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# COLLAPSING OF SIMPLICIAL COMPLEXES CROSSED WITH THE UNIT INTERVAL

Thesis for the Degree of Ph. D.

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ROLLAND LAVERNE SWANK

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ABSTRACT

COLLAPSING OF SIMPLICIAL COMPLEXES
CROSSED WITH THE UNIT INTERVAL

By

Rolland Laverne Swank

In this thesis we consider the problem of collapsing a product space $L \times I$, where $L$ is a simplicial complex and $I$ the unit interval.

By developing the concept of a simplicial relation, we find we are always able to collapse $L \times I$ onto an intermediate complex called the sewn complex. This leads us to examine some properties of the sewn complex which will allow us to conclude when it is collapsible. We find that a proper fold factorization insures collapsibility if the first complex of the factorization is collapsible.

A second technique for collapsing the product space is introduced, which, in special cases, is related to our first method. In studying this relationship, we introduce the concept of an unfoldable star. We examine, in the final section, a class of complexes possessing unfoldable stars and find that for $L$ in this class, $L \times I$ is always collapsible.

# COLLAPSING OF SIMPLICIAL COMPLEXES CROSSED WITH THE UNIT INTERVAL

By

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## SECTION I

## INTRODUCTION

We consider in this thesis the problem of collapsing the product space formed when a finite simplicial complex $K$ is crossed with the unit interval $I$. This space, $K \times I$, is certainly a convex linear cell complex and through a subdivision can be considered to be a simplicial complex.

To fix some notation, if $K$ is a complex, $|K|$ will denote the underlying point set of $K$. $s_1 \in K$ denotes the fact that the m-simplex (or cell) $s_1$ lies in $K$, while $|s_1| \subset K$ will imply that the underlying point set of $s_1$ is contained in $|K|$. $a*s^n$ is the join of the point $a$ with $s^n$. If $s^m$ is a face of $s^n$, we will denote this fact by $s^m < s^n$. $K'$ will denote a subdivision of $K$, and we define the star of the subcomplex $L$ in $K$ to be

$$st_K(L) = \{s \in K | s \cap |L| \neq \phi\}.$$

Collapsing in a simplicial complex was first defined by Whitehead [3]. Let $K_1$ and $K_2$ be simplicial complexes. If

$$K_1 = K_2 \cup a*s^n \quad \text{and}$$
$$K_2 \cap a*s^n = a*\dot{s}^n$$

then $K_1$ simplically collapses onto $K_2$ (denoted $K_1 s\searrow K_2$). If there is a sequence of simplicial collapses from $K_1$ to $L$, i.e., if $K_1 s\searrow K_2 s\searrow K_3 \ldots s\searrow K_n = L$, then $K_1$ collapses to $L$ and this is again denoted by $K_1 s\searrow L$. If $L = \langle v\rangle$, then $K_1$ is said to be collapsible.

Whitehead's simplicial definition was later extended to the polyhedral category where, for example, Zeeman [5] gives the following definition of an elementary collapse.

If $K_1$ and $K_2$ are polyhedra, we say there is an elementary collapse from $K_1$ to $K_2$ if there exists an n-cell $s$ with an $n-1$ face $t$ such that

$$K_1 = K_2 \cup s \quad \text{and} \quad t = K_2 \cap s.$$

If there is a sequence of elementary collapses from $K_1$ to $K_2$, we say $K_1$ collapses to $K_2$ and denote this by $K_1 \searrow K_2$. If $K_2$ is a point, then $K_1$ is said to be collapsible.

Certainly if $K_1 s\searrow K_2$, then $K_1 \searrow K_2$, and it is well known, see [5], that if $K_1 \searrow K_2$, then there exist subdivisions of $K_1$ and $K_2$, say $K_1'$ and $K_2'$, such that $K_1' s\searrow K_2'$. For this reason, the collapsing used primarily in this thesis is of the second type, where it is understood that (if it is needed) simplicial collapsing could be used through the proper subdivisions.

The motivation for the investigations undertaken in this dissertation can be found in Zeeman's paper [4], where several properties of the dunce hat are considered. Of particular interest for us, is Zeeman's proof in that paper that while the dunce hat $D$ is not collapsible, $D \times I$ is collapsible. Generalizing this leads us to investigate the collapsing of product spaces $L \times I$, where we will take $L$ to be a simplicial complex.

## SECTION II

## RELATIONS AND SIMPLICIAL RELATIONS

Definition 2.1: A relation $R$ on $K$ is a set of ordered tuples of the form $(s,s', h)$, where $s$ and $s'$ are m-simplexes of $K$, and $h$ is a linear homeomorphism from $|s|$ to $|s'|$ such that the following conditions are satisfied.

1. The first two co-ordinates of the tuples, considered as ordered pairs form an equivalence relation $R$ on the set of open simplexes of $K$. We denote the fact that two m-simplexes $s$ and $s'$ are related by $sRs'$.
2. $h$ is equal to the identity on $|s| \cap |s'|$. For $(s,s', h)$ and $(s', s, h')$ in the relation, we require that $h' = h^{-1}$.
3. If two m-simplexes are related, then all their faces are related each to each as follows. For the tuple $(s,s',h)$, if $t < s$, and $t' < s'$ such that $h(|t|) = |t'|$, then $tRt'$ and for the tuple $(t,t',h_1)$, $h_1 = h$ restricted to $t$.

A relation on $K$ is considered to induce an equivalence relation $R$ on the underlying point set $K$ as follows, if $a \in |s|$, and $b \in |s'|$ where $sRs'$ and $h(a) = b$, then $aRb$.

The topological space $|L|$ formed from a complex $K$ with a relation $R$ is the space $|K|/R \doteq |L|$. $f: K \rightarrow L$ is the identification function. Note that in general $|L|$ with the induced simplicial structure may not be a simplicial complex. (The induced structure $L$ on $|L|$ is given by the following: if $s \in K$, then $f(s) \in L$ and $f(s)$ will be considered a "simplex" of $L$.) For example, if $K = \{v_1, w, v_2, <w,v_1>, <w,v_2>\}$, and the relation $R$ identifies $v_1$ and $v_2$ to a single vertex $v$, then $L = \{v, w, <w,v>_1, <w,v>_2\}$, and we have two distinct one-simplexes sharing the same vertices. However, the following theorem shows in what sense the space $|L|$ may be considered a simplicial complex.

<u>Theorem 2.2</u>: Let $K$ be a simplicial complex with relation $R$, and $f: K \to L$ be the induced identification map. If $K''$ is the second barycentric subdivision of $K$ and $L''$ the induced "simplicial" structure on $|L|$, then $L''$ is a simplicial complex. This result will follow from the lemmas discussed below.

Essentially, we are asking why the "simplicial" structure $L$ may not be sufficient to make $|L|$ a true simplicial complex. It can be seen to fail in possibly two senses. First, two distinct simplexes in $L$ may intersect in more than a single common face. Such a situation will be called an improper intersection. The second possibility is that an "m-simplex" of $L$ may have some of its faces identified so that it is not homeomorphic to a standard m-simplex. We will show that the structure $L''$ has neither of these properties, so that $|L|$ with this structure can be considered a true simplicial complex embedded in some $R^n$.

<u>Lemma 2.3</u>: Let $s^n$ be a simplex of $L$. Then $s^n$ has at least two faces identified iff there exists a "one-simplex" in $s^n$ with its vertices identified. Such a one-simplex will be called a loop of type 1.

<u>Proof</u>: If $s^n$ has faces $s^i$ and $s^j$ identified, then the pre-image simplex in K has a vertex $v \in f^{-1}(s^i)$ and a distinct vertex $w \in f^{-1}(s^j)$ such that under R, $v = w$. The one-simplex $\langle v,w \rangle$ will under R be identified to a loop of type 1 in $s^n$. Conversely, if $s^n \in L$ has a loop of type 1, we know that at least two 0-faces of $s^n$ have been identified.

A loop of type 2 is defined to be two distinct m-simplexes with their boundaries identified but not their interiors. Thus a one-dimensional loop of type 2 consists of two one-simplexes with their vertices identified. The "sphere" formed by identifying the boundaries of two 2-simplexes would be a two-dimensional example.

Observe that if $s_1 \cup s_2$ is a loop of type 2 in any dimension, i.e., $s_1 \cup s_2$ is a "sphere", a first barycentric subdivision of $s_1 \cup s_2$ can contain no such loops.

<u>Lemma 2.4</u>: Let $s^m$ and $s^n$ be two "simplexes" of L of dimensions m and n respectively, and suppose neither contains loops of type 1. Then $s^m \cap s^n$ is improper iff $s^m \cup s^n$ contains loops of type 2.

<u>Proof</u>: Suppose $s^m \cap s^n$ is improper and let $\{v_i\}_0^p$ be the vertices of $s^m$ that lie in $s^m \cap s^n$, and let $\{w_i\}_0^p$ be the vertices of $s^n$ that lie in the intersection. (Thus $v_i = w_i$ for $i = 0,1,\ldots,p$.) Let $t_1$ be the p-face of $s^m$ that spans $\{v_i\}_0^p$ and let $t_1'$ be the p-face of $s^n$ that spans $\{w_i\}_0^p$. Since we have assumed an improper intersection, $t_1 \neq t_1'$. Now if the boundaries of $t_1$ and $t_1'$ are identified, we have a p-dimensional loop of type 2. If this is not the case, there exists $t_2$, a p-1 face of $t_1$, and $t_2'$, a p-1 face of $t_1'$, such that the vertices of $t_2$ are identified with the

vertices of $t'_2$ but $t_2 \neq t'_2$.

Again we either have a loop of type two or can find two non-identified lower dimensional faces. An inductive argument, assuming no higher dimensional loops of type 2 are found, leads us finally to two one-simplexes with their vertices identified, obviously a loop of type 2.

For the other direction, it is immediate that the existence of type 2 loops implies an improper intersection.

If $L$ contains loops of type 1, $L'$ can contain no such loop types, but could contain loops of type 2. However $L''$, the second barycentric subdivision of $L$, will break up all possible loops of either type. This concludes the proof of Theorem 2.2.

The map $f: K'' \to L''$ has the property that the image of an m-simplex of $K''$ is an m-simplex of $L''$, and the map can be considered to be between two simplicial complexes.

This map $f$ induces an equivalence relation $R$, called a <u>simplicial relation</u>, on $K''$ defined by relating two m-simplexes $s_1$ and $s_2$ of $K''$ iff $f(s_1) = f(s_2)$. If $s_1$ and $s_2$ are related by $R$, we will denote this by $s_1 R s_2$. The simplicial map $f$ also induces an equivalence relation, again called $R$, on the underlying set $|K|$ defined by relating two points $p$ and $q$ of $|K|$ iff $f(p) = f(q)$, and we will denote this fact by $pRq$. (In fact, any simplicial map from one complex onto another that has the dimension preserving property can be considered to induce a simplicial relation $R$ on the domain complex.)

Our results thus far show that we really need only consider simplicial relations and maps between simplicial complexes, rather than the more general concept of relation. The following definitions, however, are given (where possible) in terms of relations rather than simplicial relations to allow more freedom for some later considerations.

Definition 2.5: The set of simplexes of $K$ which are related to a given simplex $s_1$ by a relation $R$ will be denoted by $Ss_1$. Thus $Ss_1 = \{s \in K: sRs_1\}$.

Definition 2.6: The set of points of $|K|$ which are related to a given point $p$ by a relation $R$ will be denoted by $Sp$. Thus $Sp = \{x \in |K|: xRp\}$.

Definition 2.7: The order of the relation $R$ on a simplex $s_1$ of $K$ is the cardinality of $Ss_1$. The order of the relation is the maximum of the simplicial orders, taken over all simplexes of $K$. We consider only relations of finite order.

Definition 2.8: The set of points of $|K|$ related to more than one point is called the seam $S$ of $K$. Thus $S = \cup|Sq|$, where the cardinality of $Sq$ is greater or equal to two. $\bar{S}$, the closure of $S$, is called the closed seam of $K$.

Definition 2.9: In a relation $R$, if $s$ and $t$ are simplexes of $K$ such that $sRt$ and $s \cap t = r$, a non-empty face of each, the relation is said to fold $s$ across the crease $r$ onto $t$.

Note: if the relation $R$ folds $s$ onto $t$ and relates no other simplex to the crease $r$, then the points of $|s-r|$ are in $S$ and the points of $|t-r|$ are in $S$, but the points of $|r|$ will not be in $S$ as they are related only to themselves. However, $|r|$ will be contained in $\bar{S}$, for if all the points in the interior of a simplex are contained in a closed set, all its faces must also be in the closed set and $r$ is a face of both $s$ and $t$.

Note, too, that if two simplexes are related, they are either disjoint or one of them folds onto the other. In either case, all the points of their union will lie in $\bar{S}$. These observations lead immediately to the following lemma.

<u>Lemma 2.10</u>: $\bar{S} = \cup|Ss|$, where the union is taken over all simplexes related to more than one simplex of K; and $\bar{S}$, with the induced simplicial structure of K, is a subcomplex of K.

<u>Lemma 2.11</u>: If sRr, R a simplicial relation, and if the vertices of s are $\langle v_0, v_1, \ldots, v_m \rangle$ and the vertices of r are $\langle w_0, w_1, \ldots, w_m \rangle$, and if $v_i R w_i$ for all i, then any point p in s having barycentric co-ordinates $(d_0, d_1, \ldots, d_m)_s$ is related to that point q in r having the "same" barycentric co-ordinates. That is, $q = (d_0, d_1, \ldots, d_m)_r$.

