_o < \eta$ and $0 < \theta_o - \theta < \eta$ respectively.

Theorem 3: If $f(z)$ is a single-valued (real or complex) function in D, then

$$C_{Br}(f, e^{i\theta}) = C_{Bl}(f, e^{i\theta}) = C(f, e^{i\theta})$$

except perhaps for a countable set of points $e^{i\theta} \in C$. (Collingwood, 4)

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The <u>right-hand cluster set</u> $C_R(f,e^{i\theta})$ is defined to be the set of points α such that as $n \to \alpha$ $\lim f(r_n e^{i\theta_n}) = \alpha$ where $\lim r_n = 1$ and $\lim \theta_n = \theta$ with $\theta_n \leq \theta_{n+1} \leq \cdots$. The <u>left-hand cluster set</u> $C_L(f,e^{i\theta})$ is defined in the same way except that $\theta_n \geq \theta_{n+1} \geq \cdots$. Actually the right-hand cluster set is the cluster set $C_G(f,e^{i\theta})$ where G is the semi-disk closed relative to D with diameter from $-e^{i\theta}$ to $e^{i\theta}$ and to the right of it. The left-hand cluster set is defined in a similar way but is to the left of the diameter.

<u>Corollary</u>: If f(z) is single-valued in D, then

$$C_R(f,e^{i\theta}) = C_L(f,e^{i\theta}) = C(f,e^{i\theta})$$

except perhaps for a countable set of points $e^{i\theta} \in C$. (Collingwood and Lohwater, 1, Corollary, p.83)

<u>Theorem 4</u> (Bagemihl Ambiguous-Point Theorem): If f(z) is a complex function defined in D, then the set of points $e^{i\theta}$ on C with the property that there exist two boundary arcs r_1 and r_2 at $e^{i\theta}$ such that

$$C_{r_1}(f,e^{i\theta}) \cap C_{r_2}(f,e^{i\theta}) = \phi$$

is at most countable. (Bagemihl, 1)

The points $e^{i\theta}$ defined in Theorem 4 are called ambiguous points.

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RESULTS RELATED TO BAGEMIHL'S AMBIGUOUS-POINT THEOREM

Researchers, such as Bagemihl, H. Mathews and McMillan, have proved many theorems related to the Bagemihl Ambiguous-Point Theorem.

Let α be an arc lying in $\overline{D} - \{p\}$ except for one end point at p. The <u>extended arc cluster set</u> of f at p, $EC_{\alpha}(f,p)$, is defined to be the set $\bigcap \overline{\bigcup C(f,q)}$ where the intersection is taken over all neighborhoods N of p and the union over all q on $\alpha \cap N$ for $q \neq p$. The point p is called an <u>extended ambiguous point</u> for f if there exist arcs α and β in $\overline{D} - \{p\}$ such that $EC_{\alpha}(f,p)$ and $EC_{\beta}(f,p)$ are disjoint.

<u>Theorem 5</u>: If f is an arbitrary function defined in D and if a point p on C is an extended ambiguous point for f, then p is an ambiguous point for f. (H. Mathews, 1, Theorem 1, p.138)

Since Mathew's proof only holds when f is continuous, Stebbins (1) recently published the following proof. Let α be any arc in $\overline{D} - \{p\}$ such that α tends to p. It is sufficient to find an arc $\alpha' \subset D$ which tends to p such that $C_{\alpha'}(f,p) \subseteq EC_{\alpha}(f,p)$. By using points $q \in \alpha \cap C$ and the method of Gross (1), we construct a "wedge" Z in D such that every sequence of points $\{z_k\}$ in Z tends to p and $\{f(z_k)\}$ has limit points only in $\bigcap \overline{\bigcup C(f,q)}$ where $q \in \alpha \cap C$ for $q \neq p$ and the intersection is taken over all neighborhoods of p.

<u>Corollary</u>: An arbitrary function from D into W can have at most a countable number of extended ambiguous points. (H. Mathews, 1, Theorem 2, p.139)

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This corollary follows immediately from Theorem 5 and the Ambiguous Point Theorem.

If G is a simply connected region in the extended complex plane, then we denote the <u>set of boundary points</u> of G by F(G). If $e^{i\theta} \in F(G)$ and if there exists an arc in G with an end point at $e^{i\theta}$, then $e^{i\theta}$ is called an <u>accessible point</u> of F(G).

A Jordan arc which lies in G except for its two endpoints or a Jordan curve which lies in G except for one point is called a <u>crosscut</u> of G. A sequence $q_1, q_2, \ldots, q_n, \ldots$ of crosscuts of G is called a <u>chain</u> if the following conditions are satisfied:

(i) No two of them have any point, including their endpoints, in common;

(ii) q_n separates G into two domains, one of which contains q_{n-1} and the other q_{n+1}. The domain containing q_{n+1} is denoted by d_n;

(iii) The diameter of q_n tends to zero as n tends to infinity.

Two chains $Q = \{q_n\}$ and $Q' = \{q_n'\}$ in G are <u>equivalent</u> if, for all values of n, the domain d_n contains all but a finite number of the crosscuts q_n' and the domain d_n' contains all but a finite number of the crosscuts q_n. The class of all chains equivalent to a given chain is an equivalence class. A <u>prime end</u> of G is an equivalence class of chains in G.

A <u>curve</u> Λ in G <u>at the prime end</u> P means a simple continuous curve $z = z(t)$, $0 \leq t \leq 1$, such that $z(t) \in G$ and every sequence of points on Λ that approaches F(G) also converges to P in the sense that all but a finite number of the members of the sequence are contained in each d_n. If $e^{i\theta} \in F(G)$ and there exist distinct prime ends $P_1, P_2 \in G$

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and curves r and s at P_1 and P_2 respectively such that r and s are also arcs at $e^{i\theta}$, then $e^{i\theta}$ is a multiply accessible point of F(G). If an accessible point is not multiply accessible, it is simply accessible.

If Λ is an arc at a point $e^{i\theta} \in F(G)$ (or a curve at a prime end P of G), then the cluster set of f at $e^{i\theta}$ (or at P) on Λ will be denoted by $C_\Lambda(f,e^{i\theta})$ [or $C_\Lambda(f,P)$]. If P is a prime end of G and there exist two curves r and s at P such that $C_r(f,P) \cap C_s(f,P) = \phi$, then P is called an ambiguous prime end of f.

Theorem 6: A necessary and sufficient condition that a simply connected region G must satisfy, in order that every function defined in G have no more than countably many ambiguous points from different prime ends, is that at most countably many accessible points of F(G) be multiply accessible from G. (Bagemihl, 5, Theorem 8, p.203)

Proof: Suppose that the set M of all points of F(G) multiply accessible from G is more than countably many. Let $w = f(z)$ map G in a one-to-one conformal manner onto D. This mapping induces a correspondence between F(G) and C under which every point of M corresponds to at least two points of C. Thus f has more than countably many ambiguous points

Assume that F(G) contains at least two points. Let $z = \phi(w)$ map D in a one-to-one conformal manner onto G. If a function g(z) in G has an ambiguous point $e^{i\theta}$ that is simply accessible from G, then the function $h(w) \equiv g(\phi(w))$ in D has an ambiguous point at the point w on C that corresponds to $e^{i\theta}$ under the mapping ϕ. It now follows from the Ambiguous Point Theorem that g(z) has no more than countably many ambiguous points.

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<u>Theorem 7</u>: Let $f(z)$ be an arbitrary function in a simply connected region G with at least two boundary points. Then f has at most enumerably many ambiguous prime ends. (Bagemihl, 5, Theorem 9, p.203)

<u>Proof</u>: Let $z = \phi(w)$ be a one-to-one conformal mapping of D onto G. By Caratheodory's Theorem (Caratheodory, 3 and 4) this mapping induces a one-to-one correspondence between the points of C and the prime ends of G such that, if P is a prime end of G and Λ_P is a curve at P, then the preimage of Λ_P under the mapping is an arc Φ_T at the point T of C that corresponds to the prime end P. If f has more than enumerably many ambiguous prime ends, then the function $h(w) \equiv f(\Phi(w))$ in D would have more than enumerably many ambiguous points, which is impossible.

<u>Theorem 8</u>: Suppose f is continuous, S is a closed subset of W and the set $B(f,S)$ of points $e^{i\theta}$ for which there exists an arc σ at $e^{i\theta}$ such that $C_\sigma(f,e^{i\theta}) \subset S$ is uncountably dense on an arbitrary closed arc λ on C. Then the set $B^*(f,S)$ of points $e^{i\theta}$ such that for any arc ϑ at $e^{i\theta}$ $C_\vartheta(f,e^{i\theta}) \cap S \neq \Phi$ is residual on λ. (McMillan, 2, Theorem 5, p.188)

The theorem is proved by showing that $B^*(f,S) \cap \text{Interior}(\lambda)$ relative to $C = \bigcap_n \{ e^{i\theta}$ in the interior of λ such that there exists a crosscut T at $e^{i\theta}$ with diameter less than $1/n$ such that $f(T) \subset \{w: \rho(w,S) < 1/n\}\}$ where n is a positive integer and $\rho(w,S)$ denotes the Euclidean distance between w and S.

If f is any function that is defined in D and takes its values in some metric space, then a <u>boundary function</u> for f is a function ϕ on C such that for every $x \in C$ there exists a simple arc λ having one endpoint at x for

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Theorem 9: Every function f defined in D has at most $2^{\aleph_0}$ boundary functions. (Bagemihl and Piranian, 1, Theorem 1, p.201)

Proof: By the Ambiguous Point Theorem, f has at most countably many ambiguous points. At each ambiguous point f has at most $2^{\aleph_0}$ asymptotic values. Therefore, f has at most $\left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0}$ boundary functions.

RESULTS ON BOUNDARY FUNCTIONS

In 1965 Kaczynski published a paper on boundary functions for functions defined in D. It includes descriptions of boundary functions in terms of honorary Baire class functions.

Theorem 10: If f is a homeomorphism of D onto itself and ϕ is a boundary function for f, then there exists a countable set N such that ϕ_o is continuous where ϕ_o is the restriction of the boundary function to C - N. (Kaczynski, 1, Theorem 1, p.590)

Let S* be a base of open sets in R^2 and let acc(E) denote the set of all points on C which are accessible by arcs in E. Then the above theorem is proved by showing that for any $S \in S^*$

(i) $\operatorname{acc}(f^{-1}(D \cap S)) = \operatorname{acc}(f^{-1}(D \cap S)) \cap \overline{f^{-1}(D - S)} \cup (C - \overline{f^{-1}(D - S)})$

(ii) if U is any open set which can be expressed in the form $U = \bigcup S_n$ where $S_n \in S^*$ and $\overline{S}_n \subseteq U$, then $\phi_o^{-1}(U) = \bigcup \operatorname{acc}(f^{-1}((S_n \cap D)) - N$ where N consists of all of the ambiguous points accessib

Lemma 1: Let f be

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Lemma 1: Let f be a continuous real-valued function in D and λ be a finite-valued boundary function for f. Let r and t be real numbers with $r < t$. Then

(A) there exists a G_δ set G and a countable set N such that

$$\lambda^{-1}([r,+\infty)) \supseteq G \supseteq \lambda^{-1}([t,+\infty)) - N$$

where a G_δ set is the intersection of a countable number of open sets and

(B) there exists a G_δ set H and a countable set M such that

$$\lambda^{-1}((-\infty,t]) \supseteq H \supseteq \lambda^{-1}((-\infty,r]) - M.$$

(Kaczynski, 1, Lemma 3, p.592)

Proof: Let n be any positive integer. Let $\epsilon = (t-r)/2$, $C_n = \{z \in R^2: |z| = 1 - 1/n\}$, $A_n = \{z \in R^2: 1 > |z| > 1 - 1/n\}$, $E_n = \{x \in C$: there exists an arc γ at x having one endpoint on C_n with $\gamma - \{x\} \subseteq f^{-1}((-\infty,r))\}$, and $K = \{x \in C$: there exists an arc γ at x with $\gamma - \{x\} \subseteq f^{-1}((t-\epsilon, +\infty))\}$. For a fixed n and any point x in K we can find a simple arc γ_x at x such that $\gamma_x - \{x\} \subseteq A_n \cap f^{-1}([t-\epsilon,+\infty))$. Then $\gamma_x - \{x\}$ is a connected set. So $\gamma_x - \{x\}$ must be contained entirely within one component of the open set $A_n \cap f^{-1}((t-\epsilon,+\infty))$. Let O_x denote this component and let T be the set of all points of K which are two-sided limit points of $\bar{E}_n$.

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We want to show that if $x,y \in T$ and $x \neq y$, then $O_x \cap O_y$ is the empty set. Suppose on the contrary there exists an element z in $O_x \cap O_y$. We choose points x' and y' in $\gamma_x - \{x\}$ and $\gamma_y - \{y\}$ respectively. Then we join x to x' by a subarc of γ_x and join x' to z by an arc in O_x. Similarly we join z to y' by an arc in O_y and join y' to y by a subarc of γ_y. Putting these arcs together, we obtain an arc α with endpoints at x and y such that $\alpha - \{x,y\} \subseteq A_n \cap f^{-1}((t-\epsilon,+\infty))$. If α is not a simple arc, we replace it by a simple arc α' contained in α having endpoints at x and y and rename the simple arc α. α is a crosscut of D. Let L_1 and L_2 be the two open arcs of C determined by x and y. According to Newman (1, Theorem 11.8, p.119), $D - \alpha$ has two components V_1 and V_2 whose boundaries are $L_1 \cup \alpha$ and $L_2 \cup \alpha$ respectively. Because C_n is connected and does not intersect α, it is contained entirely within one component of $D - \alpha$. By symmetry we may assume that C_n is contained in V_2. Since x is a two-sided limit point of $\overline{E}_n$, L_1 must contain a point of $\overline{E}_n$ and hence a point of E_n. Suppose w is an element of $L_1 \cap E_n$. There exists a simple arc β joining w to some point on C_n with $\beta - \{w\} \subseteq f^{-1}((-\infty,r))$. But $\beta - \{w\}$ cannot have a point in common with α because $\alpha - \{x,y\} \subseteq f^{-1}((t-\epsilon, +\infty))$ and $f^{-1}((-\infty, r)) \cap f^{-1}((t-\epsilon,+\infty)) = \phi$. Thus $C_n \cup (\beta - \{w\})$ is a connected set not meeting α while meeting V_2, and so is contained in V_2. Consequently w is in the boundary of V_2. However, this is a contradiction because $w \in L_1$ and the boundary of V_2 is $L_2 \cup \alpha$. Consequently if $x,y \in T$ and $x \neq y$, then $O_x \cap O_y = \phi$.

T is countable since any family of disjoint nonempty open sets is countable. Also t

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countable. Also the set S of all points of $\bar{E}_n$ which are not two-sided limit points of $\bar{E}_n$ is countable. Again let n be any positive integer. Then $K \cap \bar{E}_n = [K \cap S] \cup [K \cap (\bar{E}_n - S)] = (K \cap S) \cup T$. So for any n the intersection of K and E_n is countable. Therefore $N = K \cap \cup \bar{E}_n = \cup (K \cap \bar{E}_n)$ is countable. Let the G_δ set G be the set C minus the union of all $\bar{E}_n$'s. Since $\lambda^{-1}((-\infty,r))$ is contained in the union of the E_n's and therefore the $\bar{E}_n$'s, $C - \lambda^{-1}((-\infty,r)) \supseteq C - \cup \bar{E}_n = G \supseteq K - N$. But $C - \lambda^{-1}((-\infty,r))$ is equal to $\lambda^{-1}([r,+\infty))$ and K contains $\lambda^{-1}((t-\varepsilon,+\infty))$ which contains $\lambda^{-1}([t,+\infty))$; so $\lambda^{-1}([r,+\infty))$ contains G which contains K - N which contains $\lambda^{-1}([t,+\infty)) - N$.

(B) follows from (A) by replacing f and λ by -f and $-\lambda$.

Let S and T be metric spaces. A function f is of <u>Baire class 1(S,T)</u> if and only if

(i) domain g = S,

(ii) range f is contained in T,

(iii) there exists a sequence of continuous functions f_n each mapping S into T such that f_n approaches f pointwise on S.

A function g is of <u>honorary Baire class 2(S,T)</u> if and only if

(i) domain g = S,

(ii) range g is contained in T,

(iii) there exists a function f of Baire class 1(S,T) and a countable set N such that the restriction of f to the set S - N is equal to the restriction of g to S - N.

Any function f from S into the reals is of <u>Baire class 0</u> if and only if it is continuous. For any ordinal number α greater than zero, f is of <u>Baire class α</u> if and only if f is the pointwise limit of a sequence of func

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<u>Theorem 11</u>: If f is a continuous real-valued function in D and λ is a finite-valued boundary function for f, then λ is of honorary Baire class 2(C,R). (Kaczynski, 1, Theorem 2, p.594)

<u>Proof</u>: For each pair of rational numbers r and t with $r < t$, we can choose, from Lemma 1, G_δ sets G(r,t), H(r,t) and countable sets N(r,t), M(r,t) such that $\lambda^{-1}([r,+\infty))$ contains G(r,t) which contains $\lambda^{-1}([t,+\infty)) - N(r,t)$ and $\lambda^{-1}((-\infty,t])$ contains H(r,t) which contains $\lambda^{-1}((-\infty,r]) - M(r,t)$. Let N be the union over r and t of $N(r,t) \cup M(r,t)$ where r is smaller than t. Thus N is countable. Let λ_o be the restriction of λ to C - N and G*(r,t) be G(r,t) - N. Since every countable set is an F_σ set, G*(r,t) is a G_δ set. $\lambda_o^{-1}([r,+\infty)) = \lambda^{-1}([r,+\infty))$ - N which contains G*(r,t) which contains $\lambda^{-1}([t,+\infty))$ - N which is equal to $\lambda_o^{-1}([t,+\infty))$. If t is a fixed rational number, let r_n be elements of a strictly increasing sequence of rational numbers converging to t. Then

$$\bigcap_{n=1}^{\infty} \lambda_o^{-1}([r_n,+\infty)) \supseteq \bigcap_{n=1}^{\infty} G^*(r_n,t) \supseteq \lambda_o^{-1}([t,+\infty)) = \bigcap_{n=1}^{\infty} \lambda_o^{-1}([r_n,+\infty)).$$

And consequently for every rational t, $\lambda_o^{-1}([t,+\infty))$ is a G_δ set.

If u is any real number, choose a strictly increasing sequence of rational numbers t_n converging to u. Then $\lambda_o^{-1}([u,+\infty))$ is equal to the intersection over n of $\lambda_o^{-1}([t_n,+\infty))$. Thus $\lambda_o^{-1}([u,+\infty))$ is a G_δ set. Similarly $\lambda_o^{-1}((-\infty,u])$ is a G_δ set for each real u. Therefore $\lambda_o^{-1}((u,+\infty))$ is the intersection of an F_σ set with C - N where an F_σ set is any set which is the union of a countable number of closed sets. By a theorem of Hausdorff (1,

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of Hausdorff (1, p.309), λ_o can be extended to a real-valued function λ_1 on C such that for every real number u, $\lambda_1^{-1}([u,+\infty))$ is a G_δ set and $\lambda_1^{-1}((u,+\infty))$ is an F_σ set. By Hausdorff (1, Theorem IX) λ_1 is of Baire class 1(C,R). Since $\lambda(x) = \lambda_1(x)$ except for $x \in N$, λ is of honorary Baire class 2(C,R).

Corollary: Let f be continuous. If $f : D \rightarrow R^N$ where R^N is the product of the reals with itself N times and $\lambda : C \longrightarrow R^N$ is a boundary function for f, then λ is of honorary Baire class $2(C,R^N)$. (Kaczynski, 1, Corollary, p.595)

Proof: We express f and λ in terms of their components: $f = < f_1, f_2, \ldots, f_N >$ and $\lambda = <\lambda_1, \lambda_2, \ldots, \lambda_N>$. λ_i is a boundary function for f_i and so is of honorary Baire class 2(C,R). Now we choose a function g_i of Baire class 1(C,R) that agrees with λ_i except on a countable set M_i. Setting $g = <g_1, g_2, \ldots, g_N>$ we see that g is of Baire class $1(C,R^N)$ and that g agrees with λ except on the countable set which is the union of M_i for i = 1,..,N. Hence λ is of honorary Baire class $2(C,R^N)$.

Lemma 2: Suppose g is a continuous function mapping C into R^3, q is a point of R^3, and ϵ is a positive real number. Then there exists a continuous function $g^*: C \longrightarrow R^3$ such that q does not lie in the range of g^* and for all $x \in C$, $|g(x) - q| \geq \epsilon$ implies $g(x) = g^*(x)$. (Kaczynski, 1, Lemma 4, p.596)

Proof: Let S be the set of points y in R^3 for which $|y - q|$ is smaller than ϵ. If the image of C by g is contained in S, let $g^*: C \longrightarrow R^3$ be any continuous fu

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any continuous function whose range does not include q. Otherwise the preimage of S, $g^{-1}(S)$, is a proper open subset of C. Hence it can be expressed in the form $g^{-1}(S) = \cup I_k$ where I_k is the set of elements e^{it} for which $a_k < t < b_k$ and $k \neq 1$ implies that I_k and I_1 are disjoint. Since $g^{-1}(\{q\})$ is a closed compact subset of $g^{-1}(S)$, it is covered by a finite number of the I_k's, say the union of $I_1, I_2, \ldots, I_n$. The end-points e^{ia_k} and e^{ib_k} of I_k are not in $g^{-1}(\{q\})$. So there exists, for each k, a continuous function g_k from $\bar{I}_k$ into R^3 such that $g_k(e^{ia_k}) = g(e^{ia_k})$, $g_k(e^{ib_k}) = g(e^{ib_k})$ and q is not in the range of g_k. We define

$$g^*(x) = g(x) \text{ if } x \in C - (I_1 \cup I_2 \cup \ldots \cup I_n)$$

$$g^*(x) = g_k(x) \text{ if } x \in I_k,\ k = 1,2,\ldots,n.$$

Thus $g^*: C \longrightarrow R^3$ as required.

<u>Theorem 12</u>: If f is a continuous function mapping D into the Riemann sphere W and λ is a boundary function for f, then λ is of honorary Baire class 2(C,W). (Kaczynski, 1, Theorem 3, p.596)

<u>Proof</u>: Since W is a subset of R^3, the Corollary of Theorem 11 shows that λ is of honorary Baire class $2(C,R^3)$. Let g be a function of Baire class $1(C,R^3)$ which differs from λ only on a countable set N. Then $g(C) - W$ is countable. Thus there exists a point q inside of W which is not in the range of g. Let g_n be an element of a sequence of continuous functions converging to g. By Lemma 2 there exists, for each n, a continuous function $g_n^*: C \longrightarrow R^3$ such that q does not lie in the range of g_n^* and, for all $x \in C$, $|g_n(x) - q| \geq 1/n$ implies $g_n(x) = g_n^*(x)$.

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We now want to define a function P. If $a \in R^3 - \{q\}$, let l be the unique ray with endpoint at q that passes through a and P(a) be the point of intersection of l with W. P is a continuous mapping of $R^3 - \{q\}$ onto W and P fixes every point of W. Therefore, $P(g(x)) = \lambda(x)$ if $x \notin N$. $P(g_n^*(x))$ is a continuous function from C into W and $P(g_n^*(x)) \to P(g(x))$ as $n \to \infty$.

Theorem 13: If the function f has a boundary function λ that is a Baire function, then every boundary function for f is a Baire function. If λ is of Baire class $\alpha \geq 3$, then every boundary function for f is of Baire class α. (Bagemihl and Piranian, 1, Theorem 3, p.202)

Proof: Let λ be of class α and suppose that λ_1 is another boundary function for f. By the Ambiguous Point Theorem, λ_1 differs from λ at no more than countably many points; therefore, λ_1 is of Baire class β where β is less than or equal to the maximum of 2 and α according to Hahn (1, Theorem VII, p.352). By a similar argument α is less than or equal to the maximum of 2 and β.

SOME SPECIAL TYPES OF CLUSTER SETS

Frequently mathematicians have investigated special cluster sets of $\bar{D}$. For example, in our introduction of this chapter we mentioned boundary cluster sets and right-hand and left-hand cluster sets. In this section we will consider another type: the outer angular cluster set. In later sections we will consider some others.

A Stolz angle is a domain bounded by an arc of C and two chords of the unit circle ea

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Lemma 3: Let f b

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the unit circle each having $e^{i\theta}$ as an endpoint. The <u>outer angular cluster set</u> $C_A(f,e^{i\theta})$ is defined to be the union of all of the cluster sets $C_\Delta(f,e^{i\theta})$ where Δ is a Stolz angle with vertex at $e^{i\theta}$.

<u>Lemma 3</u>: Let f be an arbitrary complex-valued function in D. Suppose a and b are two fixed real numbers satisfying the condition $-\pi/2 < a < b < \pi/2$. For each $e^{i\theta} \in C$ let $\Delta(\theta)$ be the set of z's in D for which $a < \arg\left[1 - (z/e^{i\theta})\right] < b$. Then there exists a subset F(a,b) of C such that F(a,b) is a set of linear measure zero and for each $e^{i\theta} \in C - F(a,b)$ $C_{\Delta(\theta)}(f,e^{i\theta}) = C_A(f,e^{i\theta})$. (Lappan, 8, Lemma, p.1060)

<u>Proof</u>: Let V_n be an element in a countable base for the open sets of W and S_n be an element in the collection of all finite unions of the sets V_n. For each positive integer j, let $\Delta(\theta,j) = \left\{ z \in D: -\pi/2 + 1/j < \arg\left[1 - (z/e^{i\theta})\right] < \pi/2 - 1/j \right\}$ and $E(r,j,n) = \left\{ e^{i\theta} \in C: f(\Delta(\theta) \cap \{z : |z| > r\}) \subset S_n \text{ and } C_{\Delta(\theta,j)}(f,e^{i\theta}) \text{ is not contained in } \overline{S}_n \right\}$. We want to show that for each pair of positive integers j and n and each real number r in the interval $0 < r < 1$, E(r,j,n) is of linear measure zero. So suppose that there exists a triple r,j,n such that E(r,j,n) is not of linear measure zero. Since E(r,j,n) is measurable and not of measure zero, there exists a subset E* of E(r,j,n) such that E* is closed, has no isolated points and has positive measure. Let G be the set $\{z : |z| \leq r\} \cup \{\Delta(\theta) : e^{i\theta} \in E^*\}$, where $\Delta(\theta)$ is as defined above. By an argument of Noshiro (1, p.71), the boundary of G is a rectifiable Jordan curve. So there exists a subset E' of E* such that E' has positive measure and the boundary of G has a tangent at each point of E' and this tangent is the tangent to C at this

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point. For any point $e^{i\theta} \in E'$ and some $\epsilon > 0$, $\Delta(\theta,j) \cap \{z : |z - e^{i\theta}| < \epsilon\}$ is contained in G. But for each point G for which $|z| > r$, $f(z) \in S_n$. Therefore, $C_{\Delta(\theta,j)}(f,e^{i\theta})$ is contained in $\bar{S}_n$ which contradicts the definition of $E(r,j,n)$. Consequently $E(r,j,n)$ must have linear measure zero.

Suppose $C_{\Delta(\theta)}(f,e^{i\theta}) \neq C_A(f,e^{i\theta})$. Then there must exist some j such that $C_{\Delta(\theta,j)}(f,e^{i\theta}) \neq C_{\Delta(\theta)}(f,e^{i\theta})$. Since each of these cluster sets is compact, there exists an integer n such that $C_{\Delta(\theta)} \subset S_n$ and $C_{\Delta(\theta,j)}(f,e^{i\theta})$ is not contained in $\bar{S}_n$. So for some r, $e^{i\theta} \in E(r,j,n)$. Let $F(a,b)$ denote the union of all of the $E(r,j,n)$'s over all rational numbers r between 0 and 1 and all pairs of positive integers n and j. Since $F(a,b)$ is the countable union of sets of linear measure zero, it is of linear measure zero. If $e^{i\theta} \in C - F(a,b)$, then $C_{\Delta(\theta)}(f,e^{i\theta})$ is equal to $C_A(f,e^{i\theta})$.

Theorem 14: Let f be an arbitrary complex-valued function in D. Then there exists a subset F of C, where F is a set of linear measure zero, such that for each point $e^{i\theta} \in C - F$ and each Stolz angle Δ with vertex at $e^{i\theta}$, $C_\Delta(f,e^{i\theta}) = C_A(f,e^{i\theta})$. (Lappan,8, Theorem 1, p.1060; Brelot and Doob, Theorem 7, p.409)

Proof: Let the elements of two sequences of rational numbers, denoted by a_n and b_n respectively, satisfy the conditions $-\pi/2 < a_n < b_n < \pi/2$ and for each pair of real numbers c and d satisfying the condition $-\pi/2 < c < d < \pi/2$ there exists an integer n such that $c < a_n < b_n < d$. Let $F = \bigcup_{n=1}^{\infty} F(a_n,b_n)$. If $e^{i\theta} \in C - F$ and if Δ is any Stolz angle with vertex at $e^{i\theta}$, then there exists a positive integer n such that

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$$\Delta'(\theta) = \left\{ z \in D : a_n < \arg\left[1 - (z/e^{i\theta})\right] < b_n \right\}$$

and $\Delta'(\theta)$ is contained in Δ. Since $e^{i\theta} \notin F(a_n, b_n)$, $C_{\Delta'(\theta)} = C_A(f, e^{i\theta})$ and so $C_{\Delta}(f, e^{i\theta}) = C_A(f, e^{i\theta})$. Furthermore, F is the countable union of sets of linear measure zero. So F is also of linear measure zero.

In the next paragraph we will give an example of a function which satisfies the conditions of Theorem 14 such that F is uncountable and has positive capacity. First we will explain the term <u>capacity</u>. Let $\mu(E)$ be a non-negative additive set function defined on all the Borel sets in the plane. Let F be a closed bounded set in the plane having a connected complement G and M* be the set of all set functions μ with the property

$$\int_{\zeta \in F} d\mu(\zeta) = 1.$$

We now define the function

$$u(z) = \int_{\zeta \in F} \log(1/|z - \zeta|)\, d\mu(\zeta),$$

and the quantity

$$V_F = \inf_{\mu \in M^*} (\sup_{z \in G} u(z)).$$

Then the capacity of the set F is defined to be cap F $= e^{-V_F}$, and the capacity of any Borel set E is

$$\text{cap } E = \sup_{F \subset E} (\text{cap } F).$$

We now give the following example which is found in (Lappan, 8, p.1062). Let U be the upper half plane, P be the Cantor middle third set on the closed

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set on the closed interval $[0,1]$, and I_n be an element in the collection of open intervals which are complementary to P in (0,1). For each n let T_n be the triangular region bounded by the equilateral triangle in $\bar{U}$ having $\bar{I}_n$ as its base. Let $T = \bigcup_{n=1}^{\infty} T_n$ and $V = U - T$. We define the function f in U to be as follows:

$$f(z) = 0 \quad \text{for} \quad z \in V,$$

$$f(z) = 1 \quad \text{for} \quad z \in T.$$

If F is the subset of C of linear measure zero mentioned in Theorem 14, then $P \subset F$; therefore, F is uncountable and has positive capacity.

HOROCYCLES

In the study of cluster sets of special subsets of $\bar{D}$, one of the most important types of subsets has been the horocycle. A <u>horocycle</u> at a point $e^{i\theta} \in C$ is defined to be a circle internally tangent to C at the point $e^{i\theta}$. The horocycle is denoted by $h_r(e^{i\theta})$ or just h_r where r $(0 < r < 1)$ is the radius of the horocycle. The point $e^{i\theta}$ is not considered to be part of h_r. A point $w \in W$ is a horocyclic cluster value of f at $e^{i\theta}$ if there exists a sequence with elements z_n lying between two horocycles at $e^{i\theta}$ such that $\lim z_n = e^{i\theta}$ and $\lim f(z_n) = w$.

Given a horocycle h_r at a point $e^{i\theta} \in C$, the region interior to h_r is denoted by Ω_r. The half of h_r lying to the right of the radius at $e^{i\theta}$ as viewed from the origin is denoted by $h_r^+(e^{i\theta})$ and is called the <u>right horocycle</u> at $e^{i\theta}$ with radius r. The <u>left horocycle</u> is defined analogously. In addition Ω_r^+ and Ω_r^- denote the right and left half respectively of Ω_r.

Suppose $0 < r_1 < r_2 < 1$ and r_3 $(0 < r_3 < 1)$ is so large that the circle $|z| = r_3$

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circle $|z| = r_3$ intersects both of the horocycles h_{r_1} and h_{r_2}. Then the <u>right horocyclic angle</u> $H^+_{r_1,r_2,r_3}$ at $e^{i\theta}$ with radii r_1, r_2 and r_3 is defined to be

$$H^+_{r_1,r_2,r_3} = \text{comp}\left[\overline{\Omega^+_{r_1}}\right] \cap \Omega^+_{r_2} \cap \left\{ z : |z| \geq r_3 \right\},$$

where the bar denotes closure and "comp" denotes complement, both with respect to the plane. The corresponding <u>left horocyclic angle</u> is denoted by $H^-_{r_1,r_2,r_3}$. H_{r_1,r_2,r_3} denotes a horocyclic angle at $e^{i\theta}$ without specifying whether it is right or left. If we do not wish to specify r_1,r_2,r_3, then the notation is simplified to H.

We now wish to define special types of cluster sets for horocycles. The <u>right outer horocyclic angular cluster set</u> of f at $e^{i\theta}$ is $C_{U^+}(f,e^{i\theta}) = \cup\, C_{H^+}(f,e^{i\theta})$, and the <u>right inner horocyclic angular cluster set</u> of f at $e^{i\theta}$ is $C_{I^+}(f,e^{i\theta}) = \cap\, C_{H^+}(f,e^{i\theta})$, where the union and the intersection are taken over H^+ which ranges over all right horocyclic angles at $e^{i\theta}$. $C_{U^-}(f,e^{i\theta})$ and $C_{I^-}(f,e^{i\theta})$ are defined analogously. The <u>outer horocyclic angular cluster set</u> of f at $e^{i\theta}$ is defined to be $C_U = C_{U^+} \cup C_{U^-}$, and the <u>inner horocyclic angular cluster set</u> of f is defined to be $C_I = C_{I^+} \cap C_{I^-}$. The <u>right principal horocyclic cluster set</u> of f at $e^{i\theta}$ is defined to be $\Pi^+_w = \cap\, C_{h^+_r}$ while the <u>left principal horocyclic cluster set</u> is defined by changing the + signs to - signs. The <u>principal horocyclic cluster set</u> is the intersection of the right and the left principal horocyclic cluster sets and is similar to the <u>principal chordal cluster set</u> which is defined as the intersection of $C_X(f,e^{i\theta})$ over X and denoted by $\Pi_X(f,e^{i\theta})$. C_X is the cluster set of f at $e^{i\theta}$ on the chord X. The inner horocyclic angular cluster set is similar to the <u>inner angular cluster set</u> $C_B(f,e^{i\theta})$ which is the intersection of C

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intersection of $C_\Delta(f,e^{i\theta})$ over all Stolz angles at $e^{i\theta}$.

