IMPLICATIONS

OF THE DEVELOPMENT OF MATHEMATICAL

PROBLEM SOLVING, 1894-1983

by

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Administration and Curriculum

1984

ABSTRACT

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Since problem solving in mathematics has been described, questioned, discussed, researched, and criticized frequently over the past century, one could question why such organizations as the National Council of Teachers of Mathematics, as well as individual mathematics educators, have taken so long to recognize its importance to the extent of making problem solving the focus of mathematics instruction over an entire decade, the 1980s.

With such importance being placed on problem solving in mathematics during the 1980s, this investigation was developed to consider the following questions:

1. What ideas, concerns, and approaches helped formulate problem solving in mathematics?

2. What ideas, concerns, and approaches helped elevate problem solving to its current level of importance in mathematics education?

3. Did past practices provide any indication for a successful problem-solving movement for the remainder of the 1980s and the future?

The focus of this historical study was to trace the development of mathematical problem solving over a ninety-year

period. Hundreds of ideas, concerns, and approaches to mathematical problem solving were considered for use in this investigation. Each--in its own way--contributed to the development of problem solving.

In summarizing the conclusions, it was found that concerns for teaching and learning mathematical problem solving similar to those found sixty years ago continue to exist. Problem-solving models developed decades ago continue to be utilized in mathematical problem solving. The following "model" is a composite of problem-solving models developed over the past ninety years:

1. Read the problem carefully.
2. Look for the "known" facts contained in the problem.
3. Determine the "unknown" portion of the problem.
4. Use the "known" and "unknown" portions to determine a procedure for solving the problem.
5. Estimate the final answer.
6. Solve the problem (computation).
7. Compare the answer in the solution with the estimated answer (look for "reasonableness").
8. Label the final answer.

It was further concluded that mathematical problem solving can be taught--successfully. For this to happen, there must be a willingness on the part of classroom teachers, building principals, curriculum coordinators, and school superintendents to fully implement a problem-solving approach to teaching mathematics. Based upon the conclusions, this investigation provides fourteen recommendations for the future teaching of mathematical problem solving.

This work is dedicated to
four special people:

My parents, Martin (1915-1965)
and Wanda (1919-1983)

Monsignor Andrew A. Andrey,
for giving me
the first teaching job of my career

Mrs. Evelvia Baeckler,
for making me
solve problems many years ago

This is for them.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge Dr. Calhoun C. Collier for his assistance, advice, and constructive criticism throughout this investigation and during my doctoral program.

A special thank you to the remaining members of my graduate committee, Dr. William L. Cole, Dr. Walter G. Hapkiewicz, and Dr. William J. Walsh. Their thoughts and concerns will always be a part of my professional makeup.

For critiquing the initial draft of this study, I express my graditude to Dr. Floyd G. Parker and Mr. Michael A. Nash. To Mike and his wife, Jane, a special thanks for the many years of unwavering friendship.

Deepest appreciation to my step-father, José, my mother-in-law, Nancy, and my father-in-law, Joseph, for their encouragement throughout my doctoral program.

Thank you to my brother, Paul, and my sister-in-law, Maribeth, for their love and support at a time when I really needed it.

Finally, I share the joy of completing this study with my wife, Barbara. For her assistance in organizing data, typing and proofreading, for her love, support, and encouragement, and for putting up with my periodic outbursts of "weird" behavior, I simply say "Luv!"

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CHAPTER I

THE PROBLEM

Introduction

In the 1922 opening statement to the teachers in Michigan, Superintendent of Public Instruction, Thomas E. Johnson wrote:

> The subject of arithmetic is a troublesome one because of the fact that we have had so many movements which appear now to have been rather of the nature of fads (p. i).

This statement prefaced a bulletin produced by the Michigan Department of Public Instruction which attempted to improve the course of study in arithmetic. Since that time there have been numerous attempts to improve arithmetic learning and instruction. A number of individuals, organizations, and committees have suggested changes in mathematics instruction both within Michigan and the United States.

In 1980, the National Council of Teachers of Mathematics (NCTM) made eight recommendations for improving school mathematics within its An Agenda for Action: Recommendations for School Mathematics of the 1980s. The Council recommended that:

1. problem solving be the focus of school mathematics in the 1980s;
2. basic skills in mathematics be defined to encompass more than computational facility;
3. mathematics programs take full advantage of the power of calculators and computers at all grade levels;
4. stringent standards of both effectiveness and efficiency be applied to the teaching of mathematics;
5. the success of mathematics programs and student learning be evaluated by a wider range of measures than conventional testing;
6. more mathematics study be required for all students and a flexible curriculum with a greater range of options be designed to accommodate the diverse needs of the student population;
7. mathematics teachers demand of themselves and their colleagues a high level of professionalism;
8. public support for mathematics instruction be raised to a level commensurate with the importance of mathematical understanding to individuals and society (p. 1).

Furthermore, the Council recommended the following "actions" to make "problem solving" the focus of school mathematics in the 1980s:

1. The mathematics curriculum should be organized around problem solving.
2. The definition and language of problem solving in mathematics should be developed and expanded to include a broad range of strategies, processes, and modes of presentation that encompass the full potential of mathematical applications.
3. Mathematics teachers should create classroom environments in which problem solving can flourish.
4. Appropriate curricular materials to teach problem solving should be developed for all grade levels.

5. Mathematics programs of the 1980s should involve students in problem solving by presenting applications at all grade levels.
6. Researchers and funding agencies should give priority in the 1980s to investigations into the nature of problem solving and to effective ways to develop problem solvers (pp. 2-5).

By making "problem solving" the focus of the 1980s, it suggests that the National Council of Teachers of Mathematics is emphasizing some new concept or idea. In truth, "problem solving" ideas, concerns, and approaches have long been described, discussed, and applied in mathematics instruction. Hence, a review of the history of problem solving, based upon the questions in the problem statement, is the focus of this investigation.

Statement of the Problem

Since problem solving in mathematics has been described, questioned, discussed, researched, and criticized for nearly a century, a reader could question why such organizations as the National Council of Teachers of Mathematics, as well as individual mathematics educators, have taken so long to recognize its importance. The magnitude of this importance is evidenced by the NCTM statement that problem solving should be the focus of instruction over an entire decade.

With such importance being given to the emphasis on problem solving in mathematics during the 1980s, this study was developed to consider the following questions:

1. What are the historical ideas, concerns, and approaches that have helped formulate problem solving in school mathematics?
2. What are the historical ideas, concerns, and approaches that have helped elevate problem solving to its current level of importance in mathematics education?
3. Compared to events and influences of the past, what practices indicate a successful problem-solving movement for the remainder of the 1980s and the future?

Historical Background

A number of educational philosophers, mathematicians, psychologists, and teachers have influenced mathematics instruction. For example, the philosophy of Johann Heinrich Pestalozzi centering on the child's perceptions, the social-utility theory of John Dewey, Edward L. Thorndike's connectionist-theory of psychology, William A. Brownell's meaningful arithmetic, and George Polya's questioning techniques included in his problem-solving model have affected the teaching of mathematics in some way.

In addition to many individuals affecting the study of arithmetic, numerous organizations, committees, and projects have provided further direction and influence. A list of organizations providing guidance through their membership would include the National Council of Teachers of Mathematics (NCTM) and the National Society for the Study of Education (NSSE).

During the past century, a number of exemplary projects have provided alternative methods and materials for use in mathematics education. Enjoying common professional visibility were the efforts of the Greater Cleveland Mathematics Program, the School Mathematics Study Group, the Cambridge Conference on School Mathematics, the Nuffield Mathematics Project, the Madison Project, and the Unified Science and Mathematics for Elementary Schools project.

Finally, committees were established at the national level to provide direction and to suggest changes in the teaching of mathematics. These committees include the Committee of Ten on Secondary School Studies, the Committee of Fifteen on Elementary Education, and the Committee of Seven on Grade-Placement in Arithmetic.

Each of the aforementioned educators, organizations, projects, and committees affected mathematics instruction. Within the realm of mathematics, numerous innovations occurred over the past ninety years. As a result, change also took place in mathematical problem solving.

In teaching and learning mathematics, the idea of problem solving is crucial. How did the problem-solving "movement" get its start? Why has problem solving become an increasingly popular area of research? How has problem solving maintained an aura of popularity--and mystique--in mathematics instruction?

Beginning with John Dewey's (1910) ideas on "reflective thinking," each decade since has seen a rise in the number of studies, books, and other sources of commentary related to mathematical problem solving. Edward L. Thorndike accelerated the idea of "problem solving" in The Thorndike Arithmetics (1917), The Psychology of Arithmetic (1922), The Psychology of Algebra (1923), and the Mathematics Teacher (1922). In The Psychology of Arithmetic (1922), Thorndike called for a clearer definition of problem solving. In discussing the general function of problem solving, he wrote:

> The aim of the elementary school is to provide for correct and economical response to genuine problems, such as knowing the total due for certain real quantities at certain real prices, knowing the correct change to give or get, keeping household accounts, calculating wages due, computing areas, percentages, and discounts, estimating quantities needed of certain materials to make certain household or shop products, and the like (p. 9).

Thus, Thorndike believed "problems should be solved in school to the end that pupils may solve the problems which life offers" (p. 11).

In its Third Yearbook (1928), the National Council of Teachers of Mathematics included an entire section devoted to "Problem-Solving in Arithmetic" (pp. 223-267).

In this yearbook, Lucie L. Dower noted the specific preparations that must be met by providing boys and girls the perfect fundamentals of arithmetic, and the abilities to "solve the quantitative situations that arise in everyday activities" (p. 223). In Dower's discussion of problem solving, the following topics were considered: sources of problems, essentials of good problems, classification of problems, emphasis on problem-solving (difficulties), degree of difficulty (of a problem), process of problem-solving, causes of failures in problem-solving, recent investigations and experiments in problem-solving, objectives of problem-solving, psychology of problem-solving, standard tests, and selected topics on experimental work, remedial instruction, and problem-solving analysis (pp. 224-263). A prime objective of the section was to "train boys and girls to solve problems met in actual life situations" (p. 249).

Harry Grove Wheat, in <u>The Psychology and Teaching of Arithmetic</u> (1937), referred to computation and problem solving as "The Persistent Concerns" (p. 126). In comparing the "modern" curriculum with the "old" curriculum, he wrote:

> In content, the modern curriculum differs markedly from the curriculum of the past. Internally, however, the influence of the past still persists in the modern school. Computation and problem-solving were the interests of earlier days; computation and problem-solving remain as interests

> of the present. The extremes to which earlier concern in computation and problem-solving were carried have led to sharp reactions in later days (pp. 126-127).

For Wheat, the purpose of instruction went beyond teaching children to solve problems. The following observation was made in 1937:

> ...the purpose is to provide them with methods of thinking, with ideas of procedure, with meanings inherent in number relations, with general principles of combination and arrangement, in order that the quantitative situations of life may be handled intelligently and without doubt and uncertainty (p. 140).

In its Forty-first Yearbook (1942), the National Society for the Study of Education devoted Chapter XII to the topic of problem solving. In writing this chapter on "problem solving," William A. Brownell noted:

> The reasons for the relative paucity of educational research on problem solving are not hard to find: On the one hand, the problems set in the classroom are exceedingly complicated, more so, in all probability, than those set in the psychological laboratory; on the other hand, children's behavior in the face of problem situations is so variable from problem to problem and from child to child that exact and comprehensive quantitative descriptions are most difficult to attain (pp. 419-420).

George Polya (1945) gave attention to the idea of problem solving with the publication of <u>How to Solve It: A New Aspect of Mathematical Method</u>. In this book, Polya describes the following four-phase procedure for solving problems:

1. Understanding the problem.
2. Devising a plan.
3. Carrying out the plan.
4. Looking back (pp. inside-front and inside-back covers).

The National Council of Teachers of Mathematics included Polya's How to Solve It problem-solving model, with its extensive questioning, as an insert for the inside-front and inside-back covers of its 1980 Yearbook, Problem Solving in School Mathematics.

In 1953, the National Council of Teachers of Mathematics focused part of its attention on "problem solving" by including a chapter on "Problem-Solving in Mathematics" in its Twenty-first Yearbook, The Learning of Mathematics: Its Theory and Practice. In opening the chapter, Kenneth B. Henderson and Robert E. Pingry (1953) stressed the importance of teachers' understanding the theory of problem solving when they wrote:

> The present chapter on problem-solving in mathematics is written on the assumption that mathematics teachers should understand the basic theory of problem-solving which is derived from research in the subject and also see clearly the implications of this theory for methods and procedure in the classroom. Both are necessary. Theory apart from the implications and consequences is largely sterile. Methods and procedures apart from a conceptual framework become little more than a bag of tricks (p. 228).

In a summary of research on teaching elementary school mathematics, C. Alan Riedesel and Paul C. Burns (1973) noted an increase in the number of research studies completed in mathematical problem solving. Although the quality of research design was quite low, Riedesel and Burns indicated "the improvement of problem-solving skills has been the topic for more research studies than any other single topic" (p. 1160). They cited the "practical answers" provided by research as having a greater effect upon the improvement of problem solving "than for any other area of the elementary-school mathematics curriculum" (p. 1160).

Further consideration of the development of mathematical problem solving was highlighted by Arlene Gilda Luc Dowshen (1980). She found that:

> The number of research studies in problem solving by decade remained fairly consistent until the 1960s when there was a marked increase in the number of studies concerning problem solving and this increase continued into the 1970s. In the first half of the 1970s there were as many studies on problem solving as there were in the 1950s and 1960s combined (p. 149).

An increase in the number of doctoral dissertations was given as the major reason for the overall increase in the number of studies cited in the area of mathematical problem solving (p. 149).

The findings of Brownell (1942), Riedesel and Burns (1973), and Dowshen (1980), indicate concerns for mathematical problem solving. These concerns have now become the focus of an entire decade, not just a section in an educational encyclopedia or a chapter in a yearbook. This investigation takes into account these concerns and considers their influences on mathematical problem solving over a ninety-year period.

Methodology

This historical investigation utilizes both primary and secondary sources of reference material. Primary sources are defined as original documents, such as textbooks, manuscripts, and school records (Ary, Jacobs and Razavieh, 1972; Best, 1981; Borg and Gall, 1979). Textbooks, such as *The Psychology of Algebra* (1923) and *How to Solve It: A New Aspect of Mathematical Method* (1945), are examples of primary sources. Secondary sources are defined as published bibliographies, referenced works, histories, and encyclopedias (Ary, Jacobs, and Razavieh, 1972; Best, 1981; Borg and Gall, 1979; Gay, 1976). For example, writing in the *Encyclopedia of Educational Research* (1969) is considered to be a secondary source.

Both primary and secondary sources are considered to be important to the development of this study. Depending on the type of reference, journal articles are considered either primary or secondary sources. However, for purposes of this investigation, journal articles found in *School Science and Mathematics* (1902-1983), the *Mathematics Teacher* (1910-1983), and the *Arithmetic Teacher* (1954-1983) are considered to be primary sources even though the format and style of the journal was changed in a reprinting process.

Theoretical Framework

The National Council of Teachers of Mathematics has given considerable emphasis to the importance of "problem solving" in several of its yearbooks (1928, 1953, and 1980). The Council, in its An Agenda for Action: Recommendations for School Mathematics of the 1980s (1980), gave top priority to the teaching of problem solving. How this priority, given by the National Council of Teachers of Mathematics, evolved, provides the basic framework for this investigation.

Organization of the Study

This study traces the mathematical problem-solving movement through three periods over the past ninety years, 1894-1983. Chapter I includes the introduction, statement of the problem, historical background, methodology, theoretical framework, limitations, assumption, and research questions. Chapters II, III, and IV present reviews of mathematical problem solving for the three periods: 1894-1928, 1929-1953, and 1954-1983. The data included in Chapters II, III, and IV are presented in sequential order based upon the year in which the sources were published or cited. Hence, the development of these three chapters is chronological. Chapter V includes the summary, conclusions, recommendations, and implications for future research in mathematical problem solving.

Limitations

Several limitations were imposed on this study in order to provide certain parameters from which this research evolved. The starting date for the data covered in this investigation was determined by the curricular changes influenced by the Report of the Committee of Ten on Secondary School Studies (1894). The three periods were selected according to the emphasis given mathematical problem solving in the National Council of Teachers of Mathematics yearbooks for the years 1928, 1953, and 1980. Thus, the ending date for the first period was determined by the date of the 1928 yearbook. The next year, 1929, established the starting date for the second period. The ending date was determined by the date of the 1953 yearbook. The third period was established by the remaining years included in the interval, 1954 through 1980. Hence, the three periods are: Period I (1894-1928), Period II (1929-1953), and Period III (1954-1980). For purposes of making this study as current as possible, the years 1981-1983 were added to the third period, changing Period III from 1954-1980 to 1954-1983. Finally, although "general" ideas on problem solving were not ignored, major attention was given to the ideas, concerns, and approaches more closely aligned with mathematical problem solving.

Assumption

The assumption of this study is that ideas on problem solving found for any particular period are based upon the development of problem solving from the preceding period. Any attempt to make recommendations for the future teaching of problem solving in mathematics is predicated on the previous development. Thus, ideas for Period II are based upon the findings found from Period I; and ideas for Period III are based upon the findings found from Period II. Finally, recommendations for the future development--and teaching--of mathematical problem solving are based upon the findings found from Period III.

Research Questions

In reviewing mathematical problem solving, this investigation examines the following research questions:

1. What ideas, concerns, and approaches on problem solving have had an impact on the teaching of mathematics?
2. What are the peak periods during the past ninety years in which the development of problem solving caused changes to take place in mathematics instruction?
3. What problem-solving models have been most popular in mathematics instruction over the past ninety years?

4. What ideas, concerns, and approaches on problem solving developed in the past continue to affect the teaching of mathematics?

CHAPTER II

A REVIEW OF MATHEMATICAL PROBLEM SOLVING
PERIOD I: 1894-1928

Development of problem solving in mathematics during the period 1894 to 1928 was influenced by the curricular changes of the Committee of Ten and the works of educators such as David Eugene Smith, John Dewey, and Edward Lee Thorndike. Prior to this period, much of the teaching and learning of arithmetic was considered to be less than "practical," revolving mostly around "principles, rules and facts." Education, in general, consisted of learning the three R's: reading, 'riting and 'rithmetic. As such, work in arithmetic consisted mainly of drill and practice.

> On a fair estimate, not less than one-fourth of the pupil's time, for the first eight or ten years of his school life, is given to the study of this subject; but the results are too often quite inadequate to this large expenditure of time, the most that can generally be claimed being a tolerable familiarity with the processes of the fundamental rules, common fractions, and denominate numbers, with a very imperfect knowledge even of the processes of decimal fractions, proportion, evolution, and the business rules of arithmetic (Kiddle and Schem, 1877, p. 40).

However, during the 1890s, educators began seriously questioning the education process in America. In 1892, the National Education Association (NEA) appointed a committee to analyze the school curriculum throughout the country. The appointment of the Committee of Ten, chaired by Charles W. Eliot, President of Harvard, typified two forces in mathematics education at this time:

> (1) the concern of persons with a major initial interest in education as a whole for the specialized subject-matter field of mathematics and (2) the influence of national committee reports as stimulators of reform (NCTM, 1970, p. 5).

In 1893, the Committee of Ten endorsed the Herbart method because it believed the bottom line to teaching mathematics was "...to store the mind with clear conceptions of things and their relations" (Kramer, 1970, p. 23).

The Report of the Committee of Ten on Secondary School Studies (1894) provided a number of curricular changes that would have an effect upon education for many years. Recommendations for mathematics included the need for shorter and more enriched assignments, the use of concrete solutions to problem solving, an earlier introduction to algebra, an earlier and more intuitive approach to geometry, and the integration of arithmetic with other subjects in the curriculum (Sizer, 1964, pp. 232-235).

Even with the Committee Report, change did not come about immediately. Mathematics teachers were slow to change, and drill in computation was still a major focus in teaching arithmetic. In the "Editor's Introduction" to The Teaching of Elementary Mathematics by David Eugene Smith (1900), Nicholas Murray Butler wrote:

> Perhaps no single subject of elementary instruction has suffered so much from lack of scholarship on the part of those who teach it as mathematics. Arithmetic is universally taught in schools, but almost invariably as the art of mechanical computation only. The true significance and the symbolism of the processes employed are concealed from pupil and teacher alike. This is the inevitable result of the teacher's lack of mathematical scholarship (p. ix).

With such lack of scholarship, the suggested changes set forth by the Committee of Ten led to curricular scrutiny by liberal arts colleges and universities. The aim of the Committee's analysis was the academic program of the normal school with its elementary content. "The criticism became particularly serious when the normal schools began to consider it their legal and professional responsibility to provide programs for the preparation of teachers for the public high schools as well as for the elementary schools" (NCTM, 1970, p. 308).

During the 1890s, many theories of teaching began making their way to the front in American education. Though not a new theory, the principle of "apperception," developed decades earlier by Johann Fredrich Herbart, played an important role during this period of time. The theory, whereby learning takes place by "teaching the new in terms of the old," began as a four-step process: preparation, presentation, comparison, and generalization. Proponents of Herbartian Philosophy later expanded the process to include the following five steps:

1. Preparation--a review of related materials.
2. Presentation--new material was presented, analyzed, and clarified.
3. Association or comparison--new material was compared to old material; an organization of facts.
4. Generalization or conclusion--generalizations made from stated facts.
5. Application--an application of the principles learned (Kramer, 1970, pp. 22-23; Meyer, 1967, p. 23).

John Dewey provided a philosophy quite different from that of Herbart. He believed that the learning of arithmetic should be related to daily living experiences. Furthermore, Dewey advocated that the learning of numbers should become an active process by which children actually measure or count (Baur and George, 1976, p. 20; Kramer, 1970, p. 24). In summary, Baur and George (1976) wrote:

> The net result of these influences led to a period in education commonly called the social-utility period. During this period those aspects of mathematics which were not directly applicable to everyday living were dropped from the curriculum. The tool (sic) dimension of mathematics was brought to a peak as children measured to bake cookies and played "grocery store" at school. The "how" of the operations became more important than the "why." Obtaining the correct answer was of utmost importance. Once again, children were taught the sequence of steps to a procedure and then they were drilled on their learning. This approach to teaching corresponded to the theories of learning being set forth by Edward Thorndike and other connectionist psychologists of the period (p. 20).

Change in the teaching of mathematics in the early 1900s came about as a result of many influences. According to David Eugene Smith (1909), these changes occurred at both the elementary and secondary levels. There was a great interest evolving in the psychological development of children. The influence of business also had an effect upon curriculum and textbook development (p. 209). Foreign influences in commerce, practical psychology, and science, and the "dogma of thoroughness," played important roles in changing mathematics instruction in the United States prior to 1909. Smith (1909) provided several suggestions for improvement including one of calling for a mathematics curriculum that would link the teaching of mathematics in grades 8 through 12 by combining algebra, concrete geometry, and arithmetic (pp. 212-217).

Commerce, business, and life itself played important parts in the development of the mathematics curriculum during the first two decades of the twentieth century. As Myrtie Collier wrote, in 1914:

> Arithmetic should be taught in such a way that the number facts learned may be usable in the life of the child, and also usable in the everyday life of the individual after leaving school. To secure this result in our school work two factors of arithmetic need special emphasis, namely:
>
> 1. The removal of all mystery in the fundamental operations and principles of number relations.
> 2. The application of the number facts through practical problems to bring the child into vital touch with actual business and industry (p. 294).

The focus of Collier's comments was to suggest ways of developing principles of arithmetic so that the "mystery in mechanical operations" could be eliminated.

Collier highlighted the inductive approach by illustrating the five steps in the Herbartian plan with a goal: To get children to learn a principle, verbally, and apply it in number form (p. 296).

Entering the 1920s, considerable emphasis on mathematical problem solving was provided by Edward Lee Thorndike. In The New Methods in Arithmetic (1921), the author summarized "the older system" of organizing arithmetic content. In calling the older system a beautiful thing "to look at, but very hard to learn by" (p. 83), the author included a list of concepts that "the pupil was supposed to learn" (p. 83). The entire list of Thorndike's "organization of learning" is contained in Appendix A.

For Thorndike, to teach arithmetic was to teach life itself. "Life organizes its arithmetical demands, not so much by the nature of the processes as by the situations involved" (p. 96). Furthermore, in discussing "the new methods," he allowed for the organization of arithmetic to focus on those situations in which the learner frequently found himself. The list of "organized centers for arithmetical training" for fourth grade children is included in Appendix B, "Titles of Lessons."

According to Thorndike (1921), an important limitation in providing "appropriate" work in arithmetic for children was found in the classroom itself because mathematics problems were frequently described in words rather than solved in "actual" situations. As was noted at the time:

> One important limitation due to the conditions of classroom teaching is that the facts of the problem can so seldom be presented to sense--must so often be described in words. The problems of life are most often questions about situations or facts actually existing before the pupil's eyes, less often questions which the person puts to himself in connection with his past affairs or future plans, and least often questions put to him in words by another. In proportion as we can escape this limitation and actually present the situations, we not only are surer of preparation for life, but also find it easier to teach the pupils how to attain correct solutions (p. 126).

In order to compensate for the "word limitations," Thorndike provided a three-pronged problem-solving model. The three main elements in this model were: "(1) to know just what the question is, (2) to know what facts you are to use to answer it, (3) to use them in the right relations" (p. 126).