<u>Proof</u>: Note that f is a simplicial map inducing the relation R and thus $f(s) = f(r)$. By the properties of simplicial maps, since p and q have the same barycentric co-ordinates, $f(p) = f(q)$ and hence pRq.

## SECTION III

## EMBEDDINGS AND SIMPLICIAL TILT EMBEDDINGS

We consider in this section embeddings of $K$ in $K \times I$ or, more precisely, embeddings of $|K|$ in $|K \times I|$. As a simplification we will avoid, where possible, using $|K|$ for the underlying subspace and instead use $K$ for both the simplicial structure and for the underlying space. The nature of the discussion will make clear which interpretation is being used, although in most cases we are considering the underlying point set.

An embedding is denoted by $i: K \to K \times I$, and $iK$ denotes the image of $K$ under $i$. $K \times I$ is considered a square with base $K$ and vertical $I$. The "level" function $l: K \times I \to I$ is defined as follows. For any $z = (x,t)$, $x \in K$ and $t \in I$, define $l(z) = t$. The point $z$ is said to have level $t$ in $K \times I$ iff $l(z) = t$.

Definition 3.1: A point $z$ of $K \times I$ is said to be above a point $w$ of $K \times I$ if $l(z) \geq l(w)$. A point $z$ is said to be over a point $w$ if $z$ is above $w$ and if $p(z) = p(w)$, where $p: K \times I \to K$ is the projection map onto $K$.

Definition 3.2: The vertical distance between two points $z$ and $w$ in $K \times I$ is the absolute value of $l(z) - l(w)$.

Definition 3.3: A slice at level $t$ for some fixed $t \in I$, is denoted by $K_t$ and is the set of all points $z$ of $K \times I$ having level $t$. Thus $K_t = \{z \in K \times I: l(z) = t\}$.

<u>Definition 3.4</u>: An embedding $iK$ of $K$ in $K \times I$ is called a projective embedding if for each $x \in K$, $p(i(x)) = x$.

<u>Definition 3.5</u>: An embedding $iK$ of a simplicial complex $K$ with a simplicial relation $R$ in $K \times I$ is said to be relation-sliceable, or R-sliceable, if, for all $t$ in $I$, $|p(K_t \cap iK)|$ contains no pair of points related by $R$.

For projective embeddings of $K$ in $K \times I$ the following notation is used for points of $iK$. Since a point $x$ in an open m-simplex $|\overset{\circ}{s}|$ of $K$ has barycentric co-ordinates $(d_0,d_1,d_2,\ldots,d_m)_s$ and since the point $ix$ is over $x$ at some level $t$, $ix$ is denoted:

$$ix = [(d_0,d_1,\ldots,d_m)_s, t] \in K \times I$$

A projective embedding $iK$ of $K$ in $K \times I$ can be determined by specifying the level of each vertex $iv$ where $v \in K$, then extending the map linearly between the images of the vertices. Thus, if $s$ is a simplex of $K$ having vertex set $\langle v_0,v_1,v_2,\ldots,v_m\rangle$, then $i(s)$ will be its image, and if $x \in s$ has barycentric co-ordinates $(d_0,d_1,\ldots,d_m)_s$, then $ix = [(d_0,d_1,\ldots,d_m)_s, d_0t_0 + d_1t_1 +\ldots+ d_mt_m]$, where $l(i\,v_j) = t_j$ for each $j$.

This type of projective embedding will be called a <u>simplicial tilt embedding</u>, and the image of any point $x \in K$ is completely determined by the image, $iv$, of the vertices $v$ of $K$.

Unless specified otherwise, all embeddings discussed from now on are simplicial tilt embeddings, and we will assume that for any vertex $v$ of $K$, $l(iv) \neq 0$ or $1$. It is obvious that $iK \doteq K$ with the homeomorphism given by the projection from $K \times I$ onto $K$ restricted to $iK$.

Let $i$ be a simplicial tilt embedding of $K$ in $K \times I$ and let $i(s)$ be the image of a simplex $s$ of $K$. $s \times I$ is the subset of $K \times I$ consisting of all the points above the simplex $s$. Obviously $i(s)$ is contained in $s \times I$.

Definition 3.6: Let $z$ and $w$ be two points of $s \times I$. Then $E = t(z) + (1-t)(w)$ denotes the maximal line segment in $s \times I$ running through the points $z$ and $w$. Note that the line $E$ is not just the set of points "between" $z$ and $w$, but that it is defined to be the whole line segment out to the "edges" of $s \times I$. Note, too, that as $i(s)$ is convex in $s \times I$, if $z$ and $w$ are points of $i(s)$, the line $E$ will lie entirely within $i(s)$.

Lemma 3.7: The level $l(r)$ of any point $r \in E$, where $E$ is the line through $z$ and $w$ and $r = t^*(z) + (1-t^*)(w)$, for some fixed $t^*$ a real number, is given by $l(r) = t^*(l(z)) + (1-t^*)(l(w))$.

Proof: It is obvious.

Corollary 3.8: If the points $z$ and $w$ in $s \times I$ are at the same level, then all points on the line segment $E$ through $z$ and $w$ are on the same level.

Lemma 3.9: If $K_t \cap i(s)$ is an m-cell, where $s$ is an m-simplex of $K$ and $i$ a simplicial tilt embedding, then $i(s)$ is contained in $K_t$.

Proof: Let $x \in \operatorname{int} K_t \cap i(s)$, and select an $m+1$ spherical neighborhood $T_x$ about $x$ in $s \times I$ such that $T_x \cap K_t = U_x$, an m-spherical neighborhood contained in the m-cell $K_t \cap i(s)$. Let $iv$ be any embedded vertex of $s$ and consider the line segment $E$ through $iv$ and $x$. Since $i(s)$ is convex, there exists $y \in E \cap U_x$, $y \neq x$, and $y$ is on the same level as $x$. The line segment

$E' = t(y) + (1-t)(x)$ contains $iv$ since in fact $E = E'$. Thus, from the corollary above, $l(iv) = l(x)$ and hence, $iv \in K_t$. Since $iv$ was an arbitrary vertex, all embedded vertices of $i(s)$ are contained in $K_t$, and it follows that $i(s)$ is contained in $K_t$.

Corollary 3.10: If, in a simplicial tilt embedding, the vertices of an m-simplex, $m > 0$, are embedded each at distinct levels in $K \times I$, then $K_t \cap i(s)$ is a cell of dimension strictly less than $m$ for all $t$ in $I$.

Proof: This follows from the above lemma, for if the intersection were an m-cell for some $t$, then all the vertices of $i(s)$ would have level $t$, but as the vertices are at distinct levels, this cannot occur.

Lemma 3.11: Given a complex $K$ with a simplicial relation $R$, there exists a simplicial tilt embedding $iK$ of $K$ in $K \times I$ such that for each slice $K_t$ and for each $Sq$, either $i(Sq) \cap K_t = i(Sq)$ or $i(Sq) \cap K_t = \emptyset$.

Proof: Divide the set of vertices of $K$ into disjoint subsets $B_1, B_2, \ldots, B_k$, where each $B_i$ is an equivalence class of related vertices. Then define for all $v \in B_j$, $i(v) = [v, j/k+1]$. Thus, related vertices are embedded at the same level, while unrelated vertices are embedded at distinct levels. The mapping is then extended linearly to a simplicial tilt embedding $i$. Now note that, for two related m-simplexes $s$ and $r$, if $a \in s$ and $b \in r$ such that $aRb$,

$$ia = [(d_0, d_1, \ldots, d_m)_s,\ d_0 t_0 + d_1 t_1 + \ldots + d_m t_m] \quad \text{and}$$

$$ib = [(d_0, d_1, \ldots, d_m)_r,\ d_0 t'_0 + d_1 t'_1 + \ldots + d_m t'_m] \quad \text{and}$$

as $t_i = t'_i$ for all $i$, then the levels of $a$ and $b$ are the same.

Hence for each point $q$ of $K$, $Sq$ is embedded at the same level, and it follows that $K_t \cap i(Sq) = \phi$ or $i(Sq)$.

Definition 3.12: Given two distinct simplexes $s$ and $r$ of $K$ such that $sRr$ and a simplicial tilt embedding $iK$ of $K$ in $K \times I$, the m-simplex $i(s)$ is said to be above the m-simplex $i(r)$, written $i(s) \geq i(r)$, if for each pair of distinct related vertices $v_j \in s$ and $w_j \in r$, $l(iv_j) > l(iw_j)$. If the relation is a fold of $s$ onto $r$, then some of the vertices of $s$ are the same as some of the vertices of $r$, and for these common vertices, say $v_j = w_j$, of course $l(iv_j) = l(iw_j)$. If all the vertices of $s$ are embedded strictly higher than their related counterparts in $r$, then $s$ is said to be strictly above $r$, and this is indicated $i(s) > i(r)$.

Theorem 3.13: A simplicial tilt embedding of $K$ in $K \times I$ is R-sliceable iff for all pairs of related simplexes $r$ and $s$, either $i(s) \geq i(r)$ or $i(r) \geq i(s)$.

Proof: The "if" direction will be proved by induction. Note that if $z$ and $w$ are related 0-simplexes for the relation to be R-sliceable either $l(iz) > l(iw)$, or $l(iw) > l(iz)$, and the theorem holds for 0-simplexes.

Consider now two distinct 1-simplexes $s$ and $r$, $s = \langle w,x \rangle$ $r = \langle y,z \rangle$, where $wRy$ and $xRz$.

Case 1: If the relation is a fold, i.e., suppose $x = z$, then either $l(iy) > l(iw)$ or $l(iw) > l(iy)$, for if the levels were equal, we would contradict the theorem for 0-dimensional simplexes. Thus either $i(s) \geq i(r)$ or $i(r) \geq i(s)$.

Case 2: If all four vertices are distinct, we note first that the images of related vertices must each be at different levels, or we have a contradiction to the theorem for 0-simplexes. Without loss of generality, let $iw$ be the point at the lowest level. We show that $i(s) < i(r)$.

We know that $l(iw) < l(iy)$, and it remains to be shown that $l(ix) < l(iz)$. Suppose by way of contradiction that $l(ix) > l(iz)$. Then the number $a = l(iw) - l(iy)$ is negative and so too is the number $b = l(iz) - l(ix)$. Consider the following statement:

$$* \quad d_0(l(iw) - l(iy)) + d_1(l(ix) - l(iz)),$$

where $d_1 = 1 - d_0$, and $0 < d_i < 1 \quad i = 0,1$.

If this statement is equal to 0 for some choice of the $d_i$, we know that the point $ig \in i(s)$ with these $d_i$'s as barycentric co-ordinates will be at the same level as the point $ih \in i(r)$ with the same barycentric co-ordinates. This follows for if $ig$ has barycentric co-ordinates $d_0$ and $d_1$, that is if

$$ig = [(d_0,d_1)_s, d_0(l(iw)) + d_1(l(ix))]$$

then the level of $ig$, $l(ig) = d_0(l(iw)) + d_1(l(ix))$, and similarly for $ih \in l(r)$ with the same barycentric co-ordinates $d_0$ and $d_1$, $l(ih) = d_0(l(iy)) + d_1(l(iz))$. If the levels are equal, i.e., if $l(ig) = l(ih)$, we then get the following equation:

$$d_0(l(iw) - l(iy)) + d_1(l(ix) - l(iz)) = 0$$

Hence we will show our assumption that $l(iw) < l(iy)$ and $l(ix) > l(iz)$ implies that * has a solution for the $d_i$'s, which implies that the relation is not R-sliceable, a contradiction.

Statement * becomes after substituting $1-d_1$ for $d_0$, setting the result equal to zero and solving for $d_1$,

$$** \quad d_1 = \frac{l(iz)-l(ix)}{l(iw)-l(iy) + l(iz)-l(ix)}$$

If $0 < d_1 < 1$, we know the relation is not R-sliceable. Our assumptions on the levels of the vertices imply that

$$d_1 = \frac{a}{b + a}$$

where a and b are both negative. Thus $0 < d_1 < 1$, and we have arrived at a contradiction.

Hence $l(iz) > l(ix)$ and for the 1-simplexes r and s in a simplicial tilt embedding, $i(s) < i(r)$.

Suppose now that u and v are two related m-simplexes in the R-sliceable relation and that neither $i(u) \leq i(v)$ nor $i(v) \leq i(u)$. Suppose, without loss of generality, that a vertex iw of $i(u)$ has the lowest level of all the embedded vertices of both simplexes. Hence if wRy, $l(iw) \leq l(iy)$. Since our assumption is that $i(u) \not\leq i(v)$, there exists a pair of related vertices x of u and z of v such that $x \neq z$ and $l(ix) > l(iz)$. Consider now the one simplexes $\langle w,x \rangle$ and $\langle y,z \rangle$. These are related, as they are the faces of related simplexes, and their images in $i(u)$ and $i(v)$ must be R-sliceable. But we have shown that for two related R-sliceable one-simplexes, one of them must be embedded greater or equal to the other. Hence, our assumption that $l(iw) \leq l(iy)$ and $l(ix) > l(iz)$ implies a contradition.

Thus for the related m-simplexes u and v, $i(u) \leq i(v)$. This concludes the proof of the theorem in one direction.

Suppose now for each pair of related simplexes u and v that either $i(u) \leq i(v)$ or $i(v) \leq i(u)$. Without loss of generality, suppose that for the related simplexes u and v that $i(u) \leq i(v)$.