Finally we wish to define special types of points on C. Any point $e^{i\theta} \in C$ is called a right horocyclic Fatou point of f with right horocyclic Fatou value $w \in W$ whenever C_{U^+} is equal to the set consisting of the single w. A point $e^{i\theta}$ is called a right horocyclic Plessner point of f if $C_{I^+} = W$, and $e^{i\theta}$ is called a right horocyclic Meier point of f provided $\Pi^+_w(f,e^{i\theta}) = C(f,e^{i\theta})$ is properly contained in W. The sets of right horocyclic Fatou points, right horocyclic Plessner points and right horocyclic Meier points of f are denoted by $F^+_w(f)$, $I^+_w(f)$ and $M^+_w(f)$ respectively. The corresponding left horocyclic sets $F^-_w(f)$, $I^-_w(f)$ and $M^-_w(f)$ are defined in an analogous manner. Finally the sets of horocyclic Fatou points, horocyclic Plessner points and horocyclic Meier points of f are denoted by F_w, I_w and M_w respectively and defined as follows: $e^{i\theta} \in F_w$ if C_U is a singleton; $e^{i\theta} \in I_w$ if $C_I = W$, that is, $I_w = I^+_w \cap I^-_w$; $e^{i\theta} \in M_w$ if $\Pi_w = C(f,e^{i\theta})$ which is properly contained in W, that is M_w is the intersection of M^+_w and M^-_w. F(f), I(f) and M(f) denote respectively the sets of Fatou points, Plessner points and Meier points. These points are quite similar to those including "horocyclic" in their names since $e^{i\theta} \in F(f)$ if C_A is a singleton and lim f(z) exists uniformly as z approaches $e^{i\theta}$ in any Stolz angle; $e^{i\theta} \in I(f)$ if $C_\Delta = W$ for every angle Δ; $e^{i\theta} \in M(f)$ if for any chord $\rho(\phi)$ of C passing through $e^{i\theta}$ and making an angle ϕ with the radius to $e^{i\theta}$, $-\pi/2 < \phi < \pi/2$, $C_{\rho(\phi)} = C(f,e^{i\theta})$ which is properly contained in W.

Lemma 4: Let f(z) be an arbitrary function from D into W. Then $C_B(f,e^{i\theta}) = C_I(f,e^{i\theta}) = C(f,e^{i\theta})$ for a residual G_δ subset of C. (Dragosh, 2, Lemma 2, p.60)

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Proof: For any $e^{i\theta} \in C$, let $\Delta_{n,r}(e^{i\theta})$ be the Stolz angle at $e^{i\theta}$ with aperture $\pi/2^n$, where the bisector of $\Delta_{n,r}(e^{i\theta})$ at $e^{i\theta}$ makes a rational angle r, $-\pi/2 < r < \pi/2$, with the radius at $e^{i\theta}$. If a_m is the annulus $1 - 1/m < |z| < 1$ where $1 - 1/m > \sin(|r| + \pi/2^{n+1})$, then let $\Delta_{n,r,m} = \Delta_{n,r}(e^{i\theta}) \cap a_m$. Let $\Sigma(e^{i\theta})$ be the countable collection of all $\Delta_{n,r,m}$ at $e^{i\theta}$ and $\Sigma_w(e^{i\theta})$ be the countable collection of H_{r_1,r_2,r_3} at $e^{i\theta}$ with rational radii r_i.

For each $\Delta \in \Sigma(e^{i\theta})$, $C_\Delta(f,e^{i\theta}) = C(f,e^{i\theta})$ for a residual G_δ subset of C (remark after Theorem 2). The intersection of countably many of these residual G_δ subsets is again a residual G_δ subset E_1 of C such that

$$C(f,e^{i\theta}) = \bigcap_{\Delta \in \Sigma(e^{i\theta})} C_\Delta(f,e^{i\theta}) = C_B(f,e^{i\theta}) \qquad \text{for } e^{i\theta} \in E_1.$$

Also

$$C(f,e^{i\theta}) = \bigcap_{H \in \Sigma_w(e^{i\theta})} C_H(f,e^{i\theta}) = C_I(f,e^{i\theta}) \qquad \text{for } e^{i\theta} \in E_2$$

where E_2 is another G_δ subset and $E_1 \cap E_2$ is the required subset of C.

$S_1, S_2 \subseteq C$ are topologically equivalent if $S_1 - S_2$ and $S_2 - S_1$ are of first category.

Theorem 15: Let $f : D \to W$. Then the sets $I(f)$, $I_w^+(f)$, $I_w^-(f)$, and $I_w(w)$ are topologically equivalent. (Bagemihl, 3, Theorem 4, p.13)

Proof: Since $C_I(f,e^{i\theta}) = C_{I^+}(f,e^{i\theta}) \cap C_{I^-}(f,e^{i\theta})$, $e^{i\theta} \in C$, $C_B(f,e^{i\theta}) = C_{I^+}(f,e^{i\theta}) = C_{I^-}(f,e^{i\theta}) = C(f,e^{i\theta})$ for a residual set of points $e^{i\theta}$ on C.

Theorem 16: If f is an arbitrary function from D into W, then the sets $F(f)$, $F_w^+(f)$, $F_w^-(f)$ and $F_w(f)$ are topologically equivalent. (Bagemihl, 3, Remark 3, p.16)

Proof: By definition we have the following conditions: $C_B \subseteq C_A$, $C_{I^+} \subseteq C_{U^+}$, $C_{I^-} \subseteq C_{U^-}$ and $C_I \subseteq C_U$ for any point on C. Consequently Lemma 4 implies that $C_A = C_{U^+} = C_{U^-} = C_U$ for a residual set of points on C.

In contrast to Theorems 15 and 16, the sets M(f) and $M_w(f)$ are not necessarily topologically equivalent. (Dragosh, 2, Remark 3, p.61) For example, let S be a countable dense subset of C. We define f(z) in D as follows: f(0) = 0, f(z) = 1 for $z \in h^+_{\frac{1}{2}}(e^{i\theta})$ for $e^{i\theta} \in S$ and f(z) = 0 for $z \in h^+_{\frac{1}{2}}(e^{i\theta})$ for $e^{i\theta} \in C - S$. Since S and C - S are both dense on C, Π_w is the element 0 for $e^{i\theta} \in C - S$ and Π_w is the set with a single element 1 for $e^{i\theta} \in S$. Thus M(f) = C, but $M_w(f) = \Phi$.

Lemma 5: If f(z) is an arbitrary function from D into W, then for any set $L(e^{i\theta})$ for which there exists a Stolz angle at $e^{i\theta}$ containing $L(e^{i\theta})$ C_L is contained in C_B except for a set on C which is of measure zero and of first category. (Dragosh, 2, Lemma 3, p.61)

Proof: Let E denote the set of points $e^{i\theta} \in C$ for which C_L is not contained in C_B. Then for each $e^{i\theta} \in E$, there exists a set $L(e^{i\theta})$ lying inside of a Stolz angle at $e^{i\theta}$ for which C_L is not contained in C_Δ for some Stolz angle Δ at $e^{i\theta}$. So there exists a disk Q_p on W such that C_L and Q_p are not disjoint while C_Δ and $\overline{Q}_p$ are. Using the notation in Lemma 4, we can find a Stolz angle $\Delta_{n,r,m,p} \in \Sigma(e^{i\theta})$ such that $\overline{f(\Delta_{n,r,m})}$ and Q_p are disjoint. So we can express E as $\cup E_{n,r,m,p}$ over all subscripts where $e^{i\theta} \in E_{n,r,m,p}$ if there exists at least one set $L(e^{i\theta})$ lying in a Stolz angle at $e^{i\theta}$ such that C_L and Q_p are not disjoint while

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while $\overline{f(\Delta_{n,r,m})}$ and Q_p are disjoint.

Suppose there exists a set $E_{n,r,m,p}$ which has positive outer measure. Then $\overline{f(\Delta_{n,r,m})}$ and Q_p are disjoint for $e^{i\theta} \in \overline{E_{n,r,m,p}}$.

If $G = \cup \Delta_{n,r,m}$ over points $e^{i\theta} \in \overline{E_{n,r,m,p}}$, then G is composed of finitely many open simply connected subsets $G_1,\ldots,G_N$ of D because $C - \overline{E_{n,r,m,p}}$ contains only a finite number of arcs with length exceeding a fixed number between 0 and 2π. Privalow (1, p.220) has shown that each G_k, for $1 \leqslant k \leqslant N$, has a rectifiable Jordan curve J_k as its boundary.

Since $E_{n,r,m,p}$ is assumed to have positive outer measure, the intersection of $E_{n,r,m,p}$ and J_k must have positive exterior measure for at least one J_k. The tangent to J_k at almost every point of $C \cap J_k$ coincides with the tangent to C. Consequently there exist points in $E_{n,r,m,p}$ belonging to $C \cap J_k$ at which the tangent to J_k coincides with the tangent to C. At any such point each Stolz angle at that point has its terminal portion contained in G_k. So there exist points $e^{i\theta}$ in $E_{n,r,m,p}$ such that C_L is contained in $\overline{f(G_k)}$ for each set $L(e^{i\theta})$ at $e^{i\theta}$ which is contained in a Stolz angle at $e^{i\theta}$. Since $\overline{f(\Delta_{n,r,m})}$ and Q_p are disjoint for any point in $\overline{E_{n,r,m,p}}$ and G is the union over points in $\overline{E_{n,r,m,p}}$ of $\Delta_{n,r,m}$, $\overline{f(G_k)}$ and Q_p are disjoint. However, the definition of $E_{n,r,m,p}$ says that for each point in $E_{n,r,m,p}$ $C_L \cap Q_p \neq \phi$ for at least one set $L(e^{i\theta})$ lying in a Stolz angle at $e^{i\theta}$ which is a contradiction. Therefore, each set $E_{n,r,m,p}$ has measure zero, and so E also has measure zero.

By a similar argument it can be shown that each $E_{n,r,m,p}$ is of first category, and consequently E is of first category.

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<u>Theorem 17</u>: Let f(z) be an arbitrary function from D into W and let K(f) denote the set of points $e^{i\theta} \in C$ for which $C_{\Delta_1}(f,e^{i\theta}) = C_{\Delta_2}(f,e^{i\theta})$ for any pair of Stolz angles Δ_1 and Δ_2 at $e^{i\theta}$. Then K(f) is residual and of measure 2π on C. (Dragosh, 2, Theorem 2, p.63)

<u>Proof</u>: At each point $e^{i\theta} \in C - K(f)$, there exists a Stolz angle Δ such that C_{Δ} is not contained in C_B. By Lemma 5, C - K(f) is of measure zero and of first category.

This theorem is a very important result as it generalizes Theorems 2 and 14.

<u>Theorem 18</u>: Let f(z) be an arbitrary function from D into W and let $K_w(f)$ denote the set of points $e^{i\theta} \in C$ for which $C_{H_1} = C_{H_2}$ for any pair of horocyclic angles H_1 and H_2 at $e^{i\theta}$. Then K_w is residual and of measure 2π on C. (Dragosh, 2, Theorem 3, p.67)

The theorem can be proved in a manner very similar to that of Lemma 5.

Two sets S_1 and S_2 are called <u>metrically equivalent</u> if and only if measure $(S_1 - S_2)$ = measure $(S_2 - S_1) = 0$.

<u>Corollary</u>: If f(z) is an arbitrary function from D into W, then the sets F_w^+, F_w^- and F_w are metrically equivalent and the sets I_w^+, I_w^- and I_w are metrically equivalent. (Dragosh, 2, Corollary 1, p.68)

<u>Proof</u>: Suppose $e^{i\theta}$ belongs to at least one of the sets F_w^+, F_w^- and F_w but not to all of them. Then there exists a pair of horocyclic angles

H_1 and H_2 at $e^{i\theta}$ such that $C_{H_1} \neq C_{H_2}$. By Theorem 18 the set of such points $e^{i\theta} \in C$ is of measure zero. So F_w^+, F_w^- and F_w are metrically equivalent. The proof for I_w^+, I_w^- and I_w is identical.

F(f) and $F_w(f)$ need not be metrically equivalent. For example (Dragosh, 2, Theorem 5, p.69), we are able to define the Blascke product

$$B(z) = \prod_{n=1}^{\infty} \frac{(\zeta_n)^{2^n} + (z)^{2^n}}{(1 + \zeta_n z)^{2^n}} \quad \text{where } \zeta_n = 1 - (n^2 2^n)^{-1}$$

for any positive integer n which has zeros at the points

$$z_{n,k} = \zeta_n e^{i(2k-1)2^{-n}\pi}, \quad k = 1,2,\ldots,2^n \text{ and } n > 0.$$

For each point $\zeta \in C$ and each horocycle h_r for $0 < r < 1$ at the point, there exist sequences of these zeros lying interior to Ω_r^+ and Ω_r^-. Thus for each point in C, $0 \in C_{\Omega_r^+}$ for $0 < r < 1$ and similarly for $C_{\Omega_r^-}$. A Blaschke product has a Fatou value of modulus one at any point of C except for a set of measure zero. Let ζ be a Fatou point of B that has the Fatou value a with $|a| = 1$. If ζ is a right horocyclic Fatou point of B, then $C_{\Omega_r^+}$ is the set with the single element 0 for $0 < r < 1$. Since this contradicts the fact that C_Δ is the set with the single element a for each Stolz angle Δ at ζ, the set of right horocyclic Fatou points of B is of measure zero. By the corollary following Theorem 18, $F_w(f)$ has measure zero.

I(f) and $I_w(f)$ also need not be metrically equivalent. Dragosh (1, Theorem, p.41) constructs a function f(z) holomorphic in D such that every point of C is a horocyclic Plessner point of f and almost every point of C is a Fatou point of f.

<u>Lemma 6</u>: If $f(z)$ is an arbitrary function from D into W, then for any set $H^*(e^{i\theta})$ for which there exists a disk Ω_r at $e^{i\theta}$ containing $H^*(e^{i\theta})$ $C_{H^*}(f,e^{i\theta})$ is contained in $C_I(f,e^{i\theta})$ except for a set on C which is of measure zero and of first category. (Dragosh, 2, Lemma 6, p.67)

<u>Proof</u>: Much of the proof of this lemma is analogous to the proof of Lemma 5. We replace Stolz angles by horocyclic angles and the region G by a region G^W, which is defined as follows; let P be a perfect nowhere dense subset of C and $H_{r_1,r_2,r_3}(e^{i\theta})$ be a fixed horocyclic angle, then G^W is the union of all of the H_{r_1,r_2,r_3}'s for $e^{i\theta}$ in P. According to Bagemihl (3, Lemma 1) G^W is composed of finitely many simply connected subregions $G_1^W,\ldots,G_k^W$ having as their respective boundaries the rectifiable Jordan curves $J_1^W,\ldots,J_k^W$. So the tangent to J_n^W for $1 \leq n \leq k$ at almost every point $e^{i\theta} \in C \cap J_n^W$ coincides with the tangent to C. We must now show that except for a set of measure zero contained in the set $C \cap J_n^W$, each horocyclic angle H at $e^{i\theta}$ has a terminal portion which lies in G_n^W because the tangent to H at $e^{i\theta}$ also coincides with the tangent to C.

In order to verify the last statement, we will first show that if P is a perfect nowhere dense subset of $[0, 1]$, then for almost every point $p \in P$ for which a sequence of open intervals (a_n,b_n) in $[0, 1] - P$ converges to p, $|a_n - p|/(b_n - a_n)$ tends to positive infinity. If E is any Lebesgue measureable set in R^1 for which the upper and lower limits of the quotient

$$\frac{\text{meas }(E \cap (x-\delta, x+\delta))}{2\delta}$$

are equal, then their common value is called the metric density of E at x. According to Hobson (p.194), in our case the metric density exists and is equal to 1 at almost every point $p \in P$. Let $p \in P$ be a point with metric density equal to 1 and suppose that the sequence $\{(a_n, b_n)\}$ converges to p from the right. Then

$$\lim_{n\to\infty} \frac{\text{meas}(P \cap (p,b_n))}{\text{meas}(p,b_n)} = \lim_{n\to\infty} \frac{\text{meas}(P \cap (p,a_n))}{\text{meas}(p,a_n)} = 1$$

and
$$\lim_{n\to\infty} \frac{\text{meas}(P \cap (p,b_n))}{(a_n - p) + (b_n - a_n)} \longrightarrow 1.$$

Since $P \cap (p,b_n) = P \cap (p,a_n)$, $\displaystyle\lim_{n\to\infty} \frac{\text{meas}(P \cap (p,b_n))}{a_n - p} \longrightarrow 1$.

Also since $\text{meas}(P \cap (p,b_n))$, $a_n - p$ and $b_n - a_n$ are each greater than zero, these conditions imply that $\lim\left[(b_n - a_n)/(a_n - p)\right]$ approaches zero. Consequently $(a_n - p)/(b_n - a_n) \to +\infty$, and in general $|a_n - p|/(b_n - a_n) \to +\infty$.

Now we will show that except for a set of measure zero contained in the set $C \cap J_n^w$, each horocyclic angle $H(e^{i\theta})$ for $e^{i\theta} \in J_n^w$ has a terminal portion which lies in G_n^w. By means of a bilinear transformation $L(z)$, it is possible to map D onto the upper half plane and to prove this result there. Let P be a nowhere dense set on the finite interval I on the real axis and $\{(a_n, b_n)\} \subset I - P$. We now choose circles C_1 : $(x - a_n)^2 + (y - R)^2 = R^2$ and $C_2 : (x - b_n)^2 + (y - r)^2 = r^2$ where

$$0 < R_1 \leqslant r \leqslant R_2 < R_3 \leqslant R \leqslant R_4. \tag{3}$$

We choose r and R in this manner so that the two horocycles h_{r_1} and h_{r_2} forming part of H_{r_1,r_2,r_3} are mapped by $L(z)$ onto circles of the form C_1 and C_2 as $e^{i\theta}$ ranges over $P \subset C$. At the left and right endpoints of each interval in $I - P$ we construct the circles C_1 and C_2 respectively. We will now prove that at almost every point $p \in P$ and for any sequence of arcs with elements (a_n, b_n) in I - P converging to p, the point $(x_n, y_n) \in C_1 \cap C_2$ closest to p lies interior to any given circle tangent to the x-axis at p for at most a finite number of n's. Our method of proof will be to show that if the previous condition holds, then $|a_n - p|/(b_n - a_n)$ tends to positive infinity. By the previous paragraph this limit is valid at almost every point of P. Suppose there exists a point $p \in P$ for which every sequence of open intervals in $[0,1] - P$ converging to p satisfies the condition $|a_n - p|/(b_n - a_n)$ tends to infinity, but for which there exists a sequence of elements (a_n, b_n) such that the point $(x_n, y_n) \in C_1 \cap C_2$ lies interior to a given circle for an infinite number of n's. Without loss of generality, we assume that $p = 0$. So we are assuming that there exists a circle: $x^2 + (y - \rho)^2 = \rho^2$ such that $x_n^2 + (y_n - \rho)^2 < \rho^2$ for infinitely many n. Since $|a_n|/(b_n - a_n)$ tends to infinity and signum (a_n) = signum (b_n), $|b_n + a_n|/(b_n - a_n)$ also tends to infinity.

Let L_1 denote the line which passes through the points of intersection of C_1 and C_2. L_1 satisfies the equation

$$(x - a_n)^2 + (y - R)^2 - R^2 - \left[(x - b_n)^2 + (y - r)^2 - r^2\right] = 0$$

or

$$x = \frac{R - r}{b_n - a_n} y + (b_n + a_n)/2.$$

Therefore,

$$x_n = \frac{R - r}{b_n - a_n} y_n + (b_n + a_n)/2. \tag{4}$$

By solving Equation (4) and the equation for C_1 simultaneously for y_n, we have

$$\left(\frac{R - r}{b_n - a_n} y_n + (b_n + a_n)/2 - a_n\right)^2 + (y_n - R)^2 = R^2,$$

which can be rewritten as

$$y_n(R - r)^2/(b_n - a_n)^2 + (b_n - a_n)^2/y_n = R + r - y_n.$$

As y_n approaches 0 from the right, we have $y_n = O((b_n - a_n)^2)$ where O indicates the order of the function. So $y_n < K(b_n - a_n)^2$ for $0 < K$ and n sufficiently large. Substituting Equation (4) into the condition $x_n^2 + (y_n - \rho)^2 < \rho^2$ we have

$$\left(\frac{R - r}{b_n - a_n}\right)^2 y_n + (R - r)\left(\frac{b_n + a_n}{b_n - a_n}\right) + \left(\frac{b_n + a_n}{2}\right)^2 \frac{1}{y_n} + y_n < 2\rho. \tag{5}$$

The left-hand side of this inequality is greater than

$$(R - r)\left(\frac{b_n + a_n}{b_n - a_n}\right) + \left(\frac{b_n + a_n}{2}\right)^2 \frac{1}{y_n}$$

which by Condition (3) and the condition for y_n is greater than

$$(R_3 - R_2)\left(\frac{b_n + a_n}{b_n - a_n}\right) + \left(\frac{b_n + a_n}{2}\right)^2 \frac{1}{K(b_n - a_n)^2} = \frac{b_n + a_n}{b_n - a_n}\left[R_3 - R_2 + \frac{b_n + a_n}{4K(b_n - a_n)}\right].$$

Since $\left|b_n + a_n\right|/(b_n - a_n)$ tends to infinity, the lower bound on Inequality (5) also tends to infinity. Hence $x_n^2 + (y_n - \rho)^2 < \rho^2$ can hold at most for a finite number of n's.

<u>Theorem 19</u>: If f(z) is an arbitrary function from D into W, then for any point $\zeta \in C$, C_Δ is contained in C_H for every Stolz angle Δ and every horocyclic angle at ζ except for a set on C which is of measure zero and of first category. (Dragosh, 2, theorem 4, p.68)

<u>Proof</u>: If ζ is a point where C_Δ is not contained in C_H, then C_Δ is not contained in C_I for some Stolz angle Δ at ζ. So this theorem follows immediately from Lemma 6.

Theorem 20: If f is a arbitrary function from D into W, then almost every horocyclic Fatou point of f is a Fatou point of f and almost every Plessner point of f is a horocyclic Plessner point of f. (Dragosh, 2, Corollary 2, p.68)

Proof: If $\zeta \in F_W(f)$, then there exists a horocyclic angle $H(\zeta)$ at ζ and a point $w \in W$ such that C_H is the set containing only the point w. From Theorem 19, it follows that C_A also contains only the point w for almost every point $\zeta \in F_W(f)$.

If $\zeta \in I(f)$, then $C_B = W$. According to Theorem 19 C_B is contained in C_I for almost every point $\zeta \in C$. So $C_I = W$ for almost every point in $I(f)$.

ORICYCLIC CLUSTER SETS

Oricycles are another type of special subsets of $\bar{D}$ which have been studied in the field of cluster sets. Let ζ be any point on C and $U(\zeta)$ denote the inscribed disk at ζ such that $U = \{z: |z - \rho\zeta| < 1 - \rho\}$ where ρ is a constant such that $0 < \rho < 1$. Then the oricyclic cluster set is defined to be $C_o(f,\zeta) = \bigcap_U C_U(f,\zeta)$. A UU-singular point is any point $\zeta \in C$ such that there exists a pair of inscribed disks U' and U'' for which $C_{U'}(f,\zeta) \neq C_{U''}(f,\zeta)$. Let V be any open angle with vertex at ζ. Then a VV-singular point is any point of C such that there exists a pair of angles V' and V'' for which $C_{V'}(f,\zeta) \neq C_{V''}(f,\zeta)$. The set of all UU (or VV) -singular points is called the UU (or VV) -singular set and is denoted by $E_{UU}(f)$ (or $E_{VV}(f)$). A GU (or GV) -singular point is any point $\zeta \in C$ for which there exists a U (or V) for which $C_U \neq C(f,\zeta)$ (or $C_V \neq C(f,\zeta)$). The GU (or GV) -singular set is denoted by $E_{GU}(f)$

(or $E_{GV}(f)$). A UV-singularity is defined analogously.

For any $\epsilon > 0$ let $N_\epsilon(\zeta)$ denote the neighborhood consisting of elements z such that $|z - \zeta| < \epsilon$. Suppose we are given a set E in C and a point ζ on C. Let $r(\zeta,\epsilon) = r(\zeta,\epsilon,E)$ be the largest of the lengths of arcs contained in $N_\epsilon \cap C$ and not intersecting E. Then for any α, $0 < \alpha \leq 1$, the set E is said to have porosity (α) at ζ if $\overline{\lim}\,(r(\zeta,\epsilon))^\alpha/\epsilon > 0$ as $\epsilon \to 0$. E is said to have porosity (α) on C if each point ζ in E has porosity (α). A set which is a countable sum of sets of porosity (α) is called a <u>σ-porosity</u> (<u>α</u>) set.

Yanagihara (1) has shown that E_{UU} and E_{UV} are $G_{\delta\sigma}$ sets and of σ-porosity for some α (Theorems 22 and 23) while Dolženko (1) has shown that E_{VV} is a $G_{\delta\sigma}$ set and E_{GV} is an F_σ set (Theorems 21 and 24).

<u>Theorem</u> <u>21</u>: If f(z) is an arbitrary function, not necessarily single-valued, then E_{VV} is of $G_{\delta\sigma}$ type and of σ-porosity for some α. (Dolženko, 1, Theorem 1, p.3)

<u>Proof</u>: Let $\{\alpha_n\}$ denote a sequence consisting of all rational numbers between $-\pi/2$ and $\pi/2$, and let $\{\overline{D_n}\}$ be a sequence consisting of all closed circles in $W - \{\infty\}$ having rational radii r_n and centers at the points a_n with rational coordinates. If $\zeta \in C$, then $V_{p,q}$ denotes the open angle of size α_p with vertex at ζ and with bisector forming an angle α_q with the interior normal to C at ζ. We define $E_{n,p,q}$ to be the set of all points $\zeta \in C$ such that if $z \in D$, $\rho(z,C) < 1/p$, and for z in $V_{p,q}$, the values of f(z) lie at a distance $\geq r_n$ from $\overline{D_n}$ where r_n is the radius of $\overline{D_n}$. For m,n,s,k any positive integers, let $F_{n,m,k,s}$ be the set of all points $\zeta \in C$ for which the set

< [illegible]

$$\left\{ f(z) : z \text{ is in } D \cap V_{m,k} \text{ and } 1/(3s) < \rho(z,C) < 1/s \right\}$$

has points in common with the disk $\overline{D_n}$. Then each $E_{n,p,q}$ is closed and each $F_{n,m,k,s}$ is open on C. We set $F_{n,m,k} = \bigcap_{t=1}^{\infty} \bigcup_{s=t}^{\infty} F_{n,m,k,s}$.

We will now show that $E_{VV} = \bigcup_{n,p} \bigcup_{q,m,k} (F_{n,m,k} \cap E_{n,p,q})$. Suppose $\zeta \in E_{VV}(f)$. Then there exist angles V' and V" such that $C_{V'} \neq C_{V''}$. Suppose $C_{V'} - C_{V''} \neq \phi$. Then we can choose m and k such that $V_{m,k} \supset V'$. So $C_{V_{m,k}} - C_{V''} \neq \phi$ and there exists a disk $\overline{D_u}$ such that for some p and q

$$\overline{D_u} \cap C_{V_{m,k}} \neq \phi \text{ and } \rho(\overline{D_u}, C_{V_{p,q}}) > 5r_u.$$

Consequently we can find positive integers p and q such that if z is in $V_{p,q}$ and $\rho(z,C) < 1/p$, then $\rho(\overline{D_u}, f(z)) > 4r_u$. Let n denote the index of the disk $\overline{D_n}$ which has the same center as $\overline{D_u}$ and radius $r_n = 2r_u$. So $\rho(\overline{D_n}, f(z)) > r_n$ if z is in $V_{p,q}$ and $\rho(z,C) < 1/p$. Due to the choice of $\overline{D_u}$, there exists a sequence of elements $\hat{z}$ in $V_{m,k}$ which approaches ζ and a corresponding sequence of elements $f(\hat{z})$ which approaches a point $a \in \overline{D_n}$. So for an infinite set of positive numbers, there exist points $\hat{z}$ in $V_{m,k}$ for $1/(3s) < \rho(\hat{z},C) < 1/s$ such that $\overline{D_n}$ and $\{f(\hat{z})\}$ are not disjoint. Thus $\zeta \in \bigcup_{s=t}^{\infty} F_{n,m,k,s}$ for all t and is therefore also in $F_{n,m,k}$.

Now suppose $\zeta \in F_{n,m,k}$ and $E_{n,p,q}$. Then by the definition of $E_{n,p,q}$, $C_{V_{p,q}}$ and $\overline{D_n}$ are disjoint. Since ζ is in $F_{n,m,k}$, it is in $F_{n,m,k,s}$ for an infinite number of s's. From the definition of $F_{n,m,k,s}$, it follows that $C_{V_{m,k}}$ and $\overline{D_n}$ are not disjoint and so $C_{V_{m,k}}$ is not equal to $C_{V_{p,q}}$.

We will now show that E_{VV} is σ-porous. Suppose on the contrary there exists a point in $F_{n,m,k}$ and $E_{n,p,q}$ which is not σ-porous with

respect to C. Then the angle $V_{m,k}$ close to its vertex is covered by a union of angles $V^{\eta}_{p,q}$ for η in $F_{n,m,k}$ and $E_{n,p,q}$. So by the definition of $E_{n,p,q}$ at points z in $V_{m,k}$ which are sufficiently close to the point, the values of f(z) are at a distance $\geq r_n$ from $\overline{D_n}$. Therefore $C_{V_{m,k}}$ and $\overline{D_n}$ are disjoint and the point is not in $F_{n,m,k}$ and $E_{n,p,q}$. Thus $F_{n,m,k} \cap E_{n,p,q}$ is porous on C and E_{VV} is σ-porous.

This theorem is closely related to the Collingwood Maximality Theorem (Collingwood, 3), which states that for an arbitrary single-valued function f(z) defined in D and any Stolz angle Δ with vertex at ζ, $C_{\Delta}(f,\zeta) = C(f,\zeta)$ except for a set of first category. It is also related to Theorem 14 which states that for an arbitrary single-valued function f(z) defined in D the outer angular cluster set $C_A = C_{\Delta}$ except for a set of measure zero.

<u>Theorem</u> <u>22</u>: If f(z) is an arbitrary function, then E_{UU} is of $G_{\delta\sigma}$ type and of σ-porosity for some α. (Yanagihara, 1, Theorem 1, p.424)

The proof of this theorem is quite similar to that of Theorem 21.

<u>Theorem</u> <u>23</u>: If f(z) is an arbitrary function, then E_{UV} is of $G_{\delta\sigma}$ type and of σ-porosity (α). (Yanagihara, 1, Theorem 2, p.425)

Yanagihara (1, Theorem 4, p.426) has shown that there exists a bounded holomorphic function f(z) for which E_{UV} is of measure 2π. For example, we pick an inscribed disk $U(1) = \{z : |z - \rho| < 1 - \rho\}$ for $0 < \rho < 1$. Then there exists a constant b such that an arc $\lambda = \{z = re^{i\theta} : \theta = b\sqrt{1 - r}\}$ is contained in U(1). In addition we choose t_n

such that $0 < t_n < 1$, is strictly increasing to 1 and $\sum_{n=1}^{\infty}\sqrt{1 - t_n} < \infty$.

Let

$$f(z) = \prod \frac{z^{k_n} - t_n^{k_n}}{(t_n z)^{k_n} - 1}$$

where the integers k_n are determined by $k_n = \left[3\pi/b\sqrt{1 - t_n}\right] + 1$. This product converges because $\sum_{n=1}^{\infty} k_n(1 - t_n)$ is finite. For every point $\zeta \in C$, $U(\zeta)$ contains an infinite number of zeros of $f(z)$ and C_U contains 0, but $f(z)$ has angular limits of modulus 1 at almost every point of C. Thus E_{UV} has measure 2π.

Theorem 24: For an arbitrary function, not necessarily single-valued, $E_{GV}(f)$ is of F_σ type and of first category. (Dolzenko, 1, Theorem 3, p.9)

The proof of this theorem uses much of the notation and style of Theorem 21. As before each $E_{n,p,q}$ is closed. $E_{GV} = \bigcup_{n,p,q} E_{n,p,q}$. If E_{GV} is not of first category, then there would exist a set E_{n_o,p_o,q_o} such that on an open arc λ contained in E_{n_o,p_o,q_o}, $C(f,\zeta)$ would be at a distance at least r_{n_o} from D_{n_o} for any ζ in λ, which contradicts the property that $E_{n_o,p_o,q_o} \cap C(f,\zeta) \neq \phi$.

SELECTOR OF ARCS

Γ is called a selector of arcs if it is a correspondence which associates with each point in C a nonempty collection Γ of arcs at that point. If Γ is a selector of arcs, then the Γ-principal cluster set of f at a point $e^{i\theta}$ is defined to be the set $\Pi_\Gamma(f,e^{i\theta}) = \bigcap_{\alpha} C(f,e^{i\theta},\alpha)$ where $C(f,e^{i\theta},\alpha)$ denotes the arc cluster set of f at $e^{i\theta}$ along α and the intersection is taken over all α in $\Gamma(\theta)$. If the intersection is taken over all arcs at $e^{i\theta}$, then the notation $\Pi(f,e^{i\theta})$ is used.

If μ is any positive number and $e^{i\theta_0}$ is any point on C, then let C_μ denote the set $\{e^{i\theta} \in C : 0 < |e^{i\theta} - e^{i\theta_0}| < \mu\}$. For any function f(z) defined in D, let $\Pi_\Gamma(f,e^{i\theta_0},\mu) = \cup\, \Pi_\Gamma(f,e^{i\theta})$ where the union is over $e^{i\theta}$ which ranges over all points in C_μ and let Π_Γ^* denote the closure of Π_Γ in the Riemann sphere. Then the <u>boundary</u> <u>Γ-principal</u> <u>cluster</u> <u>set</u> of f at $e^{i\theta_0}$ is defined to be the set

$$B\Pi_\Gamma(f,e^{i\theta_0}) = \bigcap_{\mu>0} \Pi_\Gamma^*(f,e^{i\theta_0},\mu).$$

If Γ contains all arcs at $e^{i\theta_0}$, then the notation $B\Pi(f, e^{i\theta_0})$ is used.

For any subset S contained in D, a point $e^{i\theta} \in C$ is called <u>almost</u> <u>Γ-accessible</u> <u>through</u> <u>S</u> if for every open set G with $S \subseteq G \subseteq D$ there exists an arc $\alpha \in \Gamma$ such that $\alpha \subseteq G$. This definition is abbreviated to $e^{i\theta}$ is <u>almost</u> <u>accessible</u> <u>through</u> <u>S</u> in the case that Γ is the collection of all arcs at $e^{i\theta}$ which is a point of C. Let E be contained in C and γ be a correspondence which associates with each point in E an arc $\gamma(e^{i\theta})$ in $\Gamma(e^{i\theta})$. Let $\overline{S}(\gamma,E)$ denote the relative closure in D of the set $S(\gamma,E) = \cup\, \gamma(e^{i\theta})$ where the union is over all $e^{i\theta}$ in E. Then Γ is a <u>smooth</u> <u>selector</u> <u>of</u> <u>arcs</u> if for every set E of second category in C and every arc γ, there exists a subarc $A \subseteq C$ such that E is dense in A and every point of A is almost Γ-accessible through $\overline{S}(\gamma,E)$.

