THE EQUIVALENCE OF GAUSSIAN STOCHASTIC PROCESSES

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ABSTRACT

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by Hiroshi Oodaira

This thesis is concerned with the problem of the equivalence or singularity of two probability measures induced by two Gaussian stochastic processes. First, we consider the general case and obtain a set of necessary and sufficient conditions for equivalence in terms of mean functions and covariance functions of the processes. A proof of the equivalence-or-singularity dichotomy is obtained simultaneously. The method and techniques used in the present thesis are that of reproducing kernel Hilbert spaces. We derive several equivalent forms of necessary and sufficient conditions and show the equivalence of our results and other criteria obtained by E. Parzen and by J. Feldman. Gaussian measures in abstract Hilbert space are also considered. Next, we apply our conditions in the general case to special cases, and obtain some generalizations of A. V. Skorokhod' result in the additive case and of D. E. Varberg's result in the case of Gaussian processes with covariance kernels of

triangular form. We state conditions for the equivalence of stationary Gaussian processes in terms of their spectral distribution functions and, finally, consider a particular case of the equivalence problem of stationary Gaussian processes on finite intervals.

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INTRODUCTION

Suppose that two measures P and Q are defined on a measurable space $(\Omega, \mathcal{F})$. P is called absolutely continuous with respect to Q if P (A) = 0 for every $A \in \mathcal{F}$ for which Q (A) = 0. If P and Q are absolutely continuous with respect to one another, then they are called equivalent. If there is a set $B \in \mathcal{F}$ such that P (B) = 0 and $Q(\Omega - B) = 0$, then P and Q are said to be singular (or orthogonal or perpendicular). The equivalence and singularity of two measures represent two opposite extreme cases.

The problem of the equivalence or singularity of two probability measures induced by two Gaussian stochastic processes has recently received considerable attention, because of its importance in statistical inference theory as well as in structural problems of stochastic processes. It has been proved, by many authors in varying degrees of generality, that two Gaussian probability measures are either equivalent or singular. The proofs of the existence of such a dichotomy in the general case has been given by J. Hájek [11] and, independently, by J. Feldman [6], [7].

Since Gaussian probability measures are completely determined by the mean functions and covariance functions of the processes, it should be expected that necessary and sufficient conditions for equivalence are stated directly in terms of mean functions and covariance functions. From this point of view it seems that reproducing kernel Hilbert space is the most natural setup to formulate these conditions. The purpose of this thesis is twofold: (1) to obtain sets of necessary and sufficient conditions for equivalence, using the method and techniques of reproducing kernel Hilbert spaces; (2) applying these conditions, to give a unified treatment of results obtained by various authors, concerning additive, Markov, or stationary Gaussian processes.

The contents of this thesis are as follows. Chapter I deals with necessary and sufficient conditions for equivalence in the general case. In Section 1.1 we summarize known properties of reproducing kernels which we require and also give several lemmas. The statements and proofs of the main theorems are given in Section 1.2. An alternative proof of the dichotomy is also obtained. Several other equivalent forms of n.s. conditions are derived from the main theorems in Section 1.3 and 1.4. In particular, we derive E. Parzen's criterion [19] in Section 1.3 and J. Feldman's result [6] in Section 1.4. In Section 1.5 Gaussian measures in abstract

Hilbert space are considered. In Section 1.6 the relationship between our results and other recent work is discussed and various available methods are compared with each other. Chapter II is devoted to the study of the equivalence problem for important classes of Gaussian processes. In Sections 2.1 and 2.2 we specialize our general theorems to the cases of Gaussian additive processes and of Gaussian processes with covariance kernels of triangular form and obtain some generalizations of known results due, respectively, to A. V. Skorokhod [25] and D. E. Varberg [27], [28]. In Section 2.3 we discuss the stationary case. A particular case of Slepian-Feldman's result [8] is considered.

CHAPTER I

NECESSARY AND SUFFICIENT CONDITIONS FOR EQUIVALENCE IN THE ABSTRACT FORMULATION OF REPRODUCING KERNEL HILBERT SPACES

1.1 Preliminaries

Let $(\Omega, \mathcal{F})$ be a measurable space, where $\mathcal{F}$ is the σ field generated by a class of random variables $\{X(t),\ t \in T\}$, and let P and Q be two Gaussian measures on $(\Omega, \mathcal{F})$, i.e., probability measures such that $\{X(t),\ t \in T,\ P\}$ and $\{X(t),\ t \in T,\ Q\}$ are Gaussian processes. Throughout this paper we shall assume that the index set T is either countable or a separable metric space and, in the latter case, both processes are continuous in quadratic mean. Without any loss of generality, we may assume that the mean function of the process $\{X(t),\ t \in T,\ Q\}$ is zero. The mean function of $\{X(t),\ t \in T,\ P\}$ will be denoted by $m(t)$, and the covariance functions of both processes will be denoted by $\Gamma_P(s,t)$ and $\Gamma_Q(s,t)$ respectively, i.e.,

$$m(t) = E_p X(t) = \int_\Omega X(t)\, dP,$$

$$\Gamma_p(s,t) = E_p X(s)\, X(t) - m(s)m(t)$$

$$= \int_\Omega (X(s) - m(s))(X(t) - m(t))dP,$$

$$\Gamma_Q(s,t) = E_Q X(s)\, X(t) = \int_\Omega X(s)X(t)dQ.$$

We also write

$$\Lambda(s,t) = E_p X(s)X(t) = \int_\Omega X(s)X(t)dP, \text{ and}$$

$$M(s,t) = m(s)m(t).$$

As mentioned in Introduction, our principal technique is the theory of reproducing kernels. In this section we list several propositions which will be used constantly in the present thesis. For the details of this theory we refer to N. Aronszajn's papers [2] and [3].

Let $R(\cdot, \cdot)$ be a nonnegative definite kernel.

The reproducing kernel (r.k.) Hilbert space $H(R)$ with reproducing kernel (r.k.) $R(\cdot, \cdot)$ is a (real) Hilbert space, consisting of a class of (real valued) functions defined on a certain index set T, with the following properties:

(1) for every $t \in T$, $R(\cdot, t) \in H(R)$,

(2) (The reproducing property of R) for every $t \in T$ and every $f \in H(R)$,

$$f(t) = \langle f(\cdot), R(\cdot, t) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in H(R).

To construct H(R), consider the class of functions f of the form

$$f(\cdot) = \sum_{i=1}^{n} a_i R(\cdot, t_i), \quad t_i \in T,$$

and define the norm of f by

$$|||\, f \,|||^2 = \sum_{i,j=1}^{n} a_i a_j R(t_i, t_j).$$

H(R) is obtained by completing the class of functions of the above form with respect to the given norm. As easily seen from the construction, H(R) is spanned by functions $\{R(\cdot, t_i), t_i \in T\}$ and it is unique up to congruence. We shall find it convenient to denote by F the class of all elements in H(R), without topology, and to call R the r.k. of the class F.

We shall consider several different r.k. Hilbert spaces in this thesis, such as $H(\Gamma_Q)$, $H(\Gamma_P)$, etc. Their products and norms will be denoted by $\langle \cdot, \cdot \rangle_Q$, $|||\cdot|||_Q$, $\langle \cdot, \cdot \rangle_P$, $|||\cdot|||_P$, etc.

For any two kernels R_1 and R_2 we shall write $R_1 \ll R_2$ if $R_2 - R_1$ is nonnegative definite.

<u>Proposition 1.</u> If R and R_1 are the r.k.'s of the classes F and F_1 with the norms $|||\cdot|||$, $|||\cdot|||_1$ and if there is a

finite constant c such that $R_1 \ll cR$, then $F_1 \subset F$ (in particular, $R_1(\cdot,t) \in F$) and $|||\cdot|||_1^2 \geq c^{-1}|||\cdot|||^2$.

<u>Proposition 2.</u> If K_1 is a bounded linear operator on H(R), then there corresponds a kernel R_1 such that $R_1(\cdot,t)\in H(R)$ for every t and

$$K_1 f(t) = \langle f(\cdot), R_1(\cdot,t)\rangle .$$

<u>Proposition 3.</u> For any arbitrary symmetric kernel R_1, the necessary and sufficient condition that it correspond to a bounded self-adjoint operator on H(R) with lower bound $\geq c > -\infty$ and upper bound $\leq c' < +\infty$ is that $cR \ll R_1 \ll c'R$.

Let $T' = \{t_1, t_2, \ldots, t_m\}$ be any finite subset of T, and let $R_{T'}$ denote the m x m positive definite matrix $(R(t_i,t_j))_{1\leq i,j\leq m} = (R_{ij})_{1\leq i,j\leq m}$. The inverse matrix of $R_{T'}$ is denoted by $R_{T'}^{-1} = (R^{ij})_{1\leq i,j\leq m}$.

<u>Proposition 4.</u> The norm of the finite dimensional r.k. Hilbert space with r.k. $R_{T'}$ is given by

$$|||f_{T'}|||_{T'}^2 = \sum_{i,j=1}^{m} f_i R^{ij} f_j = {}^t f_{T'} R_{T'}^{-1} f_{T'} ,$$

where $f_{T'} \in H(R_{T'})$ and ${}^t f_{T'} = (f_1, f_2, \ldots, f_m)$ stands for the transposed vector (i.e., row vector).

<u>Proposition 5.</u> If R is the r.k. of the class F of functions defined on T with the norm $|||\cdot|||$, then R restricted to a

subset $T_1 \subset T$ is the r.k. of the class F_1 of all restriction of functions of F to T_1. For any such restriction, $f_1 \in F_1$, the norm $|||\, f_1 \,|||_1$ is the minimum of $|||\, f \,|||$ for all $f \in F$ whose restriction to T_1 is f_1.

Let $H(R_1)$ and $H(R_2)$ be r.k. Hilbert spaces with norms $|||\cdot|||_1$, $|||\cdot|||_2$. The direct product $H = H(R_1) \otimes H(R_2)$ is the completion of the class of functions $g(\cdot,*)$ of the form

$$g(\cdot,*) = g_1(\cdot)g_2(*),$$

where $g_1(\cdot) \in H(R_1)$ and $g_2(*) \in H(R_2)$, with respect to the norm

$$|||\, g \,|||^2 = |||\, g_1 \,|||^2 \cdot |||\, g_2 \,|||^2.$$

<u>Proposition 6.</u> The direct product $H = H(R_1) \otimes H(R_2)$ is a r.k. Hilbert space with r.k. $R(s_1,s_2,t_1,t_2) = R_1(s_1,t_1)R_2(s_2,t_2)$.

If $\{f_k\}$ and $\{g_\ell\}$ are complete orthonormal (c.o.n.) systems in $H(R_1)$ and $H(R_2)$, respectively, then $h_{k\ell}(\cdot,*) = f_k(\cdot)g_\ell(*)$ is a c.o.n. system in $H(R) = H(R_1) \otimes H(R_2)$, and any element Φ in $H(R)$ can be written in the form

$$\Phi(\cdot,*) = \sum_{k,\ell} \alpha_{k\ell} h_{k\ell}(\cdot,*) = \sum_{k,\ell} \alpha_{k\ell} f_k(\cdot) g_\ell(*)$$

with

$$\sum_{k,\ell} |\alpha_{k\ell}|^2 < \infty,$$

and vice versa.

The following theorem will be used in the proof of main theorem as well as in its specializations.

<u>Congruence theorem</u> ([16]). Let H_1 and H_2 be two abstract Hilbert spaces with inner products $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$, respectively. Let $\{f(t),\ t\in T\}$ be a class of elements which span H_1, and, similarly, let $\{g(t),\ t\in T\}$ be a class of elements which span H_2. If, for every s, $t\in T$,

$$(f(s),\ f(t))_1 = (g(s),\ g(t))_2,$$

then there exists a congruence (an isometric isomorphism) from H_1 onto H_2 such that, for every $t\in T$,

$$\Phi f(t) = g(t).$$

<u>Proposition 7</u>. (L_2-representation of a r.k. Hilbert space $H(R)$) ([16]). If there are a measure space $(B, \mathcal{B}, \mu)$ and a class of functions $\{\Phi(t),\ t\in T\}$ in $L_2(B, \mathcal{B}, \mu)$ such that for all s,t T

$$R(s,t) = \int_B \Phi(s)\Phi(t)d\ \mu,$$

then $H(R)$ is congruent to the Hilbert subspace $L_2(\Phi(t),\ t\in T)$ of $L_2(B, \mathcal{B}, \mu)$ spanned by $\{\Phi(t),\ t\in T\}$. That is, any element $f(\cdot)\in H(R)$ is represented in the form

$$f(t) = \int_B \Phi(t) g d\mu\ ,$$

for all $t \in T$, with $g \in L_2(\Phi(t), t \in T)$, and, conversely, any element $g \in L_2(\Phi(t), t \in T)$ determines an element $f \in H(R)$ by the above relation.

Examples of L_2-representation will be given in §§2.1 and 2.2.

We shall also consider Hilbert spaces spanned by stochastic processes. First, let $\mathcal{L}(X)$ denote the class of all finite linear combinations of $\{X(t), t \in T\}$, i.e., the class of all random variables Y of the form

$$Y = \sum_{i=1}^{n} a_i X(t_i),$$

where $a_1, a_2, \ldots, a_n$ are real constants and $t_1, t_2, \ldots, t_n \in T$. Let $\mathcal{L}_2(X,P)$ denote the class of all random variables Z such that there exists a sequence of random variables Y_n in $\mathcal{L}(X)$ converging to Z in quadratic mean with respect to P. Define $L_2(X,P)$ to be the set of all equivalence classes of random variables in $\mathcal{L}_2(X,P)$ modulo the class of random variables $Z \in \mathcal{L}_2(X,P)$ with $E_p Z^2 = 0$. We denote by $\overline{Z}^P$ the equivalence class in $L_2(X,P)$ to which Z belongs. $\mathcal{L}_2(X,Q)$, $L_2(X,Q)$, $\overline{Z}^Q$ are defined in the similar manner. Clearly $L_2(X,P)$ and $L_2(X,Q)$ are separable Hilbert spaces with norms $\|\overline{Z}^P\|^2 = E_p Z^2$, $\|\overline{Z}^Q\|^2 = E_Q Z^2$. Their inner products will be denoted by $(\cdot,\cdot)_p$ and $(\cdot,\cdot)_Q$.

We introduce another Hilbert space $L_2^*(X)$ in the following manner. Suppose that there are finite positive constants c and c' such that for every $Y \in \mathcal{L}(X)$

$$(1.1) \qquad cE_Q Y^2 \leq E_p Y^2 \leq c' E_Q Y^2.$$

Let $\mathcal{L}_2^*(X)$ be the class of all random variables Z such that there exists a sequence of random variables Y_n in $\mathcal{L}(X)$ converging to Z in quadratic mean with respect to both P and Q, and let $\mathcal{N}$ be the class of random variables Z in $\mathcal{L}_2^*(X)$ with $E_Q Z^2 = 0$. Note that from assumption (1.1) and the definition of $\mathcal{L}_2^*(X)$ it follows that $E_Q Z^2 = 0$ implies $E_p Z^2 = 0$ and vice versa. Define $L_2^*(X) = \mathcal{L}_2^*(X)/\mathcal{N}$, the set of all equivalence classes modulo $\mathcal{N}$ of elements in $\mathcal{L}_2^*(X)$. Denote by $\bar{Z}$ the equivalence class in $L_2^*(X)$ to which Z belongs. It may be shown that $L_2^*(X)$ is a separable Hilbert space with inner product $(\bar{Z}_1, \bar{Z}_2) = E_Q Z_1 Z_2$ and norm $\| \bar{Z} \|^2 = E_Q Z^2$. The verification does not offer any difficulty, but it is somewhat lengthy and is therefore omitted. In passing, it should be observed that the class of elements in $L_2^*(X)$ can be endowed with different (but equivalent) norm $\| \bar{Z} \|_p^2 = E_p Z^2$.

We shall write $H_1 \cong H_2$ if two Hilbert spaces H_1 and H_2 are congruent.