We must show that the relation is R-sliceable through these embedded simplexes. To fix some notation, let

$$u = \langle w_0, w_1, w_2, \ldots, w_m \rangle \text{ with } l(iw_j) = t_j, \text{ and}$$

$$v = \langle z_0, z_1, z_2, \ldots, z_m \rangle \text{ with } l(iz_j) = t_j', \text{ and}$$

for all $j$ let $w_j R z_j$.

Consider the point $a$ in the interior of $u$, $a$ with barycentric co-ordinates $(d_0, d_1, d_2, \ldots, d_m)_u$, with $0 < d_1 < 1$ for all $i$. We know by a previous lemma that $a$ is related to the point $b$ in $v$ with the same barycentric co-ordinates. To show R-sliceability, we must show that $l(ia) \neq l(ib)$. From previous discussions we note that

$$l(ia) = d_0 t_0 + d_1 t_1 + \ldots + d_m t_m, \text{ and that}$$

$$l(ib) = d_0 t_0' + d_1 t_1' + \ldots + d_m t_m.$$

If, by way of contradiction, $l(ia) = l(ib)$, then

$$d_0(t_0 - t_0') + d_1(t_1 - t_1') + \ldots + d_m(t_m - t_m') = 0.$$

But this clearly cannot occur because $t_i - t_i' \leq 0$ for all $i$ with at least one term strictly negative, (as $u$ and $v$ have at least one pair of distinct related vertices), and for all $i$, $d_i > 0$. Hence we have shown that if $i(u) \leq i(v)$ the relation is R-sliceable through the embedded simplexes. This concludes the proof of the theorem.

<u>Corollary 3.14</u>: Given an R-sliceable simplicial tilt embedding and two simplexes $u$ and $v$ such that $uRv$. If for the vertices $x \in u$ and $y \in v$, $xRy$ and $l(ix) < l(iy)$, then $i(u) < l(iv)$.

<u>Proof</u>: Immediate from the theorem.

<u>Corollary 3.15</u>: In an R-sliceable simplicial tilt embedding, if $u$ and $v$ are two related simplexes with $i(u) > i(v)$ and $r$ and $s$ are two distinct related simplexes such that $r \cap u \neq \phi$ and $s \cap v \neq \phi$, then $i(r) \geq i(s)$.

<u>Proof</u>: Immediate, as we observe that there exists a vertex $w$ of $r$ and a related vertex $z$ of $s$ such that $l(iw) > l(iz)$. Note if $r \cap s = \phi$, then $i(r) > i(s)$.

## SECTION IV

## THE SEWN COMPLEX

In the following section we assume that $iK$ is an R-sliceable simplicial tilt embedding of a complex $K$ in $K \times I$, where the simplicial relation $R$ is induced by a simplicial map $f: K \to L$. A complex called the sewn complex, denoted by $sK$, is constructed in $L \times I$ which contains a homeomorphic image of $iK$. This sewn complex will play a special role in studying the collapsing properties of $L \times I$ and of $L$.

Consider for the point $x \in S$ (recall $S$ is the seam of $K$) the set $Sx$. Since for $y$ and $z$ in $Sx$ with $y \neq z$, $iy$ and $iz$ are at distinct levels in $K \times I$ (assume that $iy$ is above $iz$), it is possible to erect a one-cell in $K \times I$ such that one end of the cell is at $iy$, the other end of the cell is at the <u>level</u> of $iz$, and the whole cell lies in the vertical above $y$. We will denote this cell by $yIz$. Likewise, from the point $iz$, a one-cell $zIy$ can be constructed with one end at $iz$, and the other end at the level of $iy$.

Since the embedding is R-sliceable, the process can be carried out at every point in the embedded seam $i(S)$. A cell from $iy$, $yIz$, which lies below $iy$ is called a root, and a cell which lies above $iy$ is called a stalk. If we consider two related simplexes $uRv$, we note that if a point of $i(u)$ is above its related point in $i(v)$, then all points of $i(u)$ are above their related points in $i(v)$. Thus each point of $i(u)$ has a root, and each point of $i(v)$ has a stalk. We note, too, that if $u$ is an m-simplex, $i(u)$ united with all its roots and stalks is an $m+1$ cell. A given simplex can have roots to one related simplex and stalks to another, or stalks to two other related simplexes with two stalks or roots on the same point overlapping. (See Figure 1.) In any case, the embedded simplex united with all its roots and stalks is a cell one dimension higher than the simplex. Figure 2 shows the situation we are discussing for three related points.

<u>Definition 4.1</u>: We call $iK$ united with all roots and stalks the sprouted complex and denote it by $rK$.

We now extend the relation $R$ on $K$ to a relation $R_I$ on $K \times I$ by defining $(x,t)R_I(y,t)$ iff $xRy$, where $x$ and $y$ are related points in $K$, and $t$ is a point of $I$. Let $f'\colon K \times I \to K \times I/R_I \doteq L \times I$.

<u>Definition 4.2</u>: $f'(rK) = sK \subset L \times I$, and we call $sK$ the sewn complex.

Note that $sK$ contains a homeomorphic image of $iK$, namely $f'(iK)$, with additional cells "sewn" across the images in $iK$ of related simplexes of $K$. For example, if $uRv$, where $u$ and $v$ are m-simplexes of $K$, and if in $iK$, $i(u) \geq i(v)$, then the roots from $i(u)$ down to the level of $i(v)$ and the stalks above $i(v)$ up to the level of $i(u)$ will, under $R_I$, be identified to form in $sK$ a prismatic $m+1$ cell with base $f'(i(v))$ and top $f'(i(u))$. (See Figure 3.)

If $u$,$v$, and $w$ are three related m-simplexes of $K$ and if $i(u) \geq i(v) \geq i(w)$, then in $sK$ there will be a "path" of two $m+1$ cells with top $f'(i(u))$, bottom $f'(i(w))$, and $f'(i(v))$ embedded

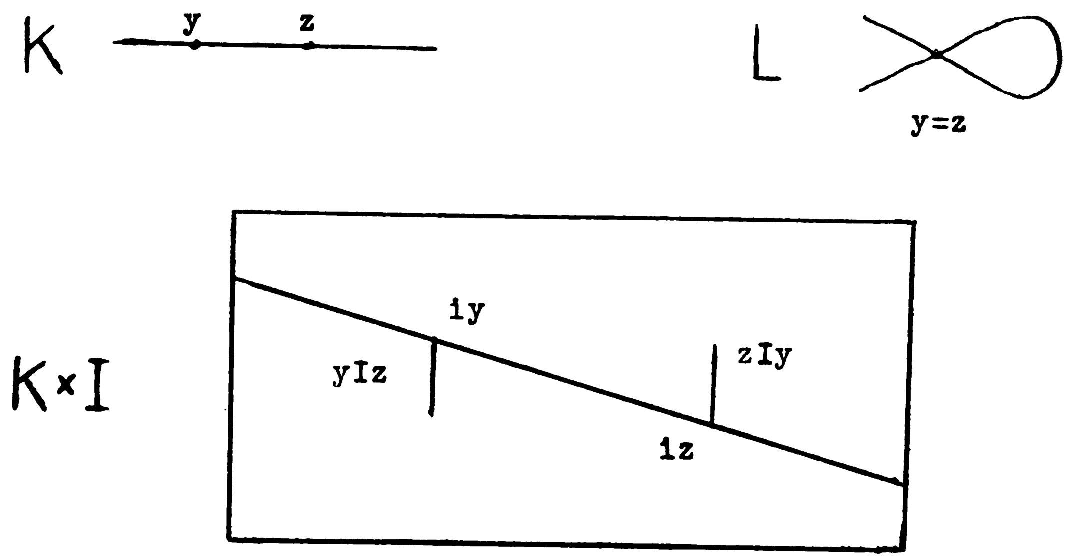



Figure 1

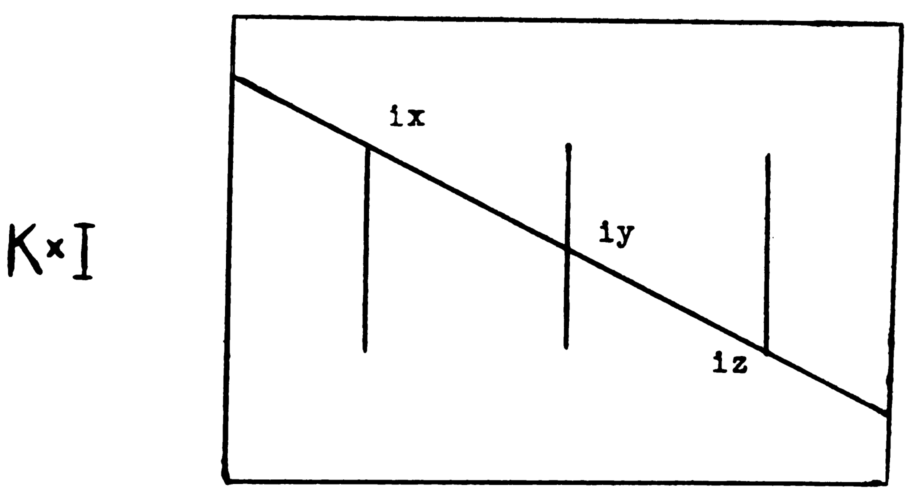



Figure 2

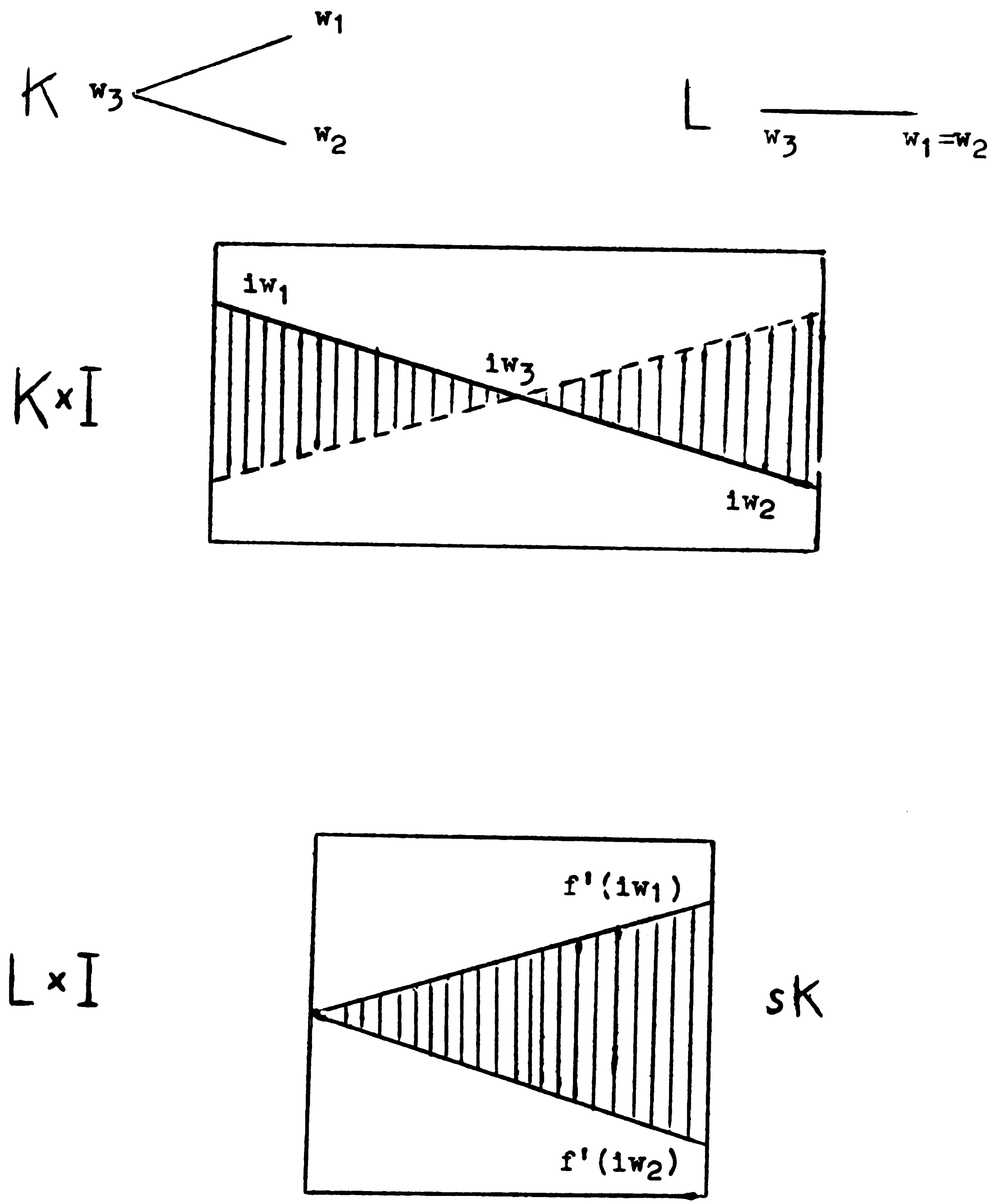



Figure 3

in the middle, separating the two cells. (See Figure 4.)

Definition 4.3: If $u$ and $v$ are two related simplexes in $K$, $uIv$ will denote the prismatic cell in $sK$ between $f'(i(u))$ and $f'(i(v))$.

Definition 4.4: $C$ contained in $sK$ is the union of all the prismatic cells in $sK$. Thus, $C = \bigcup_{i,j} s_i I s_j$, where $s_i$ and $s_j$ are related simplexes of $K$.

Theorem 4.5: Let $iK$ be an R-sliceable, projective embedding of $K$ in $K \times I$. If $sK$ is the sewn complex in $L \times I$ formed by the relation $R_I$, then $L \times I \searrow sK$.