If Γ is a selector of arcs, then a new selector of arcs Γ^* called the <u>completion</u> <u>of</u> <u>Γ</u> is defined by $\{\alpha : \alpha \subseteq \beta \in \Gamma(e^{i\theta})\}$. Finally Γ is called an <u>admissible</u> <u>selector</u> <u>of</u> <u>arcs</u> if Γ^* is a smooth selector of arcs.

The theorems which we prove in this section will lead to the major result stated in Theorem 29 that if f is a continuous function in D, then $\Pi(f,e^{i\theta}) = B\Pi(f,e^{i\theta})$ and $\Pi_X(f,e^{i\theta}) = B\Pi_X(f,e^{i\theta})$ where X denotes the collection of all chords at $e^{i\theta}$ except for a set of first category.

Theorem 25: If f is an arbitrary complex-valued function defined in D and $\mathbf{\Gamma}$ is any selector of arcs, then there exists a selector of arcs $\mathbf{\Gamma}_o$ such that for each $e^{i\theta} \in C$, $\mathbf{\Gamma}_o(e^{i\theta})$ is a finite or countable subset of $\mathbf{\Gamma}(e^{i\theta})$ and $\mathbf{\Pi}_{\mathbf{\Gamma}}(f,e^{i\theta}) = \mathbf{\Pi}_{\mathbf{\Gamma}_o}(f,e^{i\theta})$. (Gresser, 2, Theorem, 7, p.11)

Proof: Let γ be any arc of $\mathbf{\Gamma}$. Then $B_\gamma = W - C_\gamma(f,e^{i\theta})$ is open in W. So $\bigcup_{\gamma\in\Gamma} B_\gamma = W - \mathbf{\Pi}_{\mathbf{\Gamma}}(f, e^{i\theta})$ and by the Lindelof covering property there is a countable subcovering with elements B_{γ_n} of $W - \mathbf{\Pi}_{\mathbf{\Gamma}}(f, e^{i\theta})$. Consequently $\bigcup_{n=1}^{\infty} B_{\gamma_n} = W - \mathbf{\Pi}_{\mathbf{\Gamma}}(f, e^{i\theta})$ and $\mathbf{\Pi}_{\mathbf{\Gamma}}(f,e^{i\theta}) = \bigcap_{n=1}^{\infty} C_{\gamma_n}(f,e^{i\theta}) = \mathbf{\Pi}_{\mathbf{\Gamma}_o}(f,e^{i\theta})$.

Theorem 26: If f(z) is an arbitrary complex-valued function defined in D and $\mathbf{\Gamma}$ is any selector of arcs, then $\mathbf{\Pi}_{\mathbf{\Gamma}}(f,e^{i\theta}) \subseteq B\mathbf{\Pi}_{\mathbf{\Gamma}}(f,e^{i\theta})$ for all except for at most a countable number of points $e^{i\theta}$ in C. (Gresser,2, Theorem 4, p.6)

Proof: For any positive integer j, let T_j be a finite collection of compact neighborhoods on W which cover W and such that using the usual metric for W, we have diameter (G) < 1/j for G any subset of T_j. Choosing a finite number of G's for each j, we let $T_j = \bigcup_n G_{n,j}$ for each j and define P to be $\left\{e^{i\theta} \in C : \mathbf{\Pi}_{\mathbf{\Gamma}}(f,e^{i\theta}) \not\subseteq B\,\mathbf{\Pi}_{\mathbf{\Gamma}}(f,e^{i\theta})\right\}$. Let

$$P_{n,j} = \left\{e^{i\theta} \in P : G_{n,j} \cap \mathbf{\Pi}_{\mathbf{\Gamma}}(f,e^{i\theta}) \neq \phi\right\}.$$

Each $G_{n,j}$ is contained in $W - B\,\mathbf{\Pi}_{\mathbf{\Gamma}}(f,e^{i\theta})$ for each positive integer j. If $e^{i\theta} \in P$, then there exists a point $w \in \mathbf{\Pi}_{\mathbf{\Gamma}}(f,e^{i\theta})$ such that

$w \notin B\,\Pi_\Gamma(f,e^{i\theta})$ which is closed in W. Thus there exist n and j such that $w \in G_{n,j} \cap B\,\Pi_\Gamma(f,e^{i\theta}) = \phi$. Therefore $e^{i\theta} \in P_{n,j}$ and $P = \bigcup_{n,j} P_{n,j}$.

Now we wish to show that each $P_{n,j}$ is at most countable. In order to show this, we fix j and n, and let $e^{i\theta} \in P_{n,j}$. If $e^{i\theta}$ is not an isolated point of $P_{n,j}$, then there exists a sequence $\{\zeta_k\}$ of points in $P_{n,j}$ that converges to $e^{i\theta}$. Since each $\zeta_k \in P_{n,k}$, $G_{n,j} \cap \Pi_\Gamma(f,\zeta_k) \neq \phi$ for each positive integer k. So for each $\mu > 0$ $G_{n,j} \cap \Pi_\Gamma^*(f,e^{i\theta},\mu) \neq \phi$. Let $\{\mu_k\}$ be a decreasing sequence of positive real numbers which converges to zero. Then

$$G_{n,j} \cap B\,\Pi_\Gamma(f,e^{i\theta}) = \bigcap_{k=1}^{\infty} (G_{n,j} \cap \Pi_\Gamma^*(f,e^{i\theta},\mu_k)) \neq \phi$$

because W is compact. This contradicts the assumption that $e^{i\theta} \in P_{n,j}$. So each $e^{i\theta} \in P_{n,j}$ is an isolated point and the set P is at most countable.

Theorem 27: Let f be defined in D and Γ be a selector of arcs. If G is any open subset of W such that for some $e^{i\theta} \in C$, $G \cap B\,\Pi_\Gamma(f,e^{i\theta}) \neq \phi$, then there exists a sequence $\{\zeta_j\}$ of points in C which converges to $e^{i\theta}$ such that $G \cap \Pi_\Gamma(f,\zeta_j) \neq \phi$ for each j. (Gresser, 2, Lemma 5, p.8)

This theorem follows easily from the definition of $\Pi_\Gamma(f,e^{i\theta})$.

Lemma 7: Let f be continuous in D and Γ be an admissible selector of arcs. For each point $e^{i\theta} \in C$, let β be an arc in $\Gamma(e^{i\theta})$. Then $B\,\Pi_\Gamma(f,e^{i\theta}) \subseteq C_\beta(f,e^{i\theta})$ except for at most a set of first category. (Gresser, 2, Lemma 6, p.8)

<u>Proof</u>: Suppose the lemma is false. Then the set P denotes the set of points $e^{i\theta} \in C$ for which $B\,\Pi_\Gamma(f,e^{i\theta}) \not\subseteq C_\beta(f,e^{i\theta})$ is of second category in C. Let ϵ be an arbitrary positive number and $S(e^{i\theta},\epsilon)$ denote the set of all points in W whose spherical distance from $C_\beta(f,e^{i\theta})$ does not exceed ϵ. Since $C_\beta(f,e^{i\theta})$ is closed in W, it follows that for each point $e^{i\theta} \in P$ there exists an $\epsilon(e^{i\theta}) > 0$ such that $B\Pi_\Gamma(f,e^{i\theta}) - S(e^{i\theta},\epsilon) \neq \phi$. Let $\{\epsilon_j\}$ be a decreasing sequence of positive numbers which converges to zero and $P_j = \left\{e^{i\theta} \in P : B\Pi_\Gamma(f,e^{i\theta}) - S(e^{i\theta},\epsilon_j) \neq \phi\right\}$. Since $P = \cup P_j$ and is of second category, there exists a J such that P_J is of second category. We choose a finite collection $\{G_1,\ldots,G_m\}$ of open sets each of diameter $< \epsilon_J/4$. For $\mu \leq m$ let $P_J(\mu) = \left\{e^{i\theta} \in P_J : G \cap (B\Pi_\Gamma - S(e^{i\theta},\epsilon_J)) \neq \phi\right\}$. Since P_J is a union of the $P_J(\mu)$'s and P_J is of second category, there exists an M such that $P_J(M)$ is of second category.

For two subsets A and B of W, let a be any point in A and b be any point in B. Then the <u>spherical distance $\chi(A,B)$</u> between the sets A and B is defined to be the infimum of the spherical distances between points a and b. From the definition of $P_J(\mu)$ it follows that $\chi(G_M, C_\beta) \geq 3\epsilon_J/4$ for any point $e^{i\theta}$ in $P_J(M)$. According to Theorem 27 every point $e^{i\theta}$ in $P_J(M)$ is a limit point of the set $Q = \left\{e^{i\theta} \in C : G_M \cap \Pi_\Gamma(f,e^{i\theta}) \neq \phi\right\}$.

We will now show that $\chi(G_M,C_\beta) \ngeq 3\epsilon_J/4$ is valid for a subarc of C which violates the definition of Q. For each $e^{i\theta} \in P_J(M)$ let $\gamma(e^{i\theta})$ be a terminal subarc of β such that $f(\gamma(e^{i\theta})) \subseteq S(e^{i\theta},\epsilon_J/4)$. If S denotes the set $\cup\gamma(e^{i\theta})$ where the union is over $e^{i\theta} \in P_J(M)$, then $\chi(G_M,S) \geq \epsilon_J/2$ since $\chi(G_M,C_\beta(f,e^{i\theta})) \geq 3\epsilon_J/4$ for $e^{i\theta} \in P_J(M)$. From the continuity of f, $\chi(G_M,\overline{S}) \geq \epsilon_J/2$ where $\overline{S}$ denotes the relative closure in D of S. Let G be an open set such that $f(\overline{S}) \subseteq G$ and $\chi(G_M,G) \geq \epsilon_J/4$. By the continuity of f, the set $U = f^{-1}(G)$ is open in D and contains $\overline{S}$. Since Γ is an admissible selector of arcs, there exists a subarc $A \subseteq C$ such that each point of A is almost Γ^*-accessible through $\bar{S}$ for Γ^* the completion of Γ. So for every point $e^{i\theta} \in A$, there exists an arc $\alpha \in \Gamma^*(e^{i\theta})$ such that $\alpha \subseteq U$. Thus by the definition of U, $C(f,\alpha) \subseteq \bar{G}$ for $e^{i\theta} \in A$. Since α is a terminal subarc of β, an arc in $\Gamma(e^{i\theta})$, the two arc-cluster sets are the same. Therefore, $\Pi_\Gamma(f,e^{i\theta}) \subseteq \bar{G}$ for $e^{i\theta} \in A$ and $\chi(G_M,\Pi_\Gamma) \geqslant \epsilon_J/4$, a contradiction to the definition of Q.

<u>Theorem</u> <u>28</u>: Let f be a continuous function in D and Γ be an admissible selector of arcs. Then $\Pi_\Gamma(f,e^{i\theta}) = B\Pi_\Gamma(f,e^{i\theta})$ for nearly every point $e^{i\theta} \in C$. (Gresser, 2, Theorem 8, p.11)

<u>Proof</u>: According to Theorem 25 for each $e^{i\theta} \in C$ there is a finite or countable subset of $\Gamma(e^{i\theta})$, say $\left\{\alpha_j(e^{i\theta})\right\}$ such that

$$\Pi_\Gamma(f,e^{i\theta}) = \bigcap_{j=1}^{\infty} C(f,e^{i\theta},\alpha_j(e^{i\theta})).$$

If the set $\left\{\alpha_j(e^{i\theta})\right\}$ is finite, we repeat one of the arcs infinitely often. For each j, let P_j denote the set $\left\{e^{i\theta} \in C : B\Pi \not\subseteq C_{\alpha_j}(f,e^{i\theta})\right\}$. By Lemma 7 each of the sets P_j is of first category in C. Therefore the set $P = \bigcup_{j=1}^{\infty} P_j$ is of first category and $B\Pi_\Gamma(f,e^{i\theta}) \subseteq \Pi_\Gamma(f,e^{i\theta})$ for $e^{i\theta} \in C - P$. The proof is completed by using Theorem 26.

<u>Theorem</u> <u>29</u>: If f is a continuous function in D, then $\Pi(f,e^{i\theta}) = B\Pi(f,e^{i\theta})$ and $\Pi_\chi(f,e^{i\theta}) = B\Pi_\chi(f,e^{i\theta})$, where χ denotes the collection of all chords at $e^{i\theta}$, except for at most a set of first category. (Gresser, 2, Theorem 9, p.11)

<u>Proof</u>: In order to apply Theorem 28 we must show that $\mathbf{A}(e^{i\theta})$ the collection of all arcs at $e^{i\theta}$ is an admissible selector of arcs. Let E be a second category subset of C and $\lambda(e^{i\theta}) \in \mathbf{A}(e^{i\theta})$ for each $e^{i\theta} \in E$. For any positive integer j, we define $E_j = \{e^{i\theta} \in E : \lambda$ intersects the circle $|z| = 1 - 1/j\}$. Then $E = \bigcup_{j=1}^{\infty} E_j$ and so there exists an N such that E_N is of second category on C. Therefore, there is an open subarc $A \subseteq C$ such that E_N is dense in A. Let $S = \cup\, \lambda(e^{i\theta})$ where the union is taken over $e^{i\theta} \in E_N$. Let G be an open set such that $\overline{S} \subseteq G \subseteq D$. We let $e^{i\theta_0}$ be an arbitrary point in A and D_o be an open disk centered at $e^{i\theta_0}$ having radius $r \leq 1 - 1/N$. Let $\{\zeta_n\}$ be a sequence of distinct points in $E \cap D_o$ which converges to $e^{i\theta_0}$. Then for each n, let λ_n be the component of $\lambda(\zeta_n) \cap D_o$ which forms a terminal subarc of $\lambda(\zeta_n)$. We will show that there is a component G_o of $G \cap D_o$ such that $\lambda_n \subseteq G_o$ for infinitely many n. Let λ_{n_k} be a subsequence of λ_n's which converges to a limit set L. If $L \cap D_o \cap D \neq \phi$, let $z \in L \cap D_o \cap D$. Then $z \in \overline{S} \cap D_o$ so that z is contained in some component G_o of $G \cap D_o$. Since $z \in L \cap G_o$ and G_o is open, it follows from the definition of limits that there exists an M such that $G_o \cap \lambda_{n_k} \neq \phi$ for all $k > M$. Thus since λ_{n_k} is a connected subset of $G \cap D_o$, $\lambda_{n_k} \subseteq G_o$ for all $k > M$. So suppose $L \cap D_o \cap D = \phi$. Let $e^{i\theta} \in L \cap E_N \cap D_o$ and α be the component of $\lambda(e^{i\theta}) \cap D_o$ which forms a terminal subarc of $\lambda(e^{i\theta})$. By the definition of convergence, there exists an M such that $\alpha \cap \lambda_{n_k} \neq \phi$ for all $k > M$. Since α is a connected subset of $G \cap D_o$, α is contained in a component G_o of $G \cap D_o$. Furthermore, λ_{n_k} is a connected subset of $G \cap D_o$. So $\lambda_{n_k} \subseteq G_o$ for $k > M$. Therefore, we have established that there exists a component G_o of $G \cap D_o$ such that $\lambda_n \subseteq G_o$ for infinitely many n.

For each positive integer k let D_k denote the open disk centered at $e^{i\theta_0}$ having radius $(1 - 1/N)/k$. Now we will construct a sequence $\{G_k\}$ of open connected subsets such that $G \supseteq G_1 \supseteq G_2 \ldots$ and each $G_k \subseteq D_k$. If $e^{i\theta} \in D_k$, let α^k denote the component of $\alpha \cap D_k$ which forms a terminal subarc of α. If $e^{i\theta} \notin D_k$, we let $\alpha^k = \phi$. Let $\{\zeta_m\}$ be a sequence of distinct points in E_N which converges to $e^{i\theta_0}$. Since there exists a component G_o such that $\lambda_n \subseteq G_o$ for infinitely many n, we can select an infinite subset T_1 of $\{\lambda(\zeta_m)\}$ and a component G_1 of $G \cap D_1$ such that $\phi \neq \alpha^1 \subseteq G_1$ for each $\alpha \in T_1$. Inductively we can define sequences $\{G_k\}$ and $\{T_k\}$ for each positive integer k such that G_k is a component of $G \cap D_k$, $\{\lambda(\zeta_m)\} \supseteq T_1 \supseteq T_2 \ldots$, $\phi \neq \alpha^k \subseteq G_k$ for each $\alpha \in T_k$. We fix k and let $T_{k+1} \subseteq T_k$. Since $\phi \neq \alpha^{k+1} \subseteq \alpha^k$, and $\alpha^{k+1} \subseteq G_{k+1}$ and $\alpha^k \subseteq G_k$, it follows that $G_{k+1} \cap G_k \neq \phi$. But $G_{k+1} \subseteq G \cap D_{k+1} \subseteq G \cap D_k$. Since G_{k+1} is connected and G_k is a component of $G \cap D_k$ which intersects G_{k+1}, $G_{k+1} \subseteq G_k$. Finally using the G_k's which are arcwise connected, it is possible to construct an arc at $e^{i\theta_0}$ which lies in G. Consequently $\Lambda(e^{i\theta})$, the collection of all arcs at $e^{i\theta}$ is an admissible selector of arcs.

Now we will prove that the theorem is true for χ, the collection of all chords at $e^{i\theta}$. Let $\{\Delta_j\}$ be a countable collection of closed Stolz angles at $e^{i\theta_0} = 1$ such that each chord at $e^{i\theta_0}$ is contained in at least one of the Δ_j's. For each positive integer j, let $\Delta_j(e^{i\theta})$ be the closed Stolz angle at $e^{i\theta} \in C$ obtained by rotating Δ_j about the origin. Then for each j and each $e^{i\theta} \in C$, let $\chi_j(e^{i\theta})$ be the collection of all chords at $e^{i\theta}$ which are contained in $\Delta_j(e^{i\theta})$. By an argument completely analogous to that for $\Lambda(e^{i\theta})$, χ_j is an admissible selector of arcs. Consequently by Theorem 28 for each j there exists a set E_j of first category in C such that $B\Pi_{\chi_j}(f, e^{i\theta}) = \Pi_{\chi_j}(f, e^{i\theta})$ for $e^{i\theta} \in C - E_j$.

Since $B\Pi_\chi(f,e^{i\theta}) \subseteq B\Pi_{\chi_j}(f,e^{i\theta})$ for each j and $e^{i\theta} \in C$, we have $B\Pi_\chi(f,e^{i\theta}) \subseteq \Pi_{\chi_j}(f,e^{i\theta})$ for $e^{i\theta} \in C - E_j$. The set $E = \bigcup_{j=1}^{\infty} E_j$ is of first category in C. So $B\Pi_\chi(f,e^{i\theta}) \subseteq \Pi_{\chi_j}(f,e^{i\theta})$ for $e^{i\theta} \in C - E$. Finally by Theorem 26 $\Pi_\chi(f,e^{i\theta}) \subseteq B\Pi_\chi(f,e^{i\theta})$ except for at most a countable number of points in C, since $B\Pi_\chi \subseteq \bigcap_{j=1}^{\infty} \Pi_{\chi_j} = \Pi_\chi$ for $e^{i\theta} \in C - E$.

THEOREMS FOR SPECIAL TYPES OF FUNCTIONS

As one might expect, there are numerous theorems relating to the theory of cluster sets which are only valid for special types of functions. In the remaining sections of this chapter we will consider some of the more important results for various types of functions including those which are continuous, light interior and locally univalent.

Continuous Functions

The <u>set of curvilinear convergence of a function f</u> is defined to be the set $\left\{x \in C : \text{there exists an arc } \gamma \text{ at } x \text{ and a point } p \text{ in some metric space such that } \lim_{\substack{z \to x \\ z \in \gamma}} f(z) = p\right\}$.

<u>Theorem 30</u>: If f is a continuous function from D into W, then the set of curvilinear convergence of f is a $F_{\sigma\delta}$ set. (McMillan, 1, Theorem 5, p.302)

First we wish to define special subsets F(n,j,k) of D. For each positive integer n let $\left\{\Delta(n,j)\right\}_{j=1}^{\infty}$ be an enumeration of the open disks each having its center at b, a point of W whose stereographic projection has rational real and imaginary parts and such that the set

$\{z \in D: (f(z),b) < 4^{-n}\}$ contains points arbitrarily close to C. For each pair of natural numbers n and j, let $\{D(n,j,k)\}$ be an enumeration of the components of the nonempty open set $f^{-1}(\Delta(j,n)) \cap \{1 - 1/n < |z| < 1\}$. Then $F(n,j,k)$ is defined to be $\overline{D}(n,j,k) \cap \overline{A}$ where A is the set of curvilinear convergence of f. Let N denote the set of points $e^{i\theta} \in C$ for which there exist an $n > 1$ and integers j_1, j_2, k_1, k_2 with the following properties. If $e^{i\theta} \in F(n,j_1,k_1) \cap F(n,j_2,k_2)$, then either $\overline{\Delta}(n,j_1) \cap \overline{\Delta}(n,j_2) = \phi$ or there exist j_o, k' and k'' with $k' \neq k''$ such that

$$\overline{\Delta}(n,j_1) \cup \overline{\Delta}(n,j_2) \subset \Delta(n-1,j_o),$$
$$D(n,j_1,k_1) \subset D(n-1,j_o,k'),$$
$$D(n,j_2,k_2) \subset D(n-1,j_o,k'').$$

Then Theorem 30 is proved by verifying that N is countable and that $\bigcap_n (\bigcup_{j,k} F(n,j,k)) - (N \cup N') \subset A \subset \bigcap_n (\bigcup_{j,k} F(n,j,k))$ where N' is the countable set of points of $\overline{A}$ that are not two-sided accumulation points of A.

An <u>analytic arc</u> is an arc described by parametric equations $x = \gamma(t)$, $y = \psi(t)$ for $0 < t < 1$ where the functions γ and ψ can be represented in some neighborhood of t for $0 < t < 1$ by a power series with real coefficients and throughout this neighborhood at least one of the derivatives γ' and ψ' is nonzero.

Let t_1, t_2 and t_3 be Jordan arcs contained in $D \cup \{p\}$. If there exist Jordan arcs t_4, t_5 and t_6 in $D \cup \{p\}$ such that $t_1 \subset t_4$, $t_2 \subset t_5$ and $t_6 \subset t_3$, where $t_4 \cup t_5$ is a Jordan curve and $t_6 - \{p\}$ is contained in the bounded region whose boundary is $t_4 \cup t_5$, then t_3 is said to be between t_1 and t_2.

Theorem 31: If f is a continuous function from D into W and $p \in C$, then there exist analytic arcs α, β and γ each ending at p such that

(A) $C_\alpha(f,p) = C(f,p)$, $C_\beta(f,p) = C_{B1}(f,p)$ and $C_\gamma(f,p) = C_{Br}(f,p)$,

(B) α, β and γ are mutually disjoint except for their common endpoint p and

(C) α lies between β and γ.

(H.T.Mathews, 2, Theorem 1, p.1265)

Proof: We may assume without loss of generality that $p = 1$. Let $\{w_i\}_{i=1}^{\infty}$ be a countable dense subset of $C(f,1)$. Then there exist in D sequences $\{z_{ij}\}$ such that $z_{ij} \to 1$ as $j \to \infty$ and $f(z_{ij}) \to w_i$. From the z_{ij}'s we form a sequence $\{z_j\}$ such that $z_j \to 1$, Real $(z_j) <$ Real (z_{j+1}) and w_i is the limit of a subsequence of $\{f(z_j)\}$. We pick open disks E_1, E_2, ... in D sufficiently small so that f assumes only values close to $f(z_j)$ and with centers $z_1, z_2, \ldots$ respectively such that Real (a) < Real (b) for each $a \in E_j$ and $b \in E_{j+1}$. In addition if $\{x_j\}$ is any sequence with $x_j \in E_j$, then $x_j \to 1$ and each w_j is the limit of a subsequence of $\{f(x_j)\}$. Let L_j denote that part of the vertical line passing through z_j that lies in $D - E_j$ and L denote the slit disk $D - \bigcup_j L_j$. Then L is a simply connected domain and so by the Riemann mapping theorem there exists a conformal mapping ψ of D onto L. Moreover, it may be assumed that, when extended to the boundary, ψ takes -1 and 1 onto themselves. If α is the image under ψ of the real line segment $[-1,1]$, then α is an analytic arc ending at 1. Since α must pass through each disk E_j, $C_\alpha(f,1) = C(f,1)$.

Let A denote the set of all points q on C such that $0 \leq \arg(q) \leq \pi/4$.

According to Gross (1, pp.248 - 250) there exists an arc δ ending at 1 such that δ lies between α and A and if $\{z_j\}$ is any sequence of points lying between δ and A such that $z_j \to 1$ and $f(z_j) \to w$, then $w \in C_{B1}$. Let λ be an arc in D joining a point on δ to a point on A so that the domain Δ bounded by δ, λ and a subarc of C is a Jordan domain containing 1 in its closure. Let h be the restriction of f to Δ. Then $C(h,1) = C_{B1}(f,1)$. Since the preceding paragraph can be extended to Jordan domains, by conformal mappings there exists in Δ an analytic arc β ending at 1 such that $C_\beta(h,1) = C(h,1)$. Thus $C_\beta(h,1) = C_{B1}(f,1)$. The arc γ can be constructed in a similar manner.

If α_1 and α_2 are asymptotic paths of an arbitrary function $f : D \to W$ for the values a_1 and a_2 respectively, then $d(\alpha_1,\alpha_2)$ denotes the infimum of rational numbers δ such that some disk Δ, whose diameter is δ and whose center has a stereographic projection with rational real and imaginary parts, has the properties (i) $\{a_1, a_2\} \subset \Delta$ and (ii) α_1 and α_2 are eventually in the same component of $f^{-1}(\Delta) \cap \{1 - \delta < |z| < 1\}$. Any path $\beta = z(t)$ for $0 \leq t \leq 1$ such that $|z(t)| \to 1$ as $t \to 1$ is <u>eventually</u> in the subset $S \subset D$ provided that there exists a t_o for $0 \leq t_o \leq 1$ such that $z(t) \in S$ whenever $t_o \leq t \leq 1$.

If f is a continuous function from D into W, then two asymptotic paths α_1 and α_2 are <u>equivalent</u>, denoted by $\alpha_1 \sim \alpha_2$ if and only if $d(\alpha_1,\alpha_2) = 0$. Let C* denote the set of equivalence classes of asymptotic paths determined by the relation $\sim$ and $[\alpha]$ denote the element of C* to which the asymptotic path α belongs. For $[\alpha_1]$, $[\alpha_2]$ in C*, set $\rho([\alpha_1],[\alpha_2]) = d(\alpha_1,\alpha_2)$. For each $[\alpha] \in C^*$, let $\nu[\alpha]$ denote the limit value of f on α. Then both $\nu[\alpha]$ and ρ are well-defined.

Theorem 32: The metric space (C^*, ρ) is separable and complete. (McMillan, 1, Theorem 1, p.300)

Proof: In order to show separability we need to define a countable dense set D^*. This can be done in the following manner. We choose a disk Δ whose center has a stereographic projection with rational real and imaginary parts and whose diameter is a rational number δ. If for some point in Δ there exists an asymptotic path which is eventually in the component U of the set $f^{-1}(\Delta) \cap \{1 - \delta < |z| < 1\}$, then we pick one such asymptotic path and denote it by $\alpha(U)$. Then D^* is defined to be the set of all $[\alpha(U)]$ where $\alpha(U)$ is defined. So D^* is countable and dense.

Suppose $\{[\alpha_n]\}_{n=1}^{\infty}$ is a Cauchy sequence of elements in C^*. By condition (i) in the preceding definitions, $\{\nu[\alpha_n]\}$ is a Cauchy sequence in W. So $\{\nu[\alpha_n]\}$ must converge to some point $a \in W$. Let $\{\Delta_j\}$ be a sequence of disks such that each one has a rational radius and a center whose stereographic projection has rational real and imaginary parts, and the Δ_j's satisfy the conditions $\Delta_j \supset \Delta_{j+1}$ for $j \geq 1$ and $\bigcap_{j=1}^{\infty} \Delta_j = \{a\}$. Let δ_j denote the diameter of Δ_j. Then for each j there exists a component U_j of $f^{-1}(\Delta_j) \cap \{1 - \delta_j < |z| < 1\}$ and a positive integer n_j such that if $n \geq n_j$, then α_n is eventually in each U_j. Since $U_j \supseteq U_{j+1}$ for $j \geq 1$, there exists a boundary path α that is eventually in each U_j. Since $\bigcap_{j=1}^{\infty} \Delta_j = \{a\}$, α is an asymptotic path of f for the value a and $\rho([\alpha_n], [\alpha]) \to 0$ as $n \to \infty$.

M-Topology for Continuous Functions

Suppose f is a continuous complex-valued function defined in D. Then we let T(p) denote the set of all Jordan arcs contained in $D \cup \{p\}$ and having one endpoint at p, and let $G_f(p) = \{C_t(f,p) : t \in T(p)\}$. In order to define the metric M, we choose two nonempty closed sets A and B in W and set $M(A,B) = \max(\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b))$ where $d(w_1,w_2)$ is the chordal distance between w_1 and w_2. Then this metric M topologizes the set $G_f(p)$ with what is called the M-topology.

Any sequence $\{t_n\}$ of Jordan arcs in T(p) is said to be a directed sequence if for each positive integer n, the arc t_{n+1} lies between t_n and t_{n+2}.

In this section we will include some of the results of Belna and Lappan related to the M-topology. For example, if f is a continuous function in D and $p \in C$ is not an ambiguous point of f, then $G_f(p)$ is compact in the M-topology (Theorem 33, below). Additional results for normal functions will be included in Chapter II.

Theorem 33: Suppose f is a continuous function in D and p is a point in C which is at the same time not an ambiguous point of f. Then $G_f(p)$ is a compact set in the M-topology. (Belna and Lappan, 1, Theorem 1, p. 211)

Proof: Suppose $G_f(p)$ is not compact in the M-topology. Then there exist a sequence of continua $\{K_n\}$ and a continuum K such that $K_n \in G_f(p)$ for each positive integer n, $K \notin G_f(p)$ and $M(K_n,K) \to 0$. For each positive integer n, let $H_n = \{z \in D : d(f(z),K_n) < 1/n \text{ and } |z - p| < 1/n\}$. Since $K_n \in G_f(p)$, there exist a component G_n of H_n and an arc $t_n \in T(p)$

such that $C_{t_n}(f,p) = K_n$ and $t_n \subset G_n \cup \{p\}$.

Suppose $G_n \cap G_{n+1} \neq \phi$ for each n. There exists a Jordan curve $t_o \in T(p)$ such that t_o passes through the consecutive G_n and such that $M(\overline{f(t_o \cap G_n)}, K_n) \to 0$. But then $C_{t_o}(f,p) = K$ in violation of the assumption $K \notin G_f(p)$. Therefore, there exists an integer n such that $G_n \cap G_{n+1} = \phi$. For this integer n the boundary of the component G_n contains a set L such that $L \cup \{p\}$ is a closed connected set. Since f is uniformly continuous on each compact subset of D, there exists a sequence $\{s_j\}$ of points on L such that $s_j \to p$ and such that for each point z on any rectilinear segment $[s_j, s_{j+1}]$ the condition $d(f(z), K_n) > 1/2n$ is satisfied. Some subset of the union of segments $[s_j, s_{j+1}]$ constitutes an element of T(p). Since $C_s(f,p) \cap C_{t_n}(f,p) = C_s(f,p) \cap K_n = \phi$, p is an ambiguous point of f.

<u>Corollary</u>: Let f be a continuous function in D and E be the set of points p for which $G_f(p)$ is not compact in the M-topology. Then E is a countable set. (Belna and Lappan, 1, Corollary 1, p.212)

This corollary follows immediately from Theorem 33 and the Bagemihl Ambiguous Point Theorem (Theorem 4).

<u>Theorem 34</u>: Suppose f is a continuous function in D and $p \in C$. If $\{t_n\}$ is a directed sequence of arcs in T(p) such that $C_{t_n}(f,p) = K_n$ and if K is a continuum such that $M(K_n, K) \to 0$ but $K \notin G_f(p)$, then there exists a directed sequence of arcs $\{s_k\}$ in T(p) and $\epsilon > 0$ such that for each integer $k > 0$ there exists an integer $n_k > 0$ such that s_k is between t_{n_k} and $t_{n_{k+1}}$ and $d(C_{s_k}(f,p), K) > \epsilon$. (Lappan, 11, Lemma 1, p.88)

Proof: We will prove this theorem by assuming that it is false and then showing that we obtain a contradiction. If this theorem is false, then for each positive integer k there exists an integer N_k such that for each $\delta > 0$ which is sufficiently small, $n > N_k$ implies that all of the sets $t_n \cap \{z \in D : |z - p| < \delta\}$ lie in the same component of $\{z \in D : d(f(z),K) < 1/k, |z - p| < \delta\}$. Therefore, for each $n > N_k$ and each $\delta > 0$ there exists a Jordan arc q_n leading from a point of t_n to a point of t_{n+1} such that $q_n \subset \{z \in D : |z - p| < \delta \text{ and } d(f(z),K) < 1/k\}$. So we may choose a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $n_k > N_k$ for each positive integer k. Then for each k there exists a Jordan arc p_k leading from a point on t_{n_k} to a point on $t_{n_{k+1}}$ such that $p_k \subset \{z \in D : |z - p| < 1/k \text{ and } d(f(z),K) < 1/k\}$, and the portion t'_k of t_{n_k} between the terminal point of p_{k-1} and the starting point of p_k satisfies the relationship $M(f(t'_k),K) < 1/k$. Without loss of generality we may assume that p_k meets t_{n_k} and $t_{n_{k+1}}$ in exactly one point each. Then letting t be the Jordan arc obtained by splicing together all of the arcs t'_k and p_k, we have $C_t(f,p) = K$ contradicting the hypothesis $K \notin G_f(p)$.

Theorem 35: Suppose f is a continuous function in D and p is a point in C such that $G_f(p)$ is not compact in the M-topology. Then there exist directed sequences $\{t_n\}$ and $\{s_n\}$ of arcs in T(p), $\epsilon > 0$ and a continuum K such that if $K_n = C_{t_n}(f,p)$ and $L_n = C_{s_n}(f,p)$, then for each $n > 0$ $M(K_n,K) < 1/n$, $d(L_n,K) > \epsilon$ and the arc s_n is between t_n and t_{n+1}. (Lappan, 11, Lemma 2, p.89)

Proof: Let $\{t_n\}$ be a sequence of arcs in T(p) satisfying the conditions $C_{t_n}(f,p) = K_n$ and $M(K_n,K) < 1/n$ where $K \notin G_f(p)$. If the arcs are not mutually disjoint, they can be shortened individually so that an infinite subset of the shortened arcs are mutually disjoint. If this was not true, there would exist an arc $t \in T(p)$ where t is contained in the union of the t_n's and $C_t(f,p) = K$ which contradicts the assumption on K. Now we can choose a directed subsequence of the t_n's. In addition we can select an appropriate continuum K since $G_f(p)$ is not compact. So the conclusion of this theorem follows from Theorem 34.