Lemma 1.1.1

(1.2) $\qquad L_2(X,P) \cong H(\Lambda)$

(1.3) $\qquad L_2(X,Q) \cong H(\Gamma_Q)$,

and, if there exist finite positive constants c and c' with

$$c\Gamma_Q << \Lambda << c'\Gamma_Q ,$$

then (the space $L_2^*(X)$ can be defined, and)

(1.4) $\qquad L_2^*(X) \cong H(\Gamma_Q) \cong L_2(X,Q).$

Proof. (1.2): Since

$$(\overline{X(s)}^P, \overline{X(t)}^P)_p = \Lambda(s,t) = \langle \Lambda(\cdot,s), \Lambda(\cdot,t) \rangle_p$$

and both $\{\overline{X(t)}^P\}$ and $\{\Lambda(\cdot,t)\}$ span $L_2(X,P)$ and $H(\Lambda)$, respectively, the assertion (1.2) is obtained by applying the congruence theorem. The proofs of (1.3) and 1.4) are quite similar.

Since

$$\begin{aligned} \sum_{i,j=1}^{n} a_i a_j M(t_i,t_j) &= \sum_{i,j=1}^{n} a_i a_j m(t_i) m(t_j) \\ &= (\sum_{i=1}^{n} a_i m(t_i))^2 \\ &\geq 0 , \end{aligned}$$

the kernel $M(s,t)$ is nonnegative definite, i.e., $0 << M$.

Hence, $0 << M = \Lambda - \Gamma_p$, i.e.,

$$\Gamma_p << \Lambda.$$

Assume the existence of positive finite constants c_1 and c_2 with

$$c_1\Gamma_Q << \Gamma_p << \Lambda << c_2\Gamma_Q .$$

Then, $\Gamma_p(\cdot,t)\in H(\Gamma_Q)$ and, by Proposition 3, Γ_p determines a bounded self-adjoint linear operator S on $H(\Gamma_Q)$ by the relation $Sf(t) = \langle f(\cdot), \Gamma_p(\cdot,t)\rangle_Q$. Note that

$$\begin{aligned} S\Gamma_Q(s,t) &= \langle \Gamma_Q(\cdot,t), \Gamma_p(\cdot,s)\rangle_Q \\ &= \langle \Gamma_p(\cdot,s), \Gamma_Q(\cdot,t)\rangle_Q \\ &= \Gamma_p(s,t). \end{aligned}$$

Our assumption enables us to define $L_2^*(X)$ and, by Lemma 1.1.1, there is a congruence Φ from $H(\Gamma_Q)$ to $L_2^*(X)$.

<u>Lemma 1.1.2</u> For $f_1, f_2 \in H(\Gamma_Q)$,

$$\begin{aligned} \langle Sf_1, f_2\rangle_Q &= \mathrm{Cov}_p(Z_1,Z_2) , \\ &= E_pZ_1Z_2 - (E_pZ_1)(E_pZ_2) , \end{aligned}$$

where $Z_1\in\Phi f_1$ and $Z_2\in\Phi f_2$.

Proof. If f_1 and f_2 are of the form

$$f_1(\cdot) = \sum_{i=1}^{m} a_i\Gamma_Q(\cdot,s_i) , \quad f_2(\cdot) = \sum_{j=1}^{n} b_j\Gamma_Q(\cdot,t_j),$$

then

$$\langle Sf_1, f_2\rangle_Q = \langle \langle f_1(t), \Gamma_P(t,\cdot)\rangle_Q, \; f_2(\cdot)\rangle_Q$$

$$= \langle \sum_{i=1}^{m} a_i \Gamma_P(\cdot, s_i), \; \sum_{j=1}^{n} b_j \Gamma_Q(\cdot, t_j)\rangle_Q$$

$$= \sum_{i,j=1}^{m,n} a_i b_j \Gamma_P(s_i, t_j)$$

$$= \mathrm{Cov}_P \left(\sum_{i=1}^{m} a_i X(t_i), \; \sum_{j=1}^{n} b_j X(t_j)\right) .$$

For any $f \in H(\Gamma_Q)$ there is a sequence $\{f_n\}$ of elements of the above form which converges to f in norm. This implies that $Z_n \in \Phi f_n$ converges to $Z \in \Phi f$ in $\| \cdot \|_Q$-norm, and hence, in $\| \cdot \|_P$-norm. This proves the lemma.

We need the following known fact on which the proof of main theorem is based. (See [6]).

<u>Lemma 1.1.3</u> Let Z_k's be random variables that are independent, normally distributed with mean m_k and variance $v_k > 0$ with respect to P and independent, normally distributed with mean 0 and variance 1 with respect to Q. Let $\mathcal{A}_n$ denote the σ-field generated by $Z_1, Z_2, \ldots, Z_n$ and $\mathcal{A}$ the minimal σ-field containing the union of all $\mathcal{A}_n$, i.e., $\mathcal{A} = \bigvee_{n=1}^{\infty} \mathcal{A}_n$.

If

$$\sum_{k=1}^{\infty} (1-v_k)^2 < \infty \text{ and } \sum_{k=1}^{\infty} m_k^2 < \infty ,$$

then P and Q are equivalent on $\mathcal{A}$. If

$$\sum_{k=1}^{\infty} (1-v_k)^2 = \infty \text{ or } \sum_{k=1}^{\infty} m_k^2 = \infty ,$$

then P and Q are singular on $\mathcal{A}$.

The lemma can be proven by applying the following S. Kakutani's effective criterion for the equivalence of infinite product measures.

Let μ and ν be two probability measures on a measurable space $(\Omega, \mathcal{F})$, and let τ be a measure which dominates both μ and ν. Consider

$$\rho(\mu,\nu) = \int_{\Omega} \left(\frac{d\mu}{d\tau}\right)^{1/2} \left(\frac{d\nu}{d\tau}\right)^{1/2} d\tau.$$

It may be shown that $\rho(\mu,\nu)$ is independent of the choice of τ.

Let $(\Omega_n, \mathcal{F}_n)$ be a sequence of measurable spaces, and assume that, for each n, there exists two equivalent probability measures μ_n and ν_n on $(\Omega_n, \mathcal{F}_n)$. Consider infinite product measures $\mu = \otimes \mu_n$ and $\nu = \otimes \nu_n$ defined on $(\Omega = \otimes\Omega_n, \mathcal{F} = \otimes \mathcal{F}_n)$.

<u>Theorem</u> ([13]) μ and ν are either equivalent or singular

according as $\prod_{n}^{\infty} \rho\ (\mu_n, \nu_n)$ is > 0 or $= 0$. Moreover

$$\rho(\mu, \nu) = \prod_{n=1}^{\infty} \rho(\mu_n, \nu_n).$$

1.2 Main Theorems

Theorem 1.2.1 If P and Q are not singular, then the following conditions are necessary.

(1) $m(\cdot) \in H(\Gamma_Q)$.

(2) There exist finite positive constants c_1 and c_2 with $c_1\Gamma_Q << \Gamma_P << c_2\Gamma_Q$.

(3) S has a pure point spectrum.

(4) The eigenvalues $\{\lambda_k\}$ of S satisfy the relation $\sum_{k=1}^{\infty} (1-\lambda_k)^2 < \infty$.

Proof. First we prove (2): It suffices to prove the first part, $c_1\Gamma_Q << \Gamma_P$, since the other part can be proved analogously. Suppose that there is no such a positive constant c_1. Then, for each n, there are a sequence of vectors $a^n = (a_1^n, a_2^n, \ldots, a_{k_n}^n)$ and a sequence of finite sets $T^n = (t_1^n, t_2^n, \ldots, t_{k_n}^n) \subset T$ such that

$$\sum_{i,j=1}^{k_n} a_i^n a_j^n \Gamma_P(t_i^n, t_j^n) < n^{-1} \sum_{j=1}^{k_n} a_i^n a_j^n \Gamma_Q(t_i^n, t_j^n) .$$

Put

$$b_i^n = a_i^n \left(\sum_{i,j=1}^{k_n} a_i^n a_j^n \Gamma_Q(t_i^n, t_j^n) \right)^{-1/2}.$$

Then $Y_n = \sum_{i=1}^{k_n} b_i^n X(t_i^n)$ is normally distributed with mean

$m_n = \sum_{i=1}^{k_n} b_i^n m(t_i^n)$ and variance $\sigma_n^2 = \sum_{i,j=1}^{k_n} b_i^n b_j^n \Gamma_p(t_i^n, t_j^n) < 1/n$ with

respect to P and normally distributed with mean 0 and variance

$1 = \sum_{i,j=1}^{k_n} b_i^n b_j^n \Gamma_Q(t_i^n, t_j^n)$ with respect to Q.

Let $C_n = \{\omega : | Y_n(\omega) - m_n | < \sigma_n^{1/2}\}$. Then we have

$$P(C_n) = \frac{1}{\sqrt{2\pi}\,\sigma_n} \int_{|t-m_n| < \sigma_n^{1/2}} e^{-(t-m_n)^2/2\sigma_n^2} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{|s| < \sigma_n^{-1/2}} e^{-s^2/2} \, ds \to 1,$$

as $n \to \infty$, and

$$Q(C_n) = \frac{1}{\sqrt{2\pi}} \int_{|t-m_n| < \sigma_n^{1/2}} e^{-t^2/2} \, dt \to 0,$$

as $n \to \infty$. This implies that P and Q are singular.

(1): It is sufficient to show the existence of a finite constant c with $M \ll c\Gamma_Q$. For, if $M \ll c\Gamma_Q$, it implies $M(\cdot,t) = m(\cdot)m(t) \in H(\Gamma_Q)$, and hence, $m(\cdot) = m(t)^{-1}M(\cdot,t) \in H(\Gamma_Q)$ for t with $m(t) \neq 0$. If $m(t) = 0$ for all t, $m(\cdot) = 0 \in H(\Gamma_Q)$.

Now, assume that there is no finite constant c with $M \ll c\Gamma_Q$, that is, for every n there exists a sequence of vectors $a^n = (a_1^n, a_2^n, \ldots, a_{k_n}^n)$ and a sequence of finite sets $T^n = (t_1^n, t_2^n, \ldots, t_{k_n}^n) \subset T$ such that

$$n^2 \sum_{i,j=1}^{k_n} a_i^n a_j^n \Gamma_Q(t_i^n, t_j^n) < \sum_{i,j=1}^{k_n} a_i^n a_j^n M(t_i^n, t_j^n).$$

Put

$$b_i^n = n^{-1/2} \left(\sum_{i,j=1}^{k_n} a_i^n a_j^n \Gamma_Q(t_i^n, t_j^n) \right)^{-1/2} a_i^n .$$

Let $Y_n = \sum_{i=1}^{k_n} b_i^n X(t_i^n)$. We may assume $m_n = E_p Y_n = \sum_{i=1}^{k_n} b_i^n m(t_i^n) > 0$, since, if necessary, we may take $-Y_n$ instead of Y_n. Y_n is normally distributed with mean 0 and variance $1/n$ with respect to Q, and normally distributed with mean m_n and variance σ_n^2 with respect to P. Note that $m_n > n^{1/2}$ and $\sigma_n^2 \leq c_2/n$, where c_2 is the constant in (2). Letting $C_n = \{\omega : |Y_n(\omega) - m_n| < \sigma_n^{1/2}\}$, we have, as $n \to \infty$,

$$P(C_n) = \frac{1}{\sqrt{2\pi}\sigma_n} \int_{|t-m_n| < \sigma_n^{1/2}} e^{-(t-m_n)^2/2\sigma_n^2} dt \to 1 ,$$

and

$$Q(C_n) = \frac{n^{1/2}}{\sqrt{2\pi}} \int_{|t-m_n| < \sigma_n^{1/2}} e^{-nt^2/2} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{s \geq n^{1/2} m_n - n^{1/4} c_2^{1/4}} e^{-s^2/2} ds \quad \to 0.$$

This is a contradiction.

(3): In view of (1), if P and Q are not singular, we can define the operator S corresponding to Γ_P, and, since $0 << M << c\Gamma_Q$ (the proof of (2)), we have $c_1\Gamma_Q << \Lambda << c'\Gamma_Q$ taking $c' = c_2 + c$. Hence, we can define the space $L_2^*(S)$, and, by Lemma 1.1.1, there is a congruence Φ from $H(\Gamma_Q)$ onto $L_2^*(X)$.

The spectrum of S is a nonempty bounded closed set, since S is bounded and self-adjoint. By a limit point of the spectrum of S we mean a point of the continuous spectrum of S, a limit of eigenvalues of S or an eigenvalue of S of infinite multiplicity ([22]). Denote by $\{E_\lambda\}$ (λ real) the resolution of the identity determined by S. We write $E(\Delta) = E_{b-0} - E_a$ if $\Delta = (a,b)$.

First we show that any point $\alpha \neq 1$ cannot be a limit point of the spectrum of S. If the contrary is true, then there is a monotone decreasing sequences of intervals $\Delta_k = (a_k, b_k)$ containing α with $a_k \longrightarrow \alpha$ and $b_k \longrightarrow \alpha$ such that we can choose η_k from $E(\Delta_k)H(\Gamma_Q)$ with $\langle \eta_k, \eta_j \rangle_Q = \delta_{kj}$ (δ_{kj} is Kronecker's delta) and $\langle S\eta_k, \eta_i \rangle_Q = 0$ for $k \neq j$. Then we have

$$a_k \leq \langle S\eta_k, \eta_k \rangle_Q = \int_{\Delta_k} \lambda d(E_\lambda \eta_k, \eta_k) \leq b_k .$$

Let $Z_k \in \Phi\eta_k$, $k = 1,2,\ldots,$ Then Z_k's are independent, normally distributed with mean $m_k = E_p Z_k$ and variance $v_k = \langle S\eta_k, \eta_k \rangle_Q$ with respect to P (Lemma 1.1.2) and independent, normally distributed with mean 0 and variance 1 with respect to Q. Since a_k and b_k tend to α as $k \longrightarrow \infty$, v_k goes to α which is not 1 by assumption. Hence $\sum_{k=1}^{\infty} (1-v_k)^2 = \infty$, which shows that P and Q are singular (Lemma 1.1.3).

Thus, if 1 is not a point of the spectrum of S, the spectrum of S consists of a finite number of eigenvalues of finite multiplicity. On the other hand, if 1 is a point of the spectrum of S, then it is either an eigenvalue of S or a limit of eigenvalues of S. For, if 1 is not a limit of eigenvalues, there exists an interval $(1-\epsilon, 1+\epsilon)$ such that

$E_{1-\epsilon} = E_{1+\epsilon}$ and every point in the set $(1-\epsilon,1)\cup(1,1+\epsilon)$ belongs to the resolvent set of S, since any point different from 1 is not a limit point of the spectrum. This implies $E_1 \neq E_{1-0}$, i.e., 1 is an eigenvalue of S. Hence, the spectrum of S consists of either eigenvalues or eigenvalues and their limit points (in fact, 1 is the only possible limit of eigenvalues), i.e., it is a pure point spectrum.

(4): Let $\{\lambda_k\}$ be the set of all eigenvalues of S and let $\{\eta_k\}$ be corresponding orthonormal eigenvectors, i.e., $\langle\eta_k,\eta_j\rangle_Q = \delta_{kj}$. Note that if $\lambda = \lambda_{k+1} = \dots = \lambda_{k+\ell-1}$, where ℓ is the multiplicity of the eigenvalue λ, then $\eta_k,\eta_{k+1},\dots,\eta_{k+\ell-1}$ span the subspace $(E_\lambda - E_{\lambda-0})H(\Gamma_Q)$ and $\{\eta_k\}$ is a c.o.n. system in $H(\Gamma_Q)$. Let $Z_k \epsilon \Phi\eta_k$, $k = 1,2,\dots$ The relations $\langle\eta_k,\eta_j\rangle_Q = \delta_{kj}$ and $\langle S\eta_k,\eta_j\rangle = \lambda_k\delta_{kj}$ imply that Z_k's are independent, normally distributed with mean $m_k = E_p Z_k$ and variance λ_k with respect to P (Lemma 1.1.2) and independent, normally distributed with mean 0 and variance 1 with respect to Q. The assertion then follows from Lemma 1.1.3. This completes the proof of the theorem.

<u>Theorem 1.2.2.</u> If conditions (1)-(4) of Theorem 1.2.1 are fulfilled, then P and Q are equivalent.

