ON THE EXISTENCE OF LIPSCHITZ SOLUTIONS TO SOME FORWARD-BACKWARD PARABOLIC EQUATIONS

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ABSTRACT

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In this dissertation we discuss a new approach for studying forward-backward quasilinear diffusion equations. Our main idea is motivated by a reformulation of such equations as non-homogeneous partial differential inclusions and relies on a Baire's category method. In this way the existence of Lipschitz solutions to the initial-boundary value problem of those equations is guaranteed under a certain density condition. Finally we study two important cases of anisotropic diffusion in which such density condition can be realized.

The first case is on the Perona-Malik type equations. In 1990, P. Perona and J. Malik [35] proposed an anisotropic diffusion model, called the Perona-Malik model, in image processing

$$u_t = \operatorname{div}\left(\frac{Du}{1+|Du|^2}\right)$$

for denoising and edge enhancement of a computer vision. Since then the dichotomy of numerical stability and theoretical ill-posedness of the model has attracted many interests in the name of the Perona-Malik paradox [28]. Our result in this case provides the model with mathematically rigorous solutions in any dimension that are even reflecting some phenomena observed in numerical simulations.

The other case deals with the existence result on the Höllig type equations. In 1983, K. Höllig [20] proved, in dimension $n = 1$, the existence of infinitely many L^2-weak solutions to the initial-boundary value problem of a forward-backward diffusion equation with

non-monotone piecewise linear heat flux, and this piecewise linearity was much relaxed later by K. Zhang [45]. The work [20] was initially motivated by the Clausius-Duhem inequality in the second law of thermodynamics, where the negative of the heat flux may violate the monotonicity but should obey the Fourier inequality at least. Our result in this case generalizes [20, 45] to all dimensions.

For my family

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First of all, I would like to deeply thank my advisor, Dr. Baisheng Yan for his encouragement and guidance during this work. Without his support and endless patience, all of this work and much more would have been impossible. I believe he has shown me the best example of an advisor one could be and hope I can resemble him if I have a chance to help someone in the future.

I also would like to express my sincere gratitude to my committee members Dr. Casim Abbas, Dr. Peter W. Bates, Dr. Keith S. Promislow and Dr. Zhengfang Zhou for their time, efforts and valuable suggestions.

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Introduction

The evolution process of many quantities in applications can be modeled by a diffusion partial differential equation of the form

$$u_t = \operatorname{div}(A(Du)) \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $T > 0$ is any fixed number, and $u = u(x,t)$ is the density of some quantity at position x and time t, with $Du = (u_{x_1}, \cdots, u_{x_n})$ and u_t denoting its spatial gradient and rate of change, respectively. The vector function $A\colon \mathbb{R}^n \to \mathbb{R}^n$ here represents the *diffusion flux* of the evolution process. The usual heat equation corresponds to the case of isotropic diffusion given by the Fourier law: $A(p) = kp$ $(p \in \mathbb{R}^n)$, where $k > 0$ is the diffusion constant.

For standard diffusion equations, the flux $A(p)$ is assumed to be *monotone*; namely,

$$(A(p) - A(q)) \cdot (p - q) \geq 0 \quad (p,\ q \in \mathbb{R}^n).$$

In this case, equation (1.1) is parabolic and can be studied by the standard methods of parabolic equations and monotone operators. For example, when the flux $A(p)$ is given by

$A(p) = D_p W(p)$ for some smooth convex function $W : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$|D_p^2 W(p)| \le \Lambda, \quad \sum_{i,j=1}^{n} W_{p_i p_j}(p)\xi_i \xi_j \ge \lambda |\xi|^2 \quad (p,\, \xi \in \mathbb{R}^n),$$

where Λ, λ are positive constants, (1.1) can be viewed and thus studied as a certain gradient flow generated by the *convex* energy functional

$$I(u) = \int_\Omega W(Du(x))\, dx$$

in the context of non-linear semigroup theory and monotone operators; see, e.g., Brezis [6]. In regard to classical solutions, if the flux $A(p)$ satisfies the uniform ellipticity condition

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{n} A^i_{p_j}(p)\xi_i \xi_j \le \Lambda |\xi|^2 \quad (p,\, \xi \in \mathbb{R}^n),$$

the existence and properties of solutions to (1.1) can be examined by establishing various *a priori* estimates and appealing to the Leray-Schauder fixed point theorem; see, e.g., Ladyženskaja *et al.* [29] and Lieberman [30].

However, for some applications of the evolution process in certain important physical problems, underlying diffusion fluxes may not be monotone, yielding non-parabolic equations (1.1). In this dissertation, we study the diffusion equation (1.1) with certain non-monotone fluxes $A(p)$ satisfying Fourier's inequality: $A(p) \cdot p \ge 0$ ($p \in \mathbb{R}^n$). We focus on the initial-boundary value problem

$$\begin{cases} u_t = \operatorname{div}(A(Du)) & \text{in } \Omega_T, \\ A(Du)\cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u = u_0 & \text{on } \Omega \times \{t=0\}, \end{cases} \tag{1.2}$$

where $\Omega_T = \Omega \times (0,T)$, $\mathbf{n}$ is the outer unit normal on $\partial\Omega$, $u_0 = u_0(x)$ is a given initial datum, and the flux $A(p)$ is of the form

$$A(p) = f(|p|^2)p \quad (p \in \mathbb{R}^n), \tag{1.3}$$

given by a function $f\colon [0,\infty) \to \mathbb{R}$ with *profile* $\sigma(s) = sf(s^2)$ having one of the graphs in Figures 1.1 and 1.2 below. (Precise structural assumptions on $\sigma(s) = sf(s^2)$ will be given in Chapter 2.)