Proof: Order the simplexes of $K$ in decreasing dimension. If $s_1$ is a simplex not contained in the closed seam $\bar{S}$, then $i(s_1)$ is an m-simplex embedded in $K \times I$ on neither the base $K \times 0$ nor the top $K \times 1$. Hence there will be two $m+1$ cells in $K \times I$, one above $i(s_1)$ with $i(s_1)$ as its floor and one below the embedded simplex with $i(s_1)$ on its top. (See Figure 5.) Under the identification $R_I$, these two cells are sent by $f'$ to corresponding $m+1$ cells in $L \times I$, as identifications can only occur on the boundaries of these cells. Thus both can be collapsed onto $f'(i(s_1))$ and onto the "walls" of the cylinder in $L \times I$, since $f'(s_1 x1)$ and $f'(s_1 x0)$ will be free faces.

Thus in $L \times I$ collapse out all cells which are over or under embedded simplexes of this type (i.e., simplexes not contained in $\bar{S}$) proceeding down from the highest dimensional cells.

Next consider each cell path in $sK$, $s_1 I s_2 I \ldots I s_m$, where the $s_i$ are m-simplexes in $\bar{S}$. Above $f'(i(s_1))$ there is an $m+1$ cell in $L \times I$, and below $f'(i(s_m))$ there is a similar $m+1$ cell. (See Figure 6.)

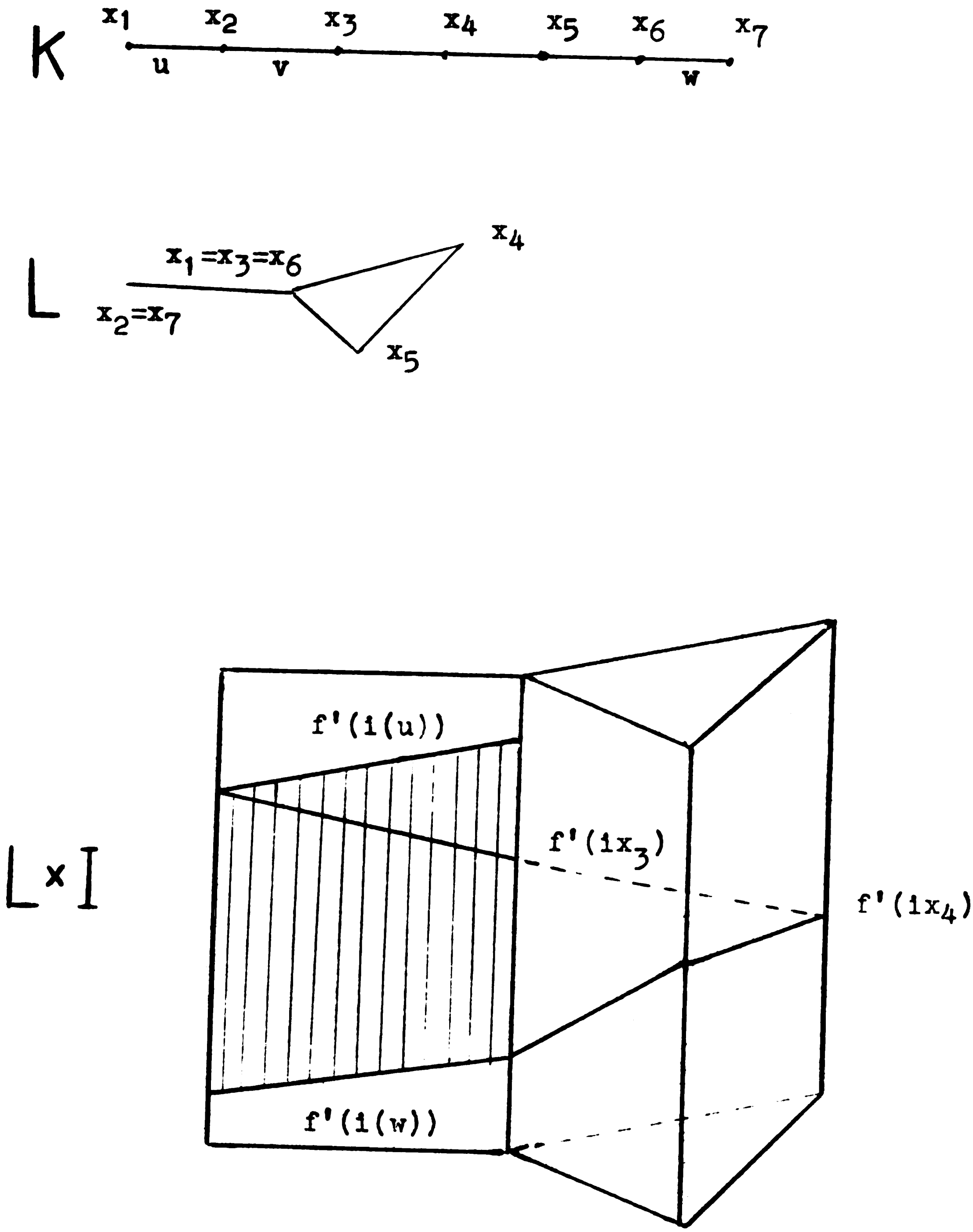



Figure 4

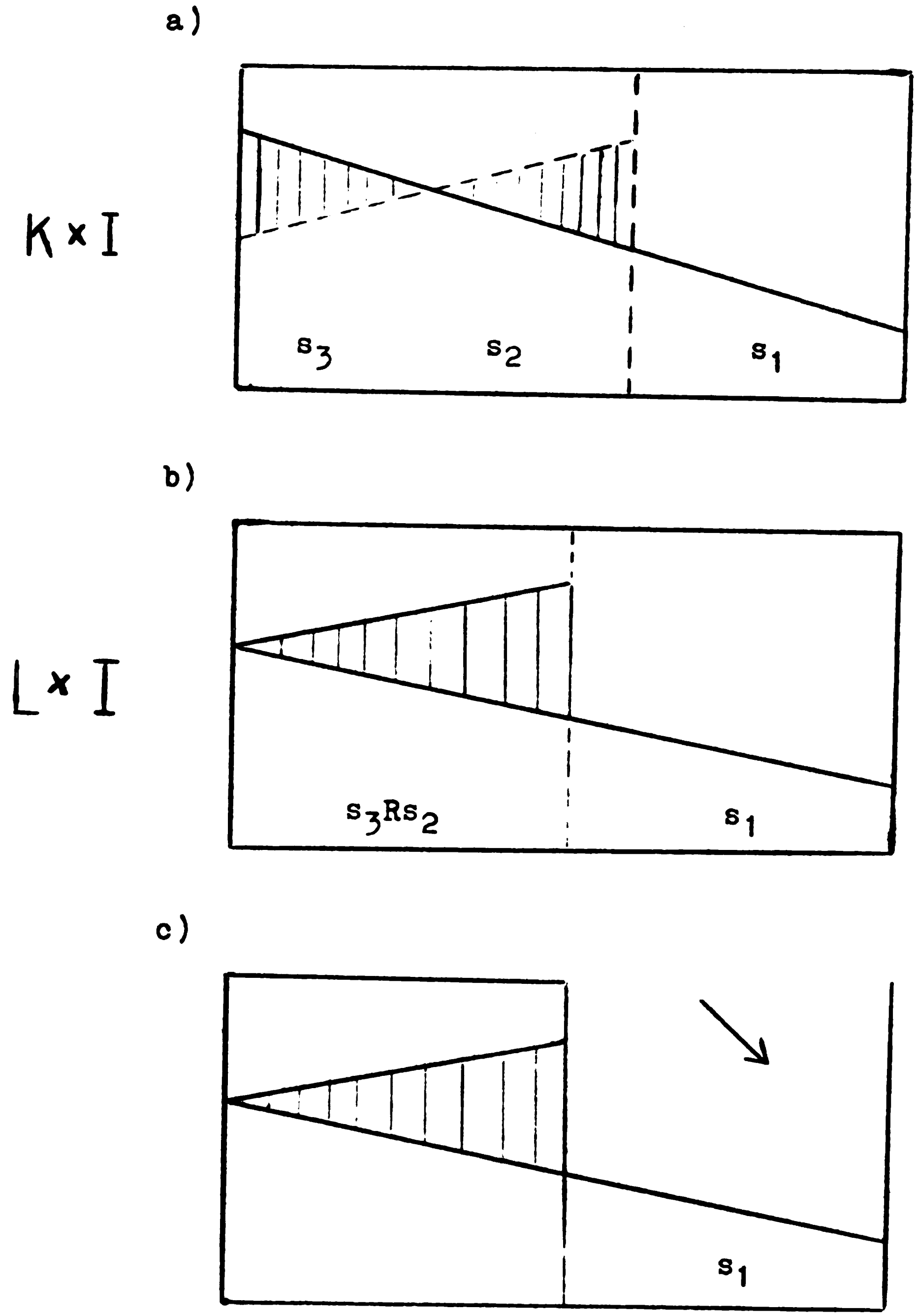



Figure 5

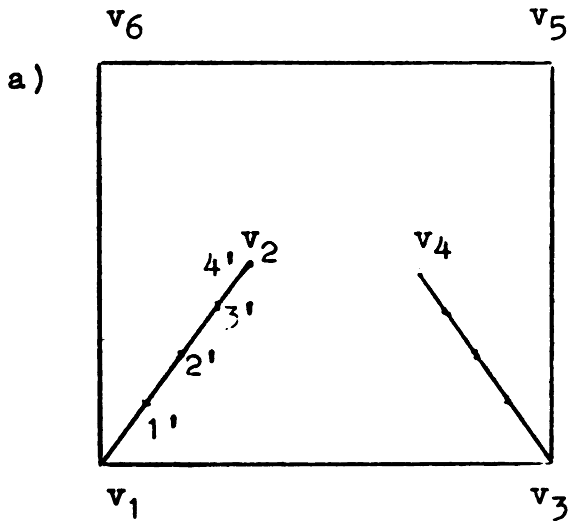
In the vertical cell path contained in $L \times I$, collapse down onto the top embedded simplex of the cell path in $sK$, in the order of decreasing dimension. This is always possible as a free face is found at the top of $L \times I$. The same procedure is followed from the bottom of $L \times I$, collapsing upward onto the base of a cell path. The end result gives a method of collapsing $L \times I$ onto $sK$.

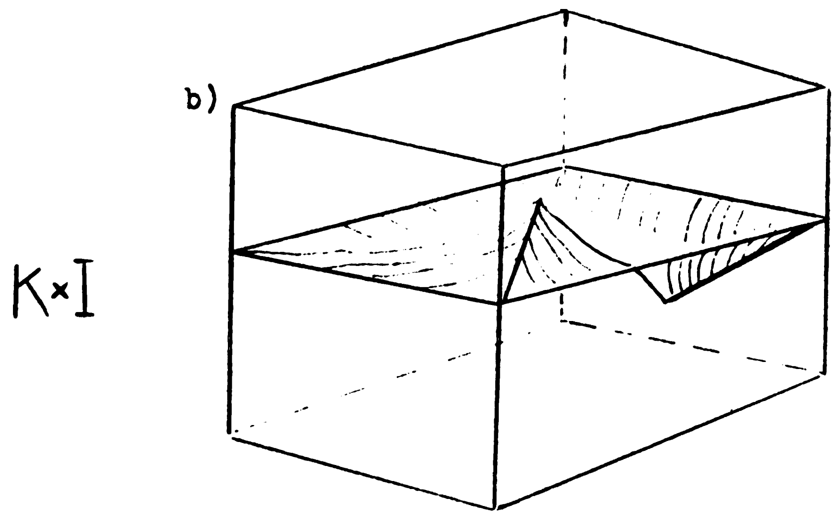
<u>Theorem 4.6</u>: If $L$ is collapsible then $sK$ is collapsible.

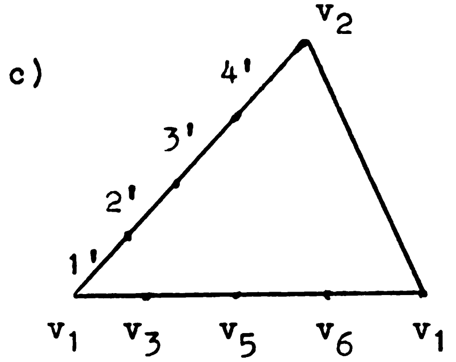
<u>Proof</u>: Consider the projection map $p: L \times I \to L$. As $sK$ is embedded in $L \times I$ consider $p$ restricted to $sK$, i.e., $p: sK \to L$. By the nature of the embedding, $p$ is a strongly pointlike map, as $p^{-1}(x)$ is a single point if $x$ is not in $f(S)$ and $p^{-1}(x)$ is a chain of one cells if $x$ is in $f(S)$. In either case $p^{-1}(x)$ is collapsible. Thus by the characterization of strongly pointlike maps [2] we get that if $L$ is collapsible, then $sK$ is collapsible.

The following is an example of how theorem 4.5 can be used to show that a given $L \times I$ is collapsible. The complex $L$ is constructed from the two-cell $K = I \times I$ in the following manner. (See Figure 7a.) We identify the "diagonal" elements $\langle v_1, v_2 \rangle$ and $\langle v_3, v_4 \rangle$ with the boundary of $K$. That is to say $\langle v_1, v_2 \rangle$ is broken into 4 one-cells, say 1', 2', 3', and 4', and we then identify 1' with the edge $\langle v_1, v_3 \rangle$. 2' is identified with the edge $\langle v_3, v_5 \rangle$, 3' with $\langle v_5, v_6 \rangle$ and 4' with $\langle v_6, v_1 \rangle$. Likewise diagonal element $\langle v_3, v_4 \rangle$ is broken into 4 one-cells, and the first is identified with $\langle v_3, v_5 \rangle$, the second with $\langle v_5, v_6 \rangle$, etc. The complex that results from these identifications is $L$.

