# ON PROPERTIES OF A CLASS OF SYSTEMS OF DIFFERENTIAL EQUATIONS AND CORRESPONDING HIGHER-ORDER DIFFERENTIAL EQUATIONS

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

Robert Hampton Rogers

1963

This is to certify that the

thesis entitled

ON PROPERTIES OF A CLASS OF SYSTEMS OF DIFFERENTIAL EQUATIONS AND CORRESPONDING HIGHER-ORDER DIFFERENTIAL EQUATIONS

presented by

Robert Hamptom Rogers

has been accepted towards fulfillment of the requirements for

Ph.D. degree in Elect. Engrg.

D. P. Brown
Major professor

Date August 8, 1963

O-169

## ABSTRACT

# ON PROPERTIES OF A CLASS OF SYSTEMS OF DIFFERENTIAL EQUATIONS AND CORRESPONDING HIGHER-ORDER DIFFERENTIAL EQUATIONS

by Robert Hampton Rogers

Procedures have recently been presented for formulating time-domain models of linear and nonlinear systems in the form

$$\frac{d}{dt}x_i = f_i\,(x_1, \ldots, x_n\,;\, t)\,, \quad i = 1, 2, \ldots, n.$$

This set of n differential equations is referred to as the normal system or state model. Procedures for determining the solution of the system and hence analyzing the system performance presently employ analog computers, digital computers, and/or functions of matrices. The choice of mathematical procedures to apply in the design of a particular physical system varies from problem to problem. However, in all cases the objective is to gain information or knowledge pertaining to the inter-relationship of the system parameters to the system performance.

A knowledge of this inter-relationship is obtained in the thesis by formally developing the mathematical properties which relate the parameters in the normal system of linear and a class of nonlinear differential equations to the parameters in an r-order ($r \leq n$) differential equation.

The mathematical foundation of the thesis is established by developing the mathematical properties which relate the solution of the normal system to the solution of an r-order ($r \leq n$) differential equation obtained from the system. This development is based on deriving the r-order equation from the normal system by means of a certain nonsingular transformation. In this development, conditions on the parameters of the normal system are determined so that an n-order differential equation is obtained from the system. In the proof of these results a technique for formulating a nonsingular transformation is given which allows the determination of the solution of the normal system in terms of the solution of an n-order differential equation.

The mathematical properties developed in the thesis are applied in the formulation of methods for the design of physical systems. The design methods necessitate constructing: (1) a function $y(t)$ from the specification of a desired system performance and (2) a normal system of differential equations having the function as a component of the system solution.

Two methods of constructing the linear system, to have a specified solution, are given. One method consists of determining the coefficients and initial conditions of the normal system in terms of the coefficients and initial conditions of an n-order differential equation. A second method relates the coefficient matrix in the normal system directly to the specified solution $y(t)$ by means of a certain matrix transformation. If the normal system is nonhomogeneous then an explicit formula is given for determining the nonhomogeneous part of the system in terms of the specified solution $y(t)$. Similar results are developed for constructing a special class of nonlinear differential equations having a specified solution.

The design methods, proposed in the thesis, are illustrated in the design of amplifiers and oscillators in the time-domain.

# ON PROPERTIES OF A CLASS OF SYSTEMS OF DIFFERENTIAL EQUATIONS AND CORRESPONDING HIGHER-ORDER DIFFERENTIAL EQUATIONS

By

Robert Hampton Rogers

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering

1963

G29120
7/8/64

## ACKNOWLEDGEMENT

The author wishes to express his appreciation to Dr. Dave P. Brown for his advice and guidance in preparing this thesis. He also wishes to thank Dr. William A. Blackwell for his patience in its editing.

Finally, the writer is indebted to his wife, Sandra and family for their understanding, patience and moral support during graduate study.

**********

TABLE OF CONTENTS

## LIST OF APPENDICES

# I. INTRODUCTION

The design of a physical system to meet a given performance specification necessitates decisions based on a knowledge of the inter-relationship of the system components (parameters) to the system performance. A major portion of any design procedure is devoted to the process of relating the system parameters to the system performance.

Brown[1], Wirth[2], Koenig, Tokad, and Kesavan[3] have recently presented methods of formulating the mathematical model of a physical system into the form

$$\frac{dx_i}{dt} = f_i\,(x_1, \ldots, x_n\,; t)\;, \quad i = 1, 2, \ldots, n. \tag{1.1}$$

The set of differential equations (1.1) is commonly referred to as the normal system, the linear form of this system will be denoted by

$$\dot{X} = AX + Q(t) \tag{1.2}$$

where $X' = [x_1, x_2, \ldots, x_n]$ , $A = [a_{ij}]$ a square matrix of order n, $Q'(t) = [q_1(t), q_2(t), \ldots, q_n(t)]$ and the prime denotes the transpose.

One method of approaching the design problem, when presented with a mathematical model of the system in the normal form, is to view the system performance as one specified component $x_i(t)$ of the vector solution. Then relate the parameters in the normal system to the system performance $x_i(t)$. One technique of obtaining these parameter-solution relationships (design equations) is to (1) convert the normal system into a higher-order differential equation in $x_i$ by a reduction method such as the one proposed by Murray and Miller [4, p. 126], and (2) determine the inter-relationship of the system performance $x_i$ to the system parameters in the higher-order differential equation. An obstacle to obtaining the desired parameter-solution relationships by this technique is that the reduction method may convert the normal system into a differential equation of r-order, where the number r is less than the number of equations in the normal system, i.e. $r \leq n$. Moulton [5, p. 9] presents, as an example, a normal system of three equations which converts into a second-order equation in any component, $x_i$. In seeking these parameter-solution relationships, many questions have arisen concerning the mathematical properties which relate the normal system (1.1) to an r-order ($r \leq n$) differential equation

$$\frac{d^r y}{dt^r} = f(y, \frac{dy}{dt}, \ . \ . \ . \ , \frac{d^{r-1}}{dt^{r-1}} y \ ; \ t) \qquad (1.3)$$

obtained from the system.

The objective of this thesis is to develop the mathematical properties which relate the normal system (1.1) to an r-order ($r \leq n$) differential equation (1.3). The mathematical properties, thus obtained, are to afford new tools for the design of electrical networks.

When the mathematical model has the linear form (1.2) certain questions arise as to how the solution of the normal system is related to the solution of the r-order ($r \leq n$) equation (1.3) obtained from the system.

Coddington and Levinson [6, p. 21] as well as others [7, p. 33] have applied the nonsingular transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{bmatrix} \qquad (1.4)$$

to the normal system (1.2), where matrix A has the special form of a companion matrix (2.2.4) and $Q'(t) = [0, 0, \ldots, 0, q_n(t)]$ to obtain an n-order differential equation. The n-order equation obtained from the system by the transformation (1.4) is referred to by Coddington and Levinson as "associated" with the system. The nonsingular transformation (1.4) thus specifies the manner in which the solution of the normal system and the solution of the n-order differential equation associated with the system are related. A generalization of this concept of the associated differential equation is required in Chapter II (homogeneous) and Chapter III (nonhomogeneous) to develop the general mathematical properties which link the solution of the normal system (1.2) to the solution of the r-order differential equation obtained from the system.

The later parts of Chapters II and III develop techniques for relating the parameters in the system (1.2) to one specified component $x_i(t)$ of the vector X(t) solution. The parameter-solution relationships developed in these sections have not been formally investigated and made available to the network designer until now. These relationships, as shown in Chapter V, offer new tools for the design of electrical networks.

In Chapter IV mathematical relationships are developed which relate the solution of a class of nonlinear systems (1.1) to a solution of an n-order differential equation (1.3), obtained from the system. A portion of the results obtained in Chapter IV are applied to the design of tunnel-diode amplifiers and oscillators in Chapter V. The design equations obtained in Chapter V are shown to be a generalization of results obtained by Kim [8, p.416] by a different method.

# II. PROPERTIES OF SYSTEMS OF HOMOGENEOUS DIFFERENTIAL EQUATIONS AND ASSOCIATED HIGHER ORDER DIFFERENTIAL EQUATIONS

## 2.1 Introduction

The general mathematical properties that relate the solution of the normal system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots\cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where the entries are constants or in symbolic form

$$\dot{X} = AX \qquad (2.1.1)^*$$

to the solution of the r-order ($r \leq n$) homogeneous differential equation

$$\frac{d^r y}{dt^r} = \sum_{i=1}^{r} a_i \frac{d^{r-i} y}{dt^{r-i}} \qquad (2.1.2)$$

obtained from the system are developed as a result of deriving the r-order equation from the system by means of the transformation

$$X = C^{-1} Y \qquad (2.1.3)$$

where $X' = [x_1, x_2, \ldots, x_n]$, $Y' = [0, 0, \ldots, 0, y, y^{(1)}, \ldots, y^{(r-1)}, 0, 0, \ldots 0]$.

---

* The number indicates the section and the number of the equation in the section. Thus (2.1.1) indicates Sec. 2.1, Eq. 1.

and the -1 superscript indicates the inverse of matrix C.

In the investigation of the general solution properties a number of problems have arisen for which the solution will be of interest to the system designer.

System design methods are proposed in Chapter V that necessitate constructing: (1) a function y(t) from the specification of a desired system performance and (2) a normal system (2.1.1) having the function y(t) as a component of the system solution, X(t).

The construction of the normal system (2.1.1) to have the component y(t) as a solution necessitates a knowledge of how the component y(t) is related to : (1) the solution of the normal system X(t) and (2) the $a_{ij}$ entries in matrix A in the normal system.

It is shown in the development of this chapter that these parameter-solution relationships can be obtained by viewing the function y(t) as the solution of the r-order differential equation (2.1.2).

In Section 2.2, the solution of the normal system is shown to be related to the solution of the r-order differential equation by the transformation (2.1.3). The formal development of this transformation allows the system designer to establish the initial conditions on the normal system in terms of the specified component y(t). In this development it is shown that y(t) is the component of the system solution denoted by $x_n(t)$.

The problem of relating the $a_{ij}$ entries in the matrix A in the normal system (2.1.1) to the solution y(t) is developed in

Section 2.3. One method requires constructing the r-order differential equation to have the specified solution y(t), and then relating the coefficients $a_j$ $j = 1, 2, \ldots, r$ of the r-order differential equation to the entries $a_{ij}$ in the normal system. A second method relates the coefficient matrix A in the normal system (2.1.1) directly to the specified solution y(t) by means of a certain matrix transformation.

The parameter-solution relationships formally developed in this section are referred to as design equations when applied to system design in Chap. V.

## 2.2 Systems of First Order Homogeneous Differential Equations and Associated Higher Order Differential Equations

In this section it is shown that when the r-order equation (2.1.2) is obtained from the normal system (2.1.1) by a nonsingular transformation of the form (2.1.3), that the solution of the system and the solution of the r-order equation are linked by two well-defined properties.

Property (a) The solution of the r-order equation is given by r entries in some row of matrix triple product $CF_mD$ when $C, F_m$ and D are nonsingular and each column of $F_m$ is a solution of the normal system.

Property (b) A solution of the system is given by $C^{-1}Y$ where Y is a vector containing the solution and derivatives of the solution of the r-order differential equation.

The r-order equation defined by Properties (a) and (b) is said to be "associated" with the system. These properties are stated mathematically by the following definition:

Definition 2.2.1: Consider the homogeneous system of differential equations:

$$\dot{X} = AX \tag{2.2.1}$$

where $A = [a_{ij}]$, $X' = [x_1, x_2, \ldots, x_n]$.

An r-order homogeneous differential equation

$$\frac{d^r y}{dt^r} = \sum_{i=1}^{r} a_i \frac{d^{r-i} y}{dt^{r-i}} \tag{2.2.2}$$

is associated with (2.2.1) if,

(a) for $F_m$ a fundamental matrix* of (2.2.1), there exists non-singular matrices C and D such that the $(i,i)$, $(i, i+1), \ldots,$ $(i, i+r-1)$ entries in $CF_mD$, for some i, $i = 1, 2, \ldots, n$, are a fundamental set of (2.2.2) and

(b) for $y_1, y_2, \ldots, y_r$ a fundamental set of (2.2.2), the vectors $C^{-1}Y_j$ where $Y_j' = [0, 0, \ldots, 0, y_j, y_j^{(1)}, \ldots, y_j^{(r-1)} 0, 0, \ldots, 0]$, $j = 1, 2, \ldots, r$, are r linearly inuependent solution on the open interval I defined by $I = [t : t_1 < t < t_2]$, $t_1$ and $t_2$ are constants, of (2.2.1).

---

* For the definitions of terminology and notations used throughout, see Appendix A.

The mathematical properties specified in Def. 2.2.1 are first encountered in Thm. 2.2.1 when a transformation of the form (2.1.3) is applied to a normal system (2.2.1) to obtain a set of s differential equations, $1 \leq s \leq n$, of the form (2.2.2). This set of s, higher-order differential equations, are ascertained to be associated with the system.

Before proving the existence of a set of s higher-order differential equations of the form (2.2.2) associated with the system (2.2.1), some basic relationships must be established between a fundamental matrix of the system (2.2.1) and a fundamental matrix of the transformed system (2.2.3). Lemma 2.2.1 follows immediately from results established in Thm. 2.2 [6, p. 69].

<u>Lemma 2.2.1</u>: If $F_m$ is a fundamental matrix of the system (2.2.1) and $Y = CX$, where $C^{-1}$ exists, then $CF_m$ is a fundamental matrix of

$$\dot{Y} = CAC^{-1}Y \tag{2.2.3}$$

<u>Proof</u>: Let $f_j$, $j = 1, 2, \ldots, n$ be the j-column of $F_m$. Since $f_j$ is a solution on I of (2.2.1) and (2.2.3) is obtained by substituting $X = C^{-1}Y$ in (2.2.1), $Cf_j$ is a solution of (2.2.3) on I. Since C and $F_m$ are nonsingular by hypothesis, $CF_m$ is nonsingular. The lemma follows.

Lemma 2.2.2: Suppose $CAC^{-1}$ of Lemma 2.2.1 is

$$CAC^{-1} = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_s \end{bmatrix}$$

where $B_i$ is of order $r_i$, $i = 1, 2, \ldots, s$. There exists a nonsingular matrix D such that a fundamental matrix of $\dot{Y}_i = B_i Y_i$, $i = 1, 2, \ldots, s$ where $Y' = [Y_1', Y_2', \ldots, Y_s']$, is the submatrix of $CF_m D$ consisting of the entries in rows $\sum_{j=1}^{i-1} r_j + 1$, $\sum_{j=1}^{i-1} r_j + 2, \ldots, \sum_{j=1}^{i} r_j$ and columns $\sum_{j=1}^{i-1} r_j + 1$, $\sum_{j=1}^{i-1} r_j + 2, \ldots, \sum_{j=1}^{i} r_j$.

Proof: If $G_i$ is a fundamental matrix of $\dot{Y}_i = B_i Y_i$, $i = 1, 2, \ldots, s$, then a fundamental matrix of (2.2.3) is

$$G_{1s} = \begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & G_s \end{bmatrix}$$

By Lemma 2.2.1 $CF_m$ is a fundamental matrix of (2.2.3). Therefore, by Thm. A.1, there exists a nonsingular matrix D such that

$$G_{1s} = CF_m D$$

Since the entries in $G_i$ are the entries in rows $\sum_{j=1}^{i-1} r_j + 1$, $\sum_{j=1}^{i-1} r_j + 2, \ldots, \sum_{j=1}^{i} r_j$ and columns $\sum_{j=1}^{i-1} r_j + 1$, $\sum_{j=1}^{i-1} r_j + 2, \ldots, \sum_{j=1}^{i} r_j$ of $CF_m D$, the lemma follows.

Theorem 2.2.1: There exists a set of $s$, homogeneous differential equations, $1 \leq s \leq n$ of order $r_i$, $i = 1, 2, \ldots, s$, associated with the system (2.2.1) such that $\sum_{i=1}^{s} r_i = n$.

Proof: Substitute $X = C^{-1}Y$ in (2.2.1) so that (2.2.3) results where

$$CAC^{-1} = \begin{bmatrix} B_1 & 0 & \ldots & 0 \\ 0 & B_2 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & B_s \end{bmatrix}$$

and

$$B_i = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 \\ a_{r_i r_i} & a_{r_i r_i - 1} & a_{r_i r_i - 2} & \ldots & a_{r_i 1} \end{bmatrix}$$

It has been shown [9, p.49] that such a transformation exists.

For any subset of equations of (2.2.3) of the form

$$\dot{Y}_i = B_i Y_i \tag{2.2.4}$$

where $Y_i' = [y_{i1}, y_{i2}, \ldots, y_{ir_i}]$, calculate $r_i - 1$ successive derivatives of the first row of (2.2.4) eliminating each time from the right hand side the first derivatives of $y_{i2}, y_{i3}, \ldots, y_{ir_i}$ by means of the last $r_i - 1$ equations of (2.2.4). This process results in

$$\begin{bmatrix} y_{i1} \\ y_{i1}^{(1)} \\ \cdot \\ \vdots \\ y_{i1}^{(r_i-1)} \end{bmatrix} = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \cdot \\ \vdots \\ y_{ir_i} \end{bmatrix} \tag{2.2.5}$$

Substituting (2.2.5) into (2.2.4) gives,

$$\begin{bmatrix} y_{i1}^{(1)} \\ y_{i1}^{(2)} \\ \vdots \\ y_{i1}^{(r_i-1)} \\ y_{11}^{(r_i)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots\cdots\cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ a_{r_ir_i} & a_{r_ir_i-1} & a_{r_ir_i-2} & \cdots & a_{r_i1} \end{bmatrix} \begin{bmatrix} y_{i1} \\ y_{i1}^{(1)} \\ y_{i1}^{(2)} \\ \vdots \\ y_{i1}^{(r_i-1)} \end{bmatrix} \tag{2.2.6}$$

The last row of (2.2.6)

$$\frac{d^{r_i} y_{i1}}{dt^{r_i}} = \sum_{j=1}^{r_i} a_{r_ij} \frac{d^{r_i-j} y_{i1}}{dt^{r_i-j}} \tag{2.2.7}$$

is associated with (2.2.1). For if $F_m$ is a fundamental matrix of (2.2.1), by Lemmas 2.2.1 and 2.2.2 and the initial part of the proof, there exists nonsingular matrices C and D such that a submatrix of $CF_mD$, $G_i$, (Lemma 2.2.2) is a fundamental matrix of (2.2.4). By (2.2.5) and (2.2.6) each entry in row one of $G_i$ is a solution on I of (2.2.7) and the j-row of $G_i$ is the j-1 derivative of the first row for $j = 1, 2, \ldots, r_i$. From Thm. A.2, since the determinant of $G_i$ is the Wronskian of solutions on I of (2.2.7) and since $G_i$ is nonsingular, the entries in row one of $G_i$ are a fundamental set of (2.2.7).

If $y_1, y_2, \ldots, y_{r_i}$ is a fundamental set of (2.2.7), by (2.2.6) and (2.2.5), $r_i$ linearly independent solutions on I of (2.2.4) are $[y_j, y_j^{(1)}, \ldots, y_j^{(r_i-1)}]'$, $j = 1,2,\ldots, r_i$. This implies that the vectors $Y_j' = [0,\ldots,0, y_j, y_j^{(1)}, \ldots, y_j^{(r_i-1)}, 0, \ldots, 0]$, $j = 1,2,\ldots, r_i$ are $r_i$ linearly independent solutions on I of (2.2.3). Since $X = C^{-1}Y$, $r_i$ linearly independent solutions on I, of (2.2.1), are $C^{-1}Y_j$, $j = 1,2,\ldots, r_i$. Therefore, (2.2.7) is associated with (2.2.1). Corresponding to each matrix $B_i$, of order $r_i$, $i = 1,2,\ldots,s$, there exists an associated differential equation of order $r_i$. Since matrix A is of order n, then $\sum_{i=1}^{s} r_i = n$.

---

Theorem 2.2.1 establishes that, (2.2.1), an arbitrary system of n differential equations explicit in the first derivative, may be converted into a set of s, $1 \leq s \leq n$, homogeneous differential equations of order $r_i$, $i = 1,2,\ldots,s$, where $\sum_{i=1}^{s} r_i = n$ ($r_i$ may be less than n). This is a generalization of results found by Murray and Miller [4, p.129].

In Theorem 2.2.2 which follows, the matrix product $CAC^{-1}$ is assumed to have the form of a set of s, $1 \leq s \leq n$, companion matrices $B_i$, $i = 1,2,\ldots,s$ as in the proof of Thm. 2.2.1. It is then proved that the zeros of the characteristic equation of (2.2.2) an r-order differential equation, associated with a system of n homogeneous differential equations, are also zeros of the characteristic equation of (2.2.1) the system of differential equations. These properties are obtained by applying well-known results from matrix algebra.

Theorem 2.2.2: Suppose

$$\frac{d^r}{dt^r} x = \sum_{j=1}^{r} a_j \frac{d^{r-j}}{dt^{r-j}} x \tag{2.2.8}$$

is associated with

$$\dot{X} = AX \tag{2.2.9}$$

as in the proof of Thm. 2.2.1. Then if $\lambda_o$ is a zero of $L(\lambda) = (-1)^r [\lambda^r - \sum_{j=1}^{r} a_j \lambda^{r-j}]$ then $\lambda_o$ is also a zero of $\det[A-\lambda I] = 0$.

Proof: By the method of the proof of Thm.2.2.1 $\det [CAC^{-1} - \lambda I] = \prod_{i=1}^{s} \det [B_i - \lambda I]$. By Lemma [6, p.88], the characteristic polynomial for $B_i$ is $\det [B_i - \lambda I] = (-1)^{r_i} [\lambda^{r_i} - \sum_{j=1}^{r_i} a_j \lambda^{r_i - j}]$. Since C is nonsingular, the zeros of $\det [A - \lambda I] = 0$ and $\det [CAC^{-1} - \lambda I] = 0$ are the same. Therefore if $\lambda_o$ is a zero of $L(\lambda)$ it also must be a zero of $\det [A - \lambda I] = 0$.