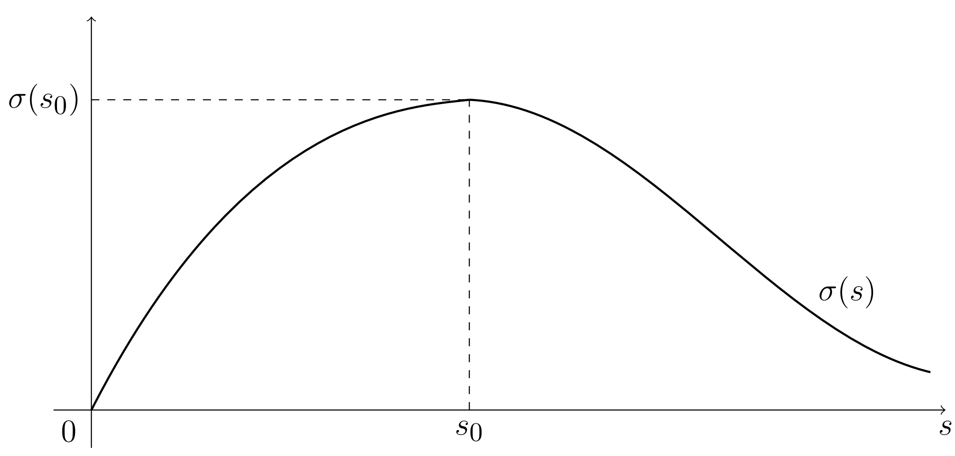


Figure 1.1: **Case I:** Perona-Malik type profile $\sigma(s)$.

The two cases in Figures 1.1 and 1.2 correspond to the applications in image processing proposed by Perona and Malik [35] and in phase transition of thermodynamics studied by

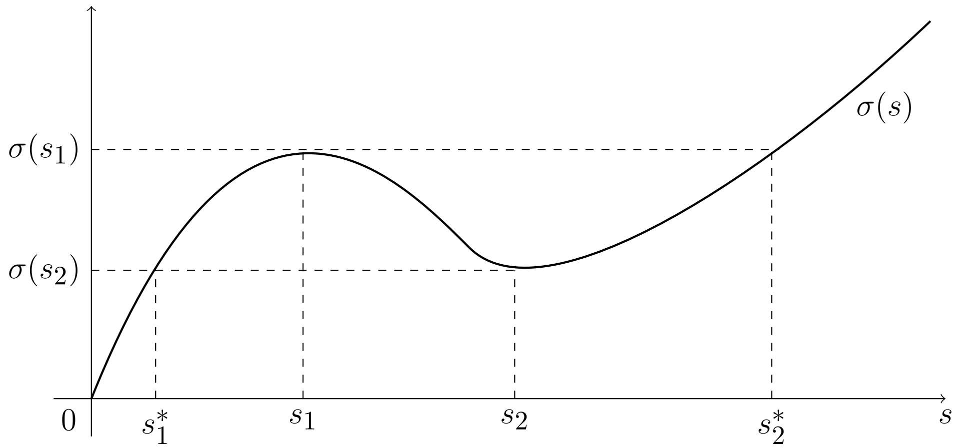


Figure 1.2: **Case II:** Höllig type profile $\sigma(s)$.

Höllig [20], respectively. For these diffusion equations (1.1), we have

$$\sigma'(\sqrt{s}) = f(s) + 2sf'(s) < 0 \quad \text{for some values of } s > 0.$$

In these cases, the diffusion is *anisotropic* since the diffusion matrix $(A^i_{p_j}(p))$, where

$$A^i_{p_j}(p) = f(|p|^2)\delta_{ij} + 2f'(|p|^2)p_ip_j \quad (i,\ j = 1, 2, \cdots, n),$$

has the eigenvalues $f(|p|^2)$ (of multiplicity $n-1$) and $f(|p|^2)+2|p|^2f'(|p|^2)$; hence the diffusion coefficients could be also negative. In such cases, problem (1.2) becomes *forward-backward parabolic*. Moreover, setting

$$W(p) = \int_0^{|p|} \sigma(r)\, dr, \quad I(u) = \int_\Omega W(Du)\, dx,$$

the initial-boundary value problem (1.2) becomes a L^2-gradient flow of the energy functional $I(u)$; however, $I(u)$ is *non-convex*. Consequently, neither the standard methods of parabolic equations and monotone operators nor the non-linear semigroup theory can be applied to study (1.2).

We now introduce the notion of a weak solution to problem (1.2) reflecting the initial and boundary conditions as follows.

Definition 1.0.1. We say that a function $u \in W^{1,\infty}(\Omega_T)$ is a *Lipschitz solution* to (1.2) provided that equality

$$\int_\Omega (u(x,s)\zeta(x,s) - u_0(x)\zeta(x,0))dx = \int_0^s \int_\Omega (u\zeta_t - A(Du)\cdot D\zeta)dxdt \tag{1.4}$$

holds for each $\zeta \in C^\infty(\bar{\Omega}_T)$ and each $s \in [0,T]$.

Before stating the main results of this dissertation, we begin with a literature review on forward-backward diffusion problems.

1.1 Review of the literature

In this review, we generally assume the flux $A(p)$ is non-monotone. However, we impose *at least* the Fourier inequality: $A(p)\cdot p \geq 0$ for all $p \in \mathbb{R}^n$, which is consistent with the Clausius-Duhem inequality in the theory of thermal conductors.

1.1.1 Young measure solutions

A *measure-valued* or *Young measure* solution to equation (1.1) is a pair (u,ν) of a function u in a suitable Sobolev space and a parametrized family $\nu = (\nu_{x,t})_{(x,t)\in\Omega_T}$ of probability measures on $\mathbb{R}^n$ generated by the spatial gradients of a sequence in the same space, satisfying

$$\int_0^T \int_\Omega (\langle \nu, A\rangle \cdot D\zeta + u_t \zeta) dxdt = 0$$

for all $\zeta \in C_c^\infty(\Omega_T)$, where

$$\langle \nu, A\rangle = \int_{\mathbb{R}^n} A(p)\nu(dp) \quad \text{a.e. in } \Omega_T.$$

In addition, the pair (u, ν) is required to satisfy

$$Du = \langle \nu, \mathrm{id}\rangle = \int_{\mathbb{R}^n} p\, \nu(dp) \quad \text{a.e. in } \Omega_T,$$

where $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity function.

Note that any Lipschitz solution $u \in W^{1,\infty}(\Omega_T)$ to equation (1.1) in the sense of Definition 1.0.1 (without initial and boundary conditions) is a Young measure solution with its corresponding parametrized family $\delta_{Du} = (\delta_{Du(x,t)})_{(x,t)\in\Omega_T}$ of point masses at Du.