We assume that $K$ is suitably subdivided such that the mapping $f: K \to L$ is a simplicial mapping.




A triangular cell sewn onto K across the diagonal $\langle v_1, v_2 \rangle$ and the boundary of K.

Figure 7

Now it is obvious that $L$ is not collapsible, as $L$ has no free edges from which a collapse may be started. However, we shall show that $L \times I$ is collapsible.

First we will show that the relation on $K$ is R-sliceable. This is obvious if we consider the following projective embedding of $K$ in $K \times I$. We embed the boundary of $K$ at level $\frac{1}{2}$, and the diagonal element $<v_1,v_2>$ linearly such that $iv_2$ is at level 3/4. The second diagonal element is "bent downward", i.e., we embed $iv_4$ at level $\frac{1}{4}$. (See Figure 7b.)

This embedding $iK$ of $K$ is obviously R-sliceable. Now we consider $sK$. It can be seen that $sK$ will be homeomorphic to $K$ with two additional "triangular" two-cells sewn on in the following manner. The top cell is sewn on in such a way that one edge of the "triangle" is identified with $<v_1,v_2>$ and the second edge with the boundary of $K$. The remaining edge is free in $sK$. (See Figure 7c.) The other two-cell is sewn onto $K$ in a similar manner but across the edge $<v_3,v_4>$ and the boundary of $K$. It is obvious that $sK \searrow iK$ as these two added triangular cells can each be collapsed, starting from the free edges. Thus $L \times I \searrow sK \searrow iK \searrow 0$, as $K$ is a two-cell.

## SECTION V

## BEADED COMPLEXES

The techniques we have introduced thus far lead us to consider in this section the following problem. We have been studying the collapsing of $L \times I$ by looking at a complex $K$ which maps onto $L$, then forming from $K$ a sewn complex $sK$. We can ask what types of complexes $K$ we can find for a given $L$. In particular, could we always find a collapsible complex $K$ such that the relation induced on $K$ by the simplicial map $f: K \to L$ is always R-sliceable?

Definition 5.1: A principal simplex $s$ in a complex $K$ with the property that for each distinct principal simplex $t$ in $K$, $s \cap t = \phi$, or $s \cap t = v$, ($v$ a vertex), is said to be vertex joined in $K$, and $v$ is called a joining vertex. $V$ will denote the set of joining vertices of such a simplex.

Definition 5.2: A principal simplex $s$ of $K$ is proper if:

1. It is vertex joined in $K$.
2. It meets at most two distinct principal simplexes in $K$.
3. If $s$ meets exactly two distinct principal simplexes, then the set $V$ of joining vertices contains exactly two distinct points.

Definition 5.3: In a proper simplex with joining vertex set $V = \{v,w\}$, the one-simplex $\langle v,w \rangle$ is called the distinguished face of $s$. If $V = \{w\}$, then the vertex $w$ is the distinguished face of $s$.

Note: if $V = \emptyset$, then $K = s$, i.e., $K$ contains only one principal simplex and in this case an arbitrarily chosen vertex of $s$ will be called the distinguished face. Thus a proper simplex in a complex $K$ has one distinguished face, which is either a vertex or a one-simplex.

Definition 5.4: A complex $K$ is a beaded complex if all principal simplexes of $K$ are proper and if the union of all their distinguished faces in $K$ is either:

1. the homeomorphic image under a map $g: I \to K$ of the unit interval, or
2. the union is a single point. (Such a complex will be called a degenerate beaded complex.)

Notation: $bK$ denotes a beaded complex. (See Figure 8.)

Definition 5.5: A principal simplex in a beaded complex is called a bead. For non-degenerate beaded complexes, the image of $I$ under $g$ is called the string of the beaded complex, and it well orders the principal simplexes of $K$ as follows. For $s$ and $t$ two beads in $K$ with joining vertex sets $V$ and $V'$ respectively, $s$ precedes $t$, written $s \leq t$, if there exists $v \in V$ such that $g^{-1}(v)$ is less than or equal to $g^{-1}(w)$ for all $w \in V'$. As a beaded complex is connected and finite, this well orders $bK$.

Lemma 5.6: If $bK$ is a degenerate beaded complex, either $bK$ contains only two distinct principal simplexes $s$ and $t$ such that $s \cap t = v$ or $bK = s$ for some simplex $s$.

Proof: Obvious.

A degenerate complex's beads can be well ordered trivially.

Definition 5.7: An extreme bead in a beaded complex is a bead with vertex set $V$ containing only one point. A beaded complex (non-degenerate) contains exactly two extreme beads, and all other beads will have two points in their joining vertex set.

Definition 5.8: Given a beaded complex bK containing two or more beads, a cut divides bK into two disjoint beaded complexes $bK_1$ and $bK_2$. This is done by selecting any two beads s and t of K such that $s \cap t = v$, then setting $bK_1$ homeomorphic to the subcomplex of bK consisting of all beads that precede s, (which includes s itself), while $bK_2$ is a complex homeomorphic to the subcomplex of K consisting of t and all beads that follow t. Two disjoint beaded complexes of dimension greater than 0, say $bK_1$ and $bK_2$, may be joined to form a beaded complex bK containing a homeomorphic image of each by selecting extreme beads of each, say s of $bK_1$ and t of $bK_2$, and identifying a non-distinguished vertex of s with a non-distinguished vertex of t. (A non-distinguished vertex of a bead is a vertex not contained in a distinguished face.)

Lemma 5.9: A beaded complex is collapsible.

Proof: Obvious.

Theorem 5.10: If L is a connected complex, there exists for each chosen "starting" vertex v of L a beaded complex bK and a simplicial map $g: bK \to L$ such that:

1. g is surjective
2. g preserves dimension (hence induces a simplicial relation on bK)
3. bK is made up of exactly one n-bead for each principal n-simplex of L, $n \geq 2$, and an arbitrary number of one-beads which map into the 1-skeleton of L.

4. $v$ is the image of a non-distinguished vertex of an extreme bead of $bK$.

<u>Proof</u>: This theorem will be proved by induction on the number of simplexes in $L$. If $L$ contains one simplex, then $L$ is a beaded complex as $L = \{v\}$, and the identity map on $L$ can be taken for the map $g$. Suppose any complex containing $k$ or fewer simplexes can be "covered" by a beaded complex which satisfies conditions 1-4.

Let $L$ contain $k+1$ simplexes. If $L$ is a complex of dimension 1 and $L$ is not a tree, there exists a principal one-simplex $s$ of $L$ such that $L-s$ is connected and contains $k$ simplexes. Hence $L-s$ can be covered by a beaded complex by the inductive hypothesis, and we select the starting vertex $w_1$ to be a face of $s$ in $L-s$. If $L$ is a tree, there exists a one-simplex $s = \langle w_1, w_2 \rangle$ of $L$ with the property that $L-s-\{w_2\}$ is connected and contains $k-2$ simplexes. Again $L-s-\{w_2\}$ can be covered by the inductive hypothesis with selected starting vertex $w_1$.

If $\dim(L) = n$, where $n$ is strictly greater than one, there exists a principal simplex $s$ of $L$ with dimension $n$; and $L-s$, containing only $k$ simplexes, can again be covered by a beaded complex with starting vertex $w_1$ a face of $s$ in $L-s$.

Thus in all cases, given an $L$ containing $k+1$ simplexes, we have removed a principal simplex $s$ and covered $L-s$ with a beaded complex, i.e., we have a $bK$ and a $g: bK \to L-s$ such that $bK$ satisfies conditions 1-3 with the starting vertex $w_1$ a face of $s$ in $L-s$. Now in $bK$ select the extreme bead $t$ whose non-distinguished vertex $x$ is mapped onto $w_1$, i.e., $g(x) = w_1$. As $t$ is an extreme bead in $bK$, it has a single joining vertex $z$, and we modify $bK$ to form a new beaded complex to cover $L$ as follows.

Select a copy $s'$ of the principal simplex $s$ in $L$ that was removed, and let $h: s' \to L$ be a simplicial map such that $h(s') = s$. Select the vertex $h^{-1}(w_1)$ of $s'$ and attach $s'$ to $bK$ by identifying $h^{-1}(w_1)$ with $x = g^{-1}(w_1)$. This forms a new beaded complex $bK \cup s'$, which covers $L$ by extending the map $g: bK \to L{-}s$ onto $bK \cup s'$ by setting $g(s') = h(s')$. Finally, replace only the beads in $bK$ which map onto the faces of $s$ by their distinguished faces. Note that this final beaded complex satisfies conditions 1,2, and 3.

For condition 4, a "tail" of one simplexes (i.e., a beaded complex of dimension 1) can be added onto $bK \cup s'$ at a non-distinguished vertex $u$ of $s'$ such that the tail maps into the one-skeleton of $L$ and forms a path that connects $g(u)$ with any selected vertex $v$ of $L$.

Definition 5.11: A beaded complex satisfying the conditions of the above theorem will be called a covering beaded complex of $L$.

Corollary 5.12: Any beaded complex mapped onto a complex $L$ by a dimension preserving map can be modified to consist of only one bead covering each principal simplex in $L$ of dimension greater than one and connecting one-beads which map into the one-skeleton of $L$.

Proof: The part concerning the principal simplexes can be met by selecting one bead of $bK$ to cover each principal simplex of $L$ and then replacing all other simplexes of $bK$, i.e., all of those not selected as covering a principal simplex with their distinguished faces. The modified complex will then satisfy the conditions of the corollary.

Corollary 5.13: Every connected n-complex is the simplicial image of a collapsible n-complex.

Proof: Obvious, as a covering beaded complex is collapsible.

Theorem 5.14: Given a beaded complex bK covering a complex L by a simplicial map f, then the simplicial relation R induced on bK by f is such that bK has an R-sliceable simplicial-tilt embedding in bK $\times$ I.

Proof: Let n equal the number of beads in bK, and let $s_1 \leq s_2 \leq \ldots \leq s_n$ be the ordering of the beads. Divide the interval [1/n+2, n+1/n+2] up into n subintervals, $I_i = [i/n+2, i+1/n+2]$, $i = 1,2,\ldots,n$. Now $s_1$ is embedded projectively in $s_1 \times I$ such that its distinguished vertex is a level 2/n+2 and all other vertices $v_i$ of $s_1$ are at levels chosen arbitrarily but such that they lie strictly within the interval $I_1$. The second distinguished vertex of $s_2$ is embedded at level 2/n+2, as the first is shared with $s_1$ and has already been embedded at level 2/n+2. The remaining vertices of $s_2$ are embedded projectively at levels strictly between 2/n+2 and 2/n+2. A simple inductive argument embeds $s_i$ in bK $\times$ I between levels i/n+2 and i+1/n+2, $i = 1,2,\ldots,n$.

This embedding of bK in bK $\times$ I is R-sliceable and projective, since any two related simplexes in the seam S can not both be faces of some $s_i$, they must lie on distinct beads and their images in bK $\times$ I lie one strictly above the other by the nature of the embedding. Hence by Theorem 3.13, the relation is R-sliceable.

## SECTION VI

## FACTORIZATIONS OF RELATIONS

We begin with a lemma which provides the motivation for the investigations undertaken in this section. This lemma leads us to ask questions concerning the homological relationships that exist between K, sK and L; and these considerations lead in turn to the concept of fold factorizations. After developing the factorization concept, we will conclude with some theorems on "folding" and homology.

<u>Lemma 6.1</u>: Let L be a two-complex, and let K be a two-complex mapped onto L by a simplicial map f. Let the closed seam $\bar{S}$ of K contain only two one-simplexes, say $s_1$ and $s_2$, which are identified by f. If K is homologically trivial and L is homologically trivial, then the identification must be a fold, (see 2.9).

<u>Proof</u>: Let $d(H_i(K))$ be the Betti number of the $i^{th}$ homology group of K. Then $X(K) = d(H_0(K)) - d(H_1(K)) + d(H_2(K)) = 1-0+0 = 1$, and likewise $X(L) = d_0(H_0(L)) - d(H_1(L)) + d(H_2(L)) = 1-0+0 = 1$. Let $V_i$ be equal to the number of $i^{th}$ dimensional simplexes in K, $i = 0,1,2$ and $V_i'$ represent the number of $i^{th}$ dimensional simplexes in L, $i = 0,1,2$. It is well known that $X(K) = V_0 - V_1 + V_2 = 1$, and likewise for the $V_i'$. Now as f identifies no two simplexes, $V_2' = V_2$. If the relation were not a fold, i.e., if $s_1 \cap s_2 = \phi$, then $V_0' = V_0-2$ and $V_1' = V_1-1$, since disjoint one-simplexes and their vertices are identified. This implies that $V_0' - V_1' + V_2' = (V_0-2) - (V_1-1) + V_2$

$= 1-1 = 0$, a contradiction. Thus the relation must be a fold, i.e., $s_1 \cap s_2$ is a crease point.

Attempting to generalize this result by allowing more simplexes in $\bar{S}$, hoping to arrive at a "sequence" of folds, proved fruitless. However, approaching from the "converse" direction leads to the following:

<u>Definition 6.2</u>: Let $K$ be a simplicial complex with a relation $R$. A factorization is a sequence of "complexes" $K_i$ and maps $f_i$ such that $K = K_1 \xrightarrow{f_1} K_2 \xrightarrow{f_2} K_3 \to \ldots \to K_n = L = K/R$, where $K_{i+1}$ is formed from $K_i$ by identifying "simplexes" in $K_i$ and $f_i$ is the identification map. We require, of course, that $f_{n-1} \circ f_{n-2} \circ \ldots \circ f_1 = f\colon K \to L$. Certainly $f\colon K \to L$ is a sequence which forms a trivial factorization. For notation, $\{K_i, f_i\}_1^n$ denotes a factorization.