Light Interior Functions

A function f from D into W is called a <u>light interior function</u> if f is a continuous open map which does not take any continuum into a single point. It has been shown that f has a factorization $f = g \circ h$ where h is a homeomorphism of the unit disk onto itself or onto the finite complex plane and g is a nonconstant meromorphic function.

Let A(f) denote the set of all $e^{i\theta}$ for which there exists an asymptotic path of f in D which includes $e^{i\theta}$ in its end and let $A_p(f)$ denote the set of all $e^{i\theta}$ for which there exists an asymptotic path of f in D which ends at the point $e^{i\theta}$. For any homeomorphism h of D onto D, we define B(h) to be the set of all $e^{i\theta}$ for which there exists an asymptotic path of h in D with end E and $e^{i\theta}$ is contained in the interior of E.

<u>Theorem 36</u>: Suppose f is a light interior function with factorization $f = g \circ h$. If A(g) is dense on C, then $A(f) \cup B(h)$ is dense on C. Furthermore, if $A_p(g)$ and $A_p(h)$ are dense on C, then $A_p(f) \cup B(h)$ is dense on C. (J. Mathews, 1, Theorem, p.79)

Proof: We will prove this theorem by assuming that it is false and show that we have a contradiction. Let the arc $(\psi_1,\psi_2) \subset C - A(f)$ be arbitrary and $[\theta_1,\theta_2] \subset (\psi_1,\psi_2)$ with $0 < \theta_2 - \theta_1 < 2\pi$. Let Γ_1 and Γ_2 be Jordan arcs in D ending at $e^{i\theta_1}$ and $e^{i\theta_2}$ respectively with $\Gamma_1 \cap \Gamma_2 = \{0\}$. Then h maps the domain Δ bounded by $\Gamma_1 \cup \Gamma_2$ and the arc $[\theta_1,\theta_2]$ onto a domain Δ_1 in D.

Case (i) $[\bar{\Delta}_1 \cap C] = C_{\Gamma_1}(h,\theta_1) \cup C_{\Gamma_2}(h,\theta_2)$.

Then there exist a point $e^{i\alpha} \in C_{\Gamma_1}(h,\theta_1) \cap C_{\Gamma_2}(h,\theta_2)$ and sequences $\{z_n\}$ and $\{z_n'\}$ in Γ_1 and Γ_2 respectively with $h(z_n) \to e^{i\alpha}$ and $h(z_n') \to e^{i\alpha}$. Let Λ be a Jordan arc at $e^{i\theta}$ which passes consecutively through the points $h(z_1)$, $h(z_1')$, $h(z_2)$, $h(z_n')$,.... According to Collingwood and Cartwright (Lemma 1, p.93), either $[\theta_1,\theta_2] \subset C_\Lambda(h^{-1},\alpha)$ or $[\theta_2,\theta_1+2\pi] \subset C_\Lambda(h^{-1},\alpha)$. Therefore, either $(\theta_1,\theta_2) \subset B(h)$ or $(\theta_2,\theta_1+2\pi) \subset B(h)$ and $(\psi_1,\psi_2) \cap B(h) \neq \phi$.

Case (ii) $[\bar{\Delta}_1 \cap C] \supset C_{\Gamma_1}(h,\theta_1) \cup C_{\Gamma_2}(h,\theta_2)$, with a proper inclusion.

Then $E = [\bar{\Delta}_1 \cap C] - [C_{\Gamma_1}(h,\theta_1) \cup C_{\Gamma_2}(h,\theta_2)]$ is a nonempty open subarc of C. Let $e^{i\alpha}$ be in both E and A(g). Then $e^{i\alpha}$ is in the end of an asymptotic path Λ of g. But $C(h^{-1},\alpha) \subset [\theta_1,\theta_2]$ so that $h^{-1}(\Lambda)$ is an asymptotic path of f whose end intersects $[\theta_1,\theta_2]$. Therefore, $[\theta_1,\theta_2] \cap A(f) \neq \phi$, a contradiction.

Consequently both cases lead to contradictions. Since (ψ_1,ψ_2) was arbitrary $A(f) \cup B(h)$ is dense on C. The second part of the theorem is proved similarly.

Locally Univalent Functions

Any function $f(z)$ meromorphic in D is called locally univalent if $f(z)$ has at most simple poles and $f'(z) \neq 0$. The function has Koebe arcs if there exist curves $J_n \subset D$ such that for some $\alpha < \beta < \alpha + 2\pi$ and some constant c which is possibly ∞

(i) J_n intersects the radii $\arg z = \alpha$ and $\arg z = \beta$ for each n,

(ii) $|z| \to 1$ for $z \in J_n$ as $n \to \infty$,

(iii) $|f(z) - c| < \epsilon$ for $z \in J_n$ as $n \to \infty$.

For any set G, the boundary of G is denoted by ∂G.

Theorem 37: Let $f(z)$ be a meromorphic locally univalent function without Koebe arcs. Then $f(z)$ has three distinct asymptotic values on each arc of C. (McMillan and Pommerenke, Theorem, p.31)

Proof: Suppose that there exists an arc A of C on which there is at most one asymptotic value. So we may assume without loss of generality that $f(z)$ has no finite asymptotic value on A. Let $d(z)$ denote the radius of the largest disk around $f(z)$ having no branch points on the Riemann image surface F. Since f is locally univalent there is a boundary point on the periphery of this disk. Seidel and Walsh (p.133) have shown that $d(z) \leq (1 - |z|^2)|f'(z)|$ for $|z| < 1$. There exists a sequence $\{z_n\}$ converging to some interior point ζ of A such that $f'(z)$ is bounded. Consequently $d(z_n) \to 0$. Assume $f(z_n) \to c$ where c is possibly ∞. Let P_n be the pre-image of the segment on F from $f(z_n)$ to the nearest boundary point b_n. Thus $f(z) \to b_n$ for $z \in P_n$ as $|z| \to 1$. Since there are no Koebe arcs, P_n ends at a point, say ζ_n. $|f(z) - f(z_n)| < d(z_n) \to 0$, $f(z_n) \to c$ and $z_n \to \zeta$ for $z \in P_n$ as $n \to \infty$.

Then $\zeta_n \to \zeta$ because there are no Koebe arcs on which $f(z) \to c$. Therefore, $f(z)$ has the finite asymptotic value b_n at $\zeta_n \in A$.

Now suppose there are no asymptotic values on the arc A except 0 and ∞. From the preceding paragraph it follows that 0 and ∞ are asymptotic values on a dense subset of A. Let $a \in A$ be a point at which there is the asymptotic value 0. Hence there is a path P ending at a such that $f(z) \to 0$ as $z \to a$ for $z \in P$. Let $G(\lambda)$ denote the component of $\{z : |f(z)| < \lambda, \lambda > 0\}$ that contains the part of P near a. Then $C \cap \partial G(\lambda) \subset A$ for small positive λ because there are no Koebe arcs on which $f(z) \to 0$. For such a value of λ, $G(\lambda)$ does not contain any asymptotic path for values $\neq 0$, but it does contain the path P on which $f(z) \to 0$. Since the Riemann image surface F does not contain branch points it follows that $f(z)$ maps $G(\lambda)$ onto a copy of the universal covering surface of $\{0 < |w| < \lambda\}$. This construction can be performed infinitely often to obtain disjoint domains $G_k \subset D$ that are mapped by $f(z)$ onto the universal covering surface of $\{0 < |w| < \lambda_k\}$. In addition this construction can be arranged so that a_k lies on a fixed closed subarc A' of A. Let H_k denote the maximal domain that contains G_k and is mapped by $f(z)$ onto the universal covering surface of $\{0 < |w| < \rho_k, \rho_k \geq \lambda_k\}$. Since F is of hyperbolic type, $\rho_k < \infty$. So there exists an asymptotic value which is not 0 and ∞ at $\zeta_k \in C \cap \bar{H}_k$. By assumption $\zeta_k \notin A$. Because of the local univalence, the domains H_k are disjoint. It may be assumed that $\rho_k \to \rho$ where $0 \leq \rho \leq \infty$. Since $\zeta_k \notin A$ and $a_k \in A'$, there exist arcs of ∂H_k that converge to an arc of $A - A'$ and on which $|f(z)| \to \rho$ as $k \to \infty$. This contradicts the fact that $f(z)$ has both 0 and ∞ as asymptotic values on a dense subset of A.

Holomorphic Functions

For any point ζ in C, let $h(\zeta,\psi)$ denote the chord at ζ which makes the angle ψ, $-\pi/2 < \psi < \pi/2$, with the radius $h(\zeta,0)$ drawn through ζ and let $\Delta(\zeta,\psi_1,\psi_2)$ denote the angle at ζ between the chords $h(\zeta,\psi_1)$ and $h(\zeta,\psi_2)$. For any two points z_1 and z_2 belonging to D, we let $\sigma(z_1,z_2)$ be the non-Euclidean hyperbolic distance between them. If f(z) is a holomorphic function and S is any set contained in D, then

$$M^*(f,S) = \sup_{z \in S} \left\{(1 - |z|^2)\frac{|f'(z)|}{1+|f(z)|^2}\right\}.$$

Using this notation in Theorem 38, we are able to obtain sufficient conditions for a holomorphic function to be a constant function.

Let $\{z_n\}$ be a sequence of points such that $z_n \in D$ and $\lim_{n\to\infty}|z_n| = 1$. Then the z_n's are called a <u>ρ-sequence</u> for a meromorphic function f(z) in D if for any real sequences $\{\epsilon_v\}$ and $\{L_v\}$ having the properties $0 < \epsilon_{v+1} < \epsilon_v$ for any positive integer v, $\lim_{v\to\infty}\epsilon_v = 0$, $1 < L_v < L_{v+1}$ and $\lim_{v\to\infty} L_v = \infty$, there exists a subsequence $\{z_{n_v}\}$ such that for every v the function f(z) takes in the disk $\{z : \sigma(z,z_{n_v}) < \epsilon_v\}$ all values of w in $|w| < L_v$ with the possible exception of a set whose diameter is less than $2/L_v$.

<u>Theorem</u> <u>38</u>: Let f(z) be holomorphic in D and γ be an arc contained in C. Suppose there exists a set A of second category on γ such that at every point $\zeta \in A$ there exists a chord $h(\zeta,\psi)$ containing a sequence of points $\{z_n\}$ satisfying the following conditions:

(i) $\lim\limits_{n \to \infty} z_n = \zeta$

(ii) $\overline{\lim\limits_{n \to \infty}} \sigma(z_n, z_{n+1}) < \infty$

(iii) $\infty \notin C_{\{z_n\}}(f,\zeta)$

(iv) $M^*(f, \Delta(\zeta, \psi_1, \psi_2) < \infty$ for $\psi_1 < \psi < \psi_2$

(v) There exists a value $a \in W$ and a set N metrically dense in γ such that for every $\zeta \in N$, $a \in C_{h(\zeta,\psi)}(f,\zeta)$ for at least one $h(\zeta,\psi)$.

Then $f(z) \equiv a$. (Krishnamoorthy, 1, Theorem 7, p.99)

<u>Proof</u>: Let ζ be an arbitrary point of A. We assume that ζ is a Plessner point, which will lead to a contradiction. In every angle $\Delta(\zeta,\psi_1,\psi_2)$ there exists a sequence of points $\{z'_{v,\Delta}\}$ with $\lim\limits_{v \to \infty} z'_{v,\Delta} = \zeta$ along which the sequence $\{f(z'_{v,\Delta})\} \to \infty$. From these sequences, we can choose a sequence of points $\{z'_v\}$ with $\lim\limits_{v \to \infty} z'_v = \zeta$ along which $\{f(z'_v)\} \to \infty$ in such a way that there is a corresponding sequence of points $\{\tilde{z}_v\}$ on $h(\zeta,\psi)$ such that $\lim\limits_{v \to \infty} \sigma(\tilde{z}_v, z'_v) = 0$. By the application of condition (ii) we can choose a subsequence $\{z_{n_v}\}$ of $\{z_v\}$ so that $\overline{\lim\limits_{v \to \infty}} \sigma(z_{n_v}, \tilde{z}_v) < \infty$. So we have two sequences $\{z_{n_v}\}$ and $\{z'_v\}$ tending to ζ such that $\overline{\lim\limits_{v \to \infty}} (z_{n_v}, z'_v) < \infty$ and $\lim\limits_{v \to \infty} f(z'_v) = \infty$ while the sequence $\{f(z_{n_v})\}$ is bounded. According to Gavrilov (1), there exists a ρ-sequence $\{\xi_v\}$ for the function $f(z)$ lying in the non-Euclidean segments joining the corresponding pairs of points z_{n_v} and z'_v. After some messy calculations involving bounds on $|f'(z)|/(1+|f(z)|^2)$, condition (iv) is violated. So ζ cannot be a Plessner point.

According to a theorem of Meier (Collingwood and Lohwater, Theorem 8.9, p.154), nearly all points of A are Meier points. For each Meier point, $C(f,\zeta)$ is a proper subset of W. So by Fatou's Theorem there exists a subarc γ_o of γ around ζ with the property almost all of its points are Fatou points. Let $F_{\gamma_o}(f)$ denote the set of Fatou points on γ_o. Then the set $N_o = N \cap F_{\gamma_o}$ is a set of Fatou points whose linear measure is positive and its angular limit is a. From the Lusin-Privalov Theorem (Noshiro, 1, p.60), $f(z) \equiv a$.

<u>Theorem 39</u>: Let $g(z) = \prod_{j=1}^{\infty}\left\{1 - \left(\frac{z}{1-a^{-j}}\right)^{a^j}\right\}$ where a is an integer > 4.

Let $\Delta_{j,l'}$ denote the disk with center at the zero $z_{j,l'} = \left(1 - \frac{1}{a^j}\right)e^{2\pi i l'/a^j}$ for $l' = 0,1,\ldots,a^j - 1$ and radius $1/(j^2 a^j)$. Then there exists a j_o such that for all $j \geqslant j_o$ the interiors of the disks $\{\Delta_{j,l'}\}$ are disjoint, and $g(z) \to \infty$ uniformly as $z \to 1$ within $D - (\bigcup_{j=j_o}^{\infty} \bigcup_{l'} \Delta_{j,l'})$. (Krishnamoorthy, 1, Theorem 1, p.94)

<u>Proof</u>: Let $z_o \in D - (\bigcup_{j=j_o}^{\infty} \Delta_{j,l'})$ and z_o near C. Then there exists a k such that $1 - a^{-k} \leqslant |z_o| < 1 - a^{-k-1}$. We will decompose g(z) into four products P_1, P_2, P_3 and P_4 which we will specify below in order to obtain a lower bound of $|g(z_o)|$. Let

$$P_1(z) = \prod_{j=1}^{k-1}\left\{1 - \left(\frac{z}{1-a^{-j}}\right)^{a^j}\right\}. \quad \text{Then}$$

$$|P_1(z_o)| = \prod_{j=1}^{k-1}\left|1 - \left(\frac{z_o}{1-a^{-j}}\right)^{a^j}\right| \geqslant \prod_{j=1}^{k-1}\left\{\left(\frac{1-a^{-k}}{1-a^{-j}}\right)^{a^j} - 1\right\} > \prod_{j=1}^{k-1}\left\{e(1-a^{-k})^{a^{k-1}} - 1\right\} > \prod_{j=1}^{k-1}\left\{e^{-\frac{2}{a}+1} - 1\right\}$$

Let $P_2(z) = 1 - \left(\dfrac{z}{1 - a^{-k}}\right)^{a^k}$, which is holomorphic in the whole complex plane. Then $|P_2(z)|$ has its minimum in $D - \left(\bigcup_{j=j_o}^{\infty} \Delta_{j,1'}\right)$ on one of the circles $\Delta_{k,1'}$ enclosing its zeros. Therefore,

$$|P_2(z_o)| = \left|\left(\frac{z_o}{1-a^{-k}}\right)^{a^k} - 1\right| > \left|\left(e^{2\pi i 1'/a^k} + \frac{e^{i\alpha}}{k^2(a^k-1)}\right)^{a^k} - 1\right| > \left|e^{2\pi i 1'}\left(1 + \frac{e^{i\alpha}}{k^2}\frac{a^k}{a^k-1}\right) + 0\left(\frac{1}{a^{2k}}\right)\right| \geq \frac{C_2}{k^2}.$$

Letting $P_3(z) = 1 - \left(\dfrac{z}{1-a^{-k-1}}\right)^{a^{k+1}}$, we have $|P_3(z_o)| = \left|\left(\dfrac{z_o}{1-a^{-k-1}}\right)^{a^{k+1}} - 1\right| > \dfrac{C_3}{(k+1)^2}$.

The last factor $P_4(z)$ must then be $\prod_{k+2}^{\infty}\left|1 - \left(\dfrac{z}{1-a^{-j}}\right)^{a^j}\right|$. Then

$$|P_4(z_o)| = \prod_{k+2}^{\infty}\left|1 - \left(\frac{z_o}{1-a^{-j}}\right)^{a^j}\right| > \prod_{k+2}^{\infty}\left\{1 - \left(\frac{1-a^{-k-1}}{1-a^{-j}}\right)^{a^j}\right\}$$ and

$$\left(\frac{1 - a^{-k-1}}{1 - a^{-j}}\right)^{a^j} < 4\left(1 - a^{-k-1}\right)^{a^j} < 4e^{-a^{j-k-1}}.$$ Consequently

$$|P_4(z_o)| > \prod_{k+2}^{\infty}\left(1 - 4e^{-a^{j-k-1}}\right) = \prod_{u=2}^{\infty}\left(1 - 4e^{-a^{u-1}}\right) = C_4 > 0.$$ Therefore,

$$|g(z_o)| > \frac{C_1}{k^2(k+1)^2}\left(e^{1 - 2/a} - 1\right)^{k-1}$$ which approaches ∞ as $k \to \infty$.

Lappan (13) has recently used Theorem 39 to construct an example of two analytic functions $f(z)$ and $g(z)$ such that the spherical distance $\chi(f(z), g(z)) \to 0$ uniformly as $|z| \to 1$ and $f(z) \not\equiv g(z)$. Let

$$H(z) = \prod_{j=1}^{\infty}\left\{1 - \left(\frac{z}{1-(1/s)^j}\right)^{s^j}\right\}$$

where s is a positive integer greater than 4. Using Theorem 39's notation and conclusion we have that $H(z)$ is an analytic function in D such that $H(z) \to \infty$ uniformily as $|z| \to 1$ in $D - (\bigcup_{j=j_o}^{\infty} \bigcup_{j} \Delta_{j,1'})$. We now want to construct an analytic function $K(z)$ in D such that $K(z) \to \infty$ uniformly as $|z| \to 1$ in $\bigcup_{j,1'} \Delta_{j,1'}$ and such that $K(z)$ has no zeros in D. For $j \geqslant 2$ let $D_j = \{z : |z| < 1 - (1/s)^j - 2/(j^2 s^j)\}$ and $\Delta_j = \bigcup_{1'=0}^{s^j-1} \Delta_{j,1'}$. By the Runge approximation theorem (Hille, 1, p.303), there exists a sequence of polynomials $\{P_j(z)\}$ such that for each integer $j \geqslant 2$ $|P_j(z)| < (1/2)^j$ for $z \in D_j$ and $|P_2(z) + P_3(z) + \ldots + P_j(z) - j| < (1/2)^j$ for $z \in \Delta_j$. Setting $L(z) = \sum_{j=2}^{\infty} P_j(z)$, we have that $L(z)$ is an analytic function in D and that for each $j \geqslant 2$, $|L(z) - j| < 1$ for $z \in \Delta_j$. Then $K(z) = \exp(L(z))$ has the properties that $K(z) \to \infty$ uniformly as $|z| \to 1$ in $\bigcup_{j,1'} \Delta_{j,1'}$ and that $K(z)$ has no zeros in D. In addition $|H(z)|^2 + |K(z)|^2 \to \infty$ uniformly as $|z| \to 1$. Let $f(z) = H(z)/K(z)$ and $g(z) = (H(z) - 1)/K(z)$. Then $f(z)$ and $g(z)$ are analytic functions in D, $f(z) \neq g(z)$ and $\chi(f(z),g(z)) = 1/\sqrt{[|H(z)|^2 + |K(z)|^2][1 + |g(z)|^2]}$. So $\chi(f(z),g(z)) \to 0$ uniformly as $|z| \to 1$.

CHAPTER II

NORMAL FUNCTIONS

SUFFICIENT CONDITIONS FOR A FUNCTION TO BE NORMAL

A family F^* of functions f defined in a region Ω is said to be normal if every sequence $\{f_n\}$ of functions in F^* contains a subsequence $\{f_{n_k}\}$ which either converges uniformly or tends uniformly to ∞ on each compact subset of Ω. A function f(z) is called normal in a simply connected region if the family $\{f(S(z))\}$ is normal where $S(z)$ denotes an arbitrary conformal map of Ω onto itself.

Noshiro (1, pp.87-88) cites the following conditions for a function to be normal.

Theorem 1: A non-constant function f(z), meromorphic in D, is normal if and only if $\alpha(f(z))|d(z)| \leq K\, d\sigma(z)$ holds at every point of D where $\alpha(f(z)) = \dfrac{|f'(z)|}{1+|f(z)|^2}$, $d\sigma(z) = \dfrac{|dz|}{1-|z|^2}$ and K is a fixed positive constant. (Lehto and Virtanen, 1)

Corollary: A non-constant function meromorphic in D is normal if and only if $\alpha(f(S(0))$ is bounded for all conformal mappings S. (Lehto and Virtanen, 1)

Theorem 2: Let $f(z)$ be meromorphic in D, $A(r,f)$ denote the spherical area of the Riemannian image of the disk $|z| < r$ and $L(r,f)$ denote the spherical length of the image of the circumference $|z| = r$. If $A(r,f(S(z)) \leq KL(r,f(S(z))$ for $0 < r < 1$ where $S(z)$ denotes an arbitrary conformal mapping of D onto itself and K is a fixed constant independent of S and r, then $f(z)$ is normal in D. (Ahlfors, 1)

Theorem 3: Let $f(z)$ be meromorphic in D and $\Delta_1, \Delta_2, \ldots, \Delta_q$ for $q \geq 3$ be mutually disjoint closed Jordan domains on the Riemann sphere. For $j = 1,2,\ldots,q$, let μ_j denote the minimum of the numbers of sheets of islands of R above Δ_j where R is the covering surface generated by $f(z)$. If there is no island of R above Δ_j, then $\mu_j = +\infty$. If $\sum_{j=1}^{q} (1 - \frac{1}{\mu_j}) > 2$, then $f(z)$ is normal in D. (Ahlfors, 1)

Corollary: A function $f(z)$ meromorphic in D is normal if one of the following conditions is satisfied:

(i) $f(z)$ omits three values in D,

(ii) the covering surface F has no univalent island above five mutually disjoint Jordan closed domains on W.

(Ahlfors, 1)

Other mathematicians have proved additional criteria for a function to be normal. These include the following results.

Theorem 4: A complex-valued function $f(z)$ in D is normal if and only if for each pair of sequences $\{z_n\}$ and $\{z'_n\}$ in D such that $\sigma(z_n, z'_n) \to 0$ the convergence of $\{f(z_n)\}$ to a value $\alpha \in W$ implies the convergence of

$\{f(z_n')\}$ to α. (Bagemihl and Seidel, 2, Lemma 1, p.10)

Since a normal function must be continuous, this theorem follows from a well-known result that a family of continuous functions in D is normal if and only if the functions are equicontinuous on each compact subset of D. (Hille, 1, Theorem 15.2.2, p.244)

The sum of two analytic normal functions need not be normal as the next example will show. However, Theorem 6 will give a sufficient condition for the sum of two meromorphic functions to be normal.

In order to construct two analytic normal functions whose sum is not normal (Lappan, 1, Theorem 3, p.190), we will first show that if $f(z)$ is a normal holomorphic function in D, then for any two sequences $\{z_n\}$ and $\{z_n'\}$ in D such that $\sigma(z_n, z_n') < M$, $\lim_{n\to\infty} f(z_n') = \infty$ if $\lim_{n\to\infty} f(z_n) = \infty$. If this conclusion is false, then without loss of generality we may assume that $\lim_{n\to\infty} f(z_n') = 0$. Let $S_n(z) = (z + z_n')/(1 + \bar{z}_n' z)$. Since $S_n(z)$ is a linear transformation of D onto itself, the sequence of functions $\{f(S_n(z))\}$ forms a normal family. Since $\lim_{i\to\infty} f(S_{n_i}(0)) = 0$, the limit of the sequence $\{f(S_{n_i}(z))\}$, which we will denote by $F(z)$, must be holomorphic in D. So there exists a positive constant L such that $|F(z)| < L$ in the disk $\sigma(0,z) \leq M$. Then there exists a positive integer N such that $|f(S_{n_i}(z))| < L + 1$ for all $n_i > N$ and all z in the disk $\sigma(0,z) \leq M$. However, setting $z_n'' = S_n^{-1}(z_n)$, we have $\sigma(0,z_n'') = \sigma(S_n(0), S_n(z_n'')) = \sigma(z_n', z_n) < M$ and $f(S_n(z_n'')) = f(z_n)$. So $\lim_{i\to\infty} f(S_{n_i}(z_{n_i}''))$ must be equal to ∞.

Now we will let $f(z)$ denote a normal holomorphic function which is unbounded in D, and we will construct a Blaschke product $B_f(z)$ in D such that $g(z) = f(z)B_f(z)$ is not normal. Let $\{z_n''\}$ be a sequence of points in D such that $\lim_{n\to\infty} f(z_n'') = \infty$ and $\sum_{n=1}^{\infty}(1-|z_n''|) < \infty$. We pick a subsequence $\{z_n\}$ of $\{z_n''\}$ such that for each $j < n$, $\sigma(z_j, z_n) > 3(n-j)M'$ where $M' > M$, the constant in the preceding paragraph. Then we choose a sequence $\{z_n'\}$ such that $\sigma(z_n', z_n) = M'$ and $B_f(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}$. So $B_f(z_n) = 0$ and $g(z_n) = 0$. Since $\lim_{n\to\infty} f(z_n) = \infty$, $\lim_{n\to\infty} f(z_n')$ also equals ∞. $|B_f(z_n')| \geq a > 0$ by comparison with the Blaschke product in Example 4 of Bagemihl and Seidel (2, p.11). So $\lim_{n\to\infty} g(z_n') = \infty$ and $g(z)$ is not normal. Finally we define $h(z) = \frac{1}{2}(B_f(z) - 2)f(z)$ and $G(z) = f(z) + h(z)$. By direct verification $h(z)$ is normal, and $G(z) = \frac{1}{2}f(z)B_f(z)$.

A holomorphic function $f(z)$ in D is <u>uniformly normal</u> if, for each $M > 0$ there exists a finite number $K > 0$ such that for each $z_o \in D$, $\sigma(z, z_o) < M$ implies that $|f(z) - f(z_o)| < K$. If $\{z_n\}$ and $\{z_n'\}$ are two sequences of points in D such that $\sigma(z_n, z_n') \to 0$, then $\{z_n\}$ is <u>close</u> to $\{z_n'\}$, or $\{z_n\}$ and $\{z_n'\}$ are called close sequences.

<u>Lemma 1</u>: *Suppose $f(z)$ is meromorphic in D and there exist two close sequences $\{z_n\}$ and $\{z_n'\}$ such that $f(z_n) \to \alpha$ and $f(z_n') \to \beta$ with $\alpha \neq \beta$. Then for each complex number δ with possibly two exceptions, there exists a sequence $\{z_k^\delta\}$ close to a subsequence of $\{z_n\}$ such that $f(z_k^\delta) = \delta$. (Lappan, 3, Theorem 4, p.44)*

<u>*Proof*</u>: *Let S_n be a linear transformation of D onto itself mapping 0 into z_n and let $F_n(z) = f(S_n(z))$. Since $S_n^{-1}(z_n') \to 0$, no subsequence of $\{F_n(z)\}$ converges continuously at $z = 0$ and no subsequence of $\{F_n(z)\}$ is a normal family in any neighborhood of $z = 0$. Suppose this lemma is false. Then there exists a neighborhood N of $z = 0$ and three complex*

numbers a, b and c such that for each n in an increasing sequence of positive integers, $F_n(z)$ omits a, b and c in N. However, by a theorem of Montel (Hille, 1, Theorem 15.2.8, p.248), this subsequence of functions is a normal family in N.

<u>Theorem 5</u>: A uniformly normal function is normal. (Lappan, 3, Theorem 8, p.46)

<u>Proof</u>: Let f be uniformly normal and $\{z_n\}$ be a sequence of points in D such that $\{f(z_n)\}$ converges to a value a which may be infinite. Given $M > 0$ there exists $K > 0$ such that for each n $\sigma(z,z_n) < M$ implies that $|f(z) - f(z_n)| < K$. If $a = \infty$, then $f(z_n') \to \infty$. If a is finite, then $\{f(z_n')\}$ is bounded. So there exist three complex numbers δ_i $(i = 1,2,3)$ such that there is no subsequence $\{z_{n_{k_i}}\}$ of $\{z_n\}$ having the property that there exists a sequence $\{z_k^{\delta}\}$ close to $\{z_{n_{k_i}}\}$ such that $\{z_k^{\delta}\}$ converges to δ_i. So from the contrapositive of Lemma 1, $f(z_n') \to a$, and by Theorem 4, f(z) is normal.

<u>Theorem 6</u>: *If f(z) and g(z) are uniformly normal functions in D, then h(z) = f(z) + g(z) is uniformly normal.* (Lappan, 3, Theorem 9, p.46)

<u>*Proof:*</u> *If $M > 0$ is given, then there exist constants K_f and K_g such that for each $z \in D$, $\sigma(z,z_o) < M$ implies $|f(z) - f(z_o)| < K_f$ and $|g(z) - g(z_o)| < K_g$. Let $K = K_g + K_f$. Then for each $z_o \in D$, $\sigma(z,z_o) < M$ implies $|(f(z) + g(z)) - (f(z_o) + g(z_o))| < K$.*

Theorem 7: If u and v are harmonic functions such that $f(z) = u(z) + iv(z)$ is uniformly normal, then u and v are both normal. (Lappan, 4, Theorem 6, p.158)

Proof: Let $\{S_n\}$ be a sequence of conformal mappings of D onto itself. Let $z_n = S_n(0)$ and $M > 0$ be given. Then the family $\{F_n(z) = f(S_n(z) - f(z_n)\}$ is uniformly bounded in $\{z \in D : \sigma(z,0) \leq M/2\}$. A subsequence $\{F_{n_k}\}$ can be chosen so that it converges uniformly on each compact subset of D. So $\{u(z_{n_k})\}$ converges to a limit which may be either finite or infinite. Since $F_n(0) = 0$ for each positive integer n, $F(z) = \lim_{k\to\infty} F_{n_k}(z)$ is a holomorphic function in D. If $F(z) = U(z) + iV(z)$, then $\lim_{k\to\infty} u(S_{n_k}(z)) = U(z) + \lim_{k\to\infty} u(z_{n_k})$. If $\lim_{k\to\infty} u(z_{n_k})$ is finite, then $u(S_{n_k}(z))$ converges uniformly on each compact subset of D to $U(z) + \lim_{k\to\infty} u(z_{n_k})$ while if $\lim_{k\to\infty} u(z_{n_k}) = \infty$, $u(S_{n_k}(z))$ converges uniformly to ∞ on each compact subset of D. Therefore, u(z) is normal. Similarly v(z) is also normal.

A special type of uniformly normal functions consists of the Bloch functions. A function f which is analytic in D is called a Bloch function if $f(0) = 0$ and it satisfies one of the following conditions:

(i) $\sup_{z \in D} d_f(z) < \infty$ where $d_f(z)$ denotes the radius of the largest single-valued disk with center $f(z)$ on the Riemann surface $f(D)$.

(ii) $\sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty$.

(iii) $f(\psi(z)) - f(\psi(0))$ where $\psi(z) = c\frac{z+\zeta}{1+\zeta z}$, $|\zeta| < 1$, $|c| < 1$, form a finitely normal family where ∞ is not allowed as a limit function.

(iv) there exists a univalent analytic function $g(z)$ in D such that $f(z) = \lambda \log g'(z)$ for some constant $\lambda > 0$.

<u>Theorem</u> <u>8</u>: The above four conditions are equivalent. (Pommerenke, 1, p.79)

<u>Proof</u>: (i) $\Longrightarrow$ (ii): For any $z_1 \in D$ we form

$$f^*(z) = \left[f\left(\frac{z + z_1}{1 + \bar{z}_1 z}\right) - f(z_1)\right] \Big/ \left[(1 - |z_1|^2) f'(z_1)\right].$$

So
$$\frac{d_f(z_1)}{(1 - |z_1|^2)|f'(z_1)|} > 0.$$

(ii) $\Longrightarrow$ (i): From Schwarz's Lemma (Hille, 1, Theorem 15.1.1, p.235) $d_f(z) \leq (1 - |z|^2)|f'(z)|$ for $|z| < 1$ whenever f is analytic in D.

(ii) $\Longleftrightarrow$ (iii): This follows from Montel's Theorem (Hille, 1, Theorem 15.3.1, p.251) and the fact that $(1 - |z|^2)|f'(z)|$ is invariant under ψ.

(iv) $\Longrightarrow$ (ii): $(1 - |z|^2)|f'(z)| = \lambda(1 - |z|^2)\left|\frac{g''(z)}{g'(z)}\right|$, which is bounded. (Hille, 1, Lemma 17.4.1, p.351)

(ii) $\Longrightarrow$ (iv): $\sup_{|z|<1} (1 - |z|^2)\left|\frac{g''(z)}{g'(z)}\right| < \infty$, which implies g(z) is univalent by Nehari (1).

If f(z) is analytic in D and $f'(z_o) \neq 0$ for $z_o \in D$, then the maximal domain containing z_o that is mapped by f(z) one-to-one onto a single-valued domain starlike with respect to $f(z_o)$ is called the <u>Gross star domain</u> $G(z_o)$ of f. <u>Rays</u> of $G(z_o)$ are defined to be the preimages of the rays of the starlike image domain. If R is a ray of $G(z_o)$ then either R is a Jordan arc in D that goes from z_o to a point $z_1 \in D$ where $f'(z_1) = 0$ or R is a Jordan arc in D except for its endpoint $e^{i\theta} \in C$, where f(z) has an asymptotic value.

<u>Lemma 2</u>: If $f(z)$ is analytic in D and without Koebe arcs, then for any point $e^{i\theta} \in C$ either $f(z)$ has an asymptotic value at $e^{i\theta}$ or diameter $G(z) \to 0$ as $z \to e^{i\theta}$. (Pommerenke, 1, Theorem 7, p.90)

<u>Proof</u>: Suppose $G(z) \not\to 0$. Then we have $\{z_n\} \to e^{i\theta}$ with dia $G(z_n) \geq r_o > 0$. So there exist rays R_n of $G(z_n)$ such that dia $R_n \geq r_o$. We have two cases.