There have been extensive studies on Young measure solutions to diffusion equations (1.1) and their properties under different assumptions on the flux $A(p)$ and Dirichlet or Neumann boundary conditions. Two early works were accomplished independently by Slemrod [39] and by Kinderlehrer and Pedregal [27]. In [39], equation (1.1) under Dirichlet or Neumann boundary conditions is approximated by a sequence of regular and singularly perturbed problems whose solutions are used to generate a Young measure solution. On the other hand, the work [27] combines the explicit methods for solutions to evolution equations with variational methods used to incorporate the oscillatory behavior. Such combination then leads to the existence of Young measure solutions to evolution problems that may be of forward-backward type. However the differences between these two works are subtle. In [39], the flux $A(p)$ and initial datum u_0 are assumed to be sufficiently smooth, $A(p)$ has strictly sub-quadratic growth, and (1.1) is satisfied in the sense of distributions. In [27], $A(p)$ is continuous and of linear growth, $u_0 \in H_0^1(\Omega)$, and (1.1) is satisfied in $H^{-1}(\Omega)$.

Following the approach in [27], Demoulini [12] established the existence of a *unique* Young measure solution to equation (1.1) with flux $A(p)$ as the gradient of some C^1 potential $\phi(p)$ satisfying a certain growth condition and under Dirichlet boundary condition. Her method was further explored by Yin and Wang [43] to extend the existence result involving other growth conditions on $\phi(p)$.

Focusing on the Perona-Malik flux $A(p) = \frac{p}{1+|p|^2}$ (see below), existence and properties of *infinitely many* Young measure solutions to problem (1.2) had been studied by Taheri *et al.* [40] and by Chen and Zhang [7] for dimensions $n = 1$ and $n = 2$, respectively. Non-uniqueness of solutions here is inevitable due to the intensity of forward-backward nature of the Perona-Malik flux $A(p)$. This is in sharp contrast to the uniqueness result in [12] as a rather mild backwardness is inherent in the fluxes treated in that paper.

1.1.2 Perona-Malik model and spatial regularizations

In the original paper of Perona and Malik [35], they proposed an anisotropic diffusion model (1.2), called the Perona-Malik model, for denoising and edge enhancement of a computer vision, where $\Omega \subset \mathbb{R}^2$ is a square and the flux $A(p)$ is given by (see Figure 1.1 for the shape of profile $\sigma(s)$)

$$\text{either} \quad A(p) = \frac{p}{1+|p|^2/s_0^2} \quad \text{or} \quad A(p) = \exp\Big(-\frac{|p|^2}{2s_0^2}\Big)p \tag{1.5}$$

with a fixed threshold $s_0 > 0$ according to some experimental purposes.

In this model, $u(x,t)$ represents an improved version of the initial gray level $u_0(x)$ of a noisy picture. The anisotropic diffusion $\operatorname{div}(A(Du))$ is forward parabolic in the *subcritical* region where $|Du| < s_0$ and backward parabolic in the *supercritical* region where $|Du| > s_0$. The expectation of the model is that disturbances with small gradient in the subcritical region will be smoothed out by the forward parabolic diffusion, while sharp edges corresponding to large gradient in the supercritical region will be enhanced by the backward parabolic equation. Such expected phenomenology has been implemented and observed in some numerical experiments; see e.g., Esedoglu [13], showing stability and effectiveness of the model. On the other hand, many analytical works have shown that the model is highly ill-posed when the initial datum u_0 is *transcritical* in Ω; namely, there are subregions in Ω where $|Du_0| < s_0$ and where $|Du_0| > s_0$, respectively. For such transcritical initial data, due to the backward parabolicity, even a proper notion and the existence of well-posed solutions to problem (1.2) have remained largely unsettled; see Kichenassamy [23] in this regard.

There have been many works trying to define a suitable notion of weak solution to problem (1.2) reflecting expected phenomenology of the model. One way is to study Young measure solutions to (1.2) as in [40, 7] explained above. Another way is to investigate the *nice* solutions to regularized problems and the limiting behaviors of such solutions as the regularization parameter approaches 0. For this discussion, let us take $s_0 = 1$ in (1.5) and focus on the flux $A(p) = \frac{p}{1+|p|^2}$.

As a perturbation of the original problem (1.2), a mild regularization was proposed by Guidotti [19] including a viscous term ($\delta > 0$):

$$u_t = \operatorname{div}\Big(\big(\frac{1}{1+|Du|^2} + \delta\big)Du\Big), \tag{1.6}$$

which is still of forward-backward type at least for $\delta < 1/8$. It is the formal gradient flow of the energy functional

$$I_\delta(u) = \frac{1}{2}\int_\Omega \big(\log(1+|Du|^2) + \delta|Du|^2\big)dx.$$

This functional has a non-trivial convexification which *uniquely* determines a Young measure solution by means of approximate weak Young measure solutions. The construction of these approximate solutions was carried out by following the approach of [12]. While Young measure solutions in [40, 7] are not unique, those in [12, 19] are unique due to the reason mentioned above. In a dynamical viewpoint, regularization (1.6) seems to replace the staircasing effect of the Peorna-Malik equation with a micro-ramping phenomenon by which the center of mass (that is, Young measure) solution is Lipschitz continuous while its gradient exhibits a micro-structure composed of gradients of small and large size. One of the main results of this dissertation actually verifies such phenomenon for *exact* Lipschitz solutions to the Perona-Malik type equations having profiles $\sigma(s)$ as in Figure 1.1 (including the Perona-Malik equation itself) without regularization (1.6).

In dimension $n = 1$, fourth order regularization has been studied by Bellettini *et al.* [3] and by Bellettini and Fusco [2]. These papers studied the singular perturbation ($\epsilon > 0$):

$$u_t = -\epsilon^2 u_{xxxx} + \Big(\frac{u_x}{1+u_x^2}\Big)_x$$

whose associated energy functional is given by

$$I_\epsilon(u) = \frac{1}{2}\int_0^1 \big(\epsilon u_{xx}^2 + \log(1+u_x^2)\big)dx.$$

In [3], it had been observed that infinitely many different evolutions may arise under the same initial datum u_0 by considering sequences u_0^ϵ of initial data that converge uniformly to u_0 as $\epsilon \to 0^+$. In [2], using the Γ-limit convergence technique with appropriate scaling, the authors could capture the long time behavior of the Perona-Malik equation with evolution of piecewise constant data.