If $s \in K_i$ and $f_i^{-1}(f_i(s)) = s$, we will consider $f_i(s) = s$, to simplify notation. If $f_i^{-1}(f_i(s)) = s \cup t \cup \ldots \cup p$, we will say that $f_i(s) = s = t = f_i(t)$, and be careful to note what stage of a given factorization we are in. Thus in the above cases, in $K_i$, $s \neq t$, but in $K_{i+1}$, $s = t$, as they have been identified by $f_i$.

<u>Lemma 6.3</u>: Given a factorization $\{K_i, f_i\}_1^n$, there exist "subdivisions" $K_i''$ of each $K_i$ such that the resulting factorization can be considered to be a sequence of simplicial complexes and simplicial maps, or a <u>simplicial</u> <u>factorization</u>.

<u>Proof</u>: This follows by a simple inductive argument, using the second barycentric subdivision of $K$ and the results of Theorem 2.2.

If $\{K_i, f_i\}_1^2$ is the factorization, then the fact that $\{K_i'', f_i\}_1^2$ is a simplicial factorization is just the conclusion of 2.2.

Let us assume that if $\{K_i, f_i\}_1^n$ is a factorization, then $\{K_i'', f_i\}_1^n$, where $K_i''$ is the second barycentric subdivision of $K_i$, is a simplicial factorization.

Let $\{K_i, f_i\}_1^{n+1}$ be a factorization. Then $K_1'' \to K_2''$ is a simplicial factorization of length 2, and $\{K_i'', f_i\}_2^{n+2}$ is a simplicial factorization of length $n$ by the inductive hypothesis. Thus $\{K_i'', f_i\}_1^{n+1}$ is a simplicial factorization.

We will assume for the next theorem that $K$ and $L$ are n-complexes with $n$ greater or equal to 2 and that $f\colon K \to L$ induces a simplicial relation on $K$ such that $\dim(\overline{S})$ is less than or equal to $n-1$, i.e., no n-simplexes in $K$ are identified. (Note: we can always cover $L$ with a beaded complex $K$ such that no n-simplexes of $K$ are identified by 5.12.)

<u>Theorem 6.4</u>: Let $\{K_i, f_i\}_1^n$ be any simplicial factorization. If $H_n(L) = 0$, then $H_n(K_i) = 0$ for all $i$.

<u>Proof</u>: Suppose for some $i$ that $H_n(K_i) \neq 0$, then it is easy to see that $H_n(K_{i+1}) \neq 0$. Consider the definition of $H_n$. As $K_i$ and $K_{i+1}$ are n-complexes, $H_n(K_i) = \ker(C_n(K_i) \xrightarrow{d} C_{n-1}(K_i))$, where $C_n$ is the free group generated by all n-simplexes, $C_{n-1}$ the free group generated by all $n-1$ simplexes and $d$ the usual boundary operator. Now as $f_i\colon K_i \to K_{i+1}$ identifies no n-simplexes, $f_i$ induces an isomorphism from $C_n(K_i) \to C_n(K_{i+1})$. Now consider the following commutative diagram.

$$\begin{tikzcd}
C_n(K_i) \arrow[r, "f_i"] \arrow[d, "d"'] & C_n(K_{i+1}) \arrow[d, "d"'] \\
C_{n-1}(K_i) \arrow[r, "f_i"] & C_{n-1}(K_{i+1})
\end{tikzcd}$$

If $d: C_n(K_i) \to C_{n-1}(K_i)$ is not 1 : 1, i.e., if $H_n(K_i) \neq 0$, then $d: C_n(K_{i+1}) \to C_{n-1}(K_{i+1})$ has to have a non-zero kernel as:

$$\begin{aligned} H_n(K_{i+1}) &= \ker(d: C_n(K_{i+1}) \to C_{n-1}(K_{i+1})) \\ &= \ker(f_i o d: C_n(K_i) \to C_{n-1}(K_{i+1})) \\ &= \ker(d o f_i: C_n(K_i) \to C_{n-1}(K_{i+1})) \\ &\neq 0 \end{aligned}$$

But $H_n(K_{i+1}) \neq 0$ implies inductively that $H_n(L) \neq 0$, a contradiction.

<u>Lemma 6.5</u>: If $f: K \to L$ induces an R-sliceable relation on $K$ and $\{K_i, f_i\}_1^n$ is a simplicial factorization, then the relation $R_j$ induced on $K$ by $f_j o f_{j-1} o \ldots o f_1: K \to K_{j+1}$ is R-sliceable.

<u>Proof</u>: There exists an R-sliceable embedding, say $iK$, of $K$ in $K \times I$, and it is obvious that $iK$ will be R-sliceable for the relation $R_j$ as $R_j$ is "essentially" a subset of the relation $R$.

<u>Definition 6.6</u>: Let $\{K_i, f_i\}_1^n$ be a factorization. $f_i: K_i \to K_{i+1}$ is called an m-fold of n-simplexes, or an $(m,n)$ fold, if the "closed seam" in $K_i$ induced by $f_i$ is a set $\{s\}$ of $m$ n-simplexes which intersect in a common p-face, (the crease), and which are identified by $f_i$ to one n-simplex in $K_{i+1}$. The faces of a fold simplex $s \in Ss$ not contained in the crease will be called the free faces of $s$. If $K_i$ and $K_{i+1}$ are simplicial complexes, the fold is a <u>simplicial</u> <u>fold</u>.

<u>Theorem 6.7</u>: Let $K$ be a simplicial complex and let $f: K \to L$ be a fold of the simplex $s_1 = \langle w, v_1 \rangle$ onto $s_2 = \langle w, v_2 \rangle$. If $|st(v_1)| \cap |st(v_2)| = w$, then $L$ is a simplicial complex.

Proof: From Theorem 2.2 we know that if $L$ contains no loops of type 1 or 2 then $L$ will be a simplicial complex with the structure induced from $K$. The only possibility for a loop of type 1 to occur is if there exists a one-simplex in $K$ of the form $\langle v_1, v_2\rangle$. But since the intersection of the vertex stars contains only $w$, no such one-simplex exists.

The only place for a loop of type 2 to occur, since $K$ is a simplicial complex, is if it is of dimension 1 and is of the form $\langle v,x\rangle_1 \cup \langle x,v\rangle_2$, where $v = v_1 = v_2$ in $L$ and $x$ is some other vertex of $K$, $x \neq w$. Again the intersection of the stars implies that $x = w$, and this loop cannot occur.

Corollary 6.8: If $K$ is a simplicial complex and $f: K \to L$ is a $(m,1)$ fold of the simplexes $s_i = \langle w,v_i\rangle$, $i = 1,2,\ldots,m$, onto a single one-simplex and if for all pairs $i$ and $j$, $|st(v_i)| \cap |st(v_j)| = w$, $(i \neq j)$, then $L$ is a simplicial complex.

Theorem 6.9: If $K$ is a simplicial complex and $f: K \to L$ is a $(2,1)$ fold of $s_1 = \langle w,v_1\rangle$ onto $s_2 = \langle w,v_2\rangle$ and if $L$ is not a simplicial complex, then there exists a subdivision $K''_1$ of $K$ and a simplicial factorization $K''_1 \to K''_2 \to K''_3 \to K''_4$, where $|K''_4| = |L|$ and each stage $K''_i \to K''_{i+1}$ is a simplicial fold.

Proof: First take the 2nd barycentric subdivision of $K$ and call it $K''_1$. Let us assume that $s_1$ is broken into the four one-simplexes $\{\langle x_i,x_{i+1}\rangle\}_1^4$, and $s_2$ into the four one-simplexes $\{\langle y_i,y_{i+1}\rangle\}_1^4$, where $x_1 = y_1 = w$, $x_5 = v_1$ and $y_5 = v_2$.

Now $K''_{i+1}$ is formed from $K''_i$ by folding $\langle x_i,x_{i+1}\rangle$ onto $\langle y_i,y_{i+1}\rangle$. Note that $K''_4$ is a simplicial complex and that $|K''_4| = |L''|$ by 2.2. It follows that each stage $K''_i$ is also a simplicial complex for if some $K''_i$ were not, it would contain loops of type 1 or 2, and it is obvious from the nature of the subsequent identification that $K''_{i+1}$ would also contain such loops. Since $K''_4$ is a complex, this cannot occur.

Corollary 6.10: If $\{K_i, f_i\}_1^n$ is a $(m_i, 1)$ fold factorization but is not a simplicial factorization, then there exists a simplicial one-fold factorization $\{K''_i, f_i\}_1^m$ where $m = 4n$ and $|K_i| = |K''_{4i}|$, $i = 1, 2, \ldots, n$.

Theorem 6.11: If an R-sliceable relation on $K$ has a simplicial factorization as a sequence of $(m_i, 1)$ folds, say $\{K_i, f_i\}_1^n$, and if

1. for each one-simplex $s$ of $\bar{S}$ where $s$ in an element of some $m_i$ fold, $Ss$ in $K$ consists of exactly $m_i$ one-simplexes, and
2. if for each $v$ an element of $\bar{S}$ such that $v$ is a free face of some $m_i$ fold, $Sv$ consists of exactly $m_i$ points,

then $sK \searrow iK$.

Proof: We will show this result by induction on the length of the factorization. If $K_2$ is the final stage, then the entire relation on $K$ consists of a single $(m, 1)$ fold, and $sK$ will consist of $m-1$ 2-cells sewn across the fold. Between any two of the one-simplexes in the fold, one embedded directly above the other in $K \times I$, there will be, in $sK$, a triangular two-cell having a free edge running through the free faces of the two one-simplexes. This free edge allows a collapse of the two-cell. Similarly all of the $m-1$ two cells can be collapsed, leaving us with $iK$.

Assume that in a fold factorization of length $n$ meeting conditions 1 and 2 that $sK$ can be collapsed back to $iK$, and let us suppose that $\{K_i, f_i\}_1^{n+1}$ is a fold factorization. The $m_n$ one-simplexes identified on the last fold will be such that through their free faces will be found, in sK, a sequence of free edges which will allow the triangular two-cells sewn across these one-simplexes to be collapsed. The resulting complex is homeomorphic to the sewn complex which would be formed by mapping K onto $K_n$ through an n-step fold factorization using the original R-sliceable embedding (see 6.5) and omitting the last fold in the given n+1 stage factorization. This complex is collapsible by the inductive hypothesis onto iK so $sK \searrow iK$.

Definition 6.12: A fold factorization is called proper if for each y a point in some $(m_i, n_i)$ fold (y not in a crease) Sy has order $m_i$.

Note: we have with the above definition "abstracted" conditions 1 and 2 of Theorem 6.11 into higher dimensions.

Theorem 6.13: If an R-sliceable relation on K induced by a map $f: K \to L$ has a simplicial factorization as a sequence of proper $(m_i, n_i)$ folds, then $sK \searrow iK$ and if $K \searrow 0$ then $L \times I \searrow 0$.

Proof: Dimension 1 is just a restatement of 6.11, and the higher dimensional cases would be proven in a similar manner. Since $K \doteq iK$, the final conclusion is obvious, as it follows from 4.5.

To illustrate the above theorem we will consider an example. We will construct a complex L through a sequence of proper 1-folds from a collapsible complex K. Corollary 6.10 implies that the initial proper fold factorization can be subdivided into a simplicial fold factorization, and it is obvious that this resulting factorization is also proper. Thus we will satisfy the hypotheses of 6.13. (We should make note of the fact that 6.10 does not appear to have higher dimensional analogs, at least the second barycentric argument does not carry through. One must be careful in applying 6.13 to insure that the complexes considered satisfy all conditions described.)

Let $K$ be the two-cell $I \times I$, (see Figure 9), and let $L$ be formed from $K$ by identifying the $n$ diagonal elements $\langle w_0, v_i \rangle$ $i = 1,2,\ldots,n$, with the boundary of $K$ in the following manner. For each $i$, $\langle w_0, v_i \rangle$ is broken into 4 1-cells, say $1_i'$, $2_i'$, $3_i'$, and $4_i'$; and identify $1_i'$ with $\langle w_0, w_1 \rangle$, $2_i'$ with $\langle w_1, w_2 \rangle$, $3_i'$ with $\langle w_2, w_3 \rangle$ and $4_i'$ with $\langle w_3, w_0 \rangle$.

An R-sliceable embedding of $K$ in $K \times I$ is given by embedding the boundary of $K$ at level $1/n+2$ and the vertices $v_i$ at levels $(i+1)/n+2$, $i = 1,2,\ldots,n$. This embedding of $K$ will be R-sliceable for any subdivision of $K$.

Now a proper fold factorization of $f\colon K \to L$ is given by letting $K_1 = K$ and then identifying all the $1'$ cells with $\langle w_0, w_1 \rangle$ to get $K_2$. $K_3$ is formed from $K_2$ by identifying all the $2'$ cells with $\langle w_1, w_2 \rangle$, and $K_4$ is formed from $K_3$ by identifying the $3'$ cells. The $4'$ identification yields $K_5$ which is $L$. Each of these stages is obviously a proper fold and 6.10 implies we can find a proper simplicial fold factorization. Thus $L \times I$ is collapsible.

