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INFINITELY DIVISIBLE MEASURES ON

MULTI-HILBERTIEN SPACES AND

A LÉVY-ITO DECOMPOSITION

By

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ABSTRACT

INFINITELY DIVISIBLE MEASURES ON MULTI-HILBERTIEN SPACES AND A LÉVY-ITO DECOMPOSITION

By

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In this work, we give a representation of infinitely divisible (ID) laws on duals of multi-Hilbertien spaces and discuss the convergence. These results give a unified approach to the existant work on infinitely dimensional Hilbert spaces and on nuclear spaces. The convergence of ID laws can be used to prove the weak convergence of homogeneous processes with independent increments. This is applied to a problem from Neurobiology, and the results obtained are generalization and improvement of a recent work of G. Kallianpur [16]. The last chapter is devoted to the processes with independent increments on duals of multi-Hilbertien spaces. Lévy-Ito decomposition on Hilbert spaces is obtained.

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Many memorable hours are spent with my friends, graduate assistants; among other things, drinking American coffee and eating Indian food is what makes life exciting.

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0. INTRODUCTION

The infinitely divisible laws (ID) in the context of infinite-dimensional Hilbert spaces were studied by S.R.S. Varadhan ([23]). For the case of nuclear spaces, a Lévy-Khinchine representation was given by Fernique [6]. Our purpose here is to give a representation of ID laws on a multi-Hilbertien space and discuss the convergence. These results give unified approach to Varadhan's and Fernique's work and extend Fernique's work to include convergence of ID laws. Such a unified approach for Bochner theorem for multi-Hilbertien spaces is recently given by K. Ito [14]. Our results are based on arguments of Fernique for the representation and Levy continuity theorem for the multi-Hilbertien spaces.

We note that the method of Fernique for the Lévy continuity theorem in nuclear spaces fails in the multi-Hilbertien spaces due to an example of Sazonov (cf. 2.1.2).

The convergence of ID laws can be used to prove the weak convergence of additive (homogeneous independent increments) processes. This is shown in the context of an example from Neurobiology. This work includes and improves recent work of G. Kallianpur [16]. In fact, our work is applicable to wider class of examples.

The last chapter is devoted to the Lévy-Ito decomposition of infinite-dimensional processes with independent increments (PII). We are unable at this point to prove this theorem for general multi-Hilbertien spaces. In fact, as the reader can see, the results seem to need some work even for the Hilbert space case. However, for the additive processes, such a decomposition follows from the relation of convolution semigroups to ID laws. This, along with the convergence of the related processes, is studied in Chapter 3. Clearly, the results there allow us to extend the work of Kallianpur to multi-neuronal models.

The convergence of PII in this context remains an open problem. But our decomposition is the basic technique needed for this study. In future work we intend to study this problem as a generalization of a recent work of Jacod (ZWV 63, 109-136, 1983).

1. PRELIMINARIES

1.1. Vector Spaces and Seminorms

R(N) will denote the set of real (natural) numbers. Throughout this work the phrase "vector space" will mean a real vector space.

Let E be a vector space. A real-valued function p defined on E is called a seminorm if for all $x, x_1, x_2 \in E$ and $a \in R$:

(i) $p(x) \geq 0$

(ii) $p(ax) = |a| p(x)$

(iii) $p(x_1 + x_2) \leq p(x_1) + p(x_2)$, and

(iv) $p(x) > 0$ for some x .

A seminorm p on E is called a Hilbertien seminorm if

$$(\forall x_1, x_2 \in E) \quad p^2(x_1 + x_2) + p^2(x_1 - x_2) = 2(p^2(x_1) + p^2(x_2))$$

For a Hilbertien seminorm p we define a symmetric bilinear form $p(\cdot, \cdot)$ on E by

$$p(x_1, x_2) = \frac{1}{4} (p^2(x_1 + x_2) - p^2(x_1 - x_2))$$

$p(x_1, x_2)$ is called the inner product corresponding to p .

A seminorm p is said to be separable if there is a countable set $D \subset E$ such that for each $x \in E$ and ε positive, there exists a d in D such that $p(x - d) \leq \varepsilon$.

1.2 Operators on Hilbert Spaces

Let H_i $(i = 1, 2)$ be separable real Hilbert spaces with norms $\|\cdot\|_i$ and inner products $\langle \cdot, \cdot \rangle_i$. Let A be a linear mapping from H_1 to H_2 .

A is a <u>compact operator</u> if for every x in H_1 :

(1) $$Ax = \sum_n t_n \langle x, e_n \rangle h_n$$

where $0 < t_n \downarrow 0$, and e_n , h_n are orthonormal sets in H_1 and H_2 respectively.

A is a <u>Hilbert-Schmidt operator</u> if (1) holds and

(2) $$\sum_n t^2 < \infty$$

A is a <u>nuclear operator</u> if (1) holds and

(3) $$\sum_n t_n < \infty$$

Let $H_1 = H_2 = H$. Operator A is said to be a <u>trace class operator</u> if

(4) $$\sum_n |\langle Ae_n, e_n \rangle| < \infty$$

for any orthonormal basis $\{e_n\}$ in H .

For a linear mapping $A : H \to H$ we define the <u>adjoint mapping</u> A^* by

(5) $$\langle A^* x_1, x_2 \rangle = \langle x_1, A^* x_2 \rangle$$

We say that A is <u>self-adjoint</u> if $A = A^*$.

A is said to be <u>positive</u> (non-negative definite) if

(6) $$\sum_{i,j} a_i \bar{a}_j \langle x_i, Ax_j \rangle \geq 0$$

for every finite set of complex numbers $\{a_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^n$ a finite subset of H .

In the following theorem we give a summary of some well-known facts (see, for example [9], Chapter 1.)

<u>Theorem 1</u>. (i) If A and B are Hilbert-Schmidt operators, then AB

is a nuclear operator.

Conversely, every nuclear operator is a product of two Hilbert-Schmidt operators.

(ii) A compact positive operator $A : H \to H$ is nuclear if and only if it is a trace operator.

(iii) If A is nuclear (or Hilbert-Schmidt) operator, then A^* is an operator of the same type.

(iv) If $A : H \to H$ is a positive nuclear operator then $A^{\frac{1}{2}}$ is positive Hilbert-Schmidt operator.

(v) A positive compact operator $A : H \to H$ has the representation.

$$Ax = \sum t_n \langle x, e_n \rangle e_n$$

where e_n is an orthonormal basis for H consisting of eigenvectors of A and t_n, $0 < t_n \downarrow 0$ are eigenvalues of A, $Ae_n = t_n e_n$.

(vi) Let H_1 and H_2 be Hilbert spaces and $A : H_1 \to H_2$ be a linear mapping. A is a Hilbert-Schmidt operator if and only if

$$\sum_n \|Ae_n\|_2^2 < \infty$$

for at least one (equivalently: for all) orthonormal basis $\{e_n\}$ in H_1. A is nuclear operator if and only if

$$\sum_n |\langle Ae_n, e_n \rangle_2| < \infty .$$

In the following section we describe multi-Hilbertien spaces, following Ito [14].

1.3. Multi-Hilbertien Spaces

Let E be a vector space.

The family of all separable Hilbertien seminorms on E will be denoted by HSN.

It is easy to see that if $p, p_1, \ldots, p_n \in$ HSN then for every positive c, $cp \in$ HSN and $(p_1^2 + \ldots + p_n^2)^{1/2} \in$ HSN .

For $p, q \in$ HSN we define the relation $\overset{\bullet}{<}$: $p \overset{\bullet}{<} q$ if and only if $p(x) \leq cq(x)$ for some $c \in R$ and all $x \in E$.

Let E_p denote E with topology given by the seminorm p .
Let $\ker p = \{x \in E : p(x) = 0\}$. Then $E_p/\ker p$ is a pre-Hilbert space; its completion $\bar{E}_p$ is a Hilbert space.

If $p \overset{\bullet}{<} q$ then the identity map $E \to E$ can be extended to a continuous linear operator $i_{p,q} : \bar{E}_2 \to \bar{E}_p$. Notice that $i_{p,q}(x) = x$ on E , and the indices p, q are just pointing the topology for completion.

Define the relation $\underset{HS}{<}$: $p \underset{HS}{<} q$ if and only if $p \overset{\bullet}{<} q$ and $i_{p,q}$ is a Hilbert-Schmidt operator.

In view of Theorem 3.1.(vi), $p \underset{HS}{<} q$ if and only if

$$\sum p^2(e_n) < \infty$$

where $\{e_n\}$ is an orthonormal basis in $\bar{E}_q$.

Let $P = \{p_d\}_{j \in J}$ be a subfamily of HSN on E such that

(i) If $p \in P$ and $p \overset{\bullet}{<} q$ or $q \overset{\bullet}{<} p$ then $q \in P$

(ii) If $p,q \in P$, $p(x_n) \to 0$, x_n is q-Cauchy then $q(x_n) \to 0$.

Define a locally convex topology τ on E by its neighborhood basis at zero:

(1) $$\{x \in E : p_{j_1}(x) < 1, \ldots, p_{j_k}(x) < 1 , \ j_1, \ldots, j_k \in J\}$$
$$k = 1, 2, \ldots.$$

The topology τ given by (1) is called a <u>multi-Hilbertien topology</u> generated by P .

Space E with a multi-Hilbertien topology τ , (E, τ) is called a <u>multi-Hilbertien space</u>.

If a multi-Hilbertien topology τ is generated by $P = \{p_j\}_{j\in J}$ we will often write $\tau = \{p_j\}_{j\in J}$.

If $\tau = \{p_j\}_{j\in J}$ where J is a countable set. we say that (E, τ) is a <u>countably Hilbert space</u>. In this case, without loss of generality, we may assume

(2) $$p_1 < p_2 < p_3 < \dots$$

For, if not, then the topology given by $q_j = (p_1^2 + p_2^2 + \dots + p_j^2)^{\frac{1}{2}}$, $j = 1, 2, \dots$ coincide with the topology given by $\{p_i\}$ and $\{q_j\}$ satisfies condition (2).

In a countably Hilbert space we will use notations E_n, $\bar{E}_n$, E_n' , rather than E_{p_n} , etc. ...

If the topology τ is given by countably many <u>norms</u> $\{p_n\}$ (i.e. ker $p_n = \{0\}$) , then we can define a metric d by

(3) $$d(x_1, x_2) = \sum_n 2^{-n} \frac{p_n(x_1-x_2)}{1+p_n(x_1-x_2)}$$

E is said to be <u>complete</u> if (E, d) is complete.

E is complete if and only if ([27])

(4) $$E = \bigcap_n \bar{E}_{p_n}$$

We say that (E,τ) is a nuclear space if for every $p \in \tau$ there is a $q \in \tau$ such that $p \underset{HS}{\leq} q$. Notice that, by Theorem 3.1.(i), E is nuclear if and only if for every $p \in \tau$ there is a $q \in \tau$ such that $i_{p,q}$ is nuclear operator.

A nuclear space in which condition (4) holds is called a <u>nuclear Fréchet space</u>.

Suppose that $\{e_j^n\}_{j=1}^{\infty}$ is a complete orthonormal basis is $\bar{E}_n$, $n = 1, 2, \ldots$.

Then E is a nuclear space if and only if for every $n = 1, 2, \ldots$ there is an $m > n$ such that (Theorem 3.1.(vi)).

$$\sum_j p_m^2(e_j^n) < \infty \tag{5}$$

<u>Example</u>. Let $C^{\infty}(R)$ be the set of all infinitely differentiable functions defined on R . The Schwartz space s is a locally convex subspace of $C^{\infty}(R)$, topology defined by norms $q_n(f) = \max_{0 \leq k \leq n} \sup_t |(1+t^2)^n f^{(n)}(t)|$.

It can be shown that this topology is equivalent to the one given by Hilbertien norms:

$$p_n(f) = \Big(\sum_{k=0}^{n} \int_R (1+t^2)^n |f^{(k)}(t)|^2 dt\Big)^{\frac{1}{2}}$$

Let $\tau = \{p_n\}_{n=1}^{\infty}$. Then (s, τ) is a nuclear space, with $E_n = \{f \in C^{\infty}(R) : p_n(f) < \infty\}$ and the inner products:

$$p_n(f_1, f_2) = \sum_{k=0}^{n} \int_R (1+t^2)^n \, f_1^{(k)}(t) \, f_2^{(k)}(t) dt$$

<u>Remark</u>. A separable Hilbert space H is nuclear if and only if $\dim H < \infty$.

Let now (E, τ) be a multihilbertian space, $\tau = \{p_j\}_{j \in J}$. We say that a set $B \subset E$ <u>is bounded</u> if for each $j \in J$ the set of numbers

$$\{p_j(x) : x \in B\}$$

is bounded.

1.4 Dual Spaces

Let (E, τ) be a multi-Hilbertien space. The set of all (τ-) continuous linear functionals on E is denoted by E'_τ (or E' if there is no confusion about the topology on E).