1.1.3 Classical solutions

Let us assume for the moment that $\Omega \subset \mathbb{R}^n$ is a bounded C^1 domain and that $A(p) = \frac{p}{1+|p|^2}$. Given a point $x \in \bar{\Omega}$, we say that the initial datum $u_0 \in C^1(\bar{\Omega})$ is *subcritical* at x if $|Du_0(x)| < 1$, *supercritical* at x if $|Du_0(x)| > 1$, and *critical* at x if $|Du_0(x)| = 1$. The initial datum u_0 is *transcritical* in Ω if there are two points $x, y \in \Omega$ with $|Du_0(x)| < 1$ and $|Du_0(y)| > 1$.

Existence of global or local classical solutions to problem (1.2) depends heavily on the initial datum u_0. Kawohl and Kutev [22] showed that a global classical solution exists in any dimension if u_0 is subcritical on $\bar{\Omega}$. However, in this case, the convexity of Ω is required to guarantee such global existence as pointed out by Kim [24]. In [22], they also proved that (1.2) cannot admit a global classical solution for $n = 1$ if u_0 is transcritical in Ω under some technical assumptions, and these assumptions were completely removed later by Gobbino [17]. Concerning the Perona-Malik type equations, it had been the general belief that classical solutions can only exist if the initial data are smooth, even analytic, at supercritical points; this was formally streamlined by Kichenassamy [23]. In regard to the class of suitable initial data for classical solutions, Ghisi and Gobbino [14] established that for $n = 1$, the set of initial data for which problem (1.2) has a local classical solution is dense in $C^1(\bar{\Omega})$.

The situation concerning the existence of a global classical solution to (1.2) with a transcritical initial datum for $n \geq 2$ turns out to be quite different from the case $n = 1$. The first existence result of global classical solutions with u_0 transcritical for $n \geq 2$ was obtained by Ghisi and Gobbino [15], where they constructed a class of global radial $C^{2,1}$ solutions with suitably chosen radial initial data transcritical on an annulus centered at the origin; these solutions also have the property of finite-time extinction of supercritical region. In contrast to the one-dimensional results [22, 17] mentioned above, their result exhibited a quite different feature of the higher dimensional problem. On the other hand, in the radial case, Ghisi and Gobbino [16] also proved that a global C^1 solution cannot exist if the gradient of initial datum u_0 is very large at a point. Therefore, requirement of regularity for solutions (e.g., classical or C^1) would prevent the existence of such solutions if the initial data should be arbitrarily given and transcritical.

1.1.4 Lipschitz solutions in dimension $n = 1$

Let the flux $A(p)$ be given by (1.3) with its profile $\sigma(s)$ as in Figures 1.1 or 1.2. When the initial datum u_0 is any given smooth function (satisfying certain compatibility conditions on $\partial\Omega$), it seems natural to lower the expectation on the regularity of solutions by finding plausible weak solutions to problem (1.2). Even under the lowering of regularity have enormous difficulties occurred on the existence of suitable weak solutions as we discussed above. To our best knowledge, Zhang [44, 45] was the first to successfully prove that, for $n = 1$, there are infinitely many Lipschitz solutions to (1.2) for any given non-constant smooth initial data u_0 (with an extra assumption when the profile $\sigma(s)$ is as in Figure 1.2); his pivotal idea was to reformulate the one-dimensional Perona-Malik or Höllig type equations into 2×2 non-homogeneous partial differential inclusions and then to prove the existence using a modified method of convex integration following the ideas of Kirchheim [28] and of Müller and Sverák [32]. Before Zhang's work [45], in the pioneering work of Höllig [20], it was proved that for $n = 1$, there are infinitely many L^2-weak solutions to (1.2) when the profile $\sigma(s)$ is piecewise linear as in Figure 1.3; however, the method of Höllig cannot be applied to generalized profiles $\sigma(s)$ as in Figure 1.2.

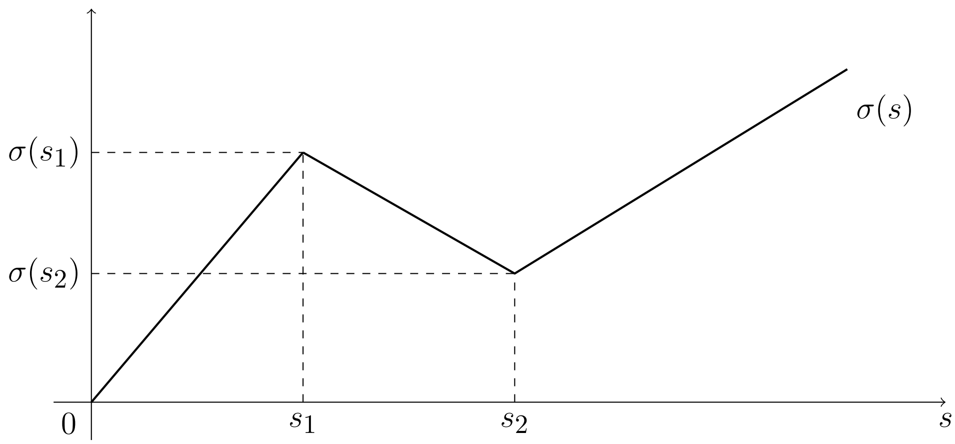


Figure 1.3: Höllig's piecewise linear profile $\sigma(s)$.

1.1.5 Lipschitz solutions in all dimensions

Recently, Kim and Yan [25] extended Zhang's method [44] to study the Perona-Malik type equations in all dimensions n for balls $\Omega = \{x \in \mathbb{R}^n : |x| < R\}$ and non-constant radially symmetric smooth initial data u_0. In this case the n-dimensional equation for radial solutions can still be reformulated as a 2×2 non-homogeneous partial differential inclusion. However, for general domains and initial data, the n-dimensional problem (1.2) can only be reformulated as an $(1 + n) \times (n + 1)$ non-homogeneous partial differential inclusion that has some uncontrollable gradient components, making the construction of Lipschitz solutions to this differential inclusion hopeless. In a very recent work of Kim and Yan [26], this difficulty was overcome by developing a suitable density method, still motivated by the method of differential inclusion but based on Baire's category method. The result is that for all smooth *convex* domains $\Omega \subset \mathbb{R}^n$ and arbitrary smooth initial data $u_0 \in C^{2+\alpha}(\bar{\Omega})$ with $Du_0 \cdot \mathbf{n} = 0$ on $\partial\Omega$, there exist infinitely many Lipschitz solutions to (1.2) with the *exact* Perona-Malik diffusion flux $A(p) = \frac{p}{1+|p|^2}$ $(p \in \mathbb{R}^n)$. The proof heavily relies on the explicit formula for the rank-one convex hull of the matrix set defined by this special function; such explicit formula for the general flux function $A(p)$ is impossible.