We conclude this section by returning to some homological considerations examined in light of our folding results.

<u>Theorem 6.14</u>: If $f\colon K \to L$ is a strongly pointlike map [2], i.e., a map between simplicial complexes such that for all $y$ in $L$, $f^{-1}(y)$ is collapsible, then $H(K) = H(L)$.

<u>Proof</u>: Since for each $y \in L$, $f^{-1}(y)$ is collapsible, the conditions of the Vietoris mapping theorem [1] are met, and thus $f$ induces

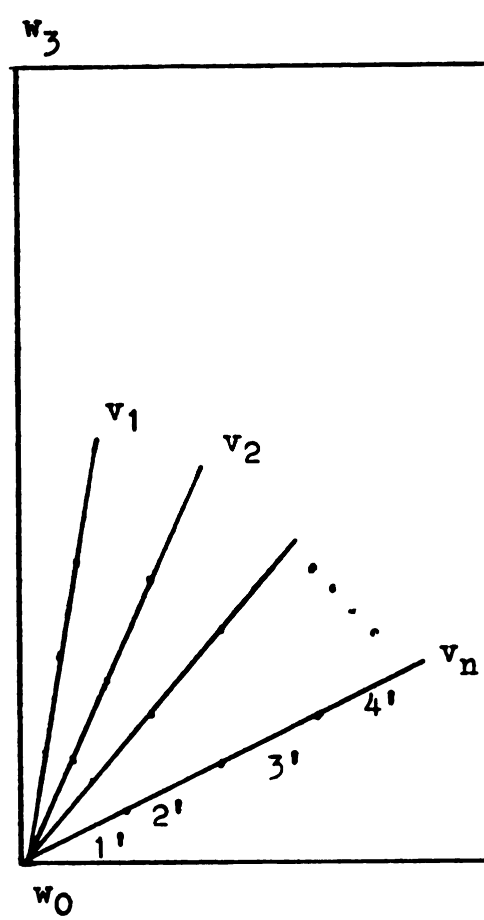



Figure 9

an isomorphism from $H(K)$ to $H(L)$.

Corollary 6.15: If $f: K \to L$ induces an R-sliceable relation on $K$, then $H(sK) = H(L)$.

Proof: Immediate as the projection $p$ (see 4.6) is a strongly pointlike map from $sK$ to $L$.

Theorem 6.16: If $f: K \to L$ is an $(m,n)$ simplicial fold, then $H(K) = H(sK)$.

Proof: Any $(m,n)$ fold induces an R-sliceable relation on $K$ and is certainly proper; thus $sK$ is defined. Since $sK \searrow K$, i.e., $sK \searrow iK$ which is homeomorphic to $K$, $sK$ and $K$ have the same homotopy type and thus $H(sK) = H(K)$.

Corollary 6.17: If $\{K_i, f_i\}_1^n$ is a simplicial fold factorization of an R-sliceable relation, then $H(sK) = H(K_i) = H(L)$, for all $i$.

Proof: Immediate from the above results.

## SECTION VII

## FURTHER TECHNIQUES

We will develop in this section some new techniques for collapsing $L \times I$. However, similarities between these new methods and the methods of the previous sections will become apparent, and we will conclude with an investigation of a few of these relationships.

Definition 7.1: A simplicial covering of a complex $L$ is a finite set of complexes $K_i$ and maps $f_i$ such that $f_i: K_i \to L$ is a simplicial homeomorphism into $L$ and $\bigcup_1^n f_i(K_i) = L$. We will let $[K_i,f_i]_1^n$ denote a simplicial covering of $L$.

Lemma 7.2: Any complex $L$ has a simplicial covering $[K_i,f_i]_1^n$ such that for all $i$, $K_i \searrow 0$.

Proof: The set of principal simplexes $\{s_i\}_1^n$ of $L$ (considered as a disjoint set) together with the embedding maps $f_i: s_i \to L$ such that $f(s_i) = s_i \subset L$, constitutes such a covering.

Definition 7.3: If $L$ has a simplicial covering $[K_i,f_i]_1^n$ such that for all $i$, $K_i \searrow 0$, we will call the covering of $L$ a collapsible covering.

Definition 7.4: For $s$ a simplex of $L$, $Ss = \{K_i: s \subset f_i(K_i)\}$. We can well order the set of complexes in $Ss$ by saying that if $K_i$ and $K_j$ are in $Ss$, $K_i < K_j$ iff $i < j$.

Lemma 7.5: If $[K_i,f_i]_1^n$ is a covering of $L$, then $L \times I \searrow cK$, where $cK$ is defined as follows:

$$cK = \bigcup_{1}^{n}(f_i(K_i) \times i/n+1)) \cup (\bigcup_{i,j} (p^{-1}(f_i(K_i) \cap f_j(K_j) \cap L \times [i/n+1,\ j/n+1])$$

Proof: Order the simplexes of $L$ in decreasing dimension. Then proceeding from the highest dimensional simplexes, collapse cells of the form $s \times [0,i/n+1] \searrow s \times (i/n+1) \cup \dot{s} \times [0,i/n+1]$, where $K_i$ is the first complex in $Ss$ and collapse cells of the form $s \times [1,\ j/n+1] \searrow s \times (j/n+1) \cup \dot{s} \times [1,j/n+1]$, where $K_j$ is the last complex in $Ss$. The remaining set is $cK$.

Lemma 7.6: If $[K_i,f_i]_1^n$ covers $L$ and $cK$ is defined as in the preceding lemma, then there exists a map $m: cK \to M$, where $M$ is a one-cell complex defined in the proof below.

Proof: First we consider $L_t \cap cK$ for $t = i/n+1$, $i = 1,2,\ldots,n$, and at each level, shrink the connected sets lying in $L_t \cap cK$ to points. Now we note that between any two levels $i/n+1$ and $i+1/n+1$ in the so modified $cK$ we will have a structure that consists of points on the "ends", i.e., at the levels $i/n+1$ and $i+1/n+1$ and connecting segments running between the two levels. Now if a point on level $i/n+1$ is connected to a point on level $(i+1)/n+1$, then the connecting segment is shrunk to a one-cell between the two points. We will consider this shrinking to preserve "levels". Thus, the inverse image of the point halfway between the endpoints of the one-cell will be a set lying on a segment in the modified $cK$ all of whose points have level $(i+\frac{1}{2})/n+1$.

It is conceivable that two points on different levels may have more than one distinct connecting segment, so in the final result, the two points will be connected by as many <u>one-cell</u> segments as there are "connecting" segments in the modified cK.

The end result is a one-cell complex M, and the map $m: cK \to M$ is merely the composition of all the shrinking steps. Because of the construction, we can meaningfully speak of the level of a point belonging to the cell complex M.

Lemma 7.7: If $[K_i, f_i]_1^n$ is a collapsible covering of L such that for all t, $L_t \cap cK$ is the disjoint union of collapsible complexes, then $m: cK \to M$ is strongly pointlike.

Proof: $m^{-1}(b)$, where b is a point of M at level t, is collapsible, as it is one of the collapsible sets in the disjoint union of $L_t \cap cK$.

Theorem 7.8: If $H(L) = 0$ and $[K_i, f_i]_1^n$ is a collapsible covering of L such that for all t, $L_t \cap cK$ is the disjoint union of collapsible complexes, then $L \times I \searrow 0$.

Proof: $L \times I \searrow cK$ and $m: cK \to M$ is strongly pointlike. By 6.14, as $H(L) = 0$, $H(cK) = 0$ and $H(M) = 0$. Now as M is a one-cell complex with zero homology, M is a tree; and thus it is collapsible. Hence $cK \searrow 0$ by the characterization of strongly pointlike maps [2].

Corollary 7.9: If $[K_i, f_i]_1^n$ is a collapsible covering of L such that $K_i \cap K_{i+1}$ is collapsible for all $i = 1,2,\ldots,n-1$, and $K_i \cap K_j = \emptyset$ if $j \neq i+1$, then $L \times I$ is collapsible.

Proof: The map $m: cK \to M$ is a strongly pointlike map onto a chain of n 1-simplexes.

An application of Corollary 7.9 is given below. Let $L_j$ be a copy of Bing's house with two rooms, and let L be made of 6 copies of $L_j$ arranged as pictured in Figure 10b, (Bing's Hotel perhaps).

We divide $L$ into 4 sets $K_i$, $i = 1,2,3$ and 4, by cutting $L$ with planes perpendicular to the $x,y$ plane and parallel to the $y$ axis passing through the points $x = 1,2$, and 3. Thus each $K_i$ is collapsible, and $K_i \cap K_{i+1}$ is either homeomorphic to a 2-cell or to the object shown in Figure 10d. In either case the intersection is collapsible, and thus Corollary 7.9 implies that $L \times I \searrow 0$. (Note that the same argument can be applied to an $m \times n$ array.)

Suppose we now consider the following situation. Let $L$ be a non-collapsible two-complex and suppose $L$ can be divided into two disjoint collapsible subsets $L_1$ and $L_2$ such that $L_1 \cap L_2$ is a tree. Corollary 7.9 would then imply that $L \times I$ is collapsible. However, we will now show that $L \times I$ is also collapsible by the methods of the previous section; i.e., we will find a collapsible complex $K$ and a simplicial map $f: K \to L$ such that the R-sliceable relation induced on $K$ has a proper fold factorization. Thus $L \times I$ is collapsible by Theorem 6.11, and, in this case at least, there is a relation between the various collapsing methods.

Theorem 7.10: If $L$ is a two-complex and $L$ can be separated by a tree $T$ into two collapsible subcomplexes, $L_1$ and $L_2$, then $L$ is the simplicial image under a map $f$ of a collapsible complex $K$ and $f: K \to L$ has a factorization as a sequence of (proper) (2,1) folds.

Proof: Select an extreme vertex $v_1$ of the tree and define $K$ to be the complex formed by identifying $L_1$ with $L_2$ at this extreme vertex. Certainly $K$ is collapsible as each "half" can be collapsed to the identified vertex $v_1$. $f: K \to L$ is merely the identification on the two copies of the tree that exist in both "halfs". The relation is R-sliceable since we can embed all the vertices of one half at level ¼ and the remaining vertices in the second half (excluding the identified extreme vertex) at level 3/4. The fact that we can realize the identification through a sequence of folds follows from the construction below.

Order the extreme vertices $\{v_i\}_1^n$ of the tree with that vertex selected as the identifying point of the two "halfs" as the first vertex $v_1$ in this ordered set. Now select the unique path in the tree that runs from the first vertex to the second vertex, the path that runs from the first to the third vertex, the first to the $i^{th}$ vertex, etc. Each of these paths is homeomorphic to a one-cell, and any two paths intersect in a subcomplex of the tree that is also homeomorphic to a one-cell and this subcomplex contains the first vertex. These facts follow from the properties of trees.

Order the set of one-simplexes $s$ of the tree $T$ as follows. $s_1, s_2, s_3, \ldots, s_i$ is the path from $v_1$ to $v_2$. Now $s_{i+1}$ is the first one-simplex in the path from $v_1$ to $v_3$ that has not already been ordered. $s_{i+2}, \ldots, s_j$ are the remaining simplexes in the path to $v_3$. $s_{j+1}$ is the first simplex in the path from $v_1$ to $v_4$ that has not been ordered. A simple inductive argument orders the one-simplexes of the tree.

Now $L_1$ contains a copy of the tree and so does $L_2$. For notation, $s_i$ will denote an ordered simplex of the tree in $L_1$, and $s_i'$ will denote a copy of the same simplex in $L_2$. In this notation the first complex of the factorization, $K_1$, has been formed by identifying $v_1$ of $s_1$ with $v_1'$ of $s_1'$. Inductively form $K_{i+1}$ from $K_i$ by folding $s_i$ onto $s_i'$. It is obvious that this construction gives a fold factorization of K onto L.

We conclude by asking if, given a non-collapsible two-complex L such that $L \times I$ is collapsible, the collapsing of the product space could always be done through the techniques of this section, say by an application of 7.9. A partial answer is obtained in the following theorem, though, in general, the question is unanswered. We include the theorem, however, because it leads into section VIII.

Theorem 7.11: The dunce hat is not separated by a (non-trivial) tree.

Proof: Suppose the dunce hat $D = K_1 \cup K_2$, where $K_1 \cap K_2 = T$, a tree. We consider the extreme points of T and note that for T to cut D "in two", T must cut the neighborhoods in D of any point of T "in two". In particular, consider the neighborhoods in D of the extreme points of T. By the structure of D we note that the neighborhoods of vertices v of D are of three possible forms. They are either homeomorphic to a two-cell, a three book, or to a pair of cones joined by a two cell. (See Figure 11.) Only one vertex of D can have a neighborhood of the third type.

Now since a neighborhood of an extreme vertex v of T must be separated by a one simplex which terminates at v, and since there is only one such neighborhood in D, namely the third one discussed above, we have contradicted our hypothesis, for a non-trivial tree must have at least two-extreme vertices.

This theorem shows that we cannot collapse $D \times I$ by using 7.9 with two collapsible subsets, even though we know $D \times I$ is collapsible.