Finally, Thorndike (1921) included the idea of problem solving in an effort to provide the proper tools included in the "new methods." The "new methods" provided so-called "real" situations to every learning experience.

> Consequently the newer methods try (1) to provide real situations or projects where that is feasible, and (2) to encourage the pupil to identify himself with the person whom the problem represents as acting or planning. If the reality cannot be supplied, and if the sense of personal participation cannot be aroused, they try at least (3) to free the problem from difficulties due (a) to its vocabulary and structure or (b) to lack of experience by the pupils of the facts described (p. 127).

Middlesex A. Bailey (1923) summarized Thorndike's influences on the philosophy of teaching arithmetic. In doing so, he reviewed the general principles and instructional ideas set forth by Thorndike in <u>The Psychology of Arithmetic</u> (1922), <u>The New Methods of Arithmetic</u> (1921), and <u>The Three-Book Series of Arithmetic</u> (1917). In the summarization, Bailey (1923) outlined Thorndike's system of solving problems in the following four-step process:

1. Introduce each process and principle by a problem illustrating its need.
2. Tell the learner what to do.
3. Require him to verify his answers from known facts.
4. Expect him to conclude that the procedure is right "Because doing so always gives the right answer" (pp. 129-130).

In critiquing Thorndike's system, Bailey analyzed each step, made several modifications, and proposed the following outline:

1. Introduce each process and principle by a problem illustrating its need.
2. Ask the learner to solve the problem by a method known to him.
3. Invite him to join with you in finding a shorter process.
4. Require him to fix the new process in mind by solving one or more other problems, first by the old method and then by the new.
5. Require the solution of other problems by the new method without statement of the rule, or by the statement of the rule and its application. In either case, require a proof of the answer by some check (pp. 134-135).

Bailey was concerned about the way in which children not only learned arithmetic but how much of the problem-solving process was actually retained. Thus, his proposed outline was an effort to provide children with a series of steps in problem solving that could include more "reasoning skills" in external situations. As Bailey noted in 1923:

> At present, pupils on leaving school are almost helpless to think out by themselves the solution of problems that depart from type, or even to solve type problems when stated in unusual forms (p. 140).

The author, however, was not alone in attempting to provide more "reasoning" skill in the problem-solving process.

Ability in thinking skill was becoming more prominent in mathematics education during the mid-1920s. The literature in arithmetic began to include topics on questioning (Gould, 1923), and relationships between thinking and memorization (Atkins, 1923). Clarence G. Gould included the following comment in an article written on the "Art of Questioning" in 1923:

> Reflective thinking is one of the most important things in mathematics. In the form of problem-solving, and the working of examples, reflective thinking plays a large part (p. 52).

Thus, mathematics educators began looking at the notion of problem solving as meaning something more than just "finding an answer."

In getting children to develop independent reasoning skill, Gould (1923) suggested that teachers include "reflective thinking" as a part of the process in solving problems. Children were encouraged to organize their thoughts, formulate principles, and to recall the principles that had been learned earlier (p. 53). K. W. Atkins (1923), in stating that "Thinking is problem solving," provided the following description:

> The individual is confronted with some situation which he <u>must</u> meet by a series of appropriate acts. (The need for the solution must be keenly felt before thinking takes place. Otherwise, the solution will be dodged or delayed.) When the problems becomes real to the individual, various solutions will be recalled from previous experiences with somewhat similar problems. These are each tried out to see if they will meet the confronting situation. Finally, some combination of experiences is found which will effect (sic) a solution to the problem. Then, and then only, may the thinking process be said to be complete (p. 762).

In 1925, John R. Clark and E. Leona Vincent compared two different methods of analyzing--and solving--mathematical problems. Included in this analysis was the following "conventional" problem-solving method:

1. What is asked for in the problem.
2. What facts are given in the problem.
3. How should these facts be used to secure the answer.
4. What is the answer to the problem (p. 226).

In contrast to this four-step method, the two authors introduced a "Graphical Method" in which pupils had to "determine what is to be found in the problem, what it depends upon, what each of these dependents in turn depends upon, and so on until he has unravelled the essential facts and relationships in the problem" (p. 226). In comparing the two methods, Clark and Vincent (1925) found that children who used the "Graphical Analysis Method" improved during practice more than those children who used the "Conventional Analysis Method" (pp. 232-233).

In 1925, Clifford Brewster Upton, in "A Plea for Professionalized Subject Matter," called for providing practice in problem solving (p. 415). Along with a "provision for bringing each normal-school student up to some standard skill in the fundamental operations," Upton called the problem-solving provision a "complicating factor of considerable importance" (p. 415). He suggested connecting practice in problem solving with work done in socio-economic arithmetic (business or personal arithmetic).

J. M. Kinney (1925), in writing about "The Value of the Verbal Problem," considered the use of such problems in teaching algebra and cited the following four reasons for including verbal problems.

1. To give a concrete application of an abstract theory.
2. To develop the ability to translate a quantitative relationship, expressed in words, into the symbolic language of algebra.
3. To develop the ability to think.
4. To develope (sic) the ability to solve the problems of a quantitative nature that may rise in the various fields of human endeavor (pp. 267-268).

Finally, in discussing the "ability to think," Kinney wrote, "Of course to solve any problem is to think. And, no doubt, most of us feel that practice in thinking improves the ability to think" (p. 267).

In the monograph, Diagnostic Studies in Arithmetic (1926), G. T. Buswell discussed the failures in elementary school arithmetic and attributed most of the failure in the elementary school to arithmetic. In the introduction, he noted:

> The failures in the elementary school are caused more frequently by arithmetic than by any other subject in the curriculum. The failures due to arithmetic are to be traced to three conditioning factors, namely, (1) the materials of arithmetic, consisting of textbooks, practice exercises, and special devices; (2) the teacher's methods of instruction and her manner of presenting arithmetic to the pupils; and (3) the methods and mental processes of the pupils (p. 1).

Much of the monograph considered work related to the four operations--addition, subtraction, multiplication, and division. The author believed that proper methods of work should be emphasized before children became involved in drill. However, in terms of ability in working with the four operations, Buswell (1926) wrote:

> ...a mastery of these fundamentals alone does not guarantee ability to meet all the demands of the school or of society so far as arithmetic is concerned. After the fundamental operations have been mastered, there is still the difficulty of applying them to the solution of problems. Problem-solving is, in a very real sense, the test of one's ability in arithmetic. There need be no conflict, however, between the emphasis on the fundamentals and the emphasis on problem-solving. Both must be adequately taught. While a mastery of the fundamentals does not guarantee ability to solve problems, a lack of knowledge of the fundamentals very seriously interferes with problem-solving. Pupils frequently fail to solve problems correctly because their methods of manipulating the four fundamental processes are so clumsy that their attention is diverted from the problems to the details of the number combinations (p. 195).

Buswell was not alone in his belief that problem solving should be an integral part of the mathematics program. The purpose of arithmetic and the methods by which it was taught were continually being discussed, updated, and improved. Problem-solving improvement was important in this updating process.

In the First Yearbook entitled <u>A General Survey of Progress in the Last Twenty-five Years</u>, the National Council of Teachers of Mathematics provided a general review of the mathematics taught during the first quarter of the 1900s. In a survey of advances in arithmetic, David Eugene Smith (1926) wrote:

> Perhaps the most important change of all is seen in the purpose of teaching arithmetic. A quarter of a century ago it was felt that the subject should be hard in order to be valuable, and it sometimes looked as if it did not make so much difference to the school as to what a pupil studied so long as he hated it. The old idea that this was good for the mind and soul was not at that time fully discarded. There was also prevalent the idea that as many applications of arithmetic should be introduced as the time allowed, irrespective of whether they were within the mental horizon of the pupil or within the probable needs of his life after leaving school. This view has now been changed; the purpose of teaching arithmetic has come to be recognized as the acquisition of power to calculate within the limits of the needs of the average well-informed citizen. It has also come to be recognized that the problem is primarily designed to show a need for computation, by giving applications that add to the interest in calculation and by introducing the puzzle element of problem-solving, which may add further interest. A secondary purpose of the problem is the imparting of some knowledge of the economic conditions, that the pupil will find in daily life, this being presented to him in a simple manner that will make it seem interesting and worth while (p. 19).

The Second and Third Yearbooks accorded a more in-depth look at specific aspects of the mathematics curriculum and of selected topics in teaching the subject of arithmetic. One of the concepts reviewed and scrutinized was problem solving in mathematics.

The Second Yearbook of the Council focused upon the Curriculum Problems in Teaching Mathematics. In it, F. B. Knight (1927) assembled information on nine specific topics in mathematics instruction. One of those, Discussion Nine, concentrated on "Remedial Drill on Arithmetic Problem Solving." In reacting to the question, "What Can Be Done for Pupils Who Need Remedial Work?" the author included the work of H. A. Greene. In doing so, a five-step problem-solving model was displayed (see Appendix C). The five steps--comprehension, analysis and organization, recognition, solution, and verification--closely resembled an analysis of the thinking process attributed to John Dewey (p. 65). In discussing the plight of "verbal problems," Knight (1927) provided the following discussion:

> Most problems now supplied to children are too difficult for them. For example, the counterpart of problems known to be useful in determining the intellectual difference of twelve- and thirteen-year-old pupils can frequently be found in material supplied to ten- and eleven-year-old pupils. Our present practice relative to problem material is based on a gross over-estimation of the child's ability to reason. This practice doubtless is largely responsible for the frequent over-helpfulness of teachers on problem material, which is a crutch few would defend. It is also probably responsible for the nervous and often ill-formed attempts to use devices to teach

> children how to solve problems. It is moreover probably responsible in part for the criticism of the public relative to problem solving in the schools.... We lead the public to think that a child can solve problems of a given nature since we give him these problems to solve. But either the child is too immature to solve these problems or the available methods for teaching problem solving are so inferior that the child cannot work them. The public notes the persistent gap between what we ask the child to do and what he actually can do, and blames the school (p. 20).

Finally, in discussing effective methods for improving skills, Knight (1927) reported a study by O. S. Lutes, the purpose of which was to discover effective techniques for teaching problem solving in the elementary grades. Children's errors in problem solving were found to fall under three main headings: "(1) ignorance of principle, or wrong operation, (2) comprehension difficulties, (3) computation errors" (p. 49). Furthermore, of the five comments in the conclusions of the study, two specifically made note of problem solving in mathematics. These comments stated that "all pupils of normal intelligence can profit from instruction in problem solving" and "motivation is an important factor in securing improvement in problem solving ability" (p. 51).

G. T. Buswell also made an extensive contribution to the Second Yearbook. In observing that the subject of arithmetic had become "a productive field of research," Buswell (1927) discussed five general problems found in arithmetic. They were:

1. the difficulties in reading encountered in arithmetic;
2. the teaching contribution of arithmetic textbooks;
3. the treatment given to the number system as such;
4. the preparation of drill exercises keyed to specific needs as revealed by diagnoses of pupils' work; and
5. the grade location of the different arithmetical processes (p. 73).

In elaborating on the second problem regarding "the teaching contribution of arithmetic textbooks," Buswell (1927) included the following discussion on problem solving:

> Problem-Solving. The mental processes employed in problem-solving have not been subjected to the same degree of analysis as the processes employed in working with the four fundamental operations. Ultimately, such detailed diagnoses will be made. It has been assumed quite generally that children solve problems by the logical methods which are supposed to be characteristic of adult thinking. This logical procedure is described ordinarily in such steps as the following: (a) defining the problem; (b) recalling related facts which bear upon it; (c) setting up and evaluating hypotheses as to its solution; (d) selection of one hypothesis; and (e) the final verification of the hypothesis and the solution of the problem. If one will observe carefully the mental processes employed by children in the actual solution of problems, he will find that there are many deviations from this so-called logical method. Children make frequent short cuts in their thinking. They frequently act upon the first hypothesis which comes to their minds rather than make a careful evaluation of several hypotheses. More commonly still, they take their cue from certain words in the problem or from certain forms of expression and proceed with very little logical thinking at all. While the logical steps by which the process of reasoning has been described may represent the way one should think, they certainly do not represent the way in which many children do think. An investigation which is badly needed is a detailed individual diagnosis of the actual thinking carried on by children in

> solving the ordinary problems presented in arithmetic. Until such a diagnostic survey of children's reasoning is available, it will be difficult to supply suitable instructional material for problem-solving (pp. 85-86).

The inclusion of, and emphasis on, verbal problems were seen as important to the mathematics curriculum. In discussing the history and significance of problems in algebra, Vera Sanford (1927) discussed the use of the verbal problem as a means for developing skill in problem solving. The following comments are noteworthy:

> One of the most important reasons for including verbal problems in our work in elementary algebra is to provide practice in analyzing a given situation to see what mathematical relationships are present and how they may be manipulated to answer a quantitative question. This involves the ability to read intelligently and to organize given facts.
>
> It is this problem-solving attitude that will, it is hoped, transfer to other situations in so far as the student sees them to have like elements (p. 3).

In conclusion, Sanford (1927) added:

> The object, however, is not convincing the student that things happen in the world as they do in algebraic problems, but providing him with work of such a degree of reasonableness that he will not be distracted from the purpose of problem solving by unnatural and irrelevant situations (p. 7).

Furthermore, E. H. Taylor (1927) related problem-solving techniques within the realm of algebra. The author provided the following eight-step process in "teaching pupils how to solve problems":

1. Read the problem carefully.
2. Decide what is to be found.
3. Decide what facts needed in the solution are given in the problem.
4. Decide what other facts are needed, and find these facts.
5. Determine the processes needed in the solution.
6. Estimate the result.
7. Perform accurately the necessary computation.
8. Check the results (pp. 106-107).

The eight-step process described by Taylor is not limited to algebraic solutions. Similar processes can also be found in general arithmetic.

In discussing "The Systematic Solution of Arithmetic Problems," Paul Ligda (1928) summarized the so-called "directions" given by textbook writers for solving problems. The following set of "directions" were cited:

1. Read the problem with understanding.
2. State what is given.
3. State what is to be found.
4. State the processes you will use.
5. Solve the problem.
6. Check (p. 24).

In response to the six-step procedure which Ligda (1928) referred to as an "analysis," the following five-step process was suggested:

1. Read the problem.
2. State the main quantitative thoughts as briefly as possible.
   Or: Make statements of comparison between sets of quantities.
   Or: Write verbal equations.

3. Identify the quantities with the terms of (2).
4. Substitute if necessary, then do the arithmetic work indicated.
5. Interpret the result, check, and prove (p. 25).

Originally, this process was designed for algebra students. However, Ligda believed that it could be easily modified to meet the problem-solving needs of younger pupils.

In 1928, G. W. Myers described the three leading objectives in public school arithmetic. They were: "(a) intelligent control of number and space procedures, (b) calculatory skill, and (c) problem-solving power" (p. 281). Myers believed that although problem solving was the highest objective of all, it was too high to be effective for children. In calling for greater control in the teaching of problem solving, the writer made the following observation:

> We must however keep up the unremittent struggle for some appreciable measure of problem-solving power, but let us not forget that there can be high social mastery of arithmetic without any considerable degree of mastery of such problem-skill as the customary "word" problems of current texts represent. Excessive preoccupation with this type of problem-solving skill to the extent of making it the ruling objective cannot but continue to result in futility (p. 282).

As an alternative to "teaching by drill," Myers (1928) included the following eight steps in teaching an arithmetic process or topic:

1. Motivate the topic.
2. Develop the concepts.
3. Assimilate the ideas.

4. Practice for skill.
5. Apply the new skill.
6. Problem tactics and strategy.
7. Consolidate new with old skills.
8. Keep recalling old skills to hold them (p. 283).

Although a certain amount of drill was necessary throughout the procedure, Myers looked for the establishment of a routine and an emphasis on recall to develop skill in problem solving or for what was referred to as assimilation, skill-building, and skill-holding.

In an address read before the National Council of Teachers of Mathematics Conference in Boston (February 24, 1928), Walter F. Downey reiterated the progress made by pupils in mathematics during the preceding twenty-five years. In doing so, he cited the major changes that had transpired in mathematics education. At the top of the list was an effort to focus more on what the author referred to as "the human phase of the subject." In considering the human factor in education, Downey made the following three important points regarding the processes in education:

> (1) that the pupils should be given opportunity to be problem finders as well as problem solvers, because problem finding and solving are infinitely more productive in the development of vital minds than is problem solving alone; (2) that whatever activity is undertaken, whether it be academic study, mechanic arts, practical arts, fine arts, or athletics, the principle should be accepted and followed that if the thing is worth assigning and is properly assigned it is worth MASTERING one hundred percent, not sixty percent or seventy percent only, before passing on to the next bit of work; and (3) that, before considering any problem

> as completed, the pupil should feel sure in his own mind, through the use of checks and other means, that his work is correct (p. 242).

Downey believed that by accomplishing problem solving through guidance of the three steps, students would accomplish "power and habit" in their problem-solving development.

The National Council of Teachers of Mathematics analyzed mathematical problem solving in its Third Yearbook (1928). By devoting an entire chapter to the concept of problem solving, NCTM accorded mathematics educators with a comprehensive look at the strengths and weaknesses found in the teaching of arithmetic. In the chapter on "Problem-Solving in Arithmetic," Lucie L. Dower (1928) included a two-fold description of how children acquired problem-solving ability. First, teachers provided children experience "by giving specific preparation for the kinds of problems the pupils will meet in life," and second, "by giving general preparation for all kinds of problems" (p. 223). For Dower, preparation in the latter form, general problem solving, was considered more important than teaching for specific types of problems. She believed that through a general preparation of problem solving, children would "acquire the ability to judge a given problem on its own merits, to study the relationships that exist between the various quantities involved, and to think out the solution" (p. 223).

In considering a general preparation toward the teaching of mathematical problem solving, Dower (1928) noted:

> This approach aims to develop skill in planning the solution as well as ability to execute the plan. This mastery of problem-solving comes only from meeting many different kinds of problems, from seeing many relations, and from reasoning out each problem in terms of the relationships that are involved. By this method, pupils are more likely to recognize similar problems in any new situation and to apply the correct solution (p. 223).

With the aforementioned comments, she set the tone for the entire chapter in which the idea that preparation of general problem-solving ability was an essential part of the education process.

In analyzing mathematical problem solving, Dower (1928) divided the chapter into three major components. The first part investigated the elements and facts that made up problems and analyzed the facts involved in problem solving. The second part provided an investigation, described research completed, noted important psychological aspects involved, and provided a list of tests that could be used in testing children's ability in problem solving. The third part provided a description of experiments (activities) that had been carried out in a normal school setting.

In addition to some of the ideas and theories on problem solving in mathematics already mentioned in this study, Dower discussed other studies, ideas, and theories on problem solving that were pertinent during the 1920s. In a summary of the causes of failure in problem solving described in the Second Yearbook of the National Council of Teachers of Mathematics, Dower (1928) provided the following list:

1. The language used in the problems is too difficult. It is beyond the reading standard for the grades in which it occurs.
2. The pupils have not had sufficient training in interpreting thought from silent reading.
3. The pupils lack understanding of the technical terms involved.
4. The situations described by the problems are not understood by the pupils, because they are outside the range of the pupils' experiences.
5. The fundamental combinations, facts, or processes called for in the solution of the problems have not been habituated.
6. The pupils are unable to see the relations between the steps called for in the solution of the problem.
7. The pupils are so burdened with undue labeling and elaborate indication of steps that their minds are diverted from the real process of solution (p. 239).

In another summary taken from the Fourth Yearbook of the Department of Superintendence, Dower (1928) noted the following eight causes of failure in problem solving:

1. Lack of general ability in silent reading.
2. Lack of familiarity with technical terms in arithmetic.
3. Carelessness in reading.
4. Lack of experiences necessary to understand the setting of the problem.
5. Inadequate skill in computation.

6. Lack of knowledge of such essential facts as tables of weights and measures.
7. Inability to see the relationships in the problems so as to choose the proper operation.
8. Inability to do reflective thinking (p. 239).

Dower provided mathematics educators with a number of summaries of works by individuals who were investigating mathematical problem solving in the 1920s. For example, she reported P. R. Stevenson's description of failure in problem solving:

1. Physical defects.
2. Lack of mentality.
3. Lack of skill in fundamentals.
4. Inability to read, which of necessity affects the ability to read arithmetic problems.
5. Lack of general and technical vocabulary.
6. Lack of proper methods or technique for attacking problems (p. 239).

Finally, in acknowledging work completed by F. B. Knight and G. T. Buswell in the Second Yearbook of the National Council of Teachers of Mathematics, Dower (1928) cited six suggestions concerning the future of research in mathematical problem solving. In quoting Knight and Buswell, she wrote:

1. We should find out what types of problems children can do and should be required to do. We constantly over-estimate the child's ability to reason. Most problems given to children are too difficult for them.
2. We should discover the most effective classroom technique for teaching the skill of problem-solving in elementary school arithmetic.

3. We should attempt a standardization of the vocabularies used in the different books and a correlation, grade by grade, between the vocabularies used in arithmetics and the vocabularies encountered in general reading in the elementary schools.
4. There is need of a detailed individual diagnosis of the actual thinking carried on by children in solving the ordinary problems presented in arithmetic; perhaps, to begin with, those dealing with one particular unit only. Until such a survey of children's reasoning is available, it will be difficult to supply suitable instructional material for problem-solving.
5. There is need of the preparation of a textbook which is composed essentially of explanatory material, with an accompanying manual of practice exercises. The space which has been previously used for examples and problems may well be given to detailed instructions to the pupils relating to methods of procedure, an explanation of arithmetical operations, and a presentation of social situations in which arithmetic is to be applied.
6. The following are possibilities for the development and application of remedial instruction units:
   a. Exercises stressing vocabulary.
   b. Exercises stressing problem comprehension.
   c. Exercises stressing what is given in the problem.
   d. Exercises stressing what is called for in the problem.
   e. Exercises stressing the estimation of answers.
   f. Exercises stressing choice of procedure.
   g. Exercises stressing relationships in problems (pp. 248-249).

The Third Yearbook, sponsored by the National Council of Teachers of Mathematics, summarized the problems facing mathematics educators in the 1920s. Lucie L. Dower (1928), in examining ideas on problem solving, provided a "new challenge" for those involved in mathematics education.

Summary of Period I: 1894-1928

During the period, 1894-1928, a number of ideas, concerns, and approaches on problem solving came to prominence in mathematics education. The influences of John Dewey and Edward Lee Thorndike played important roles during this period in which problem solving became a viable concept to teach and a powerful ally in the mathematical repertoire of students in the elementary and secondary schools. Certainly, the notion of problem solving was not new, but the resurgence of theories and the argument of proponents helped popularize the teaching of problem solving.

The development of problem solving during this period can be seen initially in the extension of the Herbartian Model--preparation, presentation, comparison, and generalization--into a five-step process with the addition of "application." The recommendations of the Committee of Ten played an important role in encouraging teachers to "teach the new in terms of the old."

Change in arithmetic came slowly in the early 1900s. Teaching arithmetic in terms of "daily living" was popular during the initial part of this period, and problem-solving tasks related to business, commerce, and science were provided. Dewey's approach, albeit different from the Herbartian four-step process, advocated an "active approach" to learning arithmetic. Along with ideas on "reflective thinking," the arithmetic-in-life theme was prevalent throughout the period; and although "drill" was still seen as an important criteria, practice was made more applicable by the organization of learning centered on "practical arithmetic." The lessons established by Thorndike during the second decade and the early 1920s provided an organized mechanism for both teachers and students to follow.

During this period, skill in reading became an initial step for many problem-solving models. For example, Paul Ligda (1928) called for reading with understanding while Lucie Dower (1928) observed that the vocabulary used in problem solving tended to be too difficult for many children. In an extensive list, F. B. Knight (1927) included a set of factors which the author believed played an important role in encouraging children to read arithmetic successfully and to comprehend language indigenous to mathematics.

The development of the "Conventional Model" became an important prototype from which many other models would evolve. This process, whereby several questions are asked, is highlighted in the following illustration:

1. What is asked for in the problem?
2. What are the facts?
3. How are the facts to be used in answering the question (or problem)?
4. What is the answer?

While reading is implied in step 1, step 4 includes aspects of computation. Later models added a step requiring a check of the final answer. As ideas on mathematical problem solving became more prominent, the four-step model evolved into a more elaborate and expansive process. Aspects related to "reading the question" became important criteria. Emphasis on looking for main ideas and thoughts, determining "what was to be found" and "what was given," were other important additions to the processes of problem solving.

Finally, the attention given to the "human phase" of problem solving and the inclusion of "everyday activities" played major roles in the development of problem solving during the 1920s. The concern for problem solving was highlighted by the amount of consideration given it by the National Council of Teachers of Mathematics in its 1928 Yearbook. The NCTM Yearbook included a number of ideas on mathematical problem solving. These ideas provided a framework for experimentation, research, and development on problem solving in mathematics for years. The importance of such concern for solving problems set the tone for the development of mathematical problem solving for Period II, 1929-1953.