Corollary 2.2.1: With the same hypothesis and $r = n$, $a_j$ is $(-1)^{j+1}$ times the sum of the principal minors of order j of A.

Proof: This follows from Thm. 2.2.2, Thm. A.3 and the fact that the coefficient of $\lambda^n$ in $\det [A - \lambda I]$ is $(-1)^n$.

Corollary 2.2.1 provides an explicit relationship between the coefficients $a_j$ $j = 1, 2, \ldots, n$ of (2.2.8) where $r = n$ and coefficients $a_{ij}$ of A in (2.2.9). However, these results are only applicable when it is known that $CAC^{-1}$ has the form of a companion matrix.

From a practical standpoint, the existence Thm. 2.2.1 might at first be considered to be of academic interest only. However, in the proof of that theorem, a new method for converting a system of n equation (2.2.1) into a higher-order differential equation by transform techniques is given, i.e. $X = C^{-1}Y$ (Thm. 2.2.1). Before the transform technique can be applied the practical problem of determining a transformation of the type $X = C^{-1}Y$ which converts the system into an r-order differential equation, must be solved. A transformation of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = G^{-1} \begin{bmatrix} x_n \\ x_n^{(1)} \\ \vdots \\ x_n^{(n-1)} \end{bmatrix}$$

where G is a nonsingular matrix, is determined in the following:

Consider the system $\dot{X} = AX$ (2.2.1) partitioned as

$$\frac{d}{dt} \begin{bmatrix} X_1 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \\ A_{21} & a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ \\ x_n \end{bmatrix} \qquad (2.2.10)$$

where $X_1' = [x_1, x_2, \ldots, x_{n-1}]$. Take n-1 successive derivatives of $A_{21}X_1 = -a_{nn}x_n + \dot{x}_n$ the last row in (2.2.10) eliminate each time the first derivative of the vector $X_1$ by means of $\dot{X}_1 = A_{11} X_1 + A_{12}x_n$ the first row in (2.2.10). This formulation results in the following matrix form:

$$\begin{bmatrix} A_{21} \\ A_{21}A_{11} \\ \vdots \\ A_{21}A_{11}^{n-2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} -a_{nn} & 1 & 0 & \cdots & 0 & 0 \\ -A_{21}A_{12} & -a_{nn} & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & & & \\ -A_{21}A_{11}^{n-3}A_{12} & -A_{21}A_{11}^{n-4}A_{12} & -A_{21}A_{11}^{n-5}A_{12} & \cdots & -a_{nn} & 1 \end{bmatrix} \begin{bmatrix} x_n \\ x_n^{(1)} \\ \vdots \\ x_n^{(n-1)} \end{bmatrix}$$

This last equation can be written in symbolic form as

$$BX_1 = PX_d \qquad (2.2.11)$$

where the i row of (2.2.11) is

$$A_{21}A_{11}^{i-1}\, X_1 = -A_{21}A_{11}^{i-2}\, A_{12}x_n - \cdots - A_{21}A_{11}^{0}A_{12}x_n^{(i-2)} - a_{nn}x_n^{(i-1)} + x_n^{(i)}$$

$i = 1, 2, \ldots, n-1$. By attaching ones and zeros to (2.2.11) form

$$\begin{bmatrix} 0 & 1 \\ B & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ x_n \end{bmatrix} = \begin{bmatrix} U \\ P \end{bmatrix} \begin{bmatrix} X_d \end{bmatrix} \qquad (2.2.12)$$

where $U = [1, 0, 0, \ldots, 0]$.

If $B^{-1}$ exists, the coefficient matrix on the left side of (2.2.12) is nonsingular. The coefficient matrix on the right side of (2.2.12)

$$L = \begin{bmatrix} U \\ P \end{bmatrix}$$

is nonsingular, since matrix L is lower triangular with ones on the main diagonal, det (L) = 1. Let

$$G = \begin{bmatrix} U \\ P \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ B & 0 \end{bmatrix} = L^{-1}B_{11} \qquad (2.2.13)$$

If the $a_{ij}$ entries of matrix A of the system (2.2.1) satisfy the condition det (B) $\neq 0$ then $B_{11}^{-1}$ exists in (2.2.13) and the matrix G defined by (2.2.13) is nonsingular. This result is stated by the following lemma:

Lemma 2.2.3: If, corresponding to the normal system (2.2.1) the $a_{ij}$ entries of the matrix A satisfy det (B) $\neq 0$, where the i-row of matrix B is

$$[a_{n1} \quad a_{n2} \cdots a_{n,n-1}] \begin{bmatrix} a_{11} & a_{12} & & a_{1,n-1} \\ & \cdots\cdots & & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{bmatrix}^{i-1}$$

$i = 1, 2, \ldots, n-1,$

then there exists a nonsingular matrix G such that

$$X_d = GX \qquad (2.2.14)$$

where $X_d' = [x_n, x_n^{(1)}, \ldots, x_n^{(n-1)}]$.

---

Lemma 2.2.3 is a significant result of this section with respect to application. In the technique for constructing the nonsingular transformation (2.2.14) it is shown that the condition det (B) $\neq 0$, on the $a_{ij}$ entries of the matrix A, must be satisfied for the existence of the transformation matrix G. Theorem 2.2.3 establishes that the transformation (2.2.14) converts the normal system into an n-order differential equation which is associated with the system. The hypothesis of the theorem necessitates the same conditions on the $a_{ij}$ entries of matrix A in the system (2.2.1) as specified by the hypothesis of Lemma 2.2.3.

Theorem 2.2.3: If $\det(B) \neq 0$, where matrix B is defined in the hypothesis of Lemma 2.2.3, then there exists an n-order differential equation associated with the system (2.2.1).

Proof: For $\dot{X} = AX$, there exists a nonsingular matrix G such that $X_d = GX$ by Lemma 2.2.3. Substituting $X = G^{-1}X_d$ into (2.2.1) results in

$$X_d = GAG^{-1}X_d \tag{2.2.15}$$

The last row of (2.2.15) is an n-order differential equation associated with (2.2.1). For if $F_m$ is a fundamental matrix of (2.2.1), by Lemma 2.2.1, $GF_m$ is a fundamental matrix of (2.2.15). By (2.2.14), the entries in the first row of $GF_m$ are solutions on I of the n-order differential equation in (2.2.15). In addition, the entries in the j-row of $GF_m$, $j = 2, 3, \ldots, n$ are the j-1 derivatives of the entries in the first row. Therefore, the determinant of $GF_m$ is the Wronskian of solutions on I of the n-order differential equation in (2.2.15) and since G and $F_m$ are nonsingular, $GF_m$ is nonsingular. It follows from Thm. A.2 that the entries in the first row of $GF_m$ are also a fundamental set for the n-order differential equation in (2.2.15).

Suppose $y_1, y_2, \ldots, y_n$ is a fundamental set for the n-order differential equation of (2.2.15). Then the matrix $Y_j' = [y_j, y_j^{(1)}, \ldots, y_j^{(n-1)}]$, $j = 1, 2, \ldots, n$ is composed of n linearly independent vector solutions of (2.2.15). Since substitution of $X_d = GX$ of Lemma 2.2.3 into (2.2.15) results in (2.2.1) the vectors $G^{-1}Y_j$, $j = 1, 2, \ldots, n$ are n linearly independent solutions on I of (2.2.1). The theorem follows.

The proof of Thm. 2.2.3 suggests a new method of deriving n-order differential equations. If the coefficients $a_{ij}$ of matrix A (2.2.1) satisfy the condition det (B) $\neq$ 0 of Lemma 2.2.3, then an n-order equation can be formed by substituting $X = G^{-1}X_d$ into $\dot{X} = AX$. The n-order equation is the last row of the resulting system of equations (2.2.15). Note that the coefficients appearing in the n-order equation have not been given explicitly in terms of $a_{ij}$ entries of matrix A.

Techniques for obtaining the higher order differential equation presented by Moulton [5, p. 6] and others [4, p.126] do not include the nonsingular transformation, $X = G^{-1}X_d$, nor do other methods restrict the final equation to one of n-order. Moulton, for example, presents a system of three equations which converts into a second order equation in any variable.

Corollary 2.2.3 has been included to illustrate a class of system models $\dot{X} = AX$ which can be converted into an n-order differential equation, that is associated with the system. Note, in the proof of the corollary, how the coefficients $a_{ij}$ of matrix A are subjected to the test of det (B) $\neq$ 0 (Lemma 2.2.3). This test establishes the existence of an n-order homogeneous differential equation associated with the system model.

<u>Corollary 2.2.3:</u> Suppose matrix A of the system (2.2.1) is

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 & a_{1n} \\ 0 & a_{22} & \cdots & 0 & a_{2n} \\ & & \cdots\cdots & & \\ 0 & 0 & \cdots & 0 & a_{n-1n} \\ a_{n1} & a_{n2} & \cdots & a_{nn-1} & a_{nn} \end{bmatrix}$$

where $a_{ii}$ distinct $i = 1, 2, \ldots, n$ and $a_{nj} \neq 0$ for $j = 1, 2, \ldots, n$. Then there exists an n-order homogeneous differential equation associated with the system (2.2.1).

Proof: That $\det(B) \neq 0$ follows from the fact that the i-row of matrix B is

$$b_i = [a_{n1}\ a_{n2} \cdots a_{nn-1}] \begin{bmatrix} a_{11}^{i-1} & 0 & \cdots & 0 \\ 0 & a_{22}^{i-1} & \cdots & 0 \\ & \cdots & \cdots & \\ 0 & 0 & \cdots & a_{n-1n-1}^{i-1} \end{bmatrix}$$

and therefore

$$\det(B) = \det[b_1', b_2', \ldots, b_n'] = a_{n1} a_{n2} \cdots a_{nn-1} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_{11} & a_{22} & \cdots & a_{n-1n-1} \\ a_{11}^{n-2} & a_{22}^{n-2} & \cdots & a_{n-1n-1}^{n-2} \end{vmatrix}$$

$= a_{n1}\, a_{n2} \cdots a_{nn-1} \prod_{i>j} (a_{ii} - a_{jj})$ which by hypothesis is not zero. Therefore the corollary follows.

## 2.3 Additional Properties of Associate Differential Equations

The mathematical properties established in Sec. 2.2 are applied in the development of mathematical techniques for relating the $a_{ij}$ entries in matrix A of (2.1.1) to a specified solution of (2.1.2), associated with the system. The techniques developed here relate the $a_{ij}$ entries in matrix A of the system (2.1.1) to one specified component $x_i$ of the vector X of the system.

The relationship between the $a_{ij}$ entries of matrix A in the system (2.3.2) and a specified solution to the r-order differential equation (2.3.1), associated with the system, is established from the solution properties specified in Def. 2.2.1.

<u>Theorem 2.3.1:</u> If $y_1, y_2, \ldots, y_r$ is a fundamental set for

$$\frac{d^r y}{dt^r} = \sum_{j=1}^{r} a_j \frac{d^{r-j} y}{dt^{r-j}} \qquad (2.3.1)$$

and (2.3.1) is associated with

$$\dot{X} = AX \qquad (2.3.2)$$

where $A = [a_{ij}]$, $X' = [x_1, x_2, \ldots, x_n]$. Then there exists an explicit relation between the $r^2$ entries of A and the entries of C and $Y_j$, $j = 1, 2, \ldots, r$, (Def. 2.2.1 for notation).

<u>Proof:</u> Since (2.3.1) is associated with (2.3.2) the vectors $C^{-1}Y_j$ where $Y' = [0, \ldots, y_j, y_j^{(1)}, \ldots, y_j^{(r-1)}, 0, \ldots, 0]$, $j = 1, 2, \ldots, r$ are r linearly independent solutions on I of (2.3.2). Let $F_m = [Y_1, Y_2, \ldots, Y_r]$ and $F_r$ be the (nonsingular) submatrix of $F_m$ containing the columns $[y_j, y_j^{(1)}, \ldots, y_j^{(r-1)}]'$, $j = 1, 2, \ldots, r$. Since the matrix $C^{-1}F_m$ satisfies (2.3.2),

$$C^{-1}\dot{F}_m = A\,C^{-1}F_m$$

Multiplying the above equation on the right by $[0, F_r^{-1}, 0]$ results in a system of equations in the following form:

$$C_1^{-1}\dot{F}_r F_r^{-1} = A\,C_1^{-1}$$

where $C_1^{-1}$ is the set of columns of $C^{-1}$ corresponding to the position of the entries in $\dot{F}_r$ $(F_r)$ in $\dot{F}_m(F_m)$. Since the columns of $C_1^{-1}$ are linearly independent, there exists a nonsingular submatrix, $C_r^{-1}$, of $C_1^{-1}$ or order r. Multiplication on the right by $C_r$ results in a system of equations which contains a subset of equations in the form

$$A_r = C_r^{-1} \dot{F}_r F_r^{-1} C_r \qquad (2.3.3)$$

Since $A_r$ is square and therefore contains $r^2$ entries of A, the theorem follows.

Corollary 2.3.1: With the same hypothesis and $r = n$.

$$A = C^{-1} \dot{F}_m F_m^{-1} C \qquad (2.3.4)$$

Proof: This is a direct consequence of Thm. 2.3.1.

---

The matrix $F_m$ as specified in (2.3.4) is

$$F_m = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_n^{(1)} \\ & \cdots\cdots & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \qquad (2.3.5)$$

It is proved in the following corollary that $\dot{F}_m F_m^{-1}$ has the special form of a companion matrix.

<u>Corollary 2.3.2:</u> With the same hypothesis and $r = n$ the $\dot{F}_m F_m^{-1}$ of Cor. 2.3.1 is

$$\dot{F}_m F_m^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots\cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_1 \end{bmatrix} \qquad (2.3.6)$$

<u>Proof:</u> Let T represent the companion matrix on the right of (2.3.6). Consider the matrix product $T\,F_m$. If the n equations

$$b_n y_1 + b_{n-1} y_1^{(1)} + \cdots + b_1 y_1^{(n-1)} = y_1^{(n)}$$

$$b_n y_2 + b_{n-1} y_2^{(1)} + \cdots + b_1 y_2^{(n-1)} = y_2^{(n)}$$

$$\cdots\cdots \qquad \vdots$$

$$b_n y_n + b_{n-1} y_n^{(1)} + \cdots + b_1 y_n^{(n-1)} = y_n^{(n)}$$

are satisfied then $T\,F_m = \dot{F}_m$.

The above system of n equations in matrix form is

$$\begin{bmatrix} y_1 & \dot{y}_1 & \cdots & y_1^{(n-1)} \\ y_2 & \dot{y}_2 & \cdots & y_2^{(n-1)} \\ & \cdots\cdots & & \\ y_n & \dot{y}_n & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_1 \end{bmatrix} = \begin{bmatrix} y_1^{(n)} \\ y_2^{(n)} \\ \vdots \\ y_n^{(n)} \end{bmatrix} \qquad (2.3.7)$$

From Thm. A.2, $F_m$ is nonsingular. Since the matrix on the left of (2.3.7) is $F_m'$, the lemma follows.

Corollary 2.3.1 establishes a relationship (2.3.4) for determining the $a_{ij}$ entries of the matrix A in (2.3.2) in terms of one specified component $x_i(t)$ of the vector X(t) of (2.3.2). When Thm. 2.2.3 is applied to Cor. 2.3.1 it is determined that if the $a_{ij}$ entries in matrix A satisfy det (B) $\neq$ 0 then the vector component $x_n(t)$ of X(t) in the system (2.3.2) can be specified as consisting of n linearly independent parts. This specification in turn restricts the $a_{ij}$ entries of matrix A by

$$A = G^{-1} \dot{F}_m F_m^{-1} G \qquad (2.3.8)$$

Equation (2.3.8) is a principal result of this section and is referred to as a "design equation" when applied to the design of linear oscillators in Chapter V.

The results of Cor. 2.3.1 are extended one step further in Cor. 2.3.2 by proving the matrix product $\dot{F}_m F_m^{-1}$ in (2.3.4) or (2.3.8) to have the form of a companion matrix (2.3.6). An important consequence of this, which is applied in Thm. 2.3.2, is that the matrix products $CAC^{-1}$ (2.3.4) and $GAG^{-1}$(2.2.15), (2.3.8) have the form of a companion matrix. This implies that the characteristic polynomial of matrix A, det $[A - \lambda I]$ must be equal to the minimum polynomial [10, p. 149] of a matrix A.

Gantmacher [10, p. 159] shows that the matrix C in the similarity transformation $CAC^{-1}$, which produces the companion matrix, is not unique. Faddeeva [11, p. 201] presents a method developed by Danilevsky which brings a matrix A into companion matrix form by means of (n-1) similarity transformations. The primary difference in the method for obtaining the companion matrix in this thesis and other methods is that here it is obtained under specified conditions det (B) $\neq 0$ on the $a_{ij}$ entries of matrix A by a particular matrix product $GAG^{-1}$, which has the desired form of a companion matrix. This is not the case in the other methods which have been established for the primary purpose of bringing the characteristic determinant det $[A - \lambda I]$ of a matrix A into polynomial form. What is even more significant in the development of this thesis is the presentation of the transform in the form $X = G^{-1} X_d$ (Lemma 2.2.3) which links the solution of an n-order differential equation (2.3.9) to the solution of the system of differential equations (2.3.2). The result of $GAG^{-1}$ having the form of a companion matrix is applied in the following theorem.

<u>Theorem 2.3.2:</u> If det (B) $\neq 0$, where matrix B is defined in the hypothesis of Lemma 2.2.3, then

$$\frac{d^n}{dt^n} x_n = \sum_{i=1}^{n} a_i \frac{d^{n-i}}{dt^{n-i}} x_n \qquad (2.3.9)$$

where $a_j$ is $(-1)^{j+1}$ times the sum of the principal minors of order j of A, is associated with

$$\dot{X} = AX \qquad (2.3.10)$$

where $A = [a_{ij}]$ and $X' = [x_1, x_2, \ldots, x_n]$.

<u>Proof:</u> Theorem follows from Cor. 2.3.2 and Thm. 2.2.3.

Theorem 2.3.2 provides a simple, yet effective procedure for establishing a mathematical relationship between the $a_{ij}$ entries of the matrix A in the system (2.3.10) and the $a_j$ entries in the n-order equation (2.3.9). If the matrix A is given and det (B) $\neq$ 0, or if matrix A contains arbitrary entries and det (B) $\neq$ 0 is specified, then the mathematical relationships are given in the theorem. The results of Thm. 2.3.2 provides a new tool for system design in a later section.

Equation (2.3.8) is the result of specifying only one component $x_n$ in the vector X of the system (2.3.10). On the other hand, Thm. 2.3.2 relates only the $a_{ij}$ entries of the matrix A in the system (2.3.10) to the $a_j$ entries in the n-order equation (2.3.9). The additional step of relating the $a_{ij}$ entries to the specification on $x_n$ is made possible by the following theorem.

<u>Theorem 2.3.3:</u> Given the n-order differenial equation

$$\frac{d^n}{dt^n} x = \sum_{j=1}^{n} a_j \frac{d^{n-j}}{dt^{n-j}} x \qquad (2.3.11)$$

(1) If a solution on I of (2.3.11) is $y(t) = \sum_{i=1}^{k} P_{m_i - 1}(t)\, e^{\lambda_i t}$ (B.2) then $a_j$, $j = 1, 2, \ldots, n$ is given by (B.1.1), (B.1.2) $a_j = -b_j$ for $m_i = 1$ for all i.

(2) If a solution on I of (2.3.11) is $y(t) = \sum_{i=1}^{r} c_i e^{\lambda_i t}$, where $c_i \neq 0$, and $\lambda_i \neq 0$ and distinct, then r coefficients of (2.3.11) are given by (B.1.4) where $a_j = -b_j$.

(3) If a solution of I of (2.3.11) is $y(t) = \sum_{i=1}^{m} c_i e^{\alpha_i t} \cos(\omega_i t + \theta_i)$ and $2m = r$, then $r$ coefficients of (2.3.11) are given by (B.1.4) where $a_j = -b_j$.

Proof: The theorem follows from the general form of solution given in (B.2).

---

Theorem 2.3.3 establishes parameter-solution relationships between the coefficients $a_j$ of (2.3.11) and a specified solution. In (2) it is interesting to note (Thm. B.1.2) that if a solution with $r$ linearly independent parts is specified, $r$ coefficients of the n-order differential equation can be expressed in terms of the remaining n-r coefficients.

Theorems 2.3.2 and 2.3.3 are sufficient to interrelate the $a_{ij}$ entries of the matrix A of the normal system (2.3.10) and the component $x_n(t)$ of the vector $X(t)$ in the system (2.3.10). These mathematical relations are referred to as "design equations" when applied to system design in Chapter V.

# III. PROPERTIES OF SYSTEMS OF NONHOMOGENEOUS DIFFERENTIAL EQUATIONS AND ASSOCIATED HIGHER-ORDER DIFFERENTIAL EQUATIONS

## 3.1 Introduction

Mathematical properties parallel to those in Chapter II are developed for relating the solution of the nonhomogeneous system

$$\dot{X} = AX + Q(t) \tag{3.1.1}$$

where $X' = [x_1, x_2, \ldots, x_n]$, $Q'(t) = [q_1(t), q_2(t), \ldots, q_n(t)]$, $A = [a_{ij}]$ to the solution of the r-order($r \leq n$) differential equation

$$\frac{d^r}{dt^r} y = \sum_{i=1}^{r} a_i \frac{d^{r-i}}{dt^{r-i}} y + F(t) \tag{3.1.2}$$

For instance, the problem of determining a transformation of the form

$$X = C^{-1} [Y_s - L^{-1} H(t)] \tag{3.1.3}$$

where $X' = [x_1, x_2, \ldots, x_n]$, $Y_s' = [0, 0, \ldots, 0, y, y^{(1)}, \ldots, y^{(r-1)}, 0, 0, \ldots, 0]$, $H(t)$ is a vector function of t, and C and L are nonsingular matrices, which links the solution of the normal system (3.1.1) to the solution of the r-order equation (3.1.2), is encountered. The solution of this problem will allow the system designer to determine the initial condition of the physical system in terms of one component $x_i(t)$, of the system solution $X(t)$.