Proof. By condition (1), $M(\cdot,t) = m(\cdot)m(t)\epsilon H(\Gamma_Q)$ for every $t\epsilon T$. Then the relation $\langle f(\cdot),M(\cdot,t)\rangle_Q = \langle f(\cdot),m(\cdot)\rangle_Q m(t)$

defines a linear operator K on $H(\Gamma_Q)$, that is, $Kf(\cdot) = \langle f(\cdot), m(\cdot)\rangle_Q m(\cdot)$. K is bounded and self-adjoint, since $|||kf|||_Q^2 \leq |||f|||_Q^2 \cdot |||m|||_Q^4$ and $\langle Kf_1, f_2\rangle_Q = \langle f_1, m\rangle_Q \langle f_2, m\rangle_Q = \langle f_1, kf_2\rangle_Q$. Hence $0 << M << |||m|||_Q^2 \Gamma_Q$. Since $\Lambda = \Gamma_p + M$, this relation and condition (2) imply

(2.1) $$c_1 \Gamma_Q << \Lambda << c' \Gamma_Q ,$$

where $c' = c2 + |||m|||_Q^2$. This enables us to define the space $L_2^*(X)$. By Lemma 1.1.1, there is a congruence Φ from $H(\Gamma_Q)$ onto $L_2^*(S)$.

Conditions (2) makes it possible to define the operator S on $H(\Gamma_Q)$. Conditions (3) then implies that the normalized eigenvectors $\{\eta_k\}$ of S is a c.o.n. system in $H(\Gamma_Q)$. Let $Z_k \in \Phi\eta_k$, k=1,2,... Let $\mathcal{A}_n$ be the σ-field generated by $Z_1, Z_2, \ldots, Z_n$, and let $\mathcal{A} = \bigvee_{n=1}^{\infty} \mathcal{A}_n$. Consider an arbitrary element Y in $\mathcal{L}_2^*(X)$ and define

$$Y_n = \sum_{k=1}^{n} (\overline{Y}, \overline{Z}_k)_Q Z_k = \sum_{k=1}^{n} (Y, \Phi\eta_k)_Q Z_k .$$

Y_n's are $\mathcal{A}_n$-measurable and, a fortiori, $\mathcal{A}$-measurable. Since $\{\eta_k\}$ is a c.o.n. system, so is $\{\Phi\eta_k\}$, and we have

$$\| \overline{Y} - \overline{Y}_n \|_Q = \| \overline{Y} - \sum_{k=1}^{n} (\overline{Y}, \Phi\eta_k)\Phi\eta_k \|_Q \longrightarrow 0 ,$$

as $n \to \infty$, and, by (2.1),

$$\| \bar{Y} - \bar{Y}_n \|_p \to 0$$

as $n \to \infty$. Since $\| \bar{Y}_n - \bar{Y}_m \|_Q \to 0$ as $n, m \to \infty$, there is a subsequence $\{Y_{n_k}\}$ converging to an $\mathcal{A}$-measurable function Y^* on the set $D \in \mathcal{A}$, where $D = \{\omega: \overline{\lim}\, Y_{n_k}(\omega) = \underline{\lim}\, Y_{n_k}(\omega)\}$. $Y^*(\omega) = \lim Y_{n_k}(\omega)$ on D, 0 elsewhere, and $Q(D) = 1$. Observe that $\| \bar{Y}_n - \bar{Y}^* \|_Q \to 0$ as $n \to \infty$. Again by (2.1), there is a subsequence $\{Y_{m_j}\}$ of the sequence $\{Y_{n_k}\}$ which converges to an $\mathcal{A}$-measurable function Y^{**} on the set $\bar{D} = \{\omega: \overline{\lim}\, Y_{m_j}(\omega) = \underline{\lim}\, Y_{m_j}(\omega)\}$ of P-measure 1 and $Y^{**}(\omega) = \lim Y_{m_j}(\omega)$ on $\bar{D}$ and 0 elsewhere. Since $\{Y_{m_j}\}$ is a subsequence of $\{Y_{n_k}\}$, $D \subset \bar{D}$, and, hence, $Q(\bar{D}) = 1$ and $Y^{**}(\omega) = Y^*(\omega)$ on D.

Therefore,

$$\| \bar{Y}_n - \bar{Y}^{**} \|_Q^2 = \int_D (Y_n - Y^*)^2 dQ = \| \bar{Y}_n - \bar{Y}^* \|_Q^2 \to 0.$$

as $n \to \infty$. Furthermore, by (2.1),

$$\| \bar{Y}_n - \bar{Y}^{**} \|_p \leq \| \bar{Y}_n - \bar{Y}_{m_j} \|_p + \| \bar{Y}_{m_j} - \bar{Y}^{**} \|_p \to 0$$

as $n \to \infty$. Hence

$$\| \bar{Y} - \bar{Y}^{**} \|_Q = \| \bar{Y} - \bar{Y}^{**} \|_p = 0. \tag{2.2}$$

Define $\mathcal{A}^* = \{A \cup N : A \in \mathcal{A},\ N \in \mathcal{F},\ P(N) = Q(N) = 0\}$. (2.2)

shows that every random variable Y in $\mathcal{L}_2^*(X)$ is $\mathcal{A}^*$-measurable and, hence, $\mathcal{F} \subset \mathcal{A}^*$.

Since $Z_k \in \mathcal{L}_2^*(X)$, there is a sequence of random variables $\{Y_n^k\}$ in $\mathcal{L}(X)$, that is, of the form

$$Y_n^k = \sum_{i=1}^{\ell(k,n)} a_{ni}^k\, X\,(t_{ni}^k),$$

converging to Z_k in quadratic mean with respect to both P and Q.

Noting that $\Phi^{-1}\bar{Y}_n^k = \sum_{i=1}^{\ell(k,n)} a_{ni}^k\, \Gamma_Q(\cdot, t_{ni}^k)$ converges to $\Phi^{-1}\bar{Z}_k = \eta_k$,

we have

$$m_k = E_p Z_k = \lim_n E_p Y_n^k$$

$$= \lim_n \sum_{i=1}^{\ell(k,n)} a_{ni}^k\, E_p X\,(t_{ni}^k) = \lim_n \sum_{i=1}^{\ell(k,n)} a_{ni}^k m(t_{ni}^k)$$

$$= \lim_n \sum_{i=1}^{\ell(k,n)} a_{ni}^k \langle m(\cdot), \Gamma_Q(\cdot, t_{ni}^k)\rangle_Q$$

$$= \lim_n \langle m(\cdot), \sum_{i=1}^{\ell(k,n)} a_{ni}^k \Gamma_Q(\cdot, t_{ni}^k)\rangle_Q$$

$$= \langle m(\cdot), \eta_k(\cdot)\rangle_Q.$$

Since $m(\cdot)\in H(\Gamma_Q)$ (condition (1)),

$$(2.3) \qquad \sum_{k=1}^{\infty} m_k^2 = \sum_{k=1}^{\infty} \langle m(\cdot), \eta_k(\cdot)\rangle_Q^2 = |||\, m(\cdot)\, |||_Q^2 < \infty .$$

z_k's are independent, normally distributed with mean m_k and variance λ_k with respect to P (Lemma 1.1.2), and independent, normally distributed with mean 0 and variance 1 with respect to Q. The positivity of λ_k follows from condition (2). Condition (3), the relation (2.3) and Lemma 1.1.3 together imply that P and Q are equivalent on $\mathfrak{A}$. If $E = A \cup N \in \mathfrak{A}^*$ and $P(E) = 0$, then $Q(E) \leq Q(A) + Q(N) = 0$ since $P(A) = 0$. Similarly $Q(E) = 0$ implies $P(E) = 0$. Hence P and Q are equivalent on $\mathfrak{A}^*$, and, hence, on $\mathfrak{F}$. This completes the proof.

Summing up, we obtain the following main theorem.

<u>Theorem 1.2.3</u> P and Q are either equivalent or singular. For the equivalence of P and Q it is necessary and sufficient that

(1) $m(\cdot) \in H(\Gamma_Q)$,

(2) there exist finite positive constants c_1 and c_2 such that

$$c_1\Gamma_Q << \Gamma_P << c_2\Gamma_Q,$$

(3) S has a pure point spectrum, and

(4) the eigenvalues $\{\lambda_k\}$ of S satisfy

$$\sum_{k=1}^{\infty} (1-\lambda_k)^2 < \infty .$$

<u>Remark.</u> Conditions (3) and (4) may be replaced by

(3') I - S is a Hilbert-Schmidt operator.

<u>Theorem 1.2.4</u> The following condition is equivalent to (2)-(4) of Theorem 1.2.3.

Γ_p has a representation of the form

$$\Gamma_p(s,t) = \sum_{k=1}^{\infty} \mu_k g_k(s) g_k(t),$$

with

$$\sum_{k=1}^{\infty} (1-\mu_k)^2 < \infty \text{ and } \mu_k > 0 \text{ for all } k,$$

where $\{g_k\}$ is a c.o.n. system in $H(\Gamma_Q)$.

Proof. Assume (2)-(4) of Theorem 1.2.3. The second half of (2) implies $\Gamma_p(\cdot,t) \in H(\Gamma_Q)$ for every $t \in T$. Let $\{\eta_k\}$ be normalized eigenvectors corresponding to the eigenvalues $\{\lambda_k\}$ of S. It forms a c.o.n. system in $H(\Gamma_Q)$ by (3). Then

$$\Gamma_p(\cdot,t) = \sum_{k=1}^{\infty} \langle \Gamma_p(\cdot,t), \eta_k(\cdot) \rangle_Q \eta_k(\cdot)$$

$$= \sum_{k=1}^{\infty} \langle S\Gamma_Q(\cdot,t), \eta_k(\cdot) \rangle_Q \eta_k(\cdot)$$

$$= \sum_{k=1}^{\infty} \langle S\eta_k(\cdot), \Gamma_Q(\cdot,t) \rangle_Q \eta_k(\cdot)$$

$$= \sum_{k=1}^{\infty} \lambda_k \langle \eta_k(\cdot), \Gamma_Q(\cdot,t) \rangle_Q \eta_k(\cdot) = \sum_{k=1}^{\infty} \lambda_k \eta_k(t) \eta_k(\cdot)$$

for every $t \in T$. Since norm convergence in $H(\Gamma_Q)$ implies pointwise convergence,

$$\Gamma_p(s,t) = \sum_{k=1}^{\infty} \gamma_k \eta_k(s)\eta_k(t)$$

for every $s,\ t \in T$. The requirements on constants $\mu_k = \lambda_k$ are fulfilled by the first half of (2) and (4).

Assume the condition of theorem. Noting that conditions that $\sum_{k=1}^{\infty} (1-\eta_k)^2 < \infty$ and $\mu_k > 0$ for all k imply the existence of finite constants c_1 and c_2 with $c_1 \leq \mu_k \leq c_2$ for all k, define on $H(\Gamma_Q)$ an operator S by

$$S = \sum_{k=1}^{\infty} \mu_k P_k,$$

where P_k is the projection on the one-dimensional subspace spanned by g_k. S is a bounded, self-adjoint linear operator with upper bound c_2 and lower bound c_1, and has a pure point spectrum (μ_k's are the eigenvalue of S). Since

$$S\Gamma_Q(\cdot,t) = \sum_{k=1}^{\infty} \mu_k g_k(\cdot) g_k(t)$$

for all $t \in T$,

$$S\Gamma_Q(s,t) = \sum_{k=1}^{\infty} \mu_k g_k(s) g_k(t),$$

which equals $\Gamma_p(s,t)$ for all $s,\ t \in T$. Hence

$$\Gamma_p(\cdot,t) = S\Gamma_Q(\cdot,t) \in H(\Gamma_Q),$$

and

$$Sf(t) = \langle Sf(\cdot), \Gamma_Q(\cdot,t)\rangle_Q$$
$$= \langle f(\cdot), \Gamma_P(\cdot,t)\rangle_Q .$$

Therefore, $c_1\Gamma_Q \ll \Gamma_P \ll c_2\Gamma_Q$, concluding the proof.

1.3 Equivalent forms of n.s. conditions, I.

Since I - S is Hilbert-Schmidt, it is expected to replace (3) and (4) of Theorem 1.2.3 by a condition stated in the direct product of two r.k. Hilbert spaces.

Theorem 1.3.1 Conditions

(α) There are finite positive constants c_1 and c_2 such that

$$c_1\Gamma_Q \ll \Gamma_P \ll c_2\Gamma_Q$$

(β) $$\Gamma_Q - \Gamma_P \in H(\Gamma_P) \otimes H(\Gamma_Q)$$

are equivalent to (2)-(4) of Theorem 1.2.3.

Proof. Assume (α) and (β). From (α) it follows that

$$(1-c_2)\Gamma_Q \ll \Gamma_Q - \Gamma_P \ll (1-c_1)\Gamma_Q.$$

Hence, there is a bounded, self-adjoint linear operator K on $H(\Gamma_Q)$ corresponding to $\Gamma_Q - \Gamma_P$, and K = I - S.

Let $\{f_k\}$ and $\{g_\ell\}$ be c.o.n. systems in $H(\Gamma_P)$ and $H(\Gamma_Q)$, respectively. Then by (β),

$$\Gamma_Q(\cdot,*)-\Gamma_P(\cdot,*) = \sum_{k,\ell} \alpha_{k\ell} f_k(\cdot)g_\ell(*)$$

with

$$\sum_{k,\ell} |\alpha_{k\ell}|^2 < \infty .$$

Note that $\sum_{\ell}\alpha_{k\ell}f_k(\cdot)\in H(\Gamma_P)$ and, by (α), $\in H(\Gamma_Q)$, for every k.

Since

$$\sum_k |||\, Kg_k(*)\, |||_Q^2 = \sum_k |||\, \langle g_k(\cdot), \Gamma_Q(*,\cdot)-\Gamma_P(*,\cdot)\rangle_Q \,|||_Q^2$$

$$= \sum_k |||\, \sum_{\ell}\alpha_{k\ell}f_k(*)\, |||_Q^2$$

$$= c_2^{-1}\sum_k |||\, \sum_{\ell}\alpha_{k\ell}f_k(*)\, |||_P^2$$

$$= c_2^{-1}\sum_k \sum_{\ell}|\alpha_{k\ell}|^2$$

$$< \infty,$$

$K = I - S$ is Hilbert-Schmidt. This implies (3) and (4). (2) is the same as (α).

Conversely, assume (2)-(4). Then I-S is Hilbert-Schmidt and $\Gamma_Q - \Gamma_P$ is the corresponding kernel. Let $\{\mu_k\}$ be the eigenvalues of I-S and let $\{g_k\}$ its corresponding eigen-vectors forming a c.o.n. system in $H(\Gamma_Q)$. Noting that $g_k\in H(\Gamma_P)$, by (2), define

$$K(\cdot,*) = \sum_{j,k}\mu_k\langle g_k, f_j\rangle_P f_j(\cdot)g_k(*),$$

where $\{f_j\}$ is a c.o.n. system in $H(\Gamma_P)$. Denote by $|||\cdot|||_{P\otimes Q}$ the norm in $H(\Gamma_P)\otimes H(\Gamma_Q)$.

$$|||\, K(\cdot,*)\, |||^2_{P\otimes Q} = \lim_{m,n}\sum_{k=1}^{n}|\mu_k|^2\sum_{j=1}^{m}\langle g_k, f_j\rangle_P^2$$

$$\leq \lim_{n} \sum_{k}^{n} |\mu_k|^2 \sum_{j}^{\infty} \langle g_k, f_j \rangle_P^2$$

$$\leq \sum_{k}^{\infty} |\mu_k|^2 \| g_k \|_P^2$$

$$\leq c_1^{-1} \sum_{k}^{\infty} |\mu_k|^2$$

$$< \infty.$$

Hence $K(\cdot,*) \in H(\Gamma_p) \otimes H(\Gamma_Q)$. Also

$$K(t,*) = \sum_{j,k} \mu_k \langle g_k, f_j \rangle_p f_j(t) g_k(*) \in H(\Gamma_Q)$$

for every $t \in T$. For,

$$\sum_k \left| \sum_j \mu_k \langle g_k, g_j \rangle_p f_j(t) \right|^2 = \sum_k |\mu_k|^2 \left| \sum_k \langle g_k, f_j \rangle_p f_j(t) \right|^2$$

$$= \sum_k |\mu_k|^2 |g_k(t)|^2$$

$$= \sum_k |\mu_k|^2 |\langle g_k(\cdot),\ Q(\cdot,t) \rangle_Q|^2$$

$$\leq \sum_k |\mu_k|^2 \Gamma_Q(t,t)$$

$$< \infty.$$

Since, for every g_k,

$$\langle g_k(\cdot), K(t,\cdot)\rangle_Q = \sum_j \mu_k \langle g_k, f_j\rangle_P f_j(t)$$

$$= \mu_k g_k(t)$$

$$= (I-S) g_k(t)$$

$$= \langle g_k(\cdot), \Gamma_Q(t,\cdot)-\Gamma_P(t,\cdot)\rangle_Q,$$

$$K(t,\cdot) = \Gamma_Q(t,\cdot)-\Gamma_P(t,\cdot).$$

Hence, $K(s,t) = \Gamma_Q(s,t)-\Gamma_P(s,t)$ for all s, $t \in T$. Therefore, $\Gamma_Q(\cdot,*) - \Gamma_P(\cdot,*) \in H(\Gamma_P) \otimes H(\Gamma_Q)$, which is (β). (α) is identical with (2). This concludes the proof.