CHAPTER III

A REVIEW OF MATHEMATICAL PROBLEM SOLVING
PERIOD II: 1929-1953

The problem-solving approach in mathematics was greatly enhanced by a number of new ideas and models developed during the period, 1929-1953. Although a number of "new" theories on problem solving were fostered by the principles set forth by John Dewey and Edward L. Thorndike, there were other ideas that came to the front in mathematics education during the period. A sampling of educators who influenced theories on problem solving in mathematics during the period, 1929-1953, includes Harry Grove Wheat, Robert Lee Morton, William A. Brownell, George Polya, Leo J. Brueckner, and Foster E. Grossnickle.

In 1929, Paul R. Hanna, in a dissertation entitled "Arithmetic Problem Solving," provided a comparison of three methods of problem solving: the Dependencies Method, which utilized "graphics and diagrams" whereby children followed a particular thought pattern; the Conventional Method, in which children followed a four-step plan; and the Individual Method, in which children used any method they so desired. Included in the analysis was a summary of the problem-solving steps found in several fourth-grade and seventh-grade textbooks. Hanna (1929) summarized the following steps found in the <u>Alexander-Sarratt Arithmetics</u> (grade 4):

1. What does the problem tell me?
2. What does it ask me to find?
3. What operations must I use?
4. What will my estimated answer be?
5. How can I check my work? (p. 46)

Children were encouraged to use the following three-step plan in the <u>Standard Service Arithmetics</u> (grade 4):

1. What does it ask you to find?
2. What facts do you use to find the answer?
3. Do you add, subtract, multiply, or divide? (p. 46)

In a third series, <u>Strayer-Upton Arithmetics</u> (grade 4), the writer found the following four-step model:

1. What does the problem mean?
2. What is to be found?
3. How can the numbers given in the problem be used to get the right answer?
4. Is the work right? (p. 46)

Hanna (1929) utilized the following four-step model as one of the three models used in his experiment:

1. What is asked for?
2. What facts are given?
3. What operations are necessary?
4. What is the answer? (p. 47)

By citing the four-step "Conventional Model," he suggested using a general procedure for problem-solving in mathematics.

In assessing problem-solving steps included in "courses of study" for teachers, Hanna (1929) found five of the nine programs studied provided no technique for problem solving while three suggested the "Conventionall Method." The author found two alternative methods that were used in two of the courses. The first, "course of study in arithmetic," relied on the following six steps:

1. Read the problem.
2. What does the problem ask you to find?
3. What does the problem tell you?
4. What process will you need to use?
5. Find your answer.
6. Is your answer a reasonable one? (p. 48)

In the second course, the "Tentative Course of Study in Arithmetic," Hanna (1929) found a three-step model for problem solving. Those steps included:

1. Silent reading of problems by whole class.
2. Clearing up language difficulties if necessary.
3. Giving the answer if within primary facts, indicating what is to be done if process work is required (p. 48).

He included an analysis of "professional literature" and concluded that twelve of the sixteen references studied gave preference to the four-step "Conventional Method." Overall, Hanna (1929) concluded that the Conventional Method was the technique most widely recommended in textbooks, courses of study, and professional literature. Although the "Conventional Method" was discovered to be the most popular method, he concluded that its utilization was not always in the best interest of the learner. In summary, Hanna (1929) noted:

> The conventional-formula method of problem solving was found to give the least gain in ability when compared with the individual or the dependencies methods. When the fourth and seventh grades are viewed as a whole, there seems to be no difference between the results of the dependencies and individual methods. In the final analysis of the results of this experiment, one would not be justified in advocating the use of the conventional-formula method nor could one say definitely whether the dependencies or the individual method is of greater value in aiding children to solve arithmetic problems (p. 53).

The models discussed so far were applicable to most mathematical situations. The emphasis on skill in solving verbal problems also played an important role in the development of problem solving at all levels of mathematics including algebra.

The inclusion of verbal problems was cause for concern in mathematics because reading played an important role in the development of skills found in problem solving. In discussing "Teaching the Verbal Problem in Intermediate Algebra," Barnet Rudman summarized a guide to solving the verbal problem. In doing so, Rudman (1929) described the following five-step procedure:

1. Read the whole problem carefully.
2. Represent the quantity to be found by x or some other letter.
3. If more than one quantity is to be found, represent the smallest quantity by x and the others in terms of x.

4. Translate the remaining statement of the problem into algebra in the form of an equation.
5. Solve the equation and check in (sic) the conditions of the problem (p. 85).

Rudman noted that the problem with the five-step guide was its inadequacy when data had to be interpreted. Recommendations were made for a more "analytic approach" to the solving of verbal problems and for a more systematized arrangement of problem material. For many educators, specific difficulties were found in the descriptions and language related to problem solving.

In an effort to clarify the meaning of problem solving, Harry Grove Wheat (1929) provided insight into its meaning and inclusion in arithmetic. In discussing "The Purpose of Problem Solving in Arithmetic," the author made the following observation:

> The language of the problem should be so clear as to leave no doubt in the child's mind as to what the situation really is. The language should not be difficult nor strange. Unusual terms have no place (p. 13).

Wheat (1929) also provided a definition of problem solving. In doing so, the difference between "problem" and "problem solving" was noted. He defined the two in the following citation:

> By "problem" is meant "practice exercise," and by "problem-solving" is meant the mental process of recognizing the general ideas of addition, subtraction, multiplication, and division (p. 14).

By defining terminology, Wheat helped clarify certain aspects of problem solving. However, difficulties in dealing with processes of problem solving continued to exist.

The idea of "improving problem solving in arithmetic" was popularized following its inclusion and emphasis in the 1928 Yearbook sponsored by the National Council of Teachers of Mathematics. In 1929, Leo J. Brueckner, writing in the Elementary English Review, made the following four recommendations for improving mathematical problem solving:

1. Increased use should be made of problematic situations that arise naturally in the activities of the school and community in which number is needed in its normal setting, in order that the pupils may learn to apply processes being taught as they are applied in life.
2. Stress should be placed on accuracy in computation and comprehension of the meaning and function of arithmetic processes. This can be accomplished by the use of carefully constructed instructional materials in which special consideration is given to the known difficulties of pupils in computation.
3. A systematic attack should be made on the teaching of problem solving. This would include the use of specially constructed reading exercises on elements in problem solving, the teaching of techniques in problem solving, and the solution of many problems that are within the experiences and comprehension of the pupils and in the solution of which they would be interested. Little is known as to the relative effectiveness of various types of exercises. It is therefore recommended that as wide a variety of exercises in problem solving be used as is practicable.
4. Obviously, standard tests of problem solving have an important place in such a program since they aid the teacher to determine the needs of the class, their level of ability, and to select the pupils in need of special help in

> problem solving. The results of experimental work suggest that pupils whose work in problem solving is below standard profit much from the training received through the use of exercises in problem solving (p. 139).

Not only has problem solving played an important role in the development of elementary mathematics but in the development of mathematics at all levels.

In 1930, William Betz included "problem solving" in a list of objectives for an introductory course in algebra. The following six objectives were considered to be the "central core" of the teaching of algebra:

1. The language and the ideas of algebra.
2. The formula.
3. The equation.
4. The graph.
5. The fundamental principles and processes.
6. Problem-solving (p. 120).

In his writing, Betz stressed the ideas of "understanding," real "application," and actual "thinking."

Orlie M. Clem and Bertha Adams Hendershot (1930) provided further insight into the inclusion of problem solving in algebra. In discussing the difficulties involved in solving verbal problems, the two authors made the following five suggestions for the inclusion of verbal problems in algebra.

1. The general opinion that problem solving in algebra causes more difficulty than mechanical manipulation, seems justified.
2. Special emphasis should be given to the teaching of problem solving in elementary algebra.

3. A significant ability demanded in problem solving is the ability to think things through.
4. ...Teacher training courses in mathematics should give more specific attention to problem solving.
5. The most common causes of failure in solving verbal problems are: lack of preparation and knowledge of techniques on the part of the teacher; on the part of the pupils,--inability to read, lack of logical reasoning ability, poor labeling, lack of statements, lack of knowledge of arithmetic, and lack of proper checking (p. 147).

A description of how children solve problems was provided by Paul Ligda (1930). In an attempt to "help children out of their bewilderment," he described the problem-solving process in the following manner:

> The solution of a problem may be considered to consist of four somewhat overlapping parts: the analysis, the synthesis, the translation into symbols, and the symbolic solution. The analysis consists in breaking the verbal statement into distinct and separate parts and finding the relationships among these parts. The synthesis consists in rearranging the results of analysis in such a way that equations are obtained. The last two parts do not need discussion (pp. 514-515).

In discussing "The Skills Involved in Problem Solving in Elementary School Arithmetic," Guy A. West summarized some of the frustration found in the teaching of problem solving. In 1930, West justified the idea of problem solving with the following comments:

> The fundamental skills of arithmetic have been analyzed by several authorities, but little has been accomplished toward setting up a reliable and complete analysis of the skills involved in problem solving. No one would argue that the teaching of problem solving is similar in principle to the teaching of the basic facts of arithmetic. On the other hand, few would doubt that a

> detailed analysis of the skills involved in the former would have considerable value for the teacher who finds that John Jones can not solve certain problems. His inability to do so remains the unsolved mystery to many teachers (p. 379).

West (1930) investigated the utilization of "problem solving" found in a dozen textbooks on the teaching of arithmetic. The following list was included as an "example" of the problem-solving process:

1. Grasp the conditions (Comprehension).
2. Separate into smaller units.
3. Plan the solution.
4. Solve in the best way.
5. Check the results (pp. 379-380).

Leo J. Brueckner (1930) included the chapter "Diagnosis in Problem Solving," in a textbook titled Diagnostic and Remedial Teaching in Arithmetic. In discussing the nature of pupil difficulties in problem solving, Brueckner noted the five-step process of W. S. Monroe for solving arithmetic problems:

1. Reading the statement of the problem with understanding.
2. Recalling of principles applicable to the problem.
3. Formulating of a plan of procedure concerning the operations to be performed, this being based upon the elements of meaning and the recalled principles.
4. Verifying of the procedure which generally does not constitute an explicit step.
5. Performing of the operation which is also, strictly speaking, not a step in the reasoning process (p. 266).

In describing the causes of difficulty in problem solving, the author made the following comments on children's inability to clearly "see" a problem.

> In other words, the problem is not concrete to the child unless he is able to form a clear mental picture of the situation described. It may be that he has not read the problem carefully or that he may have read it carefully and yet, through lack of experience in the situation described, he may not be able to form a picture of the situation (p. 271).

In summarizing the chief causes of difficulty found in problem solving, Brueckner (1930) compiled many of the ideas found in the major investigations on problem solving initiated during this period of time. The six major causes are provided in the following list:

1. Lack of ability to perform the necessary computations accurately or to select the operation needed.
2. Lack of systematic method of attack in solving a problem.
3. Careless reading or lack of vocabulary.
4. Lack of knowledge of essential facts, data, or principles involved.
5. Failure to complete the problem.
6. Failure to comprehend the problem in whole or in part (p. 308).

In joint authorship with Ernest O. Melby, Brueckner further discussed the idea of problem solving in a textbook titled <u>Diagnostic and Remedial Teaching</u>. In eleborating on "the elements in problem solving," Brueckner and Melby (1931) provided the following description of the factors involved in problem solving:

> The solution of verbal problems found in textbooks involves four major factors: (1) the ability to comprehend the meaning of the statements in the problem and the situation that is presented; (2) the knowledge of essential facts and principles needed to arrive at the solution; (3) the ability to select the processes to be used in solving the problem; (4) and the ability of the pupil to perform the necessary computations accurately (p. 222).

In suggesting ways to detect weaknesses of pupils involved in problem solving, Brueckner and Melby (1931) included a list of twenty-eight skills that were considered important to problem solving, such as the "ability to tell what facts are given" and the "ability to tell what question the problem asks." Appendix D contains the complete list of twenty-eight skills.

In 1932, the National Council of Teachers of Mathematics focused its entire yearbook on The Teaching of Algebra. In discussing the "Recent and Present Tendencies in the Teaching of Algebra in the High Schools," Joseph Jablonower (1932) cited Thorndike and made the following observation in a section on "problem solving" in algebra:

> Thorndike has given us, in his Psychology of Algebra, the most exhaustive study of the psychology of problem solving. He points out the need that the problem be genuine and that it have the tang of reality. But his chief contribution is his emphasis on the true nature of the problem and his consequent catalogue of problems. A problem is a task in connection with which the individual has to select his tools and processes. A problem necessarily involves novel elements, or a novel situation to which familiar elements must be applied in a novel way. This notion of the problem is broader than is the one which makes the problem synonymous with the verbal problem. While the verbal problem has its uses, it is not the whole of the story. Pupils may have

> difficulty with the verbal problem and yet have acceptable mastery of algebra in its more important aspect as a tool for representing quantitative and functional relations (p. 14).

During this time, Edward L. Thorndike provided numerous ideas on problem solving in mathematics. The thoughts of John Dewey also had their impact.

In 1933, Dewey, in the revised edition of How We Think, included a description of the "Essential Functions of Reflective Activity." In essence, he claimed that there were two limits--pre-reflective and post-reflective--to every unit of thinking. In between these two units were five states of thinking. They were described in the following manner:

> 1. Suggestions, in which the mind leaps forward to a possible solution.
> 2. An intellectualization of the difficulty or perplexity that has been felt (directly experienced) into a problem to be solved, a question for which the answer must be sought.
> 3. The use of one suggestion after another as a leading idea, or hypothesis, to initiate and guide observation and other operations in collection of factual material.
> 4. The mental elaboration of the idea or supposition as an idea or supposition (reasoning, in the sense in which reasoning is a part, not the whole, of inference).
> 5. Testing the hypothesis by overt or imaginative action (p. 107).

In 1934, Paul Klapper described six factors that determined ability to solve problems. The author, in defining "intelligence" as the "native ability to recognize quantitative relationships" (p. 440), called it the first and most important factor in determining ability to solve problems. The remaining five factors included:

1. Skill in silent reading.
2. Familiarity with the technical language of arithmetic.
3. Understanding of situations that give rise to arithmetical problems.
4. A high degree of skill in the fundamental operations.
5. An attitude towards accuracy that leads to questioning and checking of answers (p. 440).

Klapper (1934) differentiated "problems" from both "exercises" and "examples." In discussing the differences, the author defined an "exercise" as providing "a direction to perform an operation" and an "example" as "an exercise clothed in words so that the task seems more social and becomes, therefore, more interesting" (p. 439). Accordingly, the author defined a problem as "a challenging situation that invites solution. It cannot be resolved without careful analysis and planning" (p. 439).

In discussing problem solving, Klapper (1934) included a series of five steps for the purpose of solving a problem. The five steps were identified in the following manner:

1. Grasping the situation.
2. Ascertaining (a) what is to be sought, the unknown, and (b) what is given, the known.
3. Planning the solution.
4. Carrying out the plan.
5. Checking the answer (p. 458).

Finally, Klapper (1934) made the following comment regarding the important role problem solving played in the arithmetical process:

> Problem-solving vitalizes arithmetic by infusing purpose into its computations. Problems intensify both the utilitarian and disciplinary values of arithmetic by providing quantitative interpretations of social, economic and civic experiences. In problem-solving the child finds the real challenge of arithmetic (p. 439).

For better than a decade, since the influence of Edward L. Thorndike in the early 1920s, the teaching of mathematics had undergone a considerable amount of scrutiny. Many critics in the mid-1930s found it fashionable to judge the merits of mathematics. Perhaps Arthur E. Robinson (1935) summarized the situation with the following comments:

> ARITHMETIC was introduced into the elementary school in 1548. For 400 years it has held a place of importance in the elementary school curriculum second only to that of reading. Yet in spite of its venerable age, the subject has at no time in its long history been subjected to such severe criticism as that of the past decade (p. 215).

In discussing the education of teachers, the author noted:

> ...few studies or investigations have been directed at what would seem to be one of the first and most vital problems in the teaching of arithmetic, the professional equipment of the prospective teacher in the field of elementary school arithmetic (p. 216).

In analyzing pedagogical aspects, Robinson (1935) concluded that "Most of the teaching of arithmetic observed impresses one of the fact that the theory of teaching and the practice of teaching are still quite unrelated" (p. 219).

In examining "The Curriculum and the Mastery of Academic Skills" for the previous one hundred years, Harold Rugg provided a look at various subjects in the curriculum. As he noted, most of the arithmetic that had been included in the curriculum focused on the development of skill in addition, subtraction, multiplication, and division of integers and fractions. Rugg (1936) made specific references to the teaching of problem solving. In doing so, the following comments provided insight into the teaching of arithmetic:

> On the side of problem-solving there was the same emphasis upon mechanics rather than upon interpretation and understanding. It was implicitly assumed that skill in the use of numerical techniques could be abstracted from "life situations" and developed apart. Arithmetical "problems" were merely word descriptions of number situations, not actual social situations in which a child would naturally use numbers. Furthermore, the problems, as they appeared in arithmetic books, were outlandish and unreal. Trained in the doing of these isolated, unreal word-problems, by some mysterious process the pupils were to be able later in life to transfer their skill to actual situations requiring arithmetical techniques (p. 142).

In writing about "Problem Solving in Algebra," D. McLeod and Daniel McIntyre (1937) provided insight into some of the frustration associated with the teaching of problem solving during the 1930s. The authors put the idea of problem solving into perspective with the following comment:

> No apology is needed for re-introducing in the pages of The Mathematics Teacher a topic which may be time-worn. The subject of problem solving is always in place and ever calls for solution (p. 371).

McLeod and McIntyre (1937) cited a concern of mathematics educators when the following question was posed:

> Immediately the question arises: Is this condition universal? What is so difficult about problem-solving as compared with simplifying fractions, factoring, deriving formulas, or finding the root of an equation? (p. 371)

The fact that the teaching of problem solving has historically created difficulties was made more prevalent by Harry Grove Wheat (1937) in a textbook titled The Psychology and Teaching of Arithmetic. In discussing "The Development of Arithmetic," the writer stated:

> The early interests in computations and in problems persist in present-day arithmetic, which divides into two parts; namely, computation and problem-solving. Ancient interests dominate the arithmetic of the modern school (p. 102).

Later, in comparing "The Relation of Present versus Past Conditions to Problem-Solving," Wheat (1937) added:

> In view of the present organization of the curriculum in arithmetic with its emphasis upon problems and problem-solving, what has just been said may seem heretical in the extreme. But we should remember that the school continues to emphasize problem-solving in arithmetic, because problem-solving was, and had to be, a major activity in the earliest developments of the subject (p. 141).

What was it that he was calling heretical? It was a call for teachers to do more than teach children simple solutions to arithmetic problems. He claimed that the purpose of education was to get children to think. Wheat (1937) described a response to the question in the following manner:

> The purpose is so to order and systematize the child's methods of dealing with combination and arrangement of objects that he may go through life freed from the necessity of confronting problems of an arithmetical nature. The purpose rests upon the assumption that the individual has a higher function to perform in life than to expend his energy in solving what were once problems in arithmetic but are problems no longer. He must be set free from the necessity of ever having problems in arithmetic to solve (pp. 140-141).

Wheat (1937) defined "problem solving in arithmetic" as:

> ...practice in the recognition (a) of general ideas in familiar situations, and (b) of new situations with which are involved the general ideas that now should be familiar. Thus, the double purpose of "problem-solving" is distinguished (p. 211).

Robert Lee Morton (1937), in a mathematics methods textbook titled Teaching Arithmetic in the Elementary School (Volume I, Primary Grades), described problem solving in the following citation:

> The processes of addition and subtraction are not of importance in themselves. They are a means to an end; the end is problem solving (p. 107).

In discussing "the method of formal analysis" as a procedure for solving problems, Morton (1938) cited (Volume II, Intermediate Grades) the ideas of educators from the 1920s. In describing the so-called "Conventional Method," he included the problem-solving models of Edward Lee Thorndike and Fletcher Durell. The three-step model provided by Thorndike (1921) differed little in theory from the following six-step model presented by Durell in 1928:

1. State what is given.
2. State what is to be found.
3. Make a list of the operations to be performed.
4. Estimate the answer.
5. Make the computations.
6. Check the answer (Morton, 1938, p. 468).

A two-fold reason for utilizing the "Conventional Method" was given: first, children were forced to thoroughly read through a problem; and second, children were trained to think critically.

In 1940, the idea of problem solving was given an even greater dimension. W. S. Schlauch, in an article titled "The Use of Calculating Machines in Teaching Arithmetic," suggested that interest in problem solving could be stimulated by including "calculating machines" in the mathematics curriculum at the junior and senior high school levels. Schlauch (1940) provided the following summary:

> We may conclude that the use of calculating machines in teaching arithmetic is justified by its results, and that they should be used wherever the cost of installing such machines can be met. They are used to best advantage in the upper grades of the junior high school and in the senior high school. They lend speed, accuracy, and confidence in computation, and stimulate an interest in problem solving calling for their use (p. 38).

This was not the first time that an educator had called for the use of "calculating machines" in mathematics. In 1937, Evelyn M. Horton provided a discussion of "Calculating Machines and the Mathematics Teacher" in the _Mathematics Teacher_.

In 1940, Ben A. Sueltz discussed the "Recent Trends in Arithmetic" at the conference of the Eastern-States Association of Professional Schools for Teachers. In a follow-up article Sueltz noted these four trends.

> First, subject matter continues to be important but it is redirected in terms of functional living. It is no longer an end in itself.
> Second, the modes of learning as well as the ends of learning tend to dissolve subject-matter barriers.
> Third, textbooks continue in use both as learning and reference materials, but the textbook is freely supplemented with excursions, investigations, and interviews.
> Fourth, a more serious attempt is being made to develop in pupils the intelligent participation in, and the feeling of, responsibility for their own affairs (p. 270).

In 1942, the National Society for the Study of Education (NSSE) provided attention to the concept of problem solving in its Forty-first Yearbook. In this Yearbook, William A. Brownell discussed the many facets of problem solving including research in problem solving, processes, growth and the teaching of problem solving, and suggestions for the development of abilities in problem solving. Brownell (1942) highlighted the significance of the "special treatment" given to problem solving with the following comment:

> Since all the other chapters in this Yearbook deal in one way or another with learning, the present chapter might appear repetitious, if not redundant. Yet, here is a separate chapter on problem solving. Its presence attests the belief that, for the purposes of education at least, problem solving needs to be considered separately from other kinds of learning (p. 415).

Brownell (1942), in noting "(1) the rise of field theories of learning, with consequent changes in the design of experimentation, and (2) the attempt to get at the nature of problem-solving behavior without regard to any particular systematic point of view in psychology" (p. 419), discussed three changes that had occurred over a fifteen-year period. In describing the three changes in psychological research (with special reference to "problem solving"), the author wrote:

> One of these changes consists in the attempt to set problems which "mean" something to the subject (animal or man), or at least envisage the learning task as it most probably is envisaged by the subject. A second change is the tendency to concentrate research interest, not merely on errors and successes, but on the way in which the subject proceeds to attack and solve its problem. The latter trend has not meant a wholesale abandonment of objective data; after all, the systematic analysis of errors, for example, reveals much concerning the pattern of behavior involved in problem solving. Rather, it has meant that the experimenter has been willing to advance explanations and interpretations, anthropomorphic, if necessary, in the case of animal subjects, but nevertheless designed to understand what the problem and its solution mean to the subject. A third change, closely associated with the second, is the greater importance now attached to qualitative descriptions of significant behavior to supplement or to replace purely quantitative descriptions (pp. 418-419).

Children in the early grades were taught a system of problem solving that asked the following series of questions summarized by Brownell in 1942:

1. "What is asked?" (or, "What am I to find?")
2. "What is given?" (or, "What do I know?")

3. "What process or processes should I use?"
4. "What is the probable answer?" (the last question being followed by actual computation). (p. 432)

In response to the development and use of the models that had been patterned after Dewey's "Analytical Thought" method of solving problems, Brownell (1942) presented two criticisms in the following manner:

> In the first place, the method of step analysis represents a logical pattern of thinking which may or may not characterize expert thinking on the part of adults, but which certainly has not yet been shown to characterize good thinking on the part of children. According to this method of teaching, a formal abstract pattern, possibly suitable to adults, is imposed upon children before they are ready for it. A preferable procedure is first to ascertain the level of thinking which children have attained and then to lead them on to more mature and economical levels as rapidly (but only as rapidly) as they can adopt them.
>
> In the second place, this method puts too much trust in technique alone and disregards other essentials in effective problem solving (p. 432).

In a discussion on "Growth in Problem-Solving Ability," Brownell (1942) noted the work of Jean Piaget and associates at the Maison des Petits, in Geneva, Switzerland. In citing four studies completed by Piaget in the later 1920s and early 1930s, the writer made the following summation:

> Piaget represents growth in problem solving as influenced by two sets of factors. The first set is highly personal and narrowly individualistic, the result of the egocentrism of early childhood. Opposed to the first set of factors is another set, social factors, which are steadily imposed upon the child and which have the effect of leading him to substitute objective reality for

> his own subjective schemas and to replace his illogical, if personally satisfying, mental processes by others which are rational and can meet the requirements of impersonal appraisal. The conflict between the two sets of factors is resolved finally in favor of the social factors, though the egocentric factors are by no means easily, quickly, and completely surrendered (p. 428).