In contrast to the problems considered in Chapter II, a new problem that arises in this section is that of formulating the vector Q(t), of the normal system (3.1.1), in terms of one component $x_i$ of the vector solution, X. These parameter-solution relationships are illustrated in the design of amplifiers in Chapter V.

## 3.2 Systems of First-Order Nonhomogeneous Differential Equations and Associated Higher-Order Differential Equations

Properties parallel to those in Chapter II are developed here for the nonhomogeneous system (3.2.1). For instance in Thm. 3.2.1 it is proved by applying a transformation of the form (3.1.3) to a nonhomogeneous system, that there exists a set of s differential equations, $1 \leq s \leq n$, of the form (3.2.2) "associated" (Def. 3.2.1) with the system. In Thm. 3.2.2 conditions on the $a_{ij}$ entries of matrix A in the normal system are given so that there exists a differential equation of n-order associated with the system. In the proof of these results a technique for formulating a transformation of the form (3.1.3) is given (Lemma 3.2.2).

Definition 3.2.1 provides a concise description of the mathematical properties existing between the normal system and the r-order equation associated with the system. These properties are clarified in the theorems of this section.

<u>Definition 3.2.1:</u> Consider the system of n non-homogeneous equations,

$$\dot{X} = AX + Q(t) \tag{3.2.1}$$

where $A = [a_{ij}]$ , $X' = [x_1, x_2, \ldots, x_n]$, $Q'(t) = [q_1(t), \ldots q_n(t)]$, $q_i^{(n)}(t)$, $i = 1, 2, \ldots, n$ is continuous for all t on the open interval I defined by $I = [t\colon t_1 < t < t_2]$, where $t_1$ and $t_2$ are constants.

An r-order nonhomogeneous differential equation

$$\frac{d^r y}{dt^r} = \sum_{i=1}^{r} a_i \frac{d^{r-i} y}{dt^{r-i}} + F(t) \tag{3.2.2}$$

is associated with (3.2.1) if:

(a) The homogeneous part of (3.2.2) is associated (with nonsingular matrices C and D and row i) with the homogeneous part of (3.2.1) and

(b)1. For $X'(t) = [x_1(t), x_2(t), \ldots, x_n(t)]$ the solution of (3.2.1) on I such that $X(t_o) = 0$, $t_o$ on I, then row i of CX(t) is the solution of (3.2.2.) on I which is zero at $t_o$,

2. For y(t) the solution of (3.2.2) on I such that $y^{(j)}(t_o) = 0$, for $j = 0, 1, \ldots, r-1$, $t_o$ on I, and

$$F(t) = -\sum_{i=1}^{r-1} a_i \sum_{j=0}^{r-i-1} P_j^{r-i+1}(D)\, f_{r-i-j}(t) + \sum_{j=0}^{r-1} P_j^r(D)\, f_{r-j}(t), \tag{3.2.3}$$

where

$$P_j^k(D) = D^j + l_{k,k-1} D^{j-1} + l_{k,k-2} D^{j-2} + \ldots + l_{k,k-j},$$

$$D^j = \frac{d^j}{dt^j}, \quad P_o^k(D) = 1 \text{ and } \sum_{j=0}^{k-2} P_j^k(D)\, f_{k-1-j}(t_o) = 0, k = 2, 3, \ldots, r,$$

then $C^{-1}[Y_s(t) - L^{-1}H(t)]$ is the solution of (3.2.1) which, when evaluated at $t_o$, is the zero vector, where $L^{-1} = [l_{ij}]$ and

$$Y_s'(t) = [\underbrace{0, 0, \ldots, 0}_{i-1}, y(t), y^{(1)}(t), \ldots, y^{(r-1)}(t), 0, \ldots, 0],$$

$$H'(t) = [\underbrace{0, 0, \ldots, 0}_{i}, f_1(t), f_1^{(1)}(t) + f_2(t), \ldots,$$
$$f_1^{(r-2)}(t) + f_2^{(r-3)}(t) + \ldots + f_{r-1}(t), 0, \ldots, 0].$$

<u>Definition 3.2.2:</u> An r-order homogeneous differential equation

$$\frac{d^r}{dt^r} y = \sum_{i=1}^{r} a_i \frac{d^{r-i}}{dt^{r-i}} y$$

is associated with (3.2.1) if (a) and (b) of Def. 3.2.1 are satisfied.

---

The mathemtical properties specified in Def. 3.2.1 are first encountered in Lemma 3.2.1 which follows. The lemma presents a normal system of the form (3.2.1), with the coefficient matrix in the form of a companion matrix, that converts into an n-order differential equation associated with the system.

Lemma 3.2.1: Suppose matrix A of the system(3.2.1) is

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \cdots\cdots & & \\ 0 & 0 & 0 & \dots & 1 \\ a_n & a_{n-1} & a_{n-2} & \dots & a_1 \end{bmatrix} \qquad (3.2.4)$$

Then there exists an n-order differential equation, associated with the system (3.2.1).

Proof: Determine the n-1 successive derivatives of the first row of (3.2.1) eliminating each time from the right hand side, the first derivatives of $x_2, x_3, \dots, x_n$ by means of the last n-1 original equations. This process results in

$$X_1 = X + Q_1(t) \qquad (3.2.5)$$

where $X_1' = [x_1, x_1^{(1)}, \dots, x_1^{(n-1)}]$, $X' = [x_1, x_2, \dots, x_n]$ and $Q_1'(t) = [0,\ q_1(t),\ q_1^{(1)}(t) + q_2(t), \dots, q_1^{(n-2)}(t) + q_2^{(n-3)}(t) + \dots + q_{n-1}(t)]$ and

$$\frac{d^n}{dt^n} x_1 = \sum_{j=1}^{n} a_j \frac{d^{n-j}}{dt^{n-j}} x_1 + \sum_{j=0}^{n-1} P_j(D)\, q_{n-j}(t) \qquad (3.2.6)$$

where $P_j(D) = D^j - a_1 D^{j-1} - a_2 D^{j-2} - \dots - a_j$, $j = 1, 2, \dots, n-1$, $P_o(D) = 1$ and $D^j = \dfrac{d^j}{dt^j}$ .

That the homogeneous part of (3.2.6) is associated with the homogeneous part of (3.2.1), with C and D unit matrices and i = 1, follows by an argument similar to that used in the proof of Thm. 2.2.1.

If $X'(t) = [x_1(t), x_2(t), \ldots, x_n(t)]$ is the solution of (3.2.1) on I such that $X(t_o) = 0$, $t_o$ on I, then by the method of construction of (3.2.5) and (3.2.6), $x_1(t)$, the first entry of $X(t)$, is the solution on I of (3.2.6) such that $x_1(t_o) = 0$. Therefore if (3.2.6) is homogeneous, the lemma follows.

Suppose (3.2.6) is nonhomogeneous and let $y(t)$ be the solution of I of (3.2.6) such that $y^{(j)}(t_o) = 0$ for $j = 0, 1, 2, \ldots, n-1$, $t_o$ on I. It has been shown [4, p. 134] that such a solution exists. For $F(t) = \sum_{j=0}^{n-1} P_j(D)\, q_{n-j}(t)$, form $F(t) = \sum_{j=0}^{n-1} P_j(D)\, f_{n-j}(t)$, which is a special case of (3.2.3) with $l_{ij} = 0$ for $i \neq j$ and $l_{ii} = 1$ for $i = 1, 2, \ldots, n$, such that $\sum_{j=0}^{k-2} D^j f_{k-1-j}(t_o) = 0$, $t_o$ on I and $k = 2, 3, \ldots, n$. This can always be done, since if $f_i(t_o) = 0$, $i = 1, 2, \ldots, n-1$, $F(t) = f_n(t) = \sum_{j=0}^{n-1} P_j(D)\, q_{n-j}(t)$. Consider the transformation of variables $x_1(t) = y_1(t)$ and

$$\dot{Y}(t) = AY(t) + F_1(t) \qquad (3.2.7)$$

where $Y'(t) = [y_1(t), y_2(t), \ldots, y_n(t)]$, $F_1'(t) = [f_1(t), f_2(t), \ldots, f_n(t)]$ and A is given by (3.2.4). The solution of (3.2.7) is $Y(t) = Y_s(t) - F_{11}(t)$ where $Y_s'(t) = [y(t), y^{(1)}(t), \ldots, y^{(n-1)}(t)]$ and $F_{11}'(t) = [0, f_1(t), f_1^{(1)}(t) + f_2(t), \ldots, f_1^{(n-2)}(t) + f_2^{(n-3)}(t) + \ldots + f_{n-1}(t)]$ and $Y(t_o) = Y_s(t_o) - F_{11}(t_o) = 0$ for $t_o$ on I since $Y_s(t_o) = 0$ and $F_{11} = 0$ for $t_o$ on I. This implies the lemma.

<u>Corollary 3.2.1:</u> With the same hypothesis and if $\sum_{j=0}^{n-1} P_j(D)\, q_{n-j}(t) = 0$ then there exists an n-order homogeneous differential equation associated with (3.2.1).

Proof: This follows from the method of formulating (3.2.6) in the proof of Lemma 3.2.1.

---

A review of the salient features in the proof of Lemma 3.2.1 will lay the foundation for the theorems which follow. First, a nonsingular transformation of the form (3.1.3) is determined (3.2.5 in Lemma 3.2.1). This transformation relates the solution of the system to the solution of the higher-order differential equation (3.2.6).

Property (a) of Def. 3.2.1 is established in the same manner in which the results of Thm. 2.2.1 were established.

In Property (b)1 of Def. 3.2.1, the first row in the vector CX is the solution of the higher-order equation (3.2.6). In the lemma, matrix C is the unit matrix and the first entry in the vector $Q_1(t)$ is zero.

To satisfy Property (b)2 of Def. 3.2.1, it must be pointed out that there exists a solution of the higher-order equation with the property $y^{(j)}(t_o) = 0, j = 0, 1, 2, \ldots, n-1$. Next, it must be demonstrated that the nonhomogeneous part, F(t), of the higher-order equations, can be put into the form specified in (3.2.3) in such a way that it satisfies the conditions at $t_o$ specified in (3.2.3). It is shown in Lemma 3.2.1 that F(t) can always be put into the desired form to meet the specified conditions $t = t_o$. The reason for this last condition will become clear in the proof of Thm. 3.3.1 which follows later.

It may be noted at this point the condition at $t_o$ on $F(t)$ forces the vector $L^{-1}H(t)$ in the transformation (3.1.3) to be zero at $t_o$. In Lemma 3.2.1 this requires $F_{11}(t_o)$ to be zero. Similar operations are applied in Thm. 3.2.1 which follows to prove the existence of a set of s differential equations, $1 \leq s \leq n$, of order $r_i$ associated with the system.

Theorem 3.2.1: There exists a set of s differential equations, $1 \leq s \leq n$, of order $r_i$, $i = 1, 2, \ldots, s$, associated with the system (3.2.1) such that $\sum_{i=1}^{s} r_i = n$.

Proof: Consider the transformation $X = C^{-1}Y$ on (3.2.1) which is used in the proof of Thm. 2.2.1. For this case the transformed system of equations is

$$\dot{Y} = CAC^{-1}Y + CQ(t) \tag{3.2.8}$$

and

$$\dot{Y}_i = B_i Y_i + F_i(t) \tag{3.2.9}$$

where $B_i$, $i = 1, 2, \ldots, s$, is of the form of (3.2.4). By Lemma 3.2.1 there exists an $r_i$-order differential equation (3.2.6), $n = r_i$, associated (with matrices C and D and row i) with (3.2.9). That the homogeneous part of this differential equation is associated with the homogeneous part of (3.2.1) follows by an argument similar to that of the proof of Thm. 2.2.1.

If $X(t)$ is the solution of (3.2.1) such that $X(t_o) = 0$, $t_o$ on I, then $CX(t)$ is the solution of (3.2.8) which is zero at $t_o$. The vector $[y_{i1}(t), y_{i2}(t), \ldots, y_{ir_i}(t)]'$ where the component $y_{ij}(t)$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, r_i$, is the $\sum_{p=1}^{i-1} r_p + j$ entry in $CX(t)$, is the solution of (3.2.9) which when evaluated at $t_o, t_o$ on I, is zero. Therefore $y_{i1}(t)$ is the solution of the $r_i$-order differential equation, (3.2.6), $n = r_i$, for which $y_{i1}(t_o) = 0$.

If (3.2.6), $n = r_i$, is nonhomogeneous, then by an argument similar to that of the proof of Lemma 3.2.1 (see proof for notation), the solution of (3.2.9) is $Y(t) = Y_s(t) - F(t)$ which is zero at $t_o$, $t_o$ on I. Appending zeros to the vectors expressing this solution of (3.2.9) results in the solution of (3.2.8) which is zero at $t_o$. Since the solution of (3.2.1) is $X = C^{-1}Y$, the $r_i$-order differential equation established by Lemma 3.2.1 is associated with (3.2.1). The theorem follows since the above argument applies for all i, $i = 1, 2, \ldots, s$ and $\sum_{i=1}^{s} r_i = n$.

---

In Theorem 3.2.1 it is established that the normal system (3.2.1) converts into a set of s differential equations, $1 \leq s \leq n$, of order $r_i$. This is an extension of results found by Murray and Miller [4, p. 129] and others [5, p. 6]. In addition to this, from a practical viewpoint, the proof of the existence theorem affords a new method of determining a solution of a system in terms of a solution to a higher-order differential equation, i.e. $X = C^{-1}Y$. The problem of formulating a nonsingular transformation, of the form applied in the theorem is the subject of the following discussion:

Consider the normal system $\dot{X} = AX + Q(\cdot)$, partitioned as

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ \\ A_{21} & a_{nn} \end{bmatrix}\begin{bmatrix} X_1 \\ \\ x_n \end{bmatrix} + \begin{bmatrix} Q_1(\cdot) \\ \\ q_n(\cdot) \end{bmatrix} \qquad (3.2.10)$$

As in the proof of Lemma 2.2.3, take n-1 successive derivatives of the last row in (3.2.10), eliminate each time the first derivative of the vector $X_1$ by means of the first row in (3.2.10). This formulation results in the following (n-1) relations:

$$\begin{bmatrix} A_{21} \\ \\ A_{21}A_{11} \\ \cdots \\ A_{21}A_{11}^{n-2} \end{bmatrix}\begin{bmatrix} x_1 \\ \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} -a_{nn} & 1 & 0 & \cdots 0 & 0 \\ \\ -A_{21}A_{12} & -a_{nn} & 1 & \cdots 0 & 0 \\ \cdots & \cdots & & & \\ -A_{21}A_{11}^{n-3}A_{12} & -A_{21}A_{11}^{n-4}A_{12} & -A_{21}A_{11}^{n-5}A_{12} & \cdots -a_{nn} & 1 \end{bmatrix}\begin{bmatrix} x_n \\ \\ x_n^{(1)} \\ \vdots \\ x_n^{(n-1)} \end{bmatrix} +$$

$$\begin{bmatrix} 0 \\ \\ -q_n(t) \\ \cdots\cdots \\ -q_n^{(n-2)}(t) + A_{21}A_{11}^{o}Q_1^{(n-3)} + \ldots + A_{21}A_{11}^{n-3}Q_1(t) \end{bmatrix}$$

This last equation can be written in symbolic form as

$$BX_1 = PX_d - Q_2(t) \qquad (3.2.11)$$

where the i row of (3.2.11) is

$$A_{21}A_{11}^{i-1}X_1 = -A_{21}A_{11}^{i-2}A_{12}x_n - \ldots - A_{21}A_{11}^{o}A_{12}x_n^{(i-2)} - a_{nn}x_n^{(i-1)} + x_n^{(i)}$$

$$+ q_n^{(i-1)}(t) + A_{21}A_{11}^{o}Q_1^{(i-2)}(t) + \ldots + A_{21}A_{11}^{i-2}Q_1(t)$$

and $i = 1, 2, \ldots, n-1$. Let (3.2.11) be bordered with ones and zeros to form

$$\begin{bmatrix} 0 & 1 \\ B & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ x_n \end{bmatrix} = \begin{bmatrix} U \\ P \end{bmatrix} \begin{bmatrix} X_d \end{bmatrix} - \begin{bmatrix} 0 \\ Q_2(t) \end{bmatrix} \qquad (3.2.12)$$

where $U = [1, 0, 0, \ldots, 0]$.

If $B^{-1}$ exists, the coefficient matrix on the left side of (3.2.12) is nonsingular. The coefficient matrix

$$L = \begin{bmatrix} U \\ P \end{bmatrix}$$

also defined in Lemma 2.2.3 is nonsingular, since matrix L is lower triangular with ones on the main diagonal, $\det(L) = 1$.

Let

$$G = \begin{bmatrix} U \\ P \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ B & 0 \end{bmatrix} = L^{-1} B_{11} \qquad (3.2.13)$$

and

$$Q_d(t) = \begin{bmatrix} 0 \\ Q_2(t) \end{bmatrix}$$

If the $a_{ij}$ entreis of matrix A of (3.2.1) satisfy the condition $\det(B) \neq 0$ then $B_{11}^{-1}$ exists and the matrix G defined by (3.2.13) is nonsingular. These results are stated by Lemma 3.2.2 which follows.

<u>Lemma 3.2.2:</u> If corresponding to (3.2.1), $\det(B) \neq 0$, then there exists a vector $Q_d(t)$ and nonsingular matrices G and L such that

$$X_d = GX + L^{-1}Q_d(t) \qquad (3.2.14)$$

where matrix B is defined in the hypothesis of Lemma 2.2.3 and $X_d' = [x_n, x_n^{(1)}, \ldots, x_n^{(n-1)}]$.

---

Lemma 3.2.2 is one of the significant results of this section. The lemma is, in a sense, an existence theorem. That is, if the $a_{ij}$ entries of matrix A in the normal system (3.2.1) satisfy the condition det (B) $\neq 0$ then there exists a nonsingular transformation (3.2.14). Identical conditions on the $a_{ij}$ entries in the homogeneous systems were found (Lemma 2.2.3) for the transformation (2.2.14) to exist. Theorem 3.2.2 shows that the transformation (3.2.14) does convert the normal system into an n-order differential equation which is associated with the system.

Theorem 3.2.2: If det (B) $\neq 0$, where matrix B is defined in the hypothesis of Lemma 2.2.3, then there exists an n-order differential equation associated with the system (3.2.1).

Proof: Substituting the transformation (3.2.14) intothe system (3.2.1) results in

$$\dot{X}_d = GAG^{-1}X_d - GAG^{-1}L^{-1}Q_d(t) + L^{-1}\dot{Q}_d(t) + GQ(t) \qquad (3.2.15)$$

where

$$[L^{-1}\dot{Q}_d(t) + GQ(t)]' = [f_{11}(t), f_{11}^{(1)}(t) + f_{12}(t), \ldots, f_{11}^{(n-1)}(t) + f_{12}^{(n-2)} + \ldots$$

$$+ f_{1n}(t)]\ [L^{-1}]',$$

$$Q_d'(t) = [0, f_{11}(t), f_{11}^{(1)}(t) + f_{12}(t), \ldots, f_{11}^{(n-2)}(t) + f_{12}^{(n-3)}(t) + \ldots + f_{1,n-1}(t)]$$

and $f_{11}(t) = q_n(t)$, $f_{1i}(t) = A_{21}A_{11}^{i-2}Q_1(t)$, $i = 2, 3, \ldots, n$. The last row of (3.2.15) is,

$$\frac{d^n}{dt^n}x_n = \sum_{i=1}^{n} a_i \frac{d^{n-i}}{dt^{n-i}}x_n - \sum_{i=1}^{n-1} a_i \sum_{j=0}^{n-i-1} P_j^{n-i+1}(D)\, f_{1,n-i-j}(t) + \sum_{j=0}^{n-1} P_j^n(D)\, f_{1,n-j}(t) \qquad (3.2.16)$$

an n-order differential equation associated with (3.2.1), where $P_j^k(D)$ is given after (3.2.3) and $L^{-1} = [l_{ij}]$. For, by an argument similar to that of the proof of Thm. 2.2.3, the homogeneous part of (3.2.16) is associated (with matrices G and row one) with the homogeneous part of (3.2.1)

If $X'(t) = [x_1(t), x_2(t), \ldots, x_n(t)]$ is the solution of (3.2.1) such that $X(t_o) = 0$, $t_o$ on I, then the corresponding solution of (3.2.15) is given by (3.2.14). The first entry of $X_d(t)$ which is the first entry of $GX(t)$ (since $L^{-1}$ is lower triangular) is the solution of (3.2.15) which is zero at $t_o$. Therefore if (3.2.16) is homogeneous, the theorem follows.