Using similar arguments we obtain

Theorem 1.3.2 Conditions (3) and (4) can be replaced by the condition

(β') $\quad \Gamma_Q - \Gamma_P \in H(\Gamma_Q) \otimes H(\Gamma_Q)$

or

(β'') $\quad \Gamma_Q - \Gamma_P \in H(\Gamma_P) \otimes H(\Gamma_P)$.

Let T' be any finite subset of T, and let $\Gamma_{PT'}$ and $\Gamma_{QT'}$ denote the covariance matrices, i.e.,

$$\Gamma_{PT'} = (\Gamma_P(t_i,t_j))_{t_i,t_j \in T'}$$

$$\Gamma_{QT'} = (\Gamma_Q(t_i,t_j))_{t_i,t_j \in T'}.$$

Assume that $\Gamma_{PT'}$ and $\Gamma_{QT'}$ are non-singular for any finit subset $T' \subset T$, i.e., their inverse matrices $\Gamma^{-1}_{PT'}$ and $\Gamma^{-1}_{QT'}$ exist.

Under this assumption we prove the following

Theorem 1.3.3 ([19]). P and Q are equivalent if and only if

(1) $m(\cdot) \in H(\Gamma_Q)$,

and one of the following three conditions holds.

(a) $\Gamma_Q - \Gamma_P \in H(\Gamma_P) \otimes H(\Gamma_Q)$

(b) $\Gamma_Q - \Gamma_P \in H(\Gamma_Q) \otimes H(\Gamma_Q)$

(c) $\Gamma_Q - \Gamma_P \in H(\Gamma_P) \otimes H(\Gamma_P)$.

Lemma 1.3.1 Condition (a) implies the existence of finite positive constants c_1 and c_2 with $c_1\Gamma_Q \ll \Gamma_P \ll c_2\Gamma_Q$.

Proof. Let $c = |||\, \Gamma_Q - \Gamma_P \,|||_{P \otimes Q}$. Let $T' = \{t_1, t_2, \ldots, t_m\}$ be a finite subset of T and let $|||\cdot|||_{P \otimes Q, T'}$ denote the norm in the finite dimensional r.k. Hilbert space obtained by restricting functions in $H(\Gamma_P) \otimes H(\Gamma_Q)$ to $T' \times T'$. Then, by Propositions 4, 5 and 6,

$$c^2 \geqq |||\, \Gamma_Q - \Gamma_p \,|||^2_{P \otimes Q,\, T'}$$

$$= \mathrm{Trace}\,\{\Gamma_{QT'}\Gamma^{-1}_{PT'}\} - \mathrm{Trace}\,\{I_{T'}\} - \mathrm{Trace}\,\{I_{T'}\} +$$

$\mathrm{Trace}\,\{\Gamma^{-1}_{QT'}\Gamma_{PT'}\}$, where $I_{T'}$ is the $m \times m$ identity matrix.

There is a non-singular matrix $U_{T'}$ which transforms $\Gamma_{PT'}$ and $\Gamma_{QT'}$ into diagonal matrices, i.e., such that

$$^{t}U_{T'}\Gamma_{PT'}U_{T'} = I_{T'}$$

$$^{t}U_{T'}\Gamma_{QT'}U_{T'} = D_{T'} = \begin{bmatrix} d_{t_1} & & & 0 \\ & d_{t_2} & & \\ & & \ddots & \\ 0 & & & d_{t_m} \end{bmatrix},$$

where $^{t}U_{T'}$ denotes the transposed matrix of $U_{T'}$. d_{t_i}'s are the roots of $\det|\Gamma_{QT'}\Gamma_{PT'}^{-1} - xI_{T'}| = 0$ and they are positive. Since the transformation by $U_{T'}$ does not change traces,

$$c^2 \geqq \sum_{i=1}^{m} \frac{(1-d_{t_i})^2}{d_{t_i}}$$

$$\geqq \frac{(1-d_{t_i})^2}{d_{t_i}} \qquad \text{for } i = 1,2,\ldots,m.$$

Therefore, for every d_{t_i},

$$(3.1)\quad 0 < \frac{(2+c^2)-((2+c^2)^2-4)^{1/2}}{2} \leq d_{t_i} \leq \frac{(2+c^2)+((2+c^2)^2-4)^{1/2}}{2} < \infty.$$

Suppose now that there is no finite positive constant c_2 with $\Gamma_P << c_2\Gamma_Q$. Then, for any n, there exist a finite subset $T'_n = \{t_{n1}, t_{n2}, \ldots, t_{nm_n}\}$ and an m_n-dimensional vector a_n such that

$$n\,{}^t a_n \Gamma_{QT'_n} a_n \leq {}^t a_n \Gamma_{PT'_n} a_n .$$

Let $b_n = U_{T'_n}^{-1} a_n$. Then

$$n\,{}^t b_n D_{T'_n} b_n \leq {}^t b_n b_n ,$$

i.e.,

$$\sum_{i=1}^{m_n} b_{t_{ni}}^2 \geqq n \sum_{i=1}^{m_n} d_{t_{ni}} b_{t_{ni}}^2$$

$$\geqq n\cdot(\min_i d_{t_{ni}}) \sum_{i=1}^{m_n} b_{t_{ni}}^2 .$$

Hence $\min_i d_{t_{ni}} \leq 1/n$. This contradicts (3.1).

The existence of c_1 can be proved analogously.

<u>Proof of Theorem 1.3.3</u> If (a) is assumed, then conditions (α) and (β) of Theorem 1.3.1 are satisfied in view of Lemma 1.3.1. Converse is clear. It suffices to prove the equivalence of (a) and (b), since the proof of that of (a) and (c) is similar.

Assume (a). Let $\{f_j\}, \{g_k\}$ be c.o.n. systems in $H(\Gamma_P)$, $H(\Gamma_Q)$. Then $\Gamma_Q - \Gamma_P$ is represented in the form

$$\Gamma_Q(\cdot,*)-\Gamma_P(\cdot,*) = \sum_{j,k} \alpha_{jk} f_j(\cdot) g_k(*)$$

with

$$\sum_{j,k} |\alpha_{jk}|^2 < \infty .$$

Noting that $\Gamma_P << c_2\Gamma_Q$, we have

$$|||\, \Gamma_Q - \Gamma_P \,|||^2_{Q \otimes Q} = \sum_{j,k} |\alpha_{jk}|^2 \, |||\, f\, |||^2_Q \leq c_2^{-1} \sum_{j,k} |\alpha_{jk}|^2 < \infty,$$

where $|||\cdot|||_{Q \otimes Q}$ is the norm in $H(\Gamma_Q) \otimes H(\Gamma_Q)$. Hence $\Gamma_Q - \Gamma_P \in H(\Gamma_Q) \otimes H(\Gamma_Q)$, i.e., (b) holds.

Assume now (b). Let $c' = |||\, \Gamma_Q - \Gamma_P \,|||_{Q \otimes Q}$ and let $|||\cdot|||_{Q \otimes Q, T'}$ denote the norm of the r.k. Hilbert space restricted to $T' \times T'$ of $H(\Gamma_Q) \otimes H(\Gamma_Q)$. Then, for any finite subset $T' \subset T$,

$$\begin{aligned} c'^2 &\geqq |||\, \Gamma_Q - \Gamma_P \,|||_{Q \otimes Q,\, T'} \\ &= \mathrm{Trace}(I_{T'}) - \mathrm{Trace}(\Gamma_{PT'}\Gamma_{QT'}^{-1}) - \mathrm{Trace}(\Gamma_{QT'}^{-1}\Gamma_{PT'}) \\ &\quad + \mathrm{Trace}(\Gamma_{PT'}\Gamma_{QT'}^{-1}\Gamma_{QT'}^{-1}\Gamma_{PT'}) \\ &= \sum_{t \in T'} (1-d_t)^2 \\ &\geqq (1-d_t)^2 \qquad \text{for every } t \in T'. \end{aligned}$$

From this one can conclude that there is a positive constant

c'' with $c''\Gamma_Q << \Gamma_P$. Hence, if $g \in H(\Gamma_Q)$, then $g \in H(\Gamma_P)$ and $|||\, g \,|||_P^2 \leq c''^{-1} |||\, g \,|||_Q^2$. If $\{g_k\}$ is a c.o.n. system in $H(\Gamma_Q)$,

$$\Gamma_Q(\cdot,*) - \Gamma_P(\cdot,*) = \sum_{k,\ell} \beta_{k\ell} g_k(\cdot) g_\ell(*)$$

with

$$\sum_{k,\ell} |\beta_{k\ell}|^2 < \infty ,$$

and

$$|||\, \Gamma_Q - \Gamma_P \,|||^2_{P \otimes Q} = \sum_{k,\ell} |\beta_{k\ell}|^2 |||\, g_k \,|||_P^2 \leq c''^{-1} \sum_{k,\ell} |\beta_{k\ell}|^2 < \infty.$$

This shows that $\Gamma_Q - \Gamma_P \in H(\Gamma_P) \otimes H(\Gamma_Q)$, concluding the proof.

1.4 Equivalent froms of n.s. conditions, II

In this section we derive from Theorem 1.2.3 other equivalent conditions given by J. Feldman [6]. We assume that $m(\cdot) = 0$, i.e., both processes have zero mean functions. Let V^* denote the adjoint operator of a linear operator V.

Definition ([6]) A linear operator V from a Hilbert space to a Hilbert space is called an equivalence operator if V is one-to-one, onto, bounded, invertible and $I - V^*V$ is Hilbert-Schmidt.

Theorem 1.4.1 P and Q are equivalent if and only if the correspondence

$$\sum_{i}^{m} a_i \Gamma_Q(\cdot, t_i) \longrightarrow \sum_{i}^{n} a_i \Gamma_P(\cdot, t_i)$$

is induced by an equivalence operator from $H(\Gamma_Q)$ to $H(\Gamma_P)$.

Proof. We prove the equivalence of the above condition and (2)-(4) of Theorem 1.2.3.

Assume (2)-(4) of Theorem 1.2.3. Since

$$\left|\left|\left| \sum_{i}^{n} a_i \Gamma_P(\cdot, t_i) \right|\right|\right|_P^2 = \sum_{i,j}^{n} a_i a_j \Gamma_P(t_i, t_j)$$

and

$$\left|\left|\left| \sum_{i}^{n} a_i \Gamma_Q(\cdot, t_i) \right|\right|\right|_Q^2 = \sum_{i,j}^{n} a_i a_j \Gamma_Q(t_i, t_j),$$

it follows from (2) that

$$(4.1) \quad c_1 \left|\left|\left| \sum_{i}^{n} a_i \Gamma_Q(\cdot, t_i) \right|\right|\right|_Q^2 \leq \left|\left|\left| \sum_{i}^{n} a_i \Gamma_P(\cdot, t_i) \right|\right|\right|_P^2 \leq$$

$$c_2 \left|\left|\left| \sum_{i}^{n} a_i \Gamma_Q(\cdot, t_i) \right|\right|\right|_Q^2 .$$

Hence, if $\left\{ \sum_{i}^{n(k)} a_i^k \Gamma_Q(\cdot, t_i^k) \right\}_k$ is a $|||\cdot|||_Q$-Cauchy sequence, then $\left\{ \sum_{i}^{n(k)} a_i^k \Gamma_P(\cdot, t_i^k) \right\}_k$ is a $|||\cdot|||_P$-Cauchy sequence, and vice versa. For any element $f \in H(\Gamma_Q)$ there is a $|||\cdot|||_Q$-Cauchy sequence $\left\{ \sum_{i}^{n(k)} a_i^k \Gamma_Q(\cdot, t_i^k) \right\}_k$ converging to f, and, hence, there exists an element g in $H(\Gamma_P)$ to which the corresponding sequence $\left\{ \sum_i^{n(k)} a_i^k \Gamma_P(\cdot, t_i^k) \right\}_k$ converges in $|||\cdot|||_P$-norm. It may be shown that g is unique, i.e., independent of the choice of Cauchy sequences. Therefore we can define an operator $Ⓗ$ from $H(\Gamma_Q)$ to $H(\Gamma_P)$ by

$$Ⓗ f = g.$$

From (4.1) we obtain

$$c_1 |||\ f\ |||_Q^2 \leq |||\ Ⓗ f\ |||_P^2 \leq c_2 |||\ f\ |||_Q^2 . \tag{4.2}$$

Hence Ⓗ is linear, one-to-one, onto, bounded and has a bounded inverse. Now,

$$\begin{aligned}
&\left\langle S \sum_i^m a_i \Gamma_Q(\cdot, s_i),\ \sum_j^n b_j \Gamma_Q(\cdot, t_j) \right\rangle_Q \\
&= \sum_{i,j}^{m,n} a_i b_j \left\langle \Gamma_P(\cdot, s_i), \Gamma_Q(\cdot, t_j) \right\rangle_Q \\
&= \sum_{i,j}^{m,n} a_i b_j \Gamma_P(t_j, s_i) \\
&= \sum_{i,j}^{m,n} a_i b_j \left\langle \Gamma_P(\cdot, s_i), \Gamma_P(\cdot, t_j) \right\rangle_P \\
&= \left\langle \sum_i^m a_i \Gamma_P(\cdot, s_i),\ \sum_j^n b_j \Gamma_P(\cdot, t_j) \right\rangle_P .
\end{aligned}$$

That is, for functions f, g of the form

$$f = \sum_i^m a_i \Gamma_Q(\cdot, s_i) , \qquad g = \sum_i^n b_j \Gamma_Q(\cdot, t_j),$$

$$\langle Sf, g\rangle_Q = \langle Ⓗ f, Ⓗ g \rangle_p .$$

By the continuity of inner products and (4.2), we have, for any $f, g \in H(\Gamma_Q)$,

$$\langle Sf, g\rangle_Q = \langle Ⓗ f, Ⓗ g\rangle_p = \langle Ⓗ^* Ⓗ f, g\rangle_Q .$$

Hence

$$Ⓗ^* Ⓗ = S ,$$

and $I - Ⓗ^* Ⓗ = I-S$ is Hilbert-Schmidt, by (3) and (4).