(i) There exists a subsequence $\{n_k\}$ and some r, $0 < r < r_o$, such that $\min\{|z|: |z - e^{i\theta}| \leq r,\ z \in R_{n_k}\} \to 1$ as $k \to \infty$. Then some subarcs R'_{n_k} converge to an open arc A_o of C that has $e^{i\theta}$ as an endpoint. In addition $f(R'_{n_k})$ is either a line segment or a half line. We claim that $f(z_o)$ is analytic on A_o. We may assume without loss of generality that the endpoints of the segments $f(R'_{n_k})$ converge respectively to the points w' and w'' in W. Furthermore, we may assume that the directions of the segments $f(R'_{n_k})$ converge. So as $n_k \to \infty$ $f(R'_{n_k})$ converges to a rectilinear segment L joining w' and w'' (which may be the same point). By a suitable linear transformation we can make L a real segment or a single point. Let ζ_1 and ζ_2 be distinct points on A_o and choose points z'_n and z''_n on R'_{n_k} such that $z'_n \to \zeta_1$ and $z''_n \to \zeta_2$. Without loss of generality we may assume that the corresponding sequences of points $f(z'_n)$ and $f(z''_n)$ converge. Neither of these limits can be ∞ since $f(z)$ maps R'_{n_k} one-to-one onto $f(R'_{n_k})$ and $f(z)$ has no sequence of Koebe arcs for ∞. Therefore, by replacing A_o by its arc between ζ_1 and ζ_2 we may assume that L is bounded. We will now show that $f(z)$ is bounded in a neighborhood of each point of A_o. Suppose on the contrary that there exists a point $\zeta_3 \in A_o$ and a sequence of points $z_j \in D$ such that $z_j \to \zeta_3$ and $f(z_j) \to \infty$. Let L_j denote the half-line $\{Tf(z_j) : T \geq 1\}$ and let Λ_j be the component of the preimage $f^{-1}(L_j)$ that contains z_j. We choose z_j's such that $f'(z) \neq 0$ on Λ_j. Then Λ_j is a simple curve tending at one end to a point of C. For all sufficiently large j, L_j does not intersect $\cup f(R'_{n_k})$ because L is bounded. So f(z) has a sequence of Koebe arcs for ∞, a contradiction. Consequently f(z) is analytic on A_o and has an asymptotic value at $e^{i\theta}$.

(ii) We can find $c_1 \in D$ such that a subsequence of R_n comes arbitrarily close to c_1. If (i) does not hold, we can also find $c_2 \in D$ with $0 < |c_2 - e^{i\theta}| < |c_1 - e^{i\theta}|/2$ and such that another subsequence of R_n comes arbitrarily near to c_2. After continuing this process of taking subsequences we finally take the diagonal sequence. So we have points $c_k \in D$ with $c_k \to e^{i\theta}$ as $k \to \infty$ such that for each fixed k, R_n comes arbitrarily close to c_k as $n \to \infty$. Since $f(R_n)$ is a line segment or a half-line, all the points $w_k = f(c_k)$ lie on the same line L. The points c_k are all distinct since f(z) is not constant. Since f(z) maps R_n one-to-one onto $f(R_n)$, w_k converges monotonically along L to a limit w_o. Let D_k be a disk around c_k such that diameter $f(D_k) \to 0$ as $k \to \infty$. For each k, we choose n_k and a subarc A_k of R_{n_k} from $a_k \in D_k$ to $b_k \in D_{k+1}$. Then $A_1 + [a_1, b_1] + [a_2, b_2] + \ldots$ may be assumed to be a Jordan arc that lies in D except for its endpoint $e^{i\theta}$. $f(A_k)$ converges to $[w_{k-1}, w_k]$. Since $w_k \to w_o$ and diameter $D_k \to 0$, f(z) has an asymptotic value w_o at $e^{i\theta}$.

<u>Theorem 9</u>: Every Bloch function in D has finite or infinite angular limits on an uncountably dense subset of C. (Pommerenke, 1, Theorem 8, p.91)

<u>Proof</u>: Since every Bloch function is normal, every asymptotic value

is an angular limit. Let A be an arc of C. Suppose there exists an interior point $e^{i\theta}$ of A where there is no asymptotic value. By Lemma 2 there exists a Gross star domain $G(z_o)$ such that Boundary $G(z_o) \subset A$. The number of rays of $G(z_o)$ has the power of the continuum and only countably many of them can end at points where the derivative is equal to 0. All others end on C. Therefore, it is sufficient to prove that any two distinct rays end at distinct points of C. Let R_1 and R_2 be different rays of $G(z_o)$ with endpoints ζ_1 and ζ_2 in C. Since $af(z) + b$ is also a Bloch function, we may assume that the half-lines or segments $f(R_1)$ and $f(R_2)$ lie on different sides of the line $\{\text{Real } w = \text{Real } f(z_o)\}$. So the normal function $e^{f(z)}$ tends to different limits along R_1 and R_2. Consequently $\zeta_1 \neq \zeta_2$.

<u>Theorem</u> <u>10</u>: Suppose F(z) is the Blaschke-Quotient expressed in the form

$$F(z) = B_1(z)/B_2(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n}\left(\frac{a_n - z}{1 - \bar{a}_n z}\right) \Big/ \prod_{n=1}^{\infty} \frac{|b_n|}{b_n}\left(\frac{b_n - z}{1 - \bar{b}_n z}\right)$$

where $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$ and $\sum_{n=1}^{\infty}(1 - |b_n|) < \infty$. If the set of limit points of the a_n's is disjoint from the set of limit points of the b_n's, then F(z) is normal. (Cima, 1, Lemma 1, p.769)

<u>Proof</u>: $$\frac{|B_1(z)B_2'(z) - B_1'(z)B_2(z)|(1 - |z|^2)}{|B_1(z)|^2 + |B_2(z)|^2} \leqslant \frac{|B_2'(z)|(1 - |z|^2) + |B_1'(z)|(1 - |z|^2)}{|B_1(z)|^2 + |B_2(z)|^2}$$

$\underline{\lim}(|B_1(z)|^2 + |B_2(z)|^2) \geqslant 1$ as $z \to e^{i\theta}$ in D and $|B_i'(z)|(1 - |z|)$ for $i = 1,2$ is bounded for $|z| < 1$ according to Seidel and Walsh (1). So $\overline{\lim} \frac{|B_1(z)B_2'(z) - B_1'(z)B_2(z)|}{|B_1(z)|^2 + |B_2(z)|^2}(1 - |z|^2) < \infty$ as $z \to e^{i\theta}$ in D and

$$\alpha(F(z))\,|dz| = \frac{|B_1(z)B_2'(z) - B_1'(z)B_2(z)|}{|B_1(z)|^2 + |B_2(z)|^2}\,|dz| < \frac{c\,|dz|}{(1 - |z|^2)},$$

the condition in Theorem 1.

CLUSTER-SET THEOREMS FOR NORMAL FUNCTIONS

The following theorem of Lehto and Virtanen is one of the first important results in the theory of cluster sets of normal functions.

<u>Theorem 11</u>: Let $f(z)$ be a normal meromorphic function in D. If $f(z)$ has an asymptotic value α at a point z_o on C along a Jordan curve lying in D, then $f(z)$ possesses the angular limit α at z_o. (Lehto and Virtanen, 1, Theorem 1, p.49; Noshiro, 1, Theorem 6, p.86)

Bagemihl and Seidel have proved many other cluster-set properties of normal functions. These include conditions for $f(z)$ to be identically constant and conditions for $f(z)$ to have a limit at a point.

<u>Theorem 12</u>: Let $f(z)$ be a normal meromorphic function in D which omits the finite or infinite value c and let $\{z_n\}$ be a sequence of points in D which converges to a point $\zeta \in C$. If there exists a positive number M such that for every n, $\sigma(z_n, z_{n+1}) < M$ and if $\lim_{n\to\infty} f(z_n) = c$, then $f(z)$ has the angular limit c at ζ. (Bagemihl and Seidel, 2, Theorem 1, p.4)

<u>Proof</u>: The family of functions $g_n(z) = f\left(\frac{z + z_n}{1 + \bar{z}_n z}\right)$ for n any positive integer is normal in D and $\lim_{n\to\infty} g_n(0) = c$. So the functions $\{g_n(z)\}$

converge uniformly on every compact subset of D to c. Let S be the compact subset $|z| \leq \lambda$ where $1 > \lambda > \tanh M$. Since $\sigma(z_n, z_{n+1}) < M$, the non-Euclidean circle Δ_n with center z_n and radius equal to $\frac{1}{2}\log\frac{1+\lambda}{1-\lambda}$ contains the point z_{n+1} in its interior. $\lim\limits_{z \to \zeta} f(z) = c$ when z is restricted to the union of the interiors of the circles Δ_n. In particular this relation holds if z lies on the polygonal line formed by joining the points z_n and z_{n+1} by a Euclidean line for all n. So f(z) possesses the angular limit c at ζ by Theorem 11.

A boundary path is a simple continuous curve $z = z(t)$ $(0 \leq t < 1)$ in D such that $|z(t)| \to 1$ as $t \to 1$. The initial point of the boundary path Λ is the point z(0) and the end E of Λ is the set of limit points of Λ on C. In order to decide when two boundary paths are "close together", we let $D_1^*(\Lambda_1, \Lambda_2) = \lim\limits_{\substack{t \to 1 \\ z(t) \in \Lambda_1}} \sup \sigma(z(t), \Lambda_2)$, $D_2^*(\Lambda_1, \Lambda_2) = \lim\limits_{\substack{t \to 1 \\ z(t) \in \Lambda_2}} \sup \sigma(z(t), \Lambda_1)$ and $D^*(\Lambda_1, \Lambda_2) = \sup\{D_1^*(\Lambda_1, \Lambda_2), D_2^*(\Lambda_1, \Lambda_2)\}$.

If P is a prime end of D, $\{q_n\}$ is a chain belonging to P and d_n is the subdomain of D defined by q_n and containing q_{n+1}, then $\bigcap \bar{d}_n = \bigcap \bar{d}'_n$ for $\{q'_n\}$ any equivalent chain. The set $I(P) = \bigcap \bar{d}_n$, which is invariant in the equivalence class of chains constituting P, is called the impression of the prime end. Two distinct prime ends of D can have the same impression. For example, if the domain D is obtained by deletion of an end-cut γ from the unit disk, then each interior point of γ constitutes the impression of exactly two prime ends.

Theorem 13: Let f(z) be a non-constant meromorphic function in D that tends to C along a boundary path Λ whose end E contains more than one

point. Then given $\epsilon > 0$ there exist boundary paths Λ_1 and Λ_2 whose ends are contained in E such that Λ, Λ_1 and Λ_2 are mutually exclusive; $D^*(\Lambda_1,\Lambda_2) < \epsilon$; and $f(z) \to c$ along Λ_1, but not along Λ_2. (Bagemihl and Seidel, 1, Theorem 1, p.264)

Proof: Let $G = D - \Lambda$. The initial point of Λ is the impression of one prime end of G whereas every other point of Λ is the impression of two prime ends of G. If E is the impression of a prime end of P and E = C, then E is the impression of only P, but if $E \neq C$, then E is the impression of P and of another prime end P' in G.

If E = C, we map G onto D in a one-to-one conformal manner so that the initial point of Λ and the prime end P correspond respectively to the points -1 and 1. Let F(z) denote the image of f(z) under the conformal mapping. Since $f(z) \not\equiv c$, there exists a sequence of points in G tending to C on which $f(z) \to b \neq c$. So there is a sequence of points in D tending to the point 1 on which $F(z) \to b$ and there exists a segment S in D bounded by a suitable arc and chord of C both having an endpoint at 1 that contains infinitely many points of this sequence. $F(z) \to c$ as $z \to 1$ along C but not as $z \to 1$ on S. Consequently from an argument of Lehto and Virtanen (1, pp.49-52), given $\epsilon > 0$ there exist two disjoint boundary paths Λ_1' and Λ_2' in D whose ends are the point 1 such that $D^*(\Lambda_1,\Lambda_2) < \epsilon$ and $F(z) \to c$ along Λ_1' but not along Λ_2'. So under the original mapping of G onto D there exist boundary paths Λ_1 and Λ_2 that lie in G and satisfy the conditions of the theorem.

If $E \neq C$, then we map G onto D one-to-one conformally so that the initial point of Λ and the prime ends P_1 and P_2 correspond respectively to the points -1, -i and i. Let F(z) again denote the image of f(z)

under this conformal mapping. Let A, A_1 and A_2 denote the open subarcs of C which, when described once in the positive direction, have the respective initial and terminal points $-i$ and i, -1 and $-i$, i and -1. Under this conformal mapping A corresponds to the arc $C - E$ while A_1 and A_2 each corresponds to Λ minus its initial point. Therefore, as the point i or $-i$ is approached along A, the inverse of the mapping function approaches an end point of E. Then this limit is approached as $z \to i$ or $-i$ on the set $\{|z| \leq 1, \text{Real}(z) \geq \Delta\}$. Since $f(z) \not\equiv c$, according to Privalow (1, p.207) there exists a sequence of points in G tending to an interior point of E on which $f(z) \to b \neq c$. So there exists a sequence of points $\{z_n\}$ tending to i or $-i$ satisfying the conditions: $\text{Real}(z_n) < 0$ for n any positive integer and $F(z_n) \to b$ as $n \to \infty$. In addition $F(z) \to c$ as $z \to -i$ along A_1. The rest of the proof is the same as above.

<u>Theorem</u> <u>14</u>: Suppose that $f(z)$ is a normal meromorphic function in D and that Λ_1 and Λ_2 are boundary paths for which $D^*(\Lambda_1, \Lambda_2)$ is finite. If $f(z) \to c$ along Λ_1, then $f(z) \to c$ along Λ_2. (Bagemihl and Seidel, 1, Theorem 3, p.266)

<u>Proof</u>: Assume c is finite. If $c = \infty$, then we will look at the normal meromorphic function $1/f(z)$. If this theorem is false, then there exists a number c', possibly ∞, different from c and a sequence of points $\{z'_n\}$ on Λ_2 such that $\lim_{n \to \infty} |z'_n| = 1$ and $\lim_{n \to \infty} f(z'_n) = c'$. Since $D^*(\Lambda_1, \Lambda_2)$ is finite, there exists a positive number M and a sequence of points $\{z_n\}$ on Λ_1 such that $\lim_{n \to \infty} |z_n| = 1$ and $\sigma(z_n, z'_n) < M$ for n any positive integer. The family of functions $\{f(S_n(z))\}$ where $S_n(z) = (z + z_n)/(1 + \bar{z}_n z)$

is normal in D. As $n \to \infty$, $f(S_n(0)) = f(z_n) \to c$. Since c is finite, there exists a subsequence $\{f(S_{n_k}(z))\}$ which, as $k \to \infty$, converges uniformly to a meromorphic function F(z) on the closed disk Δ in D whose center is the origin and whose non-Euclidean radius is M. For all sufficiently large values of k, $S_{n_k}^{-1}(\Lambda_1)$ intersects every circle $\sigma(0,z) = L$ with $L < M$. Since $f(z) \to c$ along Λ_1, $F(z) \equiv c$. However, $\sigma(z_{n_k}, z'_{n_k}) < M$ so that $S_{n_k}^{-1}(z'_{n_k}) \in \Delta$ and since $f(z'_{n_k}) \to c' \neq c$ as $k \to \infty$, $F(z) \not\equiv c$, a contradiction.

Corollary: If a normal meromorphic function f(z) in D tends to a limit along a boundary path whose end contains more than one point, then f(z) is identically constant. (Bagemihl and Seidel, 1, Corollary 1, p.266)

In the section on Locally Univalent Functions in Chapter I we defined Koebe arcs of f(z). A Koebe sequence of arcs relative to an open arc A of C is a sequence of Jordan arcs $\{J_n\}$ in D such that

(i) for some sequence $\{\epsilon_n\}$ satisfying the conditions $0 < \epsilon_n < 1$ for n any positive integer and $\epsilon_n \to 0$ as $n \to \infty$, J_n lies in the ϵ_n-neighborhood of A,

(ii) every open sector Δ of D subtending an arc of C that lies strictly interior to A has the property that, for all but at most a finite number of n's, the arc J_n contains at least one Jordan subarc lying wholly in Δ except for its two end points which lie on distinct sides of Δ.

Theorem 15: Let f(z) be a normal meromorphic function in D. If $f(z) \to c$ along a Koebe sequence of arcs $\{J_n\}$, then it is identically equal to c.

(Bagemihl and Seidel, 3, Theorem 1, p.10)

<u>Proof</u>: Assume $c = 0$. If c is a finite non-zero complex number, then we replace $f(z)$ by $f(z) - c$; or, if $c = \infty$, we replace $f(z)$ by $1/f(z)$.

Let $\{J_n\}$ be the given sequence relative to an arc A. We define an arc $B = \{z : |z| = 1,\ q_1 < \arg z < q_2\}$ to be strictly interior to A. We denote by Δ the open sector of D with vertex at the origin and vertex angle β subtending the arc B. There is no loss in generality in assuming that for every n the arc J_n contains a Jordan subarc Γ_n lying wholly in Δ except for its endpoints $P_n^{(1)}$ and $P_n^{(2)}$ which lie on the sides s_1 and s_2 of Δ. We set $r_n = \min_{z \in \Gamma_n} |z|$ and $R_n = \max_{z \in \Gamma_n} |z|$ for n any positive integer. Then $\lim_{n \to \infty} r_n = \lim_{n \to \infty} R_n = 1$. Now we define a Jordan curve K_n for each n. Let $|z| = R_n$ intersect s_1 and s_2 respectively at the points $Q_n^{(1)}$ and $Q_n^{(2)}$. If B_n is the open arc of $|z| = R_n$ which lies in Δ and B_n^* is the complementary arc, then we define K_n to be the union of $P_n^{(1)}Q_n^{(1)}$, B_n^*, $P_n^{(2)}Q_n^{(2)}$ and Γ_n. An argument involving harmonic measure shows that if D is mapped conformally onto the interior of K_n by $z = \phi_n(w)$ where $\phi_n(0) = 0$, then for n sufficiently large the arc Γ_n is the image of an arc S on C having a length at least π times the harmonic measure $\omega(0, B_n, \{z : |z| < R_n\})$.

Since $f(z) \to 0$ along the Koebe sequence $\{J_n\}$, $\lim_{n \to \infty} f(\phi_n(w)) = 0$ uniformly on S. From Lehto and Virtanen (1, p.64), $\{f(\phi_n(w))\}$ tends to zero uniformly on every compact subset of D.

Suppose there exists a point $z_o \in D$ for which $f(z_o)$ is not zero. By the definition of a Koebe sequence relative to A, z_o is in the interior of each K_n for n sufficiently large. Let $w = \phi_n^*(z)$ denote the inverse of $z = \phi_n(w)$. Then $f(\phi_n(\phi_n^*(z_o))) = f(z_o)$ for n sufficiently

large. Since $\{f(\phi_n(w))\} \to 0$ uniformly on every compact subset of D but $f(z_o) \neq 0$, $\lim_{n\to\infty} |\phi_n^*(z_o)| = 1$. However, if ρ is fixed so that $|z_o| < \rho < 1$, then according to Schwarz's Lemma $|\phi_n^*(z_o)| \leq |z_o|/\rho < 1$ for n sufficiently large, a contradiction.

Theorem 16: Let f(z) be a normal function in D that omits the value w which is either finite or infinite and let A be an open subarc of C. If the set of Fatou points of f(z) on A is of measure zero, then A contains a Fatou point of f(z) at which the corresponding angular limit of f(z) is w. (Bagemihl, 1, Theorem 1, p.3)

Proof: Assume w is ∞. Let $\zeta \in A$. If f(z) were bounded in some neighborhood of ζ, then by a simple extension of Fatou's Theorem, the set of Fatou points of f(z) on A would be of positive measure, which is contrary to the hypothesis. So f(z) is unbounded in every neighborhood of ζ. Hence there exists a number $\delta > 0$ such that the region $H = D \cap \{z : |z - \zeta| < \delta\}$ satisfies the conditions that $\overline{H} \cap C \subset A$ and f(z) is unbounded in H. Consequently there exists a sequence of points $\{z_n\}$ in D such that $z_n \to \zeta$ and $M_n = |f(z_n)| \to \infty$ as $n \to \infty$ where $1 < M_1 < M_2 < \ldots < M_n < \ldots$. For n any positive integer, let V_n be the open set of points in D for which $|f(z)| > M_n - 1$. Let R_n denote the component of V_n that contains z_n. $|f(z)| = M_n - 1$ at all boundary points of R_n that lie in D. By the maximum principle, $\overline{R}_n \cap C$ is non-empty. Suppose the diameter of R_n does not tend to zero as $n \to \infty$. Let $r_n = \min_{z \in \overline{R}_n} |z|$. Since f(z) omits ∞ in D by assumption, $\lim_{n\to\infty} r_n = 1$ and there exists a Koebe sequence of arcs along which $f(z) \to \infty$, a contradiction of Theorem 15. Thus there exists a natural number N such that $R_N \subset H$.

We want to show that $f(z)$ is unbounded in $G_1 = R_N$. Let G_1^* be the smallest simply connected region containing G_1 and $z = \phi(w)$ be a function that maps D conformally onto G_1^*. The set $B^* = \bar{G}_1^* \cap C$ is non-empty. We denote by B_1^* the set of all points of B^* that are accessible from G_1^*. Let $\phi^*(e^{iu}) = \lim_{r \to 1} \phi(re^{iu})$ for every u for which the limit exists. By Fatou's Theorem this limit exists at almost every point of C. The set $E_1 = \{e^{iu} : |\phi^*(e^{iu})| = 1\}$ is a Borel set and hence measurable. In addition $B_1^* = \{\phi^*(e^{iu}) : e^{iu} \in E_1\}$. We want to show that the function $g(w) = f(\phi(w))$ is unbounded in D. Assume not.

Suppose $m(E_1) > 0$. Let E_o denote the Borel set of positive measure which is the subset of E_1 consisting of all the points for which $g(w)$ possesses a radial limit and B_o^* be the image of E_o under the mapping $z = \phi(w)$. From an extension of Löwner's Theorem (Tsuji, 1, p.322), B_o^* is a measurable subset of B_1^* with $m(B_o^*) > 0$. Let $\zeta_o \in B_o^*$. Then there exists a path in G_1^* terminating at ζ_o, and this path is the image under $z = \phi(w)$ of a path in D that terminates at a point $e^{iu_o} \in E_o$. $\phi^*(e^{iu_o}) = \zeta_o$ and $g(w)$ has a radial limit at e^{iu_o}; therefore, $f(z)$ tends to a limit along a path in G_1^* terminating at ζ_o. Since $f(z)$ is normal in D, ζ_o is a Fatou point of $f(z)$ (Theorem 11). Because ζ_o was an arbitrary point of B_o^*, a set of positive measure, we have contradicted the hypothesis that the set of Fatou points of $f(z)$ on A is of zero measure.

Suppose $m(E_1) = 0$. Since every boundary point of G_1^* is a boundary point of G_1 and $|f(z)| = M_N - 1$ at all boundary points of R_N that lie in D, the Fatou values of $g(w)$ are equal in modulus to $M_N - 1$ almost everywhere on C. The representation of $g(w)$ by its Poisson Integral shows that $|g(w)| \leq M_N - 1$ throughout D. So $|f(z)| \leq M_N - 1 = L$ throughout $G_1 = R_N$ which is contrary to the way R_N was defined.

Therefore, $g(w)$ is unbounded in D and so $f(z)$ is unbounded in G_1^* and G_1. The open set of points of G_1 at which $|f(z)| > L + 1$ is non-empty. Let G_2 denote a component of this set and $f(z)$ is unbounded in G_2 as before. A continuation of this process yields a sequence of nested subregions $G_1 \supset G_2 \supset \ldots$ of H. Now we choose $z_1 \in G_1$, $z_2 \in G_2 - \{z_1\}$, $z_3 \in G_3 - \{z_1, z_2\}, \ldots$, $z_n \in G_n - \{z_1, z_2, \ldots, z_{n-1}\}, \ldots$ and join z_1 to z_2 by means of a Jordan arc J_1 lying in G_1. In addition we join z_2 to z_3 by a Jordan arc J_2 lying in G_2 and having no point except z_2 in common with $J_1, \ldots,$ join z_n to z_{n+1} by a Jordan arc J_n lying in G_n and having no point except z_n in common with $J_1 \cup J_2 \cup \ldots \cup J_{n-1}, \ldots$. So $P = \bigcup_{n=1}^{\infty} J_n$ is a path in D with initial point z_1. Its end lies on C because $\lim_{n \to \infty} \min_{z \in J_n} |f(z)| = \infty$ and $f(z)$ omits ∞ in D. According to the Corollary following Theorem 14, the end of P is a single point $\zeta \in C$. Since $f(z)$ is normal in D, ζ is a Fatou point of $f(z)$ with the corresponding value ∞ by Theorem 11.

In conclusion if w is finite, we then define $F(z) = \dfrac{1}{f(z) - w}$. From the proof above, A contains a Fatou point of $F(z)$ with the corresponding angular limit ∞. So this is a Fatou point of $f(z)$ with the angular limit w.

A <u>hypercycle</u> is the locus of points whose non-Euclidean distance from a given non-Euclidean straight line is constant.

<u>Theorem 17</u>: Let $f(z)$ be a normal meromorphic function in D that omits the finite or infinite value w. If there exists a sequence $\{z_n\}$ having at least the limit points α and β on C and a constant M such that $\sigma(z_n, z_{n+1}) < M$ for every n and $\lim_{n \to \infty} f(z_n) = c$, then $c \neq w$ and $f(z) \equiv c$.

(Bagemihl, 1, Theorem 2, p.4)

Proof: Assume c = w. Then by an argument in Theorem 12's proof, there exists an asymptotic path P in D whose end contains the arc $\alpha\beta$ such that $\lim_{\substack{z \to 1 \\ z \in P}} f(z) = w$. According to the Corollary following Theorem 14, this implies that $f(z) \equiv w$, a contradiction.

Assume $f(z) \not\equiv c$. If the set of Fatou points is of measure zero, then by Theorem 16, since f(z) omits w, f(z) has a Fatou point on the arc $\alpha\beta$ at which the corresponding angular limit of f(z) is w. If the set of Fatou points is of positive measure, then by a theorem of Privalow (Noshiro, 1, Theorem 2, p.72), f(z) has a Fatou point ζ on the arc αB at which the corresponding angular limit of f(z) is $d \neq c$.

Let γ be an angle such that $0 < \gamma < \frac{1}{2}\pi$ and $M < \log\tan(\frac{1}{4}\pi + \frac{1}{2}\gamma)$ where M is the constant in the statement of this theorem. Let Δ be the subregion of D determined by the two hypercycles that form the angles γ and $-\gamma$ at ζ with the diameter of C joining ζ and $-\zeta$. In a neighborhood of ζ every point of Δ lies in a symmetric Stolz angle of opening 2γ. So $\lim_{\substack{z \to \zeta \\ z \in \Delta}} f(z) = d \neq c$. Since $\lim_{n \to \infty} f(z_n) = c$, the points z_n for all sufficiently large n do not lie in Δ. Since every point of $\alpha\beta$ is a limit point of $\{z_n\}$, for infinitely many n the points z_n and z_{n+1} lie on opposite sides of Δ. Every boundary point of Δ that lies in D is at the non-Euclidean distance $\frac{1}{2}\log\tan(\frac{1}{4}\pi + \frac{1}{2}\gamma)$ from the diameter of C joining $-\zeta$ and ζ. Therefore, for infinitely many n, $\sigma(z_n, z_{n+1}) \geqslant \log\tan(\frac{1}{4}\pi + \frac{1}{2}\gamma) > M$, a contradiction.

SUBHARMONIC NORMAL FUNCTIONS

One special class of normal functions, the subharmonic normal functions, have the property that for any u for which $\int_0^{2\pi}|u(re^{i\theta})| = O(1)$, u has Fatou points almost everywhere on C.

A continuous function u(x,y) not identically equal to zero is <u>subharmonic</u> if and only if it satisfies either of the following mean-value inequalities for each circular disk in D:

$$u(x_o,y_o) \leq \frac{1}{2\pi}\int_o^{2\pi} u(x_o + \rho\cos\theta, y_o + \rho\sin\theta)\, d\theta ,$$

$$u(x_o,y_o) \leq \frac{1}{\pi r^2}\int_o^{r} \int_o^{2\pi} u(x_o + \rho\cos\theta , y_o + \rho\sin\theta)\rho\, d\rho\, d\theta \cdot$$

If u(x,y) has continuous second partial derivatives in D, then it is subharmonic if and only if

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \geq 0$$

at every point of D.

<u>Theorem</u> <u>18</u>: If u is normal and subharmonic in D and

$$\int_o^{2\pi}|u(re^{i\theta})|d\theta = O(1)$$

for $0 \leq r < 1$, then u has Fatou points with finite Fatou values almost everywhere on C. (Meek, 1, Theorem, p.314)

<u>Proof</u>: According to Littlewood (1, Lemma 3, p.390) u has the representation $u = v + u^*$ where v has the property that if $w_\rho(z)$ is harmonic in $|z| < \rho < 1$ and $w_\rho = u$ on $|z| = \rho$, then $\lim_{\rho\to 1} w_\rho(z) = v(z)$ and u^* is a non-positive subharmonic function in D with $u^*(re^{i\theta}) \to 0$ as $r \to 1$ for

almost all $\theta \in [0,2\pi)$. In addition by a theorem of Tsuji (1, Theorem IV.16, p.147) v has Fatou points corresponding to finite Fatou values almost everywhere on C.

Let E denote the set of points on C at which u* has radial limit zero and v has a finite Fatou value. For any β, $0 \leq \beta < \pi/2$, and $e^{i\theta} \in E$, let $H(e^{i\theta},\beta)$ denote the open set in D bounded by the hypercycles from $-e^{i\theta}$ to $e^{i\theta}$ making angles β and $-\beta$ respectively with the diameter through $e^{i\theta}$ and $-e^{i\theta}$. We pick a sequence $\{z_n\}_{n=1}^{\infty} \subseteq H(e^{i\theta},\beta)$ such that $z_n \to e^{i\theta}$ as $n \to \infty$.

For each positive integer n, we denote the non-Euclidean straight line which passes through z_n and is perpendicular to the radius $\rho e^{i\theta}$, $0 \leq \rho < 1$, by E_n. Because of the invariance of the metric σ under one-to-one conformal mappings of D onto itself, it can be shown that each of the bounding hypercycles of $H(e^{i\theta},\beta)$ is at a hyperbolic distance $\sigma(0,\tan\beta/2)$ from the diameter between $e^{i\theta}$ and $-e^{i\theta}$. Therefore, for each positive integer n, $\rho_n e^{i\theta}$, the point of intersection of E_n with $\rho e^{i\theta}$, satisfies the relation $\sigma(\rho_n e^{i\theta}, z_n) \leq \sigma(0,\tan\beta/2)$. For each positive integer n, $S_n(w) = (w + \rho_n e^{i\theta})/(1 + \rho_n e^{-i\theta}w)$ is a one-to-one conformal mapping of D onto itself.

Since u is normal, there exists a subsequence, also denoted by $\{u(S_n)\}_{n=1}^{\infty}$, which converges uniformly or diverges uniformly on the compact set $K = \{w : \sigma(0,w) \leq \sigma(0,\tan\beta/2)\}$. Since $u(S_n(0)) = u(\rho_n e^{i\theta}) = v(\rho_n e^{i\theta}) + u^*(\rho_n e^{i\theta}) \to v(\theta)$, the Fatou value at $e^{i\theta}$, the subsequence cannot diverge uniformly on K. So the subsequence converges uniformly on K to a subharmonic function U.

We have $u(S_n(w)) \leq v(S_n(w))$ for $w \in K$ and any positive integer n. Since $e^{i\theta}$ is a Fatou point of v, $\{v(S_n)\}_{n=1}^{\infty}$ converges uniformly on K

to $v(\theta)$ and $U(w) \leq v(\theta)$ for $w \in K$. But $U(0) = \lim_{n \to \infty} u(S_n(0)) = v(\theta)$ and by the Maximum Principle for subharmonic functions $U(z) \equiv v(\theta)$ in K. Furthermore, E has linear measure 2π. So u has Fatou points almost everywhere on C.

<u>Corollary A</u>: Any normal subharmonic function on D which is bounded above has Fatou points with finite Fatou values almost everywhere on C. (Meek, 1, Corollary 1, p.316)

<u>Proof</u>: If u is normal, subharmonic and bounded above in D, then $v = e^u$ is also normal, subharmonic and bounded in D. In addition every Fatou point of v is a Fatou point of u. So u has Fatou points almost everywhere on C. By Arsove (1, Theorem B, p.260), a subharmonic function bounded above on D has finite radial limits almost everywhere on C. Consequently u has finite Fatou values almost everywhere on C.

<u>Corollary B</u>: If u is normal, subharmonic, bounded below and admits a harmonic majorant v on C, then u has Fatou points with finite Fatou values almost everywhere on C. (Meek, 1, Corollary 2, p.316)

<u>Proof</u>: Without loss of generality we may assume that $0 \leq u(z)$ for $z \in D$. Then $0 \leq \int_0^{2\pi} u(re^{i\theta})d\theta \leq \int_0^{2\pi} v(re^{i\theta})d\theta = 2\pi v(0)$ for $0 \leq r < 1$. So Theorem 18 applies.

In order to generalize Theorem 18, the following questions must be answered: Must a normal subharmonic function in D have any Fatou points on C? If so, is this set dense on C?

If Δ is any Stolz angle at $e^{i\theta}$, we define $\Pi_{\Delta}(f,e^{i\theta}) = \bigcap_{\tau} C_{\tau}(f,e^{i\theta})$ where τ is any simple continuous curve in Δ and $\tilde{R}_{\Delta}(f,e^{i\theta}) = \bigcap_{\Delta^*} \operatorname{int} R_{\Delta^*}(f,e^{i\theta})$ where $R_{\Delta^*}(f,e^{i\theta})$ is the range of f in any Stolz angle Δ^* that strictly contains Δ.

A function f has the <u>n-segment</u> <u>property</u> for any integer $n \geq 2$ if there exist n chords $\Gamma_1, \ldots, \Gamma_n$ terminating at $e^{i\theta}$ such that $C_{\Gamma_1}(f,e^{i\theta}) \cap \ldots \cap C_{\Gamma_n}(f,e^{i\theta}) = \phi$. In this section it will be shown that for any normal meromorphic function f in D the set of points $e^{i\theta}$ at which f possesses the n-segment property is of first category and measure 0 on C.