For the case of (3.2.15) nonhomogeneous the method of the last part of the proof of Lemma 3.2.1 is used. That is for

$$F(t) = -\sum_{i=1}^{n-2} a_i \sum_{j=0}^{n-i-2} P_j^{n-i+1}(D)\, A_{21}A_{11}^{n-i-j-2}Q_1(t) - \sum_{i=1}^{n-1} a_i P_{n-1-i}^{n+1-i}(D)\, q_n(t) + \sum_{j=0}^{n-2} P_j^n(D)\, A_{21}A_{11}^{n-j-2}Q_1(t) + P_{n-1}^n(D)\, q_n(t) \qquad (3.2.17)$$

form $F(t) = -\sum_{i=1}^{n-1} a_i \sum_{j=0}^{n-i-1} P_j^{n-i+1}(D)\, f_{n-i-j}(t) + \sum_{j=0}^{n-1} P_j^{n}(D)\, f_{n-j}(t)$ such that $\sum_{j=0}^{k-2} P_j^{k}(D)\, f_{k-1-j}(t_o) = 0$ for $t_o$ on I, $k = 2, 3, \ldots, n$. Consider the transformation of variables $x_n(t) = y(t)$ and

$$\dot{Y}(t) = GAG^{-1}Y(t) - GAG^{-1}L^{-1}Q_d(t) + L^{-1}F_1(t) \qquad (3.2.18)$$

where $Y'(t) = [y(t), y^{(1)}(t), \ldots, y^{(n-1)}(t)]$, $Q_d'(t) = [0, f_1(t), f_1^{(1)}(t) + f_2(t), \ldots, f_1^{(n-2)}(t) + \ldots + f_{n-1}(t)]$, $F_1'(t) = [f_1(t), f_1^{(1)}(t) + f_2(t), \ldots, f_1^{(n-1)}(t) + \ldots + f_n(t)]$. Substituting (3.2.14) into (3.2.18) results in (3.2.1) since $Q'(t) = [f_1(t), f_2(t), \ldots, f_n(t)]\,(G^{-1}L^{-1})'$. Therefore if $y(t)$ is the solution on I where $y(t_o) = 0$ and $y^{(j)}(t_o) = 0$, $j = 1, 2, \ldots, n-1$, $t_o$ on I, then the solution of (3.2.1) is $X(t) = G^{-1}[X_d(t) - L^{-1}Q_d(t)]$ where $X_d'(t) = [y(t), y^{(1)}(t), \ldots, y^{(n-1)}(t)]$ such that $X'(t_o) = 0$. The theorem follows.

---

The determination and application of the nonsingular transformation of Lemma 3.2.2 and Thm. 3.2.2 are unique to this thesis. The technique used in the proof of Thm. 3.2.2 offers not only a new method of formulating an n-order differential equation, but even more important a closed form relationship $X = G^{-1}[X_d - L^{-1}Q_d(t)]$ linking the solution of an n-order equation (3.2.16) to the solution of the normal system (3.2.1). Methods of obtaining higher-order equations presented by Moulton [5, p. 6] and others [4, p. 126] do not consider a transformation of the above type (3.2.14) and as a result do not restrict the final equation to one of n-order. The restriction of the final equation to one of n-order and the transformation in the special form (3.2.14) are important features in the design of electrical networks proposed in a later section.

A review of the main features of Thm. 3.2.2 shows that if the $a_{ij}$ entries of matrix A in the normal system (3.2.1) satisfy the conditions det (B) $\neq 0$, then the system converts into the n-order differential equation (3.2.16). This n-order equation, which is associated with the system (3.2.1), is the result of long and difficult derivations, i.e. the last row of (3.2.15). However, now that this derivation has been sucessfully performed for the general case it is no longer necessary to go through the complete process of substituting the transformation (3.2.14) into the system (3.2.1) and obtaining the last row of the resulting system, to arrive at the n-order equation (3.2.16). It is, however, necessary to determine some of the components of the n-order equation. The $a_i$, $i = 1, 2, \ldots, n$ components in (3.2.16) are determined by applying Thm. 2.3.2 as being $(-1)^{i+1}$ times the sum of the principal minors of order i of matrix A in (3.2.1). Closer examination of the nonhomogeneous part of the n-order equation (3.2.16) shows the only derivation yet to be made is $L^{-1} = [\ell_{ij}]$, where the matrix L is defined in the results of Lemma 3.2.2. This later derivation is relatively simple since the matrix L is lower triangular with ones on the main diagonal.

An additional result of Thm. 3.2.2 is the formulation of the vector

$$Q'(t) = [f_1(t), f_2(t), \ldots, f_n(t)]\,(G^{-1}L^{-1})'.$$

of the normal system (3.2.1) in terms of the components $f_1(t), f_2(t), \ldots, f_n(t)$ in the nonhomogeneous part F(t) of the n-order equation (3.2.2). The formulation of the vector $Q(t)$ is referred to as a "design equation" when applied to the design of amplifiers in a later section.

## 3.3 Additional Properties of Associated Differential Equations

Mathematical properties established in Sec. 3.2, which relate the solution of the normal system (3.1.3) to the solution of the r-order equation (3.1.2) associated with the system, are applied in this section. In the following development the coefficient matrix A and the vector $Q(t)$ in the normal system are related to the solution $y(t)$ and the nonhomogeneous part $F(t)$ of an n-order differential equation (3.1.2) associated with the system.

Theorem 3.3.1: Suppose $y_1(t), y_2(t), \ldots, y_n(t)$ is a fundamental set of the homogeneous part of (3.3.1) and $y(t)$ is the solution on I: $|t| > t_o$ of

$$\frac{d^n x}{dt^n} = \sum_{j=1}^{n} a_j \frac{d^{n-j}}{dt^{n-j}} x + F(t) \qquad (3.3.1)$$

such that $y^{(j)}(t_o) = 0$, $t_o$ on I, $j = 0, 1, \ldots, n-1$ and (3.3.1) is associated with

$$\dot{X} = AX + Q(t) \qquad (3.3.2)$$

then

$$\begin{aligned} A &= C^{-1}\dot{F}_m(t)\, F_m^{-1}(t)\, C \\ Q(t) &= F_m(t) \frac{d}{dt}(F_m^{-1}(t)\, Y(t)) \end{aligned} \qquad (3.3.3)$$

where $F_m(t) = [f_{ij}(t)]$, $f_{ij}(t) = y_j^{(i-1)}(t)$, $i, j = 1, 2, \ldots, n$ and $Y(t) = C^{-1}[Y_s(t) - L^{-1}H(t)]$ (see Def. 3.2.1 for notation).

Proof: By hypothesis, and Cor. 2.3.1 (3.3.3) of conclusion follows.

If $F_m(t)$ is a fundamental matrix for $\dot{X} = AX$ then, by Thm. 3.1 [6, p.74]

$$Y(t) = F_m(t) \int_{t_o}^{t} F_m^{-1}(s)\, Q(s) ds$$

t on I, is that solution of (3.3.2) satisfying $Y(t_o) = 0$. Application of "Leibnitz Rule" (Thm. A.4) to

$$\frac{d}{dt}(F_m^{-1}(t) \;\; Y(t)\,) = \frac{d}{dt}\left(\int_{t_o}^{t} F_m^{-1}(s)Q(s)\, ds\right)$$

results in n relations,

$$Q(t) = F_m(t) \frac{d}{dt}(F_m^{-1}(t) \;\; Y(t)\,)$$

Since the n-order equation (3.3.1) is associated with the system (3.2.2) then $Y(t) = C^{-1}[Y_s(t) - L^{-1}H(t)]$ is the solution of (3.3.2) which, is zero when evaluated at $t_o$, $t_o$ on I .

---

In Theorem 3.3.1, if y(t) is replaced by $x_n(t)$ and the restriction det (B) $\neq 0$ (as defined in the hypothesis of Lemma 2.2.3) is added to the hypothesis of the theorem, then the relationships

$$A = G^{-1}\dot{F}_m F_m^{-1} G \tag{3.3.4}$$

$$Q(t) = F_m(t) \frac{d}{dt}(F_m^{-1}(t)\ X(t)\ ) \tag{3.3.5}$$

are obtained. The solution of the normal system (3.3.2) is given in this restricted situation i.e. det (B) $\neq 0$, by the vector

$$X(t) = G^{-1}\ [X_d(t) - L^{-1}H(t)] \tag{3.3.6}$$

where $X_d'(t) = [x_n(t),\ x_n^{(1)}(t), \ldots, x_n^{(n-1)}(t)]$, matrices G and L are determined as in the discussion proceeding Lemma 3.2.2, and H(t) is determined as specified in Def. 3.2.1 where $r = n$. The relationship (3.3.4) has been discussed in Chapter II.

# IV. ON A CLASS OF SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS AND CORRESPONDING REDUCED DIFFERENTIAL EQUATIONS

## 4.1 Introduction

In the design of tunnel-diode amplifiers and oscillators, the state models of these systems have taken special forms. By divorcing the parameters, in these mathematical models from the parameters associated with a particular physical system, the class of systems

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_p \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1p-1} & a_{1p+1} & \cdots & a_{1n} \\ & & \cdots\cdots & & & \\ a_{p1} & \cdots & a_{pp-1} & a_{pp+1} & \cdots & a_{pn} \\ & & \cdots\cdots & & & \\ a_{n1} & \cdots & a_{np-1} & a_{np+1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_{p-1} \\ x_{p+1} \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(x_p) \\ \vdots \\ f_p(x_p) \\ \vdots \\ f_n(x_p) \end{bmatrix} + \begin{bmatrix} q_1(t) \\ \vdots \\ q_p(t) \\ \vdots \\ q_n(t) \end{bmatrix}$$

or in symbolic form

$$\dot{X} = A_1 X_{op} + N(x_p) + Q(t) \tag{4.1.1}$$

results.

The problem of relating the solution of the system (4.1.1) to the solution of the n-order differential equation

$$\frac{d^n}{dt^n} x_p(t) = \sum_{j=1}^{n-1} a_j \frac{d^{n-j}}{dt^{n-j}} x_p(t) + T(x_p) + F(t) \tag{4.1.2}$$

by means of a nonsingular transformation is encountered, as in Chapters II and III.

The additional problem of relating a specified solution, $B_o + B_1 \sin(\omega t+\emptyset)$, to the parameters in the n-order equation (4.1.2) where $T(\dot{x}_p)$ is a polynomial of m-order is considered. The solution of this problem will assist in the design of an amplifier or oscillator when the system contains a nonlinear component whose characteristics can be approximated by a polynomial of order m.

## 4.2 Formulation and Solution of Reduced Nonlinear Differential Equations

Conditions are given on the normal system (4.1.1) for the existence of a nonsingular transformation (4.2.3). The transformation, when it exists, reduces the system (4.1.1) to an n-order "reduced" (Def. 4.3.1) differential equation (4.1.2). The transformation in turn relates the solution of the normal system to the solution of the n-order reduced differential equation.

First consider the problem of formulating a nonsingular transformation.

Let the normal system $\dot{X} = A_1 X_{on} + N(x_n) + Q(t)$, (4.1.1) be partitioned as

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ x_n \end{bmatrix} = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}\begin{bmatrix} X_1 \end{bmatrix} + \begin{bmatrix} F_{12}(x_n) \\ f_n(x_n) \end{bmatrix} + \begin{bmatrix} Q_1(t) \\ q_n(t) \end{bmatrix} \tag{4.2.1}$$

As in the proof of Lemmas 2.2.3 and 3.2.2, take n-1 successive derivatives of the last row in (4.2.1) and eliminate the first derivative of the vector $X_1$ each time by means of the first row in (4.2.1). This formulation results in

$$x_n^{(i)} = A_{21}A_{11}^{i-1} X_1 + f_n^{(i-1)}(x_n) + A_{21}A_{11}^{o}F_{12}^{(i-2)}(x_n) + \ldots + A_{21}A_{11}^{i-1} F_{12}(x_n)$$

$$+ q_n^{(i-1)} + A_{21}A_{11}^{o} Q_1^{(i-2)} + \ldots + A_{21}A_{11}^{i-2} Q_1(t)$$

for $i = 1, 2, \ldots, n-1$. These $n-1$ relations in matrix form are

$$X_{1d} = BX_1 + Z_1(x_n, t) \qquad (4.2.2)$$

where $X'_{1d} = [x_n^{(1)}, \ldots, x_n^{(n-1)}]$ and $X'_1 = [x_1, x_2, \ldots, x_{n-1}]$.
Let (4.2.2) be bordered with a one and a zero to form

$$\begin{bmatrix} x_n \\ X_{1d} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ B & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ Z_1(x_n, t) \end{bmatrix}$$

This last equation can be written in symbolic form as

$$X_d = B_{11}X + Z(x_n, t)$$

If in addition, the $a_{ij}$ entries in matrix $A_1$ of the system (4.1.1) satisfy the condition $\det(B) \neq 0$ then $B_{11}^{-1}$ exists. These results are stated by the following lemma.

<u>Lemma 4.2.1:</u> If corresponding to the system (4.1.1) with $p = n$, $\det(B) \neq 0$ where matrix B is defined in the hypothesis of Lemma 2.2.3, then there exists a vector $Z(x_p, t)$ and a nonsingular matrix $B_{11}$, such that

$$X_d = B_{11}X + Z(x_p, t) \tag{4.2.3}$$

where $X_d' = [x_p, x_p^{(1)}, \ldots, x_p^{(n-1)}]$ .

---

Lemma 4.2.1 states that if the $a_{ij}$ entries of matrix $A_1$ in the normal system (4.1.1) satisfy the condition det (B) $\neq 0$ then there exists a nonsingular transformation (4.2.3). The transformation determined in the lemma is the main part of the Def. 4.2.1. The definition supplies an operational technique for determining, and hence defining, the n-order equation obtained from the normal system.

Definition 4.2.1: The nonlinear differential equation obtained as the last row of the system of differential equations generated by substituting $X = B_{11}^{-1}[X_d - Z(x_p, t)]$ , of Lemma 4.2.1, in (4.1.1) is the reduced differential equation corresponding to (4.1.1).

Consider as an example, the two normal systems in Theorems 4.2.1 and 4.2.2 that convert to a reduced differential equation.

Theorem 4.2.1: If the matrix $A_1$ of the system (4.1.1) with $p = n$ is

$$A_1 = \begin{bmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ 0 & a_{22} & \cdots & 0 & 0 \\ & & \cdots\cdots & & \\ 0 & 0 & \cdots & a_{n-1\,n-2} & a_{n-1\,n-1} \\ a_{n1} & a_{n2} & \cdots & a_{nn-2} & a_{nn-1} \end{bmatrix}$$

where $a_{ii}$ distinct $i = 1, 2, \ldots, n-1$ and $a_{nj} \neq 0$ for $j = 1, 2, \ldots, n-1$, then there exists an n-order reduced differential equation corresponding to (4.1.1).

Proof: The proof is similar to that of Cor. 2.2.3 and therefore has been omitted.

Theorem 4.2.2: If matrix $A_1$ of the system (4.1.1) with $p = 1$ is

$$A_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & \cdots\cdots & & \\ 0 & 0 & \cdots & 1 \\ a_{n-1} & a_{n-2} & \cdots & a_1 \end{bmatrix}$$

Then there exists an n-order reduced differential equation,

$$\frac{d^n}{dt^n} x_1 = \sum_{j=1}^{n-1} a_j \frac{d^{n-j}}{dt^{n-j}} x_1 + \sum_{j=0}^{n-1} P_j(D)\, f_{n-j}(x_1) + \sum_{j=0}^{n-1} P_j(D)\, q_{n-j}(t)$$

corresponding to (4.1.1), where $P_j(D) = D^j - a_1 D^{j-1} - a_2 D^{j-2} - \ldots - a_j$, $P_o(D) = 1$ and $D^j = \dfrac{d^j}{dt^j}$ .

Proof: Application of the procedure used in the proof of Lemma 4.2.1 (calculate n-1 successive derivatives of the first row of (4.1.1) eliminating $x_2, x_3, \ldots, x_n$) results in

$$X_d = X + Z(x_1, t) \qquad (4.2.4)$$

where ($B_{11}$ of Lemma 4.2.1 is the unit matrix) $X_d' = [x_1, x_1^{(1)}, \ldots, x_1^{(n-1)}]$, $X' = [x_1, x_2, \ldots, x_n]$, $Z'(x_1, t) = [0, f_1(x_1) + q_1(t), f_1^{(1)}(x_1) + f_2(x_1) + q_1^{(1)}(t) + q_2(t), \ldots$ $f_1^{(n-2)}(x_1) + \ldots + f_{n-1}(x_1) + q_1^{(n-2)}(t) + \ldots + q_{n-1}(t)]$. The theorem follows by substituting (4.2.4) into (4.1.1).

---

Lemma 4.2.1 is one of the important features of this section. The lemma is in a sense, an existence theorem. That is, if the $a_{ij}$ entries of matrix $A_1$ in the system (4.1.1) satisfy the condition det (B) $\neq 0$, then there exists a nonsingular transformation (4.2.3). Identical conditions on the $a_{ij}$ entries have been found for the transformation to exist, when the normal system is linear.

Theorem 4.2.3 shows that the transformation (4.3.3) converts the normal system (4.1.1) into an n-order reduced differential equation.

Theorem 4.2.3: Consider the system (4.1.1) with p = n, det (B) $\neq 0$, where matrix B is defined in the hypothesis of Lemma 2.2.3. Then there exists an n-order reduced differential equation

$$\frac{d^n}{dt^n} x_n = \sum_{j=1}^{n-1} a_j \frac{d^{n-j}}{dt^{n-j}} x_n + \sum_{j=0}^{n-1} P_j(D) f_{1n-j}(x_n) + \sum_{j=0}^{n-1} P_j(D) q_{1n-j}(t)$$

(4.2.5)

where $f_{11}(x_n) = f_n(x_n), f_{1i}(x_n) = A_{21}A_{11}^{i-2}F_{12}(x_n)$ $i = 2, 3, \ldots, n$, $q_{11}(t) = q_n(t)$, $q_{1i}(t) = A_{21}A_{11}^{i-2}\, Q_1(t)$ $i = 2, 3, \ldots, n$, corresponding to the system (4.1.1).

Proof: Substituting the transformation (4.2.3) into the system (4.1.1) results in

$$\dot{X}_d = B_{11}A_1B^{-1}X_{1d} - B_{11}A_1B^{-1}Z_1(x_n, t) + B_{11}N(x_n) + B_{11}Q(t) + \dot{Z}(x_n, t) \tag{4.2.6}$$

where $X_d' = [x_n, X_{1d}'] = [x_n, x_n^{(1)}, \ldots, x_n^{(n-1)}]$.

$[B_{11}N(x_n) + B_{11}Q(t) + \dot{Z}(x_n, t)]' = [f_{11}(x_n) + q_{11}(t), f_{11}^{(1)}(x_n) + f_{12}(x_n) + q_{11}^{(1)}(t) + q_{12}(t), \ldots, f_{11}^{(n-1)}(x_n) + \ldots + f_{1n}(x_n) + q_{11}^{(n-1)}(t) + \ldots + q_{1n}(t)]$, $Z_1'(x_n, t) = [f_{11}(x_n) + q_{11}(t), f_{11}^{(1)}(x_n) + f_{12}(x_n) + q_{11}^{(1)}(t) + q_{12}(t), \ldots, f_{11}^{(n-2)}(x_n) + \ldots + f_{1n-1}(x_n) + q_{11}^{(n-2)}(t) + \ldots + q_{1n-1}(t)]$ and $f_{11}(x_n) = f_n(x_n)$, $f_{1i}(x_n) = A_{21}A_{11}^{i-2}F_{12}(x_n)$ $i = 2, \ldots, n$, $q_{11}(t) = q_n(t)$, $q_{1i}(t) = A_{21}A_{11}^{i-2}\, Q_1(t)$ $i = 2, 3, \ldots, n$. The theorem follows since (4.2.5) is given by the last row of (4.2.6).

---

The determination (Lemma 4.2.2) and application (Thm. 4.2.3) of the nonsingular transformation $X_d = B_{11} + Z(x_p, t)$ as defined in the results of Lemma 4.2.1 are unique to this thesis. The technique employed in the proof of Thm. 4.2.3 offers not only a new method of formulating the n-order differential equation, but also of equal importance, a closed form relationship

$X = B_{11}^{-1}[X_d - Z(x_n, t)]$ which joins the solution of the n-order equation (4.1.2) to the solution of the normal system (4.1.1). This last property is demonstrated by Thm. 4.2.4, which follows.

Theorem 4.2.4: If (4.2.5) is the reduced differential equation corresponding to the system (4.1.1) and $x_p(t)$ is a solution on I of (4.1.2) then

$$X(t) = B_{11}^{-1}\,[X_d(t) - Z(x_p, t)] \qquad (4.2.7)$$

where $X_d'(t) = [x_p(t), x_p^{(1)}(t), \ldots, x_p^{(n-1)}(t)]$, $Z'(x_p, t) =$
$[0, f_{11}(x_p, t) + q_{11}(t), f_{11}^{(1)}(x_p, t) + f_{12}(x_p, t) + q_{11}^{(1)}(t) + q_{12}(t), \ldots,$
$f_{11}^{(n-2)}(x_p, t) + \ldots + f_{1n-1}(x_p, t) + q_{11}^{(n-2)}(t) + \ldots + q_{1n-1}(t)]$ is a solution of the system (4.1.1) on I.

Proof: By Definition 4.2.1 the last row of (4.2.6) is (4.2.5). By substituting (4.2.3) where $X_d$ and $Z(x_p, t)$ are defined after (4.2.7), in (4.2.6), (4.1.1) results since $X_{1d} = BX_{op} + Z_1(x_p, t)$, $N'(x_1, t) = [f_{11}(x_p, t), f_{12}(x_p, t), \ldots, f_{1n}(x_p, t)]\,(B_{11}^{-1})'$ and $Q'(t) = [q_{11}(t), q_{12}(t), \ldots, q_{1n}(t)]\,(B_1^{-1})'$. Thus if $x_p(t)$ is a solution of (4.2.5) on I, then $X_d'(t) = [x_n(t), X_{1d}'(t)] = [x_n(t), x_n^{(1)}, \ldots, x_n^{(n-1)}(t)]$ is a solution of (4.2.6) on I, which implies that (4.2.7) is a solution of (4.1.1) on I.

## 4.3 A Relationship Between the Parameters and a Solution of a Class of N-Order Nonlinear Differential Equations

Nonlinear differential equations of the form (4.1.2), where $T(x_p)$ is a polynomial, are considered in what follows.