1.2 Main results

The main purpose of this dissertation is to explore a new approach for the existence of Lipschitz solutions to non-parabolic problems (1.2), which is carried out in Chapter 2 as a general existence theorem under some density condition, Theorem 2.2.4. However the general existence theorem would be meaningless if such density condition cannot be realized for a given non-monotone flux $A(p)$. We indeed present two different classes of non-monotone fluxes $A(p)$ of the form (1.3) having profiles $\sigma(s)$ either as in Figure 1.1 or as in Figure 1.2 with which the density condition can be made true to extract Lipschitz solutions. To state these results precisely, let us assume the following on the domain Ω and initial datum u_0:

$$\begin{cases} \Omega \subset \mathbb{R}^n \text{ is a bounded domain with } \partial\Omega \text{ of } C^{2+\alpha}, \\ u_0 \in C^{2+\alpha}(\bar{\Omega}) \text{ is non-constant with } Du_0 \cdot \mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{1.7}$$

where $\alpha \in (0,1)$ is a given number.

Although we will repeat the statements of the concrete existence results in Chapter 2, we introduce them here as a summary of the dissertation.

1.2.1 Perona-Malik type equations

Let the flux $A(p)$ be of the form (1.3) with profile $\sigma(s)$ as in Figure 1.1. Then we have the following.

Theorem 1.2.1 (Perona-Malik type). *Let* Ω *and* u_0 *satisfy* (1.7) *with* Ω *convex, and let* $\Omega_T = \Omega \times (0,T)$ *for a given* $T > 0$. *Then there exist infinitely Lipschitz solutions* u *to* (1.2).

A detailed version of this result is available in Theorem 2.3.2 that provides the Perona-Malik model with mathematically rigorous solutions reflecting some phenomena observed in numerical simulations. Note that this result generalizes those of [44, 26].

1.2.2 Höllig type equations

Let the flux $A(p)$ be of the form (1.3) with profile $\sigma(s)$ as in Figure 1.2. Then the result is as follows.

Theorem 1.2.2 (Höllig type). *Let* Ω *and* u_0 *satisfy* (2.10) *with* $|Du_0(x_0)| \in (s_1^*, s_2^*)$ *for some* $x_0 \in \Omega$, *and let* $\Omega_T = \Omega \times (0,T)$ *for a given* $T > 0$. *Then there exist infinitely many Lipschitz solutions* u *to* (1.2).

This is to generalize the results of [20, 45] to Höllig type profiles $\sigma(s)$ illustrated in Figure 1.2 for all dimensions.

Precise structural assumptions on the profiles $\sigma(s)$ in Theorems 1.2.1 and 1.2.2 are given in Chapter 2. These theorems are completely proved in Chapters 4 and 5, respectively. Chapter 3 is reserved for preliminary results that may be of independent interest.

Chapter 2

A new approach by Baire's category method

The purpose of this chapter is to design a new *functional* approach to study problem (1.2), which is based on a *Baire's category method*. As a preliminary analytical background, we introduce a version of the Baire category theorem on Baire-one functions. We then introduce two important classes of non-monotone fluxes $A(p)$ with which the approach can be applied to (1.2). In doing so, the concrete existence results on the Perona-Malik and Höllig type equations are stated along with the coexistence result on radial and non-radial solutions for the Perona-Malik type when the domain Ω is a ball and the initial datum u_0 is radial.

2.1 Baire-one functions

In this preliminary section, we introduce a version of the *Baire category theorem* on *Baire-one functions* following the exposition of [9]. Here, let X, Y denote metric spaces with corresponding metrics d_X, d_Y.

We begin with basic terminologies.

Definition 2.1.1. Let $f : X \to Y$. We define the *oscillation* of f at a point $x_0 \in X$ by

$$\omega_f(x_0) = \lim_{\delta \to 0^+} \sup_{x,y \in B_X(x_0,\delta)} d_Y(f(x), f(y)),$$

where $B_X(x_0, \delta)$ is the open ball in X with center x_0 and radius $\delta > 0$. The function $\omega_f : X \to [0, \infty]$ is called the *oscillation* of f.

Definition 2.1.2. We say that $f : X \to Y$ is a *Baire-one function* if there exists a sequence $\{f_j\}_{j=1}^{\infty}$ of continuous functions from X into Y such that

$$\lim_{j\to\infty} f_j(x) = f(x) \quad \text{in } Y, \quad \forall x \in X.$$

It is very easy to prove the following; we skip the proof.

Proposition 2.1.3. *Let $f : X \to Y$. Then*

(i) f is continuous at a point $x_0 \in X$ if and only if $\omega_f(x_0) = 0$,

(ii) $\forall \epsilon > 0$, the set $\Omega_f^{\epsilon} := \{x \in X \,|\, \omega_f(x) < \epsilon\}$ is open in X.

Remark 2.1.4. Let $f : X \to Y$. Let $\mathcal{C}_f = \{x \in X \,|\, f \text{ is continuous at } x\}$ and $\mathcal{D}_f = X \setminus \mathcal{C}_f$. By Proposition 2.1.3, we have

$$\mathcal{D}_f = \{x \in X \,|\, \omega_f(x) > 0\} = \cup_{j\in\mathbb{N}}\{x \in X \,|\, \omega_f(x) \geq 1/j\},$$

which is an F_σ set. Also, $\mathcal{C}_f = \cap_{j\in\mathbb{N}}\{x \in X \,|\, \omega_f(x) < 1/j\}$ is a G_δ set.

Some basic definitions on metric spaces are included here.

Definition 2.1.5. (i) A set $N \subset X$ is called *nowhere dense* in X if the closure $\bar{N}$ of N contains no non-empty open subset of X, that is, the open set $\bar{N}^c = X \setminus \bar{N}$ is dense in X.

(ii) A set $F \subset X$ is said to be of the *first category* if it is the countable union of nowhere dense subsets of X.

(iii) A set $S \subset X$ that is not of the first category is said to be of the *second category.*

The Baire category theorem below is so standard that it is contained in almost all real analysis books; see e.g., [5].