If $F \in E'$, $x \in E$, then F evaluated at x is denoted by $\langle F, x\rangle$. In this dual notation it is understood that for fixed F (or x), $\langle F, x\rangle$ is a function of x (or F). Also, $\langle F, x\rangle$ separates points in both E and E' . The strong topology on E' is given by seminorms

$$\|F\|_B = \sup_{x \in B} |\langle F, x\rangle| \ , \quad B\text{-bounded set in } E .$$

Let E'_p denote the topological dual of E_p . E'_p is a Hilbert space with the norm $\bar{p}(F) = \sup_{p(x) \leq 1} |\langle F, x\rangle|$.

We have $E' = \bigcup_p E'_p$.

In the strong topology, E' is an inductive limit of $\{E'_p\}_{p\in\tau}$ [14].

If τ is countable, the set A is bounded (compact) in E' if and only if it is bounded (compact) in some E'_p , $p \in \tau$ [8].

1.5 Measures on dual spaces

Let E' be a topological dual of a topological vector space E .

Let A be a given Borel set in R^n . Let

(1) $$Z = \{F \in E' : (\langle F, x_1\rangle, \langle F, x_2\rangle, \ldots , \langle F, x_n\rangle) \in A\}$$

Set Z defined by (1) for some $x_1, \ldots ,x_n \in E$ is called the <u>cylinder set with base A and generating elements $x_1, \ldots ,x_n$</u> .

Another approach to defining a cylinder set is the following:

Let Y be a finitely dimensional subspace of E. Let Y^0 denote the annihilator of Y, i.e., the set of all $F \in E'$ such that

$$\langle F, x\rangle = 0 \qquad \text{for } x \in Y$$

Consider the factor space E'/Y°. It is isomorphic to Y', thus finitely dimensional. Let $A \subset E'/Y^{\circ}$ be a Borel set. The set of all $F \in E'$ which are carried into elements of A by the natural mapping $E' \to E'/Y^{\circ}$ is called the cylindar set with base A and generating subspace Y°.

These two definition are equivalent and define the same object ([9]). Note that a cylinder set may have more than one representation in terms of base and generating subspace.

It is easy to see that the cylinder sets form an algebra of sets—cylinder algebra.

By a cylinder set measure in E' we mean a nonnegative function M defined on the cylinder algebra with the following properties:

(i) If $Z = \bigcup_{i=1}^{\infty} Z_i$ where $Z_i \cap Z_j = \emptyset$ if $i \neq j$, and all Z_i are generated by the same set $x_1, \ldots, x_n$, then $M(Z) = \sum_{i=1}^{\infty} M(Z_i)$.

(ii) For any cylinder set Z, $M(Z) = \inf M(U)$, where U runs through all open cylinder sets containing Z.

Let M be a cylinder set measure. For $Y \subset E$ a finite dimensional subspace we define

(2) $$M_Y(A) = M(Z)$$

where A is a Borel set in E'/Y° and Z is the cylinder with base A

and generating subspace Y^o. So, (2) defines a Borel measure on E'/Y^o which is regular, i.e.

$$(3) \qquad M_Y(A) = \inf_U M(U)$$

where U runs through all open sets containing A .

Measures M_Y are <u>compatible</u> in the following sense. Let $Y_1 \subset Y_2$. Let T be the natural mapping $E'/Y_2^o \to E'/Y_1^o$. Then

$$(4) \qquad M_{Y_1}(A) = M_{Y_2}(T^{-1}(A))$$

Conversely, if the system of measures $\{M_Y\}$ is given, satisfying condition (3) and (4), then there is a unique cylinder set measure M such that (2) holds for every Y .

If M is a cylinder set measure, then for disjoint cylinder sets $Z_1, \ldots, Z_n$, we have $M(\bigcup_{i=1}^{n} Z_i) = \sum_{i=1}^{n} M(Z_i)$, which follows from (i) and the fact that for finitely many cylinder set there always exists a common generating set. However, a cylinder set measure may not be countably additive. If it is so, then it can be extended to a <u>measure</u> on <u>cylinder σ-algebra C</u>, which is defined to be the smallest σ-algebra that contains cylinder algebra.

We say that the measure M is a <u>probability measure</u> if $M(E') = 1$.

Let now E be a countably Hilbert space. Then ([14]) the cylinder σ-algebra $\mathcal{C}$ coicides with the Borel σ-algebra generated by the strongly open sets. Every probability measure on E' is regular.

Let (E, τ) be a multi-Hilbertian space.

A probability measure P on $(E'_\tau, \mathcal{C})$ is called <u>separable</u> if there exists a countably Hilbert topology $\tau' \subset \tau$ such that

$$P(E'_{\tau'}) = 1$$

A probability measure P on E' is called infinitely divisible (ID) if, for every $n \in N$, P can be represented as n-th convolution power of some other probability measure P_n. In terms of random variables, X is an ID random variable if for every $n \in N$ it can be represented as a sum of n independent identically distributed (iid) random variables.

We say that the sequence M_n of measures converges weakly to a measure M $(M_n \Rightarrow M)$ if for every continuous bounded real function f defined on E' we have, as $n \to \infty$:

$$\int_{E'} f(F)dM_n(F) \to \int_{E'} f(F)dM(F)$$

For a measure M on E' we define its characteristic functional as a complex-valued function defined on E by

$$f(x) = \int_{E'} e^{i<F,x>}dM(F) .$$

The characteristic functional of an ID random variable will be called an ID characteristic functional

A set $\mathcal{M}$ of measures on E' is said to be (weakly) relatively compact if every sequence M_n in $\mathcal{M}$ contains a weakly convergent subsequence.

A set $\mathcal{M}$ of measures on E' is called tight if for every $\varepsilon > 0$ there is a compact set K such that $M(K^c) < \varepsilon$ for every $M \in \mathcal{M}$. A result in [30] confirms the validity of Prohorov's theorem in E', i.e., a sequence M_n of separable probability measures is relatively compact if and only if it is tight.

We observe that a sequence of separable measures M_n is weakly convergent to a measure M if and only if it is relatively compact and the sequence f_n of characteristic functionals f_n of M_n converges pointwise to the characteristic function f of M. This follows from the fact that characteristic functional determines the measure uniquely.

Example: (Gaussian measures) Let C be a complex valued bilinear function defined on $E \times E$, satisfying $C(x, x) \geq 0$, $C(x_1, x_2) = \overline{C(x_2, x_1)}$, continuous in both arguments, and non-degenerate $(C(x, x) = 0 \Rightarrow x = 0)$

Let Y be a n-dimensional subspace of E. We define a measure g_Y:

$$g_Y(A) = \frac{1}{(2\Pi)^{\frac{n}{2}}} \int_A \exp\left(-\frac{1}{2} C(y, y)\right)dy$$

where dy is the Lebesgue measure in Y corresponding to the inner product C.

Finitely dimensional Euclidean space Y with inner product C is isomorphic to Y' which is isomorphic to E/Y^o. Therefore, there is natural isomorphism T_Y between Y and E'/Y^o.

Now define a measure G_Y on E'/Y^o by

$$G_Y(B) = g_Y(A_Y^{-1}(B)) \tag{2}$$

Now we have a set of finite dimensional measures. It can be shown [9] that (2) defines a compatible set of measures; thus, a cylinder set measure G on E' is determined. We call it <u>centered Gaussian cylinder measure</u>.

If E is a nuclear space, every cylinder measure is countably additive, therefore it can be extended to cylinder σ-algebra.

In the case of general countably Hilbert space, a sufficient condition for countable additivity of G is that for some n, the identity mapping from E_n into E_c is Hilbert-Schmidt operator, where E_c is E topologized by the norm $C^{\frac{1}{2}}(x, x)$.

The characteristic functional of centered Gaussian measure G defined as above is

$$f(x) = \exp\left(-\frac{1}{2} C(x, x)\right)$$

The function C is called <u>covariance</u> and, as we have seen, it uniquely determines a centered Gaussian measure.

If C is not non-degenerate, i.e., if for all x in some linear subspace X we have $C(x, x) = 0$, then C is nondegenerate on E/X , so a Gaussian measure G_1 on X^o can be constructed following the procedure defined above. Then we define a Gaussian measure on E' by

$$G(A) = G_1(A \cap X^o)$$

Finally, if G is a centered Gaussian measure, δ_F the measure that gives mass 1 to some element F of E' , then a noncentered Gaussian measure is defined by $\delta_F * G$, where $*$ denotes the convolution.

The characteristic functional of noncentered Gaussian measure $\delta_F * G$ is

$$f(x) = \exp(i\langle F, x\rangle - \frac{1}{2} C(x, x))$$

Gaussian measure is an infinitely divisible measure.

2. REPRESENTATION OF ID LAWS

2.1. $I(\tau)$--Topology and Bochner's Theorem

1. Definition. Let (E,τ) be a multi-Hilbertien space. We denote by $I(\tau)$ the Hilbert-Schmidt topology induced by all those Hilbertien seminorms which are $\underset{HS}{<}$ to some seminorm in τ, i.e.

$$I(\tau) = \{q \in HSN : q \underset{HS}{<} p \text{ for some } p \in \tau\} \tag{1}$$

(Recall that, by the convention in 1.3., (1) means that $I(\tau)$ is generated by the set of seminorms on right hand side.)

If E is a Hilbert space, then $I(\tau)$-topology coincides with so called S-topology which is proven to be of importance in studying characteristic functionals. In fact, there is a complete analogy between the role of S-topology in Hilbert spaces and the role of $I(\tau)$-topology in dual spaces.

2. Notation. Let $(H, \|\cdot\|)$ be a separable Hilbert space. Let τ, τ_H, τ_N, S, be topologies defined by:

$$\begin{aligned}
\tau &= \{\|\cdot\|\} \\
\tau_H &= \{p : p(x) = \|Ax\| , \ A \text{ is a Hilbert-Schmidt operator}\} \\
\tau_N &= \{p : p(x) = \langle Ax, x\rangle^{\frac{1}{2}} , \ A \text{ is a positive nuclear operator}\} \\
S &= \{p : p(x) = \langle Ax, x\rangle^{\frac{1}{2}}, \ A \text{ is a positive compact, trace class operator}\}
\end{aligned}$$

Operators A in the definition of S-topology are usually called S-operators.

3. Lemma. $\tau_H = \tau_N = S = I(\tau)$

Proof. Let us first show $\tau_H = \tau_N$. Let A be a Hilbert-Schmidt operator. Let $p(x) = \|Ax\|$. Then $p^2(x) = \langle Ax, Ax\rangle = \langle A^*Ax, x\rangle$, and A^*A is nuclear by Theorem 1.2.1. Conversely, let A be a positive nuclear operator. Again by Theorem 1.2.1., $A^{\frac{1}{2}}$ is positive Hilbert-Schmidt operator; so $p(x) = \langle Ax, x\rangle = \|A^{\frac{1}{2}}x\|$

$\tau_N = S$ follows from Theorem 1.2.1.(ii)

To show $I(\tau) = \tau_H$, let $p_A(x) = \|Ax\|$, where A is a Hilbert-Schmidt operator. Then $p_A(x) \leq \|A\| \cdot \|x\|$ and $\sum p_A^2(e_i) = \sum \|Ae_i\|^2 < \infty$, so $p_A \underset{HS}{<} \tau$. So, $\tau_H \subset T(\tau)$

Conversely, if $P \in I(\tau)$, $P \underset{HS}{<} \tau$, then $P(x) \leq c \cdot \|x\|$ for some $c > 0$. There is a map $A : H_p \to H$ such that $p(x) = \|Ax\|$ and $\sum \|Ae_i\|^2 = \sum p^2(e_i) < \infty$, so A is a Hilbert-Schmidt operator.

Let (Ω, F, P) be a probability space. In the space of random variables defined on it introduce the topology by the following neighborhood basis at zero:

(8) $$U(\varepsilon,\eta) = \{X : P(\omega: |X(\omega)| \geq \varepsilon) \leq \eta\}$$

The obtained topological space we shall denote by $L_o(\Omega, F, P)$. From (8) it follows that $X_n \to 0$ in L_o if and only if $X_n \to 0$ in P-probability. Without difficulties we can prove that $X_n \to 0$ in L_o if and only if $E(\min(|X|^k, 1)) \to 0$ for every $k > 0$, and if and only if $E(\frac{|X|}{1+|X|}) \to 0$.

Here $X = Y$ if and only if $X = Y$ a.e. [P], i.e. we are considering equivalence classes.

4. Definitions. Let (E,τ) be a multi-Hilbertian space.

(i) By a random linear functional we mean a linear mapping $X:(E,\tau) \to L_0(\Omega, F, P)$.

(ii) A random linear functional X is called separable if there exists a countable Hilbertien topology $\tau' \subset \tau$ such that $X \in E'_{\tau'} \subseteq E'_{\tau}$.

(iii) A random linear functional X is called regular if for every $x \in E$, $X(x)(\omega) = \langle X_\omega, x\rangle$ where $X_\omega \in E'_\tau$ for every ω.

(iv) Random linear functionals X and Y are said to be equivalent if, for every x in E, $P(X(x) = Y(x)) = 1$.

(v) We say that X is a version of Y if $P(X(x) = Y(x)$ for all $x \in E) = 1$.