A relationship between the parameters and a solution in the form of $B_o + B_1 \sin(\omega t + \phi)$ of this class of differential equations is obtained.

The parameter-solution relationships determined in Thm. 4.3.1 are referred to as design equations when applied in Chapter V to the design of tunnel-diode amplifiers and oscillators. Before considering the details of the theorem, reference should be made to Thm. C.1 in the appendix. In Thm. C.1 the nonlinear part $P_{1,n-1}(D)\, f(y)$ of (4.3.1) is derived for $f(y)$, a polynomial of order m.

<u>Theorem 4.3.1:</u> If $y(t) = B_o + B_1 \sin(\omega t + \phi)$, $B_o \neq 0$, is a solution on I: $|t| > t_o$ of

$$\frac{d^n}{dt^n} y = \sum_{i=1}^{n} a_i \frac{d^{n-i}}{dt^{n-i}} y + P_{1,n-1}(D)\, f(y) + F(t) \qquad (4.3.1)$$

where $P_{1,n-1}(D) = D^{n-1} - d_1 D^{n-2} - d_2 D^{n-3} - \dots - d_{n-1}$, $D^j = \frac{d^j}{dt^n}$, $P_{1,0}(D) = 1$ and $f(y) = \sum_{j=0}^{m} a_j y^j$, $a_m \neq 0$, $F(t) = q_o + q_1 \sin(\omega t + \theta)$ where $\omega > 0$ then

(1) $q_o = -a_n B_o + d_{n-1} b_o$

(2) $q_1 \cos\theta = B_1 \omega^n \cos(\frac{n\pi}{2} + \phi) - B_1 \sum_{i=1}^{n} a_i \omega^{n-i} \cos(\frac{(n-i)\pi}{2} + \phi) - L_1 b_1$

(3) $q_1 \sin\theta = B_1 \omega^n \sin(\frac{n\pi}{2} + \phi) - B_1 \sum_{i=1}^{n} a_i \omega^{n-i} \sin(\frac{(n-i)\pi}{2} + \phi) - S_1 b_1$

(4) $b_s = 0$, $s \neq 0, 1$ $s = 2j$ or $2j-1$.

Where $b_{2j}$ and $b_{2j-1}$ are defined in the results of Lemma C.2 and $L_1$, $S_1$ are defined in the results of Thm. C.1.

Proof: Since $y(t) = B_o + B_1 \sin(\omega t + \phi)$ is a solution of (4.3.1) on I, $|t| > t_o$, $\frac{d^n}{dt^n}(B_o + B_1\sin(\omega t+\phi)) = \sum_{i=1}^{n} a_i \frac{d^{n-i}}{dt^{n-i}}(B_o + B_1\sin(\omega t+\phi)) +$

$$P_{1,n-1}(D)\, f(B_o + B_1\sin(\omega t+\phi)) + q_o + q_1\sin(\omega t+\theta).$$

Calculating the indicated derivatives, and applying the conclusion of Thm. C.1, this equation reduces to

$$[B_1\omega^n\sin(\tfrac{n\pi}{2}+\phi) - \sum_{i=1}^{n} a_i B_1\omega^{n-i}\sin(\tfrac{(n-i)\pi}{2}+\phi) - S_1 b_1 - q_1\sin\theta]\cos\omega t +$$

$$[B_1\omega^n\cos(\tfrac{n\pi}{2}+\phi) - \sum_{i=1}^{n} a_i B_1\omega^{n-i}\cos(\tfrac{(n-i)\pi}{2}+\phi) - L_1 b_1 - q_1\cos\theta]\sin\omega t +$$

$$d_{n-1}b_o - a_n B_o - q_o - \sum_{j=1}^{k} M_j b_{2j}\cos 2j\omega t - \sum_{j=1}^{k} N_j b_{2j}\sin 2j\omega t -$$

$$\sum_{j=2}^{r} L_j b_{2j-1}\sin(2j-1)\omega t - \sum_{j=2}^{r} S_j b_{2j-1}\cos(2j-1)\omega t = 0.$$

Grouping the coefficients of $\cos\omega t$, $\sin\omega t$, $\cos 2j\omega t$, $\sin 2j\omega t$, $\cos(2j-1)\omega t$, $\sin(2j-1)\omega t$ and constants, and equating each group to zero results in (1), (2), and (3) conclusions of the theorem and the following relations:

$$M_j b_{2j} = 0 \qquad N_j b_{2j} = 0 \qquad j = 1,2,\ldots,k$$

$$L_j b_{2j-1} = 0 \qquad S_j b_{2j-1} = 0 \qquad j = 2,3,\ldots,r.$$

Since $M_j(2j\omega)$ is a polynomial in $2j\omega$ of order $n-1$, $n \geq 1$, with coefficients not all zero, $M_j(2j\omega)b_{2j} = 0$ $j = 1,2,\ldots,k$ for $\omega > 0$, implies that $b_{2j} = 0$ for $j = 1,2,\ldots,k$. The theorem follows by applying the same argument to $L_j((2j-1)\omega)b_{2j-1}$.

<u>Corollary 4.3.1:</u> If $p - q \neq 0, 2, 4, \ldots, B_1 \neq 0$ then

$$(1) \qquad B_o a_n = -q_o + d_{n-1} b_o$$

$$(2) \qquad a_p = \frac{-[W_1 \sin(\frac{(n-q)\pi}{2} + \phi) - W_2 \cos(\frac{(n-q)\pi}{2} + \phi)]}{\omega^{n-p} \sin \frac{(p-q)\pi}{2}}$$

$$(3) \qquad a_q = \frac{[W_1 \sin(\frac{(n-p)\pi}{2} + \phi) - W_2 \cos(\frac{(n-p)\pi}{2} + \phi)]}{\omega^{n-p} \sin \frac{(p-q)\pi}{2}}$$

where,
$$W_1 = \frac{q_1}{B_1} \cos\theta - \omega^n \cos(\frac{n\pi}{2} + \phi) + \sum_s a_i \omega^{n-i} \cos(\frac{(n-i)\pi}{2} + \phi) + L_1 \frac{b_1}{B_1}$$

$$W_2 = \frac{q_1}{B_1} \sin\theta - \omega^n \sin(\frac{n\pi}{2} + \phi) + \sum_s a_i \omega^{n-i} \sin(\frac{(n-i)\pi}{2} + \phi) + S_1 \frac{b_1}{B_1}$$

$\sum_s$ is the sum over all $i \neq p, q$.

<u>Proof:</u> This is a direct consequence of Thms. 4.3.1 and hence the proof is not included.

---

Consider as an example of Cor. 4.3.1 with $n = 2$

$$(1) \quad B_o a_2 = -q_o + d_1 b_o$$

$$(2) \quad a_1 = \frac{-b_1}{B_1} + \frac{q_1}{\omega B_1} \sin(\phi - \theta)$$

$$(3) \quad a_2 = -\omega^2 + d_1 \frac{b_1}{B_1} - \frac{q_1}{B_1} \cos(\theta - \phi)$$

and from the results of Lemma C.2 with $m = 3$ ( the derivation following Lemma C.2)

$$b_o = \alpha_o + B_o \alpha_1 + \alpha_2 (B_o^2 + \frac{B_1^2}{2}) + B_o \alpha_3 (B_o^2 + \frac{3}{2} B_1^2)$$

$$b_1 = B_1 [\alpha_1 + 2\alpha_2 B_o + \alpha_3 (3B_o^2 + \frac{3}{4} B_1^2)].$$

The importance of the parameter-solution relationships (1), (2) and (3) in Thm. 4.3.1 will be demonstrated when they are applied to arrive at design equations, in the design of tunnel-diode amplifiers and oscillators.

A notable feature of the n-order differential equation (4.3.1) is that the nonlinear part f(y) is a polynomial of order m. This results in $b_o$ and $b_1$, in equation (1), (2), (3) and (4) of Thm. 4.3.1, being nonlinear in the specified $B_o$ and $B_1$. When the polynomial f(y) is of order m = 3 then $b_o$ and $b_1$ are given in the example following Cor. 4.3.1.

# V. ON DESIGN METHODS AND EXAMPLES

## 5.1 Introduction

Design methods, which require the construction of a normal system of differential equations having a specified solution, are presented here. These methods are based on a portion of the mathematical properties developed in the preceding sections of this thesis.

To exhibit that the design methods apply equally well to a very large class of physical systems, the methods are applied to a normal system of equations whose parameters have been divorced from those of a particular physical system.

The approach to the design is to view the system performance as one specified component $x_i(t)$ of the vector solution, $X(t)$, of the normal system. Then, by mathematical techniques developed in the preceding sections, the mathematical relationships which must be satisfied between the parameters in the normal system and the specified solution $x_i(t)$ are determined.

## 5.2 Oscillator Design

Methods of oscillator design in terms of complex frequency domain equations (Laplace Transforms) are well-known [12, 13]. From Chapter II, the development of Theorems 2.3.2, 2.3.3 and Cor. 2.3.1 (Eq. 2.3.8) provides some new tools which can be applied to oscillator design in the time-domain.

The linear oscillator is assumed to take the mathematical form of a normal system of linear homogeneous differential equations. If $y(t)$ is to represent the oscillator response then $y(t)$ must be a non-constant solution of the normal system for all time $t$, $t > 0$. In addition, there must exist a constant $\epsilon > 0$ and time $t_o > 0$ such that

$$|y(t + nt_o) - y(t + (n+1)t_o)| < \epsilon \tag{5.2.1}$$

for all $t > T$ and $n = 0, 1, 2, \ldots$

The number $t_o$ in (5.2.1) is called the period of oscillation. The smallest values of $T$, $T_o$, for which (5.2.1) is satisfied, is the rise time of the oscillator. The amplitude of oscillation is $\lim_{n\to\infty}\left[\begin{matrix}\text{Max } y(t)\\ nt_o < t < (n+1)t_o\end{matrix}\right]$.

Two methods of designing an oscillator which has a specified frequency of oscillation, amplitude of oscillation and rise time are presented. Method 5.2.1 illustrates the results obtained in Theorems 2.3.2 and 2.3.3 whereas, Method 5.2.2 applies the results of Cor. 2.3.1 (Eq. 2.3.8).

Method 5.2.1: Oscillator Design

(a) Construct $y(t) = \sum_{i=1}^{k} P_{m_i - 1}(t)\, e^{\lambda_i t}$, $t \geq 0$, from the specifications.

(b) Apply Thm. 2.3.3 to $y(t)$ of (a) and thereby determine the coefficients $a_j$, $j = 1, 2, \ldots, n$ of

$$\frac{d^n}{dt^n} y = \sum_{j=1}^{n} a_j \frac{d^{n-j}}{dt^{n-j}} \tag{5.2.2}$$

in terms of $y(t)$.

(c) Relate the $a_j$ $j = 1,2,\ldots,n$ entries to the $a_{ij}$ entries in matrix A of the system

$$\dot{X} = AX \tag{5.2.3}$$

where $X' = [x_1, x_2, \ldots, x_n]$ , $A = [a_{ij}]$ by applying Thm. 2.3.2.

(d) Relate the initial condition $y^{(j)}(t_o) = C_{j+1}$, $j = 0,1,\ldots,n-1$ to the initial condition on the system (5.2.3) by

$$X = G^{-1}X_d \tag{5.2.4}$$

where $X' = [x_1, x_2, \ldots, x_n]$ , $X_d' = [x_n, x_n^{(1)}, \ldots, x_n^{(n-1)}]$ and matrix G is defined in the results of Lemma 2.2.3.

Method 5.2.2: Oscillator Design

(a) Construct y(t) as in (a) Method 5.2.1.

(b) Form the matrix $F_m$ from a fundamental set of y(t), i.e. if $y_1, y_2 \ldots y_n$ is a fundamental set of (5.2.2) then

$$F_m = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_n^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \tag{5.2.5}$$

(c) Determine the coefficient matrix A of the normal system (5.2.3) as

$$A = G^{-1} \dot{F}_m F_m^{-1} G. \tag{5.2.6}$$

where matrix G is defined in the results of Lemma 2.2.3.

(d) Determine the initial conditions for the normal system as in (d) of Method 5.2.1.

The parameter-solution relationships obtained by either of these two methods when associated with the parameters in systems of differential equations corresponding to known network configurations are referred to as design equations.

The basic steps of the proposed design methods are illustrated in the examples that are itemized to correspond to the design method being applied.

Example 5.2.1: (Method 5.2.1) Specifications require an oscillator with a frequency of oscillation $f = \omega/2\pi$, aplitude of oscillation c and a "small" rise time.

(a) A possible y(t) is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t} \tag{5.2.7}$$

where $c_1 = \frac{ce^{j\emptyset}}{2}$, $c_2 = \frac{ce^{-j\emptyset}}{2}$, $\lambda_1 = + j\omega, \lambda_2 = -j\omega$ and $\lambda_3 = -\alpha_3$, where $\omega$, c, $\emptyset, \alpha_3$ are real and positive and $\alpha_3$ is as large as possible.

(b) The coefficients $a_j$ of (5.2.2) for n = 3 as given in the results of Thm. 2.3.3

$$a_1 = \lambda_1 + \lambda_2 + \lambda_3 = -\alpha_3 \quad a_2 = (-\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = -\omega^2 \tag{5.2.8}$$

$$a_3 = \lambda_1\lambda_2\lambda_3 = -\alpha_3\omega^2$$

(c) Proceeding as indicated in (c) of Method 5.2.1 the condition, det (B) $\neq$ 0, in the hypothesis of Thm. 2.3.2 must be satisfied. For this condition

$$\det(B) = a_{31}(a_{33}a_{32}+a_{31}a_{12}+a_{32}a_{22}) - a_{32}(a_{33}a_{31}+a_{31}a_{11}+a_{32}a_{21}) \neq 0$$

the third-order equation (5.2.2), n = 3, is associated with the normal system (5.2.3) and hence the $a_j$ entries (5.2.8) are related to the $a_{ij}$ entries of matrix A in (5.2.3) by

$$\begin{aligned} a_{11} + a_{22} + a_{33} &= -\alpha_3 = a_1 \\ -(a_{11}a_{22}-a_{21}a_{12}+a_{11}a_{33}-a_{31}a_{13}+a_{22}a_{33}-a_{32}a_{23}) &= -\omega^2 = a_2 \\ \det(A) &= -\alpha_3\omega^2 = a_3 \end{aligned} \qquad (5.2.9)$$

(d) In addition, for det (B) $\neq$ 0, the initial conditions as specified by (5.2.4) are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} \\ a_{33}a_{31}+a_{31}a_{11}+a_{32}a_{21} & a_{33}a_{32}+a_{31}a_{12}+a_{32}a_{22} & a_{33}^2+a_{31}a_{13}+a_{32}a_{23} \end{bmatrix}^{-1} \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}$$

are in symbolic form

$$X(t_o) = G^{-1}X_d(t_o). \qquad (5.2.10)$$

Any physical system which has a mathematical model in the normal form (5.2.3), with parameters satisfying the design equations (5.2.9) and (5.2.10), will have a specified response y(t); where normal system $x_3(t) = y(t)$ specified. Note, when y(t) is given the solution of the normal system is implied by (5.2.10). The parameter-solution relationships (5.2.9) and (5.2.10) are further illustrated when applied in the design of a "Colpitts Oscillator".

A normal system of equations corresonding to the Colpitts Oscillator

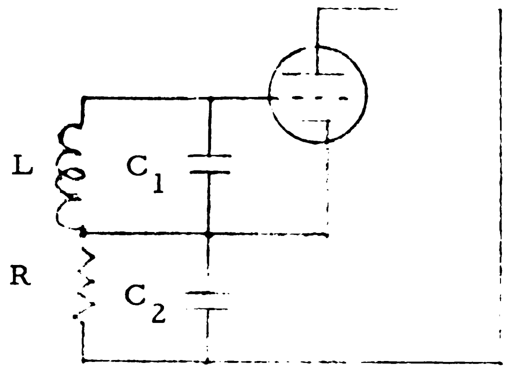


Figure 5.2.1

where $g_m$, $\mu$ and $r_p$ are well-known tube parameters is

$$\frac{d}{dt}\begin{bmatrix} V_{c_1} \\ V_{c_2} \\ I_L \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{-1}{C_1} \\ -\frac{g_m}{C_2} & \frac{-1}{C_2 r_p} & \frac{1}{C_2} \\ \frac{1}{L} & \frac{-1}{L} & \frac{-R}{L} \end{bmatrix}\begin{bmatrix} V_{c_1} \\ V_{c_2} \\ I_L \end{bmatrix} \qquad (5.2.11)$$

Specifying that $\det(B) = \frac{\mu + 1}{r_p C_2 L^2} \neq 0$. Method 5.2.1 applies and the $a_{ij}$ entries in the coeffieient matrix (5.2.11) are related to the specifications (design equations) by (5.2.9) as

$$\frac{1}{C_2 r_p} + \frac{R}{L} = a_3$$

$$\frac{1}{LC_1} + \frac{1}{LC_2} + \frac{1}{LC_2} \cdot \frac{R}{r_p} = \omega^2 \qquad (5.2.12)$$

$$\frac{g_m}{C_1 C_2 L} + \frac{1}{C_1 C_2 L r_p} = \frac{(\mu + 1)}{C_1 C_2 L r_p} = a_3 \omega^2$$

and the system (5.2.11) initial conditions are related to the specifications by (5.2.10) as

$$X(t_o) = G^{-1} X_d(t_o)$$

where $X' = [V_{c_1}, V_{c_2}, I_L]$, $X_d' = [I_L, I_L^{(1)}, I_L^{(2)}]$ and the matrix G is determined from (5.2.10).

Note, in the normal system (5.2.11) $I_L(t) = y(t)$, any other component of the system model could have been specified similarly by starting with the desired component in the last row (provided det (B) $\neq 0$).

<u>Example 5.2.2:</u> (Method 5.2.2) Specifications require an oscillator with a response as given by (a) of Ex. 5.2.1 where $c_3 = 0$.

(a) y(t) is given by (a), in Ex. 5.2.1, with $c_3 = 0$.

(b) The matrix $F_m$ as specified by Method 5.2.2, is

$$F_m = \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{bmatrix} \qquad (5.2.13)$$

(c) The matrix A of the normal system is determined from (5.2.6). First, the conditions on the $a_{ij}$ entries of matrix A in system (5.2.3) for matrix G to exist are stated, i.e. det (B) $\neq 0$. In the case of system (5.2.3) where $n = 2$, det (B) $= a_{21} \neq 0$ is required. Second, formulate the matrix G as in proof of Lemma 2.2.3 as

$$G = \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix} \tag{5.2.14}$$

Finally the matrix product $G^{-1}\dot{F}_m F_m^{-1} G$ is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -a_{22}+\lambda_1+\lambda_2 & -\lambda_1\lambda_2-a_{22}^2+a_{22}(\lambda_1+\lambda_2) \\ a_{21} & a_{22} \end{bmatrix} \tag{5.2.15}$$

(d) The specified initial conditions are derived from (5.2.4) as

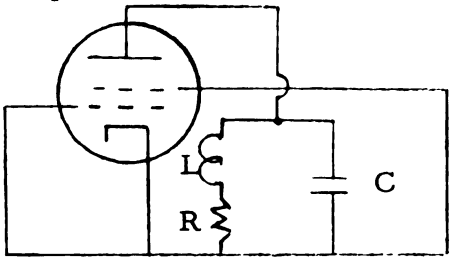
$$\begin{bmatrix} x_1(t_o) \\ x_2(t_o) \end{bmatrix} = \begin{bmatrix} \frac{-a_{22}}{a_{21}} & \frac{1}{a_{21}} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y(t_o) \\ \dot{y}(t_o) \end{bmatrix} \tag{5.2.16}$$

Hence, the final mathematical model for the oscillator, as determined from the specification of y(t), is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -a_{22} & \frac{-\omega^2-a_{22}^2}{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.2.17}$$

where $a_{21} \neq 0$, and the initial conditions are given in (5.2.16). Note, the specified y(t) corresponds to $x_2(t)$ in the normal system. The parameter-solution relationships (5.2.16) and (5.2.17) are applied in the design of a "negative resistance" oscillator.

Consider the following network and corresponding system of equations




$$\frac{d}{dt}\begin{bmatrix} V_c \\ I_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{Cr_p} & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R}{L} \end{bmatrix}\begin{bmatrix} V_c \\ I_L \end{bmatrix} \tag{5.2.18}$$

To impose the mathematical restrictions (5.2.16), (5.2.17), on the parameters of the physical system requires; first $1/L \neq 0$ then

$$-\frac{1}{Cr_p} = \frac{R}{L}, \quad -\frac{1}{C} = \frac{-\omega^2-(\frac{R}{L})^2}{1/L} \tag{5.2.19}$$

and finally

$$\begin{bmatrix} V_c(t_o) \\ I_L(t_o) \end{bmatrix} = \begin{bmatrix} R & L \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y(t_o) \\ \dot{y}(t_o) \end{bmatrix} \tag{5.2.20}$$

Note in this case $I_L(t) = y(t)$.

<u>Example 5.2.3:</u> (Method 5.2.2) Suppose the specifications on y(t) are given by (5.2.7). The fundamental matrix $F_m$ (5.2.5) determined from y(t) is

$$F_m(t) = \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & e^{\lambda_3 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} & \lambda_3 e^{\lambda_3 t} \\ \lambda_1^2 e^{\lambda_1 t} & \lambda_2^2 e^{\lambda_2 t} & \lambda_3^2 e^{\lambda_3 t} \end{bmatrix} \tag{5.2.21}$$

For det (B) $\neq 0$ as specified in (c) of Example 5.2.1 the calculation $A = G^{-1}\dot{F}_m F_m^{-1} G$ (5.2.6) is

$$A = G^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda_1\lambda_2\lambda_3 & -(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) & \lambda_1+\lambda_2+\lambda_3 \end{bmatrix} G \tag{5.2.22}$$

where the matrix G is given in (5.2.10). Design equations for the Colpitts Oscillator are obtained in this case by letting $\lambda_1 = +j\omega$, $\lambda_2 = -j\omega$, $\lambda_3 = -\alpha_3$ and relating matrix A (5.2.22) to the coefficient matrix of (5.2.11). The initial conditions are related to the Colpitts Oscillator system of equations the same as in Ex. 5.2.1. Note the companion-matrix form of $\dot{F}_m F_m^{-1}$ in (5.2.22).