To prove the other direction, suppose that Ⓗ is an equivalence operator from $H(\Gamma_Q)$ to $H(\Gamma_p)$ and

$$Ⓗ \sum_i^n a_i \Gamma_Q(\cdot, t_i) = \sum_i^n a_i \Gamma_p(\cdot, t_i).$$

Since Ⓗ is bounded and invertible, there are finite positive constants c_1 and c_2 such that

$$(4.3) \qquad c_1 ||| f |||_Q^2 \leq ||| Ⓗ f |||_p^2 \leq c_2 ||| f |||_Q^2$$

for all $f \in H(\Gamma_Q)$. In particular, for any f of the form

$$f = \sum_i^n a_i \Gamma_Q(\cdot, t_i),$$

$$c_1 \left|\left|\left| \sum_i^n a_i \Gamma_Q(\cdot,t_i) \right|\right|\right|_Q^2 \leq \left|\left|\left| Ⓗ \sum_i^n a_i \Gamma_Q(\cdot,t_i) \right|\right|\right|_P^2$$

$$= \left|\left|\left| \sum_i^n a_i \Gamma_P(\cdot,t_i) \right|\right|\right|_P^2$$

$$\leq c_2 \left|\left|\left| \sum_i^n a_i \Gamma_Q(\cdot,t_i) \right|\right|\right|_Q^2,$$

i.e.,

(4.4) $\qquad c_1\Gamma_Q << \Gamma_P << c_2\Gamma_Q,$

which is (2). The second half of (4.4) implies $\Gamma_P(\cdot,t)\epsilon H(\Gamma_Q)$ for every $t\epsilon T$. For any $f,g\epsilon H(\Gamma_Q)$ there exist sequences of functions $\left\{ \sum_i^m a_i\Gamma_Q(\cdot,s_i) \right\}$, $\left\{ \sum_j^n b_j\Gamma_Q(\cdot,t_j)\right\}$ converging to f,g, respectively.

$$\langle Ⓗ^* Ⓗ \sum_i^m a_i\Gamma_Q(\cdot,s_i), \quad \sum_j^n b_j\Gamma_Q(\cdot,t_j)\rangle_Q$$

$$= \langle Ⓗ \sum_i^m a_i\Gamma_Q(\cdot,s_i), Ⓗ \sum_j^n b_j\Gamma_Q(\cdot,t_j)\rangle_P$$

$$= \langle \sum_i^m a_i\Gamma_P(\cdot,s_i), \quad \sum_j^n b_j\Gamma_P(\cdot,t_j) \rangle_P$$

$$= \sum_{i,j}^{m,n} a_i b_j \Gamma_P(t_j,s_i)$$

$$= \sum_{i,j}^{m,n} a_i b_j \langle \Gamma_P(\cdot,s_i),\Gamma_Q(\cdot,t_j)\rangle_Q$$

$$= \left\langle \sum_i^m a_i \Gamma_P(\cdot, s_i), \quad \sum_j^n b_j \Gamma_Q(\cdot, t_j) \right\rangle_Q$$

$$= \left\langle \sum_i^m a_i \left\langle \Gamma_P(*, \cdot), \Gamma_Q(*, s_i) \right\rangle_Q, \quad \sum_j^n b_j \Gamma_Q(\cdot, t_j) \right\rangle_Q$$

$$= \left\langle \left\langle \sum_i^m a_i \Gamma_Q(*, s_i), \Gamma_P(*, \cdot) \right\rangle_Q, \quad \sum_j^n b_j \Gamma_Q(\cdot, t_j) \right\rangle_Q .$$

Using the continuity of inner product, we can conclude that

$$Ⓗ^* Ⓗ f(\cdot) = \left\langle f(*), \Gamma_P(*, \cdot) \right\rangle_Q$$

Hence, by definition of S, $Ⓗ^* Ⓗ = S$. Since $I - Ⓗ^* Ⓗ$ is Hilbert-Schmidt, so is I-S, from which (3) and (4) follow. This concludes the proof.

<u>Theorem 1.4.2</u> ([6]). P and Q are equivalent if and only if, for any $Z \in \mathcal{L}(X)$, $\overline{Z}^P = \overline{Z}^Q$ (set-theoretically) and the correspondence $\overline{Z}^Q \longrightarrow \overline{Z}^P$ is induced by an equivalence operator from $L_2(X,Q)$ to $L_2(X,P)$.

Proof. Necessity. By Lemma 1.1.1, $H(\Gamma_P) \cong L_2(X,P)$ and $H(\Gamma_Q) \cong L_2(X,Q)$. Let Φ_P denote the congruence from $H(\Gamma_P)$ to $L_2(X,P)$, and, similarly, let Φ_Q denote the congruence from $H(\Gamma_Q)$ to $L_2(X,Q)$. Note that

$$\Phi_P \quad \sum_i^n a_i \Gamma_P(\cdot, t_i) = \sum_i^n a_i \overline{X(t_i)}^P$$

and

$$\Phi_Q \sum_i^n a_i \Gamma_Q(\cdot, t_i) = \sum_i^n a_i \overline{X(t_i)}^Q.$$

Define an operator $\Xi = \Phi_P \textcircled{H} \Phi_Q^{-1}$, where $\textcircled{H}$ is the equivalence operator in Theorem 1.4.1. Since Φ_P and Φ_Q are congruences, Ξ is an equivalence operator from $L_2(X,Q)$ to $L_2(X,P)$.

From (4.3) we have

$$(4.5) \qquad c_1 \| \overline{Z}^Q \|_Q^2 \leq \| \Xi \overline{Z}^Q \|_P^2 \leq c_2 \| \overline{Z}^Q \|_Q^2$$

for all $\overline{Z}^Q \in L_2(X,Q)$. Let $Z \in \mathcal{L}_2(X,P) \cap \mathcal{L}_2(X,Q)$ and let $Y \in \overline{Z}^Q$.

Then, since P and Q are equivalent,

$$\begin{aligned} \left(\int_\Omega |Y|^2 dP \right)^{1/2} &\leq \left(\int_\Omega |Z|^2 dP \right)^{1/2} + \left(\int_\Omega |Y-Z|^2 dP \right)^{1/2} \\ &= \left(\int_\Omega |Z|^2 dP \right)^{1/2} + \left(\int_\Omega |Y-Z|^2 dQ \right)^{1/2} \\ &= \left(\int_\Omega |Z|^2 dP \right)^{1/2} < \infty . \end{aligned}$$

Hence $Y \in \mathcal{L}_2(X,P) \cap \mathcal{L}_2(X,Q)$. By (4.5),

$$\| \overline{Y}^P - \overline{Z}^P \|_P^2 = \| \Xi (\overline{Y}^Q - \overline{Z}^Q \|_P^2 \leq c_2 \| \overline{Y}^Q - \overline{Z}^Q \|_Q^2 = 0.$$

This shows $Y \in \overline{Z}^P$. Similarly, if $Y \in \overline{Z}^P$, then $Y \in \overline{Z}^Q$. In particular, this is true for any $Z \in \mathcal{L}(X)$. Therefore, for all $Z \in \mathcal{L}(X)$, the P-equivalence class $\overline{Z}^P$ and the Q-equivalence class $\overline{Z}^Q$ are the same set and the correspondence $\overline{Z}^Q \rightarrow \overline{Z}^P$ is induced by an equivalence operator Ξ.

Sufficiency. Let Ξ be an equivalence operator from $L_2(X,Q)$ to $L_2(X,P)$ such that, for all $Z \in \mathcal{L}(X)$, $\Xi \overline{Z}^Q = \overline{Z}^P$. Let $\textcircled{H} = \Phi_P^{-1} \Xi \Phi_Q$. Then $\textcircled{H}$ is an equivalence operator from $H(\Gamma_Q)$ to $H(\Gamma_P)$, and

$$\begin{aligned}
\textcircled{H} \sum_i^n a_i \Gamma_Q(\cdot, t_i) &= \Phi_P^{-1} \Xi \Phi_Q \sum_i^n a_i \Gamma_Q(\cdot, t_i) \\
&= \Phi_P^{-1} \Xi \sum_i^n a_i \overline{X(t_i)}^Q \\
&= \Phi_P^{-1} \sum_i^n a_i \overline{X(t_i)}^P \\
&= \sum_i^n a_i \Gamma_P(\cdot, t_i) .
\end{aligned}$$

Hence, by Theorem 1.4.1, P and Q are equivalent.

1.5 Gaussian measures in abstract Hilbert space.

In this section we consider the equivalence problem of Gaussian measures in Hilbert space.

Let H be a separable real Hilbert space with inner product $(\cdot,\cdot)$, let $\mathcal{B}$ be the σ-field of subsets of H generated by all continuous linear functionals on H, and let P and Q be two Gaussian measures on $(H, \mathcal{B})$. We identify, as usual, the conjugate space of H with H. Then any element f in H may be considered as a random variable, since (f,x) is

$\mathcal{B}$ -measurable. For the sake of simplicity, we assume that

$$E_P(f,x) = \int_H (f,x)dP(x) = 0$$

and

$$E_Q(f,x) = \int_H (f,x)dQ(x) = 0$$

for all $f \in H$. The operators A and B on H, defined by

$$(Af,g) = E_P(f,x)(g,x)$$

$$(Bf,g) = E_Q(f,x)(g,x),$$

are known to be bounded, self-adjoint, and positive ([21]). We shall write $V_1 \leq V_2$ for any two self-adjoint operators if $V_2 - V_1$ is a positive operator.

Define nonnegative definite kernels Γ_P and Γ_Q by

$$\Gamma_P(f,g) = (Af,g)$$

$$\Gamma_Q(f,g) = (Bf,g).$$

Let $\Omega = H$, $\mathcal{F} = \mathcal{B}$, and $T = H$. Then a version of Theorem 1.2.3 may be stated as follows.

<u>Theorem 1.5.1</u> Necessary and sufficient conditions for the equivalence of P and Q are that

(1) there exist finite positive constants c_1 and c_2 with

$$c_1 B \leq A \leq c_2 B,$$

(2) the operator S on $H(\Gamma_Q)$ corresponding to Γ_P has a pure point spectrum, and

(3) the eigenvalues of S satisfy the relation

$$\sum_{k=1}^{\infty} (1-\lambda_k)^2 < \infty .$$

Proof. We need only to show that (1) is equivalent to (2) of Theorem 1.2.3. If $c_1\Gamma_Q << \Gamma_P << c_2\Gamma_Q$, then, for every $f \in H$, $c_1\Gamma_Q(f,f) \leq \Gamma_P(f,f) \leq c_2\Gamma_Q(f,f)$, i.e.,

$$(c_1Bf,f) \leq (Af,f) \leq (c_2Bf,f),$$

which is (1), by definition.

Since

$$\sum_{i,j}^{n} a_i a_j \Gamma_P(f_i,f_j) = \sum_{i,j}^{n} a_i a_j (Af_i,f_j)$$

$$= (a \sum_{i}^{n} a_i f_i, \sum_{i}^{n} a_i f_i)$$

and, similarly,

$$\sum_{i,j}^{n} a_i a_j c_1 \Gamma_Q(f_i,f_j) = (c_1 B \sum_{i}^{n} a_i f_i, \sum_{i}^{n} a_i f_i)$$

$$\sum_{i,j}^{n} a_i a_j c_2 \Gamma_Q(f_i,f_j) = (c_2 B \sum_{i}^{n} a_i f_i, \sum_{i}^{n} a_i f_i),$$

$c_1\Gamma_Q << \Gamma_P << c_2\Gamma_Q$ follows from (1).

Conditions (2) and (3) of Theorem 1.5.1 are stated in terms of operator S on $H(\Gamma_Q)$. One can formulate these conditions on a different space. Let us recall the construction of space $L_2^*(X)$ which was used in the proof of main theorem. In the setup of this section $\mathcal{L}(X)$ is nothing but H itself. H may be considered as a subset of the intersection of $\mathcal{L}_2(H, \mathcal{B}, P)$ and $\mathcal{L}_2(H, \mathcal{B}, Q)$, where

$$\mathcal{L}_2(H, \mathcal{B}, P) = \mathcal{L}_2(P) = \left\{ h: \int_H h(x)^2 dP(x) < \infty \right\}$$

$$\mathcal{L}_2(H, \mathcal{B}, Q) = \mathcal{L}_2(Q) = \left\{ h: \int_H h(x)^2 dQ(x) < \infty \right\},$$

since, for any $f \epsilon H$, $\int_H (f,x)^2 dP(x)$ and $\int_H (f,x)^2 dQ(x)$ are finite.

Write, for $h \epsilon \mathcal{L}_2(P)$

$$\| h \|_P = (h,h)_P^{1/2} = \left[\int_H h^2(x) dP(x) \right]^{1/2},$$

and, for $h \epsilon \mathcal{L}_2(Q)$,

$$\| h \|_Q = (h,h)_Q^{1/2} = \left[\int_H h^2(x) dQ(x) \right]^{1/2} .$$

Observe that if $f \epsilon H$,

$$\| f \|_P^2 = \int_H (f,x)^2 dP(x) = (Af, f)$$

$$\| f \|_Q^2 = \int_H (f,x)^2 dQ(x) = (Bf, f),$$

so that from condition (1) of Theorem 1.5.1,

$$(5.1) \qquad c_1 \| f \|_Q^2 \leq \| f \|_P^2 \leq c_2 \| f \|_Q^2$$

for all $f \epsilon H$. Just as we defined the class $\mathcal{L}_2^*(X)$, define $\mathcal{K}$

to be the class of all real $\mathcal{B}$-measurable functions h in $\mathcal{L}_2(P) \cap \mathcal{L}_2(Q)$ such that a sequence $\{f_n\}$ exists in H with $\| f_n - h \|_P \rightarrow 0$ and $\| f_n - h \|_Q \rightarrow 0$ as $n \rightarrow \infty$. Then it follows from (5.1) that, for all $h \in \mathcal{K}$,

(5.2) $$c_1 \| h \|_Q^2 \leq \| h \|_P^2 \leq c_2 \| h \|_Q^2 .$$

Letting $\mathcal{N}$ be the class of functions in $\mathcal{K}$ with $\| h \|_Q = \| h \|_P = 0$, we define $K = \mathcal{K}/\mathcal{N}$, the set of equivalence classes modulo $\mathcal{N}$ of elements in $\mathcal{K}$. It may be verified that K is a separable Hilbert space with inner product $(\cdot,\cdot)_Q$ and norm $\| \cdot \|_Q$. This space K corresponds to $L_2^*(X)$ defined earlier. Denote by $\bar{h}$ an element of K, i.e., the equivalence class to which h belongs. From the construction it is clear that H spans K. Since, for and f, g H,

$$\begin{aligned} \langle \Gamma_Q(\cdot,f), \Gamma_Q(\cdot,g) \rangle_Q &= \Gamma_Q(f,g) \\ &= (Bf,g) \\ &= \int_H (f,x)(g,x)\,dQ(x) \\ &= (\bar{f},\bar{g})_Q , \end{aligned}$$

$H(\Gamma_Q) \cong K$, by the congruence theorem. Let Φ denote the congruence from $H(\Gamma_Q)$ onto K, and define $S^* = \Phi S \Phi^{-1}$. Note that, for $f,g \in H$,

$$(S^{*}\bar{f},\bar{g})_Q = \langle S\Phi^{-1}f,\ \Phi^{-1}g\rangle_Q$$

$$= \langle \Gamma_P(\cdot,f), \Gamma_Q(\cdot,g)\rangle_Q$$

$$= \Gamma_P(f,g)$$

$$= (Af,g)$$

$$= \int_H (f,x)(g,x)dP(x).$$

Hence, in view of (5.2), $(S^{*}\,\bar{h}_1,\bar{h}_2)_Q$, for $\bar{h}_1,\bar{h}_2 \in K$, is the covariance of h_1 and h_2 with respect to P.

We have proved the following

<u>Theorem 1.5.2</u> P and Q are equivalent if and only if

(1) there are finite positive constants c_1 and c_2 such that

$$c_1 B \leq A \leq c_2 B,$$

(2) S^{*} on K has a pure point spectrum, and

(3) its eigenvalues $\{\lambda_k\}$ satisfy

$$\sum_{k=1}^{\infty} (1-\lambda_k)^2 < \infty.$$

It might be of some interest to note that the foregoing theorem can be proved without making use of techniques of r.k. Hilbert space. We give the outline of the proof.

First, condition (1) is proved by the argument used in the proof of (2) of Theorem 1.2.1. It implies (5.1) and

(5.2), and enables us to define the space K. For $\overline{h}_1$ and $\overline{h}_2$ in K define the symetric bilinear form $L(\overline{h}_1,\overline{h}_2) = Cov_P(h_1,h_2)$, where

$$Cov_P(\overline{h}_1,\overline{h}_2) = \int_H h_1(x)h_2(x)dP(x) .$$

(Remark that, from the assumption,

$$E_P h = \int_H h(s)dP(x) = 0$$

for all $h \in K$.) The definition of L is unambiguous. L is bounded, since, by (5.2),

$$\begin{aligned} |L(\overline{h}_1,\overline{h}_2)| &= |Cov_P(h_1,h_2)| \\ &\leq \| h_1 \|_P \cdot \| h_2 \|_P \\ &\leq c_2 \| h_1 \|_Q \cdot \| h_2 \|_Q . \end{aligned}$$

Hence there is a bounded self-adjoint operator S^* on K such that $(S^*\overline{h}_1,\overline{h}_2)_Q = L(\overline{h}_1,\overline{h}_2) = Cov_P(h_1,h_2)$. The remainder of the proof, including sufficiency part, is similar to that of Theorem 1.2.3.