5. Theorem [14]. A random linear functional X has a τ-regular separable version if and only if X is $I(\tau)$-continuous (i.e. if the mapping $X: (E,I(\tau)) \to L_0(\Omega, F, P)$ is continuous).

6. Theorem (Ito, [14]--Generalized Bochner's theorem)

Let (E,τ) be a multi-Hilbertien space.

Let f be a complex-valued function defined on E such that

(i) f is positive definite,

(ii) $f(0) = 1$,

(iii) f is $I(\tau)$-continuous at 0.

Then

(iv) f is the characteristic function of a separable probability measure P on E'.

Conversely, (iv) implies (i), (ii) and (iii).

7. Remark. If E is a Hilbert space, then by Lemma 3, $I(\tau) = S$, so Theorem 8 reduces to a well-known result of Sazonov [26]. On the

other hand, when E is a nuclear space, thus $I(\tau) = \tau$, this result is given in [20].

In the next section, we generalize Lévy's continuity theorem to multi-Hilbertien spaces.

2.2 Continuity theorem

The convergence of characteristic functions on real line implies convergence of corresponding probability measures. In infinitely dimensional spaces we have to impose some conditions for relative compactness.

1. Theorem ([23], Ch. VI). Let H be a Hilbert space.
Let $\{P_k\}$ be a sequence of probability measures on H, and f_k the corresponding sequence of characteristic functionals.
$\{P_k\}$ is relatively compact if and only if for every $\varepsilon > 0$, and for every $k = 1, 2, \ldots$:

(i) There exists a S-operator S_k such that

$$1 - \mathrm{Re} f_k(x) \leq \langle S_k x, x\rangle + \varepsilon$$

(ii) $$\sup_k \sum_i \langle S_k e_i, e_i\rangle < \infty$$

(iii) $$\lim_N \sup_k \sum_{i=N}^{\infty} \langle S_k e_i, e_i\rangle = 0 ,$$

where $\{e_i\}$ is an orthonormal basis in H.

2. Example. (Sazonov, [26]) It is well known that, on real line, a convergent sequence of characteristic functions is equicontinuous. This example shows that it is not true in a Hilbert space. However, as shown in Meyer's paper [20], it remains true in a nuclear space.

Let

$$t_{ij} = \begin{cases} 1/i^2j^2 & \text{if } i \neq j \\ 1/i^2 & \text{if } i = j \end{cases} \qquad i, j = 1, 2, \ldots$$

Then

(1) $\displaystyle\sum_{j=1}^{\infty} t_{ij} < C < \infty$

(2) $\displaystyle\lim_{N} \sup_{i} \sum_{j=N}^{\infty} t_{ij} = 0$

(3) $\displaystyle\sum_{j=1}^{\infty} \sup_{i} t_{ij} = \infty$

Define operators S_k , $k = 1, 2, \ldots$ by $S_k e_k = t_{kj} e_j$, where e_j , $j = 1, 2, \ldots$ is an orthonormal basis in an infinitely dimensional Hilbert space H . Let P_k be probability measures on H with characteristic functions $f_k(x) = \exp\left(\frac{-1}{2} <S_k x, x>\right)$, $k = 1, 2, \ldots$

By Theorem 1, (2) and (3) imply weak convergence of f_k .

Suppose now that $\{f_k\}$ is equicontinuous at 0 in $S(=I(\tau))$-topology. Then for every $\varepsilon > 0$ there is an S-operator S_ε such that $<S_\varepsilon x, x> < 1$ implies $1 - f_k(x) < \varepsilon$, for all k . By definition of f_k it follows $<S_k x, x> < \eta(\varepsilon)$ whenever $<S_k x, x> < 1$ for all k , $\eta(\varepsilon) \to 0$. Let $t_{\varepsilon,j} = <\eta(\varepsilon) S_\varepsilon e_j, e_j>$.

For every real r , if $<\eta(\varepsilon) S_\varepsilon re_j, re_j> = r^2 t_{\varepsilon,j} < \eta(\varepsilon)$, then $<S_\varepsilon re_j, re_j> < 1$, but then $r^2 t_{\varepsilon,j} < \eta(\varepsilon)$ implies $r^2 t_{kj} < \eta(\varepsilon)$ for all k . Since r is arbitrary, it follows $t_{kj} < t_{\varepsilon,j}$ for all k . Since S_ε is an S-operator we have $\sum_j t_{\varepsilon,j} < \infty$, which then implies $\sum \sup_k t_{k,j} < \infty$, and this contradicts (3). So, f_k is not $I(\tau)$-equicontinuous.

3. <u>Example</u>. This example is to show that in the infinitely dimensional Hilbert space, the convergence of characteristic functionals does not imply convergence of corresponding measures.

Let t_{ij} be any set of real numbers satisfying

(4) $\qquad t_{ij} \to u_j \qquad$ as $i \to \infty$, $j = 1, 2, \ldots$

(5) $\qquad 0 < t_{ij} \leq 1$

(6) $\qquad \lim_{N\to\infty} \sup_i \sum_{j=N}^{\infty} t_{ij} > 0$

For an orthonormal sequence $\{e_i\}$ in H , define the operators S_k and S by $S_k e_j = t_{kj} e_j$; $Se_j = u_j$, $j, k = 1, 2, \ldots$

By (4) we have $<S_k x, x> \to <Sx, x>$ for every x in H . Define $f_k(x) = \exp\{-\frac{1}{2} <S_k x, x>\}$ and $f(x) = \exp\{-\frac{1}{2} <Sx, x>\}$. Then clearly $f_k(x) \to f(x)$ for every x . But by (6), the corresponding sequence of measures is not weakly convergent.

Until further notice, E will denote a milti-Hilbertien space, with topology τ .

4. <u>Theorem</u>.

Let M_k be a sequence of separable probability measures on E', let f_k be the corresponding sequence of characteristic functionals. Assume the following:

(i) There is a function f , $I(\tau)$-continuous at 0 , such that

(7) $\qquad f_k(x) \to f(x)$, for every x in E .

(ii) For every $\varepsilon > 0$ there is a sequence of $I(\tau)$-seminorms p_k and a τ-seminorm q such that $p_k \underset{HS}{<} q$ for every k and

(8) $\quad 1 - \operatorname{Re} f_k(x) \leq \varepsilon + p_k^2(x)$

(9) $\quad \sup_k \sum_i p_k^2(e_i) < \infty$

(10) $\quad \lim_N \sup_k \sum_{i=N}^{\infty} p_k^2(e_i) = 0 ,$

where $\{e_i\}$ is an orthonormal basis in E_q .

Then there is a separable probability measure M on E' such that $M_k \Rightarrow M$.

Conversely, if $M_k \Rightarrow M$, where M is a separable probability measure, then (7)-(10) hold.

Proof. Assume (7)-(10). By Theorem 1, the set of measures induced by M_k in E_q is relatively compact. So there exists a q-compact (thus τ-compact) set K such that $M_k(K^c) \leq 1 - \varepsilon$.

So, M_k is a tight sequence for which $f_k(x) \to f(x)$ pointwise. By Theorem 1.5, there is a separable probability measure M such that f is the characteristic functional of M ; thus $M_k \Rightarrow M$.

Conversely, let $M_k \Rightarrow M$. Then there is a compact set $K \subseteq E'_{\tau'}$ such that $M_k(K^c) < \varepsilon/2$ for every $k = 1, 2, \ldots$.

For every x and every k we have

$$
\begin{aligned}
1 - \operatorname{Re} f_k(x) &= \int (1 - \cos \langle F,x\rangle) dM_k(F) \\
&\leq \int_K (1 - \cos \langle F,x\rangle) dM_k(F) + \varepsilon \\
&\leq \frac{1}{2} \int_K \langle F,x\rangle^2 dM_k + \varepsilon
\end{aligned}
$$

K is compact in $E'_{\tau'}$; so there is a q such that K is compact in E'_q .

Let $p_k^2(x) = \frac{1}{2} \int_K \langle F,x\rangle^2 \, dM_K(F)$.

If $\{e_i\}$ is an orthonormal basis in E_q then

$$\sum_i p_k^2(e_i) = \frac{1}{2} \int_K \bar{q}^2(F) dM_K(F) < \frac{1}{2} \sup_{F\in K} \bar{q}^2(F) < \infty$$

because $\bar{q}$ is bounded on K , being a continuous function. Also we have

$$p_k^2(x) \leq q^2(x) \cdot \frac{1}{2} \int_k \bar{q}^2(F) dM_k(F) \leq C \cdot q^2(x) ,$$

so $p_k \underset{HS}{<} q$, for every k . Thus, (8) and (9) are proved .

To prove (10) note that

$$\sum_{i=N}^{\infty} p_k^2(e_i) = \sum_N^{\infty} \int_K \langle F,e_i\rangle^2 dM_k(F)$$
$$\leq \sup_{F\in K} \sum_N^{\infty} \langle F,e_i\rangle^2 \to 0 , \text{ as } N\to\infty ,$$

by compactness of K .

5. Corollary. Let f_k , $k = 1, 2, \ldots$, be a sequence of characteristic functionals of separable probability measures M_k , $k = 1, 2, \ldots$ on E' , which is $I(\tau)$-equicontinuous at 0 , i.e., there is a seminorm p in $I(\tau)$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $k = 1, 2, \ldots$ we have:

(11) $\qquad p(x) \leq \delta \Rightarrow 1 - \mathrm{Re} f\,(x) \leq \varepsilon$.

Then $\{M_k\}$ is a relatively compact sequence.

Proof. Assume (11). Then conditions (7)-(10) are satisfied for $p_k = \frac{1}{\delta} \cdot p$, $k = 1, 2, \ldots$, and q such that $p \underset{HS}{<} q$. Then from the proof of Theorem 4 it follows that $\{M_k\}$ is relatively compact.

6. Remark. Theorem 4 and Corollary 5 remain true for any sequence of finite separable measures M_n such that $\sup_n M_n(E'_{\tau'}) < \infty$.

2.3. Infinitely divisible probability measures

Let (E,τ) be a multi-Hilbertian space.

We have defined infinitely divisible probability measures in Chapter 1. By Theorem 1.5., a separable probability measure on E' is uniquely determined by its characteristic function, so we have

1. Theorem. Let M be a separable probability measure on E', with the characteristic function f. M is an ID measure if for every n there is a characteristic function f_n such that

$$f_n^n(x) = f(x) , \tag{12}$$

for every x in E.

The following theorem can be proved in the same way as for the real random variables.

2. Theorem. (i) ID characteristic functional never vanishes.

(ii) If f is an ID characteristic functional, then f^s is ID characteristic functional, for every $s > 0$.

(iii) The sum of finite number of ID random variables is ID.

(iv) Weak limit of ID measures is an ID measure; moreover, if only $f_k \to f$ pointwise, and f_k are ID characteristic functionals, then f is an ID characteristic functional.

3. Examples. (i) Let

$$f(x) = \exp C(\int e^{i<F,x>} dM(F) - 1) , \tag{13}$$

where M is a separable probability measure on E', and C any real number.

The function defined by (13) is positive definite, $I(\tau)$-continuous and

$f(0) = 1$. So, by Theorem 1.5 it is the characteristic functional of a separable probability measure on E'-Poisson measure. It satisfies (12), so it is ID measure.

(ii) Let

$$f(x) = \exp(i\langle F,x\rangle - \frac{1}{2} p^2(x)) , \tag{14}$$

where F is a fixed element in E' , p is a $I(\tau)$-seminorm. By Bochner's theorem there is a unique separable probability measure on E' determined by (14)--Gaussian measure. It is clearly an ID measure. In Chapter 1, the construction of a Gaussian measure, starting from cylinder measures is given.

4. Theorem. A function g is the logarithm of an ID characteristic functional f (of a separable probability measure on E') if and only if

$$g(0) = 0 \tag{15}$$

$$g \text{ is } I(\tau)\text{-continuous at } 0 \text{ , and} \tag{16}$$

for all finite sets of complex numbers $A_I = \{a_i\}_{i\in I}$ such that $\sum_{i\in I} a_i = 0$, and $X_I = \{x_i\}_{i\in I}$ $x_i \in E$, we have

$$\sum_{i,j\in I} a_i\bar{a}_j g(X_i - X_j) \geq 0 \tag{17}$$

Proof. The assertion follows directly from the Theorem 1.5. In fact, (17) is the consequence of positive definiteness of f and Schoenberg's theorem [19], (16) follows from $I(\tau)$-continuity of f , and (15) from $f(0) = 1$.

5. Remark. Making an appropriate choice of A_I and X_I in (17), one can show that for every $x \in E$

(18) $\quad \text{Re } g(x) \leq 0$

(19) $\quad g(-x) = \overline{g(x)}$.

6. Theorem. The class of ID characteristic functionals coincide with the class of Poisson characteristic functionals and their pointwise limits.

Proof. By Example 3 and Theorem 2, Poisson characteristic functionals and their limits are ID.

Conversely, let f be an ID characteristic functional.