Brownell offered four criticisms of Piaget's work on problem solving. In discussing the first criticism, the author made note of "Piaget's failure to consider sufficiently the prejudicial character of the problem tasks with which he worked" (p. 430). In the second criticism, Brownell took exception to Piaget's definition of reasoning. In commenting on Piaget's use of "reasoning" in place of "problem solving," Brownell (1942) provided the following three objections:

> ...first, that this kind of thinking is rare; second, that it overvalues verbal expression as a measure of thinking; and third, that it tends to encourage the notion that young children cannot solve problems of any kind (p. 430).

The third criticism dealt with Piaget's ideas on age levels and the development of skill in reasoning. According to Brownell (1942), it could be interpreted "that children at certain rather definite ages achieve equally definite levels of thinking" (p. 430). The following rebuttal was noted:

> ...it is probably true that the changes in problem solving which Piaget attributes to age are better explained as the effects of increases in general experience and in control over language. In this case, age makes its contribution chiefly by providing opportunity (p. 430).

The fourth and last criticism had to do with Piaget's notion that adult reasoning was different than the reasoning of children. Brownell (1942) cited the works of those who found that both children and adults displayed the same tendencies in reasoning skill (p. 431).

Brownell (1942) included twelve "Practical Suggestions for Developing Ability in Problem Solving." In doing so, he elaborated on problem situations, puzzles, learning situations, meanings and understandings, problems, mistakes, abilities, and attitudes. Three of the "suggestions" were presented in the following manner:

> d. Skill in problem solving is partly a matter of technique and partly a matter of meanings and understandings. Highly formal and abstract techniques should never be imposed upon the child. Instead, they should be viewed as the end-products of development. Teaching should start with whatever technique the child uses proficiently and should guide him in the adoption and use of steadily more mature types of problem solving.
>
> g. To be most fruitful, practice in problem solving should not consist in repeated experiences in solving the same problems with the same techniques, but should consist in the solution of different problems by the same techniques and in the application of different techniques to the same problems.
>
> k. A problem-solving attitude, an inquiring and questioning mind, is a desirable educational outcome, and it is possible of development. The practice of "learning" by cramming does not produce this outcome, nor does the practice of accepting from others truths and conclusions which ought to be established by the learner himself. The attitude <u>is</u> produced by continued experience in solving real problems, one consequence of which is that the learner comes to <u>expect</u> new problems and to look for them (pp. 439-440).

In 1944, Harry C. Johnson provided a review of literature on "Problem-Solving in Arithmetic." In the opening paragraph, he noted a recurring theme that had concerned mathematics educators for years. This theme was:

> Educators interested in the improvement of learning in the elementary school will readily agree that the teaching of problem-solving in arithmetic offers one of the greatest challenges to elementary-school teachers (p. 396).

It is noted that a majority of the references included in the Johnson review dated back to the 1920s and early 1930s. In fact, over two-thirds (28 out of 39 references) were written prior to 1934 and only three were dated in the 1940s. The "average" date of all the references included in the Johnson review of literature was 1931. Although not very scientific, the statistics illustrate the wide-spread emphasis that was given to "problem solving in arithmetic" during the latter stages of the 1920s and the early 1930s.

During the mid-1940s, the idea of "meaningful arithmetic" became important. The "meaning theory," presented a decade earlier by the National Council of Teachers of Mathematics in its Tenth Yearbook (1935), was enhanced by mathematics educators such as William A. Brownell, Leo J. Brueckner, Foster E. Grossnickle, and Harry Grove Wheat.

The "meaningful arithmetic" movement, based upon the idea "that children must understand the structure of the number system and be able to perform number operations meaningfully" (Kramer, 1978, p. 28), provided the overall theme for the Council's Sixteenth Yearbook in 1941. In addition to the individuals cited on the previous page, many of the top mathematics educators in the country--including Robert Lee Morton, Guy T. Buswell, and B. R. Buckingham--had a hand in putting together the yearbook titled <u>Arithmetic in General Education</u>. The Sixteenth Yearbook included a chapter by C. L. Thiele on "Arithmetic in the Early Grades." After discussing the socialization aspect of arithmetic, Thiele (1941) described the teaching of meaningful arithmetic. In noting the difference between the ideas contained in the Sixteenth Yearbook and those ideas found ten years earlier, the author wrote:

> The significant difference between the program of arithmetic which finds support in this Yearbook and that of a decade ago is in the extent to which children see meaning in the numbers which they use and operate. The keynote of the new arithmetic is that it should be <u>meaningful</u> rather than <u>mechanical</u> (p. 45).

Thiele included a brief description of the role "problem solving" was to play in the teaching of "meaningful arithmetic." In noting that "experience" was an important factor and that the use of dramatization, verbal description, and illustration was vital, the writer provided the following discussion:

> Problem solving. In the meaningful program of arithmetic instruction, problem solving instruction does not assume an independent role. Instead, it is ultimately bound up with the whole teaching process. The experiences with concrete settings through which abilities are developed provide

> experiences in using number for purposes of quantitative thinking. Following this, the applications of these abilities to situations which described rather than "present-to-sense," contributes to the development of problem-solving abilities (p. 78).

In 1945, Harry Grove Wheat approached the question concerning "the sense and utility of teaching meaning." In discussing the development of the "meanings approach" for children in the early grades, he wrote:

> The meanings in early arithmetic are simple; in later arithmetic, they still are simple, though each in succession is a step beyond those that precede. Why meanings in later arithmetic seem complex and beyond apprehension is that they are frequently considered in isolation and without due regard for the earlier meanings which give them support (pp. 101-102).

During the same year, Wheat (1945), in a summary of theses in arithmetic completed at West Virginia University, wrote about the "types of training" that were related to the teaching of problem solving. In analyzing six studies on "problem solving," the author distributed the "types of training" into two distinct categories. The first category included the type of training students received in preparation _for_ problem solving. The second category included the type of training "that pupils should be expected to get _from_ problem solving" (p. 31).

G. Polya (1945) produced the often-mentioned book _How to Solve It: A New Aspect of Mathematical Method_. In it, he described a four-phase approach to solving problems.

The Polya Model has been popularized and used by teachers of mathematics at all levels of instruction. The four steps were presented in the following manner:

> First. You have to <u>understand</u> the problem.
>
> Second. Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a <u>plan</u> of the solution.
>
> Third. <u>Carry out</u> your plan.
>
> Fourth. <u>Examine</u> the solution obtained (pp. inside-front and inside-back covers).

The key words underlined above, highlight the steps in the Polya Model for problem solving. Each step in the model contains a series of questions. The complete list of questions is included in Appendix E of this investigation.

In 1946, J. T. Johnson, in summarizing a study of four factors affecting problem solving, provided the following introduction:

> The solution of the problem of problem solving is a major trend right now in the teaching of arithmetic. Let us hope that the end of this decade will see a part of this solution (p. 256).

In addition to Johnson (1946), a panel of four members--Elizabeth F. Jeffords, G. T. Buswell, W. B. Storm, and Lucile B. Gates--attempted to answer the following questions concerning the factors contributing to problem solving:

1. Is It Reading?
2. Is It Number Relations?
3. Is It Meaning of Numbers and Their Operations?
4. Is It Teaching? (pp. 256-266)

A case was built in support of each of the questions cited on the previous page as having an influence on problem solving. In answering the fourth question, Lucile B. Gates (1946) included the following list of what were believed to be the chief causes of failure in problem solving (as seen by teachers of arithmetic):

> 1. Failure to read the problem correctly.
> 2. Lack of knowledge of arithmetical vocabulary.
> 3. Using the wrong computational process.
> 4. Guessing in computation.
> 5. Inability to make correct judgments concerning the problem.
> 6. Lack of knowledge of place value.
> 7. Failure to estimate answer.
> 8. Problems of little or no practical value.
> 9. Lack of number knowledge such as the number of feet in a mile, quarts in a gallon and so forth (p. 265).

In terms of problem solving, Gates (1946) provided the following six-step process in an attempt to answer the question, "What does correct solution require?"

> 1. Reading with understanding, which means that the pupil must be able to read the problem so that he will know what he is given and what he is required to find.
> 2. Deciding what processes he will have to use and using them correctly.
> 3. Estimating his answer.
> 4. Carrying out all steps until final completion.
> 5. Knowing when to stop, when he has arrived at the point, when he has answered the question, "To Find."
> 6. Proving the result (p. 265).

In 1946, Harry Karstens, in discussing the "Effective Guidance in Problem Solving," provided some insight into the inclusion of problem solving in the mathematics curriculum. Regarding the difficulties of teaching problem solving, he made the following comment:

> The teaching of problem solving, which makes greater demands on the ability and resourcefulness of a teacher than any other phase of arithmetic, is a particularly delicate and exacting procedure when handling low-ability groups (p. 172).

The author noted that "the first requirement for teaching problem solving is a clear concept of the function of problems" (p. 172). With this in mind, the idea of problem solving and its role in the computational process were noted in the following comment:

> Computation in arithmetic achieves its objective in problem solving. Computation, in itself, is meaningless; problem solving, without computational ability, is futile. Computation makes problem solving possible; problems give sense to computation. <u>Problems are the humanizing element in arithmetic</u> (p. 172).

Karstens (1946) discussed the use and teaching of problem solving. On the teaching of problem solving, he provided the following look at the concerns and steps involved in the process:

> The term "teaching problem solving" carries unfortunate implications. "Teaching problem solving" implies formal methods and set routines while the term "guidance in problem solving" connotes a sympathetic molding of the pupil's thinking and an informal directing of his work habits. The problem solving material in many texts and courses of study

> is based primarily on a pattern of formal analysis in which the pupil asks himself three questions.
>
> 1. What am I to find out?
> 2. What facts (numbers) are given?
> 3. What shall I do with the numbers? (p. 173)

As a substitute for the three questions listed above, Karstens (1946) suggested a multi-step method. In noting that "any method of formal analysis tends to stress techniques instead of understandings" (p. 173), he provided the following plan for solving problems:

> 1. What is the problem about?
> 2. What am I to find out?
> 3. What are the essential phrases or sentences?
> 4. What must I do to solve the problem?
> 5. What facts will I need?
>
> After this, the computation, checking, and evaluation of the answer should follow in due order (p. 174).

Karstens (1946) saw the ideas of "meaning" and "thinking" as important tools in the problem-solving process. In summarizing his thoughts on "effective guidance programs in problem solving," the author provided the following citation which described much of the attitude toward arithmetic instruction during the 1940s:

> ...meaning is all important in a guidance program in problem solving. The teacher's problem is to take stock of each individual's "mental tools," such as they are, and guide him into more efficient and more mature ways of using those tools. Her efforts should be devoted toward making each child a thinking individual. Since all other phases of arithmetic culminate in the problem solving program, guidance in problem solving should be the most interesting and at the same time the most challenging phase of arithmetic teaching (p. 175).

It has been mentioned frequently in this chapter that "meanings" and "understanding" played major roles in mathematics instruction during the 1940s. In discussing the "Aspects of Problem-Solving Ability," William L. Schaaf (1946) combined the ideas of meaning and understanding with problem-solving ability. In writing about the ability to solve problems, he elaborated in the following manner:

> Most of the common methods used in teaching problem-solving are unsuccessful because they do not get at the heart of the matter, which is an adequate understanding of the meaning of number and of the nature of relationships between quantities, together with a ready facility in recognizing such relationships in a wide variety of settings, verbal or otherwise (p. 495).

In suggesting a "realistic approach" to teaching problem solving, Schaaf (1946) recommended the following areas of emphasis:

1. Understanding numbers and number relations and the rationalization of operations and processes.
2. Understanding the mathematical relationships involved in a problem situation.
3. Understanding the essence of a problem situation.
4. Understanding the vocabulary used in the problem.
5. Understanding the relevance of data.
6. Recognizing the arithmetic operations or procedures to be used.
7. Analyzing two-step problems.
8. Estimating the answer.
9. Checking the answer (pp. 496-497).

In 1947, Leo J. Brueckner and Foster E. Grossnickle provided a look at instruction in meaningful arithmetic with the writing of <u>How to Make Arithmetic Meaningful</u>.

Included in the textbook was an extensive chapter on "The Place of Problem Solving in Arithmetic." In the following list, they summarized the causes of difficulty in solving problems:

1. Failure to comprehend the problem in whole or in part because of lack of experience and inability to visualize the situation.
2. Deficiencies in reading, such as inability to locate information, inability to remember what is read, inability to organize what is read, and inability to read for details.
3. Inability to perform the computations involved, either because the pupil has forgotten the procedure or has failed to learn it.
4. Lack of understanding of the process, resulting in the random trial of any process that may come to mind in order to get an answer.
5. Lack of knowledge of essential facts, rules, and formulas, such as how many inches there are in a yard or the rule for finding the perimeter of a rectangle.
6. Lack of orderliness in arranging written work.
7. Ignorance of quantitative relations due to a a a llimited vocabulary or to lack of understanding of principles, such as the relations between selling price, cost, profit, and margin.
8. Lack of interest, due to inability to solve the problems because of their difficulty, unattractiveness, and general low level of merit.
9. Level of mental ability too low to grasp the relations implied.
10. Lack of practice in solving verbal problems (p. 452).

In reviewing research, Brueckner and Grossnickle (1947) looked at "good" and "poor" achievers in problem solving. Describing the analysis, the two authors included the list on the next page which categorizes the differences between good and poor achievers in problem solving in terms of having either "significant" or "no significant" difference:

A. <u>Differences Highly Significant</u>
   1. Psychological factors
      a. General reasoning ability
      b. Non-verbal mental ability
      c. Delayed and immediate memory
      d. General language ability
      e. General reading level
   2. Computation abilities
      a. Skill in fundamental operations
      b. Ability to estimate answers of examples
      c. Ability to see relations in number series
      d. Ability to think abstractly with numbers
   3. Problem solving reading skills
      a. Steps in problem analysis
      b. Finding the key-question of the problem
      c. Estimating answers to problems
      d. Ability to read graphs, charts, tables
      e. Range of information about arithmetic uses
B. <u>No Significant Differences</u>
   1. Range of general information
   2. Gates Tests in General Reading
      a. Grasp of Central Thought
      b. Prediction of Outcomes
      c. Following Directions
      d. Reading for Details (p. 454).

Throughout the textbook, Brueckner and Grossnickle (1947) pointed out that experience played a major role in the way children learned arithmetic. In looking at "Direct Experience in Problem Solving," they noted that "the most effective way to develop the ability to use quantitative procedures is through their direct application" (p. 435). Direct experience was defined as "the actual use of number in social situations that arise in daily life in and out of school" (p. 435).

In 1947, G. T. Buswell edited the monograph, "Arithmetic 1947," which included a series of papers presented at The University of Chicago. In writing the opening chapter, L. J. Brueckner (1947) discussed "arithmetic in elementary and junior high schools." Included in the chapter was a description of the "Minnesota State Guide for the Improvement of Instruction in Arithmetic." Six major activities--problem-solving activities, construction activities, appreciation activities, creative activities, excursions, and practice activities--were described. The list of "problem-solving activities" included the following items:

a. The formulation of a problem.
b. Consideration of the scope and significance of the problem.
c. Planning a method of attack.
d. The assignment of tasks to individuals or groups.
e. The location and gathering of necessary information from persons and printed sources.
f. Research and experimentation needed to get new data.
g. The assembling, organizing, and presenting of findings.
h. Drawing conclusions and making decisions.
i. Taking steps to carry out decisions (p. 6).

Numerous ideas pertaining to problem solving appeared during the 1940s. In 1948, H. Van Engen commented on the changes that were taking place in mathematics instruction. Writing about the implications of research on the organization and learning of arithmetic, the author noted the following citation about problem solving:

> There is evidence that the teaching world is gradually changing its conception of what constitutes problem solving in arithmetic. The so-called problems found in the textbook are now, at times, being called examples. There is a growing realization that at best the book problems are exercises in using the language of arithmetic and that very little problem solving activity may accompany the procurement of answers to a page of textbook problems (p. 262).

In 1948, Lee J. Cronbach provided a working definition of problems and exercises. Writing in the monograph, "Arithmetic 1948," edited by G. T. Buswell, Cronbach included the following ideas on problem solving:

> ...the major purpose in teaching problem-solving is to prepare the pupil to solve problems when he encounters them in the flesh, far from the classroom. Moreover, any proper psychology of problem-solving should encompass problems which are non-mathematical in nature.
>
> The characteristic of problem-solving that separates it from all other behavior is that one cannot solve problems by habit (p. 32).

In summarizing the comments on "The Meanings of Problems," Cronbach noted, in the following citation, the important role that "problem solving" and "meanings" play in arithmetic instruction:

> The implications of present knowledge of problem-solving can be summarized concisely. First, meaningful problem-solving will take place if the problem has meaning for the child. The selection of sensible problems is the easiest way to increase the soundness of the pupil's thinking. Second, problems must not degenerate into exercises, since exercises make small demands on higher mental processes. Third, if arithmetic lessons must make use of puzzles remote from the pupil's experience, the topic can be made meaningful by giving the child the necessary experience, directly or vicariously. Finally, the setting of the problem and the emotional experiences of the child must be carefully considered, in order to remove factors which will distract the child from giving his full attention to the problem. If this is done, the child should have just one concern in thinking of his problem, namely, "What is the answer?" (p. 43)

Also writing in _"Arithmetic 1948,"_ Maurice L. Hartung (1948) elaborated on the "Advances in the Teaching of Problem Solving." In discussing the "behaviors" needed for solving problems, he included the following "key words:"

> 1. _Recognize_ and _formulate_ the problem.
> 2. _Collect_ and _organize_ data.
> 3. _Analyze_ and _interpret_ the data.
>
>   In arithmetic, it is often (but by no means always) useful to replace this group by a more restrictive set, namely, _choosing_ and _carrying out_ the processes.
> 4. _Draw_ and _verify_ conclusions (p. 45).

Hartung included a six-step model for solving problems that had originally been cited by Raleigh Schorling, John R. Clark, and Rolland R. Smith in the textbook _Arithmetic for Young America_ in 1944. The steps for problem solving read as follows:

> a. Read the problem carefully to learn what is given and what you are to find.
>
> b. If you do not know the meaning of any word or expression, find out its meaning.

c. Think through the steps of solution and decide what process to use in each step.

d. If possible, estimate your answer.

e. Solve the problem and check each process.

f. Check your answer for reasonableness. See that it is expressed correctly (Hartung, 1948, pp. 45-46).

In looking at the relationship between reading and problem solving, Hartung (1948) took a dim view of the types of problems that were used to teach problem solving. The author provided the following comment:

> Although attention to reading skills may improve problem-solving scores, improvement in reading, alone, will not make good problem-solvers. Much more needs to be done. Perhaps one reason that many of the efforts to improve problem-solving abilities have proved to be disappointing is that the kinds of problems to be used have been too narrowly conceived (p. 52).

Along the same line, J. T. Johnson (1949), in writing about the "nature" of problem solving, called for psychologists to better define "memory" and "reasoning." In discussing different aspects of problem solving, the author looked at the existing relationship between reading and problem solving. Johnson provided the following comment in an effort to get educators to consider something other than reading skill as the sole indicator of success in problem solving:

> For many years we have heard it stated that the reason children cannot solve problems in arithmetic is because they cannot read. This is only a half truth. It seems to fall under the mathematical category of "necessary and sufficient" reasons. In this case reading ability is the necessary but not sufficient reason for problem-solving in arithmetic.

> That there is something besides reading ability which is required in problem-solving is evidenced by the fact, well known to every elementary school teacher, that there are many good readers in our schools who are poor in problem-solving (p. 110).

Lucy Lynde Rosenquist, in Young Children Learn to Use Arithmetic, provided a four-step example of problem solving. In writing a "things-to-do" list, she relied heavily upon "Visualizing the Problem Situation." As a part of the process, children were taught to analyze their own errors and to "become conscious of the various places in the solution of a problem where mistakes may be made" (p. 86). Rosenquist (1949) stated that the following abilities were needed to solve problems:

1. Visualizing the Problem Situation
    a. Dramatizing Problems
    b. Making Pictures of Problems
    c. Clarifying Word Meanings
    d. Stating the Question for a Problem
    e. Stating Original or Personal Experience Problems
2. Selecting the Process to be Used in Computation
3. Performing the Computation
4. Checking Results (pp. 86-93).

In 1950, Millie Almy discussed the use of problems and problem-solving skills for young children. After pointing out that many educators believed problem solving to be an adult activity, she endorsed the use of problem experiences with very young children. In prefacing her comments on providing young children with problem-solving activities, Almy (1950) wrote:

> When the relationship of problem-solving to child behavior is clearly understood there can be little question as to the importance of problem-solving in the education of young children (p. 148).

Writing on "The Need for Extending Arithmetical Learnings," Ben A. Sueltz and John W. Benedick (1950) provided insight into some of the growing discontent with competence and achievement in mathematics. The following observation is particularly noteworthy:

> The current trend toward teaching arithmetic and mathematics for "meaning" and "understanding" coupled with the aim to achieve functional competence on the part of the pupil is causing reverberation among mathematics teachers. Teachers ask, "What and how much can pupils really understand?" and "Shall I teach this topic if I cannot find more than a few poor examples of functional usefulness?" While these questions do not represent the point of view of all teachers, they are indicative of a growing discontent with the achievement in mathematics at both the elementary and secondary school levels (p. 69).

In answering the question, "What is Arithmetic?," the two writers provided the following response:

> ...arithmetic is a study of the significance and uses of numbers in the social, cultural, economic, and industrial situations most commonly found in our society. This does not preclude a study of "the science of numbers" but rather provides that both the science of numbers and the algorisms of computation shall be important elements and stages in the study of arithmetic (p. 69).

In 1950, Max R. Goodson, in writing about "Problem-Solving in the Elementary School," referred to the "act" of problem-solving "as an orderly sequence of steps" (p. 145). In describing this "formalized" process, the author cited the following five-step model:

1. definition of the problem;
2. formulation of hypotheses or ideas;
3. testing hypotheses by reasoning through their logic and by experimentation;
4. applying the affirmed hypothesis in changing the conditions of the problem so as to bring about a correction or control; and
5. generalizing the affirmed hypothesis to the limits of its supporting facts for applicability to as wide a range of phenomena as possible (p. 146).

In looking at the question, "Why Problem-Solving in the Elementary School?," Goodson (1950) concluded that the "changing society" and the "maturity required by our times" provided the greatest possible reason for including problem solving as "one of the basic functions of the elementary school" (p. 147).

In 1950, Robert L. Thorndike, writing in The Forty-ninth Yearbook of the National Society for the Study in Education (NSSE), expanded the theory of problem solving in a chapter titled "How Children Learn the Principles and Techniques of Problem Solving." In the Yearbook, Learning and Instruction, Thorndike (1950) looked at three questions:

1. What is a problem?
2. What do we know about successful techniques for problem-solving?
3. What can teachers do to develop the problem-solving abilities of their students? (p. 192)

In answering the first question, Thorndike suggested that children become more "aware" of the problems when they are "involved" in the process. Interestingly, in answering the second question, he divided the section into five distinct--

and familiar--steps. For purposes of analysis, Thorndike utilized five steps described by John Dewey (1933) in How We Think. The five steps are:

1. Becoming aware of a problem.
2. Clarifying the problem.
3. Proposing hypotheses for solution of the problem.
4. Reasoning out implications of hypotheses.
5. Testing the hypothesis against experience (Thorndike, 1950, p. 196).

At this point, Thorndike (1950) expanded the thinking on problem solving. In doing so, the author suggested such factors as expanding life's experiences, bridging the gap between "the given" and "the desired," positive use of facts based upon a person's experience, understanding of individual maturity, understanding unique situations, and the ability to appraise problem situations (pp. 196-208). He called for more awareness of problem situations and improved questioning techniques on the part of teachers. The author made several noteworthy observations which included calling the nature of problem solving a "complex and variable behavior" (p. 215). Thorndike (1950) referred to the process as an "attack upon problematic situations for which the individual has no ready-made response patterns" (p. 215). His comments were summarized in the following manner:

> There is no simple pattern or routine of problem-solving which can be isolated and taught in the schools as a simple unitary skill. Rather, problem-solving is an integration of a host of more particular knowledges, skills, and attitudes with which the schools can appropriately be concerned.

> A wide range of interests and experiences, an organized and functional stock of background information, efficient skills for locating and organizing needed information, perseverance yet flexibility in attacking problem situations, a willingness to suspend judgment until the evidence is in, habits of testing critically any proposed solutions, attitudes of critical appraisal of the reliability and bias of sources, skill in "if-then" thinking--these and many more are the qualities which the school must try to develop if it is to improve problem-solving ability in its pupils (p. 215).

In 1951, C. Newton Stokes, in the textbook Teaching the Meanings of Arithmetic, included a chapter on "Problem-Solving." The concept of problem solving was defined as follows:

> ...problem-solving in arithmetic is the determination of the nature of the relations involved in a challenging quantitative situation and consequent activity that unifies properly chosen relations into a satisfactory result. Thus problem-solving is thoughtful action. It is reflective thinking in general, or it is reflective thinking in such a specific task as finding the how-many or how-much. It is the action involved in a purpose-to-end mind process, executed by analyzing (differentiating and discriminating) the quantitative relations found in a problem situation and then unifying (integrating) relevant elements into a meaningful and usable outcome (pp. 187-188).