1.6 Comparison of various methods

The problem of the equivalence of Gaussian stochastic processes has been studied by many authors under various assumptions (e.g., [6], [7], [10], [11], [12], [16], [17], [18], [20], [25], [26], [27], L. LeCam (unpublished) and C. Stein (unpublished)). E. Parzen ([16], [17], [18]) exploited the notion of reproducing kernel Hilbert space, which was introduced earlier by M. Loève in developing the theory of second order stochastic processes, and formulated a condition for equivalence (in the case of same covariance functions and different mean functions) in the form (1) of our Theorem 1.2.3. The existence of the equivalence-or-singularity dichotomy in the general case was established in 1958, independently, by J. Feldman [6] and J. Hájek [11]. Their methods of proof are entirely different. Hájek's approach is information-theoretic. More precisely, he considers "J-divergence" defined as follows. Let $\{X_1, X_2, \ldots, X_n\}$ be normal with respect to both P and Q, and let p and q denote their normal densities. Then the J-divergence of p and q is defined by

$$J = E_P \log(p/q) - E_Q \log(p/q).$$

Suppose now that $\{X_t, t \in T, P\}$ and $\{X_t, t \in T, Q\}$ are real Gaussian processes. Then the J-divergence of P and Q is equal to the supremum of the J-divergences of finite-dimensional distributions $P_{t_1,\ldots,t_n}$ and $Q_{t_1,\ldots,t_n}$ of vectors $\{X_{t_1}, \ldots, X_{t_n}\}$, i.e.,

$$J_T = \sup_{t_1,\ldots,t_n \in T} J_{t_1,\ldots,t_n}.$$

His criterion is: P and Q are equivalent if and only if $J_T < \infty$.

Feldman's method is operator-theoretic and his conditions for the equivalence of P and Q are stated in terms of "equivalence operator" from a Hilbert space to another Hilbert space (see § 1.4).

Recently, efforts have been made to simplify their proofs and to obtain more effective criteria for equivalence ([14], [19], [24], and T. S. Pitcher's unpublished work which has been reproduced in [23]). Rozanov [24] gave a simple proof of the dichotomy relying on properties of the information function and obtained necessary and sufficient conditions for equivalence which are stated by properties of an operator defined on $L_2(X,Q)$. His idea of getting conditions is to regard $\Gamma_P(s,t)$ as a positive bilinear form L on the space $L_2(X,Q)$. Rozanov's condition is that the operator corresponding to L has a pure point spectrum, its eigenvalue $\{\lambda_k\}$ satisfy the relation $\Sigma(1-\lambda_k)^2 < \infty$ and the corresponding normalized eigenvectors $\{\eta_k\}$ have expectations $E_p\eta_k = m_k$ with respect to P satisfying the requirement $\Sigma m_k^2 < \infty$. In [14] Professor G. Kallianpur and the writer considered Gaussian measures in Hilbert space and, using the same idea as Rozanov's, obtained an alternative proof of the dichotomy as well as conditions of Rozanov's form simultaneously. They also formulated conditions directly in terms of mean functions and covariance functions applying the reproducing kernel Hilbert space method (see Theorems 1.2.3 and 1.2.4). Independently, Parzen [19] observed that Hájek's J-divergence can be expressed as the norm of a function in $H(\Gamma_P) \otimes H(\Gamma_Q)$ and thus obtained conditions which are also stated in reproducing kernel Hilbert spaces (see Theorem 1.3.3).

The proof presented in § 1.2 of this thesis is essentially a reproduction of that given in [14]. However, there is a slight difference. Here we bring out the role of r.k. Hilbert space more explicitly. Also in § 1.5 we derived Theorem 1.5.2 from our main theorem as a corollary, which was proved in [14] independently. The key condition in our proof is (2) of Theorem 1.2.3, which makes it possible to define the space $L_2^*(X)$ and the operator S. As indicated in § 1.5, one can transform conditions from $H(\Gamma_Q)$ to $L_2^*(X)$. One could also state conditions in the space $L_2(X,Q)$ as Rozanov has done. However, it seems to the writer that one needs some condition like (2) to do so. The space $L_2^*(X)$ was introduced in our previous paper [14], but the writer found later that the somewhat similar idea was used in [12], which deals with the same-covariances-different-mean-functions case.

In [4] G. Baxter extended a strong limit theorem for the Wiener process, which was discovered by P. Lévy and, independently, by C. Cameron and W. T. Martin, to a wide class of Gaussian processes. His theorem is stated as follows.

Let $\{X(t), 0 \leq t \leq 1\}$ be a Gaussian process with mean function $m(t)$ having bounded first derivative and covariance kernel $\Gamma(s,t)$ which is continuous and has uniformly bounded second derivatives for $s \neq t$. Let

$$D^{+}(t) = \lim_{s \to t+} \frac{\Gamma(t,t)-\Gamma(s,t)}{t-s}$$

$$D^{-}(t) = \lim_{s \to t-} \frac{\Gamma(t,t)-\Gamma(s,t)}{t-s}$$

$$f(t) = D^{-}(t) - D^{+}(t) .$$

Then, with probability one,

$$\lim_{n \to \infty} \sum_{k=1}^{2^n} [X(\frac{k}{2^n}) - X(\frac{k-1}{2^n})]^2 = \int_0^1 f(t)dt .$$

This result of Baxter yields immediately necessary conditions for equivalence (i.e., sufficient conditions for singularity) for a large class of Gaussian processes (e.g., Slepian (1958) (see [23])and [27]). Gladyshev [9] and, more recently, Aleksev [1] generalized Baxter's theorem and applied their results to the equivalence problem for stationary Gaussian processes on finite intervals. The writer feels, however, that the proofs of sufficiency of conditions thus obtained are difficult in many cases, and, further, one needs more general limit theorems to get conditions in the general case.

In the following chapter we specialize our general theorems to important particular cases and get some generalizations of known results previously obtained by applying some limit theorems or other techniques. Feldman [8], applying his general condition to the stationary finite interval case, proved the sufficiency of Slepian's result. It is also reported (see [29]) that Rozanov simplified and generalized Feldman's theorem. For other recent work in this field we refer to A. M. Yaglom's survey paper [29].

CHAPTER II

APPLICATION TO SPECIAL CASES

2.1 <u>Gaussian processes with independent increments</u>

Suppose that $\{X(t), t \in T\}$ is a process with independent Gaussian increments, where $T = [a,b]$ or $[a,\infty]$ (a,b, are finite). Then, $m(t) = E(X(t)-X(a))$ is continuous and $V(t) = \mathrm{Var}(X(t)-X(a))$ is monotone increasing, continuous on T. Conversely, if a continuous function $m(t)$ with $m(a) = 0$ and a monotone increasing function $V(t)$ with $V(a) = 0$ are given on T, there exists a process $\{X(t), t\ T\}$ with independent Gaussian increments such that $m(t)$ and $V(t)$ are, respectively, the mean and the variance of $X(t)-X(a)$.

Let $\{X(t), t \in T, P\}$ and $\{X(t), t \in T, Q\}$ be processes with independent Gaussian increments with means $m(t)$, 0 and variances $V_P(t)$, $V_Q(t)$, respectively. The following theorem was previously obtained by A. V. Skorokhod [25]. We give an alternative proof.

<u>Theorem 2.1.1</u> For the equivalence of P and Q it is necessary and sufficient that

$$(\alpha) \quad V_P(t) = V_Q(t) \qquad \text{for all } t \in T,$$

and

(β) there exist a function $\Psi(t)$ such that

$$m(t) = \int_a^t \Psi(t)\,dV_Q(t)$$

with

$$\int_T |\Psi(t)|^2 dV_Q(t) < \infty .$$

First we seek for a prepresentation of $H(\Gamma_Q)$. Note that

$$\Gamma_Q(s,t) = \begin{cases} V_Q(s) & \text{for } s \leq t \\ V_Q(s) & \text{for } s \geq t \end{cases}$$

$$= \int_T \chi_{[a,s]}(\tau)\,\chi_{[a,t]}(\tau)\,dV_Q(\tau) ,$$

where

$$\chi_{[a,t]}(\tau) = \begin{cases} 1 & \text{for } a \leq \tau \leq t \\ 0 & \text{for } t < \tau . \end{cases}$$

Observe also that V_Q, the measure induced by function $V_Q(t)$, is nonatomic, since $V_Q(t)$ is continuous. By Proposition 7, there is a congruence Φ from $H(\Gamma_Q)$ onto $L_2(V_Q)$, where

$$L_2(V_Q) = \left\{ \phi \;\middle|\; \int_T |\phi(\tau)|^2 dV_Q(\tau) < \infty \right\}.$$

<u>Proof of the theorem.</u> From the above representation it follows immediately that condition (β) is equivalent to saying $m(\cdot) \in H(\Gamma_Q)$, which is (1) of Theorem 1.2.3. Hence it suffices

to prove the equivalence of condition (α) and (2)-(4) of Theorem 1.2.3.

Assume (2)-(4) of Theorem 1.2.3. By (2), $\Gamma_P(\cdot,t)\in H(\Gamma_Q)$ for all $t\in T$, and, hence, $\Gamma_P(\cdot,t)$ can be represented in the form

$$\Gamma_P(s,t) = \int_T \chi_{[a,s]}(\tau)p_t(\tau)dV_Q(\tau),$$

where $p_t(\tau) = \Phi\Gamma_P(\cdot,t)\in L_2(V_Q)$. Since $\{X(t),t\in T,P\}$ is also a process with independent increments, for all $s \geqq t$,

$$\begin{aligned}\int_T \chi_{[a,s]}(\tau)p_t(\tau)dV_Q(\tau) &= \Gamma_P(s,t)\\ &= \Gamma_P(t,t)\\ &= \int_T \chi_{[a,t]}(\tau)p_t(\tau)dV_Q(\tau)\\ &= \int_T \chi_{[a,t]}(\tau)p_t(\tau)\chi_{[a,s]}(\tau)dV_Q(\tau).\end{aligned}$$

Hence

(1.1) $\quad p_t(\tau) = p_t(\tau)\chi_{[a,t]}(\tau) \qquad \text{a.e.}[V_Q].$

For all $s \leqq t$,

$$\begin{aligned}&\int_T \chi_{[a,s]}(\tau)p_t(\tau)\chi_{[a,t]}(\tau)dV_Q(\tau)\\ &\qquad = \int_T \chi_{[a,s]}(\tau)p_t(\tau)\, dV_Q(\tau)\\ &\qquad = \Gamma_P(s,t)\end{aligned}$$

$$= \Gamma_p(s,1)$$

$$= \int_T \chi_{[a,s]}(\tau) p_1(\tau) dV_Q(\tau)$$

$$= \int_T \chi_{[a,s]}(\tau) p_1(\tau) \chi_{[a,t]}(\tau) dV_Q(\tau) .$$

Therefore,

(1.2) $\quad p_1(\tau)\chi_{[a,t]}(\tau) = p_t(\tau)\chi_{[a,t]}(\tau) \quad$ a.e. $[V_Q]$.

From (1.1 and 1.2) we have

$$p_t(\tau) = p_1(\tau)\chi_{[a,t]}(\tau) \qquad \text{a.e.} \quad [V_Q] ,$$

i.e., as elements in $L_2(V_Q)$, $p_t = p_1\chi_{[a,t]}$.

Define the operator S^* on $L_2(V_Q)$ by $S^* = \Phi S \Phi^{-1}$. S^* has the properties (3) and (4) of S. In particular, if there are eigenvalues different from 1, then they are of finite multiplicity. Since S maps $\Gamma_Q(\cdot,t)$ into $\Gamma_p(\cdot,t)$, S^* maps $\chi_{[a,t]}(\cdot)$ into $p_t(\cdot) = p_1(\cdot)\chi_{[a,t]}(\cdot)$. Hence $S^*\phi = p_1\phi$ for every $\phi \in L_2(V_Q)$. We shall prove that S^* is the identity operator, i.e., $p_1 = 1$.

Let λ_k be an eigenvalue of S^*, and let $\phi_{k1}, \phi_{k2}, \ldots, \phi_{km_k}$ be corresponding normalized eigenvectors, i.e., a set of eigenvalues which span the invariant subspace corresponding to λ_k. Since for all $\Psi \in L_2(V_Q)$

$$(p_1\phi_{ki},\Psi)_{L_2(V_Q)} = (S^*\phi_{ki},\Psi)_{L_2(V_Q)} = (\lambda_k\phi_{ki},\Psi)_{L_2(V_Q)} ,$$

$$(1.3)\quad \| (p_1 - \lambda_k)\phi_{ki} \|^2_{L_2(V_Q)} = \int_T |p_1(\tau)-\lambda_k|^2 |\phi_{ki}(\tau)|^2 dV_Q(\tau)$$

$$= 0.$$

Let C_k be the union of the supports of ϕ_{ki}'s $(i = 1,2,\ldots,m_k)$, i.e.,

$$C_k = \{\tau: \phi_{ki}(\tau) = 0,\ i = 1,2,\ldots,m_k\},$$

and let

$$D_k = \{\tau: p_1(\tau) = \lambda_k\}.$$

(1.3) implies that $V_Q(D_k' \cap C_k) = 0$, where D_k' is the complement of D_k, and this, in turn, implies $V_q(C_k \cap C_j) = 0$ for k,j with $\lambda_k \neq \lambda_j$, since $D_k \cap D_j = \Phi$ for $\lambda_k \neq \lambda_j$. Observe also that these relations are independent of the choice of ϕ_{ki}'s, i.e., if we choose another set of eigenvectors ϕ^*_{ki}, say, and if we denote by C^*_k the union of the supports of ϕ^*_{ki}'s, then $V_Q(D_k' \cap C^*_k) = 0$ for every λ_k and $V_Q(C^*_k \cap C^*_j) = 0$ for $\lambda_k \neq \lambda_j$. C_k must have positive V_Q-measure, otherwise ϕ_{ki}'s cannot be normalized eigenvectors. On the other hand, if $\lambda_k \neq 1$, C_k must be of V_Q-measure zero. For, if it were not, the invariant subspace corresponding to $\lambda_k \neq 1$ would be of infinite dimension, since the measure V_Q is nonatomic. This contradicts the finite multiplicity of $\lambda_k \neq 1$. Therefore, S^* cannot have eigenvalues different from 1, i.e., S^* is the

the identity operator. Condition (1) then follows immediately.

If (α) is assumed, $\Gamma_P(s,t) = \Gamma_Q(s,t)$ since the processes are additive. Conditions (2)-(4) of Theorem 1.2.3 are trivially satisfied, concluding the proof.

Remark. In case of separable Hilbert space valued random variables (see § 1.5), condition (α) may be stated in the form

$$A = B.$$

This generalizes Skorokhod's result.

2.2 Gaussian processes with covariance kernels of triangular form.

Suppose that $T = [a,b]$, a finite interval, and $\{X(t), t\in T, P\}$ and $\{X(t), t\in T, Q\}$ are Gaussian processes with mean functions $m(t)$, 0 and covariance kernels $\Gamma_P(s,t), \Gamma_Q(s,t)$ of the following form (i.e., triangular form)

$$\text{(A)} \quad \begin{aligned} \Gamma_P(s,t) &= \begin{cases} \theta(s)\phi(t) & \text{for } s \leq t \\ \theta(t)\Phi(s) & \text{for } s \geq t, \end{cases} \\ \Gamma_Q(s,t) &= \begin{cases} u(s)v(t) & \text{for } s \leq t \\ u(t)v(s) & \text{for } s \geq t, \end{cases} \end{aligned}$$

where

$$\Phi(t) > 0 \text{ and } v(t) > 0 \quad \text{for } t\in T.$$

It is known that if a Gaussian process has zero mean function and covariance kernel of triangular form, it is Markov process.

It may be easily shown ([15]) that if $\Gamma_Q(s,t)$ is of the above form, then $w(\tau)=u(\tau)/v(\tau)$ is nonnegative and non-decreasing. Let w denote the measure on T induced by function w. Define the measure μ on T by

$$\mu(\tau) = \begin{cases} u(a)/v(a) & \text{for } \tau = a \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$w^* = w + \mu.$$

Let

$$L_2(w^*) = \left\{ \Psi: \int_a^b |\Psi(\tau)|^2 dw^*(\tau) < \infty \right\}.$$

It is readily verified that

$$\Gamma_Q(s,t) = \int_a^b (v(s)\chi_{[a,s]}(\tau))\ (v(t)\chi_{[a,t]}(\tau))\,dw^*(\tau),$$

which shows that there is a congruence Φ from $H(\Gamma_Q)$ to $L_2(w^*)$. Hence any element $g(\cdot)\in H(\Gamma_Q)$ has the representation

$$g(t) = v(t)\int_a^b \chi_{[a,t]}(\tau)\Phi g(\tau)\,dw^*(\tau)$$

with $\Phi g\in L_2(w^*)$, and, conversely, any element $\Psi\in L_2(w^*)$ determines an element $g = \Phi^{-1}\Psi\in H(\Gamma_Q)$ by the above relation.