Then for all x in E ,

$$n(f^{1/n}(x) - 1) \to \log f(x) \qquad (n \to \infty)$$

so, $f(x) = \lim_n \exp\, (n(f^{1/n}(x) - 1))$. By Theorem 1, $f^{1/n}$ is a characteristic functional of some separable probability measure M_n , so

$$f^{1/n}(x) = \int e^{i<F,x>} dM_n(F)$$

and

$$f(x) = \lim_n \exp n \int (e^{i<F,x>} - 1) dM_n(F) ,$$

which is a limit of Poisson characteristic functionals.

Now we give some elementary inequalities that will be used later.

7. Lemma. For any real t,s :

(i) $|e^{it}-1| \leq 2 \min (t^2,1) \leq 2 \min (t,1)$

(ii) $|e^{it}-1-it| \leq t^2/2$

(iii) $1 - \sin t/t \geq c \cdot \min (t^2,1)$

(iv) $|(e^{is}-is) - (e^{it}-it)| \leq |s-t| \cdot \frac{|s| + |t|}{2}$

Proof: elementary.

8. Notation. For $q \in \tau$ define the following quantity (finite or not)

$$\tilde{q}(F) = \sup_{q(x)\leq 1} |<F,x>| \tag{20}$$

Let M be a positive separable measure on $E'-\{0\}$ such that

$$\int_{E'-\{0\}} \min(\tilde{q}^2(F), 1)\, dM(F) < \infty . \tag{21}$$

For a G in E' and $r \in I(\tau)$ define

$$\begin{aligned} g(x) &= g[G,r,M,q](x) \\ &= i<G,x> - r^2(x) + \int(e^{i<F,x>} - 1 - i<F,x> \cdot 1[\tilde{q}(F)\leq 1])\, dM(F) \end{aligned} \tag{22}$$

Function defined by (22) will be referred to as $g[G,r,M,q]$ or only g.

We will also use the following notation:

$$h(F,x) = e^{i<F,x>} - 1 - i<F,x> \cdot 1[\tilde{q}(F)\leq 1] \tag{23}$$

9. Lemma. g is $I(\tau)$-continuous at 0.

Proof: $G \in E'$ implies $G \in E_p'$ for some $p \in \tau$.

Let $\{e_i\}$ be an orthonormal basis in E_p.

Then $\sum <G,e_i>^2 = \tilde{p}(G)$, thus $|<G,x>|$ is a $I(\tau)$-continuous seminorm, so $i<G,x>$ is $I(\tau)$-continuous.

Since $r \in I(\tau)$, it remains to show the $I(\tau)$-continuity of the integral in (22). We have:

$$\int h(F,x)\, dM(F) = \int_{\tilde{q}(F)>1} (e^{i<F,x>} - 1)\, dM(F) + \int_{\tilde{q}(F)\leq 1} (e^{i<F,x>} - 1 - i<F,x>)\, dM(F) \tag{24}$$

Let us show that both terms above are $I(\tau)$-continuous.

By Lemma 7(i),

$$\Big|\int_{\tilde{q}(F)>1} (e^{i<F,x>} - 1)\, dM(F)\Big| \leq \int_{\tilde{q}(F)>1} \min(<F,x>^2, 1)\, dM(F) \tag{25}$$

Restricted to the set where $\tilde{q}(F)>1$, M is a positive finite measure;

let $M(\tilde{q}(F)>1) = m$. By separability of M , there is a topology σ determined by an increasing family of seminorms q_n such that $E'_{q_n} \uparrow E'_\sigma$. So, for given ε we can find n such that

$$M(F \notin E'q_n) \leq \varepsilon/2$$

i.e., if $\{e_k\}$ is q_n-orthonormal basis:

$$M(\sum\langle F,e_k\rangle^2 = \infty) \leq \varepsilon/2 .$$

Let $r = r(\varepsilon)$ be a real number such that

$$M(\sum\langle F,e_k\rangle^2 \geq r) \leq \varepsilon/2 . \tag{26}$$

Then, with $F_k = \langle F,e_k\rangle$, we have

$$\int_{\tilde{q}(F)>1} \min(\langle F,x\rangle^2,1)dM \leq \int_{\tilde{q}(F)>1} \langle F,x\rangle^2 \cdot 1[\sum F_k^2 < r]dM(F) + \int 1[\sum F_k^2 \geq r]dM(F) . \tag{27}$$

Let $p^2(\varepsilon,x) = p^2(x) = \int_{\tilde{q}(F)>1} \langle F,x\rangle^2 \cdot 1[\sum F_k^2 < r]dM(F)$

Then $\sum p^2(e_k) < r \cdot m$; so p is an $I(\tau)$-seminorm ; by (25) and (27) we have

$$\left| \int_{\tilde{q}(F)>1} (e^{i\langle F,x\rangle} - 1)dM(F)\right| \leq p^2(x) + \varepsilon ,$$

so the first term in (24) is $I(\tau)$-continuous.

By Lemma 7.(ii):

$$\left| \int_{\tilde{q}(F)<1} e^{i\langle F,x\rangle} - 1 - i \langle F,x\rangle)dM(F)\right| \leq \frac{1}{2} \int_{\tilde{q}(F)<1} \langle F,x\rangle^2 dM(F) . \tag{28}$$

Now, by definition of $\tilde{q}$, it follows

$$|\langle F,x\rangle| \leq \tilde{q}(F)q(x) ,$$

so by the assumption (21), the right hand side of (28) is finite. Moreover, on the set where $\tilde{q}(F) < 1$, we have $F \in E'_q$, so the expression on the lefthand side of (28) is $I(\tau)$-continuous.

10. Corollary. g is logarithm of an ID characteristic functional of a separable probability measure on E' .

Proof. We shall use Theorem 4. Condition (15) is satisfied; (16) follows from Lemma 9 and (17) can be easily checked.

11. Lemma. For given $q \in \tau$, there is 1-1 correspondence between functions g defined by (22) and triplets $[G, r, M]$.

Proof: Let $t \in R$. We have

$$|\mathrm{Re}\, \frac{e^{i<F,tx>}-1}{t^2}| \leq |\frac{e^{i<F,tx>}-1}{t^2}| \leq \min\,(<F,x>^2,\ 2/t^2)\ ,$$

$$\leq 2q^2(x) \cdot \min\,(\tilde{q}^2(F),\ 1)\ ,$$

for a large enough t .

So, by dominated convergence theorem,

$$t^{-2} \cdot \int_{\tilde{q}(F)>1} (e^{i<F,tx>} - 1) \cdot dM(F) \to 0 \quad \text{as} \quad t \to \infty\ .$$

Similarly,

$$t^{-2} \cdot \int_{\tilde{q}(F)\leq 1} (e^{i<F,tx>} - 1 - i<F,tx>) \quad dM(F) \to 0 \quad \text{as} \quad t \to \infty$$

So, we have

$$r^2(x) = \lim_{t\to\infty} - \mathrm{Re}g(tx) \cdot t^{-2} \qquad (\geq 0 \ \text{ by } (18))\ ,$$

which shows that r is uniquely determined by g .

Let now $x,y \in E$. The following formula holds:

$$(29) \qquad \int (1 - \cos<F,y>)e^{i<F,x>}dM(F) = r^2(y) + g(x) - \frac{g(x+y) + g(x-y)}{2}\ .$$

We shall show that M is uniquely determined by (29).

For $x = 0$ and y being fixed, we have from (29) that the measure N defined by

$$dN = (1 - \cos\langle F,y\rangle)dM \tag{30}$$

is a positive finite measure on E' (more precisely, its extension to E') Since N has a $I(\tau)$-continuous characteristic functional, it follows that N is a separable finite measure, uniquely determined by (30). So, M is uniquely determined by N on any set on which $1 - \cos\langle F,y\rangle$ is bounded away from 0. Since y runs through E, by regularity we conclude that M is uniquely determined. Finally, G is determined by M, q and r, from (22).

The following two lemmas are proved in [6], in a different context:

<u>12. Lemma</u>. Let f be an ID characteristic functional.

Let $f_n = f^{1/n}$. Suppose that, for some $\varepsilon > 0$ and some $p \in I(\tau)$, $p \underset{HS}{<} q$, we have:

$$p(x) \leq 1 \Rightarrow 1 - f(x) \leq \varepsilon . \tag{31}$$

Then

$$n(1-\mathrm{Re}\, f_n(x)) \leq 8\varepsilon(1+p(x)) , \quad \text{for all } x , \text{ and} \tag{32}$$

$$\int \min (q^2(F), 1)dU_n \leq 48\varepsilon , \quad \text{for all } n , \tag{33}$$

where U_n is the measure that corresponds to nf_n.

13. <u>Lemma</u>. Let $Q = \{F: q(F)\leq 1\} - \{0\}$. Let $K(F,x) = \tilde{q}^{-2}(F)(e^{i\langle F,x\rangle} - 1 - i \langle F,x\rangle)$. Let M_n be a sequence of positive finite measures on Q, such that $M_n(Q) \leq 1$ and $\int K(F,x)dM_n(F)$

converges pointwise to a function $w(x)$. Then there is a positive separable measure M_o on Q , such that $M_o(Q) \leq 1$ and an $I(\tau)$-seminorm r , such that $w(x) = -r^2(x) + \int K(F,x)dM_o(x)$.

14. Theorem. Let g be the logarithm of an ID characteristic functional of a separable probability measure on E' . Suppose $q \in \tau$ such that there is a $p \in I(\tau)$, $p \underset{HS}{<} q$, and

$$(34) \qquad p(x) \leq 1 \Rightarrow |g(x)| \leq \varepsilon ,$$

for some ε , $0 < \varepsilon < \frac{1}{4}$.

Then there is a triplet $[G, r, M]$ satisfying conditions of Notation 8, such that $g = g[G,r,M,q]$. This representation is unique.

Proof. Let $g = \log f$, where f is an ID characteristic functional of a separable probability measure on E' . Let $f_n = nf^{1/n}$. Let Q be as in Lemma 13. We have $\log f = g = \lim_n n(f^{1/n}-1)$. Let P_n be the measure corresponding to f_n . Let U_n be P_n restricted to Q : let V_n be P_n restricted to $Q^c-\{0\}$. Denote by $\hat{U}_n$ and $\hat{V}_n$ the corresponding characteristic functionals. Let $f(x) = 1-Z$. Then, as $n\to\infty$ we have

$$\begin{aligned} f_n(0) - f_n(x) &= n(1 - f^{1/n}(x)) = n(1 - (1-Z)^{1/n}) \\ &\doteq n(1 - (1-\tfrac{Z}{n})) \doteq -Z . \end{aligned}$$

so we conclude that $\{f_n\}$ is equicontinuous at 0 . Now we have:

$$f_n(0) - \text{Re } f_n(x) = (\hat{U}_n(0) - \text{Re } \hat{U}_n(x)) + (\hat{V}_n(0) - \text{Re } V_n(x)) ,$$

and by $\hat{U}_n(0) - \text{Re } \hat{U}_n(x) \geq 0$, we conclude that

$$f_n(0) - \text{Re } f_n(x) < \varepsilon \Rightarrow \hat{V}_n(0) - \text{Re } \hat{V}_n(x) < \varepsilon$$

i.e., $\hat{V}_n$ is equicontinuous at 0 .

By (34) and Lemma 12 we have

(35) $$V_n(E') = P_n(Q^c - \{0\}) \leq 48\varepsilon ,$$

so by Corollary 2.5. and Remark 2.6. it follows that V_n is relatively compact. Let V be a measure such that $V_{n'} \Rightarrow V$ for some subsequence n'. Then $\hat{V}_{n'} - \hat{V}_n(0) \to \hat{V} - \hat{V}_n(0)$ pointwise. By (34) and Lemma 12,

(36) $$\int \tilde{q}^2 \, dU_{n'} \leq 48\varepsilon$$

By inequality 2.7.(ii) we have, for every x in E:

$$\begin{aligned} &\left|\int(e^{i<F,x>} - 1 - i<F,x>)dU_{n'}(F)\right| \\ &\leq \int \frac{<F,x>^2}{2} \, dU_{n'}(F) \\ &\leq 12 \varepsilon q^2(x) . \end{aligned}$$

Now let $w_n(x) = \int(e^{i<F,x>} - 1 - i<F,x>)dU_{n'}$.

The above inequality shows that $w_n(x)$ is bounded for every x ; therefore, there is a function $w(x)$ such that for every x , $w_{n''}(x) \to w(x)$, for some subsequence n'' .

We have obtained so far:

$$\begin{aligned} g(x) &= \lim_n n(f_n(x) - 1) \\ &= \lim_{n''} ((\hat{V}_{n''}(x) - \hat{V}_{n''}(0)) + (\hat{U}_{n''}(x) - \hat{U}_{n''}(0))) \\ &= (\hat{V}(x) - \hat{V}(0)) + w(x) + i \lim_{n''} \int <F,x>dU_{n''}(F) \\ &= \hat{V}(x) - \hat{V}(0) + w(x) + i<G_o,x> . \end{aligned}$$

By Lemma 13, we have that there is a $r \in I(\tau)$, a separable measure M_o on Q such that

$$w(x) = -r^2(x) + \int(e^{i<F,x>} - 1 - i<F,x>) \cdot \tilde{q}^{-2}(F)dM_o$$

To conclude the proof, define G by

$$<G,x> = <G_o, x> + \int_{\tilde{q}(F)\leq 1} <F,x> \, dV(F)$$

and define measure M on $E'-\{0\}$ to be M_o on Q and V on $Q^c-\{0\}$. Clearly, the condition (21) is satisfied and $g = g[G,r,M,q]$.