Theorem 2.1.6 (Baire Category Theorem: Version I). *Let X be complete. Then any countable intersection of dense open subsets of X is dense in X.*

The theorem below is the main part of this section whose proof is provided for reader's convenience.

Theorem 2.1.7 (Baire Category Theorem: Version II). *Let X be complete. If $f : X \to Y$ is a Baire-one function, then $\mathcal{D}_f$ is of the first category; so $\mathcal{C}_f$ is dense in X.*

Proof. In view of Remark 2.1.4, it suffices to show that for each $\epsilon > 0$, the set $F_\epsilon := \{x \in X \mid \omega_f(x) \geq 5\epsilon\}$ is nowhere dense in X. So fix an $\epsilon > 0$.

Since f is a Baire-one function, we can choose a sequence $\{f_j\}_{j=1}^{\infty}$ of continuous functions from X into Y such that

$$\lim_{j\to\infty} f_j(x) = f(x) \quad \text{in } Y, \quad \forall x \in X.$$

For each $\nu \in \mathbb{N}$, define

$$E_\nu = \cap_{i,j\geq\nu}\{x \in X \mid d_Y(f_i(x), f_j(x)) \leq \epsilon\}.$$

We show that E_ν is closed in X for each $\nu \in \mathbb{N}$. To do this, let i, $j \in \mathbb{N}$. Then it is sufficient to check that $x \mapsto d_Y(f_i(x), f_j(x))$ is a continuous function from X into $[0, \infty)$. Let $x_0 \in X$

and $\eta > 0$. Since f_i, f_j are continuous (at x_0), there exists a $\delta = \delta(\eta, i, j) > 0$ such that

$$x \in X,\, d_X(x_0, x) < \delta \Longrightarrow d_Y(f_i(x_0), f_i(x)) < \eta/2,\; d_Y(f_j(x_0), f_j(x)) < \eta/2$$

$$\Longrightarrow |d_Y(f_i(x_0), f_j(x_0)) - d_Y(f_i(x), f_j(x))| \leq d_Y(f_i(x_0), f_i(x)) + d_Y(f_j(x), f_j(x_0)) < \eta.$$

Hence the function $x \mapsto d_Y(f_i(x), f_j(x))$ is continuous at x_0.

Note $E_1 \subset E_2 \subset \cdots \subset X$. We now check $X = \cup_{\nu \in \mathbb{N}} E_\nu$. Choose any $x_0 \in X$. Since $f_\nu(x_0) \to f(x_0)$ in Y as $\nu \to \infty$, there is an $N \in \mathbb{N}$ such that $d_Y(f_i(x_0), f_j(x_0)) \leq \epsilon \;\; \forall i, j \geq N$. Thus

$$x_0 \in E_N \subset \cup_{\nu \in \mathbb{N}} E_\nu.$$

Thus $X = \cup_{\nu \in \mathbb{N}} E_\nu$.

Let I be any closed set in X with interior $\mathrm{int} I \neq \emptyset$. Then

$$I = I \cap X = \cup_{\nu \in \mathbb{N}} (E_\nu \cap I),$$

where each $E_\nu \cap I$ is closed in X. If each $E_\nu \cap I$ is nowhere dense in X, then $I^c = \cap_{\nu \in \mathbb{N}} (E_\nu \cap I)^c$ is dense in X by Theorem 2.1.6, and so $I^c \cap \mathrm{int} I \neq \emptyset$, a contradiction. So there is an index $\nu_0 \in \mathbb{N}$ such that $E_{\nu_0} \cap I$ is not nowhere dense in X; that is, there exists a non-empty open set J in X with $J \subset E_{\nu_0} \cap I$. Thus for each $x \in J$, we have $d_Y(f_i(x), f_j(x)) \leq \epsilon$ for all i, $j \geq \nu_0$, and in particular,

$$d_Y(f_{\nu_0}(x), f(x)) \leq \epsilon,$$

since $f_j(x) \to f(x)$ in Y. By the continuity of f_{ν_0}, for each $x_0 \in J$, there is a neighborhood

$I(x_0)$ of x_0 in the open set J such that

$$x \in I(x_0) \Longrightarrow d_Y(f_{\nu_0}(x_0), f_{\nu_0}(x)) \leq \epsilon$$

$$\Longrightarrow d_Y(f_{\nu_0}(x_0), f(x)) \leq d_Y(f_{\nu_0}(x_0), f_{\nu_0}(x)) + d_Y(f_{\nu_0}(x), f(x)) \leq 2\epsilon$$

from the above inequality. Thus, for each $x_0 \in J$, we have $d_Y(f(x), f(y)) \leq 4\epsilon$ for every $x, y \in I(x_0)$, and so $\omega_f(x_0) \leq 4\epsilon$; so $x_0 \notin F_\epsilon$. This implies $J \subset F_\epsilon^c \cap I$.

Putting everything together, we can conclude that for any closed set $I \subset X$ with $\mathrm{int} I \neq \emptyset$, there is a non-empty open set J in X with $J \subset F_\epsilon^c \cap I$. So $F_\epsilon^c \cap O \neq \emptyset$ for any non-empty open set O in X. Therefore, F_ϵ is nowhere dense in X. $\square$

2.2 General existence theorem

To set up a general approach for studying problem (1.2), we assume, in this section, the domain Ω has a Lipschitz boundary $\partial\Omega$ and the initial datum $u_0 \in W^{1,\infty}(\Omega_T)$. Without loss of generality, we assume

$$\int_\Omega u_0(x)\, dx = 0, \tag{2.1}$$

since otherwise we can solve (1.2) for initial datum $\tilde{u}_0 = u_0 - \bar{u}_0$ with $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0\, dx$.

Our new approach is motivated by the following observation.

Proposition 2.2.1. *Suppose $u \in W^{1,\infty}(\Omega_T)$ is such that $u(x,0) = u_0(x)$ $(x \in \Omega)$, there exists a vector function $v \in W^{1,2}((0,T); L^2(\Omega; \mathbb{R}^n))$ with weak time derivative v_t satisfying*

$$v_t = A(Du) \quad a.e. \;\; in \; \Omega_T, \tag{2.2}$$

and for each $\zeta \in C^\infty(\bar\Omega_T)$ *and each* $t \in [0,T]$,

$$\int_\Omega v(x,t)\cdot D\zeta(x,t)\,dx = -\int_\Omega u(x,t)\zeta(x,t)\,dx. \tag{2.3}$$

Then u *is a Lipschitz solution to* (1.2).