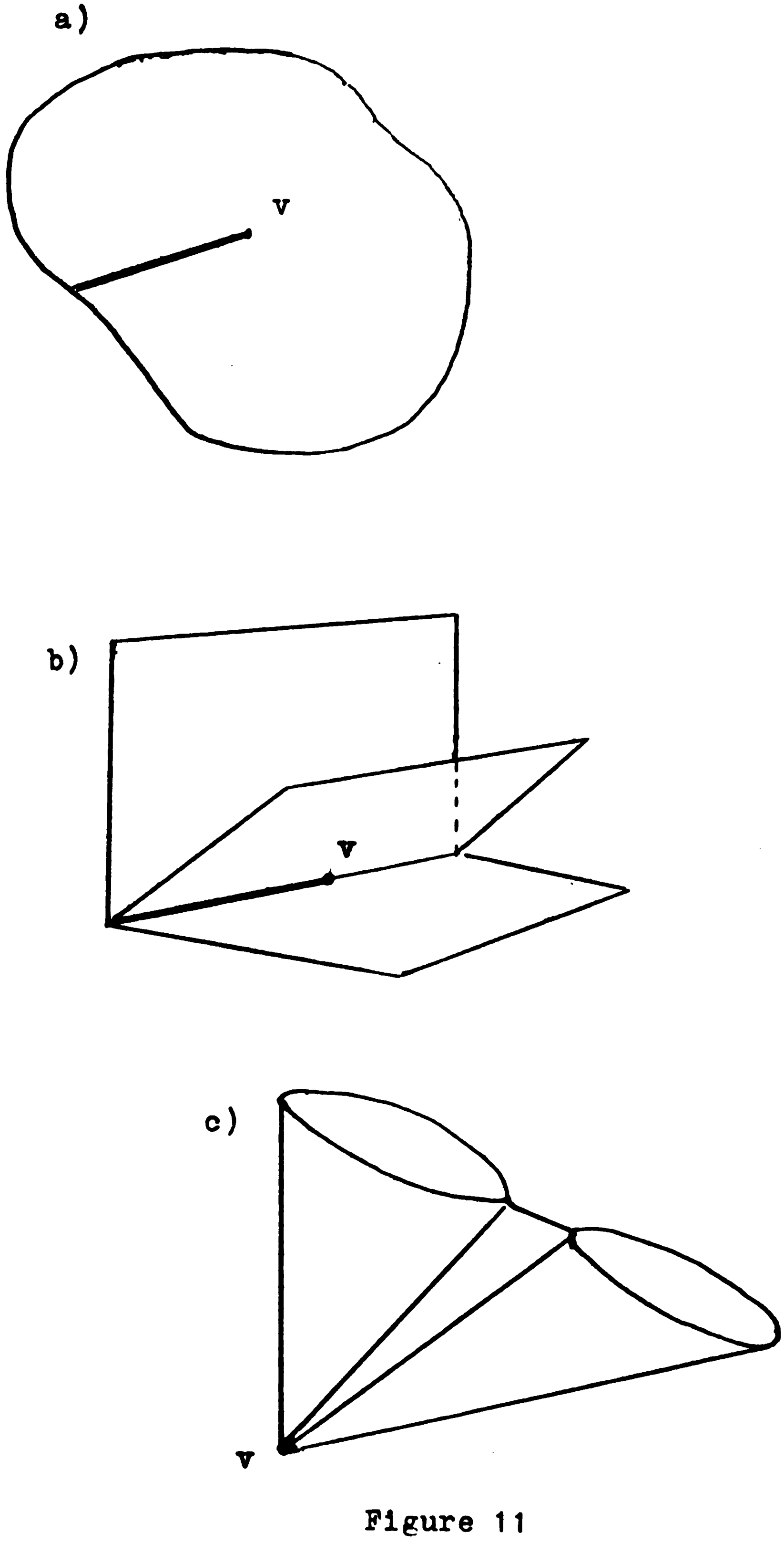



Figure 11

## SECTION VIII

## UNFOLDABLE STARS

In the proof of Theorem 7.11, we examined neighborhoods of vertices belonging to the dunce hat. One of these neighborhoods had the property that it could be separated by a single one-simplex. This phenomenon and its relationship to the collapsing of product spaces is examined in our concluding section. We will restrict ourselves to two-complexes and show that the dunce hat belongsto a class of complexes such that for any complex $L$ in this class, $L \times I$ is collapsible.

Let $K$ be a simplicial complex containing a vertex $v$.

<u>Definition 8.1</u>: $st(v)$ is said to be a proper star if $|st(v)-v|$ is connected. A complex is properly connected if each vertex has a properly connected star.

<u>Definition 8.2</u>: A properly connected star, $st(v)$ is said to be m-unfoldable if $|st(v) - s|$, $s \in st(v)$ is not connected for $s$ some m-simplex of $K$.

<u>Theorem 8.3</u>: Let $K$ and $L$ be properly connected 2-complexes and suppose $f\colon K \to L$ is a simplicial $(2,1)$ fold. Then $L$ has a 1-unfoldable star.

<u>Proof</u>: Let the fold $f$ be of $s_1 = \langle w, v_1\rangle$ onto $s_2 = \langle w, v_2\rangle$. That is $f$ identifies $s_1$ with $s_2$. We know that $|st(v_1)| \cap |st(v_2)| = w$, so let $v = v_1 = v_2$ in $L$ and consider $st(v) \subset L$.

We claim that $|st(v) - \langle w,v\rangle|$ is not connected. Since $|st(v)| \doteq |st(v_1)| \cup_f |st(v_2)|$, it is immediate that $|st(v_1)| \cup_f |st(v_2)| - |\langle w,v\rangle|$ is not connected.

Lemma 8.4: If L is a two-complex, L can be subdivided such that each two-simplex in the subdivision has the property that two of its one-faces each lie in at most only one other two-simplex.

Proof: A barycentric subdivision of L will have this property. Note that if L is properly connected, so is L'.

We will next consider the class of properly connected two complexes with the property that no one-simplex lies in more than 3 two-simplexes. From the above lemma, we can also assume that each two simplex in such a complex has at most one 1-face lying in more than two 2-simplexes. For notation let us call 2-complexes of this type 3-1 complexes.

Theorem 8.5: If L is a 3-1 complex with $st(v) \subset L$, 1-unfoldable, then there exists a 3-1 complex K and a simplicial map $f: K \to L$ such that f is a (2,1) fold and K has at least one two-simplex with a free edge.

Proof: For notation let $|L - st(v)| = Z$ and $|st(v) - \langle w,v\rangle| = Y$. As L is a proper 3-1 complex we have at most 3 disconnected subsets in Y.

Case 1: Suppose there are only two disjoint subsets in Y, say $X_1$ and $X_2$, and $\langle w,v\rangle$ is the face of only two 2-simplexes in L. Then $\bar{X}_1 \cap \bar{X}_2 = \langle w,v\rangle$. Select two disjoint copies of $\bar{X}_1$ and $\bar{X}_2$, say $\bar{W}_1$ and $\bar{W}_2$. Let $\langle w_1,v_1\rangle$ be the copy of $\langle w,v\rangle$ lying in $\bar{W}_1$, and likewise for $\langle w_2,v_2\rangle$ in $\bar{W}_2$. Join $\bar{W}_1$ to $\bar{W}_2$ by identifying the point $w_1$ with $w_2$. We can now form the complex K by attaching the space we have just formed to a copy of Z along $|lk(v)|$ in the natural way since $\overline{W}_1 \cup \overline{W}_2$ now contains a copy of $|lk(v)|$. By this construction, $f\colon K \to L$ is a (2,1) fold formed by identifying $\langle w,v_1\rangle$ with $\langle w,v_2\rangle$, and $K$ has a two simplex with a free edge.

Case 2: Suppose $Y$ contains two subsets $X_1$ and $X_2$ as before but that $\langle w,v\rangle$ is a face of three 2-simplexes in $L$. Then two of the 2-simplexes must lie in one of the pieces, say $\overline{X}_1$. Then $\overline{X}_2$ contains only one two-simplex with the edge $\langle w,v\rangle$. Proceed as in Case 1 to construct $K$.

Case 3: If $Y$ contains three subsets $X_1$, $X_2$, and $X_3$, then $\langle w,v\rangle$ is a face of 3 two-simplexes, and one lies in $\overline{X}_i$ for each $i$. As $\overline{X}_1 \cap \overline{X}_2 \cap \overline{X}_3 = \langle w,v\rangle$, let $\overline{W}_1$ be a copy of $\overline{X}_1 \cup \overline{X}_2$ and $\overline{W}_2$ be a copy of $\overline{X}_3$. The construction of Case 1 again gives the complex $K$.

For our final results we note that a triangulated 2-cell $L$ has the property that starting from any one-simplex in the boundary of the cell, $L$ can be collapsed onto its boundary less the starting simplex.

We will let $X$ denote the class of homologically trivial 3-1 complexes formed from triangulated 2-cells by 1-cell identifications on the boundary. For example, the dunce hat can be formed in this way by identifying the edges of a triangle.

Figure 12 shows how some elements of $X$ can be constructed. Observe that starting from a two-cell and folding together two one-simplexes on its boundary, we arrive at the situation pictured in part a. Thus we can get any number of "diagonal-type" one-cells by folding on the boundary of the starting two-cell. The remaining illustrations in Figure 12 assume the diagonal cells have already been formed.

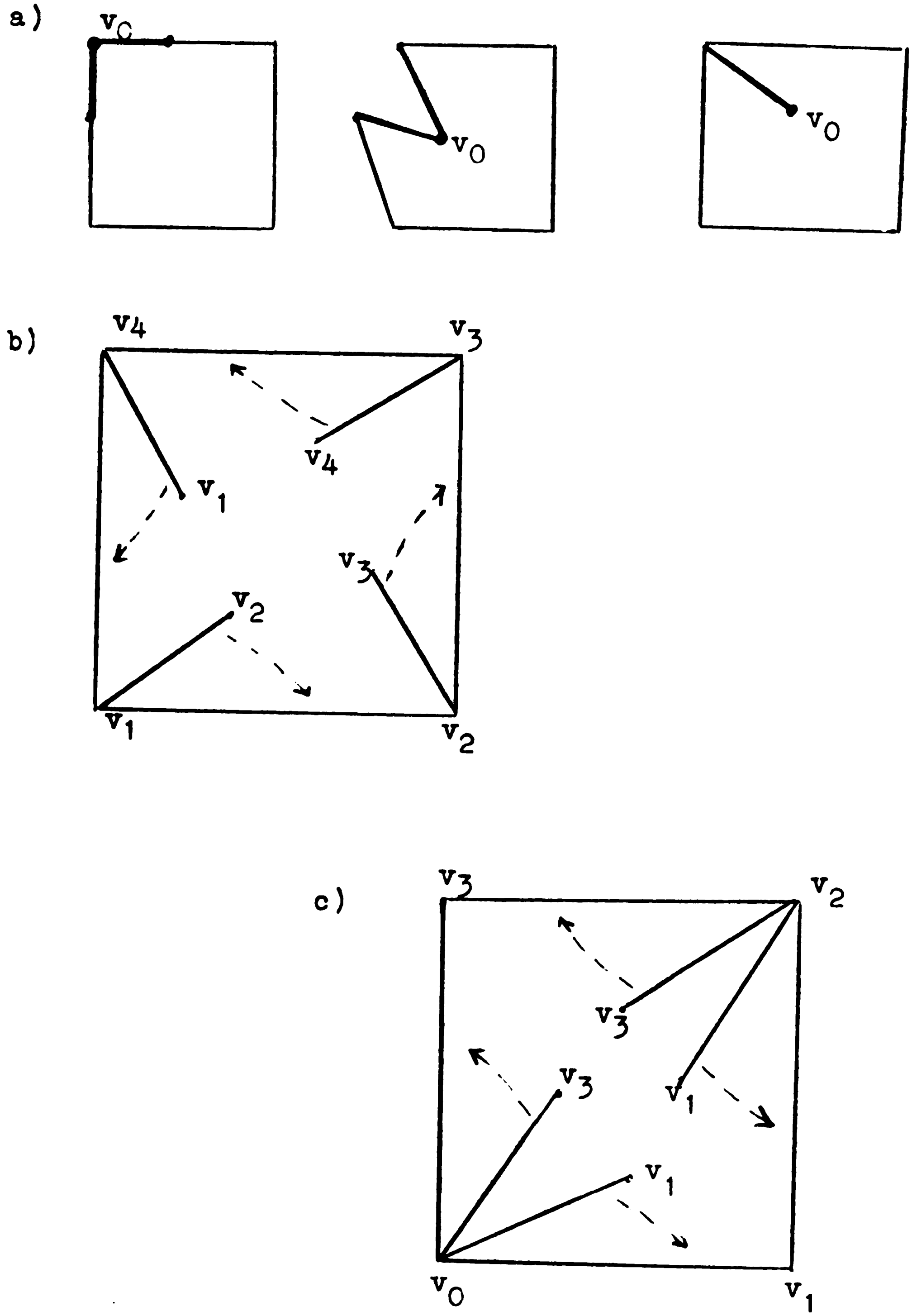



Figure 12

Theorem 8.6: If $L \in X$ and $L$ has a 1-unfoldable star, then $L \times I$ is collapsible.

Proof: By 8.5 we can find a complex $K$ with a free edge which maps onto $L$ by a fold. As $K$ is also in $X$ it can be collapsed onto a one-complex, namely that one-complex formed from the boundary identification of the original two-cell. $H(K) = 0$ implies that the one-complex is a tree $T$ and hence is collapsible. Thus $L \times I \searrow sK \searrow K \searrow T \searrow 0$, using 4.5 and 6.11 for the first two collapsings.

Corollary 8.7: If $K_1$ is a two-cell and $\{K_i, f_i\}_1^n$ is a simplicial fold factorization forming a 3-1 complex $L = K_n$ such that the closed seam of $K_1$ lies in its boundary, then $L \times I$ is collapsible.

Proof: $H(K_n) = 0$ by 6.17 and $K_n$ has an unfoldable star left by the last fold in the factorization.

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