Uniqueness is proved in Lemma 11.

3. CONVERGENCE OF ID LAWS AND HOMOGENEOUS PROCESSES WITH INDEPENDENT INCREMENTS

3.1. Convergence of ID laws

1. Definition. We say that a sequence of $I(\tau)$-seminorms $\{p_n\}$ is compact if there exists a τ-seminorm t such that $p_n \underset{HS}{<} t$ for all n and, for an orthonormal basis $\{e_i\}$ in E_t, we have:

$$\text{(1)} \qquad \sup_n \sum_i p_n^2(e_i) < \infty$$

$$\text{(2)} \qquad \lim_N \sup_n \sum_{i=N}^{\infty} p_n^2(e_i) = 0$$

2. Theorem. Let P_n and P be separable ID probability measures on E' with the characteristic functionals $g_n = g_n[G_n, r_n, M_n, q]$ and $g = g[G,r,M,q]$. Then $P_n \Rightarrow P$ if and only if

$$\text{(3)} \qquad \lim_n G_n = G$$

(4) $\qquad M_n \Rightarrow M$ on every set $\{F: \tilde{q}(F) > \varepsilon\}$

(5) $\qquad$ The sequence $\{t_n\}$ of $I(\tau)$-seminorms defined by

$$t_n^2(x) = r_n^2(x) + \int_{\tilde{q}(F)\le 1} \langle F,x\rangle^2 dM_n(F)$$

is compact,

$$\text{(6)} \qquad \text{for every } x, \quad \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \ \frac{1}{2} \int_{\tilde{q}(F)\le\varepsilon} \langle F,x\rangle^2 dM_n(F) + r_n^2(X) = r^2(x) .$$

Proof. Assume the conditions above. Then for every x in E

$$\text{(7)} \qquad g_n(x) = i\langle G_n,x\rangle + \int_{\tilde{q}(F)\ge\varepsilon} (e^{i\langle F,x\rangle} - 1 - i\langle F,x\rangle \cdot 1[\tilde{q}(F)\le 1])dM_n(F)$$
$$+ \int_{\tilde{q}(F)\le\varepsilon} (e^{i\langle F,x\rangle} - 1 - i\langle F,x\rangle)dM_n(F) - r_n^2(x) .$$

Now we let $n\to\infty$ to obtain that first term in (7) converges to $i\langle G,x\rangle$

and the second term to the corresponding expression with M. Now we take care of the remainder. Use inequality 2.3.7.(ii) and note that if $|a| < |b|$ then $|a-c| < |-b-c|$, for a,b,c complex numbers. So we have

$$(8) \quad \int_{\tilde{q}(F)\leq\varepsilon} (e^{i<F,x>} - 1 - i<F,x>)dM_n(F) - r_n^2(x) + r^2(x)| \leq$$
$$\leq \ | - \frac{1}{2} \int_{\tilde{q}(F)\leq\varepsilon} <F,x>^2 dM_n(x) \ - \ r_n^2(x) \ + \ r^2(x)| \ .$$

Letting now $\varepsilon \to 0$, and using (6) we have that for every x in E, $g_n(x) \to g(x)$, so then characteristic functionals f_n converge pointwise to the characteristic functional f of the measure M.

The condition (5), together with inequalities in the proof of Lemma 2.9., provides condition (ii) of Theorem 2.4. So, by Theorem 2.4., we conclude that $P_n \Rightarrow P$.

Conversely, let $P_n \Rightarrow P$. Using Theorem 2.4. and the proof of Theorem 5.5. of [23] we obtain conditions (3)-(6).

3.2. Homogeneous processes with independent increments

Let T be a finite or infinite interval on the real line, starting at 0.

1. Definition. (i) A process with independent increments is a family of random variables $\{X(t)\}_{t\in T}$, defined on a probability space (Ω, F, P), such that for every $t_1,\ldots,t_n$ $(0 \leq t_1 < \ldots < t_n)$, the random variables

$$X(t_o) \ , \ X(t_1) - X(t_o) \ , \ \ldots \ , \ X(t_n) - X(t_{n-1})$$

are independent.

(ii) $\{X(t)\}_{t\in T}$ is said to be a homogeneous process if the distribution of random variable $X(t) - X(s)(s < t)$ depends on

$t - s$ only.

A connection between homogeneous processes with independent increments and ID laws is immediate. Let $t \in T$ be fixed. By Definition 1, $X(\frac{kt}{n}) - X(\frac{(k-1)t}{n})$ are iid random variables $(k = 1, \ldots, n)$, and $X(t) - X(0) = \sum_{k=1}^{n} X(\frac{kt}{n}) - X(\frac{(k-1)t}{n})$ is an ID random variable. Therefore, every increment of a homogeneous process with independent increments is an ID random variable. Let P_t be the distribution of $X(t) - X(0)$, and Q be the distribution of $X(0)$. It is easy to see that $P_{t+s} = P_t * P_s$ and $P_o = \delta_o$ (Dirac distribution), so $\{P_t\}_{t \in T}$ is a semi-group of convolutions. So, the distribution of X_t is obtained as $P_t * Q$; with some additional work it can be shown ([1]) that all finitely dimensional distributions of the process X_t are determined by the semi-group $\{P_t\}$ and Q.

Let now $\{X_t\}$ be a E'-valued process, and let f_t be the characteristic functional of X_t. Until further notice assume that $\{X_t\}$ is a homogeneous process with independent increments.

By independence we have:

(1) $$f_{t+s}(x) = f_t(x) \cdot f_s(x) .$$

In a particular case when $f_t(x)$ is, for every x, continuous at $t = 0$, we have an especially simple relation.

<u>1. Theorem</u>. Suppose that for every x in E, $f_t(x)$ is continuous at $t = 0$. Then

(2) $$f_t(x) = [f_1(x)]^t$$

<u>Proof</u>: Let $s \to 0$ for fixed t in (1) to show that f is continuous

for every t . The only continuous solutions of (1) are functions of the form

$$(3) \qquad f_t(x) = \exp t\, g(x) ,$$

for some g . This proves (2).

Suppose now that f_1 is the characteristic functional of a separable probability measure on E' . Then (2) completely determines the process. If $X(0) = 0$ with probability 1, then for every t, X_t is an ID random variable; by (3) and results in Chapter 2, we know the form of $f_t(x)$, and by Theorem 1.2. we have necessary and sufficient conditions for weak convergence.

Note that f_1 is an ID characteristic functional.

3.3. An Example

In this section we present the solution to a problem in real line, which arises in a stochastical model of neuronal activity. This is an improvement and a generalization of results of Kallianpur [16] and Tuckwell [29]. We also discuss a possiblity of a generalization to infinitely dimensional spaces.

Let us first recall some facts about real valued ID random variables.

Let X be a random variable with the characteristic function f , and its logarithm g . Suppose that $\operatorname{Var} X < \infty$. Then g is represented in the form

$$(1) \qquad g(x) = iGx + \int (e^{iux} - 1 - iux)u^{-2}dK(u) ,$$

where K is distribution function of some finite measure on R . The representation (1) is unique and we write $g = g[G,K]$. $X_n \Rightarrow X$ if and only if $K_n(u) \to K(u)$, for every $u \in R$, and $G_n \to G$.

If X is normal with mean m and the standard deviation S, then $G = m$ and $K(u) = S^2 \cdot 1[u > 0]$

If X is generalised Poisson, i.e., if

(2) $$P(X = a + kh) = \frac{e^{-\lambda}}{k!} \lambda^k, \quad k = 0, 1, 2, \ldots$$

then $G = a + \lambda h$; $k(u) = \lambda h^2 \cdot 1[t \geq h]$

Let $Y_{kn}(t)$ $k = 1, 2, \ldots, p_n$, $n = 1, 2, \ldots$ be independent Poisson processes with parameters λ_{kn} ; let ε_{kn} , $k = 1, 2, \ldots, p_n$, $n = 1, 2, \ldots$ be real numbers, $0 \leq t \leq T < \infty$.

Define

(3) $$N_n(t) = \sum_{k=1}^{p_n} \varepsilon_{kn} Y_{kn}(t)$$

(4) $$\overline{N}_n(t) = (N_n(t) - EN_n(t)) \cdot \sigma_n^{-1} ,$$

where $EN_n(t) = t \cdot \sum_{k=1}^{p_n} \varepsilon_{kn} \lambda_{kn}$; $\sigma_n = (\sum_{k=1}^{p_n} \varepsilon_{kn}^2 \lambda_{kn})^{1/2}$

Then for $N_n(1)$ we have the representation:

(5) $$G_n = \sum_1^{p_n} \lambda_{kn} \varepsilon_{kn} , \quad k_n(u) = \begin{cases} \lambda_{kn} \varepsilon_{kn}^2 & \text{if } u = \varepsilon_{kn} \\ 0 & \text{otherwise ,} \end{cases}$$

where k_n is the point mass function;

$$K_n(u) = \sum_{v \leq u} k_n(v) .$$

Similarly, $\overline{N}_n(1)$ has the representation:

(6) $$G_n = 0 \; ; \quad k_n(u) = \begin{cases} (\varepsilon_{kn}/\sigma_n)^2 \cdot \lambda_{kn} & \text{if } u = \dfrac{\varepsilon_{ku}}{\sigma_n} \\ 0 & \text{otherwise} \end{cases}$$

The interest is to investigate the limit behavior of $N_n(t)$ and $\overline{N}_n(t)$.

Let $W(t)$ be a standard Brownian motion and let $X(t)$ be the Poisson process with independent increment whose distribution is given by (2) with λt in place of λ .

1. <u>Theorem</u>. In order that $\overline{N}_n \Rightarrow W$ (in the space $D(0,T)$) , it is necessary and sufficient that for every $\varepsilon > 0$:

$$(7) \qquad \sum_{k:|\varepsilon_{kn}/\sigma_n|>\varepsilon} (\varepsilon_{kn}/\sigma_n)^2 \lambda_{kn} \to 0 \qquad (n \to \infty)$$

2. <u>Theorem</u>. In order that $N_n \Rightarrow X$ (in $D(0,T)$) it is necessary and sufficient that, for every $\varepsilon > 0$:

$$(8) \qquad \sum_{k=1}^{p_n} \lambda_{kn} \varepsilon_{kn} \to a + \lambda h$$

$$(9) \qquad \sum_{k:|\varepsilon_{kn}-h|>\varepsilon} \lambda_{kn} \varepsilon_{kn}^2 \to 0 \qquad (n \to \infty)$$

$$(10) \qquad \sum_{k=1}^{p_n} \lambda_{kn} \to \lambda , \qquad \text{for some } \lambda .$$

<u>Proof of Theorem 1</u>

Let us first prove the convergence of one dimensional distributions. Without loss of generality set $t = 1$ (otherwise we may take $\lambda_{kn}t$ in place of λ_{kn}) . By (6), we have to show that (7) is equivalent to

$$(11) \qquad \sum_k f(\varepsilon_{kn}/\sigma_n) \cdot (\varepsilon_{kn}/\sigma_n)^2 \lambda_{kn} \to f(0)$$

as $n \to \infty$, for all bounded continuous functions on R . So, assume (11) and let, for ε fixed,

$$f_\varepsilon(u) = \begin{cases} 1 & \text{if } |u| \geq \varepsilon \\ 0 & \text{if } u = 0 \\ \text{linear in } (-\varepsilon,\varepsilon) & \end{cases}$$

Then $\sum_{k:|\varepsilon_{kn}/\sigma_n| > \varepsilon} (\varepsilon_{kn}/\sigma_n)^2 \lambda_{kn} \leq \sum_k f_\varepsilon(\varepsilon_{kn}/\sigma_n) \cdot (\varepsilon_{kn}/\sigma_n)^2 \lambda_{kn} \to 0$, and (7) is proved.

Assume now (7) and let f be a bounded continuous function. So for every u, $|f(u)| \leq M$ and for every η there is an ε such that $f(0) - \eta < f(u) < f(0) + \eta$ if $u \in (-\varepsilon, \varepsilon)$. From (7) it follows

$$\left|\sum_k f(\varepsilon_{kn}/\sigma_n)(\varepsilon_{kn}/\sigma_n)^2 \lambda_{kn} - f(0)\right| \leq \eta \cdot \sum_{k:|\varepsilon_{kn}/\sigma_n| \leq \varepsilon} (\varepsilon_{kn}/\sigma_n)^2 \lambda_{kn} + $$
$$+ 2M \sum_{k:|\varepsilon_{kn}/\sigma_n| > \varepsilon} (\varepsilon_{kn}/\sigma_n)^2 \lambda_{kn} \to \eta$$

Since η is arbitrary small, it proves (11).