## 5.3 Amplifier Design

Methods of amplifier design in terms of complex frequency-domain equations (Laplace Transforms) are well-known [12, 13]. To illustrate the results of Theorems 3.2.2 and 3.3.1 developed in Chapter III, two methods of amplifier design in the time-domain are given.

The mathematical model of the amplifier is assumed to take the form of a normal system of linear nonhomogeneous differential equations (5.3.4). The nonhomogeneous part of system, vector Q(t), is assumed to contain a component of the form $q_i(t) = q_i \sin(\omega t + \theta)$ $\omega > 0$, which can be associated with the amplifier input.

If $y(t)$ is to represent the amplifier response then $y(t)$ is a nonconstant solution of the normal system for all time $t$, $t > 0$. In addition there is a constant $\epsilon > 0$ and time $t_o > 0$ such that

$$|y(t + nt_o) - y(t + (n+1)\, t_o)| < \epsilon \qquad (5.3.1)$$

for all $t > T$ and $n = 0, 1, 2, \ldots$

The number $t_o$ is called the period of the amplifier response and the smallest value of $T$, $T_o$ which satisfies (5.3.1) is the transient time of the amplifier. The gain $G_m$ of the amplifier is the usual steady-state peak output divided by peak input, denoted mathematically by

$$G_m = \frac{\text{Max } y(t)}{\text{Max } q_i(t)} \qquad (5.3.2)$$

for $t > T$. The amplifier bandwidth denoted by b will be $|\omega_1 - \omega_2| \leq b$. If $y(t) = y_t(t) + y_s(t)$, and $y_t(t)$ satisfies the homogeneous part of an n-order differential equation (5.3.3) where $r = n$ then $y_t(t)$ is called the transient response of the amplifier.

Two methods of designing an amplifier which has a specified response $y(t)$ are presented. Method 5.3.1 illustrates the results of Thm. 3.2.2, whereas Method 5.3.2 applies the results of Thm. 3.3.1. The first part of Methods 5.3.1 and 5.3.2 have been given previously in Methods 5.2.1 and 5.2.2.

<u>Method 5.3.1:</u> Amplifier Design

(a) Construct $y(t) = y_t(t) + y_s(t)$, $t \geq 0$, from the specifications, where $y_t(t) = \sum_{i=1}^{k} P_{m_i - 1}(t)\, e^{\lambda_i t}$, $t \geq 0$, is the transient response of the amplifier.

(b) Apply (b) and (c) of Method 5.2.1, to relate $y_t(t)$ to the $a_{ij}$ entries in matrix A of the homogeneous system (5.2.3).

(c) Determine the nonhomogeneous part F(t) of the n-order equation

$$\frac{d^n}{dt^n} y = \sum_{j=1}^{n} a_j \frac{d^{n-i}}{dt^{n-i}} y + F(t) \qquad (5.3.3)$$

by a theorem from Sec. B.2.

(d) Formulate the nonhomogeneous part, vector Q(t), of the normal system

$$\dot{X} = AX + Q(t) \qquad (5.3.4)$$

where $X' = [x_1, x_2, \ldots, x_n]$ , $Q'(t) = [q_1(t), q_2(t), \ldots, q_n(t)]$, $A = [a_{ij}]$, by letting

$$Q'(t) = [f_1(t), \ \ldots, \ f_n(t)] \, (G^{-1} L^{-1})' \qquad (5.3.5)$$

as determined in Thm. 3.2.2. Matrices G and L are formulated in the proof of Lemma 3.2.2. Parameters $f_1(t)$, $f_2(t)$, ..., $f_n(t)$ are the components of the nonhomogeneous part F(t), (5.3.3) and satisfy Def. 3.2.1, Eq.(3.2.3).

(e) Relate the initial conditions $y^{(j)}(t_o)$, $j = 0, 1, \ldots, n-1$ to the initial conditions on the normal system (5.3.4) by

$$X = G^{-1} X_d \qquad (5.3.6)$$

which is defined in the results of Lemma 2.2.3.

Method 5.3.2: Amplifier Design

(a) Construct y(t) as in (a) Method 5.3.1.

(b) Form the matrix $F_m$ from a fundamental set of (a) as in (b), Method 5.2.2.

(c) Determine the nonhomogeneous part F(t) of the n-order equation (5.3.3) as in (c) Method 5.3.1.

(d) Determine the matrix A and vector Q(t) in the normal system (5.3.4) as

$$A = G^{-1}\dot{F}_m F_m^{-1} G$$
$$Q(t) = F_m \frac{d}{dt}(F_m^{-1} X(t)) \qquad (5.3.7)$$

where notation is defined in (3.3.4), (3.3.5) and (3.3.6).

(e) The initial condition on the normal system will be the same as in (e) Method 5.3.1.

The parameter-solution relationships obtained by either of these two methods when associated with the parameters in systems of differential equations that correspond to known networks configurations, are referred to as design equations.

The basic steps in design Method 5.3.1 are illustrated by an example that is itemized in correspondence to the design method.

Specifications require an amplifier with a transient response $y_t(t) = c_1 e^{-\alpha t}\sin(\omega t + \emptyset)$ and gain $G_m$ for a frequency range $\omega_1 \leq \omega \leq \omega_2$.

(a) A possible y(t) is

$$y(t) = c_1 e^{-\alpha t} + c_2 e^{-\alpha t}\sin(\omega t + \emptyset) + B_1 \sin\omega t \qquad (5.3.8)$$

where $c_1, c_2, \alpha, \omega$, and $B_1$ are real non-zero constants.

(b) First the coefficients $a_j$ $j = 1,2,3$ of (5.3.3) are related to $y_t(t)$ as in (b) Method 5.2.1. Next specifying det (B) $\neq 0$ (as in (c) Method 5.2.1) the $a_j$ coefficient are related to the $a_{ij}$ entries in matrix A of (5.3.4) as

$$a_{11} + a_{22} + a_{33} = -3\alpha = a_1$$

$$-(a_{11}a_{22} - a_{21}a_{12} + a_{11}a_{33} - a_{31}a_{13} + a_{22}a_{33} - a_{32}a_{23}) = -(3\alpha^2 + \omega^2) = a_2$$

$$\det(A) = -\alpha(\alpha^2 + \omega^2) = a_3. \qquad (5.3.9)$$

(c) Proceeding as indicated by Method 5.3.1, a possible F(t) for the third-order equation (5.3.3) is found in Thm. B.2.3 where m = 1 and n = 3 as

$$F(t) = q_1 \sin(\omega t + \theta) \qquad (5.3.10)$$

where $q_1 = B_1\alpha\sqrt{(3\alpha\omega)^2 + (2\omega^2 - \alpha^2)}$, $\quad \theta = \tan^{-1}\dfrac{3\alpha\omega}{\alpha^2 - 2\omega^2}$.

(d) The nonhomogeneous part, Q(t), of the normal system (5.3.4) is related to the nonhomogeneous part, F(t), of (5.3.3), n = 3, by

$$Q'(t) = [f_1(t), f_2(t), f_3(t)]\,(G^{-1}L^{-1})' \qquad (5.3.11)$$

where $\sum_{j=0}^{k-2} P_j^k(D)\, f_{k-1-j}(t_o) = 0$, $t_o$ on I, $k = 2,3$, as specified in Def. 3.2.1. This specification requires that the components of F(t) satisfy

$$P_o^2(D)\ f_1(t_o) = f_1(t_o) = 0$$

$$P_o^3(D)\ f_2(t_o) + P_1^3(D)\ f_1(t_o) = (\ D + \ell_{32})f_1(t_o) = 0$$

Let $f_1(t) = f_2(t) = 0$, $f_3(t) = q_1 \sin(\omega t + \theta)$. The matrix G is specified by (5. 2. 10) and matrix $L^{-1} = [\ell_{ij}]$ is determined, as in the proof of Lemma 3. 2. 2, as

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a_{33} & 1 & 0 \\ a_{33}^2 + a_{31}a_{13} + a_{32}a_{23} & a_{33} & 1 \end{bmatrix} \qquad (5.3.12)$$

The vector Q(t) (5. 3. 11) can now be calculated as

$$Q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} = \begin{bmatrix} \dfrac{-a_{32}q_1 \ \sin(\omega t + \theta)}{d} \\ \dfrac{a_{31}\ q_1\ \sin(\omega t + \theta)}{d} \\ 0 \end{bmatrix} \qquad (5.2.13)$$

where $d = a_{31}(a_{31}a_{12} + a_{32}a_{22}) - a_{32}(a_{31}a_{11} + a_{32}a_{21})$.

(e) Finally the specified initial conditions $y^{(j)}(t_o) = c_{j+1}$ $j = 0, 1, 2$, are related to the normal system by $X(t_o) = G^{-1}X_d(t_o)$, as implied by (5. 2. 10).

Any physical device having a mathematical model in the normal form (5. 3. 4), with parameter-solution relationships as specified in (5. 3. 9), (5. 3. 13) and (5. 2. 10) will contain the specified function y(t) as a component of its vector solution, in this case $x_3(t) = y(t)$. These parameter-solution relationships (design equations) are applied to the design of an amplifier.

The normal system (5.3.14) corresponds to the network shown in Figures 5.3.1.

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ I_4 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C_1} & \frac{-1}{C_1} \\ \frac{-1}{L_4} & \frac{-(r_p+R_4)}{L_4} & 0 \\ \frac{1}{L_3} & 0 & \frac{-R_3}{L_3} \end{bmatrix}\begin{bmatrix} V_1 \\ I_4 \\ I_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-\mu e_g(t)}{L_4} \\ 0 \end{bmatrix} \tag{5.3.14}$$

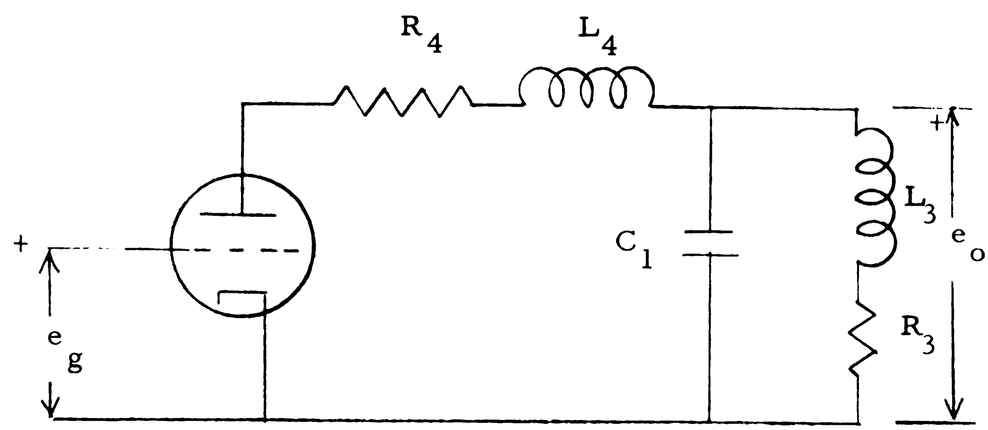



Figure 5.3.1

If $\det(B) = \frac{1}{L_3^2 C_1} \neq 0$, Method 5.3.1 applies, and the relationship between the entries in (5.3.14) and apecifications (design equations) (5.3.9), (5.3.13) and (5.2.10) are

$$\frac{(r_p + R_4)}{L_4} + \frac{R_3}{L_3} = 3a$$

$$\frac{1}{C_1}\left(\frac{1}{L_4} + \frac{1}{L_3}\right) + \frac{R_3}{L_3 L_4}(r_p + R_4) = 3a^2 + \omega^2 \qquad (5.3.15)$$

$$\frac{-(R_3 + r_p + R_4)}{L_3 L_4 C_1} = a(a^2 + \omega^2)$$

$$\frac{-\mu}{L_4} e_g(t) = L_3 C_1 q_1 \sin(\omega t + \theta).$$

The initial conditions for the system (5.3.3) are $G^{-1}[y(t_o), y^{(1)}(t_o), y^{(2)}(t_o)]'$.

The gain and frequency (bandwidth) are related by the ratio of y(t) and $e_g(t)$, from (5.3.15), as

$$G_m = \frac{-\mu}{L_3 L_4 C_1 a \sqrt{(3a\omega)^2 + (2\omega^2 - a^2)}} \qquad (5.3.16)$$

Note, in the system (5.3.14) where $I_3(t)$ is the current through $L_3$, $I_3(t) = y(t)$. Any other solution component of the normal system could have been specified similarly by starting with the desired component in the last row.

## 5.4 Nonlinear Amplifier and Oscillator Design

The mathematical properties developed in Thm. 4.3.1 and the process specified in Def. 4.2.1 for reducing the normal system (4.1.1) to an n-order differential equation of the form (4.1.2) are used in forming a method of designing nonlinear amplifiers and oscillators in the time-domain.

The mathematical models of the nonlinear amplifiers and oscillators are characterized by a discussion similar to that afforded the linear amplifier if the word "linear" is replaced by "nonlinear". In addition, it is assumed that the nonlinear oscillator contains only constant entries in the nonhomogeneous part of its mathematical model.

Suppose an amplifier (or oscillator) is to be designed with a specified response $y(t)$. A method of obtaining a normal system of nonlinear differential equations which has a solution satisfying the specification is:

(a) Construct $y(t)$ from the specifications.

(b) Apply Thm. 4.3.1 to relate $y(t)$ to (1) the coefficient $a_j$ $j = 1, 2, \ldots, n$, (2) nonlinear part $f(y)$, and (3) the nonhomogeneous part $F(t)$ of the n-order equation (4.3.1).

(c) Construct the reduced differential equation from the normal system (4.1.1) by the method specified in Def. 4.2.1. Equate the coefficients and parameters determined in (b) to corresponding parameters $a_j$ $j = 1, 2, \ldots, n-1$, $F(t)$ and $T(x_n)$ of the reduced differential equation.

(d) Relate the specified initial condition $y^{(j)}(t_o)$ $j = 0, 1, \ldots, n-1$, to the initial condition on the normal system (4.1.1) by means of the nonsingular transformation

$$X = B_{11}^{-1} [ X_d - Z(x_n, t)] \tag{5.4.1}$$

as formulated in the proof of Lemma 4.2.1 and applied in Thm. 4.2.4.

This method is illustrated on a general system of equations to obtain parameter-solution relationships which are then applied in the design of a tunnel-diode amplifier and oscillator.

Specifications require an amplifier with a gain $G_m$ for frequency range $\omega_1 \leq \omega \leq \omega_2$ .

(a) A possible y(t) is

$$y(t) = B_o + B_1 \sin(\omega t + \phi) \tag{5.4.2}$$

the same y(t) is suitable as an oscillator response where $B_1$ is the amplitude of oscillation and $f = \omega/2\pi$ the frequency of oscillation.

(b) The coefficients $a_i$, $i = 1, 2$, the parameters T(y) and F(t) of

$$\frac{d^2}{dt^2} y = \sum_{i=1}^{2} a_i \frac{d^{2-i}}{dt^{2-i}} y + T(y) + F(t) \tag{5.4.3}$$

are related to y(t) by

$$\begin{aligned} a_1 &= \frac{-b_1}{B_1} + \frac{q_1}{\omega B_1} \sin(\phi - \theta) \\ a_2 &= -\omega^2 + \frac{d_1 b_1}{B_1} - \frac{q_1}{B_1} \cos(\theta - \phi) \\ T(y) &= P_{1,n-1}(D)\, f(y) = (D - d_1) \sum_{j=0}^{m} \alpha_j y^j \\ F(t) &= d_1 b_o - B_o\left[-\omega^2 + \frac{d_1 b_1}{B_1} - \frac{q_1}{B_1}\cos(\theta - \phi)\right] + q_1 \sin(\omega t + \theta). \end{aligned} \tag{5.4.4}$$

The notation will be found in Thm. 4.3.1 and Cor. 4.3.1. Note, the nonlinear part T(y) is a polynomial of order m. The $b_o$ and $b_1$, are defined in the results of Lemma C.2 and have the form of the $b_o$ and $b_1$ in the discussion after Cor. 4.3.1, namely, they are nonlinear in the specified $B_o$ and $B_1$.

(c) Construct the reduced nonlinear differential equation corresponding to the normal system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} \\ \\ a_{21} \end{bmatrix}\begin{bmatrix} x_1 \end{bmatrix} + \begin{bmatrix} f_1(x_2) \\ \\ f_2(x_2) \end{bmatrix} + \begin{bmatrix} q_1(t) \\ \\ q_2(t) \end{bmatrix} \qquad (5.4.5)$$

as specified in Def. 4.2.1.

First, specify the condition det (B) = $a_{21} \neq 0$, formulate $X_d = B_{11}X + Z(x_n, t)$ as in the proof of Lemma 4.2.1 for this case

$$\begin{bmatrix} x_2 \\ \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \\ a_{21} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ f_2(x_2) + q_2(t) \end{bmatrix} \qquad (5.4.6)$$

Solve (5.4.6) for the vector X. Substitute the vector X (5.4.6) into the system (5.4.5) to obtain

$$\frac{d}{dt}\begin{bmatrix} x_2 \\ \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \\ 0 & a_{11} \end{bmatrix}\begin{bmatrix} x_2 \\ \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ a_{21}f_1(x_2) + (D - a_{11})f_2(x_2) + a_{21}q_1(t) + (D - a_{11})\,q_2(t) \end{bmatrix} \qquad (5.4.7)$$

the last row of which is the reduced differential equation. Equating the coefficients of the reduced equation to corresponding parts in (5. 4.4) provides the desired parameter-solution relationships (design equations).

Coefficients of matrix $A_1$:

$$a_{11} = \frac{-b_1}{B_1} + \frac{q_1}{\omega B_1}\sin(\phi - \theta) \qquad (5.4.8)$$

$$a_{21} = a_{21} \neq 0$$

Coefficients of vector $N(x_2)$:

$$f_1(x_2) = \frac{1}{a_{21}}[-\omega^2 - (\frac{b_1}{B_1} - \frac{q_1}{\omega B_1} \sin(\phi-\theta)) \frac{b_1}{B_1} -$$

$$\frac{q_1}{B_1} \cos(\theta - \phi)] y$$

$$f_2(x_2) = \sum_{j=0}^{m} a_j y^j$$

Coefficients of vector $Q_1(t)$:

$$q_1(t) = \frac{1}{a_{21}} \Big\{ - (\frac{b_1}{B_1} - \frac{q_1}{\omega B_1} \sin(\phi - \theta)) b_o -$$

$$B_o[-\omega^2 - (\frac{b_1}{B_1} - \frac{q_1}{\omega B_1} \sin(\phi - \theta)) \frac{b_1}{B_1} -$$

$$\frac{q_1}{B_1} \cos(\theta - \phi)] + q_1 \sin(\omega t + \theta) \Big\}$$

$$q_2(t) = 0.$$

Where $b_o$ and $b_1$, as defined in the results of Lemma C.2, are nonlinear in the specified $B_o$ and $B_1$. It is important to note the system parameters (5.4.8) have been obtained for the case where the nonlinearity is a polynomial of order m. The effect of the nonlinearlity is apparent in $b_o$ and $b_1$ which contain the specified $B_o$ and $B_1$ and powers of these constants up to and including $B_o^m$ and $B_1^m$.

(d) Finally the initial conditions for the normal system (5.4.5) as obtained from (5.4.6) are

$$x_1(t_o) = \frac{1}{a_{21}} [\dot{y}(t_o) - \sum_{j=0}^{m} a_j y^j(t_o)]$$

$$x_2(t_o) = y(t_o) \tag{5.4.9}$$

Any physical device having a mathematical system of equations in the normal form (5.4.5), with parameter-solution relationships (design equations) as specified in (5.4.8) and (5.4.9) will contain the specified response y(t) as a component of its vector solution (in this case $x_2(t) = y(t)$ ). The parameter-solution relationships (design equations) (5.4.8) and (5.4.9) are applied in the following to the design of a particular physical system.