<u>Theorem</u> <u>19</u>: If f is normal and meromorphic in D, then for any $e^{i\theta} \in C$ and any symmetric Stolz angle Δ_{α} of opening 2α at $e^{i\theta}$,

$$C_{\Delta_\alpha}(f,e^{i\theta}) - \tilde{R}_{\Delta_\alpha}(f,e^{i\theta}) \subseteq \Pi_{\Delta_\alpha}(f,e^{i\theta}).$$

(Rung, 1, Theorem 1, p.44)

<u>Proof</u>: Let c be any arbitrary point in $C_{\Delta_\alpha}(f,e^{i\theta}) - \tilde{R}_{\Delta_\alpha}(f,e^{i\theta})$. Then there exists a sequence $\{z_n\}$ in Δ_α with $\lim_{n\to\infty} z_n = e^{i\theta}$ and $\lim_{n\to\infty} f(z_n) = c$. Let w_n be the unique point on the diameter of C from $e^{i\theta}$ to $e^{-i\theta}$ for which $\sigma(z_n,w_n)$ equals the non-Euclidean distance of z_n to this diameter. For any β satisfying $\alpha < \beta < \pi/2$, let H_β denote the region bounded by the two hypercycles symmetric in this diameter that form the angles β and $-\beta$ with it. If β_1 is chosen so that $\alpha < \beta_1 < \beta$, then there exists a positive integer N such that $n > N$ implies

$$\sigma(z_n,w_n) < \tfrac{1}{2}\log\cot(\tfrac{1}{4}\pi - \tfrac{1}{2}\beta_1) = M \qquad (1)$$

because

$$\overline{\lim_{n \to \infty}}\ \sigma(z_n, w_n) \leq \frac{1}{2} \log \cot\left(\frac{\pi}{4} - \frac{\alpha}{2}\right).$$

We define $g_n(\zeta) = f\left(\frac{\zeta + w_n}{1 + \bar{w}_n \zeta}\right)$ for any positive integer n. Since f is normal, there exists a subsequence $\{g_{n_k}\}$ which converges uniformly in any compact subset of D to g. We also define another sequence $\{\zeta_k\}$ by the equations $z_{n_k} = \frac{\zeta_k + w_{n_k}}{1 + \bar{w}_{n_k}\zeta_k}$. Because of (1) $|\zeta_k| < \frac{e^{2M} - 1}{e^{2M} + 1} = K$, and so for any accumulation point ζ_o of $\{\zeta_k\}$, $|\zeta_o| \leq K$. Let $\{\zeta_p\}$ be a subsequence of $\{\zeta_k\}$ tending to ζ_o. Then the continuous convergence of $\{g_{n_k}\}$ implies that $\lim_{p \to \infty} g_{n_p}(\zeta_p) = \lim_{p \to \infty} f(z_{n_p}) = g(\zeta_o) = c$.

We want to show $g(\zeta) \equiv c$. Suppose not. Let D* be any fixed non-Euclidean open disk with center ζ_o which is contained in the open disk

$$|\zeta| < \frac{1}{2}\, \frac{e^{2M^*} - 1}{e^{2M^*} + 1} \quad \text{where} \quad M^* = \frac{1}{2} \log \cot\left(\frac{\pi}{4} - \frac{\beta}{2}\right).$$

Let ξ be any point in D*. By Hurwitz's Theorem (Carathéodory, 2, p.195) there exists an integer K_o and a sequence of points $\{\xi_k\}$ in D* that tend to ξ such that $g_{n_k}(\xi_k) = g(\xi)$ for all $k \geq K_o$. For the points $x_k = (\xi_k + w_{n_k})/(1 + \xi_k \bar{w}_{n_k})$ for which $k \geq K_o$, $f(x_k) = g(\xi)$. So $g(\xi) \in R_{\Delta_\beta}(f, e^{i\theta})$. Since $g(\zeta) \not\equiv c$, $c \in \tilde{R}_{\Delta_\alpha}(f, e^{i\theta})$, a contradiction. Consequently $g(\zeta) \equiv c$, which implies that f tends uniformly to c on the sequence of non-Euclidean disks with centers w_{n_k} and radii M. Each disk intersects both boundary segments of Δ_α; so $c \in \Pi_{\Delta_\alpha}(f, e^{i\theta})$.

In Chapter I we defined K(f) to be the set of points $\zeta \in C$ for which $C_{\Delta_1}(f,\zeta) = C_{\Delta_2}(f,\zeta)$ for any pair of Stolz angles at ζ and the outer angular cluster set $C_A(f,\zeta)$ to be the union of all cluster sets $C_\Delta(f,\zeta)$ for Δ any Stolz angle at ζ.

<u>Theorem 20</u>: If f is normal and meromorphic in C and $\zeta \in K(f)$, then, for any Stolz angle Δ and any chord $\psi(\alpha)$ terminating at ζ and making the angle α for $-\pi/2 < \alpha < \pi/2$ with the radius at ζ, $C_\Delta(f,\zeta) = C_{\psi(\alpha)}(f,\zeta)$. (Rung, 1, Theorem 2, p.48)

<u>Proof</u>: First we want to show that the set $\hat{C}_{\psi(\alpha)}(f,\zeta) = \bigcap_{\Delta^*} C_{\Delta^*}(f,\zeta)$, where Δ^* is any Stolz angle containing $\psi(\alpha)$, is contained in $C_{\psi(\alpha)}(f,\zeta)$. Let $c \in \hat{C}_{\psi(\alpha)}(f,\zeta)$ and $\{\Delta_n\}$ be a sequence of Stolz angles at ζ containing $\psi(\alpha)$ and satisfying the conditions $\Delta_n \supset \Delta_{n+1}$ and $\bigcap_{n=1}^{\infty} \Delta_n = \psi(\alpha)$. For each positive integer n, let $\{z_k^{(n)}\}$ be a sequence contained in Δ_n such that $z_k^{(n)} \to \zeta$ and $f(z_k^{(n)}) \to c$. We select a sequence $\{w_n = z_{k_n}^{(n)}\}$ so that $|w_n - \zeta| < 1/n$ and $|f(w_n) - c| < 1/n$. The non-Euclidean distance of w_n to $\psi(\alpha)$ tends to 0 as $n \to \infty$. If ζ_n is the point on $\psi(\alpha)$ at which this distance is assumed, then $\sigma(w_n,\zeta_n) \to 0$. By a result of Bagemihl and Seidel (2, Lemma 1, p.10), $f(\zeta_n) \to c$ as $n \to \infty$. So $c \in C_{\psi(\alpha)}(f,\zeta)$. Since $C_{\psi(\alpha)}(f,\zeta) \subseteq \hat{C}_{\psi(\alpha)}(f,\zeta)$, $\hat{C}_{\psi(\alpha)}(f,\zeta) = C_{\psi(\alpha)}(f,\zeta)$.

The condition $\zeta \in K(f)$ says that for any two Stolz angles Δ_1 and Δ_2, $C_{\Delta_1}(f,\zeta) = C_{\Delta_2}(f,\zeta)$. So if $\zeta \in K(f)$, from the preceding paragraph, we have that $C_\Delta(f,\zeta) = \hat{C}_{\psi(\alpha)}(f,\zeta) = C_{\psi(\alpha)}(f,\zeta)$.

<u>Theorem 21</u>: If f is meromorphic in D and its range of values R(f) is equal to the Frontier of R(f), then, for each $\zeta \in C$, $C_A(f,\zeta) = \bigcap_{\Delta} \Pi_\Delta(f,\zeta)$ where Δ varies over all Stolz angles at ζ. (Rung, 1, Theorem 3, p.49)

<u>Proof</u>: From the hypothesis there exist three values that f assumes at most a finite number of times in D. So f is normal by a result of Lehto and Virtanen (1, p.54). Since Interior $R_\Delta(f,\zeta) = \phi$ for any ζ and any

symmetric Stolz angle at ζ, Theorem 19 implies $C_\Delta(f,\zeta) = \Pi_\Delta(f,\zeta)$. If Δ_1 and Δ_2 are any two Stolz angles at ζ, then let Δ_3 be a symmetric Stolz angle that contains Δ_1 and Δ_2. Because of the definition of $\Pi_\Delta(f,\zeta)$ and $C_{\Delta_3}(f,\zeta) = \Pi_{\Delta_3}(f,\zeta)$, it follows that $C_{\Delta_1}(f,\zeta) = \Pi_{\Delta_2}(f,\zeta)$; therefore, $\bigcup_\Delta C_\Delta(f,\zeta) = \bigcap_\Delta \Pi_\Delta(f,\zeta)$ where Δ varies over all Stolz angles at ζ.

A function f possesses the _n-segment property_ at ζ if there exists n chords $\Gamma_1,\ldots,\Gamma_n$ at ζ such that $C_{\Gamma_k}(f,\zeta) \cap C_{\Gamma_j}(f,\zeta) = \phi$ for $1 \leq k \leq n$, $1 \leq j \leq n$ and $k \neq j$.

Theorem 22: Let f be a normal meromorphic function in D. For any integer $n \geq 2$, the set of all points ζ at which f possesses the n-segment property is a set of first category and measure zero on C. (Rung, 1, Theorem 5, p.50)

Proof: Let S(f) denote the set of all ζ at which f possesses the n-segment property. Then $S(f) \cap K(f) = \phi$ by Theorem 20 since the cluster set along any chord lying in Δ and terminating at a point of K is $C_\Delta(f,\zeta)$. Since K(f) is a residual set of measure 2π on C (Theorem 17, Chapter I), S(f) is of first category and measure zero on C.

Suppose γ is any boundary arc of D and that $0 < r < \infty$. Then we define the set $J(\gamma,r) = \{z \in D : \sigma(z,\gamma) < r\}$. If d_τ denotes the diameter of C ending at $\tau \in C$, then a boundary arc $\gamma = z(t)$ at τ _approaches τ in a non-tangential manner_ whenever there exists some $0 \leq t_o < 1$ and some $0 < r < \infty$ such that $z(t) \in J(d_\tau,r)$, $t \geq t_o$. The set of all non-tangential boundary arcs at τ will be denoted by $\Lambda(\tau)$. Finally we define the sets

$\Pi_u = \bigcap_{\gamma \in \Lambda(\tau)} C_\gamma(f,\tau)$ and $\Pi_{J(\gamma,r)}(f,\tau) = \bigcap_{\gamma^*} C_{\gamma^*}(f,\tau)$ where γ^* ranges over all boundary arcs at τ that lie in $J(\gamma,r)$.

Theorem 23 states that for any normal function f in D, any γ, $\gamma' \in \Lambda(\tau)$ and any r, $r' > 0$, $\Pi_{J(\gamma,r)}(f,\tau) = \Pi_{J(\gamma',r')}(f,\tau) = \Pi_u(f,\tau)$. In order to prove this theorem we will need the following two lemmas.

Lemma 3: Suppose f is a normal function in D and $\gamma \in \Lambda(\tau)$. Let $B(z,a) = \{z \in D : \sigma(z,z') < a\}$ and $Z_f(w,a) = \bigcup_{z'} B(z',a)$ where the union is taken over all $z' \in D$ such that $f(z') = w$. If $w \in C_\gamma(f,\tau)$ and there exists an $a > 0$ such that $\gamma \cap Z_f(w,a) = \phi$, then $w \in C_{\gamma'}(f,\tau)$ for any $\gamma' \in \Lambda(\tau)$. (Lappan and Rung, 1, Lemma 1, p.257)

Proof: Since $w \in C_\gamma(f,\tau)$, there exists a sequence $\{z_n\}$ on γ such that $z_n \to \tau$ and $f(z_n) \to w$ as $n \to \infty$. Let $S_n(\zeta) = \dfrac{\zeta + z_n}{1 + \zeta \bar{z}_n}$ for $|\zeta| < 1$ and $f(S_n(\zeta)) = g_n(\zeta)$ for any positive integer n. Because of the normalcy of f, there exists a convergent subsequence $\{g_{n_k}(\zeta)\}$. If $g(\zeta)$ denotes the limit function, then $g(0) = \lim_{k \to \infty} g_{n_k}(0) = \lim_{k \to \infty} f(z_{n_k}) = w$; but, for $|k| < \tanh a$, the equation $g_{n_k}(\zeta) = w$ is not satisfied for any value of k. So by Hurwitz's Theorem (Caratheodory, 2, p.195) $g(\zeta) \equiv w$. So for any fixed $0 < a' < \infty$, $f \to w$ as $z \to \tau$ on the set $\bigcup_{k=1}^{\infty} B(z_{n_k}, a')$. If $\gamma' \in \Lambda(\tau)$, then $\gamma' \cap B(z_{n_k}, a') \neq \phi$ for suitable values of $a' > 0$ and any positive integer k. Therefore, $w \in C_{\gamma'}(f,\tau)$.

Lemma 4: Let f be normal in D. Suppose for some $\tau \in C$ and for every positive integer n, a set of distinct points $\{\mathfrak{z}_i^{(n)}\}$, $i = 1,2,\ldots,m_n$, with the following properties exists:

(i) For some $r > 0$ and all n, $\mathfrak{z}_1^{(n)} \in J(d_\tau, r)$;

(ii) $\mathfrak{z}_1^{(n)} \to \tau$ as $n \to \infty$;

(iii) $\sigma(\mathfrak{z}_i^{(n)}, \mathfrak{z}_{i+1}^{(n)}) < K_n$ for $i = 1,2,\ldots,m_n-1$, with $K_n \to 0$ as $n \to \infty$;

(iv) there exists a positive number A independent of n such that

$\sigma(\mathfrak{z}_1^{(n)}, \mathfrak{z}_{m_n}^{(n)}) \geqslant A > 0$;

(v) $f(\mathfrak{z}_i^{(n)}) = w$ for $i = 1,2,\ldots,m_n$ and $n = 1,2,\ldots$.

Then $w \in C_\gamma(f,\tau)$ for all $\gamma \in \Lambda(\tau)$. (Lappan and Rung, 1, Lemma 2, p.258)

<u>Proof</u>: As in the previous lemma, we set $f(S_n(\zeta)) = g_n(\zeta)$ for any integer n where now $S_n(\zeta) = (\zeta + \mathfrak{z}_1^{(n)})/(1 + \bar{\mathfrak{z}}_1^{(n)}\zeta)$. Again we denote the convergent subsequence by $\{g_{n_k}(\zeta)\}$ and the limit function by $g(\zeta)$. Since $g_{n_k}(0) = f(\mathfrak{z}_1^{(n_k)}) \to w$ as $n \to \infty$, $g(0) = w$. We want to show that the set of points ζ' such that $g(\zeta') = w$ which also lie in $|\zeta| < \tanh A \equiv B$ is infinite.

Suppose there exists a ring R, $0 < r' \leq |\zeta'| \leq r'' < B$ with $r' \neq r''$, which contains none of the points ζ'. For any fixed n, the set $\{\mathfrak{z}_i^{(n)} : i = 1,2,\ldots,m_n\}$ is transformed by $S_n^{-1}(z)$ onto a set of points we call $\{\zeta_i^{(n)} : i = 1,2,\ldots,m_n\}$ which have the properties:

(i') $\zeta_1^{(n)} = 0$ and $|\zeta_{m_n}^{(n)}| \geqslant B$;

(ii') $\sigma(\zeta_i^{(n)}, \zeta_{i+1}^{(n)}) < K_n$ for $i = 1,2,\ldots,m_n - 1$;

(iii') $g_n(\zeta_i^{(n)}) = w$ for $i = 1,2,\ldots,m_n$.

There must be at most a finite number of $\zeta_i^{(n)}$ for $i = 1,2,\ldots,m_n$ and any positive integer n within R. Otherwise this set would have a limit point ζ_o and by continuous convergence of $g_n(\zeta)$ to $g(\zeta)$, $g(\zeta_o) = w$, a contradiction of the definition of R. So there exists a positive integer N such that for $n > N$ no point of the form $\zeta_i^{(n)}$ for $i = 1,2,\ldots,m_n$ lies in R. If $n_1 > N$ is chosen so that $K_{n_1} < \sigma(0,r'') - \sigma(0,r')$, this violates the properties (i') — (iii') and the definition of R.

Consequently $g(\zeta) \equiv w$ in D and the rest of the proof is the same as the last part of Lemma 3.

Theorem 23: If $f(z)$ is normal in D, γ and γ' are any two arcs in $\Lambda(\tau)$, and r and $r' > 0$, then $\Pi_{J(\gamma,r)}(f,\tau) = \Pi_{J(\gamma',r')}(f,\tau) = \Pi_u(f,\tau)$. (Lappan and Rung, 1, Theorem 1, p.259)

Proof: Using the same notation as in the statement of Lemma 3, we let $B(z',a) = \{z \in D : \sigma(z,z') < a\}$ and $Z_f(w,a) = \bigcup_{z'} B(z',a)$ where the union is taken over all $z' \in D$ such that $f(z') = w$. Then for any fixed curve $\gamma \in \Lambda(\tau)$ and fixed $r > 0$, let $Z_f'(w,1/n) = Z_f(w,1/n) \cap J(\gamma,r)$ for n any positive integer.

Suppose $\gamma \cap Z_f'(w,1/n) = \phi$ for some n. Then the conclusion of this theorem follows immediately from Lemma 3.

Now suppose $\gamma \cap Z_f'(w,1/n) \neq \phi$ for every n. Then for each n we decompose $Z_f'(w,1/n)$ into its components $\{Y_i^{(n)} : i = 1,\ldots,j_n$ where $1 \leq j_n \leq \infty\}$. Assume for each n there exists at least one component $Y_{i_n}^{(n)}$ whose boundary meets both γ and the boundary of $J(\gamma,r)$. Then there exists a finite set of points $\{\mathfrak{z}_j^{(n)} : j = 1,\ldots,h_n\}$ with the properties:

(i) $\mathfrak{z}_1^{(n)} \in J(d_\tau,r)$;

(ii) $\mathfrak{z}_1^{(n)} \to \tau$ as $n \to \infty$;

(iii) $\sigma(\mathfrak{z}_j^{(n)},\mathfrak{z}_{j+1}^{(n)}) < 2/n$ for $j = 1,\ldots,h_n - 1$;

(iv) $\sigma(\mathfrak{z}_1^{(n)},\gamma) < 1/n$ and $\sigma(\mathfrak{z}_{h_n}^{(n)}$, Frontier $J(\gamma,r)) < 1/n$ which imply $\sigma(\mathfrak{z}_1^{(n)},\mathfrak{z}_{h_n}^{(n)}) \geqslant r - 2/n$;

(v) $f(\mathfrak{z}_j^{(n)}) = w$ for $j = 1,\ldots,h_n$ and n any positive integer.

If n_o is chosen such that $2/n_o < r/2$, then for $n \geqslant n_o$ the conditions of Lemma 4 are satisfied with $A = r/2$ and $K_n = 2/n$. So the

conclusion of this theorem follows.

Finally we assume that there exists an n_o such that no $Y_i^{(n_o)}$ for $i = 1,\ldots,j_{n_o}$ has a boundary which meets both γ and the boundary of $J(\gamma,r)$. Let V denote the union of all of the components of $Z_f'(w,1/n_o)$ that meet γ and also γ itself. Since this is a connected set lying entirely in $J(\gamma,r)$, $\overline{V} \cap C = \{\tau\}$. There exists a subset β of Frontier V which is a boundary arc approaching τ within $J(\gamma,r)$ and $\beta \cap Z_f'(w,1/n_o) = \phi$. Since $w \in C_\beta(f,\tau)$, this theorem's conclusion follows from Lemma 4.

HOROCYCLIC PROPERTIES OF NORMAL FUNCTIONS

In Chapter I we proved properties of horocycles of arbitrary functions. In this section we will prove other properties of horocycles which only hold for normal functions. Here we will use some of the same definitions and notations as we used previously. In addition we will use the following definitions.

An <u>admissible</u> <u>tangential</u> <u>arc</u> at a point $\zeta \in C$ is an arc γ at ζ for which there exists a sequence $\{H_{r_1(n),r_2(n),r_3(n)}(\zeta)\}$ of nested right or nested left horocycles at ζ with $\lim_{n\to\infty}(r_2(n) - r_1(n)) = 0$ and each member of the sequence contains some terminal subarc of γ. Then $\Pi_{T_w}(f,\zeta) = \bigcap_\gamma C_\gamma(f,\zeta)$ where the intersection is taken over all admissible tangential arcs γ at ζ.

Any point $\zeta \in C$ that is both a Plessner point and a horocyclic Plessner point of f is called a <u>generalized</u> <u>Plessner</u> <u>point</u> of f.

Let $\Omega_r(\zeta)$ denote the interior of the horocycle $h_r(\zeta)$. Then the <u>primary-tangential</u> <u>cluster</u> <u>set</u> of f at ζ is defined to be the set

$$C_\Omega(f,\zeta) = \overline{\bigcup_{0<r<1} C_{\Omega_r(\zeta)}(f,\zeta)}.$$

For any function $f : D \rightarrow W$, a <u>primary-tangential</u> <u>pre-Meier</u> <u>point</u> is any point $\zeta \in C$ such that $\Pi_{T_W}(f,\zeta) = C_\Omega(f,\zeta) \subset W$, where $\subset$ denotes proper inclusion. The term "pre-Meier" is used because the condition

$$C_{h_r^-}(f,\zeta) = C_{h_r^+}(f,\zeta) \subset W \text{ for } 0 < r < 1 \text{ and } 0 < r' < 1$$

is fulfilled at each primary-tangential pre-Meier point of f, and this is a necessary condition for a point $\zeta \in C$ to be a horocyclic Meier point. If it is also true that $C_\Omega(f,\zeta) = C(f,\zeta) \subset W$, then ζ is actually a horocyclic Meier point of f. We will show in Theorem 25 that if f is any normal meromorphic function in D, then almost every point $\zeta \in C$ is either a primary-tangential pre-Meier point or a point at which $\Pi_{T_W}(f,\zeta) = W$.

<u>Lemma</u> <u>5</u>: If $f(z)$ is a normal meromorphic function in D and $\zeta \in K_W(f)$, the set of points on C such that $C_{H_1}(f,\zeta) = C_{H_2}(f,\zeta)$ for any pair of horocyclic angles H_1 and H_2, then $\Pi_{T_W}(f,\zeta) = C_U(f,\zeta)$ for $C_U(f,\zeta)$ the outer horocyclic angular cluster set. (Bagemihl, 2, Lemma 4, p.16)

<u>Proof</u>: Let $\alpha \in \Pi_{T_W}(f,\zeta)$. Then $\alpha \in C_\Omega(f,\zeta)$ for every admissible tangential arc Λ at ζ. By definition there exists a horocyclic angle H at ζ which contains a terminal subarc of γ. Since $C_\gamma(f,\zeta) \subseteq C_H(f,\zeta)$, $\alpha \in C_U(f,\zeta)$.

Now suppose $\alpha \in C_U(f,\zeta)$. Let γ be any admissible tangential arc at ζ. Since $\zeta \in K_W(f)$, $\alpha \in C_H(f)$ for every horocyclic angle H at ζ. Therefore, there exists a sequence of points $\{z'_n\}$ in D where $\lim_{n\rightarrow\infty} z'_n = \zeta$ and $\lim_{n\rightarrow\infty} f(z'_n) = \alpha$ such that for an appropriate sequence of points $\{z_n\}$ on γ with $\lim_{n\rightarrow\infty} z_n = \zeta$, we have $\lim_{n\rightarrow\infty} \sigma(z_n, z'_n) = 0$. By Theorem 4 this implies that $f(z_n) \rightarrow \alpha$ as $n \rightarrow \infty$, and $\alpha \in C_\gamma(f,\zeta)$. Since γ was an

arbitrary admissible tangential arc at ζ, $\alpha \in \Pi_{T_W}(f,\zeta)$.

<u>Theorem</u> <u>24</u>: Suppose $f(z)$ is a nonconstant normal meromorphic function in D and the set of asymptotic values $A(f)$ is of harmonic measure zero. Then there exists a residual subset S of C of measure 2π such that for every $\zeta \in S$, $\Pi_{T_W}(f,\zeta) = W$. (Bagemihl, 2, Theorem 9, p.17)

<u>Proof</u>: According to Theorem 15, Section I, almost every Plessner point of f is a horocyclic Plessner point of f; therefore, by Plessner's Theorem (Collingwood and Cartwright, 1, Theorem 8.2, p.147) almost every point of C is either a Fatou point or a point which is both a Plessner point and a horocyclic Plessner point. Since f is nonconstant and $A(f)$ is of harmonic measure zero, Privalow's Theorem (1, p.210) implies that the set of Fatou points of f is of measure zero. Consequently the set of horocyclic points $I_W(f)$ is of measure 2π. A horocyclic analogue of Collingwood's Theorem (1, Theorem 3, p.382) implies that $I_W(f)$ is also residual on C. If $\zeta \in I_W(f)$, then $C_U(f,\zeta) = W$. Since $I_W(f) \subseteq K_W(f)$, $\Pi_{T_W}(f,\zeta) = C_U(f,\zeta)$ by Lemma 5. This theorem is now valid by setting $S = I_W(f)$.

<u>Theorem</u> <u>25</u>: If $f(z)$ is a normal meromorphic function in D, then almost every point $\zeta \in C$ is either a primary-tangential pre-Meier point of f or a point at which $\Pi_{T_W}(f,\zeta) = W$. (Dragosh, 2, Theorem 10, p.76)

<u>Proof</u>: For any point $\zeta \in C$, $C_I(f,\zeta)$, the inner angular cluster set, satisfies $C_I(f,\zeta) \subseteq C_U(f,\zeta) \subseteq C_\Omega(f,\zeta)$. An approach similar to that used to prove Lemma 4, Section I, shows that $C_\Omega(f,\zeta) \subseteq C_I(f,\zeta)$ at almost every

point $\zeta \in C$. So at almost every point $\zeta \in C$, $C_U(f,\zeta) = C_I(f,\zeta) = C_\Omega(f,\zeta)$. Since $K_W(f)$ is of measure 2π (Theorem 18, Section I), Lemma 5 implies that $\Pi_{T_W}(f,\zeta) = C_\Omega(f,\zeta)$ at almost every point $\zeta \in C$. This theorem now follows because at every point $\zeta \in C$, either $C_\Omega(f,\zeta) \subset W$ or $C_\Omega(f,\zeta) = W$.

A FUNCTION THEORETIC CHARACTERIZATION OF NORMAL MEROMORPHIC FUNCTIONS

Let H^∞ denote the algebra of holomorphic functions bounded in D. In the study of the behavior of these function near the boundary, it is helpful to compactify D in such a way that each of them has a continuous extension to the compactification. Let M be a compact Hausdorff space such that it contains D as a dense subset. Each $f \in H^\infty$ can be extended to a continuous function $\hat{f}$ on M and each pair of distinct points in M can be separated by one of the functions $\hat{f}$. By Carleson's Corona Theorem (1), M is the maximal ideal space of H^∞. Let $\beta = M/D$ denote the ideal boundary of D. If S is any subset of D, then we set $\beta(S) = \overline{S}/D$ where $\overline{S}$ denotes the closure of S in M.

Two points m_1 and m_2 in M are in the same <u>Gleason part</u> if there exists a constant c, $0 < c < 2$, such that $|\hat{f}(m_1) - \hat{f}(m_2)| \leq c$ for $f \in H^\infty$ and $|f| \leq 1$. This is an equivalence relation and we denote by P(m) the Gleason part of the point $m \in M$. If $S \subset D$, then $P^*(S) = \cup\{P(m) : m \in \beta(S)\}$ for the set of Gleason parts generated by S. Each Gleason part P(m) consists of either a single point or the image of a one-to-one analytic map of an open disk into M (Hoffman, 1 and 2). We say that m is a <u>regular point</u> if P(m) contains more than one point and denote the set of all regular points in M by G. In Theorem 26 we will show that f is normal in D if and only if f admits a spherically continuous extension to G.

For any subsets S and T in D, we define the pseudometrics:

(i) $H_\sigma(S,T) = \inf\{\epsilon : S \subset \{z : \sigma(z,T) < \epsilon\}, T \subset \{z : \sigma(z,S) < \epsilon\}\}$ where $\sigma(z,z')$ denotes the hyperbolic distance between z and z';

(ii) $H(S,T) = \inf_r H_\sigma(S \cap \{|z| > r\}, T \cap \{|z| > r\})$;

(iii) $\lambda(S,T) = \inf_r \sigma(S \cap \{|z| > r\}, T \cap \{|z| > r\})$.

<u>Lemma 6</u>: If S and T are subsets in D, then $\beta(S) = \beta(T)$ if and only if $H(S,T) = 0$. (Brown and Gauthier, 1, Theorem 1, p.367)

<u>Proof</u>: Suppose $H(S,T) = 0$ and $m \in \beta(S)$. Let $\{x_\lambda\}$ be any net in S that converges to m. We choose $y_\lambda \in T$ such that $\sigma(x_\lambda, y_\lambda) < 2\sigma(x_\lambda, T)$. Since $|x_\lambda| \to 1$ and $H(S,T) = 0$, it follows that $\sigma(x_\lambda, y_\lambda) \to 0$. So $\{y_\lambda\}$ converges to m and $\beta(S) \subset \beta(T)$. By a similar argument we obtain the inclusion $\beta(T) \subset \beta(S)$. Therefore, $\beta(S) = \beta(T)$.

Conversely, suppose $H(S,T) > 0$. Then we may choose a Blaschke sequence $\{z_n\}$ in S such that, for each positive integer n, $\prod_{k \neq n} \left|\frac{z_k - z_n}{1 - \bar{z}_n z_k}\right| \geq \delta > 0$ and $\sigma(z_n, T) \geq \alpha > 0$. From Cima and Colwell (1, p.796) and Kerr-Lawson (2, p.532) it follows that the Blaschke product B associated with the z_n's is bounded away from zero on T. Consequently $\hat{B}(m) \neq 0$ for each $m \in \beta(T)$. Since $\{z_n\} \subset S$, there is a point $m \in \beta(S)$ such that $\hat{B}(m) = 0$. So $\beta(S) \neq \beta(T)$.

<u>Lemma 7</u>: If S and T are subsets of D, then $G \cap \beta(S) \cap \beta(T) \neq \phi$ if and only if $\lambda(S,T) = 0$. (Brown and Gauthier, 1, Theorem 3, p.368)

<u>Proof</u>: Suppose $\lambda(S,T) = 0$. We choose two sequences $\{z_n\}$ and $\{z_n'\}$ such that $\{z_n\} \in S$ and $\{z_n'\} \in T$, $\sigma(z_n, z_n') < 1/n$, and $\prod_{k \neq n} \left|\frac{z_k - z_n}{1 - \bar{z}_n z_k}\right| \geq \delta > 0$, $n > 0$. Let m be in $\beta(\{z_n\})$. We pick a subsequence $\{z_{n(\lambda)}\}$ of $\{z_n\}$ that

converges to m . Since $n(\lambda)\to\infty$ and $\sigma(z_{n(\lambda)}, z'_{n(\lambda)})\to 0$, $\{z'_{n(\lambda)}\}$ converges to m. By Hoffman (1, p.75), m is in G. So $G\cap\beta(S)\cap\beta(T)\neq\phi$.

The converse follows immediately from Hoffman (1, p.75).

<u>Theorem</u> <u>26</u>: A function f is normal in D if and only if f admits a spherically continuous extension to the set G of regular points of M. (Brown and Gauthier, 1, Theorem 4, p.368)

<u>Proof</u>: First we will show that if $m\in G$, then $C_f(m)$ is a singleton. Suppose on the contrary there exist two distinct values w_1 and w_2 in $C_f(m)$ with spherical distance $\chi(w_1,w_2) = \epsilon > 0$. For each neighborhood V of m, we choose two points z_V and z'_V in $D\cap V$ such that $\chi(f(z_V),w_1) < \epsilon/3$ and $\chi(f(z'_V),w_2) < \epsilon/3$. Let $S = \{z_V\}$ and $T = \{z'_V\}$. Then $m\in\beta(S)\cap\beta(T)\cap G$ and Lemma 7 implies that $\lambda(S,T) = 0$. By uniform continuity of f, we can pick $z_1\in S$ and $z_2\in T$ so that $\sigma(z_1,z_2) < \delta$ where δ is chosen so small that $\chi(f(z_1),f(z_2)) < \epsilon/3$. So $\epsilon = \chi(w_1,w_2) \leqslant \chi(w_1,f(z)) + \chi(f(z_1),f(z_2)) + \chi(f(z_2,w_2) < \epsilon$, a contradiction. Therefore, $C_f(m)$ is a singleton for $m\in G$ and we set $\hat{f}(m) = C_f(m)$.

If $\hat{f}$ is not continuous at m, then for some $\epsilon > 0$, each relative neighborhood $V\cap G$ of m contains a point m_V such that $\chi(\hat{f}(m_V),\hat{f}(m)) \geqslant \epsilon$. There exists a $z_V\in V\cap D$ such that $\chi(f(z_V),\hat{f}(m_V)) < \epsilon/2$. So the net $\{z_V\}$ converges to m, but $\chi(f(z),\hat{f}(m)) \geqslant \epsilon/2$, a contradiction.

Conversely, if f is not normal, then according to Lappan (3, Theorem 1, p.155) there exist two sequences $\{z_n\}$ and $\{z'_n\}$ and an $\epsilon > 0$ such that for each $n > 0$, $\sigma(z_n,z'_n)\longrightarrow 0$ but $\chi(f(z_n),f(z'_n)) \geqslant \epsilon$. We may assume that the sequence $\{z_n\}$ satisfies the condition $\prod_{k\neq n}\left|\frac{z_k-z_n}{1-\bar{z}_n z_k}\right| \geqslant \delta > 0$. So by Hoffman (1, p.75) $\beta(\{z_n\})\subset G$. Since $\sigma(z_n,z'_n)\longrightarrow 0$, $H(\{z_n\},\{z'_n\}) = 0$,

and $\beta(\{z_n\}) = \beta(\{z'_n\}) \subset G$ by Lemma 6. Suppose $m \in \beta(\{z_n\})$. Then for any subnet $\{z_{n(\lambda)}\}$ converging to m, we also have $\{z'_{n(\lambda)}\}$ converging to m. Since $\chi(f(z_{n(\lambda)}), f(z'_{n(\lambda)})) \geqslant \epsilon$, the cluster set $C_f(m)$ is not a singleton.

Theorem 27: If f is a normal meromorphic (holomorphic) function in D and $\hat{f}$ is the extension of f to the set G of regular points of M, then on each nontrivial Gleason part, $\hat{f}$ is either meromorphic (holomorphic) or identically equal to infinity. (Brown and Gauthier, 1, Theorem 5, p.369)

Proof: Let $m \in G$. Then for any $\alpha \in D$ converging to m, $L_\alpha(z) = \dfrac{z+\alpha}{1+\bar{\alpha}z}$ converges pointwise to L_m, a one-to-one mapping of D onto P(m). (Hoffman, 1, p.75) We will prove that $\hat{f} \circ L_m$ is a meromorphic (holomorphic) function. We pick a fixed point z_o in D and assume that $\hat{f} \circ L_m(z_o)$ is finite. Furthermore, we may suppose that α lies in some neighborhood of m for which $f \circ L_\alpha$ is uniformly bounded. For if $f \circ L_\alpha$ is not uniformly bounded in some neighborhood of z_o, there exist sequences $\{z_n\}$ and $\{\alpha_n\}$ such that $z_n \to z_o$ and $f \circ L_{\alpha_n}(z_n) \to \infty$. Since f is normal, $\{f \circ L_{\alpha_n}\}$ is a normal family of functions. Consequently it contains a subsequence which converges uniformly on compact subsets to a function g meromorphic in D or to ∞. Since $f \circ L_{\alpha_n} \to \infty$, $g(z_o)$ is infinite; however, $\{f \circ L_\alpha\}$ is uniformly bounded at z_o, a contradiction. The family $\{f \circ L_\alpha\}$ converges to $\hat{f} \circ L_m$ pointwise. Since $\{f \circ L_\alpha\}$ is uniformly bounded in a neighborhood of z_o, $\hat{f} \circ L_m$ is holomorphic in a neighborhood of z_o.