The convergence of finite dimensional distributions follows by independence of increments: firstly we have $(\overline{N}_n(t_1), \overline{N}_n(t_2) - \overline{N}_n(t_1)) \Rightarrow (W(t_1), W(t_2) - W(t_1))$ by one-dimensional result; then it follows $(\overline{N}_n(t_1), \overline{N}_n(t_2)) = (\overline{N}_n(t_1), \overline{N}_n(t_1) + (\overline{N}_n(t_2) - \overline{N}_n(t_1))) \Rightarrow (W(t_1), W(t_2))$, and for higher dimension by induction.

Finally, to show tightness, note that, for $t_1 \leq t \leq t_2$:

$$E((\overline{N}_n(t) - \overline{N}_n(t_1))^2(\overline{N}_n(t_2) - \overline{N}_n(t))^2) = E(\overline{N}_n(t) - \overline{N}_n(t_1))^2 E(\overline{N}_n(t_2) - \overline{N}_n(t))^2$$
$$= (t-t_1)(t_2-t) \leq (t_2-t_1)^2 ,$$

so by Theorem 15.6. in Billingsley [4], we have that $\overline{N}_n$ is tight and $\overline{N}_n \Rightarrow W$.

<u>Proof of Theorem 2</u>. Note that (9) and (10) together imply

$$(11) \qquad \sum_{k=1}^{p_n} \lambda_{kn}\, \varepsilon_{kn}^2 \to \lambda h^2$$

Condition (8) implies $G_n \Rightarrow G$; so we have to show $K_n \Rightarrow K$, i.e., for every bounded continuous f :

$$(12) \qquad \sum f(\varepsilon_{kn})\lambda_{kn}\, \varepsilon_{kn}^2 \to f(h)\lambda h^2$$

But (12) follows easily from (9) and (11). Assume now (12). Then (9) and (10) can be proved choosing appropriate f (see the proof of Theorem 1.) This proves one-dimensional convergence. The rest of the proof is same as in Theorem 1. Note that h is determined by (9), λ is determined by (10) and then a is determined by (8).

The origin of problems considered here is the following: In the absence of external stimulation, the membrane potential is in the form of impulses of small magnitude arriving at a large number of synaptic sites according to independent Poisson processes (see [10]), so it can be represented by $N_n(t)$. If conditions of Theorem 1 are satisfied, then the diffusion approximation is valid, and the membrane potential can be approximated by the solution $V(t)$ of stochastic differential equation.

(13) $$dV(t) = (-AV(t) + B)dt + \sigma dW(t) ,$$

where W is the standard Wiener process. Suppose that we want to generalize (13) to infinite dimensions. Then we replace A by a bounded operator; $N_n(t)$ becomes a process with independent increments in an infinitely dimensional space. Due to some technical consideration, we may wish to investigate the problem in the Schwartz space of distributions, or on a general multi-Hilbertian space. The convergence results in the previous chapter, the relation between ID laws and homogeneous processes with independent increments give all the necessary techniques needed to extend the example to infinite dimension in exactly similar manner. This was the main motivation for the previous results.

We now investigate purely mathematical questions related to the independent increment processes in the next chapter.

4. PROCESSES WITH INDEPENDENT INCREMENTS

4.1. Properties of paths

Let E be a countably Hilbert complete space.

Let (Ω, F, P) be a measure space, and let X_t, $0 \leq t \leq T$ be a random process with independent increments, defined on Ω and taking values in E'.

1. Definition. We say that X_t is a Lévy process if

(i) $t \to \langle X_t, x\rangle$ is continuous in probability for every x and

(ii) $X.(\omega) \in D([0,T], E')$ for almost every ω.

(iii) $X_0(\omega) = 0$ for almost every ω.

In what follows we shall consider Lévy processes only. Also, we shall assume that (ii) and (iii) hold for every ω, which is not a loss of generality.

From Definition 1, it follows that for each x, $\langle X_t, x\rangle$ is a real valued Lévy process, continuous in probability (see Ito [13] for the definitions on the real line).

By a jump (or a strong jump) of $X(\omega)$ at time t we mean the difference $X_t(\omega) - X_{t-}(\omega)$ if it is not zero.

By a weak jump of $X(\omega)$ at time t at the point x we mean $\langle X_t(\omega) - X_{t-}(\omega)$ if it is not zero.

Clearly, X has a strong jump at t if and only if X has at least one weak jump at t.

2. Lemma. If $x_s \to x$ in E' (as $s \to t$), then there is some m such that x and X_s belong to E'_m for s close enough to t, and $x_s \to x$

in E'_m .

<u>Proof</u>. Suppose not. Then, for $x_s \to x$, there exists a sequence x_{n_k} such that $x_{n_k} \in E'_{n_k}$ and $n_k \to \infty$. Then $x_{n_k} \to x$, and $x_{n_k} \in E'_{n_k}$ contradicts Theorem 34 in [8], for sequences.

3. <u>Lemma</u>. If $T < \infty$ and $X_t(\omega) \in D([0,t],E')$, then for every ω there is a norm $q \in \tau$, such that $X_t(\omega)$ belongs to E'_q for all t in $[0,T]$.

<u>Proof</u>. By the Definition 1, for every $t > 0$ we have

$$\lim_{\substack{s \to t \\ s<t}} X_s^{(\omega)} = X_{t-}^{(\omega)} .$$

Then by Lemma 2, there is an ε_t and a norm q_t such that for $s \in (t-\varepsilon_t, t)$ we have $X_s^{(\omega)} \in E'_{q_t}$. For $t = 0$ we have $X_0^{(\omega)} = 0$, and, by right continuity $X_s^{(\omega)} \to 0$ as $s \to 0$, $s > 0$. So, again by Lemma 1, there is an ε_0 such that $X_t^{(\omega)}$ belongs to some space Eq'_0 for all x in $[0, \varepsilon_0]$.

By compactness, there is a finite cover 0_i , $1 \leq i \leq n$, such that all points in 0_i belong to the space Eq'_k . But then all X_t for $t \in [0,T] = 0_1 \cup 0_2 \cup \ldots \cup 0_n$ belong to the space $Eq' = Eq' \cup \ldots \cup Eq'_n$.

4. <u>Lemma</u>. Suppose $T < \infty$ and suppose that, for every $x \in E$, the number of jumps of $\langle X_t^{(\omega)}, x \rangle$, $0 \leq t \leq T$ is finite. Then there is a constant M such that for every x , the function $\langle X_t^{(\omega)}, x \rangle$ has at most M jumps in $[0,T]$ (ω is fixed).

<u>Proof</u>. Let $f_T(x)$ be the number of jumps of $\langle X_t, x \rangle$ up to time T .

For each $x \in E$, $f_T(x)$ is finite. Let us now prove that f_T is lower semicontinuous, i.e., if $x_n \to x$ then

(1) $$\underline{\lim_{n\to\infty}} \; f_t(x_n) \geq f_t(x) \; .$$

It clearly suffices to show that, whenever $\langle X_t, x\rangle$ has a jump of size $> \eta$ then, starting with some n_o all functions $\langle X_t, x_n\rangle$ have a jump of size $> \eta$.

Suppose now that $\langle X, x\rangle$ has a jump of amount $> \eta$ at time t .

By assumption $\lim\limits_{\substack{s<t \\ s\to t}} X_s = X_{s-}$.

By Lemma 2, there is a norm q such that

(2) $$\tilde{q}(X_s - X_{t-}) < \varepsilon \quad \text{if} \;\; s \in [t-\varepsilon, t) \;\; \text{for some} \;\; \varepsilon \; .$$

Clearly, it suffices to show (1) for $q(x) < 1$ only. So, assume $x_n \to x$ and $q(x_n) < 1$ and $q(x) < 1$. Then we have from (2):

(3) $$|\langle X_s, x_n - X_{t-}, x_n\rangle| < \varepsilon \, , \quad s \in [t-\varepsilon, t) \; .$$

By τ-continuity of X_t and X_{t-} we have

(4) $$|\langle X_t, x_n\rangle - \langle X_t, x\rangle| < \varepsilon \quad \text{for} \;\; n \geq n_o$$

(5) $$|\langle X_{t-}, x_n\rangle - \langle X_{t-}, x\rangle| < \varepsilon \quad \text{for} \;\; n \geq n_o \; .$$

Then from (3), (4) and (5) it follows:

$$|\langle X_{t-}, x_n\rangle - \langle X_t, x_n\rangle| > \eta - 3\varepsilon \, ,$$

and the assertion (1) follows by arbitrarity of ε .

By Osgood's theorem [12], page 62, , there exists a τ-norm p such that

$$|f_t(x)| < M/2$$

for all x in $B_r(z) = \{x\colon p(z-x) \leq r\}$, for some $z \in E$ and $r > 0$.

The function $f_T(x)$ has the following properties:

(6) $\quad f_T(x + y) \leq f_T(x) + f_T(y)$

(7) $\quad f_T(cx) = f_T(x)$

which follow immediately from the definition of f_T . So, we deduce:

$$\begin{aligned} f_T(x) &= f_T((p^{-1}(x) \cdot rx + z - z) \cdot p(x) \cdot r^{-1}) \\ &= f_T(P^{-1}(x) \cdot rx + z - z) \\ &\leq f_T(P^{-1}(x) \cdot rx + z) + f_T(-z) \\ &\leq M . \end{aligned}$$

5. Lemma. If $T < \infty$, then X_t has only countably many jumps.

Proof. See Remark 3.3.

4.2. A decomposition of E'-valued processes.

Let E be a countably Hilbert complete space.

Let $E_0' = E' - \{0\}$. Denote by $\mathcal{C}$ the cylinder (= Borel) σ-algebra of E' , and by $\mathcal{B}$ the Borel σ-algebra of sets in $T_0 \times E_0'$, where T_0 will denote the interval $(0,T]$, for T finite or infinite.

Let $\mathcal{B}^*$ be the class of all sets A in $\mathcal{B}$ such that for some $a > 0$ and some $x_1, x_2, \ldots, x_k \in E'$ we have

(1) $\quad A \subset (0,a) \times \{F: |\langle F,x_1\rangle| > \frac{1}{a}, \ldots, |\langle F,x_k\rangle| > \frac{1}{a}\}$.

Let X_t be a process with independent increments.

Let, for an ω fixed, $I(\omega) = \{t: X_t \neq X_{t-}\}$ and $J(\omega) = \{(t, \Delta X_t), t \in I(\omega)\}$, where $\Delta X_t = X_t - X_{t-}$. Define the set function N by

$$N(A) = N(A, \omega) = \text{number of points in } A \cap J(\omega)$$

$N(A)$ is a finite random variable for $A \in \mathcal{B}^*$. The completed cylinder σ-algebra generated by $X_u - X_v$, $s \leq u$, $v \leq t$ will be denoted by $\mathcal{B}_{st}(X)$.

2. <u>Lemma</u>. If $A \in \mathcal{B}^*$ and $A \subset (s,t] \times E_0'$ then $N(A)$ is $\overline{\mathcal{B}}_{st}(X)$-measurable.

<u>Proof</u>. Let $E(s, t, x_1,\ldots,x_k, a_1,\ldots,a_k) = E(s, t, x, a) =$
$(s,t] \times \{F \in E' : \langle F,x_1\rangle > a_1 ,\ldots, \langle F,x_k\rangle > a_k\}$,
where $0 \leq s < t < T$, $a = (a_1,\ldots a_k) \in R^k$, $a_i > 0$ for $1 \leq i \leq k$, $x = (x_1,\ldots,x_k) \in E^k$.

Let Q be a countable dense subset of $(s,t]$, including t . We have:

$$\{N(E(s,t,x,a)) \geq 1\} = \{\text{for some } u \in (s,t],\ \langle X_u - X_{u-}, x_i\rangle > a_i ,\ i = 1 ,\ldots, k\}$$

$$= \bigcup_p \bigcap_q \bigcup_{\substack{r,r' \in Q \\ (s+1)/p \leq r < r' \leq t \\ r'-r<1/q}} \bigcap_{i=1}^{k} \{\langle X_{r'} - X_r, x_i\rangle \geq a + \tfrac{1}{p}\} \in \overline{\mathcal{B}}_{st}(X) .$$

Now we have

$$\{N(E(s,t,x,a)) \geq k + 1\} = \bigcup_{r\in Q \cap (s,t]} \{N(E(s,t,x,a)) \geq k\} \cap \{N(E(r,t,x,a)) \geq 1\} ,$$

so, by induction, $\{N(E(s,t,x,a)) \geq k\} \in \overline{\mathcal{B}}_{st}(X)$, thus $N(E(s,t,x,a))$ is $\overline{\mathcal{B}}_{st}(X)$-measurable.

Let $\mathcal{D}$ denote the class of all sets A in $\mathcal{B}$ such that $N(A \cap E(s,t,x,a))$ is $\overline{\mathcal{B}}_{st}(x)$-measurable. Since $N(A \cap E(x,t,x,a))$ is a bounded measure in A , it follows that $\mathcal{D}$ is a Dynkin class. By measurability of $N(E(s,t,x,a))$ we conclude that the class

$$\mathcal{M} = \{(s,\infty) \times \{F \in E' : \langle F,x_j\rangle \geq b_j\} ,\ j = 1 ,\ldots, n ,\ n \in N ,\ 0 \leq s < \infty ,\ -\infty < b_j < \infty\}$$

belongs to $\mathcal{D}$.