A normal system corresponding to the tunnel-diode network of Figure 5.4.1 is

$$\frac{d}{dt}\begin{bmatrix} I_L \\ \\ v_c \end{bmatrix} = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} I_L \\ \\ v_c \end{bmatrix} + \begin{bmatrix} 0 \\ \\ \frac{-f(v_c)}{C} \end{bmatrix} + \begin{bmatrix} \frac{v_s(t)}{L} \\ \\ 0 \end{bmatrix} \tag{5.4.10}$$

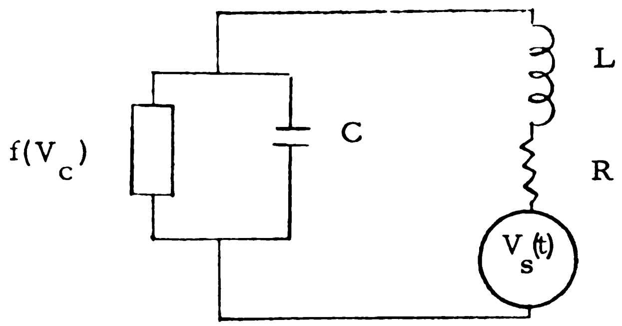



Figure 5.4.1

The tunnel diode characteristic is assumed to be

$$f(v_c) = (gv_o - hv_o^3 + I_o) + (-g + 3hv_o^2)v_c - 3hv_o v_c^2 + hv_c^3 \tag{5.4.11}$$

which is the idealized-tunnel-diode characteristic

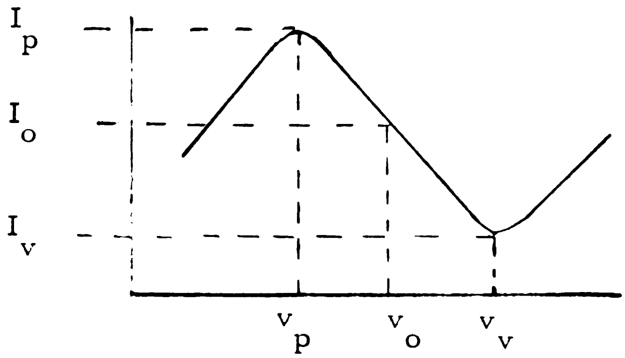



Figure 5.4.2

where:

$$I_o = \frac{I_p + I_v}{2} \quad , \qquad v_o = \frac{v_p + v_v}{2}$$

$$g = \frac{3}{2}\,\frac{(I_p - I_v)}{(v_v - v_p)} \quad , \; h = \frac{2(I_o - I_v)}{(v_v - v_o)(v_v - v_p)^2}$$

$I_p$ and $v_p$ are the peak current and voltage, $I_v$ and $v_v$ are the valley current and voltage. The inter-relationship between the entries in (5.4.8), (5.4.9), (5.4.10) and (5.4.11) are determined as

$$\frac{-R}{L} = \frac{G_d(e)}{C} + \frac{1}{\omega G_m}\sin(\emptyset - \theta)$$

$$\frac{-1}{L} = C[-\omega^2 - (\frac{G_d(e)}{C} + \frac{\sin(\emptyset - \theta)}{\omega G_m})\,\frac{G_d(e)}{C} - \frac{\cos(\theta - \emptyset)}{G_m} \qquad (5.4.12)$$

$$\frac{-f(v_c)}{C} = \sum_{j=0}^{3} a_j\, v_c^j$$

$$\frac{v_s(t)}{L} = C\Big\{\frac{-1}{C}\,(\frac{G_d(e)}{C} + \frac{\sin(\emptyset - \theta)}{\omega G_m})\,[(-g + \frac{3}{2}hB_1^2)\,e + he^3 + I_o] -$$

$$(v_o + e)\,[-\omega^2 - (\frac{G_d(e)}{C} + \frac{\sin(\theta - \emptyset)}{\omega G_m})\,\frac{G_d(e)}{C} - \frac{q_1}{B_1}\cos(\theta - \emptyset)] +$$

$$q_1 \sin(\omega t + \theta)\Big\}$$

where: $G_d(e) = g + 3he^2 + \frac{3}{4} hB_1^2$, Gain $G_m = q_1/B_1$ frequency $f = \omega/2\pi$ and $e = B_o - V_o$. The initial conditions on the normal system are then

$$I_L(t_o) = C\ B_1\ \omega \cos(\omega t_o + \emptyset) + f((v_o + e) + B_1 \sin(\omega t_o + \emptyset))$$

$$v_c(t_o) = (v_o + e) + B_1 \sin(\omega t_o + \emptyset). \tag{5.4.13}$$

The oscillator design equations ($q_1 = 0$) are:

$$\frac{-R}{L} = \frac{G_d(e)}{C}$$

$$\frac{1}{L} = C\left[\omega^2 + \frac{G_d^2(e)}{C^2}\right]$$

$$\begin{aligned} \frac{v_s}{L} = C \Big\{ & -\frac{G_d(e)}{C^2}\left[\left(-g + \frac{3}{2} hB_1^2\right) e + h\ e^3 + I_o\right] + \\ & (v_o + e)\left[\omega^2 + \frac{G_d^2(e)}{C^2}\right]\Big\}. \end{aligned} \tag{5.4.14}$$

It is important to note that three design equations are presented in (5.4.12) and (5.4.14). The addition of the equation for $V_s(t)/L$ in (5.4.12) and (5.4.14) is unique to this thesis. In addition, the nonlinear function (a function of operating point e, and a function of the specified $B_1$) $G_d(e)$ is a generalization of the results of Kim [8, p.416], who obtained by a different method $G_d(e)$ where $e = 0$. The generality presented in this design technique is apparent after examining parameter-solution relations (5.4.8).

The parameter-solution relationships (5.4.8) as viewed from the application point of view are a function of operating point and a tunnel-diode characteristic which is approximated by a polynomial of m-order.

# VI. CONCLUSION

The first parts of Chapters II and III develop the mathematical properties which relate the solution of the normal system of linear differential equations (1.2) to the solution of an r-order ($r \leq n$) differential equation, (1,3). The foundation of this development is presented in the proofs of Theorems 2.2.1 and 3.2.1. It is proved in these theorems, by applying a transformation of the form

$$X = C^{-1} [ Y_s - L^{-1} H(t) ] \qquad (6.1)$$

where $X' = [x_1, x_2, \ldots, x_n]$ and $Y_s' = [0, 0, \ldots, 0, y, y^{(1)}, \ldots, y^{(r-1)}, 0, 0, \ldots, 0]$ to the normal system (1.2), that there exists a set of s differential equations, $1 \leq s \leq n$, of r-order associated with the system. These results are extended in Theorems 2.2.3 and 3.2.2.

In Theorems 2.2.3 and 3.2.2 conditions on the $a_{ij}$ entries of matrix A in the normal system (1.2) are given so that there exists a differential equation of n-order associated with the system. In the proof of these results a technique for formulating a transformation of the form (6.1), $r = n$, is given. Note, the mathematical properties developed in these theorems allow the determination of the solution of a normal system in terms of the solution to the r-order ($r \leq n$) differential equation associated with the system.

Additional mathematical relationships are developed in the later parts of Chapters II and III to interrelate the parameters in the normal system (1.2) to the solution of the r-order ($r \leq n$) differential equation associated with the system. These relationships provide the mathematical tools for relating the system parameters to a component $x_n(t)$ of the system solution $X(t)$.

The usefulness of the mathematical properties developed in Chapters II and III are demonstrated in design methods and examples of Chapter V. One method, which illustrates some parameter-solution relationships developed in Chapter II, has two basic steps: (1) Construct (Thm. 2.3.3) an n-order homogeneous differential equation which has a specified solution, $x_n(t)$. (2) Relate the coefficients (Thm. 2.3.2) and the initial conditions, transformation (6.1) where $r=n$ and $H(t) = 0$, of the n-order differential equation to the coefficients and initial conditions of a normal system of differential equations.

In another design method of Chapter V some particularly interesting results which were developed in Thm. 3.3.1 of Chapter III are applied. The coefficient matrix A and the vector $Q(t)$ in the normal system (1.2), (3.1.1) are related to a solution $x_n(t)$ and the nonhomogeneous part $F(t)$ of the n-order differential equation (3.1.2) by an expression of the form

$$A = G^{-1} \dot{F}_m F_m^{-1} G$$

$$Q(t) = F_m \frac{d}{dt}(F_m^{-1} X(t))$$

This design method is completed by determining the initial condition on the normal system by means of a transformation (3.3.6), of the form (6.1).

Patterned after the linear case, it is proved in Chapter IV that under certain conditions (hypothesis of Thm. 4.2.3) a class of nonlinear differential equation (4.1.1) can be transformed into an n-order "reduced" (Def. 4.3.1) differential equation. A solution of the reduced differential equation and a solution of the corresponding nonlinear system are shown (in the proof of Thm. 4.2.4) to be related by

$$X_d = B_{11} X + Z(x_p, t)$$

where $X_d' = [x_p, x_p^{(1)}, \ldots, x_p^{(n-1)}]$, $X' = [x_1, x_2, \ldots, x_n]$, and $B_{11}$ and the vector $Z(x_p, t)$ are defined in the proof of Lemma 4.2.1. These results are applied in the design of tunnel-diode amplifiers and oscillators in Chapter V.

The criteria det (B) $\neq 0$ as defined in the hypothesis of Lemma 2.2.3 was proved necessary for a transformation of the form (6.1), r=n, to exist. This criteria is thus found in the mathematical tools applied to the five design methods of Chapter V. The development of other criteria for the existence of the transformation (6.1) would be a useful extension of the results presented here.

# LIST OF REFERENCES


1. Brown, D.P., Derivative explicit differential equations for RLC graphs, Journal of Franklin Institute, Vol. 275, No. 6, 1963.

2. Wirth, J.L., Time Domain Models of Physical Systems and Existence of Solutions, A Thesis, Michigan State University, 1962.

3. Koenig, H.E., Tokad, Y., and Kesavan, H.K., Analysis of Discrete Physical Systems, Class Notes for Pilot Program, E.E. Dept., Michigan State University., 1962.

4. Murray, F., and Miller, K., Existence Theorems, New York University Press, 1954.

5. Moulton, F.R., Differential Equations, Dover, 1958.

6. Coddington, E., and Levinson, N, Theory of Ordinary Differential Equations, McGraw-Hill Company, Inc., 1955.

7. Hurewicz, W., Lectures on Ordinary Differential Equations, John Wiley and Son, Inc., 1958.

8. Kim, C.S., and Brandli, A., High frequency high-power operation of tunnel diodes, I.R.E. Trans. on Circuit Theory, Vol CT-8, No. 4, 1961.

9. Turnbull, H.W., and Aitken, A.C., An Introduction to the Theory of Canonical Matrices, Dover, 1961.

10. Gantmacher, F.R., The Theory of Matrices, Vol. 1, Chelsea Publishing Company, New York, N.Y., 1959.

11. Faddeeva, V.N., Computational Methods of Linear Algebra, Dover, 1959.

12. Ryder, J.D., Electronic Fundamentals and Application, 2nd ed., Prentice-Hall, Inc., 1960.

13. Valley, Jr., G.E., and H. Wallman (Editors), Vacuum Tube Amplifiers, McGraw-Hill Book Company, Inc., 1948.

14. Hohn, F.E., Elementary Matrix Algebra, Macmillan Company, 1958.

15. Kaplan, W., Advanced Calculus, Addison-Wesley Publishing Company, Inc., 1953.

16. Hildebrand, F.B., Introduction to Numerical Analysis, McGraw-Hill Book Company, Inc., 1953.

17. Dwight, H., Tables of Integrals and Other Mathematical Data, Macmillan Company, 1961.

## APPENDIX A

## THEOREMS AND DEFINITIONS FROM REFERENCES

Definition A.1: The m n-dimensional vectors $X_j$ $j = 1, 2, \ldots, m$ are said to be "linearly independent" if the identity $\sum_{i=1}^{m} C_i X_i = 0$, where $C_i$ constant, implies $C_i = 0$, $i = 1, 2, \ldots, m, m \leq n$.

Definition A.2: A set of functions $y_j(t)$, $j = 1, 2, \ldots, r$, is said to be "linearly independent" over the open interval $I = [t : t_1 < t < t_2]$ where $t_1$ and $t_2$ are constants, if the identity $\sum_{j=1}^{r} C_j y_j(t) = 0$ for all t on I implies $C_j = 0$, $j = 1, 2, \ldots, r$.

Definition A.3: A set of functions $y_j(t)$, $j = 1, 2, \ldots, r$, which are linearly independent solutions of the r-order equation (2.1.2) on I, is called a "fundamental set" of (2.1.2).

Definition A.4: If $F_m$ is a matrix whose n column are n linearly independent solutions on I of (2.1.1), the normal system $\dot{X} = AX$, then $F_m$ is called a "fundamental matrix."

Theorem A.1: [6, p.70] If H is a fundamental matrix of (2.1.1) the normal system $\dot{X} = AX$ and D a(complex) constant nonsingular matrix, then HD is again a fundamental matrix of (2.1.1). If $H_1$ and $H_2$ are fundamental matrices then $H_1 = H_2 D_2$ where $D_2$ is nonsingular.

Theorem A.2: [7, p. 47] The r-order homogeneous differential equation (2.1.2) always has a fundamental set (Def. A.3) of precisely r solutions. A set of r solutions of (2.1.2) on I constitutes a fundamental set if and only if its Wronskian

$$W(y_1, y_2, \ldots, y_r) = \begin{bmatrix} y_1 & y_2 & \cdots & y_r \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_r^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(r-1)} & y_2^{(r-1)} & \cdots & y_r^{(r-1)} \end{bmatrix}$$

is nonsingular ($W \neq 0$) for some $t_o$ on I.

Theorem A.3: [14, p. 215] The coefficients of $\lambda^r$ in the characteristic polynomial of matrix A, det $[A-\lambda I]$ is $(-1)^r$ times the sum of the principal minors of order n-r of matrix A, where $A = [a_{ij}]$ is of n-order and I is the unit matrix. In particular, the coefficient of $\lambda^n$ is $(-1)^n$ and the constant term in the characteristic polynomial is det (A).

Leibnitz's Rule: [15, p. 219] Let the function f(x, t) be continuous and have a continuous derivative in a domain of the x-t plane which includes the rectangle $a \leq x \leq b$, $t_1 \leq t \leq t_2$. Then for $t_1 < t < t_2$

$$\frac{d}{dt} \int_a^b f(x,t)dx = \int_a^b \frac{\partial f}{\partial t}(x,t)dx \quad .$$

Theorem A.4: [15, p. 220] Let the function $f(x, t)$ satisfy Leibnitz's rule. In addition, let the functions $a(t)$ and $b(t)$ be defined and have continuous derivatives for $t_1 < t < t_2$.

Then

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = f[b(t),t]\,\frac{db(t)}{dt} - f[a(t),t]\,\frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)\,dx \ .$$

## APPENDIX B

## SOLUTIONS OF CERTAIN DIFFERENTIAL EQUATIONS

The design techniques developed in this thesis require that a specified system performance characteristic be considered a solution of an n-order differential equation. This necessitates a knowledge of the inter-relationship of the solution of an n-order differential equation to the coefficients of the n-order differential equation.

The n-order homogeneous differential equation

$$\frac{d^n}{dt^n} y = \sum_{i=1}^{n} a_i \frac{d^{n-i}}{dt^{n-i}} y \qquad (B.1)$$

is known [7, p. 65] to have the general solution

$$y(t) = \sum_{i=1}^{k} P_{m_i - 1}(t)\, e^{\lambda_i t} \qquad (B.2)$$

where (a) $P_{m_i-1}(t)$ is a polynomial in t of degree $m_i-1$, (b) $\lambda_i$, $i = 1, 2, \ldots, k$ are the distinct zeros of the polynomial $L(\lambda) = \lambda^n - \sum_{i=1}^{n} a_i \lambda^{n-i} = 0$ each of multiplicity $m_i$ and (c) $\sum_{i=1}^{k} m_i = n$.

The n-order nonhomogeneous differential equation

$$\frac{d^n}{dt^n} y = \sum_{i=1}^{n} a_i \frac{d^{n-i}}{dt^{n-i}} y + F(t) \qquad (B.3)$$

is known [6, p. 87] to have the solution

$$y(t) = y_h(t) + y_p(t) \qquad (B.4)$$

where (a) $y_h(t)$ satisfies (B.1), (b) $y_p(t) = \sum_{i=1}^{n} \phi_i(t) \int_{t_o}^{t} \frac{W_k(s)}{W(s)} F(s)ds$

and (1) $\phi_i(t)$, $i = 1, 2, \ldots, n$, is a fundamental set for (B.3) when $F(t) = 0$. (2) $W(s)$ is the Wronskian of (B.1) and (3) $W_k(s)$ is the determinant obtained from the Wronskian (2) by replacing the k-column by (0, ..., 0, 1).

When $\lambda_i$ in (B.2) are distinct, $i = 1, 2, \ldots, n$, then the particular solution $y(t)$, $y(t_o) = 0$, $t_o$ on I can be expressed by

$$y(t) = \sum_{i=1}^{n} A_i e^{\lambda_i t} \int_{t_o}^{t} F(s)\, e^{-\lambda_i s}\, ds \qquad \text{(B.5)}$$

where $A_i = \dfrac{(-1)^{n+i}}{\prod (\lambda_p - \lambda_i)(\lambda_i - \lambda_j)}$

$n \geq p > i > j > 0$

## B.1 Inter-relationships of Coefficients of N-Order Polynomial to Zeros of Polynomial

The design methods of Chapter V necessitate a knowledge of the relationships between the coefficients and a solution of an n-order homogeneous differential equation. This information is supplied in Thm. 2.3.3 as a result of developing the relationships between the coefficients and zeros of an n-order polynomial.

<u>Theorem B.1.1:</u> If $\lambda_i$, $i = 1, 2, \ldots, k$ is a zero of $L(\lambda) = \lambda^n + \sum_{j=1}^{n} b_j \lambda^{n-j}$, of multiplicity $m_i$, $\sum_{i=1}^{k} m_i = n$, then

$$
\begin{aligned}
b_1 &= -(r_1 + r_2 + \ldots + r_n) \\
b_2 &= (r_1 r_2 + \ldots + r_1 r_n + r_2 r_3 + \ldots + r_2 r_n + \ldots + r_{n-1} r_n) \\
&\;\;\vdots \\
b_n &= (-1)^n (r_1 r_2 \ldots r_n)
\end{aligned}
\tag{B.1.1}
$$

where $r_i = \lambda_j$, $i = \sum_{p=1}^{j-1} m_p + 1, \sum_{p=1}^{j-1} m_p + 2, \ldots, \sum_{p=1}^{j} m_p$, $j = 1, 2, \ldots, k$.

<u>Proof</u>: By hypothesis

$$
\begin{aligned}
L(\lambda) &= \lambda^n + \sum_{j=1}^{n} b_j \lambda^{n-j} \\
&= (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_k)^{m_k}
\end{aligned}
$$

If $r_i = \lambda_j$ where $i = \sum_{p=1}^{j-1} m_p + 1, \sum_{p=1}^{j-1} m_p + 2, \ldots, \sum_{p=1}^{j} m_p$ and $j = 1, 2, \ldots, k$,

then

$$
\begin{aligned}
L(\lambda) &= (\lambda - r_1) \ldots (\lambda - r_{m_1})(\lambda - r_{m_1+1}) \ldots (\lambda - r_{m_1+m_2}) \ldots (\lambda - r_n) \\
&= \lambda^n - (r_1 + r_2 + \ldots + r_n)\lambda^{n-1} + (r_1 r_2 + \ldots + r_1 r_n + r_2 r_3 + \ldots \\
&\quad + r_2 r_n + \ldots + r_{n-1} r_n)\lambda^{n-2} \ldots (-1)^n (r_1 r_2 \ldots r_n).
\end{aligned}
$$

This implies the theorem.

Corollary B.1.1: If $\lambda_i$, $i = 1, 2, \ldots, n$ are distinct roots of $L(\lambda) = \lambda^n + \sum_{j=1}^{n} b_j \lambda^{n-j}$ then

$$
\begin{aligned}
b_1 &= -(\lambda_1 + \lambda_2 \ldots + \lambda_n) \\
b_2 &= +(\lambda_1\lambda_2 + \ldots + \lambda_1\lambda_n + \lambda_2\lambda_3 + \ldots + \lambda_2\lambda_n + \ldots + \lambda_{n-1}\lambda_n) \\
&\vdots \\
b_n &= (-1)^n (\lambda_1\lambda_2 \ldots \lambda_n).
\end{aligned}
\tag{B.1.2}
$$

Proof: This is a direct consequence of Thm. B.1.1.

Theorem B.1.2: If $\lambda_i$, $i = 1, 2, \ldots, r$ are distinct non zero, zeros of $L(\lambda) = \lambda^n + \sum_{j=1}^{n} b_j \lambda^{n-j}$, then r coefficients of $L(\lambda)$, $b_j, b_{j+1}, \ldots, b_{j+r-1}$ are explicitly related (B.1.4) to the remaining n-r coefficients.

Proof: By hypothesis

$$L(\lambda_i) = \lambda_i^n + \sum_{j=1}^{n} b_j \lambda_i^{n-j} = 0 \quad \text{for } i = 1, 2, \ldots, r.$$

This system is written in matrix form as

$$Va = L$$

where $a' = [b_j, b_{j+1}, \ldots, b_{j+r-1}]$, $L' = [-\lambda_1^n - \sum_s b_i \lambda_1^{n-i}, \ldots, -\lambda_r^n - \sum_s b_i \lambda_r^{n-i}]$,

$\sum_s$ is the sum over all i, $i \neq j, j+1, \ldots, j+r-1$ and

$$V = \begin{bmatrix} \lambda_1^{n-j} & \lambda_1^{n-j-1} & \cdots & \lambda_1^{n-j-r+1} \\ \lambda_2^{n-j} & \lambda_2^{n-j-1} & \cdots & \lambda_2^{n-j-r+1} \\ & \cdots\cdots & & \\ \lambda_r^{n-j} & \lambda_r^{n-j-1} & \cdots & \lambda_r^{n-j-r+1} \end{bmatrix} \qquad (B.1.3)$$

For $j = 1$, $r = n$ (B.1.3) is the Vandermonde matrix [16, p. 85]. For $r \leq n$

$$\det(V) = k\, V_1$$

where $k = [\lambda_1^{n-j-r+1}, \lambda_2^{n-j-r+1}, \ldots, \lambda_r^{n-j-r+1}]$, $V_1 = \prod_{r \geq i > j} (\lambda_i - \lambda_j)$.

$V_1$ is the Vandermonde determinant [14, p.47] which is non-zero since $\lambda_i \neq \lambda_j$. Therefore

$$a = V^{-1} L \qquad (B.1.4)$$

which implies the theorem.

## B.2 Solutions of N-Order Nonhomogeneous Differential Equations

The relationships between the coefficients and a solution of an n-order nonhomogeneous differential equations are now determined. Formulas are given such that if a solution is known, some or all of the coefficients of a corresponding differential equation are specified. The relationships determined are for particular solutions in the form of power series, linear combinations of exponential functions and linear combinations of sine and cosine functions. The results of this section are applied in the design methods of Chapter V.