Proof. To verify (1.4), given any $\zeta \in C^\infty(\bar\Omega_T)$, let

$$g(t) = \int_\Omega u(x,t)\zeta(x,t)dx, \;\; h(t) = \int_\Omega u(x,t)\zeta_t(x,t)dx \quad (t \in [0,T]).$$

Then by (2.3), for each $\psi \in C_c^\infty(0,T)$,

$$\int_0^T \psi_t g dt = -\int_0^T\int_\Omega \psi_t v\cdot D\zeta dxdt, \;\; \int_0^T \psi h dt = -\int_0^T\int_\Omega \psi v\cdot D\zeta_t dxdt.$$

Since $v \in W^{1,2}((0,T);L^2(\Omega;\mathbb{R}^n))$ and $v_t = A(Du)$ a.e. in Ω_T, we have

$$\int_0^T\int_\Omega (\psi D\zeta)_t\cdot v dxdt = -\int_0^T\int_\Omega A(Du)\cdot\psi D\zeta\,dxdt.$$

As $(\psi D\zeta)_t = \psi_t D\zeta + \psi D\zeta_t$, combining the previous equations, we obtain

$$\int_0^T \psi_t g\,dt = \int_0^T \psi\left(-h + \int_\Omega A(Du)\cdot D\zeta\,dx\right)dt,$$

which proves that g is weakly differentiable in $(0,T)$ with its weak derivative

$$g'(t) = h(t) - \int_\Omega A(Du(x,t))\cdot D\zeta(x,t)\,dx \quad \text{a.e. } t \in (0,T).$$

Upon integrating this, (1.4) follows for each $s \in [0,T]$. $\square$

Condition (2.3) means that the following condition holds in the sense of distributions in Ω for each $t \in [0,T]$:

$$\operatorname{div} v(\cdot,t) = u(\cdot,t), \quad v(\cdot,t)\cdot \mathbf{n}|_{\partial\Omega} = 0.$$

If dimension $n=1$, this condition together with (2.2) implies $v \in W^{1,\infty}(\Omega_T;\mathbb{R}^1)$. However for $n \geq 2$, since it is impossible to bound $\|Dv\|_{L^\infty(\Omega_T)}$ in terms of $\operatorname{div} v$, the function v may not be in $W^{1,\infty}(\Omega_T;\mathbb{R}^n)$; this is the reason we only assume $v \in W^{1,2}((0,T);L^2(\Omega;\mathbb{R}^n))$ in Proposition 2.2.1. Nevertheless, we still try to approximate such v's by some functions in $W^{1,\infty}(\Omega_T;\mathbb{R}^n)$.

To choose suitable approximating functions, we first introduce the following definition.

Definition 2.2.2. A function $\Phi = (u^*, v^*)$, with $u^* \in W^{1,\infty}(\Omega_T)$ and $v^* \in W^{1,\infty}(\Omega_T;\mathbb{R}^n)$, is called a *boundary function* if it satisfies

$$\begin{cases} u^*(x,0) = u_0(x), & x \in \Omega, \\ \operatorname{div} v^*(x,t) = u^*(x,t), & \text{a.e. } (x,t) \in \Omega_T, \\ v^*(\cdot,t)\cdot \mathbf{n}|_{\partial\Omega} = 0, & t \in [0,T]. \end{cases} \tag{2.4}$$

Fix a boundary function $\Phi = (u^*, v^*)$. We denote by $W^{1,\infty}_{u^*}(\Omega_T)$, $W^{1,\infty}_{v^*}(\Omega_T;\mathbb{R}^n)$ the usual *Dirichlet classes* with boundary traces u^*, v^*, respectively. We also define the following.

Definition 2.2.3. A class $\mathcal{U} \subset W^{1,\infty}_{u^*}(\Omega_T)$ is called an *admissible set* provided that $\mathcal{U} \neq \emptyset$ is bounded in $W^{1,\infty}_{u^*}(\Omega_T)$ and that for each $u \in \mathcal{U}$, there exists a vector function $v \in$

$W^{1,\infty}_{v^*}(\Omega_T;\mathbb{R}^n)$ satisfying

$$\operatorname{div} v = u \quad \text{a.e. in } \Omega_T, \qquad \|v_t\|_{L^\infty(\Omega_T)} \leq r,$$

where $r>0$ is a fixed number. For an admissible set $\mathcal{U}$ and each $\epsilon>0$, let $\mathcal{U}_\epsilon$ be the set of all $u\in\mathcal{U}$ such that there exists a function $v\in W^{1,\infty}_{v^*}(\Omega_T;\mathbb{R}^n)$ satisfying

$$\operatorname{div} v = u \quad \text{a.e. in } \Omega_T, \qquad \|v_t\|_{L^\infty(\Omega_T)} \leq r,$$
$$\int_{\Omega_T} |v_t(x,t) - A(Du(x,t))|\, dxdt \leq \epsilon|\Omega_T|.$$

Our new approach is the following *general existence theorem* under the pivotal density hypothesis of $\mathcal{U}_\epsilon$ in $\mathcal{U}$, which is based on the *Baire category theorem* in the previous section.

Theorem 2.2.4. *Let $\mathcal{U}\subset W^{1,\infty}_{u^*}(\Omega_T)$ be an admissible set satisfying the* density property:

$$\mathcal{U}_\epsilon \textit{ is dense in } \mathcal{U} \textit{ under the } L^\infty\textit{-norm for each } \epsilon>0. \tag{2.5}$$

Then, given any $\varphi\in\mathcal{U}$, for each $\delta>0$, there exists a Lipschitz solution $u\in W^{1,\infty}_{u^*}(\Omega_T)$ to (1.2) *satisfying $\|u-\varphi\|_{L^\infty(\Omega_T)}<\delta$. Furthermore, if $\mathcal{U}$ contains a function which is not a Lipschitz solution to* (1.2), *then* (1.2) *itself admits infinitely many Lipschitz solutions.*

Proof. For clarity, we divide the proof into several steps.

1. Let $\mathcal{X}$ be the closure of $\mathcal{U}$ in the metric space $L^\infty(\Omega_T)$. Then $(\mathcal{X},L^\infty)$ is a non-empty complete metric space. By assumption, each $\mathcal{U}_\epsilon$ is dense in $\mathcal{X}$. Moreover, since $\mathcal{U}$ is bounded in $W^{1,\infty}_{u^*}(\Omega_T)$, we have $\mathcal{X}\subset W^{1,\infty}_{u^*}(\Omega_T)$.