$\mathcal{M}$ is a multiplicative class that generates $\mathcal{B}$; so, by Dynkin's theorem $N(A \cap E(s,t,x,a))$ is $\overline{\mathcal{B}}_{st}(X)$-measurable for every A in $\mathcal{B}$.

Now if we define $E^{-}(s,t,x,a) = (s,t] \times \{F \in E' : \langle F,x_1\rangle \leq -a_1,\ldots,$ $\langle F,x_k\rangle \leq -a_k\}$ we similarly conclude that $N(A \cap E'(s,t,x,a))$ is $\overline{\mathcal{B}}_{st}(X)$-measurable, for every A in $\mathcal{B}$.

Finally, if $A \in \mathcal{B}^*$ and $A \subset (s,t] \times E_o'$, then for some a and x we have

$$A = A \cap E(s,t,x,a) \;\cup\; A \cap E^{-}(s,t,x,a) ,$$

so $\quad N(A) = N(A \cap E(s,t,x,a)) + N(A \cap E^{-}(s,t,x,a))$ is $\overline{\mathcal{B}}_{st}(X)$-measurable.

Let $\mathcal{B}_c \subset \mathcal{B}$ denote the algebra of all sets of the form: Borel set in $T_o \times$ cylinder set in E_o' .

3. Lemma. (i) For $A \in \mathcal{B}^*$, $N(A)$ is Poisson distributed with a finite parameter.

(ii) For $A \in \mathcal{B}_c$, $N(A)$ is either identically equal to ∞ (a.s.) or is Poisson distributed with finite parameter.

Proof. Let $A(t) = A \cap (0,t] \times E_o$, $A \in \mathcal{B}^*$.

Let $N(t) = N(t,\omega) = N(A(t),\omega)$.

Clearly, $N(t,\omega)$ is a right continuous step function in t , increasing, with jumps of amount 1. From Lemma 2 and

$$N(t) - N(s) = N(A(t) - A(s)) = N(A \cap (s,t] \times E_o') ,$$

it follows that $N(t)$ is a real Lévy process with independent increments.

For every t fixed we have

$$P(N(t) - N(t^-) \neq 0) \le P(X_t - X_{t^-} \neq 0) = 0 ,$$

so N is continuous in probability.

So, by results about real processes, $N(t)$ is a Poisson process.

For sufficiently large t , $A = A(t)$, $N(A) = N(A(t))$ and so $N(A)$ is Poisson distributed.

Let $A \in \mathcal{B}_c$. Then there is an increasing sequence $A_n \in \mathcal{B}^*$ such that $A_n \uparrow A$.

Then $N(A) = \lim_n N(A_n)$. $N(A_n)$ is Poisson random variable with expectation λ_n , say. λ_n is increasing sequence. If $\lim_{\to\infty} \lambda_n = \infty$, then for every k ,

$$P(N(A) \le k) \le P(N(A_n) \le k) = e^{-\lambda_n} \cdot \sum_{j=0}^{k} \lambda_n^j/j! \to 0 ,$$

so $N(A) = \infty$ almost surely.

If $\lim_n \lambda_n = \lambda < \infty$, then $N(A)$ is Poisson with parameter λ .

Recall that (Section 1), if X_t is an E'-valued Lévy process with independent increments, then for every x , $\langle X_t, x\rangle$ is a real Lévy process with independent increments. Let $\mathcal{B}_x(T_0 \times R_0)$ denote cylinder sets in $T_0 \times R_0$ that depend on x only. If $A \in \mathcal{B}_x(t_0 \times R_0)$ then the number of strong jumps of X_t that take place in A is equal to the number of jumps of the real process $\langle X_t, x\rangle$ that take place in A . Denoting by $N(x, A)$ the number of jumps of $\langle X_t, x\rangle$ in A we have the following:

4. <u>Lemma</u>. If $A \in \mathcal{B}_x(T_0 \times R_0)$, then $N(A) = N(x, A)$.

Let $n(A) = E(N(A,\omega))$. Rewriting the Lévy-Ito decomposition for real processes, and using Lemma 3, we have

5. <u>Theorem</u>. For every x in E $\langle X_t, x\rangle = Z_t(x) + Y_t(x)$, where $Z_t(x)$ is a Gaussian process with independent increments, and $Y_t(x)$ is given by

$$Y_t(x) = \lim_{k\to\infty} \Big(\iint\limits_{\substack{0<s\leq t \\ |\langle F,x\rangle| \geq \frac{1}{k}}} \langle F,x\rangle \, dN(s,F) + \iint\limits_{\substack{0<s\leq t \\ \frac{1}{k} \leq |\langle F,x\rangle| \leq 1}} \langle F,x\rangle \, dn\,(s,F)\Big)$$

6. <u>Remark</u>. The decomposition in Theorem 5 is not a decomposition in E', because the process Y_t, and consequently Z_t may not be linear in x. In the following section we shall prove the complete decomposition in a separable Hilbert space. The general problem remains open.

4.3. <u>A Lévy-Ito decomposition</u>

Let H be a real separable Hilbert space. We assume that X_t is a H-valued Lévy process with independent increments, and $t \in [0,T]$, where T is finite or infinite.

1. <u>Theorem</u>. Let $\eta > 0$. If $X_t \in D([0,T],H)$ then X has only finitely many jumps of the norm bigger than η on any finite subinterval of $[0,T]$.

<u>Proof</u>. Let us first prove the theorem on $[0,T]$, $T < \infty$. Let A be the class of all points in $[0,T]$ such that there are only finitely many jumps of the norm $> \eta$ in $[0,t)$. Let $s = \sup\{t : t \in A\}$. By continuity of X at 0, and by $X_o = 0$, there is a neighborhood $U = (0,\varepsilon)$ such that $\|X_t\| < \eta/2$ for $t \in U$. Then for $u,v \in U$, $\|X_u - X_v\| \leq \|X_u\| + \|X_v\| \leq \eta$, so there is no jump in U of size $> \eta$. Thus $s > 0$. Suppose $s < 1$. Then again, there is a neighborhood $(s, s+\varepsilon)$ in which X does not have jumps exceeding η in the norm; so $s = T$. The above proof goes through if $[0,T]$ is replaced by an arbitrary finite interval.

2. Theorem. If $X \in D([0,T], H)$, then X has only countably many jumps.

Proof. Immediate by Theorem 1.

3. Remark. If E is a countably Hilbert space, and X_t an E'=valued Lévy process, it is proved in Lemma 1.2. that for ω fixed, $X_t(\omega)$ belongs to a Hilbert space. So, Theorem 2 holds for E'.

4. Lemma. Let $F = \{\mathcal{B}_{st}, s<t, s,t \in T\}$ be an additive family of sub-σ-algebras on H, i.e.,

(i) $\mathcal{B}_{st} \vee \mathcal{B}_{tn} = \mathcal{B}_{sn}$ a.s.,

(ii) $\{\mathcal{B}_{t_0 t_1}, \mathcal{B}_{t_1 t_2}, \ldots, \mathcal{B}_{t_{n-1} t_n}\}$ is independent for $t_0 < t_1 \ldots < t_n$.

Let X_t be a Lévy process with values in H, Y_t a real valued Lévy process of Poisson type, such that with probability one there are no common jump times for X_t and Y_t. Assume that both X_t and Y_t are adapted to F, i.e., $X_t - X_s$ as well as $Y_t - Y_s$ is $\mathcal{B}_{st}$-measurable. Then X_t and Y_t are independent.

Proof. If there are no common jump points for X_t and Y_t, then for every x, there are no common jump points for real processes $\langle X_t, x\rangle$ and Y. Then by Ito's fundamental lemma $\langle X_t, x\rangle$ and Y_t are independent. By a trivial extension of this argument, all finite dimensional processes $(\langle X_t, x_1\rangle, \ldots, \langle X_t, x_k\rangle)$ are independent of Y_t, and this gives the result.

5. Lemma. A Lévy process X_t whose sample functions $X_t(\omega)$ are continuous a.s. is a Gaussian process.

Proof. By finite dimensional case, all finite-dimensional distributions of $X_t - X_s$ are Gaussian. Therefore, for every s,t, $X_t - X_s$ is a Gaussian process.

6. Lemma. If a Lévy process X_t is Gaussian, then its sample functions are continuous almost surely.

Proof. Let X_t be a Gaussian Lévy process. Then for every x, $<X_t, x>$ is a Gaussian process, so $<X_t, x>$ is a.s. continuous (null sets depend on t).

By separability of H, there is a countable dense set $D = \{x_1, x_2, \ldots\}$. Then there is $\Omega_1 \subset \Omega$, such that for every $\omega \in \Omega_1$ and every $x_i \in \mathcal{D}$, $<X_t(\omega), x_i>$ is continuous, and $P(\Omega_1) = 1$.

Now fix $\omega \in \Omega_1$. We want to show that for every $x \in H$, $<X_t(\omega), x>$ is continuous. Since X_t is a Lévy process, there is a neighborhood U of t such that $\|X_s(\omega)\| < M$ if $s \in U$, for some M. Then, let $\varepsilon > 0$ be given, and let $x_j \in \mathcal{D}$ be such that $\|x - x_j\| < \varepsilon$. Let $s \in U$ be such that $<X_s - X_t, x_j> \ < \ \varepsilon$.

Then

$$|<X_s - X_t, x>| \leq |<X_s - X_t, x_j>| + |<X_s - X_t, x - x_j>| \leq$$

$$\leq \varepsilon + 2M\varepsilon,$$

which shows continuity of $<X_t, x>$.

Now, since for every x, $<X_t(\omega), x>$ is continuous, it follows that $X_t(\omega)$ is continuous for all $\omega \in \Omega_1$.

7. Notation. $I(\omega) = \{t : X_t(\omega) \neq X_t - (\omega)\}$

$J(\omega) = \{(t, \Delta X_t(\omega)), \ t \in I(\omega)\}$

$\Delta X_t(\omega) = X_t(\omega) - X_{t-}(\omega)$

$T = T - \{0\}$; $H_o = H - \{0\}$.

$\mathcal{B} = \mathcal{B}(T_0 \times H_0)$ is the class of all Borel subsets of $T_0 \times H_0$.

$\mathcal{B}^* = \mathcal{B}^*(T_0 \times H_0)$ is the class of all sets $A \subset \mathcal{B}$ such that

$$A \subset (0,a) \times \{F \in H : \|F\| > \frac{1}{a}\} , \quad \text{for some } a > 0 .$$

By Theorem 1, for $A \subset \mathcal{B}^*$, $A \cap J(\omega)$ is a finite set. Let $N(A) = N(A,\omega)$ denote the number of points in $A \cap J(\omega)$

The proofs of the following two lemmas are almost identical to those of 2.2. and 2.3. and therefore are omitted.

8. Lemma. If $A \in \mathcal{B}^*(T_0 \times H_0)$ and $A \subset (s,t] \times H_0$, then $N(A)$ is $\overline{\mathcal{B}}_{st}(X)$ measurable.

9. Lemma. For $A \in \mathcal{B}(T_0 \times H_0)$, $N(A)$ is either Poisson distributed or identically $= \infty$.

If $A \in \mathcal{B}^*$, then $N(A)$ is Poisson with a finite parameter.

Now consider for every $x \in H$ and $A \in \mathcal{B}^*$:

$$\begin{aligned} S(A,x) = S(A,x,\omega) &= \sum_{(t,\Delta X_t) \in A} \{\langle X_t(\omega), x\rangle - \langle X_{t^-}(\omega), x\rangle\} \\ &= \sum_{(t,F) \in A \cap J(\omega)} \langle F,x\rangle \end{aligned}$$

Clearly, for $A \in \mathcal{B}^*$ we have

$$\begin{aligned} (1) \qquad S(A,x) &= \lim_{n\to\infty} \sum_k \frac{k}{n} N(A \cap \{F : \langle F,x\rangle \in (\frac{k-1}{n}, \frac{k}{n}]\}) \\ &= \int\int_{(t,F) \in A} \langle F,x\rangle \, dN(t,F) \end{aligned}$$

Define also $S(A) = S(A,\omega) = \sum_{(t,\Delta X_t \in A} \Delta X_t$

This sum is finite, so we have $S(A,x) = \langle S(A), x\rangle$.

10. Lemma. Lemma 8 holds for $S(A,x)$ and $S(A)$ in place of $N(A)$.

Proof. For $S(A,x)$ clear from relation (1). Then for $S(A)$ it follows from

$$S(A) = \sum_{i=1}^{\infty} \langle S(A), e_i\rangle e_i$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis in H.

Set now $A(t) = A \cap [0,t] \times H_o$, $A \in \mathcal{B}^*$

$$H(t,A) = X_t - S(A(t)) .$$

11. Lemma. $H(t,A)$ is a Lévy process independent of the process $N(A(t))$.

Proof. Immediate, by Lemmas 4, 8 and 9. □

The following three lemmas can be proved the same way as for real processes (see [13], Section 1.6.5.