Theorem B.2.1: If $y(t) = \sum_{j=0}^{m} B_j t^j$ is a solution on I: $|t| > t_o$, of

$$\frac{d^n x}{dt^n} = \sum_{j=1}^{n} a_j \frac{d^{n-j}}{dt^{n-j}} x + F(t) \qquad \text{(B.2.1)}$$

where $F(t) = \sum_{j=0}^{m} q_j t^j$ then

$$q_j = \frac{(j+n)!}{j!} B_{j+n} - \frac{(j+n-1)!}{j!} B_{j+n-1} a_1 - \dots - \frac{j!}{j!} B_j a_n \qquad \text{(B.2.2)}$$

where $j = 0, 1, \dots, m$, $B_k = 0$, $k > m$.

Proof: By hypothesis $y(t) = \sum_{j=0}^{m} B_j t^j$ is a solution of (B.2.1) on I: $|t| > t_o$, therefore

$$\frac{d^n}{dt^n} \left( \sum_{j=0}^{m} B_j t^j \right) = \sum_{i=1}^{n} a_i \frac{d^{n-i}}{dt^{n-i}} \left( \sum_{j=0}^{m} B_j t^j \right) + \sum_{j=0}^{m} q_j t^j$$

Since $\frac{d^r}{dt^r} \left( \sum_{j=0}^{m} B_j t^j \right) = \sum_{j=0}^{m} (j)(j-1)\dots(j-r+1)\, t^{j-r}$.

$$\sum_{j=0}^{m} B_j (j)(j-1)(j-2)\dots(j-n+1) t^{j-n} = \sum_{i=1}^{n} a_i \sum_{j=0}^{m} B_j (j)(j-1)\dots(j-n+i+1) t^{j-n+i}$$

$$+ a_n \sum_{j=0}^{m} B_j t^j + \sum_{j=0}^{m} q_j t^j .$$

Grouping the coefficients of like powers of t and equating each to zero the theorem follows.

Corollary B.2.1: If $B_m$ of hypothesis of Thm. B.2.1 is not zero then

(1) $a_{n-m}, a_{n-m+1}, \ldots, a_n$ are explicitly expressed in terms of $q_o, q_1, \ldots, q_m$, for $m < n-1$

(2) $a_1, a_2, \ldots, a_n$ are explicitly expressed in terms of $q_o, q_1, \ldots, q_m$, for $m = n+1$

(3) $a_1, a_2, \ldots, a_n$ are explicitly expressed in terms of $q_{m-n+1}, q_{m-n+2}, \ldots, q_m$, for $m > n-1$.

Proof: The system of equations (B. 2. 2) in matrix form is

$$M a = \mathbf{q}$$

where $a' = [a_{n-m}, a_{n-m+1}, \ldots, a_n]$, $q' = [-q_o, \ldots, -q_m]$ for $m < n-1$, $a' = [a_1, a_2, \ldots, a_n]$, $q' = [-q_o, \ldots, -q_m]$ for $m = n-1$, and $a' = [a_1, a_2, \ldots, a_n]$, $q' = [-q_{m-n+1}, \ldots, -q_m]$ for $m > n-1$.
The coefficient matrix M is upper triangular in each case and has a nonzero determinent equal to $\frac{(m!\ B_m)^n}{(m-n+1)!\ (m-n+2)!\ \ldots (m)!}$ for $m = n$, $\frac{(m!\ B_m)^n}{0!\ 1!\ 2!\ \ldots\ m!}$ for $m \geq n$ and $\frac{(m!\ B_m)^{m+1}}{0!\ 1!\ \ldots\ m!}$ for $m < n-1$.

This implies the corollary.

Theorem B. 2. 2: If $y(t) = \sum_{j=1}^{m} B_j e^{\mu_j t}$ is a solution on $I: |t| > t_o$ of (B. 2. 1) where $F(t) = \sum_{j=1}^{m} q_j e^{\mu_j t}$, then

$$q_j = B_j (\mu_j^n - \sum_{i=1}^{n} a_i \mu_j^{n-i}) \qquad \text{(B. 2. 3)}$$

Proof: By hypothesis $y(t) = \sum_{j=1}^{m} B_j e^{\mu_j t}$ is a solution of (B.2.1) on I: $|t| > t_o$, therefore,

$$\frac{d^n}{dt^n}\left(\sum_{j=1}^{m} B_j e^{\mu_j t}\right) = \sum_{i=1}^{n} a_i \frac{d^{n-i}}{dt^{n-i}}\left(\sum_{j=1}^{m} B_j e^{\mu_j t}\right) + \sum_{j=1}^{m} q_j e^{\mu_j t}$$

Since $\frac{d^r}{dt^r}\left(\sum_{j=1}^{m} B_j e^{\mu_j t}\right) = \sum_{j=1}^{m} \mu_j^r B_j e^{\mu_j t}$, grouping the coefficients of $e^{\mu_j t}$ of the above equation results in

$$\sum_{j=1}^{m} \left(B_j \mu_j^n - B_j \sum_{i=1}^{n} a_i \mu_j^{n-i} - q_j\right) e^{\mu_j t} = 0.$$

Since this is true for all $|t| > t_o$, the theorem follows.

Corollary B.2.3: If $B_j \neq 0$, $\mu_j \neq 0$ and $\mu_i \neq \mu_j$ for $i, j = 1, 2, \ldots, m$, then m coefficients of (B.2.3) $a_j, a_{j+1}, \ldots, a_{j+m-1}$ are explicitly related (B.2.5) to the remaining n-m $a_i$'s.

Proof: Consider the system of m, $m \leq n$, equations (B.2.3) written in matrix form

$$Va = L$$

where $a' = [a_j, a_{j+1}, \ldots, a_{j+m-1}]$, $L' = [\frac{-q_1}{B_1} + \mu_1^n + \sum_s a_i \mu_1^{n-i}, \ldots,$ $\frac{-q_m}{B_m} + \mu_m^n + \sum_s a_i \mu_m^{n-i}]$, $\sum_s$ is the sum over all i, $i \neq j, j+1, \ldots,$ $j + m-1$ and

$$V = \begin{bmatrix} \mu_1^{n-j} & \mu_1^{n-j-1} & \cdots & \mu_1^{n-j-m+1} \\ & \cdots\cdots\cdots & & \\ \mu_m^{n-j} & \mu_m^{n-j-1} & \cdots & \mu_m^{n-j-m+1} \end{bmatrix} \tag{B.2.4}$$

Since $\det(V) = \mu_1^{n-j-m+1} \mu_2^{n-j-m+1} \cdots \mu_m^{n-j-m+1} \prod_{m \geq i > j} (\mu_i - \mu_j)$,

and by hypothesis $\mu_i \neq 0$, $\mu_i \neq \mu_j$ therefore $\det(V) \neq 0$. This implies the conclusion since,

$$a = V^{-1} L \tag{B.2.5}$$

<u>Theorem B.2.3:</u> If $y(t) = B_o + \sum_{j=1}^{m} B_j \sin(\omega_j t + \phi_j)$ is a solution on I: $|t| > t_o$ of (B.2.1) and $F(t) = q_o + \sum_{j=1}^{m} q_j \sin(\omega_j t + \theta_j)$ then

(1) $q_o = -a_n B_o$

(2) $q_j \cos\theta_j = B_j \omega_j^n \cos(\frac{n\pi}{2} + \phi_j) - B_j \sum_{i=1}^{n} a_i \omega_j^{n-i} \cos(\frac{(n-i)\pi}{2} + \phi_j)$

(3) $q_j \sin\theta_j = B_j \omega_j^n \sin(\frac{n\pi}{2} + \phi_j) - B_j \sum_{i=1}^{n} a_i \omega_j^{n-i} \sin(\frac{(n-i)\pi}{2} + \phi_j)$

where $j = 1, 2, \ldots, m$.

<u>Proof:</u> By hypothesis $y(t) = B_o + \sum_{j=1}^{m} B_j \sin(\omega_j t + \phi_j)$ is a solution of (B.2.1) on I: $|t| > t_o$, therefore

$$\frac{d^n}{dt^n}\left(B_o + \sum_{j=1}^{m} B_j \sin(\omega_j t + \phi)\right) = \sum_{i=1}^{n} a_i \frac{d^{n-i}}{dt^{n-i}}\left(B_o + \sum_{j=1}^{m} B_j \sin(\omega_j t + \phi_j)\right) +$$

$$q_o + \sum_{j=1}^{m} q_j \sin(\omega_j t + \theta_j).$$

Calculating the indicated derivatives, this equation is,

$$\sum_{j=1}^{m} B_j \omega_j^n [\sin(\frac{n\pi}{2} + \phi_j) \cos \omega_j t + \cos(\frac{n\pi}{2} + \phi_j) \sin \omega_j t ]$$

$$= \sum_{i=1}^{n-1} a_i B_o + \sum_{j=1}^{m} B_j \omega_j^{n-i} [\sin(\frac{(n-i)\pi}{2} + \phi_j)\cos\omega_j t + \cos(\frac{(n-i)\pi}{2} + \phi_j)\sin\omega_j t]$$

$$+ a_n B_o + a_n \sum_{j=1}^{m} B_j [\sin\phi_j \cos \omega_j t + \cos \phi_j \sin \omega_j t] +$$

$$q_o + \sum_{j=1}^{n} (q_j \sin \theta_j \cos \omega_j t + q_j \cos \theta_j \sin \omega_j t).$$

Grouping the coefficients of $\sin \omega_j t$ and the coefficients of $\cos \omega_j t$ and equating each to zero results in the theorem.

If $y(t) = B_o + \sum_{j=1}^{m} B_j \cos \phi_j \sin \omega_j t + \sum_{j=1}^{m} B_j \sin \phi_j \cos \omega_j t$ is the solution on I: $|t| > t_o$ of (B.2.1) and $F(t) = q_o + \sum_{j=1}^{m} q_j \cos \theta_j \sin\omega_j t + \sum_{j=1}^{m} q_j \sin \theta_j \cos \omega_j t$ then in a manner similar to that of the proof of Thm. B.2.3

(1) $\quad q_o = -a_n B_o$

(2) $$q_j \cos\theta_j = B_j \cos\theta_j [\omega_j^n \cos\frac{n\pi}{2} - \sum_{i=1}^{n} a_i \omega_j^{n-i} \cos \frac{(n-i)\pi}{2} ]$$
$$- B_j \sin\theta_j [\omega_j^n \sin\frac{n\pi}{2} - \sum_{i=1}^{n} a_i \omega_j^{n-i} \sin \frac{(n-i)\pi}{2} ]$$

(3) $$q_j \sin\theta_j = B_j \cos\theta_j [\omega_j^n \sin\frac{n\pi}{2} - \sum_{i=1}^{n} a_i \omega_j^{n-i} \sin \frac{(n-i)\pi}{2} ]$$
$$+ B_j \sin \theta_j [\omega_j^n \cos \frac{n\pi}{2} - \sum_{i=1}^{n} a_i \omega_j^{n-i} \cos \frac{(n-i)\pi}{2} ].$$

Corollary B.2.5: If $m = 1$, $p-q \neq 0, 2, 4, \ldots, \omega_1 \neq 0$, $B_1 \neq 0$ then

(1) $B_o a_n = -q_o$

(2) $$a_p = \frac{-[F_1 \sin(\frac{(n-q)\pi}{2} + \phi_1) - F_2 \cos(\frac{(n-q)\pi}{2} + \phi_1)]}{\omega_1^{n-q} \sin \frac{(p-q)\pi}{2}}$$

(3) $$a_q = \frac{[F_1 \sin(\frac{(n-p)\pi}{2} + \phi_1) - F_2 \cos(\frac{(n-p)\pi}{2} + \phi_1)]}{\omega_1^{n-q} \sin \frac{(p-q)\pi}{2}}$$

where, $$F_1 = \frac{q_1 \cos\theta_1}{B_1} - \omega_1^n \cos(\frac{n\pi}{2} + \phi_1) + \sum_s a_i \omega_1^{n-i} \cos(\frac{(n-i)\pi}{2} + \phi_1)$$

$$F_2 = \frac{q_1 \sin\theta_1}{B_1} \omega_1^n \sin(\frac{n\pi}{2} + \phi_1) + \sum_s a_i \omega_1^{n-i} \sin(\frac{(n-i)\pi}{2} + \phi_1)$$

$\sum_s$ is the sum over all i, $i \neq p, q$.

Proof: This is a direct consequence of Thm. B.2.3 for $m = 1$.

## APPENDIX C

## ON A SPECIAL INTER-RELATIONSHIP OF AN M-ORDER POLYNOMIAL TO A SINE FUNCTION

The derivation of $P_{1,n-1}(D)f\,(B_o + B_1\sin(\omega t + \phi))$ as defined in the proof of Thm. 4.3.1 is developed in this Appendix.

Lemma C.1: If $T(y) = \sum_{j=0}^{m} a_j y^j, a_m \neq 0$ and $y = y_1 + y_2$ then

$$T(y) = L\, Y_2\, Y_1 \tag{C.1}$$

where $L = [a_o, a_1, \ldots, a_m]$, $Y_1' = [1, y_1^1, y_1^2, \ldots, y_1^m]$ and

$$Y_2 = \begin{bmatrix} \binom{0}{0} & 0 & 0 & \cdots & 0 \\ y_2\binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\ y_2^2\binom{2}{0} & y_2\binom{2}{1} & \binom{2}{2} & \cdots & 0 \\ \cdots & \cdots & \cdots & & \\ y_2^m\binom{m}{0} & y_2^{m-1}\binom{m}{1} & y_2^{m-2}\binom{m}{2} & \cdots & \binom{m}{m} \end{bmatrix} \tag{C.2}$$

Proof: By hypothesis

$$T(y) = \sum_{j=0}^{m} a_j (y_1 + y_2)^j = \sum_{j=0}^{m} a_j y_2^j \left(1 + \frac{y_1}{y_2}\right)^j$$

Using the binomial theorem

$$T(y) = \sum_{j=0}^{m} a_j y_2^j \sum_{r=0}^{j} \frac{j!}{(j-r)!\; r!} \left(\frac{y_1}{y_2}\right)^r, \; r \leq j \leq m$$

$$= \sum_{j=0}^{m} a_j y_2^j \sum_{r=0}^{j} \binom{j}{r} \left(\frac{y_1}{y_2}\right)^r$$

$$= \sum_{j=0}^{m} [a_j \binom{j}{j} + a_{j+1} y_2 \binom{j+1}{j} + a_{j+2} \binom{j+2}{j} + \ldots +$$

$$a_m y_2^{m-j} \binom{m}{j} ]\, y_1^j \quad .$$

The lemma conclusion follows by writing this equation in matrix form

$$T(y) = L\, Y_2\, Y_1$$

where $L_1$, $Y_2$ and $Y_1$ are defined in the lemma conclusion.

<u>Lemma C.2:</u> If $T(y) = \sum_{j=0}^{m} a_j y^j$, $a_j \neq 0$, and $y(t) = B_1 \sin(\omega t+\phi)+B_o$, $B_o \neq 0$, then for $L = [a_o, a_1, \ldots, a_m]$, $\Phi = \omega t + \phi$,

$$T(y, t) = L R_1 R_e Y_e = \sum_{j=0}^{m/2} b_{2j} \cos 2j\Phi + \sum_{j=1}^{m/2} b_{2j-1} \sin(2j-1)\Phi$$

where m is even, $Y_e' = [1, \sin \Phi, \cos 2\Phi, \ldots, \cos m\Phi]$

$$R_1 = \begin{bmatrix} \binom{0}{0} & 0 & 0 & \ldots & 0 & 0 \\ B_0\binom{1}{0} & \binom{1}{1} & 0 & \ldots & 0 & 0 \\ B_0^2\binom{2}{0} & B_0\binom{2}{1} & \binom{2}{2} & \ldots & 0 & 0 \\ & & \cdots\cdots & & & \\ B_0^{m-1}\binom{m-1}{0} & B_0^{m-2}\binom{m-1}{1} & B_0^{m-3}\binom{m-1}{2} & \ldots & \binom{m-1}{m-1} & 0 \\ B_0^{m}\binom{m}{0} & B_0^{m-1}\binom{m}{1} & B_0^{m-1}\binom{m}{2} & \ldots & B_0\binom{m}{m-1} & \binom{m}{m} \end{bmatrix}$$

$$R_e = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & B_1 & 0 & \cdots & 0 & 0 \\ \frac{B_1^2\binom{2}{1}}{2^1 . 2} & 0 & \frac{-B_1^2\binom{2}{0}}{2} & \cdots & 0 & 0 \\ 0 & \frac{B_1^3\binom{3}{1}}{2^2} & 0 & \cdots & 0 & 0 \\ & & \cdots\cdots & & & \\ 0 & \frac{B_1^{m-1}\binom{m-1}{\frac{m}{2}-1}}{2^{m-2}} & 0 & \cdots & \frac{\pm B_1^{m-1}\binom{m-1}{0}}{2^{m-1}} & 0 \\ \frac{B_1^m\binom{m}{m/2}}{2^{m-1} . 2} & 0 & \frac{-B_1^m\binom{m}{\frac{m}{2}-1}}{2^{m-1}} & \cdots & 0 & \frac{\pm B_1^m\binom{m}{0}}{2^{m-1}} \end{bmatrix}$$

and

$$T(y,t) = L R_1 R_o Y_o = \sum_{j=0}^{\frac{m-1}{2}} b_{2j} \cos 2j\Phi + \sum_{j=1}^{\frac{m+1}{2}} b_{2j-1} \sin(2j-1)\Phi$$

where $m$ is odd, $Y_o' = [1, \sin\Phi, \cos 2\Phi, \ldots, \sin m\Phi]$ and $R_o = R_e$ with $m = m_o - 1$.

Proof: Set $y_2 = B_o$ and $y_1 = \sin\Phi$ in (C..1). It is well-known [17, p. 82] that for $k$ even

$$y_1^k = \frac{B_1^k\binom{k}{k/2}}{2^k} + \frac{B_1^k}{2^{k-1}} \sum_{v=1}^{k/2} (-1)^v \binom{k}{\frac{k}{2} - v} \cos 2v\Phi$$

and for $k$ odd

$$y_1^k = \frac{B_1^k}{2^{k-1}} \sum_{v=0}^{\frac{k-1}{2}} (-1)^v \binom{k}{\frac{k-1}{2} - v} \sin(2v+1)\Phi.$$

The lemma follows by writing (C.1) in terms of the above defined quantities ($Y_2 = R_1$ and $Y_1 = R_e Y_e$ or $R_o Y_o$).

The following relation, is an example of Lemma C.2. for $m = 3$,

$$T(y,t) = [a_o + a_1 B_o + a_2 (B_o^2 + \frac{B_1^2}{2}) + a_3 (B_o^3 + \frac{3}{2} B_o B_1^2)] +$$

$$B_1 [a_1 + 2a_2 B_o + a_3 (3B_o^2 + \frac{3}{4} B_1^2)] \sin(\omega t + \emptyset)$$

$$- \frac{B_1^2}{2} [a_2 + 3a_3 B_o] \cos 2(\omega t + \emptyset)$$

$$- a_3 \frac{B_1^3}{4} \sin 3(\omega t + \emptyset) .$$

<u>Theorem C.1:</u> If $f(y) = \sum_{j=0}^{m} a_j y^j, a_m \neq 0$, and $y(t) = B_o + B_1 \sin(\omega t + \emptyset)$. then

$$P_{1,n-1}(D) f(B_o + B_1 \sin(\omega t + \emptyset)) = -d_{n-1} b_o + \sum_{j=1}^{k} M_j b_{2j} \cos 2j\omega t +$$

$$\sum_{j=1}^{k} N_j b_{2j} \sin 2j\omega t + \sum_{j=1}^{r} L_j b_{2j-1} \sin(2j-1)\omega t + \sum_{j=1}^{r} S_j b_{2j-1} \cos(2j-1)\omega t$$

where $P_{1,n-1}(D) = D^{n-1} - d_1 D^{n-2} - d_2 D^{n-3} - \ldots - d_{n-1}, D^j = \frac{d^j}{dt^j}$, $P_{1,0}(D) = 1$,

$k = r = \frac{m}{2}$ for m even, $k = \frac{m-1}{2}$ $r = \frac{m+1}{2}$ for m odd and

$$M_j = (2j\omega)^{n-1} \cos(2j\emptyset + \frac{(n-1)\pi}{2}) - \sum_{i=1}^{n-1} d_i (2j\omega)^{n-i-1} \cos(2j\emptyset + \frac{(n-i-1)\pi}{2})$$

$$N_j = -(2j\omega)^{n-1} \sin(2j\emptyset + \frac{(n-1)\pi}{2}) + \sum_{i=1}^{n-1} d_i (2j\omega)^{n-i-1} \sin(2j\emptyset + \frac{(n-i-1)\pi}{2})$$

$$L_j = [(2j-1)\omega]^{n-1} \cos((2j-1)\emptyset + \frac{(n-1)\pi}{2}) - \sum_{i=1}^{n-1} d_i [(2j-1)\omega]^{n-i-1} \cos((2j-1)\emptyset + \frac{(n-i-1)\pi}{2})$$

$$S_j = [(2j-1)\omega]^{n-1}\sin((2j-1)\emptyset + \frac{(n-1)\pi}{2}) - \sum_{i=1}^{n-1} d_i[(2j-1)\omega]^{n-i-1}\sin((2j-1)\emptyset + \frac{(n-i-1)\pi}{2}) \quad .$$

Proof: The theorem follows by determining

$P_{1,n-1}(D)\ f(B_o+B_1\sin(\omega t+\emptyset))$ which is equal to

$$P_{1,n-1}(D)\ (\sum_{j=0}^{u} b_{2j}\cos 2j\Phi + \sum_{j=1}^{u} b_{2j-1}\sin(2j-1)\ \Phi)$$

by Lemma C.2.