In discussing the application of problem solving in arithmetic, Stokes (1951) provided a set of related concepts that had been identified by The Committee on the Function of Mathematics in General Education. In describing the following ideas, the author noted "that the child's understanding of these concepts involved in problem-solving in arithmetic should give him a more mature and a more operative ability to resolve problem situations" (p. 199). The concepts were:

a. Recognize and formulate a problem.

b. Collect needed information and data.

c. Determine the relationships involved.

d. Express the relations in symbolizations--words or number symbols involving operations.

e. Find the solution or consequence (p. 195).

Like so many educators who discussed the importance of "experience" in problem solving, Stokes (1951) alluded to the socialization aspect of solving problems. The following comment was made regarding the "source of problems":

> Problem-solving ability in arithmetic requires a background of experience. Then, to develop this ability so that it parallels growth in other fields, there must be provision for a wide variety of experiences in problem-solving. Sources must be found, both in the social and mathematical aspects of arithmetic (p. 199).

Finally, Stokes (1951) made the following observation regarding the "mathematical aspect" of problem solving:

> The mathematical aspect of problem-solving is work with symbols. Once the relations expressed in words are formulated in symbols, then thinking takes place through the manipulation of these symbolic representations of quantitative measures. The teacher must see that her problems are sufficiently comprehensive in terms of the mathematical work to be done. Every learning element must receive attention, and the amount of attention must be commensurate with the difficulty of the concepts involved in the element (p. 354).

J. Allen Hickerson provided insight into a more recent phenomenon found in the mathematics classroom. Writing in the textbook, <u>Guilding Children's Arithmetic</u>

Experiences: The Experience-Language Approach to Numbers, Hickerson (1952) included the following comment on problem solving:

> The term problem solving in arithmetic parlance has come to mean, unfortunately, not the solving of one's own problems first hand, but the solving of vicarious problems. The theory of such problem solving is that if a child can read about someone else's out-of-school problems and learn to solve them while sitting in the classroom he prepares himself for solving his own out-of-school problems when and if he meets them (p. 8).

To the concept of problem solving, Hickerson attached the significance of the term "word problems." In his language-approach to teaching arithmetic, Hickerson (1952) provided the following observation:

> The newer arithmetic textbooks and workbooks include word-problems that describe experiences and situations many children encounter. They are for the most part within the range of activity of many children. In spite of this excellence, however, the teacher who expects his children to solve these word-problems should be aware of certain things.
>
> Since solving a printed word-problem is primarily a matter of being able to read, the teacher must be sure that the child is ready to read the particular word-problem. A child may be ready to read one problem with meaning, but not another (p. 8).

In their revised textbook, Making Arithmetic Meaningful, Leo J. Brueckner and Foster E. Grossnickle (1953) expanded the material on "problem solving." In the chapter on "The Scope of Problem Solving in Arithmetic," they attempted to get teachers to "see" problem solving as an integral part of the mathematics curriculum. In opening the chapter on "problem solving," Brueckner and Grossnickle made the following comments:

> Instruction in an arithmetic program that emphasizes meaning and understanding is based largely on problem solving. Quantitative thinking is the basis of problem solving (p. 491).

As for "problem solving" playing a major role in the mathematics curriculum, Brueckner and Grossnickle (1953) added:

> The teacher should not look upon problem solving as a separate aspect of the work in arithmetic but as an integral part of all phases of the work.... The goal of the teacher should be to lead him [the student] gradually to use increasingly mature procedures that involve the use of abstract symbols and formulas, and higher levels of thinking (p. 491).

Regarding the diagnostic aspect of problem solving, Brueckner and Grossnickle included a section on "Techniques for Diagnosing Difficulties in Problem Solving." The two authors suggested using textbooks and workbooks to perform informal tests on the following items:

1. Telling what is to be found in a problem,
2. Telling what facts are given,
3. Naming the process to use in solving a problem,
4. Estimating answers of problems, and
5. Checking computations (p. 515).

In its Twenty-first Yearbook titled The Learning of Mathematics: Its Theory and Practice, the National Council of Teachers of Mathematics included a chapter on "Problem-Solving in Mathematics." In writing the chapter, Kenneth B. Henderson and Robert E. Pingry (1953) included an "Analyses of Problem-Solving." As part of their investigation, the writers presented an "analysis of reflective thinking" first described by John Dewey in 1933.

In observing the inadequacies of the Dewey analysis, Henderson and Pingry (1953) identified the processes they believed occurred "regularly" during problem solving. The two authors noted a three-step process which had been introduced by Donald Johnson in 1944. The Johnson model included the following steps: "(a) 'Orientation to the problem'; (b) 'Producing relevant material, an elaborative function'; and (c) 'Judging, a critical function'" (p. 236). The authors reworked the Johnson model and provided the following three steps in their analysis of problem solving:

1. Orientation to the problem.
2. Producing relevant thought material.
3. Testing hypotheses (p. 237).

Henderson and Pingry, in calling problem solving "a very complex process" (p. 247), made suggestions to teachers on how they could help students improve skill in solving problems. The authors suggested a greater understanding and inclusion of the psychological aspects of learning. They wrote:

> ...psychologists find it difficult to distinguish between problem-solving and learning generally. From this point of view there is a sense in which this entire Yearbook, rather than just this chapter concerns problem-solving. Motivation, attitudes, transfer of training, drill, concept formation, language, and logic, are all aspects of problem-solving. A teacher who is seeking to improve problem-solving ability must necessarily give proper emphasis to each of these aspects. A program of instruction, however, that involves these necessary phases of learning is not sufficient. It is important that specific experiences

> designed to foster problem-solving abilities be provided in the program of instruction (pp. 247-248).

In discussing the teacher's responsibility in teaching problem solving, Henderson and Pingry (1953) made the following observation:

> The teacher's task is two-fold concerning problem-solving. One aspect is that of helping the students with the problems at hand. The second aspect is that of helping students understand the problem-solving processes per se (p. 249).

The two authors, in writing about the inclusion of problem solving in the mathematics curriculum, cited the following concern:

> Of course, before the teacher can teach problem-solving the teacher must understand problem-solving. Mathematics teachers need to be students of problem-solving processes as well as students of mathematics. There is considerable evidence that many mathematics teachers do not understand what problem-solving is; or if they know, they do not have it as an objective of instruction (p. 249).

In providing suggestions for improvement in the teaching of problem solving, Henderson and Pingry (1953) noted the importance of the organization of the body of knowledge; the emphasis on reading skill (especially with verbal problems); the recognition of individual differences; the encouragement of verbal responses; a better utilization of diagramming techniques, dramatization and modeling; and the inclusion of open-ended questions to improve problem-solving skill in the classroom (pp. 248-255). The authors suggested the use of questions such as those provided by

G. Polya in How to Solve It. In conclusion, they elaborated on the problem-solving abilities of both students and teachers. In summary, they presented the following discussion:

> Mathematics teachers believe that the ability of a student to solve mathematics problems is dependent upon how deep his understanding of mathematics is. The student's ability to solve problems also depends upon the student's understandings, attitudes, and skills concerning problem-solving processes. This implies that the teacher of mathematics must understand mathematics as well as the psychological processes of problem-solving to be of help to the student. To provide such an understanding of the latter, this chapter attempted to set forth a conceptual framework of problem-solving and point out some of the implications of this for classroom procedure. The hope is that this will afford a teacher fruitful hypotheses concerning his own efforts to teach students the set of understandings, attitudes and skills conducive to solving problems (p. 268).

The chapter on problem solving was aimed at the classroom teacher. Henderson and Pingry provided information on problem solving in the hopes of changing the problem-solving behavior of teachers. Hopefully, this change in attitude would be passed on to children in the mathematics classroom. The result would be the development of more proficient problem solvers.

Summary of Period II: 1929-1953

Paul R. Hanna's dissertation on arithmetic problem solving signaled the advance of problem-solving research, theories, and ideas during the period, 1929-1953. Hanna summarized work that had been accomplished on problem solving and provided a comparison of three distinct problem-solving "models": the Dependencies Model, the Conventional Model, and the Individual Model. In addition to focusing on the problem-solving processes described in professional literature and problem-solving activities used in courses of study, Hanna studied the problem-solving models found in children's textbooks. As this period progressed, suggestions were made for more analytical approaches to problem solving and for a more systematized arrangement of problem material.

The emphasis on "meaningful arithmetic" during the late 1930s and early 1940s, the emphasis on psychological aspects of problem solving, the National Society for the Study of Education emphasis on problem solving in 1942, and the National Council of Teachers of Mathematics emphasis on problem solving in the early 1950s, provided the peak periods for this time frame. As a result of some of the work started during the late 1920s, an attempt to emphasize problem solving in algebra was important during this period.

Harry Grove Wheat, William A. Brownell, and Leo J. Brueckner provided much of the stimulus for the development of problem solving in mathematics during the 1930s. Research revealed that the three major causal factors involved in problem-solving difficulties were: failure to accurately or completely read the problems and problem statements, lack of ability to perform accurate computations, and the inability to read technical (mathematical) language.

A number of problem-solving models similar to the "conventional model" originated during the 1930s. Models presented by Fletcher Durell and Paul Klapper were especially important in establishing a foundation for the development of later models. The model by Klapper (1934) included the basic components of grasping and ascertaining the problem, planning, carrying out the plan, and checking the final answer. These steps compare quite favorably with the steps found in the model by George Polya (1945)--understanding the problem, devising the plan, carrying out the plan, and looking back (or examining). Partially because of its extensive questioning, the Polya Model continued to be important to problem solving in mathematics.

As had been the case in the late 1920s, finding a solution to the "problem" of problem solving became a dominant pursuit in the mid-1940s. Toward the end of the decade, there was a move to change some of the terminology involved in problem solving. A number of textbooks and educators referred to problems as "examples," and emphasis was put on "exercises" in textbooks. Toward the end of the 1940s and at the beginning of the 1950s, there was a renewed interest in the "scientific-method" approach to teaching problem solving. Along with the "meaningful" approach to teaching arithmetic, the emphasis on "an orderly sequence of steps" and the ideas of "experience" and "critical thinking" were seen as important parts of the problem-solving process.

Entering the 1950s, mathematical problem solving was still viewed as a complex process. Emphasis on the theory of problem solving, and providing teachers with problem-solving methods and procedures, were considered necessary parts of the learning process. Finally, it was believed that in order for teachers to teach problem solving, they must be proficient in problem-solving techniques themselves. It is noted that a new influence began to "appear" during the period 1929-1953 with "calculating machines" being recognized as devices that would someday revolutionize the way mathematics--and mathematical problem solving--would be taught. The development of problem solving for Period III, 1954-1983, would see extensive use of various "calculating machines" such as counting frames, calculators, and computers in mathematics education at all levels of instruction.

CHAPTER IV

A REVIEW OF MATHEMATICAL PROBLEM SOLVING
PERIOD III: 1954-1983

In February, 1954, the teaching of arithmetic in the elementary school was accorded additional recognition when the National Council of Teachers of Mathematics published the first volume of the Arithmetic Teacher. In writing about "The Revolution in Arithmetic," William A. Brownell reviewed some of the changes that had taken place in arithmetic during the 1900s. In noting the changes that had occurred over a fifty-year period, he discussed the "Formal Disciplines" of mathematics in the early 1900s, the "functional" approach that began around 1910, the "social aim" that prospered during the late teens and early 1920s, the move from "product" learning to "process" learning in the mid-1920s, the "meaningful" approach of the 1930s and 1940s, and the "child development" approach that evolved from the "child study" movement of the early- and mid-1920s (Brownell, 1954, pp. 1-5). In summary, Brownell (1954) suggested that the so-called "revolution" was actually change that had evolved over an extensive period of time. He made the following observation:

> Actually of course the process of change has been one of evolution, for each modification has emerged from a given status and has led to the next modification. The steadying and stabilizing influence in this period of evolution has been what I have called the search for a functional curriculum (p. 5).

In finalizing his comments, Brownell (1954) pointed out that mathematics, in order to be effective, should have a mathematical aim and a social aim. He concluded his article with the following comment:

> To be intelligent in quantitative situations children must see sense in the arithmetic they learn. Hence, instruction must be meaningful and must be organized around the ideas and relations inherent in arithmetic as mathematics. But they must also have experiences in using the arithmetic they learn in ways that are significant to them at the time of learning, and this requirement makes it necessary to build arithmetic into the structure of living itself (p. 5).

Although he was discussing mathematics in general, the same two aims--mathematical and societal--could well be applied to the teaching of problem solving in arithmetic during the 1950s.

In the same volume of the Arithmetic Teacher, Foster E. Grossnickle (1954) discussed the "Dilemmas Confronting the Teachers of Arithmetic" and observed:

> The teacher of arithmetic today frequently is caught between opposing forces in the fields of the curriculum and in the psychology of learning. The dilemma resulting from the curriculum is caused by the need for meeting standards of achievement in arithmetic so as to satisfy the demands of the business world on one hand and the operation of the policy of continuous promotion on the other. Frequently, these two forces work in opposite directions. The dilemma resulting from the psychology of learning is caused by the

> time needed for mastery of a topic in meaningful learning as compared to the time needed for manipulation of symbols by rote learning (p. 14).

Grossnickle (1954) concluded his comments with the following observation:

> The acceptance of meaningful arithmetic is in jeopardy as long as a teacher knows that an evaluation of her instruction will be measured in terms of standards set for rote learning (p. 15).

Such an evaluation, measured in terms of rote learning, might have an effect upon the development of problem solving in mathematics.

In the second issue of the Arithmetic Teacher, Charlotte Junge reviewed the mathematics curriculum of 1954 and highlighted two areas of arithmetic. First, Junge (1954) noted that the development of mental skills provided "a keen number sense, a healthy self-reliance, and the power to think with numbers" (p. 5). Second, she noted the attention given to the "development of abilities in problem-solving," with the following comment:

> Verbal problems are included as a part of the work at all grade levels--even the first, and effective problem-solving helps are systematically provided. These problem-solving activities aim at helping the child understand and see the relationships between what he wants to find out and the known facts. They seek to help the child develop his own way of solving problems and to leave him with a method of attack on quantitative situations. Consequently, modern programs in arithmetic encourage the use of problems to introduce new concepts, for practice on concepts which have been developed and for evaluation and testing of concepts learned. Problem-solving is assuming a role of major importance in programs based on the development of meanings (p. 5).

In discussing the "link" between computation and problem solving, Harry Grove Wheat (1954), criticized the teaching of mathematics during the 1950s with the following comment:

> Computation and problem solving have always been widely-separated activities in the arithmetic of the school. We do very little in present-day teaching to get our pupils able to recognize them as a single thinking procedure in different dresses (p. 5).

In the textbook, Guilding Arithmetic Learning, John R. Clark and Laura K. Eads (1954) noted that "thinking" and "concept development" were integral parts in the problem-solving process. In stating that problem-solving situations could be found "at the experience level, at the materials level, at the generalizing level, at the computation level" (pp. 258-259), Clark and Eads (1954) included the following eight steps for "determining readiness in problem solving" for individuals and for groups of children:

1. They can find the solution readily.
2. They can change the numbers in the problem to other reasonable numbers.
3. They are not distracted by extraneous data.
4. They can change the items in the problem to other reasonable items.
5. Often they can solve the problem in more than one way.
6. They can devise other problems using the same situation.
7. They can talk about the problem, tell things about it that were not stated, invent circumstances that created the problem, etc.
8. They can explain why they used the method they used in solving the problem (pp. 264-265).

In writing Practical Plans for Teaching Arithmetic in 1954, Ruth H. Drewes, Ada S. Mermer, and Winifred P. von Boenigk discussed "training" in problem analysis as a part of the development of problem-solving ability, especially in solving verbal problems. In developing skill in problem analysis, the following series of questions and directions were suggested:

1. Read the problem silently (Teacher supplies needed words.).
2. What does the problem ask?
3. What facts must we know in order to answer the question?
4. What process shall we use?
5. Estimate the answer.
6. Work the problem.
7. Label the problem.
8. Check the problem.
9. Compare the answer with the estimate (p. 99).

In 1954, Herbert F. Spitzer, in the textbook The Teaching of Arithmetic, made the following observation regarding the treatment of problem solving in the elementary classroom:

> The treatment of problem-solving in many books on the teaching of arithmetic includes extensive material on the kinds of subject matter with which verbal problems should deal. It is generally recommended that problems should deal with things and conditions that are within the experience of the elementary school child--that problems should be real. Where problems are used for illustrative purposes, as in initial instruction, this matter of having the subject matter of problems within the experience of the children is important (pp. 183-184).

In discussing the inclusion of problem solving activities and processes in materials used by children in the classroom, Spitzer (1954) summarized some of the suggestions on problem solving made by publishers. He wrote:

> The manuals and advertising which accompany pupils' textbooks make extensive claims regarding the program of teaching problem-solving included in the books. The procedures actually used in the books are not always as impressive as the statements made about them. Textbooks emphasize most often the technique of selecting the fundamental process needed to solve the problem. Other procedures frequently found in the books are these: (1) systematic analysis of problems (stating what is given, what is to be found, etc.); (2) finding hidden questions in problems; (3) estimating the answer to problems; (4) solving problems containing superfluous numbers; (5) dealing with problem situations which lack sufficient data; (6) admonition to read carefully; and (7) use of sets of problems based on one social scene, such as a State Fair, the Grocery Store, or Ranch Life (p. 188).

Spitzer (1954) called for a more elaborate plan for the development of skill in problem solving. In addition to the procedures noted in the previous citation, he presented the following activities and ideas for the development of problem-solving ability:

1. The non-pencil-and-paper, or oral solving of problems.
2. The use of diagrams or drawings.
3. Writing the number question.
4. Pupil formulation of problems.
5. Intensive study of number operations.
6. Class solution of difficult quantitative problems encountered in other school work.

7. The work or study habit whereby the pupil asks himself questions about the problem.
8. Solving the same problem by several techniques.
9. Other procedures for improving problem-solving ability (pp. 189-199).

In a texbook titled The Diagnosis and Treatment of Learning Difficulties, Leo J. Brueckner and Guy L. Bond (1955) provided insight into "Diagnosis in Problem-Solving and Quantitative Thinking." The authors suggested several types of tests "to measure the ability of pupils to solve problems and to think quantitatively" (p. 233). In 1955, Brueckner and Bond noted that in diagnosing problem solving, the measures found in computational ability and reading ability should supplement the measures found in the following five "tests":

1. the ability to solve verbal problems, one of the major skills measured by available standard tests;
2. the ability to read graphs, charts, tables, and similar materials and to answer questions based on them;
3. knowledge of vocabulary;
4. knowledge about social applications of arithmetic fundamental in problem-solving; and
5. quantitative understandings (p. 233).

In citing research, Brueckner and Bond (1955) included a list of nine areas in which good problem solvers were found to be superior to poor problem solvers. The categories were:

1. Computational ability.
2. Ability to apply the sequence of steps involved in problem-solving.

3. Ability to estimate answers to verbal problems.
4. Range of information about social uses of arithmetic.
5. Ability to read graphs, charts, tables.
6. Ability to see relations in number series.
7. General and nonverbal reasoning ability.
8. General reading level.
9. Level of mental ability (p. 290).

The authors recommended the following specific procedures for treating deficiencies in problem solving:

1. Making number operations meaningful.
2. Experience in using operations in social situations.
3. Using objects to show the meaning of processes used in solving problems.
4. Using manipulative materials to work out solutions of problems.
5. Visualizing solutions of problems.
6. Improving the quality of verbal problems.
7. Developing relationships among number processes.
8. Explaining reasons for using processes.
9. Identifying processes to use.
10. Problems without numbers.
11. Learning to sense relationships in equations.
12. Making the vocabulary of problems meaningful.
13. General program for improving problem-solving (pp. 294-300).

E. H. Taylor and C. N. Mills, in 1955, classified problem-solving difficulties into five distinct categories. Beginning with reading, the categories were listed in the following manner:

1. Difficulties with reading.
    a. Difficult vocabulary.
    b. Carelessness in reading.
    c. Technical terms.
    d. Unfamiliar units of measure.
    e. Unfamiliar forms of statements or questions.
2. Difficulties with numbers.
    a. Very large numbers.
    b. Very small numbers.
    c. Kinds of numbers--integers, fractions, decimals.
3. Difficulties with computation.
    a. Errors in computation.
    b. Inability to perform the necessary computation.
4. Difficulties with facts.
    a. Lack of knowledge of facts assumed to be known.
    b. Not enough facts given to make a problem.
    c. Confusion caused by extraneous facts.
5. Difficulties with comprehension and analysis.
    a. Inability to comprehend the meaning of the problem.
    b. Inability to analyze the problem into separate conditions.
    c. Inability to image the conditions of the problem.
    d. Inability to decide the necessary process or processes to be performed (pp. 227-228).

The authors suggested the following "directions" in solving problems:

1. Read the problem carefully.
2. Decide what is to be found.
3. Decide what facts are given in the problem.
4. Decide what other facts are needed, and determine these facts.

5. Determine the processes needed in the solution.
6. Estimate the result.
7. Perform accurately the necessary computation.
8. Check the results (pp. 242-243).

In addressing the question, "How do you teach problem solving?," Howard F. Fehr (1955) made the following suggestions:

> Perhaps most significant are (1) always look at the whole problem, the whole situation; (2) seek the relationship of the parts to the whole, and the whole to the part; (3) analyze, organize and reorganize the relationships until what is known is directly related to what is wanted, then insight will occur. Thus problem solving demands "<u>ceaseless attention to the building of clear, well interrelated arithmetic concepts in all the areas of common experience</u>" (pp. 30-31).

Regarding the solving of word problems, Fehr (1955) noted the reliance upon "active thinking" in the problem-solving process. "Estimation," "mental solutions," and "the association of language with an operation" were cited as three characteristics that needed greater attention in the problem-solving activities associated with word problems (p. 31).

In 1956, Shirley Stillinger Brewer utilized an often-used concept in order to improve the problem-solving abilities of her college-level students. In working experiments with electricity, Brewer introduced the "scientific method." The steps are:

1. Define problem
2. Research
3. Hypothesis

4. Experiment
5. Conclusion (p. 117).

Then, in applying the scientific method to solving word problems, Brewer (1956) reworked the method and provided the following five-step problem-solving model:

1. What do we want to know?
2. What do we know?
3. Best estimate.
4. Working the problem.
5. What did we find out? (p. 117)

At the conclusion of Brewer's comments in the Arithmetic Teacher, the editor said:

> EDITOR'S NOTE. Although the scientific method or approach to problem solving may be very old, each new generation should discover it. Mrs. Brewer's pupils did this by simple transfer from work in science. There are many opportunities for "tieing together" arithmetic with other areas of learning. Each of these ties tends to give greater understanding and significance to each of the areas involved. The most important element in Mrs. Brewer's development is the role of the pupils in thinking (p. 118).

In an article titled "Developing Facility in Solving Verbal Problems," Herbert F. Spitzer and Frances Flournoy (1956) elaborated on the use of problem-solving techniques utilized in the mathematics classroom with the following comments:

> The improvement of pupil achievement in verbal problem solving is an important objective of most upper grade arithmetic teachers. That this objective is not often reached with any degree of satisfaction is evident to all students of arithmetic teaching. It is also quite evident to students of arithmetic teaching that, although there are many problem-solving improvement procedures in use, the

> most widely used procedure is that of just having pupils work problems without specific directions or suggestions. The other problem-solving improvement procedures practically all suggest specific steps for pupils to engage in (p. 177).

Spitzer and Flournoy (1956) criticized instruction in problem solving. On a pessimistic note, they suggested, in the following comment, a simple modification of existing practices:

> In view of the rather long time that instructors have been concerned with problem solving, it is very doubtful whether any one entirely new procedure of merit will turn up. Improvement will, then, most likely be the result of modification and refinement of plans now in use (p. 177).

In their conclusion, Spitzer and Flournoy provided two recommendations for improving abilities in problem solving. In citing "the sketchy problem-solving improvement program offered by any one textbook, the lack of agreement on procedures among textbooks, and the omission from textbooks of what appear to be promising procedures" (p. 182), they recommended the following:

1. The typical textbook program for improving or developing problem-solving ability has to be supplemented by providing more experience with the techniques recommended and by using promising techniques not included in the textbook.
2. Students of arithmetic teaching need to make studies to determine whether or not proposed problem-solving improvement procedures actually contribute to this ability (p. 182).

Harry Peeler (1956), in an article on "Teaching Verbal Problems in Arithmetic," noted that "a meaningful approach to problem solving calls for problem solving directed toward an outgrowth of generalizations, understandings, and number facts--using these tools to lead into other phases of problem solving" (p. 244). Peeler elaborated on how the phases of problem solving are considered to be an integral part of the arithmetic program. The following comments were made:

> In a program of teaching arithmetic meaningfully, problem solving is not one separate part of the whole but is an element running through and intertwined with the whole. As concepts, skills, generalizations and understandings progress and grow so does ability in problem solving grow, and as problem solving grows so grow the skills, concepts and understandings (p. 245).