If $\hat{f} \circ L_m(z_o) = \infty$, we look at the family of functions $\{1/f \circ L_\alpha\}$. The family $\{f \circ L_\alpha\}$ is normal and so it is equicontinuous. Since the spherical metric is invariant when taking reciprocals, the family of reciprocals $\{1/f \circ L_\alpha\}$ is equicontinuous and thus normal. $1/\hat{f} \circ L_m(z_o) = 0$,

and, from the previous argument for the finite case, $1/f\circ L_m$ is holomorphic in a neighborhood of z_o. Therefore, for each point $z \in D$, $\hat{f}\circ L_m$ is either meromorphic (holomorphic) at z or identically infinite in a neighborhood of z. Consequently $\hat{f}\circ L_m$ is either meromorphic (holomorphic) in D or identically infinite.

We will now give an example of a normal meromorphic function f such that for each $m \in M/G$, $C_f(m) = R_f(m) = W$. Let f be a Schwarz triangle function (Carathéodory, 1, Part 7, pp.173-194) whose initial triangle is strictly interior to the unit circle. It is well-known that f is a normal function. Let a be any point on W, and let $\{z_n\}$ be the preimages of a. Since each triangle has the same finite ρ-diameter, there exists an $\epsilon > 0$ such that an ϵ-neighborhood of $\{z_n\}$ covers D. By a result of Hoffman (1, Corollary, p.84), $\beta(\{z_n\}) \supset M/G$. Therefore, $a \in R_f(m)$ for each $m \in M/G$. It is an open question whether for each $m \in M/G$ the cluster set is always equal to W. If this is true, then Theorem 27 is also sharp for holomorphic functions.

NORMAL HOLOMORPHIC FUNCTIONS

In this section we will continue to use the same definitions and notations used in the section of Chapter I which discusses the M-topology for arbitrary functions. Here we will show in Theorem 30 that if f(z) is a normal holomorphic function, then $G_f(p)$ is compact in the M-topology. First of all we will prove in Theorem 28 that any function f(z) which is normal and holomorphic in D belongs to the <u>class I_p</u>. This class consists of the holomorphic functions f in D that have the property that for each pair of arcs t_1, $t_2 \in T(p)$ along which $f(z) \to \infty$ as

$z \to p$, $f(z)$ is unbounded on each path t between t_1 and t_2.

Theorem 28: If f is a holomorphic normal function in D, then for each $p \in C$, f is in the class I_p. (Lappan, 11, Theorem 3, p.91)

Proof: Suppose $p \in C$ and ∞ is an asymptotic value of f at p along two disjoint paths t_1 and t_2 in $T(p)$. If t is any path in $T(p)$ between t_1 and t_2, then by a remark of Lehto and Virtanen (1, p.53) $f(z) \to \infty$ as $z \to p$ along t.

Theorem 29: If $p \in C$ and $f \in I_p$, then f may have at most two finite asymptotic values at p. (Lappan, 11, Theorem 4, p.91)

Proof: Suppose f has three distinct finite asymptotic values a_1, a_2 and a_3 at p so that there exist three disjoint arcs t_1, t_2 and t_3 in $T(p)$ such that $f(z) \to a_i$ as $z \to p$ along the t_i's. Then there exist paths q_1 and q_2 in $T(p)$ such that q_i is between t_i and t_{i+1} and $f(z) \to \infty$ as $z \to p$ along q_i for $i = 1, 2$. (Remark, MacLane, p.7) So t_2 is between q_1 and q_2 and f is bounded on t_2. Therefore, $f \notin I_p$.

Lemma 8: If $p \in C$ and f is a holomorphic function in D which is bounded in a neighborhood of p relative to D, then $G_f(p)$ is compact in the M-topology. (Lappan, 11, Theorem 1, p.89)

Proof: Suppose $G_f(p)$ is not compact in the M-topology. According to Theorem 35 in Chapter I, there exist directed sequences $\{t_n\}$ and $\{s_n\}$ of arcs in $T(p)$, a number $\epsilon > 0$, and a continuum K such that letting

$K_n = C_{t_n}(f,p)$ and $L_n = C_{s_n}(f,p)$, we have, for any $n > 0$, $M(K_n, K) < 1/n$, $d(L_n, K) > \epsilon$, and s_n is between t_n and t_{n+1}. Without loss of generality we may assume that all of the arcs s_n and t_n originate at the origin, terminate at p and no pair of arcs have any points in common except 0 and p. Finally we assume $M(K_n, K) < \epsilon/2$ for all n. Let Δ_n be the region bounded by $t_n \cup t_{n+1}$ and Δ_n' be the region bounded by $s_n \cup s_{n+1}$. It should be noted that Δ_n and Δ_n' are bounded in the complex plane. Since $s_n - \{0,p\} \subset \Delta_n$ and $t_{n+1} - \{0,p\} \subset \Delta_n'$, $L_n \subset C_{\Delta_n}(f,p)$ and $K_{n+1} \subset C_{\Delta_n'}(f,p)$. According to Collingwood and Lohwater (1, Theorem 5.2.1, p.91),

$$\text{Frontier } C_{\Delta_n}(f,p) \subset C_{t_n}(f,p) \cup C_{t_{n+1}}(f,p) = K_n \cup K_{n+1}$$

and

$$\text{Frontier } C_{\Delta_n'}(f,p) \subset C_{s_n}(f,p) \cup C_{s_{n+1}}(f,p) = L_n \cup L_{n+1}.$$

Since $M(K_k, K) < \epsilon/2$ and $d(L_k, K) > \epsilon$ for every positive integer k, there exists a point $w_o \in L_n \cup L_{n+1}$ such that $|w_o| > \sup\{|w| : d(w,K) < \epsilon/2\}$. If $w_o \in L_n$, then the fact that L_n is contained in a bounded set whose boundary is $K_n \cup K_{n+1}$ leads to the existence of a point $w_1 \in K_n \cup K_{n+1}$ such that $|w_1| > |w_o|$. However, $d(w_1, K) < \epsilon/2$ violates the choice of w_o. If $w_o \in L_{n+1}$, then there is a similar contradiction. So $K \in G_f(p)$.

Lemma 9: Let f be holomorphic in D and $p \in C$. Suppose further $\{t_n\}$ is a directed sequence of arcs in T(p), $K_n = C_{t_n}(f,p)$ for $n > 0$, and K is a continuum such that $M(K_n, K) \to 0$. Then one of the following must hold:

(i) $K \in G_f(p)$;

(ii) $\infty \in K$;

(iii) there exists q_2 between q_1 and q_3 in T(p) such that $f \to \infty$ on q_1 and q_3 as $z \to p$ and f is bounded on q_2. (Lappan, 11, Lemma 3, p.90)

<u>Proof</u>: Suppose $K \notin G_f(p)$, $\infty \notin K$ and each K_n is bounded. We want to show that (iii) holds. Let Δ_n be the region bounded by $t_n \cup t_{n+1}$. If there exists an integer N such that f is bounded in each region Δ_n for $n > N$, then $K \in G_f(p)$ because the proof of Lemma 8 only required that f be bounded on a union of three consecutive regions Δ_n. Since we are assuming $K \notin G_f(p)$, there exist positive integers n_1 and n_2 with $n_2 > n_1$ such that f is unbounded in Δ_{n_1} and Δ_{n_2}. So there exist paths q_1 and q_3 in T(p) such that $q_1 - \{p\} \subset \Delta_{n_1}$, $q_3 - \{p\} \subset \Delta_{n_2}$, and $f(z) \to \infty$ as $z \to p$ along q_1 and q_3. Letting $q_2 = t_{n_2}$, we have $C_{q_2}(f,p) = K_{n_2}$ which is bounded. So f is bounded on q_2 which is between q_1 and q_3.

<u>Theorem</u> <u>30</u>: If $p \in C$ and $f \in I_p$, then $G_f(p)$ is compact in the M-topology. (Lappan, 11, Theorem 5, p.91)

<u>Proof</u>: If $G_f(p)$ is not compact in the M-topology, then according to Theorem 35 in Chapter I, there exist directed sequences $\{t_n\}$ and $\{s_n\}$ of arcs in T(p), a number $\epsilon > 0$ and a continuum K such that letting $K_n = C_{t_n}(f,p)$ and $L_n = C_{s_n}(f,p)$, we have that for each positive integer n, $M(K_n,K) < 1/n$, $d(L_n,K) > \epsilon$ and s_n is between t_n and t_{n+1}. We may assume that $M(K_n,K) < \epsilon/2$ for each n. Since $f \in I_p$, $\infty \in K$ by Lemma 9. Then there exists a bounded set L such that $L_n \subset L$ for each n and $d(L,K) > \epsilon$. Let Δ'_n be the set bounded by $s_n \cup s_{n+1}$. f must be unbounded in Δ'_n for each n since $K_{n+1} \subset C_{\Delta'_n}(f,p)$, Frontier $C_{\Delta'_n}(f,p) \subset L_n \cup L_{n+1}$ and ∞ is in the same component of the complement of $L_n \cup L_{n+1}$ as K_{n+1}. Thus for each n, f has ∞ as an asymptotic value at p along a path q_n such that $q_n - \{p\} \subset \Delta'_n$. So s_{n+1} is between q_n and q_{n+1} for every n and f is bounded on s_{n+1}. So $f \notin I_p$.

<u>Corollary</u>: If f is a normal holomorphic function in D, then G_f is compact in the M-topology.

This corollary follows immediately from Theorems 28 and 30.

NORMAL HARMONIC FUNCTIONS

In this paragraph we will show in Theorem 33 that a harmonic normal function has Fatou points on a dense subset of C and in Theorem 34 that a harmonic normal function which does not have $+\infty$ as a Fatou value has a set of Fatou points possessing positive measure.

<u>Theorem 31</u>: If u is a harmonic normal function in D which omits the value a and if $u(z) \to a$ along a non-tangential boundary path P, then u has a as a Fatou value. (Lappan, 5, Theorem 2, p.154)

<u>Proof</u>: Suppose a is finite. Since u omits a, we may assume that $u(z) > a$ for every z in D. Let Δ be an angle containing P and ζ denote the vertex of Δ. If $\{z_n\}$ is a sequence of points in Δ such that $z_n \to \zeta$, there exists a real number M and a sequence of points $\{z_n'\}$ in P such that $\sigma(z_n, z_n') < M$. Setting $S_n(z) = (z + z_n')/(1 + \overline{z}_n' z)$, we have a subsequence of $\{u(S_n(z))\}$ converging uniformly in $\{z : \sigma(z,0) \leq M + 1\}$ to a harmonic function $U(z)$. But $u(S_n(0)) = u(z_n')$ and so $U(0) = a$ while $U(z) \geqslant a$ for $z \in D$. It follows from the minimum principle for harmonic functions that $U(z) \equiv a$. So $u(z_n) \to a$ and a is a Fatou value of u.

Suppose $a = +\infty$. Defining S_n, z_n and z_n' as above, we have $U(0) = \infty$. However, since $\{u(S_n(z))\}$ is a normal family, there exists a neighborhood N of 0 such that $u(S_n(z)) > 0$ for n sufficiently large and $z \in N$.

It follows from Harnack's Inequality (Ahlfors, 1, Theorem 6, p.183) that $U(z) = \infty$ for $z \in N$. Consequently $U(z) = \infty$ for $z \in D$. Therefore, $u(z) \to \infty$ and ∞ is a Fatou value of u. If $a = -\infty$, the argument is similar.

Lemma 10: If u is a harmonic normal function in D and v is a harmonic conjugate of u, then $f(z) = e^{u(z)+iv(z)}$ is a holomorphic normal function in D. (Lappan, 7, Lemma 1, p.110)

Proof: Let a and b be two complex numbers such that $|a| \neq |b|$ and let $\{z_n\}$ be any sequence of points in D such that $f(z_n) \to a$. Then $u(z_n) \to \ln|a|$ where $\ln 0 = -\infty$ and $\ln\infty = +\infty$. If $\{z_n'\}$ is another sequence of points in D such that $\sigma(z_n, z_n') \to 0$, then by Theorem 4, $u(z_n') \to \ln|a|$ since u is normal. Therefore, $|f(z_n')| \to |a|$ and $f(z_n') \not\to b$. Using the contrapositive of Lemma 1, we conclude that f is normal.

Theorem 32: Let u be a harmonic normal function in D and $f = e^{u(z)+iv(z)}$. Then every Fatou point of f is a Fatou point of u. (Lappan, 7, Theorem 1, p.111)

Proof: If ζ is a Fatou point of f with Fatou value a, then $f(z) \to a$ and $u(z) \to \ln a$ as $z \to \zeta$ from inside each Stolz angle at ζ. So ζ is a Fatou point of u.

Theorem 33: The set of Fatou points of a harmonic normal function in D is a dense subset of C. (Lappan, 7, Theorem 2, p.111)

Proof: Let $f(z) = e^{u(z)+iv(z)}$. Since f is a holomorphic normal

function, the set of Fatou points of f is dense on C according to Bagemihl and Seidel (3, Corollary 1, p.16). So by Theorem 32, the set of Fatou points of u is also dense on C.

<u>Theorem 34</u>: If u is a harmonic normal function in D such that u does not have $+\infty$ as a Fatou value, then the set of Fatou points of u has positive linear measure on C. (Lappan, 7, Theorem 3, p.111)

<u>Proof</u>: Let $f(z) = e^{u(z)+iv(z)}$. Since u does not have $+\infty$ as a Fatou value, f does not have ∞ as a Fatou value. Consequently according to Bagemihl and Seidel (3, Theorem 3, p.15) the set of Fatou points of f has positive measure on C. So by Theorem 32, the set of Fatou points of u has positive measure on C.

CHAPTER III

CLASS A FUNCTIONS

INTRODUCTION

Let $f(z)$ be holomorphic and non-constant in D. For any complex number a, including ∞, let A_a denote the set of points $\zeta \in C$ such that $f(z)$ has the asymptotic value a at ζ. Let $A^* = \bigcup_{a \neq \infty} A_a$ and $A' = A^* \cup A_\infty$. Then $f(z)$ belongs to Class A if and only if f is holomorphic and non-constant in D and A' is dense on C.

Let B^* denote the set of points $\zeta \in C$ such that there exists an arc Γ in D ending at ζ on which $|f|$ is bounded on Γ by some finite constant M. In general M varies as Γ and ζ vary. Set $B' = B^* \cup A_\infty$. Then $f(z)$ belongs to Class B if and only if f is holomorphic and non-constant in D and B' is dense on C.

Since $A^* \subset B^*$ and $A' \subset B'$, Class A $\subset$ Class B.

Now let S be any subset of D. For each $i > 0$, $0 < r < 1$, let S_i be the components of $S \cap \{r < |z| < 1\}$. Let $\delta_i(r) = \operatorname{dia} S_i(r)$ and $\delta(r) = \sup_i \delta_i(r)$ with $\delta(r) \equiv 0$ if no $S_i(r)$ exists. S ends at points of C if and only if $\delta(r) \downarrow 0$ and $r \uparrow 1$. For any constant $\lambda \geqslant 0$ the level set $LS(\lambda)$ is given by $LS(\lambda) = \{z : |f(z)| = \lambda\}$, and a level curve $LC(\lambda)$ is any component of $LS(\lambda)$. $f(z)$ belongs to Class L (Class L*) if and only if $f(z)$ is holomorphic and non-constant in D and every level set $LS(\lambda)$ (every level curve $LC(\lambda)$) ends at points of C.

In 1963 G.R. MacLane published a monograph (1) which contains many important properties of Class A functions. The purpose of his paper was to derive results about the asymptotic values of functions $f(z)$ holomorphic in D. We list some of these conclusions below.

Theorem 1: $A = B = L \subset L^*$ and the inclusion is proper. (MacLane, 1, Theorem 1, p.10)

PROPERTIES OF CLASS A FUNCTIONS

Theorem 2: If $f \in A$ and γ is an arc of C such that $A_\infty \cap \gamma = \phi$, then $A^* \cap \gamma$ has the power of the continuum and is of positive measure. (MacLane, 1, Theorem 2, p.14 and Theorem 11, p.25)

A tract $\{D(\epsilon), a\}$ associated with the finite value a is a set of non-empty domains $D(\epsilon)$, one for each $\epsilon > 0$, such that

(i) $D(\epsilon)$ is a conponent of the open set $\{z: |z| < 1, |f(z) - a| < \epsilon\}$

(ii) $0 < \epsilon_1 < \epsilon_2$ implies $D(\epsilon_1) \subset D(\epsilon_2)$

(iii) $\bigcap_{\epsilon > 0} D(\epsilon) = \phi$.

If $a = \infty$, the only change in the above definition is to replace $|f(z) - a| < \epsilon$ by $|f(z)| > 1/\epsilon$.

Let $K = \bigcap \overline{D(\epsilon)}$. Then K is a non-empty, connected closed subset of C and is called the end of the tract. If K is an arc, it is called an arc-tract. A tract is a global tract if and only if K is the entire circumference C and for each arc $\gamma \subset C$ there exists a sequence of arcs $\{\gamma_n\}$ such that $\gamma_n \subset D(1/n)$ and $\gamma_n \rightarrow \gamma$. This last condition is important since Theorem 5 is untrue without some condition of this type in the definition of global tracts.

If $\{D(\epsilon), a\}$ is a tract and $\Gamma: z = \psi(t)$, $0 \leq t < 1$, is a continuous curve in D such that $\psi(t) \in D(\epsilon)$ for $1 - \delta(\epsilon) < t < 1$, then Γ _belongs_ to $\{D(\epsilon), a\}$.

Theorem 3: Let $f \in A$ and $\{\gamma_n\}$ be a sequence of distinct simple arcs in D which tend to the arc γ of C with the property that $\inf_{z \in \gamma_n} |f(z)| = u_n \to \infty$ as $n \to \infty$. Then f has $\{D(\epsilon), \infty\}$ with end K such that $\gamma \subset K$ and for any $\zeta \in K$ there is a curve $T \subseteq \{D(\epsilon), \infty\}$ which ends at ζ. At any interior point of K the only asymptotic values come from this tract.
(MacLane, 1, Theorem 3, p.15)

Theorem 4: Let $f \in A$ and let $\{D(\epsilon), a\}$ for $a \neq \infty$ be a tract of f. Then the end of this tract is a single point. (MacLane, 1, Theorem 4, p.18)

Theorem 5: Let $f \in A$. Then

(i) f has a global tract if and only if f is unbounded and all level curves of f are compact;

(ii) f has a global tract if and only if f is unbounded on every curve Γ in D on which $|z| \to 1$.

(MacLane, 1, Theorem 6, p.18)

Theorem 6: If $f \in A$ and S is any Borel set on the sphere, then A(S) is measurable. (MacLane, 1, Theorem 10, p.22)

SUFFICIENT CONDITIONS FOR $f \in A$

Theorem 7: Each one of the following conditions is a sufficient condition for $f \in A$:

(i) f is a holomorphic, non-constant function in D such that there exists a set $S_\theta \subset [0,2\pi]$ that is dense in $[0,2\pi]$ such that $\int_0^1 (1-r)\log^+|f(re^{i\theta})|\,dr < \infty$ for $\theta \in S_\theta$;

(ii) f is a holomorphic, non-constant function in D such that $\int_0^1 (1-r)m(r)\,dr < \infty$ where $m(r) = \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta$, $0 \leq r < 1$;

(iii) f is a holomorphic, non-constant function in D such that $\int_0^1 (1-r)\log M(r)\,dr < \infty$ where $M(r)$ is the maximum modulus of f.

(MacLane, 1, Theorem 14, p.36 and following discussion)

It is important to notice that in condition (i) no uniformity is implied. All that is required is that each individual integral converges.

<u>Theorem 8</u>: Let $f(z)$ be non-constant and expressible in the form $f(z) = \sum_0^\infty a_n z^n$ for $|z| < 1$ and let λ be a constant such that $0 < \lambda < 2/3$ and $\log^+|a_n| < n^\lambda$ for $n > N$. Then $f \in A$. (MacLane, 1, Theorem 16, p.42)

f(z) belongs to the <u>Class</u> <u>N</u> if and only if f is holomorphic, non-constant in D and normal.

<u>Theorem</u> <u>9</u>: $N \subset A$. Also, if $f \in N$, then (i) given $\zeta \in C$, f has at most one asymptotic value at ζ. If f has the asymptotic value a at ζ, then f has the angular limit a at ζ; (ii) f has no arc-tracts. (MacLane, 1, Theorem 17, p.43)

This theorem contains the Bagemihl and Seidel results in Chapter II, Theorems 4, 12 and 14.

BARTH'S GENERALIZATIONS OF MACLANE'S RESULTS

In order to generalize MacLane's Results, Barth defined classes A_m, B_m, L_m and L_m^* which differ from MacLane's classes A, B, L and L* only in the replacement of the word "holomorphic" with the word "meromorphic" in the appropriate definitions. Theorem 12 shows that $A_m \subset B_m$ and $L_m \subset B_m$. However, there are examples to show that no other inclusion relationships exist among the classes A_m, B_m and L_m. Let $LS(\lambda) = \{z \in D: |f(z)| = \lambda\}$.

<u>Theorem</u> <u>10</u>: Let $f \in A_m$ and $\{\gamma_n\}$ be a sequence of disjoint simple arcs in D that tend to the arc γ of C with the property that there exists a complex number a such that

$$\sup_{\gamma_n} |f(z) - a| = \mu_n \to 0 \text{ as } n \to \infty \text{ if } a \neq \infty,$$

$$\inf_{\gamma_n} |f(z)| = \mu_n \to \infty \text{ as } n \to \infty \text{ if } a = \infty.$$

Then f has an arc tract $\{D(\epsilon), a\}$ with end K such that $\gamma \subset K$ and such that for each point $\zeta \in K$ some curve Γ belonging to $\{D(\epsilon), a\}$ ends at ζ. At any interior point of K, the only asymptotic values come from this tract. If $f \in L_m$, the preceding conclusions are true for $a = \infty$. (Barth, 1, Theorem 1, p.323)

<u>Proof</u>: First we assume that $f \in L_m$ and $a = \infty$. Let $\gamma = \{e^{i\theta} : \alpha \leq \theta \leq \beta\}$ and ζ be an interior point of γ. We choose α', β' such that $\alpha < \alpha' < \arg \zeta < \beta' < \beta$. Let $S(\alpha', \beta')$ denote the sector $\{\alpha' < \arg z < \beta' \text{ for } |z| < 1\}$ and let $\gamma_n' \subset \gamma_n$ be a cross-cut of $S(\alpha', \beta')$ joining a point of $\arg z = \alpha'$ to a point of $\arg z = \beta'$. By using a subsequence of γ_n if necessary, we may assume that each γ_n contains a cross-cut of γ_n' and that γ_{n+1}'

separates γ_n' from $|z| = 1$ within $S(\alpha',\beta')$. Let E_n denote the subdomain of $S(\alpha',\beta')$ which is bounded by γ_n', γ_{n+1}' and two intervals on the boundary radii of $S(\alpha',\beta')$. For λ, a fixed constant greater than zero, we choose $N(\lambda)$ such that for $n \geqslant N(\lambda)$ no γ_n intersects $LS(\lambda)$. Let the components of $LS(\lambda) \cap E_n$ be denoted by $p(n,i)$, for $1 \leqslant i \leqslant n_i$. For simplicity we pick λ so that $LS(\lambda)$ has no multiple points. Since $f \in L_m$, the maximum diameter $p(n,i)$ for $n \geqslant N(\lambda)$ approaches zero. So for $n \geqslant N_1$, any curve $p(n,i)$ which intersects the radius $R = \{\arg z = \arg \zeta\}$ is a Jordan curve contained in E_n. Therefore, any interval of R in E_n on which $|f(z)| < \lambda$ may be replaced by an arc of a level curve $p(n,i)$. By making a finite number of such replacements for any one n, we obtain a curve $\Gamma(\lambda)$ such that $\lim_{z \to 1} \inf_{z \in \Gamma(\lambda)} |f(z)| \geqslant \lambda$.

We will now construct Γ. Let $\lambda_n \uparrow \infty$ be given. Let Q_n be the intersection of R with γ_n' having $\max |z|$, and let $\Gamma(\lambda_k, n)$ be the portion of $\Gamma(\lambda_k)$ joining Q_n to ζ. We define $\Gamma = \Gamma(\lambda_1)$ from $z = 0$ to Q_{n_1} where n_1 is chosen so that $|\arg z - \arg \zeta| < 1/2$ and $|f(z)| \geqslant \lambda_2$ for $z \in \Gamma(\lambda_2, n_1)$; and for any integer $p > 1$, $\Gamma = \Gamma(\lambda_p)$ from $Q_{n_{p-1}}$ to Q_{n_p} where $n_p > n_{p-1}$ is chosen so that $|\arg z - \arg \zeta| < 1/2^p$ and $|f(z)| \geqslant \lambda_k$ for $z \in \Gamma(\lambda_p, n_{p-1})$. So $\Gamma \to \zeta$ and $f(z) \to \infty$ on Γ.

Now we assume that ζ is an endpoint of γ. Let $\{\zeta_n\}$ be a sequence of interior points of γ with $\zeta_n \to \zeta$ and Γ_n be a curve ending at ζ_n on which $f \to \infty$. By using a construction similar to the one given above, we can construct a curve Γ tending to ζ on which $f \to \infty$.

Each asymptotic path Γ to an interior point ζ of γ intersects all γ_n's for $n > N$. So there exists an integer N' such that all γ_n for $n > N'$ belong to the same domain $D(\epsilon)$ for $|f(z)| > 1/\epsilon$. Thus all paths belong to the same tract $\{D(\epsilon), \infty\}$. If the end K of this tract

contains γ as a proper subset, then we can choose arcs $\gamma'_n \subset D(1/n)$ such that $\gamma'_n \to \gamma' = K$. Since γ'_n and γ' satisfy the same hypotheses as γ_n and γ, it follows that if $\zeta \in K$, then there exists a curve Γ belonging to $\{D(\epsilon),\infty\}$ which tends to ζ.

If $f \in A_m$ and $a = \infty$, then $LS(\lambda) \cap S(\alpha,\beta)$ must also end at points of C for all $\lambda > 0$. For if this were not true, there would exist a $\lambda_1 > 0$, a subarc Δ of γ and a sequence of continuous arcs $\{\Delta_n\}$ compact in D such that $\Delta_n \subset LS(\lambda_1)$ for all n and $\Delta_n \to \Delta$ as $n \to \infty$. Let ζ be any interior point of Δ. Each curve ending at ζ must cross all but a finite number of the Δ_n's and γ_n's. Therefore, f cannot have an asymptotic value at ζ, contradicting the assumption $f \in A_m$.

Finally, if a is finite, we define the function $1/(f - a)$ and use the above proofs.

<u>Theorem 11</u>: If $f \in L_m$ and $\gamma = \{e^{i\theta} : \alpha \leq \theta \leq \beta, \alpha \neq \beta\}$ is a subarc of C such that no level curve of f ends at any point of γ, then exactly one of the following two statements is valid.

(i) For each interior point $e^{i\phi}$ $(\alpha < \phi < \beta)$ of γ, there exists a continuous curve $\Gamma(e^{i\phi}) \subset D$ ending at $e^{i\phi}$ and such that f is bounded on $\bigcup_{\alpha<\phi<\beta} \Gamma(e^{i\phi})$. Furthermore, f does not have ∞ as an asymptotic value at any interior point of γ.

(ii) There exists an arc-tract $\{D(\epsilon),\infty\}$ of f with end K such that $\gamma \subset K$.

(Barth, 1, Theorem 2, p.324)

Proof: Let $S(\alpha,\beta)$ denote the sector $\{z : |z| < 1 \text{ and } \alpha < \arg z < \beta\}$. We pick $\{\lambda_n\}$ so that $0 < \lambda_n$, $\lambda_n \to \infty$ as $n \to \infty$ and $LS(\lambda_n)$ has no multiple

points. Since $f \in L_m$, each $LC(\lambda_n)$ is either a closed Jordan curve or a crosscut of D. Suppose 0 is not a pole of f. If 0 is a pole, we may pick a different point close to 0 which is not a pole and repeat the following argument using this new point. We choose N such that $0 \in \{z : |f| < \lambda_N\}$. For any $n \geq N$, let $\Delta(\lambda_n)$ denote the component of $\{z : |f| < \lambda_n\}$ that contains 0. Since $f \in L_m$ and no level curve ends at any point of γ, at least one of the following statements is valid for any $n \geq N$.

(iii) there exists a $T_n \subset$ Boundary $\Delta(\lambda_n)$ such that T_n is a crosscut of the sector $S(\alpha,\beta)$ that joins a point of $\arg z = \alpha$ to a point of $\arg z = \beta$.

(iv) Boundary $\Delta(\lambda_n) \supset \gamma$.

If (iii) is true for all $n \geq N$, then $T_n \to \gamma$ and so by Theorem 10, f has an arc tract $\{D(\epsilon),\infty\}$ with end $K \subset \gamma$. So (ii) holds.

Now suppose there exists some $n = M$ for which (iv) holds. Let $\zeta = e^{i\phi}$ for $\alpha < \phi < \beta$ be any arbitrary point of γ. By (iv) $\zeta \in$ Boundary $\Delta(\lambda_M)$. Since $f \in L_m$ and no level curves of f end at points of γ, there exists a $\delta > 0$ depending on ζ such that each component of Boundary $\Delta(\lambda_M)$ having non-empty intersection with the set $\{z : |z - \zeta| < \delta, |z| < 1\}$ is a closed Jordan curve contained in $S(\alpha,\beta)$. Since the diameter of the set $LS(\lambda_M) \cap \{z : 1 - \epsilon < |z| < 1\} \to 0$ as $\epsilon \to 0$, 0 and ζ may be connected by a continuous curve $\Gamma(e^{i\phi}) \subset \Delta(\lambda_M) \cup \zeta$.

The last part of (i) is proved by observing that the existence of the asymptotic value ∞ at ζ implies that $LS(\lambda)$ ends at ζ for all $\lambda > \lambda_M$, which is a contradiction.

Theorem 12: $A_m \subset B_m$ and $L_m \subset B_m$. (Barth, 1, Theorem 3, p.325)

Proof: Since the generalized definitions and notations include "meromorphic functions" instead of only "holomorphic functions", $A' = A^* \cup A_\infty$ where $A^* = \bigcup_{a \neq \infty} A_a$ and $B = B^* \cup A_\infty$. So $A^* \subset B^*$ and $A' \subset B'$. Consequently $A_m \subset B_m$.

Now we want to show that $L_m \subset B_m$. Let $f \in L_m$ and $\gamma = \{e^{i\theta} : \alpha \leq \theta \leq \beta\}$ be any subarc of C. We will show that there exists a continuous curve ending at some point of γ on which f is bounded or else a continuous curve ending at some point of γ on which f has the asymptotic value ∞. If a level curve of f ends at a point of γ, we are done. So suppose not. Then Theorem 11 holds and either there exists for every interior point $e^{i\theta}$, $\alpha < \theta < \beta$, a continuous curve $\Gamma(\theta) \subset D$ that ends at $e^{i\theta}$ on which f is bounded or there is an arc-tract $\{D(\epsilon), \infty\}$ with end K containing γ. In the first case we are finished; in the second case by Theorem 10, f has the asymptotic value ∞ at each point of γ.

The Schwarz triangle function is an example of a function f such that $f \in B_m$ and $f \in L_m$, but $f \notin A_m$. This shows that $B_m \not\subset A_m$ and $L_m \not\subset A_m$.

We will now construct a function f such that $f \in A_m$, $f \in B_m$, but $f \notin L_m$. (Barth, Example 2, p.326) Let $\{r_n\}$ denote a sequence of positive numbers which are strictly increasing to 1. For $n \geq 1$, let

$$C_n = \{|z| = r_n\}$$
$$D_n = \{|z| < r_n\}$$
$$E_n = \{z : r_n \leq |z| \leq r_{n+1} \text{ and } \arg z = 2k\pi/2^n\} \text{ for } k = 0, 1, \ldots, 2^n - 1.$$

For $n > 1$, let $F_n = \overline{D}_{n-1} \cup E_{n-1} \cup C_n$. Two sequences of functions $\{f_n(z)\}$ and $\{R_n(z)\}$ are now defined inductively.

We define $f_1(z)$ and $R_1(z)$ on $\overline{D}_1$ so that $f_1 \equiv R_1(z) \equiv 1/2$. Next we

construct $f_2(z)$ so that it is continuous on F_2 and

$f_2(z) = f_1(z)$ on $\overline{D}_1$,

$f_2(z) = 5/4$ on C_2,

$f_2(z)$ is linear on each component of E_1.

F_2 is closed and it divides the plane into a finite number of regions. In addition $f_2(z)$ is continuous on F_2 and analytic on the interior of F_2. Therefore, by a remark in Mergelyan's paper (1, p.24) there exists a rational function $R_2(z)$ such that $\max\limits_{z \in F_2} |f_2(z) - R_2(z)| < 2^{-4}$. In general suppose that $f_n(z)$ is spherically continuous on F_n and that

$f_n(z) = R_{n-1}(z)$ on $\overline{D}_{n-1}$,

$f_n(z) = 1 + (-1)^n 2^{-n}$ on C_n,

$f_n(z)$ is linear on each component of E_{n-1}.

By using a remark of Mergelyan (1, p.24) we can find a rational function $R_n(z)$ such that $\max\limits_{z \in F_n} |f_n(z) - R_n(z)| < 2^{-n-2}$. A straightforward calculation shows that $\{R_n(z)\}$ converges to a meromorphic function $R(z)$ in D.

In order to show that $R(z) \notin L_m$, it is sufficient to prove that for each n some component of $\{z : |R| = 1\}$ separates C_n and C_{n+1}. This is shown by verifying that $|R(z) - (1 + (-1)^n 2^{-n})| < 2^{-n-1}$ for $z \in C_n$. Furthermore, f (the limit of $\{f_n(z)\}$) has the asymptotic value 1 on each radius of the form $\{z : 0 \leq |z| < 1$ and $\arg z = 2^{-n}k\}$ for $n > 0$ and $k = 0, 1, \ldots, 2^{n-1}$. Since these radii are dense, $f \in A_m$, $f \in B_m$ and $f \notin L_m$.

Barth has established some sufficient conditions for a function to be a member of A_m. Theorem 14 shows that the conditions of Theorem 7 can be generalized to meromorphic functions.