We impose the following conditions

(B) θ', ϕ', u' and v' exist and are continuous on $[a,b]$.

(C) $\theta'\phi - \theta\phi' > 0$ and $u'v - uv' > 0$ on $[a,b]$.

<u>Theorem 2.2.1</u> (See [27], [28],) Suppose that $\{X(t), t\in T, P\}$ and $\{X(t), t\in T, Q\}$ satisfy conditions (A), (B) and (C). Then, P and Q are equivalent if and only if

(α) there is a function $\Psi(\tau)$ such that

$$m(t) = v(t) \int_a^b \chi_{[a,t]}(\tau)\Psi(\tau)\,dw^*(\tau)$$

with

$$\int_a^b |\Psi(\tau)|^2 dw^*(\tau) < \infty ,$$

(β) $\theta'\Phi - \theta\Phi' = u'v - uv'$ on $[a,b]$, and

(γ) $u(a)$ and $\theta(a)$ are either both zero or both non-zero.

Proof. Condition (α) is nothing but a restatement of condition (1) of Theorem 1.2.3, using the representation of $H(\Gamma_Q)$. We shall prove the equivalence of (β), (γ) and (2)-(4) of Theorem 1.2.3.

Assume (2)-(4) of Theorem 1.2.3. By (2), $\Gamma_p(\cdot,t)\in H(\Gamma_Q)$ for every $t\in[a,b]$. Hence for all $s,t\in[a,b]$,

$$(2.1)\quad \Gamma_p(s,t) = v(s) \int_a^b \chi_{[a,s]}(\tau)\Phi\Gamma_p(\cdot,t)(\tau)\,dw^*(\tau).$$

Write $p_t(\tau) = \Phi\Gamma_p(\cdot,t)(\tau)$.

If $u(a) = 0$, $\mu\equiv 0$, i.e., $w^* = w$. Hence, from (2.1)

$$\theta(a)\Phi(t) = \Gamma_p(a,t) = 0.$$

Since $\phi(T) > 0$ for $t \in [a,b]$, we get $\theta(a) = 0$. Similarly, considering the representation of $H(\Gamma_p)$ and that of $\Gamma_Q(s,t)$ in it (using (2)), we obtain that if $\theta(a) = 0$, then $u(a) = 0$. This gives (γ).

One can actually show the explicit form of $p_t(\tau)$. Define, for each $t \in [a,b]$,

$$p_t^*(\tau) = \begin{cases} \begin{cases} 0 & \text{for } \tau = a \text{ and if } u(a) = 0 \\ \phi(t)\,\theta(a)/u(a) & \text{for } \tau = a \text{ and if } u(a) \neq 0 \end{cases} \\ \dfrac{\theta' v - \theta v'}{u'v - uv'}(\tau)\,\phi(t)\,\chi_{[a,t]}(\tau) \\ \qquad + \dfrac{\phi' v - \phi v'}{u'v - uv'}(\tau)\,\theta(t)\,\chi_{(t,b]}(\tau) \quad \text{for } a < \tau \leq b. \end{cases}$$

Clearly $p_t^* \in L_2(w^*)$, and

$$v(s)\int_a^b \chi_{[a,s]}(\tau)\,p_t(\tau)\,dw^*(\tau)$$

$$= v(s)\int_a^s \phi(t)\,\frac{\theta' v - \theta v'}{u'v - uv'}\,\chi_{[a,t]} + \theta(t)\,\frac{\theta' v - \theta v'}{u'v - uv'}\,\chi_{(t,b]}\,dw$$

$$+ v(s)\phi(t)\,\frac{\theta(a)}{u(a)}\cdot\frac{\dot{u}(a)}{v(a)}$$

for $s \leq t$

$$= v(s)\,\phi(t)\,[\theta(\tau)/v(\tau)]_a^s + v(s)\,\phi(t)\,\theta(a)/v(a)$$

$$= \theta(s)\,\phi(t)$$

for $s \geq t$

$$= v(s)\phi(t)\ [\theta(\tau)/v(\tau)]_a^t + v(s)\theta(t)[\phi(t)/v(\tau)]_t^s$$

$$+ v(s)\phi(t)\theta(a)/v(a)$$

$$= \theta(t)\phi(s),$$

i.e., for all $s,t \in [a,b]$,

$$= \Gamma_p(s,t).$$

Hence, $p_t(\tau) = p_t^*(\tau)$ a.e. $[w^*]$, and, as elements of $L_2(w^*)$, $p_t = p_t^*$.

Let

$$A(\tau) = \begin{cases} 0 & \text{for } \tau = a \\ \dfrac{\theta'\phi-\theta\phi'}{u'v-uv'}(\tau) & \text{for } a < \tau \leq b, \end{cases}$$

$$B(\tau,t) = \begin{cases} 0 & \text{for } \tau = a \\ \dfrac{\theta'v-\theta v'}{u'v-uv'}(\tau)\left(\dfrac{\phi}{v}(t)-\dfrac{\phi}{v}(\tau)\right) + & \\ \quad \dfrac{\phi'v-\phi v'}{u'v-uv'}(\tau)\left(\dfrac{\phi}{v}(t)-\dfrac{\phi}{v}(\tau)\right) & \text{for } a < \tau \leq b, \end{cases}$$

$$C(\tau,t) = \begin{cases} \begin{cases} 0 & \text{for } \tau = a, \text{ if } u(a) = 0 \\ \dfrac{\theta(a)}{u(a)} \cdot \dfrac{\theta}{v}(\tau) & \text{for } \tau = a, \text{ if } u(a) \neq 0 \end{cases} \\ \dfrac{\theta'v-\theta v'}{u'v-uv'}(\tau)\ \dfrac{\theta}{v}(t) & \text{for } a < \tau \leq b \end{cases}$$

Then

$$p_t(\tau)/v(t) = A(\tau)\chi_{[a,t]}(\tau) + B(\tau,t)\chi_{[a,t]}(\tau)$$

$$+ C(\tau,t)\chi_{[a,b]}(\tau)$$

Define the operator S^* on $L_2(w^*)$ by $S^* = \Phi S \Phi^{-1}$. Note that S^* maps $\chi_{[a,t]}(\tau)$ into $p_t(\tau)/v(t)$, since S maps $\Gamma_Q(\cdot,t)$ into $\Gamma_p(\cdot,t)$. Define operators S_1^*, S_2^*, and S_3^* by

$$S_1^*\chi_{[a,t]}(\tau) = A(\tau)\chi_{[a,t]}(\tau),$$

$$S_2^*\chi_{[a,t]}(\tau) = B(\tau,t)\chi_{[a,t]}(\tau), \text{ and}$$

$$S_3^*\chi_{[a,t]}(\tau) = C(\tau,t)\chi_{[a,b]}(\tau).$$

Since $\chi_{[a,t]}(\tau)$, $t\in[a,b]$ span $L_2(w^*)$, these relations determine S_1^*, S_2^* and S_3^* completely, i.e., for every $h\in L_2(w^*)$,

$$S_1^*h(\tau) = A(\tau)h(\tau),$$

$$S_2^*h(\tau) = \begin{cases} 0 & \text{for } \tau = a \\ \dfrac{\theta'v-\theta v'}{u'v-uv'}(\tau)\displaystyle\int_a^b h(s)\chi_{(\tau,b]}(s)\frac{\phi'v-\phi v'}{u'v-uv'}(s)\,dw(s) \\ \quad - \dfrac{\phi'v-\phi v'}{u'v-uv'}(\tau)\displaystyle\int_a^b h(s)\chi_{(\tau,b]}(s)\frac{\theta'v-\theta v'}{u'v-uv'}(s)\,dw(s) & \text{for } a < \tau \leq b. \end{cases}$$

$$S_3^*h(\tau) = \begin{cases} 0 & \text{for } \tau = a, \text{ if } u(a) = 0 \\ \dfrac{\theta(a)}{u(a)}\displaystyle\int_a^b h(s)\,\frac{\phi'v-\phi v'}{u'v-uv'}(s)\,dw(s) \\ \quad + h(a)\dfrac{\theta(a)}{u(a)}\cdot\dfrac{\phi(a)}{u(a)}\cdot\dfrac{u(a)}{v(a)} & \text{for } \tau = a, \text{ if } u(a) \neq 0 \end{cases}$$

$$\begin{cases} \dfrac{\theta'v-\theta v'}{u'v-uv'}(\tau) \displaystyle\int_a^b h(s)\frac{\phi'v-\phi v'}{u'v-uv'}(s)\,dw(s) & \text{for } a<\tau \le b, \text{ fi } u(a)=0 \\[2ex] \dfrac{\theta'v-\theta v'}{u'v-uv'}(\tau) \displaystyle\int_a^b h(s)\frac{\phi'v-\phi v'}{u'v-uv'}(s)\,dw(s) + \frac{\theta'v-\theta v'}{u'v-uv'}(\tau)h(a)\frac{\phi(a)}{u(a)}\,\frac{u(a)}{v(a)} & \text{for } a<\tau\le b, \text{ if } u(a)\neq 0. \end{cases}$$

Hence S_2^* and S_3^* are Hilbert-Schmidt operators with the following Hilbert-Schmidt kernels $K_2(\tau,s)$ and $K_3(\tau,s)$:

$$K_2(\tau,s) = \begin{cases} \dfrac{\theta'v-\theta v'}{u'v-uv'}(\tau)\chi_{(\tau,b]}(s)\dfrac{\phi'v-\theta v'}{u'v-uv'}(s) - \dfrac{\phi'u-\phi u'}{u'v-uv'}(\tau)\chi_{(\tau,b]}(s)\dfrac{\theta'v-\theta v'}{u'v-uv'}(s) & \text{for } a<\tau,\ s\le b \\[2ex] 0 & \text{otherwise} \end{cases}$$

$$K_3(\tau,s) = \begin{cases} \dfrac{\theta'v-\theta v'}{u'v-uv'}(\tau)\dfrac{\phi'u-\phi v'}{u'v-uv'}(s) & \text{for } a<\tau,\ s\le b \\[2ex] \begin{cases} 0 & \text{for } \tau=a \text{ or } s=a, \text{ if } u(a)=0 \\[1ex] \left.\begin{array}{ll} \dfrac{\theta'v-\theta v'}{u'v-uv'}(\tau)\dfrac{\theta(a)}{v(a)} & \text{for } a<\tau\le b,\ s=a \\[1ex] \dfrac{\theta(a)}{u(a)}\,\dfrac{\phi'v-\phi v'}{u'v-uv'}(s) & \text{for } \tau=a, a<s\le b \\[1ex] \dfrac{\theta(a)}{u(a)}\cdot\dfrac{\theta(a)}{u(a)} & \text{for } \tau=s=a. \end{array}\right\} & \text{if } u(a)\neq 0 \end{cases} \end{cases}$$

It is clear that $S^* = S_1^* + S_2^* + S_3^*$. By (3) and (4) of Theorem 1.2.3, $I-S^*$ is Hilbert-Schmidt. Hence $I-S_1^* = (I-S^*) + S_2^* + S_3^*$ is also Hilbert-Schmidt.

Consider the case $u(a) = 0$. Let

$$A^*(\tau) = \frac{\theta'\phi-\theta\phi'}{u'v-uv'}(\tau) \qquad \text{for } a \leq \tau \leq b.$$

$A^*(\tau) = A(\tau)$ a.e. $[w^*]$, since $\mu \equiv 0$ in this case. Hence

$$S_1^* h(\tau) = A^*(\tau)h(\tau)$$

for all $h \in L_2(w^*)$. Note that the measure $w^* = w$ is nonatomic. The argument used in the proof of Theorem 2.1.1 shows that $I-S_1^*$ is zero operator and $A^*(\tau) = 1$.

In case $u(a) = 0$, $\mu \not\equiv 0$, i.e., there is a point mass at the point a. Applying the same argument, we can conclude that $A(\tau) = 1$ for $a < \tau \leq b$, 1 is the only nonzero eigenvalue of $I-S_1^*$, and its corresponding invariant subspace is 1-dimensional (i.e., if $\Psi(\tau)$ is its eigenvector, then $\Psi(\tau) = 0$ for $\tau \neq a$). Condition (β) then follows immediately.

It remains to prove (2)-(4) of Theorem 1.2.3 assuming (β) and (γ). Define the functions $p_t^*(\tau)$ $(t \in [a,b])$ and the operators S^*, S_1^*, S_2^* and S_3^*. S_2^* and S_3^* are Hilbert-Schmidt, and, by condition (β), $I-S_1^*$ is also Hilbert-Schmidt (zero operator if $u(a) = 0$). Hence $I-S^* = (I-S_1^*)-S_2^* - S_3^*$ is

Hilbert-Schmidt, and so is $I-S = I-\Phi^{-1}S^{*}\Phi$. Therefore, (3) and (4) hold. By virtue of (γ), we have, for all $s,t \in [a,b]$,

$$v(s) \int_a^b \chi_{[a,s]}(\tau) p_t^{*}(\tau) dw^{*}(\tau) = \Gamma_P(s,t),$$

which shows that $\Gamma_P(\cdot,t) \in H(\Gamma_Q)$. Since S corresponds to Γ_P, there exists a finite positive constant c_2 such that $\Gamma_P \ll c_2\Gamma_Q$. By interchanging the roles of Γ_P and Γ_Q, we can show the existence of a finite positive constant c_1 with $c_1\Gamma_Q \ll \Gamma_P$. Hence (2) also holds. This concludes the proof.

<u>Remarks.</u> (i) Assume that m(t) is differentiable and its derivative m'(t) is continuous on [a,b]. Then condition (α) may be replaced ([28]) by

(α') If $u(a) = 0$, then $m(a) = 0$.

Proof. By (α), if $u(a) = 0$,

$$m(t) = v(t) \int_a^t \Phi m(\tau) dw^{*}(\tau) = v(t) \int_a^t \Phi m(\tau) dw(\tau).$$

Hence $m(a) = 0$. Conversely, assume (α'). Define $\Psi(\tau)$ by

$$\Psi(\tau) = \begin{cases} \begin{cases} 0 & \text{for } \tau = a \text{ if } u(a) = 0 \\ m(a)/v(a) & \text{for } \tau = a \text{ if } u(a) \neq 0 \end{cases} \\ \dfrac{m'v-mv'}{u'v-uv'}(\tau) & \text{for } a < \tau \leq b. \end{cases}$$

$\Psi(\tau) \in L_2(w^*)$ and

$$m(t) = v(t) \int_a^t \Psi(\tau)\,dw^*(\tau),$$

which shows $m(\cdot) \in H(\Gamma_Q)$.

(ii) Theorem 2.2.1 is an improvement of D. E. Varberg's result [27], [28], and our proof is entirely different from his. He assumes the existence of second derivatives θ'', Φ'', u'', v'' and their continuity, and his proof is based on Baxter's strong limit theorem [4].

(iii) Observe that ([4], [27])

$$\theta'\Phi - \theta\Phi'(t) = \lim_{s \to t-} \frac{\Gamma_P(s,t) - \Gamma_P(t,t)}{s-t} - \lim_{s \to t+} \frac{\Gamma_P(s,t) - \Gamma_P(t,t)}{s-t}$$

and, similarly,

$$u'v - uv'(t) = \lim_{s \to t-} \frac{\Gamma_Q(s,t) - \Gamma_Q(t,t)}{s-t} - \lim_{s \to t+} \frac{\Gamma_Q(s,t) - \Gamma_Q(t,t)}{s-t}$$

(iv) The following examples are taken from [27].