In 1957, Leo J. Brueckner, in the textbook, Improving the Arithmetic Program, cited the following procedures for improving problem-solving ability:

1. Have the children solve many easy, interesting, well-graded problems.
2. Seize opportunities that arise to have the children apply numbers in social settings.
3. Give the slower learners direct guidance in the reading and analysis of problems, tables, graphs, maps, diagrams, and so on. The more able learners can devise their own procedures.
4. Try to increase the child's understanding of the meaning of the four number processes and the situations under which they are operative.
5. Insist on the checking of computations performed in problem-solving.
6. Exercises in the carefully directed reading of problems will be of real value.
7. Vocabulary exercises to broaden and enrich meanings of words and expression are desirable.
8. Have the children demonstrate problem situations with objects, drawings, illustrations, and diagrams to make them meaningful.
9. Be sure that the children are familiar with systems and instruments of measurement.

10. Show the child the progress he is making by the results of standard and informal tests of reading.
11. Have the children try to suggest several ways of solving a problem so as to give extra practice in problem-solving and to improve quantitative thinking (pp. 78-80).

Leo J. Brueckner, Foster E. Grossnickle, and John Reckzeh (1957) included a section on the "Logical Pattern of Problem Solving" in the textbook titled Developing Mathematical Understandings in the Upper Grades. A five-step plan, called a "logical approach," was outlined in the following manner:

1. Find what the problem question is.
2. Then find what facts the problem gives.
3. Try to think of ways to find the answer to the question asked, or search for a familiar pattern or model in the problem.
4. Do the necessary computation.
5. Check the answer to see if it is sensible (p. 319).

In contrast to the above steps, the three authors suggested the utilization of the following steps for children who lack sufficient background ability in problem solving and for children who are slow learners:

1. Find what the problem question is.
2. Identify the information and numbers given in the situation the problem presents.
3. Show how the answer to the question asked depends upon what is given (p. 320).

John L. Marks, C. Richard Purdy, and Lucien B. Kinney, in the textbook Teaching Arithmetic for Understanding (1958), noted that children who follow a systematic plan were more liable to succeed in solving word problems than children who did not follow a plan. They observed that "no one systematic plan is superior" (p. 329) and included the following problem-solving procedure:

1. Read the problem and decide: What does the problem ask me to find? What does the problem give me to use?
2. Make a drawing, if needed.
3. Select the process or processes. Do I add? Subtract? Multiply? Divide? In what order? With what numbers?
4. Estimate the answer.
5. Compute and check the computations.
6. Compare the answer to the estimate (pp. 328-329).

In 1958, Robert H. Koenker authored an article titled "Twenty Methods for Improving Problem Solving." He included the following suggestions which had been "proven to be of value by research and/or clasroom practice" (p. 74):

1. Use the whole method of attacking problems.
2. Estimating answers to problems before solving.
3. Diagramming problems.
4. Dramatizing problems.
5. Orally solving problems.
6. Encourage children to work problems using different methods.
7. Differentiating problems for the various ability levels.
8. Making up problems.
9. An understanding of arithmetic is prerequisite to problem solving.
10. Using concrete objects and devices in solving problems.

11. Going over assigned problems with children.
12. Checking or proving answers.
13. Working problems without numbers.
14. Solving problems with irrelevant facts.
15. Finding missing facts in problems.
16. Finishing incomplete problems.
17. Stressing careful reading of problems.
18. Developing an understanding of arithmetic vocabulary.
19. Stressing neatness of work.
20. Mixing problems of different types (pp. 74-77).

In a critical look at "Twentieth Century Mathematics for the Elementary School," H. Van Engen highlighted some of the difficulties found in the teaching of arithmetic. Buoyed by the launching of the Russian satellites, Sputnik I and Sputnik II, the comments by Van Engen (1959) were part of a trend criticizing the teaching and learning practices in mathematics and science. Regarding problem solving, he made the following notation:

> The schools have not been successful in devising a sensible approach to problem-solving. This audience is all too familiar with the various proposals for improving the problem-solving ability of the elementary school pupil. In spite of all the proposals and the research, it is probably not too far amiss to summarize the results of present-day research by the single statement: The best way to teach children how to solve problems is to give them lots of problems to solve. Certainly a fresh approach to problem solving is needed (p. 74).

Van Engen (1959) recommended a more mathematical approach to problem solving. In calling the procedure "a mathematician's approach to problems in miniature" (p. 75), he provided the following observation:

> One first searches for the fundamental structure of a problem situation; then he finds the appropriate symbols to express this structure. Once the problem has been structured, a knowledge of previous problems and problem-solving techniques can be applied. Certainly, no "cue" method or mere admonitions to THINK hold the mathematical power that the search for the structure of the physical situation can command. The failure of the older methods over the past years should be reason enough to banish them from the classroom and search for methods with more mathematical power (p. 75).

The Twenty-fifth Yearbook sponsored by the National Council of Teachers of Mathematics included a short section on "The Psychology of Problem Solving." Embedded in a chapter on "Reading in Arithmetic," David H. Russell (1960) discussed and characterized "problem solving" as "a systematic and logical process containing from three to nine steps" (p. 216). The author noted that most of the problem-solving procedures included such activities as "*getting to understand the problem*, *search*, *suggesting solutions* and *eliminating sources of error*" (p. 216). In summarizing the research findings into what problem solving entails, he provided the following commentary:

> The research suggests that problem solving is not a unitary factor, best described by one term such as *reasoning*, but rather a complex of different abilities. While the specifics are not always clear, the essential parts of problem solving seem to be an orienting function, an elaborative and analytical function, and a critical function. The problem-solving process varies with the nature of the form of stating the problem, the methods of attack known by the solver, the personal characteristics of the solver, and the total situation in which the problem is presented (p. 216).

In 1961, Wilbur H. Dutton and L. J. Adams included a six-step problem-solving model in Arithmetic for Teachers. The steps in the model had a dual purpose--a set of rules to be used in the solving of arithmetic problems in both textbook and non-textbook situations. The six "rules" in the problem-solving technique were stated in the following manner:

1. Read the problem.
2. Decide what is given.
3. Decide what is to be found.
4. Decide which operations are necessary to take what is given and use it to find what is to be found.
5. Solve the problem.
6. Check the result (p. 178).

In commenting on the use of the above-mentioned problem-solving model, the two writers solicited the inclusion of "application" as a substitute for the so-called "realistic, practical" problems that had permeated mathematics instruction. Dutton and Adams (1961) made the following observation:

> Problems do not have to be realistic to be useful. So-called fantasy problems may generate interest more readily than realistic, practical problems, and lead to genuine desire on the part of the student to learn more about arithmetic and mathematical principles. Besides, the use of the word "practical" can lead to argument; what is practical for one may not be practical for another. It is not necessary for a problem to be of immediate value to be justifiable. Instead of describing problems as practical it may be better to describe them as applications; that is, many problems are applications in the sense that they illustrate how arithmetic may be applied in situations that are

> practical for some people and not necessarily practical for the learner at the time that he studies them. In this sense many problems illustrate how arithmetic can be used (p. 180).

"Trends and issues" in elementary school arithmetic provided the theme for the book of readings titled Improving Mathematics Programs: Trends and Issues in the Elementary School edited by M. Vere DeVault (1961). In addition to a number of citations on solving problems appearing throughout the book, a chapter written by Lowry W. Harding (1961) was devoted to "Productive Approaches to Problem Solving." He noted in the introduction that "the essence of problem solving is search and discovery" (p. 194). Harding then followed the above comment with:

> And discovery is creative--it requires originality. Thus, the teacher faces both temptation and opportunity. If he succumbs to the temptation merely to drill his pupils on routine operations he deadens their interest, inhibits their creativity and hinders their intellectual development. But, if he stimulates the interest of pupils by considering what are problems to them and challenges curiosity and effort by suggesting other questions appropriate to their ability and knowledge he may give them a desire for, and some competence in independent thinking (p. 194).

In a textbook titled The Teaching of Arithmetic, James Robert Overman (1961) described problem solving as a "thought-process." In a chapter on "Developing Patterns and Habits of Thinking Useful in Problem-Solving," he included a series of "useful thought patterns in solving problems." The following four-phase problem-solving process was cited:

1. Getting a clear understanding of the conditions of the problem.
2. Planning the solution.
3. Carrying out the plan.
4. Checking the result obtained (pp. 381-382).

Overman noted that the four phases of thinking do not always "occur separately" nor do they always occur "in a fixed order" (p. 382).

Writing about "those problem-solving perlexities," Cleata B. Thorpe (1961) made the following introductory comment about teachers and their dealing with the concept of problem solving:

> We teachers are in large measure responsible for problem-solving being the obstacle that it is to many a pupil in elementary schools. In the first place, we toss the terms "problem" and "problem-solving" about quite indiscriminately. We seem to have no clear and definite concepts for those terms in our own minds (p. 152).

After discussing the teachers' vague concepts of problem solving, she included recommendations for improved instruction in solving problems. In noting that "the use of reflective teaching in problem-solving lessons in arithmetic is most effective when certain conditions prevail" (p. 155) the author included the following provisions for the betterment of problem solving:

1. The atmosphere of the classroom must be conducive to pupils feeling accepted and at ease.
2. The attitude of the class must be cooperative and receptive to the ideas and opinions of others.
3. Suitable problem material must be provided.

4. There must be time for unhurried thought and relaxed activity.
5. Teachers and pupils remember that concepts do not come as single flashes of insight; they unfold slowly, step by step, even with guidance.
6. There is recognition of the fact that there is usually more than one correct and acceptable solution to every problem, though some may be too "round-about" for practicality.
7. Any correct solution is commended (pp. 155-156).

Thorpe also called upon teachers to do their part in improving the teaching of problem solving in the elementary school. She wrote the following summary in 1961:

> The values of such learning processes in the teaching of problem-solving in arithmetic have come to be more widely recognized than in the past, and there is every indication that they will play an increasingly important part in elementary arithmetic-learning in the immediate future. By way of summary, we may conclude that there are some very promising possibilities for vanquishing the problem-solving perplexities--if teachers will do their part (p. 156).

In 1962, George Polya produced Volume I of Mathematical Discovery on Understanding, Learning, and Teaching Problem Solving. In the Preface, he commented upon the importance and applicability of teaching problem solving in mathematics as follows:

> What is know-how in mathematics? The ability to solve problems--not merely routine problems but problems requiring some degree of independence, judgment, originality, creativity. Therefore, the first and foremost duty of the high school in teaching mathematics is to emphasize methodical work in problem solving (p. viii).

Charles F. Howard and Enoch Dumas (1963) included a chapter titled "Developing Problem-Solving Ability" in the textbook Basic Procedures in Teaching Arithmetic. In the discussion, they noted that "traditionally, the arithmetic program in the elementary school has relied heavily upon verbal problems, which are referred to often as 'word problems' or 'story problems,' to develop the pupils' problem-solving abilities" (p. 339). The authors suggested an understanding of the setting of a verbal problem (including reading and vocabulary skills), the identification of mathematical relationships in problem situations, and improved computational procedures in an effort to strengthen the problem-solving abilities of pupils in arithmetic (pp. 343-356).

In the fourth edition of Discovering Meanings in Elementary School Mathematics, Foster E. Grossnickle and Leo J. Brueckner (1963) referred to "problem solving" as "the highest level of quantitative thinking" (p. 301). In making the above comment, they noted that "it is a primary function of the arithmetic program to arrange experiences that will develop in children the ability to 'think through' problematic situations that they encounter and to deal with them intelligently and skillfully" (p. 301). In discussing the "scope of quantitative thinking," the two authors connected aspects of quantitative thinking and reflective thinking with problem solving in the comments that follow:

> Problem solving is the highest form of reflective thinking. A mathematics program...that emphasizes meaning and understanding is based largely on problem solving. Quantitative thinking is the basis of effective problem solving (p. 302).

Grossnickle and Brueckner (1963) provided insight into the development of problem-solving ability with the inclusion of five "elements of a problematic situation" which were described in the following manner:

1. A desired goal is to be attained.
2. There is a blocking of the path to be taken to attain the goal.
3. Available habitual responses are not suitable or adequate to attain the goal.
4. Various possible solutions (hypotheses) are proposed and tested.
5. A tentative conclusion is reached (p. 303).

In commenting on "Guidance in Problem Solving," Grossnickle and Brueckner (1963) included the following list of factors that contributed to the successful solving of problems:

1. The problem was of concern to the children.
2. The problem was solvable.
3. The problem was clearly defined in terms that each pupil understood.
4. The children gave suggestions as to possible solutions and evaluated them.
5. The children worked out a tentative plan for solving the problem and gathered, evaluated, and organized the necessary data.
6. A variety of operations was used to obtain the data.
7. The children were guided by the teacher in planning the solution and in reaching conclusions as the occasion demanded.

8. The children used the solution that they had arrived at in dealing with the situation the original problem presented and realized that they had been successful (p. 305).

Foster E. Grossnickle (1964), in an article titled "Verbal Problem Solving," noted that the concept of "problem solving" is taught to help the pupil discover a pattern used in solving problems. The following six-step process was included to help develop a pattern for solving problems:

1. identification of the problem question,
2. recognition of the operation to use,
3. writing the mathematical sentence to express the relationship between the numbers given,
4. finding the number which will make the sentence true,
5. checking the solution by evaluating the equation, and
6. labeling the answer (p. 14).

In "A Look at Problem Solving in Elementary School Mathematics," Kathryn V. Herlihy (1964) noted that "the development of problem-solving ability may well be the main objective of education" (p. 308). In the discussion on problem solving, she suggested a plan that was closely aligned with the problem-solving model first proposed by G. Polya in the 1940s. The steps included devising a plan, which Herlihy called "the main achievement in problem solving" (p. 310). After the plan was devised, she suggested knowledge of computational skills, a process for checking the steps, and the final solution to better develop skill in problem solving (p. 310).

J. Houston Banks, in 1964, discussed the "objective" of problem solving in the textbook _Learning and Teaching Arithmetic_ with the following comment:

> Skill in problem solving is not the sole objective of arithmetic instruction. Nor is problem solving a separate topic of arithmetic. It should be a vital part of all phases of the subject. Meaningful problems should serve to introduce each new arithmetical topic. The problem should establish a need for the new skill. For example, problems most efficiently solved by division should be used as a preliminary to learning the division algorithm. They enable the pupil to appreciate the usefulness of the process (pp. 405-406).

In elaborating on the "Improvement of Problem-Solving Ability," Banks (1964) included the "Analysis Model" which he noted was "held in high favor by many arithmetic textbook writers" (p. 414). The "Analysis Model" consisted of the following formal steps:

> Read the problem to determine (a) what is given, and (b) what is required. (c) Determine from the relationships between the quantities, given and required, what operations are necessary. (d) Estimate the answer. (e) Solve by performing the operations in (c). (f) Check the answer (p. 414).

Wilbur H. Dutton (1964), in _Evaluating Pupils' Understanding of Arithmetic_, provided the following concern regarding the change that had been taking place in mathematics instruction during the 1960s:

> No other educational problem is receiving as much attention as the new mathematics curricula. We are, in fact, in the midst of a revolution which began after World War II and which is assuming amazing proportions today. Persistent problems relating to the _kind_ and amount of mathematics to be taught in the elementary school have been accentuated. Problems pertaining to the teaching

> of arithmetic have been created by the introduction of new mathematical content and recent interpretations of the meaning theory. Because of rapid developments within the field of mathematics and the rapidity of change within our culture, it is not clear what the content or methods will be a few years from now. Thus the need for pupils and teachers to learn mathematical concepts thoroughly and to be able to adjust to additional changes in application and methods in the future seems paramount (p. 1).

In discussing the "new math," he made the following comments regarding change in elementary school arithmetic instruction:

> In methodology, emphasis is placed upon discovery and intuitive thinking. The primary aim of all teaching of the new mathematics is to help children understand the fundamental structure of mathematics and methods of mathematical reasoning (p. 13).

Enoch Dumas, in writing a section on "Solving the Problem of Problem Solving" in Updating Mathematics (edited by Francis J. Mueller), described a multi-step problem-solving process aimed at the improvement of solving word problems. Dumas (1964) provided the following procedure:

1. Recognize each word for what it is and attach the meaning intended by the author.
2. See the relationships among parts of a sentence and among a group of sentences.
3. Understand the social setting.
4. Identify the mathematical relationships and the needed computations.
5. Carry out the necessary computation.
6. Label the answer.
7. Verify the correctness of his answer (pp. 179-180).

In the same book, Gerald W. Brown (1964) described a seven-step problem-solving sequence in a discussion on "Mental Arithmetic in the Problem-Solving Process." In citing the sequence of steps taken from a number of textbook series, he noted the following procedure:

1. Be certain you understand the problem.
2. Think what you need to know to solve the problem.
3. Look for a hidden question.
4. Decide what computations to make.
5. Estimate or decide on a reasonable answer.
6. Perform the necessary computations.
7. Check your final answer by comparing it to your estimate (p. 195).

Brown advocated plenty of practice in getting children to understand the above process of problem solving.

In 1966, Leslie A. Dwight included a chapter on problem solving in the textbook, Modern Mathematics for the Elementary Teacher. The author included the "conventional" use of problem solving which was described in the following manner:

> ...problem solving refers to a set of statements, oral or written, which gives information, usually related to everyday life situations, involving quantitative data and which implies finding a quantitative answer without indicating how the quantitative answer is to be obtained (p. 470).

The conventional use of problem solving implies work with word problems, story problems, or verbal problems. Dwight (1966) provided the following seven-step procedure for solving verbal problems:

1. Read the problem to get an over-all view of the whole problem.
2. Reread the problem and identify:
   a. The problem question--what is to be found
   b. The data relevant to the problem question.
3. Formulate, in words, a sentence (or sentences) expressing the relation (or relations) of the known and unknown numbers implied by the pattern or action described in the verbal problem.
4. Write the word sentence (or sentences) in symbolic (horizontal) form in which a symbol (or symbols) is used to represent the unknown number (or numbers).
5. Find the set of numbers that makes the mathematical sentence (or sentences) of (4) a true statement (or true statements).
6. Check the solution obtained in (5).
7. State a word sentence, using the solution, to answer the problem question (p. 471).

In writing Building Mathematical Competence in the Elementary School, Peter Lincoln Spencer and Marguerite Brydegaard (1966) introduced the chapter on problem solving with the following comments:

> In a very real sense, the process of mathematicking is one of problem sensing and problem solving. It seems axiomatic that problem-solving behavior occurs only when one is confronted with a problem that he wills to solve. Hence, the first concern with the development of problem-solving abilities is that of leading the student to sense a problem and to desire to find a way to cope with it (p. 349).

In a discussion on "Developing Competency with Problem Solving," they noted the separate procedures of Leslie Beatty and E. H. C. Hildebrandt. In describing the Beatty Model, Spencer and Brydegaard (1966) included the "phases" incorporating procedures and materials that had been utilized in a 1959 project involving 2000 children. The model was described in the following manner:

1. Identifying "What is the question?",
2. Analysis of the problems independent of number,
3. Children writing their own problems,
4. Estimating and judging problems,
5. Graphic structuring,
6. Labeling the answer to a problem,
7. Identifying "On what does the answer depend?", and
8. Mental arithmetic (p. 353).

The Hildebrandt Model was described as follows:

1. Presentation,
2. Attention,
3. Observation and Exploration,
4. Classification,
5. Further Exploration,
6. Formulation,
7. Generalization, and
8. Verification and Application (p. 354).

In 1966, Robert M. Gagne discussed the idea of "Discovery in Problem Solving" in Learning by Discovery: A Critical Appraisal, edited by Lee S. Shulman and Evan R. Keislar. He made the following comment:

> A still more complex kind of learning situation within which it is appropriate to suppose that discovery may occur is called problem solving. In order to define this kind of learning, it is necessary to distinguish two things. First is the fact that problem solving typically requires the learner to acquire what may be called a higher-order principle, formed by putting together two or more simpler principles.

> The second characteristic of the problem-solving situation is a matter of method, rather than content. Problem-solving situations are usually designed to <u>require</u> discovery on the part of the learner. This is an inevitable part of their makeup; otherwise, they would probably not be called problem-solving (p. 147).

Louis S. Cohen and David C. Johnson (1967) summarized some of the behaviors found in "problem solving" in the <u>Arithmetic Teacher</u>. They provided a list of "behaviors that might be exhibited by the problem solver." The list included: "observing, exploring, decision making, organizing, recognizing, remembering, supplementing, regrouping, isolating, combining, diagramming, guessing, classifying, formulating, generalizing, verifying, and applying" (p. 261). Furthermore, the authors noted "the ability to translate a given situation into mathematical symbolism is considered to be <u>the most useful tool in problem solving</u>" (p. 262).

In the textbook <u>Multiple Methods of Teaching Mathematics in the Elementary School</u> (1968), Charles H. D'Augustine defined problem solving as "the process of reorganizing concepts and skills into a new pattern of application that opens a path to a goal. This is in contrast to the application of a habitual pattern to reach a previously attained goal" (p. 25). In the chapter on "problem solving," he cited two factors which the teacher had the responsibility for "nurturing." The two factors were "the concepts and skills that a child brings to a problem-solving situation and his repertoire of previously solved problems" (p. 26).

The October, 1969, issue of Review of Educational Research (RER) provided a review of research in science and mathematics. In summarizing the research on "Problem Solving in Mathematics," Jeremy Kilpatrick (1969) introduced the section by noting: "The preeminence of increased problem-solving ability as a goal of mathematics instruction has long been admitted; but like the weather, problem solving has been more talked about than predicted, controlled, or understood" (p. 523). In discussing the research on problem solving completed during the 1960s, he noted that most of the research in mathematics education had been completed by doctoral students working on dissertations. The following comment was made:

> Problem solving is not now being investigated systematically by mathematics educators. Few studies build on previous research; few studies have an explicit theoretical rationale. There are signs, however, that some mathematics educators are beginning to borrow ideas and techniques from recent psychological studies of higher cognitive processes. As more educational researchers appear who are trained both in mathematics and in psychology, research on problem solving in mathematics may attain a direction and cohesiveness it currently lacks (p. 523).

In 1969, Ruth S. Jacobson provided "A Structured Method for Arithmetic Problem Solving in Special Education." The following "Guide for Problem Solving" was included to help "solve the unknown":

1. Read the problem.
2. What is the action? Is it combining or separating equal or unequal groups? (In a comparison problem, omit Step 2.)
3. What is the operation?
4. Show the equation using frames
$$\bigcirc \times \triangle = \square$$
5. Fill in the knowns.
6. Compute (p. 25).

The focus of the lesson on problem solving included in the discussion was twofold: "One, to give the child who has had difficulty in solving word problems a method of attack, and two, to give the classroom teacher some ideas so that he may develop new lessons for the future" (p. 27).

In 1969, J. D. Williams considered three "determinants" in the success of solving problems. These were:

1. Factors in the problem-solving situation.
2. Factors in the problem-solver's previous training.
3. Characteristics of the problem-solver (p. 261).

In developing abilities to solve problems, Williams provided the following list of "general advice" in suggesting an eight-step process that had originally been cited in 1957 by A. N. Frandsen:

1. Determine what is wanted.
2. Find which of the given facts are relevant.
3. State in a single sentence what is wanted as a function of the data given.
4. Restate this in arithmetical language.
5. Try to recognise (sic) this statement as one of the standard operations used in arithmetic, and plan the solution.

6. Estimate an answer.
7. Make necessary computations.
8. Check the solution. (a) Does the answer approximate to the estimated answer? (b) Perform any possible arithmetical check (pp. 274-275).

The author noted the importance of "readiness" in the ability to provide "transfer" in the problem-solving process (p. 274).

Shortly after the launching of the Russian satellites Sputnik I and II in 1957, and continuing through the 1960s and into the 1970s, a number of projects were developed in an effort to improve the overall quality of learning and instruction in science and mathematics. These mathematics projects looked at changes in curricular materials and content, ideologies, and methodologies. Most of the projects incorporated some aspect of problem solving into the teaching/learning process, either through the structure of learning or through the use of materials in the project. For example, the Greater Cleveland Mathematics Program (GCMP) emphasized problem solving throughout the K-12 program (Weiss, 1978, p. 9). "The project made extensive use of the discovery approach to learning and drew heavily upon the principles of mathematics to help children learn the underlying structure of the material presented" (Mahaffey and Perrodin, 1973, p. 247). See Appendix F for a list of projects.

A number of ideas were fostered by the mathematics programs and projects started in the late 1950s and 1960s. Anne W. Schaefer and Albert H. Mauthe (1970) cited the work of the Nuffield Teaching Project. In noting its theme created by an ancient Chinese Proverb--I hear, and I forget; I see, and I remember; I do, and I understand--the authors made the following comment:

> A promising approach to the teaching of problem solving is the use of the mathematics laboratory. The purpose of the laboratory is to provide children with opportunities to discover mathematical concepts through their active involvement in solving problems. The emphasis is on learning by doing (p. 7).

In 1970, Wilbur H. Dutton, Colin C. Petrie, and L. J. Adams included an extensive chapter on "Problem Solving" in the textbook <u>Arithmetic for Teachers</u>. The chapter included: (1) types of problems; (2) children's thinking; (3) concept formation; and (4) strategies for solving problems. In noting the "types of problems," the authors described various levels of cognition. Part of the discussion involved "<u>The Taxonomy of Educational Objectives</u> for the cognitive domain" cited earlier in the <u>Taxonomy of Educational Objectives</u> by D. R. Krathwohl, B. S. Bloom, and B. B. Masia (1964). In discussing the levels of cognition, Dutton, Petrie, and Adams (1970) state that the "classification enables the curriculum developer and the teacher to select the level of cognition intended for a particular unit or lesson" (p. 141). The six levels of cognition were described in the following manner:

1. Knowledge, requiring recall of specific facts, trends and sequences, classifications, criteria, theories, and structures.
2. Comprehension, requiring understanding, translation, interpretation, and extrapolation.
3. Application, requiring the use of abstractions in particular and concrete situations.
4. Analysis, requiring a breakdown of communication into its constituent elements such that the relative hierarchy of ideas is made clear or the relations between the ideas made explicit.
5. Synthesis, requiring the putting together of elements so as to form a whole.
7. Evaluation, requiring judgments about the value of material and methods for given purposes (p. 141).