<u>Theorem 13</u>: Let g and h be holomorphic in D and let g/h be nonconstant. Suppose $g \in A$, h is bounded and $f = g/h$. Then $f \in A_m$ and $1/f \in A_m$. (Barth, Theorem 6, p.331)

<u>Proof</u>: Let γ be any subarc of C. We will show that there exists a point $\zeta \in \gamma$ and a curve ending at ζ on which f tends to a limit as $|z| \to 1$. First suppose $A_\infty(g) \cap \gamma \neq \phi$. Then there exist a point $\zeta \in \gamma$ and a curve Γ ending at ζ on which $g \to \infty$ as $|z| \to 1$. Consequently, since h is bounded, $f \to \infty$ as $|z| \to 1$ on Γ and f has the asymptotic value ∞ at ζ.

Now suppose $A_\infty(g) \cap \gamma = \phi$. If g is bounded in some neighborhood of a point ζ on γ, then f has an asymptotic value at ζ by the Fatou Theorem (Fatou, 1). So suppose $\limsup_{z \to \zeta} |g(z)| = \infty$ for all $\zeta \in \gamma$. Under these hypotheses MacLane (1, p.26) has shown that there exists a $\Delta \subset D$ with the following properties:

(i) Δ is a simply connected Jordan domain, bounded by crosscuts Γ of D on which $|g| = \lambda$ for some $\lambda > 0$ and by a nonempty subset F of γ.

(ii) $|g(z)| < N$ where N is a positive integer for all $z \in \Delta$.

(iii) There exists a nonempty subdomain Δ' of Δ such that $\lambda < |g(z)| < N$ for all $z \in \Delta'$.

Based on an argument of MacLane (1, p.27) for the proof of Theorem 2 of this chapter, it can be shown that f has asymptotic values at some points of γ. Consequently $f \in A_m$ and $1/f$ is also in A_m.

We are now ready to generalize the conditions (i), (ii) and (iii) in Theorem 7 to obtain sufficient conditions for meromorphic functions

to be in A_m. (Barth, 1, p.332) Let f be meromorphic in D.

Condition (i') Suppose there exists a complex number a, possibly ∞, and a set Θ dense on $[0,2\pi]$ such that the Nevanlinna counting function $N(r,a) = O(1)$ (Nevanlinna, 1) and $\int_0^1 (1 - r) \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| dr < \infty$ for $\theta \in \Theta$ if $a \neq \infty$. If $a = \infty$, then $\int_0^1 (1-r) \log^+ |f(re^{i\theta})| dr < \infty$, $\theta \in \Theta$.

Condition (ii') Suppose there exists a complex number a, possibly ∞, such that $N(r,a) = O(1)$ and $\int_0^1 (1-r) m(r,a) dr < \infty$ where $m(r,a) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| d\theta$ if $a \neq \infty$ and $m(r,\infty) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$.

Condition (iii') Suppose $N(r,a) = O(1)$ and $\int_0^1 (1-r) T(r) dr < \infty$ where $T(r)$ is the Nevanlinna characteristic of f.

Since Condition (ii') implies that $\int_0^1 (1-r) \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| dr < \infty$ if $a \neq \infty$ and $\int_0^1 (1-r) \log^+ |f(re^{i\theta})| dr < \infty$ in the case $a = \infty$, Condition (ii') implies Condition (i'). By Nevanlinna's First Main Theorem (Nevanlinna, 1, p.168), it can be shown that Condition (iii') implies (ii').

<u>Theorem 14</u>: If f is meromorphic and nonconstant in D and satisfies one of the preceding conditions (i'), (ii') or (iii'), then $f \in A_m$. (Barth, 1, Theorem 7, p.333)

<u>Proof</u>: Since Condition (iii') implies Condition (ii') which in turn implies (i'), it is sufficient to show that (i') implies $f \in A_m$. Suppose $a = \infty$. Let $B(z)$ denote the Blaschke product

$$B(z) = z^{\lambda} \prod_{k=1}^{\infty} \frac{|b_k| - ze^{-\beta_k i}}{1 - \bar{b}_k z}$$

where λ is the order of the pole at $z = 0$ and the rest of the poles of f are denoted by $b_k = |b_k| e^{i\beta_k}$ with a pole of order u appearing u times among the b_k's. Then the function $g(z) = B(z)f(z)$ is holomorphic in D and

$$\int_0^1 (1-r) \log^+|g(re^{i\theta})| \, dr = \int_0^1 (1-r) \log^+|B(re^{i\theta}) f(re^{i\theta})| \, dr$$

$$\leq \int_0^1 (1-r) \log^+ |B(re^{i\theta})| dr + \int_0^1 (1-r) \log^+ |f(re^{i\theta})| dr \text{ for } \theta \in \boldsymbol{\Theta}.$$

So

$$\int_0^1 (1-r) \log^+ |g(re^{i\theta})| dr \leq \int_0^1 (1-r) \log^+ |f(re^{i\theta})| dr \text{ for } \theta \in \boldsymbol{\Theta}$$

since $|B(z)| \leq 1$. Therefore,

$$\int_0^1 (1-r) \log^+ | g(re^{i\theta}) | dr < \infty \text{ for } \theta \in \boldsymbol{\Theta}$$

and $g \in A$ by Theorem 7. Thus $f = g/B$ and $f \in A_m$ by Theorem 13.

If $a \neq \infty$, the argument above implies that $1/(f-a) \in A_m$ and so $f \in A_m$.

ALGEBRAIC OPERATIONS OF CLASS A FUNCTIONS

In Theorem 16, Brannan and Hornblower prove that Class A functions are not closed under the operations of addition and multiplication. In fact every nonconstant, holomorphic function in D can be written as the sum or the product of pairs of functions in Class A. Furthermore, Barth and Schneider (1, Theorem, p.121) have constructed after much work an example of a function f in Class A such that $e^f \notin A$. However, if $f \in A$ and f has no arc-tracts, then $e^f \in A$ (Theorem 17).

Barth and Schneider (3) have recently shown that the product of a function in A with a bounded holomorphic function is not necessarily in A. First they construct a function f(z) which is holomorphic and non-zero in D and which is approximately equal to n on certain subsets Γ_n of D if n is even and approximately equal to 1 if n is odd. Much notation is required in order to specify these Γ_n's. Let $\{r_n\}$ be a sequence of real numbers such that $0 < r_o < r_1 < \ldots < r_n < \ldots \uparrow 1$ and such that $r_n - r_{n-1} \leq \pi/4n$. Furthermore, let $\{\phi_{n,k}\}$ denote the set of angles $\phi_{n,k} = 2\pi k/2^n$ for n any positive integer and $k = 1,2,\ldots,2^n-1$. The angles $\theta_{n,k}$'s are defined as follow:

(i) $\theta_{1,1} = \phi_{1,1}$

(ii) $\theta_{2,1} = \phi_{2,2}$; $\theta_{2,2} = \phi_{2,1}$; $\theta_{2,3} = \phi_{2,3}$

(iii) in general, after the $\theta_{n,k}$'s have been defined for $k=1,\ldots,2^n-1$, the $\theta_{n+1,k}$'s for $k=1,\ldots,2^n-1$ are defined in the only possible way such that all the $\theta_{m,k}$'s for $m=1,2,\ldots,n+1$ and $k=1,2,\ldots,2^n-1$ with the same second index are equal. Finally the $\theta_{n+1,k}$'s for $k=2^n,2^n+1,\ldots,2^{n+1}-1$ are defined in such a way that $\theta_{n+1,2^n} = \phi_{n+1,1}$; $\theta_{n+1,2^n+1} = \phi_{n+1,3}$; $\theta_{n+1,2^n+2} = \phi_{n+1,5}$; ... ; $\theta_{n+1,2^{n+1}-1} = \phi_{n+1,2^{n+1}-1}$.

With the above notation, many subsets of D are defined as follow:

$E_n = \{z : r_{n-1} + 3/4(r_n - r_{n-1}) \leq x \leq r_n + 3/4(r_{n+1} - r_n), |y| \leq t_n = \pi/4n\}$ where $z = x + iy$.

$\gamma_n = \{z : |z| < r_n\}$ - Interior E_n.

$\alpha_{n,i} = A[0,r_n,r_n + 3/8(r_{n+1} - r_n)] \cap S[0,\phi_{n,i},\phi_{n,i}]$ where $A[a,r',r''] = \{z : r' \leq |z-a| \leq r''\}$ and $S[a,\theta',\theta''] = \{z : \theta' \leq \arg(z-a) \leq \theta''\}$.

$\beta_{n,i} = A[0,r_n + 5/8(r_{n+1} - r_n),r_{n+1}] \cap S[0,\phi_{n,i},\phi_{n,i}]$.

$\Gamma_n = \gamma_n \cup (\bigcup_{i=1}^{2^n-1} \alpha_{n,i}) \cup (\bigcup_{i=1}^{2^{n-1}-1} \beta_{n-1,i})$ for $n \geq 2$ and $\Gamma_1 = \gamma_1 \cup \alpha_{1,1}$.

This function $f(z)$, which Barth and Schneider construct, is not in A. It is bounded away from zero on a countable set of asymptotic paths $\{\alpha_n\}$ which are tangent to those radii ending at a countable dense subset $\{t_n\}$ of C. According to Privalow (p.214), there exist nonzero functions analytic and bounded in D which have radial, and hence angular, limit zero on any pre-assigned subset N of C of measure zero. Let $b(z)$ be the particular function obtained when $N = \bigcup_{n=1}^{\infty} t_n$. $h(z) = w(z)/b(z)$ is in Class A since $\lim_{\substack{z \in \alpha_n \\ |z| \to 1}} h(z) = \infty$ for each α_n while $w(z) = h(z) \cdot b(z) \notin A$.

Recently Tse (1) has shown a condition which holds whenever a product of a bounded holomorphic function and a Class A function is not in Class A.

If $f(z)$ is a meromorphic function in D, then we define $F_f(K)$ [or $F_f^*(K)$] for $0 \leq K \leq \infty$ to be the set of Fatou points of $f(z)$ on C at which the Fatou values are greater than [or less than] K in absolute value.

Theorem 15: If $b(z)$ is a bounded holomorphic function in D and if $f(z) \in A$, but $f(z)b(z) \notin A$, then $F_b(0)$ is of first category in some subarc of C. (Tse, 1, Theorem, p.68)

Proof: Let $A_\infty(fb)$ denote the set of points ζ for which $\zeta \in C$ and fb has ∞ as its asymptotic value. Let $B^*(fb)$ denote the set of points ζ such that $\zeta \in C$ and there exists an arc Γ in D ending at ζ on which $|f|$ is bounded by some finite constant. So $B_{fb} = B^*(fb) \cup A_\infty(fb)$. Since $f(z)b(z) \notin A = B$, there exists a subarc γ of C such that $B_{fb} \cap \gamma = \phi$. By definition $F_b(0) \cap \gamma = \bigcup_{n=1}^{\infty} F_b(1/n) \cap \gamma$. We will show that $F_b(0) \cap \gamma$ is of first category. Suppose on the contrary it is of second category.

Then there exists an $n_o > 0$ such that $F_b(1/n) \cap \gamma$ is of second category. So at each point $\zeta \in F_b(1/n_o) \cap \gamma$, the radial cluster set of $b(z)$ does not contain the value 0. By Collingwood (2, Lemma 1) there exists a number $M' > 0$ such that $|1/b(z)| \leq M'$ in a neighborhood U of a subarc β of γ. Therefore,

$$0 < 1/M' \leq |b(z)| \leq M < \infty, \tag{1}$$

in U where M is the bound of $b(z)$ in D. Since $f(z) \in A$, $B_f \cap \beta \neq \phi$. By (1) $B_{fb} \cap \beta \neq \phi$, which contradicts the condition $B_{fb} \cap \gamma = \phi$. Consequently $F_b(0) \cap \gamma$ is of first category.

A set of points on C is of <u>second category evenly</u> on C if it is of second category on each subarc of C.

<u>Corollary</u>: Let $f(z)$ and $g(z)$ both be in A and $F_f(0) \cap F^*_f(\infty)$ be of second category evenly on C, then $f(z)g(z) \in A$. (Tse, 1, Corollary 1, p.68)

This follows from Theorem 15 and Collingwood (2, Lemma 2).

<u>Theorem 16</u>: Any nonconstant function $R(z)$ holomorphic in D can be represented as the sum and as the product of pairs of Class A functions. (Brannan and Hornblower, 1, Theorem 1, p.86)

<u>Proof</u>: We define $u(r) = M(r,R(z))/(1 - r)$ where $M(r,R(z))$ denotes the maximum modulus of R. According to Hornblower (1) there exists a nonconstant, nonzero function $f(z) \in A$ which, on a dense set of radii, tends to zero faster than $1/u(r)$ and tends to ∞ faster than $u(r)$. Then $\lim_{r \to 1}[R(re^{i\theta})/f(re^{i\theta})] = 0$ on a dense set of θ's and $R(z)/f(z) \in A$.

Thus $R(z)$ can be written as the product of $R(z)/f(z)$ and $f(z)$. In addition, $R(z)$ can also be written as the sum of $[R(z)+f(z)]$ and $[-f(z)]$ since $\lim_{r\to 1}[R(re^{i\theta})+f(re^{i\theta})] = \infty$ on a dense set of θ's implies that $[R(z)+f(z)] \in A$ and $f(z) \in A$ implies that $[-f(z)] \in A$.

Theorem 17: Any nonconstant function $M(z)$ meromorphic in D may be represented in each of the following three ways:

(i) the quotient of two holomorphic functions in A,

(ii) the product of a function in A and a function in $A_m \cap L_m$,

(iii) the sum of two functions in $A_m \cap L_m$.

(Brannan and Hornblower, 1, Theorem 2, p.86)

Proof: According to Heins (1, p.14) any meromorphic function in D can be represented in the form $M(z) = f_1(z)/f_2(z)$ where f_1 and f_2 are holomorphic in D. Consequently we construct the nonzero holomorphic function $f(z) = \max\{M(z,f_1),M(z,f_2)\}/(1-r)$ in A.

Then $M(z) = \{f_1/f\}/\{f_2/f\}$ where f_1/f and f_2/f are nonconstant holomorphic functions in D, which according to Hornblower (1) have 0 as a radial limit on a dense set of radii. So f_1/f and f_2/f are both in A.

In addition $M(z) = f_1/f \cdot f/f_2$ where f_1/f is again in A. Since f/f_2 has ∞ as a radial limit on a dense set of radii, $f/f_2 \in A_m$. Furthermore, $f/f_2 \in L_m$ because no level set $LS(\lambda)$ for λ finite can end on an arc of D.

Finally $M(z) = f/f_2 + f/f_2[(f_1/f) - 1]$. On the same dense set of radii f/f_2 and f_1/f have radial limits ∞ and 0 respectly. So on this dense set of radii $(f_1/f) - 1$ has radial limit -1 and the radial limit of $f/f_2[(f_1/f) - 1]$ is infinite.

<u>Theorem 18</u>: If $f \in A$ and has no arc tracts, then $e^f \in A$. (proof by MacLane in Barth and Schneider, 1, Theorem M, p.120)

<u>Proof</u>: Let γ be any subarc of C. If f has the finite asymptotic value a at $\zeta \in \gamma$, then e^f has the asymptotic value e^a at ζ. So we can assume that f has only the asymptotic value ∞ at points of γ. Let ζ be any one of these points in the interior of γ. We choose a tract $T(\epsilon)$ so that $|f| > 1/\epsilon$ near ζ and $\overline{T(\epsilon)} \cap C \subset \gamma$. We also pick $z_o \in T(\epsilon)$ and consider the Riemann surface over the w-plane corresponding to $T(\epsilon)$. There is a δ, $0 < \delta < \pi/4$, such that sector $\{w : -\delta < \arg(w-f(z_o)) < \delta\}$ or $\{w : \pi-\delta < \arg(w-f(z_o)) < \pi+\delta\}$ does not intersect $\{w : |w| < \frac{1}{\epsilon}\}$. Denote the sector S and find a θ such that the ray $\{w : w = f(z_o) + e^{i\theta}t,\ 0 \leq t < \infty\}$ is contained in S and such that the ray can be lifted into the Riemann surface corresponding to $T(\epsilon)$. Consequently Real $f \to \pm\infty$ on the preimage Γ of this ray. So $e^f \to 0$ or ∞ on Γ.

CALCULUS PROPERTIES OF CLASS A FUNCTIONS

MacLane (2) and Barth and Schneider (2) have investigated the question, "If $f \in A$, then what are sufficient conditions for $f' \in A$ or $\int_o^z f(\xi)d\xi \in A$?" The latter have also studied similar conditions for functions in Class A_m.

Let J denote any domain bounded by a Jordan curve K and lying in C. Then A[J] is the set of nonconstant functions f holomorphic in J with asymptotic values at every point of a set of points $S \subset K$ with S dense on K. If a is a finite asymptotic value along an arc Γ such that $w = f(z)$ maps Γ one-to-one onto a linear segment, then we say that this asymptotic value is linearly accessible. The set of linearly accessible

points is denoted by A_1^*.

Lemma 1: Let $f(z)$ be holomorphic in any arbitrary domain Δ in the complex plane. Suppose $b \neq \infty$ is a boundary point of Δ and $p_o(z)$, $p_1(z)$, $\ldots$, $p_{n-1}(z)$, $q(z)$ are given functions holomorphic in some disk $\Delta_o = \{|z-b| < r_o\}$. Let $\Gamma: z = \psi(u)$, $0 \leq u \leq 1$, be a continuous curve such that $\psi(1) = b$ and $\Gamma - \psi(1) \subset \Delta \cap \Delta_o = \Delta'$. If Γ satisfies the three properties:

(i) $\Gamma_s : z = \psi(u)$, $0 \leq u \leq s$, is rectifiable for any $s < 1$,

(ii) the function $\phi(z) \equiv f^{(n)}(z) + \sum_{m=o}^{n-1} p_m(z) f^{(m)}(z) + q(z)$ for $z \in \Delta'$ satisfies $\phi(\psi(u)) \to \lambda \neq \infty$ as $u \uparrow 1$,

(iii) either (a) Γ is rectifiable or (b) $\phi_1(\psi(u)) = \phi(\psi(u)) - q(\psi(u))$ is of bounded variation on $[0,1]$,

then f has a finite asymptotic value on Γ. (MacLane, 2, Lemma, p.273)

Proof: Consider the differential equation

$$w^{(n)}(z) + \sum_{m=o}^{n-1} p_m(z)\, w^{(m)}(z) = \phi(z) - q(z) \qquad \text{for } z \in \Delta' \tag{2}$$

where ϕ is the function defined in (ii). Then $f(z)$ is a solution in Δ'. Let Δ^* denote the component of Δ' which contains $\Gamma - \psi(1)$ where $\psi(1)$ is understood to be λ. Let $g_1(z)$, $\ldots$, $g_n(z)$ be a set of linearly independent solutions of the homogeneous differential equation associated with Equation (2). The functions g_i are holomorphic in Δ_o. By variation of parameters the solution of Equation (2) is given by

$$f(z) = \sum_{m=1}^{n} \{\alpha_m + \int_a^z h_m(t)[\phi(t) - q(t)]\, dt\}\, g_m(z) \qquad \text{for } z \in \Delta^*$$

where $a = \psi(0)$ and α_m are constants. h_m are functions holomorphic in Δ_o.

In Case (iii) (a), the integrand is continuous on Γ and

$$\beta_m = \int_a^z h_m(t)\,[\phi(t) - q(t)]\,dt = \int_a^z h_m(t)\,\phi_1(t)\,dt$$

has the finite asymptotic value

$$\beta_{m_1} = \int_o^1 h_m(\psi(u))\,[\phi(\psi(u)) - q(\psi(u))]\,d\psi(u).$$

In Case (iii) (b), let $H_m'(z) = h_m(z)$ in Δ_o. Then

$$\beta_m = \int_a^z \phi_1(t)\,dH_m(t) = \phi_1(z)H_m(z) - \phi_1(a)H_m(a) - \int_a^z H_m(t)\,d\phi_1(t).$$

Each of the first two terms has a finite asymptotic value on Γ. Also

$$\int_a^{\psi(s)} H_m d\phi_1 = \int_o^s H_m(\psi(u))\,d\phi_1(\psi(u)) \rightarrow \int_o^1 H_m(\psi(u))\,d\phi_1(\psi(u))$$

because $H_m(\psi(u))$ is continuous on $[0,1]$ and $\phi_1(\psi(u))$ is continuous and of bounded variation on $[0,1]$.

<u>Theorem</u> <u>19</u>: Suppose $f(z)$ is holomorphic and nonconstant in D. Let $\Delta^* = \{|z - 1| < r\}$, n be a positive integer and $p_o(z), \ldots, p_{n-1}(z)$, $q(z)$ be holomorphic functions in Δ^*. Let $\Delta = \Delta^* \cap D$ and

$$\phi(z) = f^{(n)}(z) + \sum_{m=o}^{n-1} p_m(z) f^{(m)}(z) + q(z) \qquad \text{for } z \in \Delta.$$

If $\phi \in A[\Delta]$ and there exists a finite constant c such that $\phi_1(z) = \phi(z) - q(z) \neq c$, then $f \in A[\Delta]$ and $A^*(f)$, the set of finite asymptotic values, is dense on $\Delta^* \cap C$. (MacLane, 2, Theorem 1, p.275)

<u>Proof</u>: Since Δ^* can be replaced by a smaller disk contained in Δ^* with its center on C, it is sufficient to show that f possesses a finite asymptotic value at one point on $\Delta^* \cap C$.

Let E denote the subset of points ζ on C such that for each point ζ there exists a neighborhood $U_\zeta = \{|z-\zeta| < r\} \cap D$ and a Jordan arc J_ζ such that $\phi(z)$ maps U_ζ into the complement of J_ζ. In a similar manner E_1 is defined using $\phi_1 = \phi(t) - q(t)$ instead of ϕ. We set $E_2 = E \cup E_1$.

Suppose $\Delta^* \cap C$ contains points of E_2. By shrinking Δ^* we may assume $\Delta^* \cap C$ is contained in E_2. From a simple generalization of Fatou (1) both ϕ and ϕ_1 have finite angular limits almost everywhere since $\phi(z) - \phi_1(z) = q(z)$ has angular limits almost everywhere. Using the notation in the proof of Lemma 1, we see that β_m has finite angular limits almost everywhere. So f also has finite angular limits almost everywhere on $\Delta^* \cap C$. From a theorem of Privalow (1, p.210) the asymptotic values assumed by $f(z)$ on any interval $\Delta^* \cap C$ contained in E_2 form a set containing a closed set of positive harmonic measure. Consequently this set must be infinite. If E_2 is dense on $\Delta^* \cap C$, we are finished.

So we now assume that $\Delta^* \cap C$ is contained in the complement of E_2. According to MacLane (1, Theorem 7, p.19) each asymptotic tract of ϕ_1 must end at a single point because ϕ_1 omits the value c. Suppose that the asymptotic values of ϕ_1 are bounded by a finite constant M. We choose two distinct points ζ_1 and ζ_2 on $\Delta^* \cap C$ at which ϕ_1 has asymptotic values and join ζ_1 and ζ_2 by a curve $\Gamma \subset \Delta$ which is an asymptotic path at both ζ_1 and ζ_2. Then ϕ_1 is bounded on Γ and we denote the bound by B. Let G be the domain bounded by Γ and part of $\Delta^* \cap C$. If ϕ_1 is bounded in G, then the arc $\zeta_1\zeta_2$ is contained in E_2, a contradiction. So we pick a value w_o such that $w_o = \phi_1(z_o)$ for some $z_o \in G$ and $|w_o| >$ $\max(B,M)$. Then by the lifting argument of MacLane (1, p.13, Section 2), ϕ_1 has an asymptotic value a at some boundary point of G on C satisfying the condition $\infty \geqslant a > |w_o| > M$, a contradiction. Therefore, we now assume

that ϕ_1 has two asymptotic values along Γ, whose magnitudes are greater than $2|c|$, where c is the constant defined above. So $|\phi_1 - c| \geqslant \delta > 0$ on Γ. Since ϕ_1 omits fewer values in G than a Jordan arc, there exists a $z_1 \in G$ such that $|\phi_1(z_1) - c| < \delta$. By the lifting argument of MacLane (1, p.13, Section 2), there exists $\Gamma_1 \subset G$ ending at a point of $\Delta^* \cap C$ such that $\phi_1(z)$ maps Γ_1 one-to-one onto a linear segment. By Case (iii) (b) of Lemma 1, f has a finite asymptotic value on Γ_1.

Special cases of Theorem 19 show that for any positive integer n if $f^{(n)} \in A$ and $f^{(n)} \neq c$, then $f \in A$ and $A^*(f)$ is dense on C.

<u>Theorem</u> <u>20</u>: Let $f \in A$ and $f(z) \neq c$, where c is some finite constant. Then A_1^* is dense on C. (MacLane, 2, Theorem 5, p.278)

<u>Proof</u>: We may assume without loss of generality that $c = 0$. Let γ be an arbitrary arc of C. First we suppose that there is an interior point of γ such that $\liminf_{z \to \zeta} |f(z)| = 0$. From MacLane (1, Theorem 11 and its corollary, pp.25 - 28) we can find a crosscut Γ of C from $\zeta_1 \in \gamma$ to $\zeta_2 \in \gamma$ with the properties that f has nonzero asymptotic values on Γ at both ζ_1 and ζ_2 and ζ is in the open arc from ζ_1 to ζ_2. Let Δ be the domain bounded by Γ and the open arc from ζ_1 to ζ_2. $|f(z)| \geqslant m > 0$ on Γ. We pick a point $z_o \in \Delta$ such that $|f(z_o)| < m$. By the lifting argument of MacLane (1, p.13, Section 2), the ray $[f(z_o),0)$ produces a linear asymptotic value at a point in the arc joining ζ_1 to ζ_2.

If there is no interior point ζ of γ such that $\liminf_{z \to \zeta} |f(z)| = 0$, then by shortening γ if necessary we can find a neighborhood U of γ such that $|f(z)| \geqslant m > 0$ for all $z \in U$. By the Riesz-Riesz Theorem f(z) has

finite asymptotic values on a dense set of points of γ. So there exists a crosscut Γ_1 of U with the properties that Γ_1 ends at distinct points ζ_3 and ζ_4 of γ, f has finite asymptotic values on Γ_1 at both ends, and the image of Γ_1 by $w = f(z)$ is a polygonal curve P on the Riemann surface onto which f maps U. If the curve has only a finite number of sides, we are finished. So we assume P has an infinite number of sides in both directions. Let P* denote the projection of P in the w-plane.

Suppose that every neighborhood N in the w-plane contains a subset S_N with the power of the continuum such that each point in S_N is the image of only a finite number of points in D. Let a be a finite asymptotic value of f at an interior point of γ. By choosing ϵ small enough the set $\{z \in D : |f(z) - a| < \epsilon\} = D(\epsilon,a)$ is bounded by curves in D on which f takes values on a square with center at a and sides of length ϵ (denoted by $Q(\epsilon,a)$) and interior points of γ. We choose $0 < \epsilon_1 < \epsilon$ so that a point $w \in S_{Q(\epsilon,a)}$ lies on the boundary of $Q(\epsilon_1,a)$. Let $w = f(z)$ map D onto the Riemann surface over the w-plane, and let $\Delta(\epsilon_1,a)$ denote the lifting of $D(\epsilon_1,a)$. The part of the boundary of $\Delta(\epsilon_1,a)$ which lies over $Q(\epsilon_1,a)$ can contain no closed curves because otherwise $D(\epsilon_1,a)$ would be relatively compact in D. Thus each boundary component over $Q(\epsilon_1,a)$ will be an open polygonal arc containing only a finite number of segments since w is on the boundary of $Q(\epsilon_1,a)$. The last segment of any such polygon produces a linearly accessible asymptotic value of f at some interior point of γ.

Now we assume that there exists a value w_o which f(z) assumes infinitely many times in Δ and that w_o is not an asymptotic value of ζ_3 or ζ_4. We choose any ray R from w_o to a point $w_1 = me^{i\alpha}$ where α is picked so that R is a positive distance from the asymptotic values of ζ_3 and

ζ_4, R is not parallel to P*, and there are no branch points of the Riemann surface over R. We consider all the liftings of R into the Riemann surface starting at each of the infinitely many points over w_o. The liftings will be unique because R does not contain the projection of any branch points. Some of the liftings may stop at points of P giving asymptotic values at points of Γ_1, but two distinct liftings cannot stop at the same point of P. Since R is a positive distance from the asymptotic values of ζ_3 and ζ_4, R intersects P* in only a finite number of points. So this lifting process gives a countably infinite number of finite asymptotic values with at most a finite number corresponding to points of Γ_1.

The theorem of McMillan and Pommerenke (Theorem 37, Chapter I) generalizes some of the results of MacLane (2) since any function $f \in A$ for which $f'(z) \neq 0$ is also a meromorphic locally univalent function without Koebe arcs. For example if f(z) is meromorphic locally univalent without Koebe arcs and a is a finite asymptotic value of f at $\zeta \in C$, then either a is a linearly accessible asymptotic value in the tract $\{D(\epsilon), a\}$ or there is an infinite sequence of numerically distinct linear asymptotic values $\{a_n\}$ occurring at a point $\zeta_n \in C$ such that $a_n \to a$ and $\zeta_n \to \zeta$. Another result is that if $f' \in A$ and $f'(z) \neq 0$, then f possesses at least three numerically distinct asymptotic values since $f \in A$ by Theorem 19.

For $n \geq 1$, any function f meromorphic in D and any $z \in D$, we define the "nth integral of f" as

$$F^{(-n)}(z) = \int_o^z \int_o^{\xi_1} \int_o^{\xi_2} \cdots \int_o^{\xi_{n-1}} f(\xi_n)\, d\xi_n\, d\xi_{n-1} \cdots d\xi_1$$

where the ξ_i's are dummy variables. In order to eliminate the statement, "If $f'(z)$ is nonconstant," in our theorems, we define the Class A^* to be the union of all functions in Class A and the constant functions.

Theorem 21: If f is holomorphic in D and satisfies the integral part of Condition (ii) of Theorem 7, then $f' \in A^*$. (Barth and Schneider, 2, Theorem 1, p.4)

Proof: First suppose $f(0) = \alpha \neq 0$. Using the notation

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\,d\theta,$$

we have

$$m(r,f') = m(r,f\cdot(f'/f)) \leq m(r,f) + m(r,f'/f).$$

According to the "logarithmic derivative lemma" of Nevanlinna theory (Hayman, 1, p.36)

$$m(r,f'/f) < 4\log^+ m(R,f) + 4\log^+(\log^+\frac{1}{|f(0)|}) + 5\log^+ R + 6\log^+\frac{1}{R-r} + \log^+\frac{1}{r} + 14$$

where $0 < r < R < 1$. Suppose $r > 1/2$ and let $R = (r+1)/2$. Then

$$m(r,f'/f) < 4\log^+ m((r+1)/2,f) + 6\log^+(2/(1-r)) + K$$

where K is a constant that depends on α, but not on f. Therefore,

$$\int_0^1 (1-r)m(r,f')dr \leq \int_0^1 (1-r)m(r,f)dr + \int_0^1 (1-r)[6\log^+\frac{2}{1-r} + K]dr$$

$$+ 4\int_0^1 2[1-(r+1)/2]\log^+ m(\tfrac{1}{2}(r+1),f)dr.$$

Because of the hypotheses of this theorem, all the integrals on the right hand side of the last inequality are finite. Consequently f' satisfies the integral part of Condition (ii) of Theorem 7. So $f' \in A^*$.

Actually the previous proof demonstrates that if f and f' are

nonconstant and f satisfies Condition (ii) of Theorem 7, then f' satisfies it also. So we have the following Corollary.

Corollary a: If f is holomorphic in D and satisfies Condition (ii) of Theorem 7, then $f^{(n)}(z) \in A^*$ for all $n \geqslant 0$. (Barth and Schneider, 2, Corollary 1, p.6)

Corollary b: If f is holomorphic and normal in D, then $f^{(n)}(z) \in A^*$ for $n \geqslant 0$. (Barth and Schneider, 2, Corollary 2, p.6)

Proof: According to MacLane (1, p.44) if f is holomorphic and normal in D, then $m(r,f) \leqslant C_1 \log(1/(1-r)) + C_2$ where C_1 and C_2 are constants. Hence f satisfies Condition (ii) of Theorem 7.

Theorem 22: If f is holomorphic in D and satisfies

$$(1-r)\log(1/(1-r))m(r,f)dr < \infty,$$

then f and $F^{(-1)}(z) \in A^*$. (Barth and Schneider, 2, Theorem 2, p.7)

Proof: Since the integral condition in this theorem's hypothesis is stronger than the integral part of Theorem 7's Condition (ii), $f \in A^*$. In order to establish that $F^{(-1)}(z) \in A^*$, we will use the theorem of Hayman (2, Theorem 2) which states that if F(z) is holomorphic in $|z| < R$, $f(z) = F'(z)$ has bounded characteristic and $F(0) = 0$, then for $0 < r < R$

$$m(r,F) \leqslant (1 + \frac{1}{\pi}\log\frac{R+r}{R-r})\, m(R,f).$$

Let $R = (r+1)/2$ where $0 < r < 1$. Since f(z) is bounded in

$|z| < (r+1)/2$ we can use the theorem of Hayman to obtain the inequality

$$m(r,F^{(-1)}) \leq (1+(1/\pi))\log(4/(1-r))m((r+1)/2,f).$$

Consequently

$$\int_0^1 (1-r)m(r,F^{(-1)})dr \leq \int_0^1 (1-r)m((r+1)/2,f)dr$$

$$+ \frac{1}{\pi}\int_0^1 2[1-(r+1)/2](\log\frac{4}{2[1-(r+1)/2]})m((r+1)/2,f)dr.$$

Because of the hypotheses of this theorem, both of the above integrals of the right hand side of the inequality are finite. Consequently $F^{(-1)}$ satisfies the integral part of Condition (ii) of Theorem 7. So $F^{(-1)} \in A^*$.

Theorem 21 and 22 may be generalized to meromorphic functions f for which the Nevanlinna counting function $N(r,f) = O(1)$. Let $\underline{A_m^*}$ denote the union of the functions of Class A_m and the constant functions.

<u>Theorem</u> <u>23</u>: If f is meromorphic in D, $N(r,f) = O(1)$, and $\int_0^1(1-r)T(r,f)dr < \infty$, where $T(r,f)$ is the Nevanlinna characteristic of f, then $f' \in A_m^*$. (Barth and Schneider, 2, Theorem 4, p.11)

Since $T(r,f) = m(r,f) + O(1)$ and $T(r,f') = m(r,f') + O(1)$, this proof is completely analogous to that of Theorem 21.

<u>Theorem</u> <u>24</u>: If f and $F^{(-1)}$ are meromorphic in D, $N(r,f) = O(1)$ and

$$\int_0^1(1-r)\log(1/(1-r))T(r,f)dr < \infty,$$

then f and $F^{-1}(z) \in A_m^*$. (Barth and Schneider, 2, Theorem 5, p.12)

The proof of this theorem is quite similar to that of Theorem 22 where the meromorphic form of Hayman's Theorem (2, Theorem 2) replaces the holomorphic one.

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