12. Lemma. Let $A_1, \ldots, A_n \in \mathcal{B}^*$ be disjoint. Then the following processes are independent:

$$N(A_1(t)), \ldots, N(A_n(t)), \quad H(t, \bigcup_1^n A_i) .$$

13. Lemma. Lemma 12 holds for S in place of N.

14. Lemma. Let $A_1, \ldots, A_n$ be disjoint sets in $\mathcal{B}$. Then $N(A_1), \ldots, N(A_n)$ are independent.

Now for $A \in \mathcal{B}^*$ set

$$A_{mk}(x) = \{(s,F) \in A \cap \{F : \langle F,x\rangle \in [\tfrac{k}{m}, \tfrac{k+1}{m}]\}$$

Then

$$\langle S(A),x\rangle = \int\int_A \langle F,x\rangle \, dN(s,F) = \lim_{m\to\infty} \sum_k \frac{k}{m} N(A_{mk}(x)) \tag{2}$$

Let $EN(A) = n(A)$.

15. Lemma. If $A \in \mathcal{B}^*$, then

$$E\, e^{i\langle S(A),x\rangle} = \exp \int\!\!\int_A (e^{i\langle F,x\rangle} - 1)\,dn(s,F)$$

Proof.

For each m,k , $N(A_{mk}(x))$ is Poisson distributed with parameter $n(A_{mk}(x))$. So,

$$E \exp \frac{ik}{m}\; N(A_{mk}(x)) = \exp\left(\left(\frac{ik}{m} - 1\right) n(A_{mk}(x))\right)$$

By relation (2),

$$\begin{aligned} E \exp i\langle S(A),x\rangle &= \lim_{\to\infty} E \exp \left(i \sum_k \frac{k}{m} N(A_{mk}(x))\right) \\ &= \lim_{m\to\infty} \prod_k E \exp \left(\frac{ik}{m} N(A_{mk}(x))\right) \\ &= \lim_{m\to\infty} \prod_k \exp\left(\left(e^{\frac{ik}{m}} - 1\right) n(A_{mk}(x))\right) \\ &= \lim_{m\to\infty} \exp\left(\sum_k \left(e^{\frac{ik}{m}} - 1\right) n(A_{mk}(x))\right) \\ & \exp \int\!\!\int_A (e^{i\langle F,x\rangle} - 1)\, dn(s,F) \;. \end{aligned}$$

From the above we obtain

16. Lemma. If $A \in \mathcal{B}^*$ is included in $\{(s,F) : \|F\| < m\}$ for some $m < \infty$, then

$$\begin{aligned} E \langle S(A),x\rangle &= \int\!\!\int_A \langle F,x\rangle\, dn(s,F) \\ \operatorname{Var} \langle S(A),x\rangle &= \int\!\!\int_A \langle F,x\rangle^2\, dn(s,F) \end{aligned}$$

17. Remark. If $(0,t] \times \{F : \|F\| > \varepsilon\}$ $(\varepsilon > 0)$ belongs to $\mathcal{B}^*(T_0 \times H_0)$, we have

$$\int\!\!\int_{\substack{0<s\le t \\ \|F\|>\varepsilon}} dn(s,F) < \infty ,$$

because N on sets in $\mathcal{B}^*$ is Poisson with finite parameter n .

18. <u>Lemma</u>. For every $x \in H$,

(i) $$\int\int_{\substack{0<s\leq t\\ \|F\|\leq 1}} \langle F,x\rangle^2 \, dn(s,F) < \infty$$

(ii) $$\int\int_{\substack{0<s\leq t\\ \|F\|\leq 1}} \|F\|^2 \, dn(s,F) < \infty .$$

<u>Proof</u>. (i) Let $E_\varepsilon = \{(s,F) : 0 < s \leq t \,,\ \varepsilon < \|F\| < 1\}$, $\varepsilon > 0$. Then $E_\varepsilon \in \mathcal{B}^*$ and, by Lemma 15,

$$E\, e^{i\langle S(E_\varepsilon),x\rangle} = \exp \int\int_{\substack{0<s\leq t\\ \varepsilon<\|F\|\leq 1}} (e^{i\langle F,x\rangle} - 1) \, dn(s,F) .$$

By Lemma 13, the processes $S(E_\varepsilon(s))$ and $X_s - S(E_\varepsilon(s))$ are independent. Thus, $S(E_\varepsilon)$ and $X_t - S(E_\varepsilon)$ are independent, and

$$E\, e^{i\langle X_t,x\rangle} = E\, e^{i\langle S(E_\varepsilon),x\rangle} \cdot E\, e^{i\langle X_t - S(E_\varepsilon),x\rangle}$$

So, we have

$$|E\, e^{i\langle X_t,x\rangle}| \leq |E\, e^{i\langle S(E_\varepsilon),x\rangle}|$$

$$= \exp \int\int_{\substack{0<s<t\\ \varepsilon<\|F\|\leq 1}} (\cos \langle F,x\rangle - 1) \, dn(s,F)$$

If $\|x\| \leq 1$ and $\|F\| \leq 1$, then $|\langle F,x\rangle| \leq 1$; in that case we have $\cos \langle F,x\rangle \leq 1 - \langle F,x\rangle^2/4$. So, for every x such that $\|x\| \leq 1$:

$$|E\, e^{i\langle X_t,x\rangle}| \leq \exp \{ - \int\int_{\substack{0<s\leq t\\ \varepsilon<\|F\|\leq 1}} \langle F,x\rangle^2 \, dn(s,F)\}$$

Letting $\varepsilon \downarrow 0$, taking logarithm and dividing by $\|x\|$, we have that for every $x \in H$:

(3) $$\|x\|^{-2} \int\int_{\substack{0<s\leq t\\ \|F\|\leq 1}} \langle F,x\rangle^2 \, dn(s,F) \leq -\log |E\, e^{i\langle X_t,x\rangle/\|x\|}|$$

$$< \infty ,$$

where the last inequality follows from the fact (one-dimensional) that characteristic function of $\langle X_t, x\rangle$, being an ID, never vanishes. This proves (i).

The proof of (ii) will be given in Theorem 22.

19. <u>Notations and some facts</u>.

Set

$$S_1(t,x) = S_1(t,x,\omega) = \int\int_{\substack{0<s\leq t \\ \|F\|\geq 1}} \langle F,x\rangle \, dN(s,F)$$

$$= S((0,t] \times \{F : \|F\| \geq 1\} , x)$$

$$= \sum_{\substack{0<s\leq t \\ \|\Delta X_s\|\geq 1}} \langle X_s - X_{s-}, x\rangle .$$

$$S_k(t,x) = S_k(t,x,\omega) = \int\int_{\substack{0<s\leq t \\ \frac{1}{k}\leq \|F\|<\frac{1}{k-1}}} \langle F,x\rangle \, dN(s,F) , \quad k > 1 .$$

Define $S_1(t) = \sum_{\substack{0<s\leq t \\ \|\Delta X_s\|\geq 1}} \Delta X_t$, and $S_k(t)$ analogously.

Clearly, for $k \geq 1$, $S_k(t) \in H$ and $\langle S_k(t), x\rangle = S_k(t,x)$.

Define also

$$T_k(t,x) = T_k(t,x,\omega) = S_1(t,\omega,x) + \sum_{j=2}^{k} S_j(t,x,\omega) - ES_j(t,x,\omega)$$

$$= \int\int_{\substack{0<s\leq t \\ \|F\|>1/k}} \langle F,x\rangle \, dN(s,F) - \int\int_{\substack{0<s\leq t \\ \frac{1}{k}\leq \|F\|\leq 1}} \langle F,x\rangle \, dn(s,F) .$$

Using previous results, it can be shown that $S_n(t)$, $n = 1, 2, \ldots$, are independent Lévy processes. By Lemma 18(i), $E(S_k(t,x))$ is continuous in t ; so $T_k(t,x)$ is a Lévy process for every k and x .

Using Komogorov's inequality (see [13] Section 1.7.2.) we have the following

20. Theorem. For every $x \in H$, we have the following decomposition:

$$\langle X_t(\omega),x\rangle = Z_t(x,\omega) + \lim_{k\to\infty} \Big(\iint_{\substack{0<s\leq t\\ \|F\|\geq 1/k}} \langle F,x\rangle \, dN(s,F) \tag{4}$$

$$+ \iint_{\substack{0<s\leq t\\ \frac{1}{k}\leq \|F\| \leq 1}} \langle F,x\rangle \, dn(s,F) \Big) ,$$

where, for every x, $Z_t(x)$ is a continuous Lévy process (thus Gaussian), and independent on N.

Now we can prove the following theorem on a decomposition in Hilbert space.

21. Theorem. Let X_t be a Hilbert space-valued Lévy process. Let $N(A,\omega)$ be the number of jumps of $X_t(\omega)$ in A, and let $n(A) = EN(A)$. Then N and n are measures on H and we have:

$$X_t = Z_t + Y_t \tag{5}$$

where Z_t is a Gaussian random process in H, Y_t is an H-valued random process, independent on Z_t, defined by

$$\langle Y_t(\omega),x\rangle = \lim_{k\to\infty} \Big\{ \iint_{\substack{0<s\leq t\\ \|F\|\geq 1/k}} \langle F,x\rangle \, dN(s,F) + \iint_{\substack{0<s\leq t\\ \frac{1}{k}\leq \|F\| \leq 1}} \langle F,x\rangle \, dn(s,F)\Big\}$$

Proof. For ω and k fixed, the term after the "lim" in (4) is linear in x. Notice that

$$\iint_{\substack{0<s\leq t\\ \|F\|\geq 1/k}} \langle F,x\rangle \, dN(s,F)$$

is continuous in x, as a finite sum of continuous terms.

For the other term, we have

$$\Big| \int\int_{\substack{0<s\le t\\ \frac{1}{k}\le\|F\|\le 1}} \langle F,x\rangle\, dn(s,F)\Big| \le \|x\| \cdot \int\int_{\substack{0<s\le t\\ \frac{1}{k}\le\|F\|\le 1}} \|F\|\, dn(s,F)$$

$$\le \|x\| \cdot n((0,t] \times \{F : \tfrac{1}{k} \le \|F\| \le 1) ,$$

so this is also continuous in x .

Then, by Banach-Steinhaus theorem, the limit in (4) defines a continuous linear operator in H , call it $Y_t(\omega)$.

Thus, $Z_t = X_t - Y_t$ is also in H , and, since $\langle Z_t,x\rangle$ is continuous in t for every x , it foolows that Z_t is also continuous. Then, by Lemma 5, Z_t is Gaussian on H . Z_t and N are independent by Lemma 4.

For a Lévy process X_t decomposed as in previous theorem, we define

$$M_t(x) = E\langle Z_t,x\rangle$$

$$V_t(x) = \mathrm{Var}\,\langle Z_t,x\rangle ,$$

where Z_t is Gaussian process defined by (5).

For fixed t , M_t is linear and continuous in x ; V_t is an $I(\|\cdot\|)$-seminorm.

The functions M_t, V_t, and the measure $dn(s,F)$ are called (following Ito), the three components of X_t .

22. Theorem. Let X_t be a Lévy process with three components M_t, V_t, n . Then the characteristic function of $X_{t_2} - X_{t_1}$, $t_2 > t_1$ is given by

$$(6)\qquad f_{t_1,t_2}(x) = \exp\{ i \langle M_{t_2} - M_{t_1}, x\rangle - \tfrac{1}{2}(V_{t_2}(x) - V_{t_1}(x)) + \int\int_{\substack{t_1<s<t_2\\ F\neq 0}} (e^{i\langle F,x\rangle} - 1 - i\langle F,x\rangle \cdot 1[\|F\|\le 1])\, dn(s,F)\}$$

and is subject to an infinitely divisible distributions. Also,

(7) $$\int\int_{\substack{0<s\leq t \\ \|F\|\leq 1}} \|F\|^2 \, dn(s,F) < \infty .$$

Proof. It suffices to prove (6) for $t_2 = t$, $t_1 = 0$.

By Theorem 20, we have

$$\langle X_t, x\rangle = \langle Z_t, x\rangle + \lim_{k\to\infty} T_k(t,x) ,$$

and these two terms are independent. So,

$$\begin{aligned} E\, e^{i\langle X_t, x\rangle} &= E\, e^{i\langle Z_t, x\rangle} \cdot E \lim_{k\to\infty} e^{i\, T_k(t,x)} \\ &= \exp\left(i\langle m_t, x\rangle - \frac{V(t)}{2}\right) \cdot \lim_{n\to\infty} E\, e^{i\, T_k(t,x)} \end{aligned}$$

By previous lemmas we have

$$E\, e^{i\langle T_k(t,x)\rangle} = \exp\left\{ \int\int_{\substack{0<s\leq t \\ \|F\|\geq 1/k}} \left(e^{i\langle F,x\rangle} \cdot 1\{\|F\|\leq 1\}\right) dn(s,F)\right\}$$

which proves (6).

To prove that this is an infinite divisible characteristic function, notice that in the finite-dimensional case we have (ii) of Lemma 18 trivially satisfied, and therefore, by results in Chapter 2, (6) is an ID characteristic function. By Mandrekar-Zinn theorem 2.8. in [18], we conclude that (6) is an ID characteristic functional of a probability measure in H . By Theorem 2.3.14, we have (7).

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