In terms of problem-solving techniques, Dutton, Petrie, and Adams (1970) emphasized various aspects of "discovery learning." In discussing the procedures used in the "discovery" process, the following problem-solving techniques included in a non-referenced textbook series were cited:

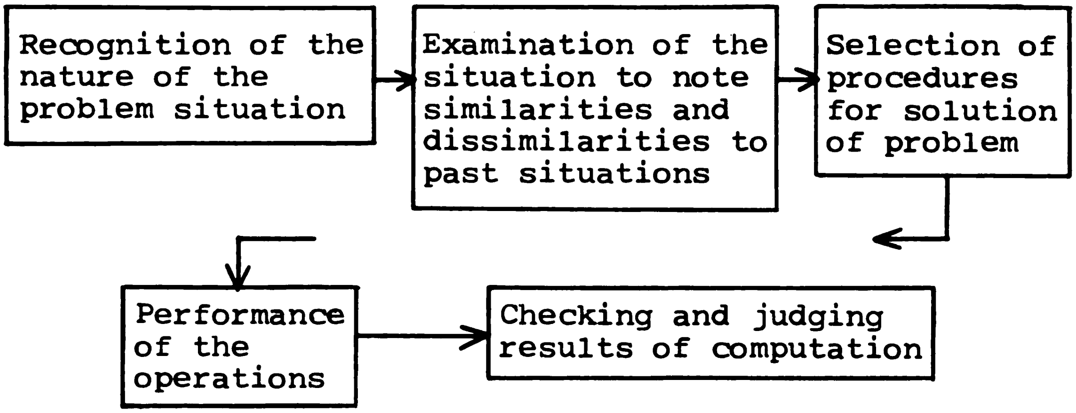
1. Using objects and pictures to begin instruction on a topic or problem,
2. Relating the new work to meaningful, everyday experiences,
3. Presenting and developing mathematical ideas and concepts before introducing abstract symbols or other formalizations,
4. Developing effective study procedures,
5. Varying teaching procedures to meet individual differences (p. 151).

Dutton, Petrie, and Adams (1970) provided the following set of rules to assist children in the problem-solving process:

1. Read the problem.
2. Decide what was given.
3. Decide what is to be found.
4. Decide which operations are necessary to take what is given and use it to find what is to be found.
5. Solve the problem.
6. Check the result (p. 151).

The authors recommended that "problem-solving work should be organized around the abilities and experiences of children" (p. 151). Thus, it is not surprising that they cited Piaget's three stages of mental structuring--(1) sensori-motor group structures, (2) concrete-operations group structures, and (3) formal mental structures--and the four principles of conceptual learning formulated by Zoltan P. Dienes--The Dynamic Principle, The Perceptual Variability Principle, The Mathematical Variability Principle, and The Constructivity Principle (pp. 143-148).

In the textbook Guiding Children to Mathematical Discovery (1970), Leonard M. Kennedy provided a broad description of problem solving. The author stated that "a problem should be considered as any situation an individual faces for which no immediate solution is apparent, but for which the possibility of solution exists" (p. 358). It was noted that word problems were "designed to supplement the real problems children encounter during their in-school and out-of-school activities" (p. 358). In a sense, word problems were "practice exercises in vicarious problem-solving" (p. 358). Kennedy (1970) included the following "schematic representation of problem-solving procedure" (p. 359).



As a part of the process, he suggested use of the following activities to assist children in the problem-solving process: mathematical sentences; concrete materials, pictures, and diagrams; tables and graphs; reading word problems; dramatizations; child-composed word problems; simplifying problems; estimating answers; using alternate methods of solution; word problems with too much or too little information, without questions, or without numbers; and mental arithmetic (pp. 359-368).

Maribeth Henney (1971) discussed a multi-step "problem analysis" in an attempt to improve verbal problem-solving ability through reading instruction. The eight steps included the reading of the "verbal problem," a look at the "main idea," asking a pertinent "question," deciding what the "important facts" are, writing a "relation sentence" followed by a "mathematical sentence," performing the "computation," and stating an "answer sentence" (pp. 226-227).

In 1973, Charles H. D'Augustine included a chapter on "problem solving" in Multiple Methods of Teaching Mathematics in the Elementary School. He provided a six-step set of "rules" that closely resembled those "rules" found in many elementary mathematics textbook series. The author referred to the use of such "rules" as the "translation approach" whereby the learner reads a story problem word-by-word, phrase-by-phrase and "translates" the words and phrases into expressions that are combined to form mathematical equations. The following "rules" were cited:

1. Find out what question you are supposed to answer.
2. Find the essential information.
3. Decide on the appropriate operation.
4. Write the equation.
5. Solve the equation.
6. Put the solution into an English sentence which answers the question (p. 50).

D'Augustine (1973) also included the following "Guidelines for Developing Problem-Solving Skills":

1. Not only should the child be given the skills necessary to solve problems, but he should also be taught how to identify and delimit problems.

2. Not only should the child be taught how to translate a problem into a mathematical sentence, but he should also be taught how to translate the problem into a simpler model of the problem.
3. Not only should the child be taught how to find alternative paths to his goal, but he should also be taught how to decide which of these is the most efficient path.
4. Not only should the child be taught how to derive a numerical answer, but he should also be taught how to interpret and use the information practically.
5. Not only should a child be taught to check his results, but he should also be taught to modify his solution as new data become available to him. In other words, he should be made aware of the fact that the answer for today may not be the answer for tomorrow.
6. Not only should a child be taught to solve problems, but he should also be taught to create problems (p. 47).

D'Augustine (1973) noted that "problem solving is the process of striving to reach a goal by a previously untraveled path. Problem solving takes many guises" (p. 52). In the process of solving problems, he stated that all or some of the following activities will occur:

1. A problem is recognized.
2. The basic structure of the problem is reduced to a simplified model.
3. Data are gathered or avenues for solution are selected.
4. A value judgment is rendered in selecting the "best" avenue for a solution.
5. The problem is solved and the choice of solution is evaluated.
6. Alternate solutions are tested (p. 52).

In 1974, Wayne A. Wickelgren wrote How to Solve Problems: Elements of a Theory of Problems and Problem Solving. In discussing "General versus Special Methods" of problem solving, the author stated:

> The relation between specific knowledge and methods, on the one hand, and general problem-solving methods, on the other hand, appears to be as follows. When you understand the relevant material and specific methods quite well and already have considerable experience in applying this knowledge to similar problems, then in solving a new problem you use the same specific methods you used before. Considering the methods used in similar problems is a general problem-solving technique. However, in cases where it is obvious that a particular problem is a member of a class of problems you have solved before, you do not need to make explicit, conscious use of the method: simply go ahead and solve the problem, using methods that you have learned to apply to this class. Once you have this level of understanding of the relevant material, general problem-solving methods are of little value in solving the vast majority of homework and examination problems for mathematics, science, and engineering courses (p. 3-4).

In the first chapter, he discussed the methodology involved in the problem-solving process. The following summary on the development of skill in problem solving was provided:

> There is no particular reason to engage in this careful, conscious analysis of a problem when you can immediately get some good ideas on how to solve it. Just go ahead and solve the problem "naturally." However, after you solve it or, even better, while you are solving it, analyze what you are doing. It will greatly deepen your understanding of problem-solving methods, and you might discover new methods or a new application of an old method.
>
> As you get extensive practice in using these problem-solving methods you should become so skilled in their use that the process becomes less conscious and more automatic or natural. This is the way of all skill learning, whether driving a car, playing tennis, or solving mathematical problems (p. 6).

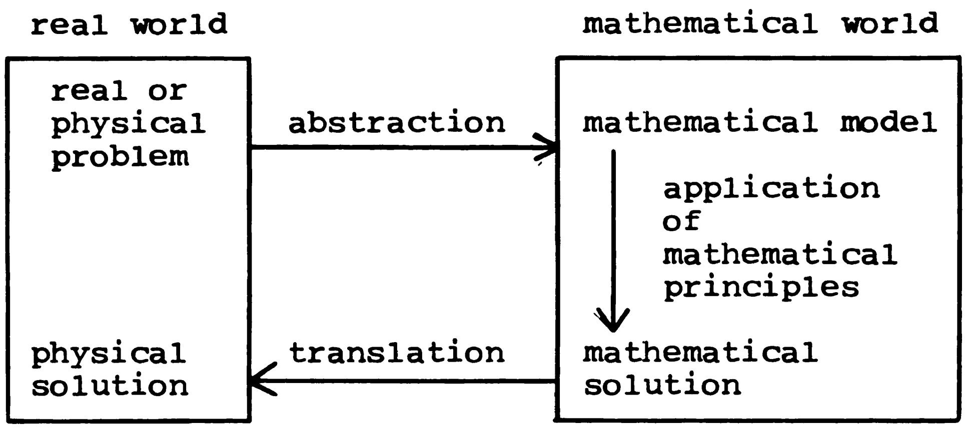
Stephen Willoughby (1975), in writing about "Lessons Learned and Pitfalls to be Avoided in the Continuing Curricular Reform Efforts in Mathematics Education," included a brief discussion on problem solving. In reference to the practical value of problems in the "new math" programs, he provided the following remarks:

> The most common valid criticism of "the new mathematics" is that it really doesn't teach children to solve real problems. But this has also been the easiest valid criticism to make of any mathematics education program in history. When a program does not have enough good work with problem solving it is not because the authors of the program are evil or even stupid. It's because having enough good problem-solving work is undoubtedly one of the hardest possible things to do in mathematics (p. 6).

In summarizing the section on the practicality of solving problems, Willoughby (1975) wrote:

> As we develop problem-solving material, I suggest that it is desirable, where possible, to allow the student to learn about the problem without reading it in a book. The reason for this suggestion is that many children find their greatest difficulty with mathematics to be the reading of "word problems" and this is not what we really want them to be able to solve anyway. It often seems to be the case that the words used to simulate the real problem are what keep the pupil from solving the problem (p. 6).

Jack C. Gill and Morris I. Beers (1976) provided a "look" at the relationship between the "real world" and the "mathematical world." In describing the complexity of the problem-solving process, they included the following diagram (p. 217):



For purposes of illustration, Gill and Beers (1976) focused on "verbal" problem solving and suggested the following four strategies to assist the problem solver in the learning process: estimation, graphs or tables, pattern recognition, and equations or number sentences (pp. 217-218).

In 1977, Kenneth J. Travers, Len Pikaart, Marilyn N. Suydam, and Garth E. Runion approached the teaching of problem solving through "heuristics." The authors described the teaching of heuristics as "general student activities to help in solving problems" (p. 140). In separating the "heuristics" into two parts--Initiating Heuristics and Looking Back Heuristics--the four authors included the following list:

Initiating Heuristics

1. Select appropriate notation.
2. Make a drawing, figure, or graph.
3. Identify wanted, given, and needed information.
4. Restate the problem.
5. Write an open sentence.
6. Draw from a cognitive background.
7. Construct a table.

8. Guess and check.
9. Systematize.
10. Make a simpler problem.
11. Construct a physical model.
12. Work backwards.

Looking Back Heuristics

13. Generalize.
14. Check the solution.
15. Find another way to solve it.
16. Find another result.
17. Study the solution process (pp. 150-152).

Frank K. Lester, Jr. discussed "Research on Mathematical Problem Solving" in the Professional Research Series presented by the National Council of Teachers of Mathematics in 1980. The plight of problem solving was provided in the following comments:

> The fact that problem solving has been the object of so much research, a focal point for several curriculum-development efforts, and the subject of innumerable books, articles, and conference reports attests to its importance in the study of mathematics. Indeed, there is substantial support for the notion that the ultimate aim of learning mathematics at every level is to be able to solve problems. Despite this well-recognized importance, the role problem solving should play in the mathematics curriculum is less clear. A cursory look at the most popular mathematics textbooks gives ample evidence of the lack of generally accepted tenets about the role of problem solving (p. 287).

Lester (1980), in noting "that problem solving is a very personal type of activity" (p. 287), provided the following citation on solving problems:

> ...the way a problem is presented and the type of information provided may significantly influence success in solving it. A few of the many factors that play a role in mathematical problem solving

> include previous experience, mathematical background, level of interest, motivation, and problem structure. In short, problem solving is an extremely complex area of human behavior (p. 287).

In making the observation that "mathematical problem solving appears, to a certain extent, to be so complex and subtle as to defy description and analysis" (p. 288), Lester (1980) listed the following four types of factors associated with problem solving:

> The Problem Itself: Task Variables
>
> Characteristics of the Individual: Subject Variables
>
> Problem-Solving Behavior: Process Variables
>
> Environmental Features: Instruction Variables
> (pp. 288-289)

The author also provided seven key issues in his discussion on problem solving. The issues are highlighted in the following list:

> Theory-related Issues
>
> Issue 1 Past problem-solving research in mathematics has suffered from the absence or neglect of theory.
>
> Issue 2 Problem solving is a chaotic area of inquiry largely because of the widely diverse types of tasks used. The tasks used in problem-solving research significantly affect the generalizability of results.
>
> Issue 3 Characteristics of problem solvers greatly affect behavior and consequently severely limit the generalizability of results. What kinds of subject to use in problem-solving research is a topic of much discussion.
>
> Instruction-related Issues
>
> Issue 4 There is little agreement regarding how best to improve problem-solving performance beyond the obvious fact that <u>attempting</u> to solve problems is a necessary ingredient.

Issue 5 In addition to a lack of consensus regarding the best ways to enhance problem solving, there is no accord about the nature of improvement in problem solving. Some researchers interested in problem-solving instruction have focused on the improvement of students' abilities to use particular strategies or skills, and others have considered improvement only in terms of an increase in the number of correct solutions.

Issue 6 The extent of instructional treatments in recent research varies from about one week to several months, with relatively short treatments being the most common. Treatments should be extensive enough not only to allow for full explication of ideas and procedures but also to provide ample opportunity for students to practice the procedures they are being taught.

Research Methodology Issue

Issue 7 No generally accepted methods or instruments for measuring performance or observing behavior during problem solving are clearly reliable and valid. Thus, the kind of instrumentation that is appropriate for a particular purpose remains an issue (pp. 315-316).

Thomas P. Carpenter, Henry Kepner, Mary Kay Corbitt, Mary Montgomery Lindquist, and Robert E. Reys summarized the "Results and Implications of the Second NAEP Mathematics Assessment: Elementary School" in 1980. The authors offered a pessimistic review of the results in problem-solving skill in the elementary school:

> If it were necessary to single out one area that demands immediate attention, it would clearly be problem-solving. At all age levels, and in virtually every content area, performance was extremely low on exercises requiring problem-solving or applications of mathematical skills. In general, <u>respondents demonstrated a lack of even the most basic problem-solving skills</u>. Rather than attempting to reason a problem through and figure out what

> needs to be done to solve the problem, most respondents simply tried to apply a single arithmetic operation to the numbers in the problem. The results indicate that students are not familiar with such basic problem-solving strategies as drawing a picture of a figure described in a problem, substituting smaller numbers in a problem to attempt to find a solution method, or checking the reasonableness of a result (p. 47).

Writing in the book, Problem Solving: A Handbook for Teachers, Stephen Krulik and Jesse A. Rudnick (1980) included the following set of "workable heuristics":

1. Read
   - 1a. Note key words.
   - 1b. Get to know the problem setting.
   - 1c. What is being asked for?
   - 1d. Restate the problem in your own words.
2. Explore
   - 2a. Draw a diagram, or construct a model.
   - 2b. Make a chart. Record the data.
   - 2c. Look for patterns.
3. Select a strategy
   - 3a. Experiment.
   - 3b. Look for a simpler problem.
   - 3c. Conjecture/guess.
   - 3d. Form a tentative hypothesis.
   - 3e. Assume a solution.
4. Solve
   - 4a. Carry through your strategy.
5. Review and extend
   - 5a. Verify your answer.
   - 5b. Look for interesting variations on the original problem (pp. 20-29).

The two authors also provided the following list of "activities" in "The Pedagogy of Problem Solving":

1. Create an atmosphere of success.
2. Encourage your students to solve problems.
3. Teach students how to read the problem.
4. Involve your students in the problem.
5. Require your students to create their own problems.
6. Have your students work together in pairs or small groups.
7. Encourage the use of freehand drawings.
8. Suggest alternatives when the present approach has apparently yielded all possible information.
9. Raise creative, constructive questions.
10. Emphasize creativity of thought and imagination.
11. Encourage your students to use a calculator.
12. Use strategy games in class.
13. Have your students flow-chart their own problem-solving process.
14. Don't teach new mathematics while teaching problem solving (pp. 37-63).

In 1980, the National Council of Teachers of Mathematics (NCTM) devoted its entire yearbook to the teaching of problem solving. The yearbook, titled Problem Solving in School Mathematics, was written for those individuals who have the greatest impact upon the teaching of problem solving--the classroom teacher. This importance was noted by Editor Stephen Krulik in the following passage extracted from the Preface of the 1980 Yearbook:

> Although Problem Solving in School Mathematics was never intended to be a definitive work on problem solving (in keeping with the concept of this series of yearbooks), the papers were all written with an eye to the classroom teacher. Here are ideas to be used in the classroom at all levels of instruction. Here are problems, examples, and illustrations. And if these ideas are used, then the

> authors will have done their job well! Problem solving should be the major focus of all mathematics instruction; Problem Solving is an attempt to help the classroom teacher in this important effort (p. xiv).

With the emphasis on problem solving, the National Council of Teachers of Mathematics embarked upon a journey it prescribed for all individuals and organizations involved in the teaching of mathematics: Problem solving was to be the focus of mathematics instruction during the 1980s.

In Chapter 6 of the yearbook, Alan Osborne and Margaret B. Kasten (1980) discussed the "Opinions about Problem Solving in the Curriculum for the 1980s: A Report." The authors described the results of a survey of nine populations: subscribers to Arithmetic Teacher and Mathematics Teacher, members of the American Mathematical Association of Two-Year Colleges, members of the Mathematical Association of America, teacher educators, mathematics supervisors, principals, school board presidents, and presidents of parent-teacher associations. Osborne and Kasten (1980) reported the following conclusions:

> CONCLUSION 1. Problem solving should receive more emphasis in the school mathematics program during the coming decade.
>
> CONCLUSION 2. There is broadly based unanimity for the aims or purposes of instruction on problem solving.
>
> CONCLUSION 3. The strategies of (1) translating a problem to an equation, (2) constructing a table and searching for patterns, (3) drawing pictures or diagrams to represent a problem, and (4) solving a simple problem first and extending the solution to the original problem were identified as appropriate content for both the elementary and secondary school levels.

CONCLUSION 4. No extreme positions concerning teaching methodology for problem solving were preferred by any group.

CONCLUSION 5. Using a problem as the means or the vehicle to develop and introduce mathematical topics is a preferred methodology.

CONCLUSION 6. Problem solving is important for all students and should begin early in their mathematical experience.

CONCLUSION 7. A modification of the mathematics curriculum to provide unique problem-solving experiences for special groups, such as women, the college bound, or ethnic minorities, is perceived as inappropriate (pp. 51-60).

Nicholas A. Branca wrote in Chapter 2 of the 1980 Yearbook, "Problem Solving as a Goal, Process, and Basic Skill." Both the difficulty and the importance of understanding the multiple interpretations of problem solving were discussed. The following comments summarize the ideas of the chapter:

> We as teachers must realize the importance of problem solving with respect to each of the three interpretations. We must be aware that the students who enter school during this decade will spend the majority of their productive lives solving the problems of the twenty-first century. Although we have no way of knowing what these problems will be like, considering the different interpretations of problem solving can help us prepare for them.
>
> Considering problem solving as a basic skill can help us organize the specifics of our daily teaching of skills, concepts, and problem solving. Considering problem solving as a process can help us examine what we do with the skills and concepts, how they relate to each other, and what role they play in the solution of various problems. Finally, considering problem solving as a goal can influence all that we do in teaching mathematics by showing us another purpose for our teaching.

> Each of these interpretations is important, but they are different. When we encounter the term <u>problem solving</u>, we should consider which interpretation (or interpretations) is intended. The multiple meanings for the term can easily lead a writer to ambiguity and a reader to misunderstanding. Problem solving has too many facets for us always to look at it from the same angle (p. 7).

In discussing the "general nature of problem solving," Marilyn N. Suydam provided a composite list of steps in the problem-solving process in Chapter 5 of the 1980 Yearbook. These steps are:

1. Understanding the problem--an awareness of the problem situation that stimulates the person to generate a statement of the problem in writing, orally, or merely in thought.
2. Planning how to solve the problem.
   a. Break down the components; enumerate data; isolate the unknown.
   b. Recall information from memory; associate salient features with promising solution procedures.
   c. Formulate hypotheses or a general idea of how to proceed.
3. Solving the problem.
   a. Transform the problem statement into a mathematical form, or construct representations of the problem situation.
   b. Analyze the statement into subproblems for which the solution is more immediate.
   c. Find a provisional solution.
4. Reviewing the problem and the solution.
   a. Check the solution against the problem.
   b. Verify whether the solution is correct; if not, reject the hypotheses, the method of solution, or the provisional solution.
   c. Ascertain an alternative method of solution (pp. 38-39).

In 1981, Francis M. Fennell, in the Elementary Mathematics Diagnosis and Correction Kit, cited the importance of teaching problem solving in the following manner:

> Problem solving transcends all other strands or topic areas within the elementary mathematics curriculum. Developing the ability to solve relevant problems is a major objective in all areas (p. 361).

He described fifteen approaches to better meet the problem-solving abilities of children. They were: oral problems, open-ended problems, illustrated problems, interdisciplinary examples, problems without numbers, mini-problems, pupil problems, challengers, experiencing charts, deletion of extra information, card file, calculator, bulletin boards, interpretation interviews, and the problem of the day (pp. 362-364).

In terms of diagnosis, Fennell (1981) included a "staircase" of problem solving. The diagram lends itself to a "Problem-Solving Process Checklist" in which the teacher keeps track--via yes or no responses--of the problem-solving abilities of children (pp. 365-366). The "staircase" of steps--What's the problem?, Make a Plan, Do It!, and Checkup--is illustrated in Appendix G.

Frank K. Lester, Jr. and Joe Garofalo edited Mathematical Problem Solving: Issues in Research in 1982. In the introduction, the following comment regarding interest in and research of problem solving was noted:

> The high level of interest in problem solving and thinking among educators and psychologists has been accompanied by a corresponding level of concern for developing a stable and useful body of knowledge about the phenomena associated with these highly complex areas of human behavior. There is no question that our understanding of problem-solving behavior has increased greatly in recent years. However, for various reasons... the research on the whole has been rather unsystematic and has lacked clarity of purpose and focus (p. ix).

In a section titled "Building Bridges Between Psychological and Mathematics Education Research on Problem Solving," Lester (1982) cited three reasons for what was called "the rather chaotic state" of problem-solving research. The three factors were:

1. a neglect of theory to guide systematic inquiry;
2. the exteme complexity of the nature of problem solving; and
3. the rudimentary state of the research methodologies employed (p. 55).

In addition to the three factors, the author noted the following issues:

1. The role of theory in problem-solving research;
2. The types of research tasks to use;
3. The relative emphasis to place on developing competency models or performance models of problem-solving behavior;
4. Teaching problem solving (if in fact it can be taught) and what the teacher's role should be;
5. The nature of problem-solving performance changes;
6. The types of research methodologies to employ (p. 55).

Lester (1982) also considered a series of seven questions, each of which was related to one or more of the issues listed on the previous page. The seven questions were:

> 1. Can problem solving be taught?
> 2. What is the role of understanding in problem solving?
> 3. To what extent does transfer of learning occur in problem solving?
> 4. What are the primary task variables that affect problem solving?
> 5. What is the role of metacognitive behavior in problem solving?
> 6. How do successful and unsuccessful (good versus poor, expert versus novice) problem solvers differ with respect to their problem-solving behavior?
> 7. What are the most appropriate research methodologies? (p. 56)

Eugene D. Nichols and Merlyn J. Behr (1982) wrote Elementary School Mathematics and How to Teach It. In a chapter on "problem solving," they provided the following description:

> When we speak of problem solving, we are referring to what are commonly called by teachers and students story problems or verbal problems. A substantial part of elementary school mathematics instruction is devoted to teaching students how to solve such problems.
>
> The central emphasis given to the matter of problem solving is correctly placed, for this is important in everyday life as well. Many problems encountered in everyday activities lend themselves to solution by mathematical methods. It is therefore necessary that these methods be developed systematically in a mathematics curriculum (p. 277).