<u>Example 1</u> Let $\{X(t),\ 0 \le t < b \le 1, P\}$ be the process with mean 0 and covariance kernel

$$\Gamma_P(s,t) = \begin{cases} cs(1-t) & \text{for } s \le t \\ ct(1-s) & \text{for } s \le t, \end{cases}$$

where c is a positive constant, and let $\{X(t),\ 0 \le t \le b < 1, Q\}$ be the process with mean 0 and covariance kernel

$$\Gamma_Q(s,t) = \min(s,t) = \begin{cases} s & \text{for } s \leq t \\ t & \text{for } s \geq t. \end{cases}$$

P and Q are equivalent if and only if c = 1.

Example 2. (Ornstein-Uhlenbeck processes) Let $\{X(t),\ 0 \leq t \leq 1, P\}$ and $\{X(t),\ 0 \leq t \leq 1, Q\}$ be Ornstein-Uhlenbeck processes with means 0 and covariance kernels

$$\Gamma_P(s,t) = \sigma_P^2 \exp(-\beta_P |s-t|)$$

and

$$\Gamma_Q(s,t) = \sigma_Q^2 \exp(-\beta_Q |s-t|).$$

Then P and Q are equivalent if and only if $\sigma_P^2 \beta_P = \sigma_Q^2 \beta_Q$.

Any element $g \in H(\Gamma_Q)$ has the following representation

$$g(t) = 2\sigma_Q^2 \beta_Q \int_0^1 e^{-\beta_Q t} \chi_{[0,t]}(\tau) \Psi(\tau) e^{2\beta_Q \tau}\, d\tau + \sigma_Q^2 e^{-\beta_Q t} \Psi(0),$$

with $\Psi \in L_2(w^*)$, where $w^* = w + \mu$, $dw = 2\sigma_Q^2 \beta_Q e^{2\beta_Q \tau}\, d\tau$ and $\mu(0) = \sigma_Q^2$.

2.3 Stationary Gaussian processes

Let $\{X(t),\ -\infty < t < \infty, P\}$ and $\{X(t), -\infty < t < \infty, Q\}$ be stationary Gaussian processes with mean functions zero and covariance kernels $\Gamma_P(s,t)$ and $\Gamma_Q(s,t)$, and let $F_P(\lambda)$ and $F_Q(\lambda)$ denote their spectral distribution functions. It is well known that Γ_Q (similarly for Γ_p) can be represented in the form

$$\Gamma_Q(s,t) = \int_{-\infty}^{\infty} e^{-i(s-t)\lambda} dF_Q(\lambda).$$

This relation gives immediately a representation of $H(\Gamma_Q)$, i.e., $H(\Gamma_Q) \cong L_2(F_Q)$.

In this case we have the following complete characterization for the equivalence of P and Q in terms of their spectral distribution functions. The proof is similar to that of Theorem 2.2.1 and is, therefore, omitted.

Let $F_P^c(\lambda)$, $F_P^d(\lambda)$, $F_Q^c(\lambda)$ and $F_Q^d(\lambda)$ denote the continuous parts and the discontinuous parts of $F_P(\lambda)$ and $F_Q(\lambda)$, respectively. Let

$$a_i = F_P^d(\lambda_i) - F_P^d(\lambda_i - 0)$$

$$b_i = F_Q^d(\lambda_i) - F_Q^d(\lambda_i - 0),$$

i.e., point mass at the discontinuity point λ_i.

<u>Theorem 2.3.1</u> ([8]). P and Q are equivalent if and only if

(α) $F_P^c(\lambda) \equiv F_Q^c(\lambda)$, and

(β) their discontinuity points are the same, and

$$\sum_i \{(a_i/b_i)-1\}^2 < \infty.$$

If, in particular, both processes have continuous spectral distribution functions, i.e., $F_P \equiv F_P^c$ and $F_Q \equiv F_Q^c$, conditions (α) and (β) are reduced to $F_P = F_Q$. It might be of some interest to note that there is a very simple proof to this special case.

If F_P and F_Q are continuous, then, by Maruyama's theorem, P and Q are ergodic. Hence they are either identical or singular. (This is a trivial consequence of Birkhoff's ergodic theorem.)

Suppose now that $T = [-a,a]$, i.e., a finite interval, $\hat{\mathcal{F}}$ is the σ-field generated by a subclass of random variables $\{X(t), t\in T\}$, and $\hat{P}$ and $\hat{Q}$ are the restrictions of P and Q on $\hat{\mathcal{F}} \subset \mathcal{F}$. No general criterion for the equivalence of $\hat{P}$ and $\hat{Q}$ is known at the present time. However, many partial results have been obtained (e.g., [1], [8], [19] and [29]). To indicate the applicability of our method, we consider a very special case in the remainder of this section.

Assume that $\{X(t),P\}$ has rational spectral density and $\{X(t),Q\}$ is an Ornstein-Uhlenbeck process with covariance kernel $\Gamma_Q(s,t) = \sigma^2 \exp(-\beta|s-t|)$. Since $\Gamma_Q(s,t)$ is of the triangular form, the result stated in §2.2 applies to this case, and we have $H(\Gamma_Q) \cong L_2(w^*)$, where $w^* = w + \mu$, $dw = 2\sigma^2\beta e^{2\beta\tau}d\tau$ and $\mu(-a)=\sigma^2 e^{-2\beta a}$ (see Example 2 in § 2.2).

To obtain a necessary condition for the equivalence of P and Q, we assume (2)-(4) of Theorem 1.2.3. The argument is similar to that used in the proof of Theorem 2.2.1. By (2), we have $\Gamma_P(\cdot,t)\in H(\Gamma_Q)$ for all $t\in[-a,a]$. Hence, letting Φ denote the congruence from $H(\Gamma_Q)$ to $L_2(w^*)$, for all $s,t\in[-a,a]$,

$$\Gamma_P(s,t) = 2^{\sigma 2\beta}\int_{-a}^{a} e^{-\beta s}\chi_{[-a,s]}(\tau)\Phi\Gamma_P(\cdot,t)(\tau)e^{2\beta\tau}d\tau + \sigma^2 e^{-2\beta a}\, e^{-\beta s}\Phi\Gamma_P(\cdot,t)(-a).$$

Noting that $\Gamma_P(s,t)$ is infinitely times differentiable for $s \neq t$, since $\{X(t),P\}$ has rational spectral density, define

$$\frac{\partial}{\partial\tau^-}\Gamma_P(\tau,t) = \begin{cases} \dfrac{\partial}{\partial\tau}\Gamma_P(\tau,t) & \tau < t \\ \lim\limits_{s\to t-}\dfrac{\Gamma_P(s,t)-\Gamma_P(t,t)}{s-t} & \tau = t \end{cases}$$

$$\frac{\partial}{\partial\tau^+}\Gamma_P(\ ,t) = \begin{cases} \dfrac{\partial}{\partial\tau}\Gamma_P(\tau,t) & \tau > t \\ \lim\limits_{s\to t+}\dfrac{\Gamma_P(s,t)-\Gamma_P(t,t)}{s-t} & \tau = t \end{cases}$$

$$\frac{\partial^2}{\partial u^+ \partial \tau^-} \Gamma_p(\tau,u) = \begin{cases} \dfrac{\partial^2}{\partial u \partial \tau} \Gamma_p(\tau,u) & u > \tau \\[2em] \lim\limits_{u \to \tau+} \dfrac{\dfrac{\partial}{\partial \tau^-} \Gamma_p(\tau,u) - \dfrac{\partial}{\partial \tau^-} \Gamma_p(\tau,\tau)}{u - \tau} & u = \tau \end{cases}$$

$$\frac{\partial^2}{\partial u^- \partial^+} \Gamma_p(\tau,u) = \begin{cases} \dfrac{\partial^2}{\partial u \partial \tau} \Gamma_p(\tau,u) & u < \tau \\[2em] \lim\limits_{u \to \tau^-} \dfrac{\dfrac{\partial}{\partial \tau^+} \Gamma_p(\tau,u) - \dfrac{\partial}{\partial \tau^+} \Gamma_p(\tau,\tau)}{u - \tau} & u = \tau \end{cases}$$

Define $p(t,\tau)$, for each $t \in [-a,a]$, by

$$\begin{aligned} P(\tau,t) = {} & \sigma^{-} e^{\beta a} \Gamma_p(-a,t) & \tau = -a \\ & \frac{e^{-\beta\tau}}{2\sigma^2\beta} \left\{ \beta \Gamma_p(\tau,t) + \frac{\partial}{\partial \tau^-} \Gamma_p(\tau,t) \right\} \chi_{[-a,t]}(\tau) & \\ & + \frac{e^{-\beta t}}{2\sigma^2\beta} \left\{ \beta \Gamma_p(\tau,t) + \frac{\partial}{\partial \tau^+} \Gamma_p(\tau,t) \right\} \chi_{(t,a]}(\tau) & \\ & & -a < \tau \leq a. \end{aligned}$$

Then

$$2\sigma^2\beta \int_{-a}^{a} e^{-\beta s} \chi_{[-a,s]}(\tau) p(\tau,t) e^{2\beta\tau} d\tau + \sigma^2 e^{-2\beta a} e^{-\beta s} p(-a,t)$$

for $s \leq t$

$$= e^{-\beta s} \int_{-a}^{a} \left\{ \beta \Gamma_p(\tau,t) + \frac{\partial}{\partial \tau^-} \Gamma_p(\tau,t) \right\} e^{\beta\tau} d\tau + e^{-\beta(s+a)} \Gamma_p(-a,t)$$

$$= e^{-\beta s}\left\{\Gamma_p(s,t)e^{\beta s}-\Gamma_p(-a,t)e^{-\beta a}\right\} + e^{-\beta(s+a)}\Gamma_p(-a,t)$$

$$= \Gamma_p(s,t)$$

for $s \geq t$

$$= e^{-\beta s}\int_t^s \left\{\beta\Gamma_p(\tau,t)+ \frac{\partial}{\partial\tau^{+}} \Gamma_p(\tau,t)\right\} e^{\beta\tau}d\tau$$

$$+ e^{-\beta s}\int_{-a}^{t}\left\{\beta\Gamma_p(\tau,t)+ \frac{\partial}{\partial\tau^{-}}\Gamma_p(\tau,t)\right\} e^{\beta\tau}dt + e^{-\beta(s+a)}\Gamma_p(-a,t)$$

$$= \Gamma_p(s,t).$$

Hence, $p(\tau,t) = \Phi\Gamma_p(\cdot,t)(\tau)$ a.e. [w*]. Rewrite $p(\tau,t)$, for $-a < \tau \leq a$, as follows.

$$p(\tau,t) = \frac{1}{2\sigma^2\beta}\left\{\frac{\partial}{\partial\tau^{-}} \Gamma_p(t,t)- \frac{\partial}{\partial\tau^{+}} \Gamma_p(t,t)\right\} e^{-\beta t}\,\chi_{[-a,t]}(\tau)$$

$$+ \frac{e^{-\beta\tau}}{2\sigma^2\beta}\left\{\beta\Gamma_p(\tau,t) + \frac{\partial^*}{\partial\tau}\Gamma_p(\tau,t)\right\}\chi_{[-a,a]}(\tau),$$

where

$$\frac{\partial^*}{\partial\tau}\Gamma_p(\tau,t) = \begin{cases} \dfrac{\partial}{\partial\tau^{-}} \Gamma_p(\tau,t) - e^{\beta(\tau-t)}\left\{\dfrac{\partial}{\partial\tau^{-}} \Gamma_p(t,t)- \dfrac{\partial}{\partial\tau^{+}} \Gamma_p(t,t)\right\} & \tau \leq t \\ \dfrac{\partial}{\partial\tau^{+}} \Gamma_p(\tau,t) & \tau \geq t. \end{cases}$$

Noting the stationarity of $\{X(t), P\}$, write

$$D = \frac{\partial}{\partial\tau^-}\Gamma_p(t,t) - \frac{\partial}{\partial\tau^+}\Gamma_p(t,t)$$

$$= \Gamma_P^{(1)}(0-) - \Gamma_P^{(1)}(0+)$$

Define

$$P_1(\tau,t) = \begin{cases} 0 & \tau = -a \text{ and } t < \tau \leq a \\ (2\sigma^2\beta)^{-1} D\, e^{-\beta t} & -a < \tau \leq t \end{cases}$$

$$P_2(\tau,t) = \begin{cases} \sigma^{-2} e^{\beta a}\, \Gamma_P(-a,t) & \tau = -a \\ \dfrac{e^{-\beta\tau}}{2\sigma^2\beta}\left\{\beta\Gamma_p(\tau,t) + \dfrac{\partial^*}{\partial\tau}\Gamma_p(\tau,t)\right\} & -a < \tau \leq a \end{cases}$$

Then $p(\tau,t) = p_1(\tau,t) + p_2(\tau,t)$. Define $S^* = \Phi S \Phi^{-1}$.

Since $S\Gamma_Q(\cdot,t) = \Gamma_P(\cdot,t)$, $S^*\chi_{[-a,t]}(\tau) = e^{\beta t}p(\tau,t)$. Define operators S_1^* and S_2^* by

$$S_1^*\chi_{[-a,t]}(\tau) = (2\sigma^2\beta)^{-1}\, D\chi_{(-a,t]}(\tau)$$

$$S_2^*\chi_{[-a,t]}(\tau) = e^{\beta t}p_2(\tau,t) .$$

Clearly $S^* = S_1^* + S_2^*$. Define

$$K_2(\tau,u) = \begin{cases} (2\sigma^2\beta)^{-2} e^{-\beta\tau} e^{-\beta u} \Big\{ \beta^2 \Gamma_p(\tau,u) \\ \quad + \beta \Big\{ \dfrac{\partial}{\partial u^-} \Gamma_p(\tau,u) \chi_{[-a.\tau]}(u) + \dfrac{\partial}{\partial u^+} \Gamma_p(\tau,u) \chi_{(\tau,a]}(u) \Big\} \\ \quad + \beta \Big\{ \dfrac{\partial}{\partial \tau^+} \Gamma_p(\tau,u) \chi_{[-a,\tau]}(u) + \dfrac{\partial}{\partial t^-} \Gamma_p(\tau,u) \chi_{(\tau,a]}(u) \Big\} \\ \quad + \beta \Big\{ \dfrac{\partial^2}{\partial u^- \partial \tau^+} \Gamma_p(\tau,u) \chi_{[-a,\tau]}(u) \\ \quad + \dfrac{\partial^2}{\partial u^+ \partial \tau^-} \Gamma_p(\tau,u) \chi_{(\tau,a]}(u) \Big\} \Big\} \\ \qquad\qquad \text{for } -a < \tau,\ u \leq a \ , \\ \dfrac{e^{2\beta a}}{\sigma^2} \cdot \dfrac{e^{-\beta(\tau+a)}}{2\sigma^2\beta} \Big\{ \beta\, \Gamma_p(\tau,-a) + \dfrac{\partial}{\partial \tau^+} \Gamma_p(\tau,-a) \Big\} \\ \qquad\qquad \text{for } -a < \tau,\ -a < t \leq a,\ u = -a, \\ \dfrac{e^{2\beta a}}{\sigma^2} \cdot \dfrac{e^{-\beta(u+a)}}{2\sigma^2\beta} \Big\{ \beta \Gamma_p(-a,u) + \dfrac{\partial}{\partial u^+} \Gamma_p(-a,u) \Big\} \\ \qquad\qquad \text{for } \tau = -a\ ,\ -a < u \leq a\ , \\ \sigma^{-4} e^{2\beta a} \Gamma_p(-a,-a) \\ \qquad\qquad \text{for } \tau = u = -a\ . \end{cases}$$

Elementary but somewhat tedious calculation shows that $K_2\ (t,u)$ is the Hilbert-Schmidt kernel of S_2^*. Since I-S is Hilbert-Schmidt, so is $I-S_1^*$. Again applying the argument used in the proof of Theorem 2.1.1, we can conclude that

$$D = 2\sigma^2\beta,$$

i.e.,

$$\Gamma_P^{(1)}(0-) - \Gamma_P^{(1)}(0+) = \Gamma_Q^{(1)}(0-) - \Gamma_Q^{(1)}(0+).$$

This condition is equivalent to the following requirements (see [5]).

If $\left(\sum_0^m A_k\lambda^k\right) / \left(\sum_0^n B_k\lambda^k\right)(2\pi)$ is the spectral density of $\{X(t), P\}$, then $n - m = 2$ and $|A_m / B_n| = 2\sigma^2\beta$.

This is a special case of Slepian-Feldman's result (see [8]).

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