The following activities were suggested in teaching skill in problem solving: writing equations (translation), writing and solving equations, solving number puzzles, writing miniproblems, pictures in problem solving (diagrams), using logical reasoning, asking "What additional information is needed?," looking for "extra information," estimating answers, solving "multi-step problems," making up problems, reading a chart, and using flow charts (pp. 277-296). In terms of "heuristics," the authors recommended using the Polya Model (pp. 296-298). Finally, they suggested using the calculator in problem-solving activities because that "accomplishes at least two things: it frees the child to think about the problem-solving methods rather than concentrating on the computations, and it forces the child to think about the accuracy of the solution and the reasonableness of the answer" (pp. 323-324).

In 1983, Daniel T. Dolan and James Williamson, in Teaching Problem-Solving Strategies, offered the following caution:

> One must be extremely careful not to fall into the trap of defining problem solving as another basic skill which can be treated as an algorithm. The National Assessment of Educational Progress findings on mathematics achievement made in 1979, reported that "evidence shows that students proceed mechanically and thoughtlessly through problems--if they forget the rule, then they are unable to do the problem on their own." This analysis of results of achievement in problem solving is consistent with the usual methodology of teaching it. Problem solving in mathematics has become a set of words which are wrapped around some computational exercises (p. ix).

In recommending the teaching of strategies, the authors included the following activities: guessing and checking, making a table, patterning, making a model, eliminating, and simplifying (pp. 3-105).

The 1983 Yearbook of the National Council of Teachers of Mathematics was written in response to the recommendations included in An Agenda for Action: Recommendations for School Mathematics of the 1980s. Titled The Agenda in Action, the 1983 Yearbook provided a "framework for exploring current developments leading to the improved teaching of mathematics at all levels" (p. ix). More than a third of the yearbook highlighted the first recommendation made by the Council in 1980, that "problem solving must be the focus of school mathematics in the 1980s." Several citations taken from the 1983 Yearbook are particularly noteworthy.

Writing a section on "Problem Solving as a Focus: How? When? Whose Responsibility?," Peggy House, Martha L. Wallace, and Mary A. Johnson (1983) stated the following:

> Problem solving is a process, not a step-by-step procedure or an answer to be found; it is a journey, not a destination. Successful problem solvers can be identified by the processes or the attitudes of mind they display. Four characteristics that help identify good problem solvers are desire, enthusiasm, facility, and ability (p. 10).

The writers discussed each of the four characteristics and concluded:

> The scenario of a future in which problem solving is truly the focus of mathematics instruction is exciting and challenging. However, the degree to which the Agenda actually becomes action depends on the

> degree to which each mathematics teacher, teacher educator, and supervisor accepts the responsibility to create that future. There is no question that we have the facility and the ability to do this and to make it stick. The real question is, do we have the desire and the enthusiasm? (p. 19)

Richard Brannan and Oscar Schaaf, writing in the 1983 Yearbook, considered the topic of "An Instructional Approach to Problem Solving." In their opening remarks, the two authors summarized the concern for teaching mathematical problem solving with the following comment:

> Problem solving has been a subject of research by mathematics educators, educational psychologists, and philosophers since the 1930s. Yet tests, including the recent mathematics tests of the National Assessment of Educational Progress, indicate that our school students and even graduates have serious deficiencies in problem-solving ability. An examination of current mathematics texts and the results of classroom visitations reveal that problem solving is only a minor part of the mathematics instruction in both elementary and secondary schools (p. 41).

Brannan and Schaaf (1983) made several recommendations for making "problem solving the focus of mathematics education." In the summary, they provided the following:

1. The mathematics curriculum should be organized around problem solving.
2. Appropriate curricular materials to teach problem solving should be developed for all grade levels.
3. Mathematics teachers should create classroom environments in which problem solving can flourish (p. 59).

The criteria listed above provide a foundation for making problem solving the focus of mathematics instruction for the remainder of the 1980s. Over the next seven years, teachers must recognize such propositions--and those that preceded them--so that the concerns for teaching mathematical problem solving so prevalent in the 1980s will not be repeated during the 1990s.

Summary of Period III: 1954-1983

Changes in curriculum and the psychology of learning played important roles in the development of mathematical problem solving during the period, 1954-1983. "Thinking" and "conceptual development" became integral parts of the problem-solving process. Noteworthy was the fact that in teaching problem solving there were two aims, societal and mathematical. Although a number of problem-solving models continued to reflect the "conventional method" of the 1920s, most models retained the initial stages of "reading the problem thoroughly or carefully" and "comprehending the problem." Verbal problem solving continued to be an issue.

During this period, problem solving was incorporated throughout the field of mathematics. Research played an important function in the teaching of problem solving, and mathematics educators were encouraged to include theoretical research in the instructional process.

A number of mathematics educators contributed to the development of mathematical problem solving during Period III. However, the problem-solving model presented by George Polya in the course of Period II continued to gain in popularity. Toward the end of Period III, the Polya Model was cited widely in journal articles, mathematics methods' textbooks, and in the 1980 Yearbook of the National Council of Teachers of Mathematics. The importance of this fact is recognized because the decade of the 1980s was--and still is--viewed by the Council as the decade of problem solving, and the Polya Model, with its extensive questioning has been frequently highlighted.

In addition to many individuals, numerous mathematics programs and projects, developed shortly after the launching of Sputnik I and II, affected the teaching of problem solving in mathematics. This period spawned a series of mathematics programs initiated by private and governmental agencies. During the 1960s, there was renewed interest in providing for extensive training in the area of problem solving with a variety of techniques suggested. In numerous attempts to translate word problems to number problems as a part of the problem-solving process, children were encouraged to solve problems "naturally." Discovery became an important ingredient in the process and students were invited to discover new problem-solving strategies and to apply them in settings other than the problem itself.

Further development in the teaching of mathematical problem solving during this period can be found in the emphasis accorded problem analysis, critical thinking skill, and "mental" arithmetic. Ideas prevalent during Period III include the use of diagnosis in problem solving and continued emphasis on a meaningful arithmetic program (Brueckner and Bond, 1955), a more mathematical approach to problem solving (Van Engen, 1959), emphasis on the methodical work found in mathematical problem solving (Polya, 1962), emphasis placed on discovery and intuitive thinking in the "new math" program (Dutton, 1964), emphasis on Piaget's stages of mental structuring and Dienes' Principles of Conceptual Learning (Dutton, Petrie, and Adams, 1970), the relationships--and differences--between general and specific methods of solving problems (Wickelgren, 1974), and the recommendation that problem solving be the focus of mathematics instruction for an entire decade, the 1980s (NCTM, 1980).

Finally, with the pessimistic outlook cited during the late 1970s, and the NCTM focus on problem solving for the 1980s, it is clear that many problems associated with the teaching of mathematical problem solving continue to exist. In the final analysis, the gap between the theory of problem solving and the inclusion of problem solving in the mathematics classroom might be wider than most educators realize.

CHAPTER V

SUMMARY, CONCLUSIONS, RECOMMENDATIONS, AND IMPLICATIONS

This chapter (1) summarizes the problem; (2) tenders the conclusions; (3) provides recommendations for the future teaching of problem solving in mathematics; and (4) states the implications for future research in mathematical problem solving.

Summary

This investigation considered the following questions:

1. What are the historical ideas, concerns, and approaches that have helped formulate problem solving in school mathematics?
2. What are the historical ideas, concerns, and approaches that have helped elevate problem solving to its current level of importance in mathematics education?
3. Compared to events and influences of the past, what practices indicate a successful problem-solving movement for the remainder of the 1980s and the future?

Teaching problem solving in mathematics has been a matter of interest since the 1920s. Yet, with continuing emphasis through the intervening decades some of the problems found in the 1920s continue to exist in 1983.

The focus of this study is to trace the development of mathematical problem solving over a ninety-year period and make recommendations for the future teaching and learning of problem solving. Hundreds of ideas, concerns, and approaches on mathematical problem solving have been discussed in this investigation. Each--in its own way--has contributed to the development of problem solving in mathematics.

Improved research techniques, greater interest in methodology, and the incorporation of various ideas and theories of learning into the instructional process have had an impact on current practices found in the teaching of mathematical problem solving. But with the ideas, thoughts, theories, models, approaches, and research included during the past ninety years, the development, improvement, and implementation of problem solving in mathematics continues to be a source of concern. The following section provides a list of conclusions gathered from this investigation.

Conclusions

Based upon the data included in this study, the following conclusions are cited:

1. Concern for the teaching of mathematical problem solving has been a source of major interest in the mathematics community for over ninety years. Serious consideration for teaching problem solving in mathematics began in the 1920s. Concerns similar to those found sixty years ago continue to exist. The National Council of Teachers of Mathematics provided extensive material on mathematical problem solving in its 1928, 1953, and 1980 Yearbooks. The goal was to encourage classroom teachers to better understand the basic ideas and theories of mathematical problem solving.
2. During the past nine decades, numerous ideas, concerns, and approaches have provided teachers with a number of descriptions and suggestions on how <u>best</u> to implement problem solving into the mathematics curriculum. Some descriptions can be confusing to classroom teachers. For example, in certain cases, problem-solving descriptions are limited strictly to word or story problems. In other cases, descriptions of problem solving are used in conjunction with mathematical problems of all types, for example,

$2 + 7 = n$, $45.6 - 7.8 + 8.99 = z$, and $5\frac{1}{4} \times 3\frac{1}{2} = m$, and are not limited to story or word problems.

3. Models developed decades ago continue to be utilized in mathematical problem solving. For example, the Polya Model and the "conventional" models that preceded it, continue to appear in textbooks, journal articles, and other professional literature. Thousands of steps have been included in hundreds of problem-solving models. Most of the steps "fit" into eight basic categories. The following "model" is a composite representation of the problem-solving models developed over the past ninety years:

   a. Read the problem carefully.
   b. Look for the "known" facts contained in the problem.
   c. Determine the "unknown" portion of the problem.
   d. Use the "known" and "unknown" portions to determine a procedure for solving the problem.
   e. Estimate the final answer.
   f. Solve the problem (computation).
   g. Compare the answer in the solution with the estimated answer (look for "reasonableness").
   h. Label the final answer.

4. The "problem" of problem solving has been a source of concern for classroom teachers throughout the ninety-year period. The "problem" includes difficulties with:

a. teachers' inability to implement problem-solving approaches and techniques in mathematics instruction. The recent data indicate that problem solving occupies only a minor part of the mathematics programs found in elementary and secondary schools; and

b. children's inability to read and understand the technical language of mathematics and to comprehend various problem-solving strategies. Recent literature indicates that children lack an understanding of the basic problem-solving skills.

5. Problem solving can be taught! The recent literature indicates that a successful problem-solving movement can be realized. For this to happen, there must be a greater awareness on the part of classroom teachers in providing for alternative approaches, strategies, and materials used in mathematical problem solving. Furthermore, there must be a willingness on the part of all classroom teachers, building principals, curriculum coordinators, and school superintendents to fully implement a problem-solving approach to teaching mathematics. This includes daily problem-solving activities in mathematics, in-service programs that center on mathematical problem solving, and a mathematics curriculum that follows the National Council of Teachers of Mathematics recommendation that

problem solving be the focus of mathematics instruction during the 1980s. The next section provides specific recommendations for implementing particular procedures in teaching problem solving in mathematics.

Recommendations

Based upon the conclusions of this investigation, the following recommendations for the future teaching of mathematical problem solving are suggested:

1. There should be greater emphasis placed on mathematical problem-solving activities--such as the use of drawings, tables, and charts, guessing, trial-and-error, and estimation--to solve problems in the early grades. The strategies learned in the early grades should be expanded as children progress through the curriculum. At each level, teachers should recognize children's readiness in problem-solving skill. Pupils should be strongly encouraged to use a variety of problem-solving strategies and to look for alternative solutions to problems. They should be provided with specific problem-solving experiences such as working with brainteasers, product-purchasing problems, and other mathematical situations that reflect not only mathematical growth but also growth in their ability to solve problems in the real world as well.

2. Teachers should stress and integrate mathematical problem-solving practice with other subject-matter areas such as science, social studies, physical education, industrial arts, and other parts of the curriculum. Such problem-solving activities include: longitudinal and latitudinal problems in geography; relating statistical concepts to predicting outcomes for local, state, and national elections; applying mathematical principles of distance, speed, and time to physical education activities; using newspapers and magazines to solve comparison-buying problems; preparing alternative classroom designs for seating arrangements and learning centers; and developing large-scale problem-solving activities involving the entire class (for example, determining the expenses involved in purchasing a computer for the classroom including software, hardware, and miscellaneous materials).
3. Teachers should provide for the development of critical thinking skills such as analysis, synthesis, and evaluation which are important to understanding mathematical problem-solving processes at all levels of mathematics. Skill in critical thinking is an important factor in recognizing a problem, comprehending the question(s) in a problem, determining computational procedures, estimating the final

answer, and determining the "reasonableness" of the final solution. These skills must be developed if students are to comprehend the scope of mathematical problem solving.

4. Children should be challenged to ask questions when solving problems in mathematics. The teacher should encourage children to question various strategies, provide time for follow-up questions, and recognize unusual solutions that are both mathematically correct and important to the problem solver.

5. Various problem-solving strategies and techniques should be included in preparing pre-service teachers in mathematics. A problem-solving approach to teaching mathematics is recommended. Mathematics programs should reflect the theme of the Nuffield Mathematics Teaching Project: I hear, and I forget; I see, and I remember; I do, and I understand.

6. In order to teach mathematical problem solving, teachers must understand processes of problem solving, exhibit confidence in their own problem-solving abilities, and utilize alternative methods in presenting problem-solving material. These practices should begin in the preschool and continue through all levels of the mathematics curriculum.

7. All classroom teachers should become familiar with the suggestions and recommendations made by organizations such as the National Council of Teachers of Mathematics. The recommendation that during the 1980s mathematics instruction focus on problem solving must be known and understood by each teacher, building principal, curriculum coordinator, and school superintendent. The ideas contained in various journals, yearbooks, and bulletins provide the classroom teacher with numerous problem-solving strategies which should be incorporated into mathematics instruction at every opportunity.
8. Local school districts should adopt a problem-solving approach to teaching mathematics. In-service programs on mathematical problem solving should be provided several times during the school year. In these in-service programs, teachers should experience and understand various problem-solving strategies and approaches. They should become familiar with current literature and research on problem solving in mathematics.
9. Textbooks should reflect, emphasize, and utilize various problem-solving strategies and approaches beginning in the early grades and continuing through college-level courses in mathematics. They should

make use of the various problem-solving models, and provide for extensive questioning techniques similar to those found in the Polya Model.

10. Reading continues to be a source of concern in mathematical problem solving. Teachers should be aware of the reading abilities of both good and poor readers in mathematics. Children should be helped to recognize the type of reading found in mathematical symbolism and relate this type of language to mathematical story or word problems. Teachers must stress the reading aspects of all types of mathematical problems. Students must realize that the initial step in any problem-solving model or procedure is to read the problem <u>carefully</u>.

11. In working with a variety of problem-solving strategies, children should use concrete aids and other manipulative devices such as nuts, bolts, sticks, straws, geometric shapes, abaci, counting frames, and calculators. Each student should be aware of not only manipulative aids but also the various types of strategies used in solving problems. This list includes the use of drawings, diagrams, charts, tables, mathematical sentences, and pictures in the problem-solving process.

12. The development of mathematical problem solving must include the use of computers. A variety of computer

experiences should be provided for students at all levels of instruction. Children should become familiar with computer literacy in the primary grades. The use of computer material should not be limited strictly to drill and practice. Exercises must include elements of strategy and provide for alternative solutions. Teachers and curriculum coordinators should select "software" materials that reflect a variety of approaches to problem solving and incorporate unique activities that will provide students with various experiences in solving problems.

13. The Polya Model continues to be used extensively in mathematics education. Because of this, teachers should become familiar with and utilize the questioning techniques included in the model. Students should be able to apply the questions in the Polya Model to all types of problems in mathematics and not just to story or word problems.

14. The Polya Model should not be the only model used in the mathematics classroom. Teachers should become familiar with general problem-solving steps similar to those cited earlier in this chapter. Those steps are:

a. Read the problem carefully.

b. Look for the "known" facts contained in the problem.

c. Determine the "unknown" portion of the problem.

d. Use the "known" and "unknown" portions to determine a procedure for solving the problem.

e. Estimate the final answer.

f. Solve the problem (computation).

g. Compare the answer in the solution with the estimated answer (look for "reasonableness").

h. Label the final answer.

It is recommended that the teacher add at least two steps to the above procedure. They are:

i. Double check the answer to guarantee correctness.

j. Look for alternative methods of solution to verify the final answer.

Implications for Future Research

Based upon the findings of this investigation, the following implications for future research in mathematical problem solving are noted:

1. Research on the historical development of mathematical problem solving for the 1970s and 1980s should be expanded and include an in-depth look at the impact of various "calculating machines" on problem solving in mathematics.

2. Research on the degree of impact that various individuals, projects, committees, and organizations had on mathematical problem solving should be more closely investigated.

3. Research on the historical development of problem solving in mathematics should compare and contrast the development of problem solving found internationally, for example, Great Britain, Japan, and the Soviet Union.

APPENDICES

APPENDIX A

THORNDIKE'S ORGANIZATION OF LEARNING

1. To read, write, and understand integers.
2. To add with integers.
3. To subtract with integers.
4. To multiply with integers.
5. To divide with integers.
6. To read, write, and understand United States money.
7. To add with United States money.
8. To subtract with United States money.
9. To multiply with United States money.
10. To divide with United States money.
11. To read, write, and understand fractions.
12. To reduce them to higher and lower terms.
13. To find the least common multiple.
14. To add with common fractions, then with mixed numbers.
15. To subtract with common fractions, then with mixed numbers.
16. To multiply with common fractions, then with mixed numbers.
17. To divide with common fractions, then with mixed numbers.
18. To read, write, and understand decimal numbers.
19. To reduce common fractions to decimals and vice versa.
20. To add with decimal fractions and decimal mixed numbers.

(continued)

APPENDIX A - continued

21. To subtract with decimal fractions and decimal mixed numbers.

22. To multiply with decimal fractions and decimal mixed numbers.

23. To divide with decimal fractions and decimal mixed numbers.

24. To understand denominate numbers.

25. To reduce them, "ascending" and "descending."

26. To add with denominate numbers.

27. To subtract with denominate numbers.

28. To multiply with denominate numbers.

29. To divide with denominate numbers.

30. To read, write, and understand percents.

31. To manipulate the "three cases" of percentage:
    I. Multiplying by a percent.
    II. Dividing one number by another and expressing the result as a percent.
    III. Dividing a number by a percent to find what number it is that percent of.

32. To understand the uses of percents in computing interest, discounts, insurance premiums, taxes, dividends, yields of bonds, etc.

33. To understand and compute square root and cube root.

34. To compute the areas of certain surfaces and the volumes of certain solids or the contents of certain receptacles (Thorndike, 1921, pp. 83-84).

APPENDIX B

THORNDIKE'S TITLES OF LESSONS

1. Vacation Activities
9. School Supplies
14. Playing "How Far"
15. Playing "Saving"
18. Telegrams, Express, and Freight
19. Playing "Cashier"
20. House Plans
21. Drawing to Scale
24. The School Program
45. Weighing
46. Buying Candy
51. School Marks and Averages
53. Keeping Accounts
54. Buying Fruit
58. Henry's Orchard
60. How Lewis Earns Money
61. How Elsie Earns Money
67. At the Fish Market
72. A Christmas Party
74. Earning and Saving
79. At the Butcher Shop
87. Buying in Quantity
98. Report Cards
99. Earning and Saving

(Thorndike, 1921, p. 96)

APPENDIX C

KNIGHT'S ANALYSIS OF STEPS IN THE PROCESS OF PROBLEM SOLVING

Steps in Problem Solving	Factors Underlying Problem Solving	Types of Drill Provided
COMPREHENSION	Vocabulary Ability to read numerals Ability to read rapidly Ability to comprehend a. Follow directions b. Make generalizations c. Select potent elements d. Discard irrelevant elements e. Determine problem setting as a unit f. Determine the outcome of the problem g. Grasp significance of problem cues	Vocabulary drill Comprehension drills a. Directions exercise b. Completion exercise c. Multiple choice exercises
ANALYSIS AND ORGANIZATION	Selection of potent factors Selection of processes involved Determining what the problem calls for Determining what is given in the problem Determining process relationships	What is called for Process analysis What is given Problem relationships

(continued)

APPENDIX C - continued

Steps in Problem Solving	Factors Underlying Problem Solving	Types of Drill Provided
RECOGNITION	Choice of procedure Determining problem conditions Determining purpose of the problem Determining relevant elements in problem	Process analysis What is given What is called for
SOLUTION	Selection of processes Organization of processes in order Knowledge of combinations Problem relationships	Process analysis Problem relationships Problem scales
VERIFICATION	Probable form of answer Probable magnitude of answer	Probable answer

(Knight, 1927, p. 66)

APPENDIX D

SKILLS INVOLVED IN PROBLEM SOLVING BY BRUECKNER AND MELBY

1. Ability to tell what facts are given.
2. Ability to tell what question the problem asks.
3. Ability to select essential facts.
4. Ability to estimate answers.
5. Ability to tell how to check answers.
6. Ability to name the process to use in solving one-step problems.
7. Ability to name process in order used to solve two-step problems.
8. Knowledge of vocabulary.
9. Knowledge of essential denominate numbers and units of measure.
10. Knowledge of essential principles and concepts.
11. Ability to judge absurdities.
12. Ability to check true and false statements.
13. Ability to assemble essential data.
14. Ability to read accurately and exactly.
15. Ability to follow directions.
16. Ability to attack the solution of a problem in a systematic manner.
17. Ability to apply processes in local situations.
18. Ability to interpret tables found in reference books.
19. Ability to use the index, table of contents, etc., as aids in studying.

(continued)

APPENDIX D - continued

20. Ability to understand quantitative concepts in map reading.

21. Range of information in application of arithmetic.

22. Ability to make analogies.

23. Ability to answer specific questions about problems.

24. Ability to formulate problems from given data.

25. Ability to illustrate uses of processes.

26. Ability to detect cues in solving problems.

27. Knowledge of historical background.

28. Ability to restate a problem in the words of the pupil (Brueckner and Melby, 1931, p. 227).

APPENDIX E

POLYA'S MODEL: THE QUESTIONS

First. UNDERSTANDING THE PROBLEM

<u>What is the unknown</u>?

<u>What are the data</u>?

<u>What is the condition</u>?

Is it possible to satisfy the condition?

Is the condition sufficient to determine the unknown?

Or is it insufficient?

Or redundant?

Or contradictory?

Draw a figure.

Introduce suitable notation.

Separate the various parts of the condition.

Can you write them down?

Second. DEVISING A PLAN

Have you seen it before?

Or have you seen the same problem in a slightly different form?

<u>Do you know a related problem</u>?

Do you know a theorem that could be useful?

<u>Look at the unknown</u>!

And try to think of a familiar problem having the same or a similar unknown.

<u>Here is a problem related to yours and solved before</u>.

<u>Could you use it</u>?

Could you use its result?

Could you use its method?

Should you introduce some auxiliary element in order to make its use possible?

Could you restate the problem?

Could you restate it still differently?

Go back to definitions.

(continued)

APPENDIX E - continued

If you cannot solve the proposed problem try to solve first some related problem.

Could you imagine a more accessible related problem?

A more general problem?

A more special problem?

An analogous problem?

Could you solve a part of the problem?

Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary?

Could you derive something useful from the data?

Could you think of other data appropriate to determine the unknown?

Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other?

Did you use all the data?

Did you use the whole condition?

Have you taken into account all essential notions involved in the problem?

Third. CARRYING OUT THE PLAN

Carrying out your plan of the solution, <u>check each step</u>.

Can you see clearly that the step is correct?

Can you prove that it is correct?

Fourth. LOOKING BACK

Can you <u>check the result</u>?

Can you check the argument?

Can you derive the result differently?

Can you see it at a glance?

Can you use the result, or the method, for some other problem? (Polya, 1945, pp. inside-front and inside-back covers)

APPENDIX F

MATHEMATICS PROJECTS AND PROGRAMS

1. The Boston College Mathematics Institute

2. The Cambridge Conference on School Mathematics

3. The Developmental Project in Secondary Mathematics at Southern Illinois University

4. The Greater Cleveland Mathematics Program (GCMP)

5. The University of Illinois Committee on School Mathematics (UICSM)

6. The University of Maryland Mathematics Project (UMMaP)

7. The Minnesota Mathematics and Science Teaching Project (MINNEMAST)

8. The Nuffield Mathematics Project

9. The Ontario Mathematics Commission

10. The School Mathematics Study Group (SMSG)

11. The Stanford Program in Computer-Assisted Instruction (CAI)

12. The Southern Illinois Project-Comprehensive School Mathematics Project (CSMP)

13. The Syracuse University-Webster College Madison Project

14. Unified Science and Mathematics for Elementary Schools (USMES)

(National Council of Teachers of Mathematics, 1970, p. 284; Weiss, 1978, pp. 6-9)

APPENDIX G

FENNELL'S STAIRCASE OF PROBLEM SOLVING

4. Checkup.
   a. Answer labeled?
   b. Solution reasonable?
   c. Solution checked.

3. Do it!
   Carry out your plan.

2. Make a Plan.
   How to solve the problem.

1. What's the Problem?
   a. Read.
   b. What is known?
   c. What is unknown?

(Fennell, 1981, p. 365)

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