2. Let $\mathcal{Y}=L^1(\Omega_T;\mathbb{R}^n)$. For $h>0$, define $T_h\colon\mathcal{X}\to\mathcal{Y}$ as follows. Given any $u\in\mathcal{X}$,

write $u = u^* + w$ with $w \in W_0^{1,\infty}(\Omega_T)$ and define

$$T_h(u) = Du^* + D(\rho_h * w),$$

where $\rho_h(z) = h^{-N}\rho(z/h)$, with $z = (x,t)$ and $N = n+1$, is the standard h-mollifier in $\mathbb{R}^N$, and $\rho_h * w$ is the usual convolution in $\mathbb{R}^N$ with w extended to be zero outside $\bar{\Omega}_T$. Then, for each $h > 0$, the map $T_h\colon (\mathcal{X}, L^\infty) \to (\mathcal{Y}, L^1)$ is continuous, and for each $u \in \mathcal{X}$,

$$\lim_{h\to 0^+} \|T_h(u) - Du\|_{L^1(\Omega_T)} = \lim_{h\to 0^+} \|\rho_h * Dw - Dw\|_{L^1(\Omega_T)} = 0.$$

Therefore, the spatial gradient operator $D\colon \mathcal{X} \to \mathcal{Y}$ is the pointwise limit of a sequence of continuous functions $T_h\colon \mathcal{X} \to \mathcal{Y}$; hence $D\colon \mathcal{X} \to \mathcal{Y}$ is a *Baire-one map*. By the Baire category theorem, Theorem 2.1.7, there exists a *residual set* $\mathcal{G} \subset \mathcal{X}$ such that the operator D is continuous at each point of $\mathcal{G}$. Since $\mathcal{X} \setminus \mathcal{G}$ is of the *first category*, the set $\mathcal{G}$ is *dense* in $\mathcal{X}$. Therefore, given any $\varphi \in \mathcal{X}$, for each $\delta > 0$, there exists a function $u \in \mathcal{G}$ such that $\|u - \varphi\|_{L^\infty(\Omega_T)} < \delta$.

3. We now prove that each $u \in \mathcal{G}$ is a Lipschitz solution to (1.2). Let $u \in \mathcal{G}$ be given. By the density of $\mathcal{U}_\epsilon$ in $(\mathcal{X}, L^\infty)$ for each $\epsilon > 0$, for every $j \in \mathbb{N}$, there exists a function $u_j \in \mathcal{U}_{1/j}$ such that $\|u_j - u\|_{L^\infty(\Omega_T)} < 1/j$. Since the operator $D\colon (\mathcal{X}, L^\infty) \to (\mathcal{Y}, L^1)$ is continuous at u, we have $Du_j \to Du$ in $L^1(\Omega_T; \mathbb{R}^n)$. Furthermore, from the definition of $\mathcal{U}_{1/j}$, there exists a function $v_j \in W_{v^*}^{1,\infty}(\Omega_T; \mathbb{R}^n)$ such that for each $\zeta \in C^\infty(\bar{\Omega}_T)$ and each

$t \in [0, T]$,

$$\int_\Omega v_j(x,t) \cdot D\zeta(x,t) dx = -\int_\Omega u_j(x,t)\zeta(x,t)\, dx,$$
$$\|(v_j)_t\|_{L^\infty(\Omega_T)} \le r, \quad \int_{\Omega_T} |(v_j)_t - A(Du_j)|\, dxdt \le \frac{1}{j}|\Omega_T|. \tag{2.6}$$

Since $v_j(x,0) = v^*(x,0) \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ and $\|(v_j)_t\|_{L^\infty(\Omega_T)} \le r$, it follows that both sequences $\{v_j\}$ and $\{(v_j)_t\}$ are bounded in $L^2(\Omega_T;\mathbb{R}^n) \approx L^2((0,T);L^2(\Omega;\mathbb{R}^n))$. So we may assume $v_j \rightharpoonup v$ and $(v_j)_t \rightharpoonup v_t$ in $L^2((0,T);L^2(\Omega;\mathbb{R}^n))$ for some $v \in W^{1,2}((0,T);L^2(\Omega;\mathbb{R}^n))$, where $\rightharpoonup$ denotes the weak convergence. Upon taking the limit as $j \to \infty$ in (2.6), since $v \in C^0([0,T];L^2(\Omega;\mathbb{R}^n))$ and $A \in C^0(\mathbb{R}^n;\mathbb{R}^n)$, we obtain

$$\int_\Omega v(x,t) \cdot D\zeta(x,t)\, dx = -\int_\Omega u(x,t)\zeta(x,t)\, dx \quad (t \in [0,T]),$$
$$v_t(x,t) = A(Du(x,t)) \quad a.e.\ (x,t) \in \Omega_T.$$

Consequently, by Proposition 2.2.1, u is a Lipschitz solution to (1.2).

4. Finally, assume $\mathcal{U}$ contains a function which is not a Lipschitz solution to (1.2); hence $\mathcal{G} \neq \mathcal{U}$. Then $\mathcal{G}$ cannot be a finite set, since otherwise the L^∞-closure $\mathcal{X} = \overline{\mathcal{G}} = \overline{\mathcal{U}}$ would be a finite set, making $\mathcal{U} = \mathcal{G}$. Therefore, in this case, (1.2) admits infinitely many Lipschitz solutions.

The proof is complete. $\square$

In fact, only when problem (1.2) is non-parabolic (that is, $A(p)$ is non-monotone) could Theorem 2.2.4 yield the non-uniqueness result.

Corollary 2.2.5. *Assume the density property* (2.5) *holds for some admissible set* $\mathcal{U} \subset W^{1,\infty}_{u^*}(\Omega_T)$. *Suppose* $A\colon \mathbb{R}^n \to \mathbb{R}^n$ *is monotone. Then any function* $u \in \mathcal{U}$ *must be a*

Lipschitz solution to (1.2)*; in this case,* $\mathcal{U}$ *contains precisely one function.*

Proof. We follow the proof of Theorem 2.2.4. The monotonicity of the flux $A(p)$ implies that there exists *at most* one Lipschitz solution to (1.2). Since $\mathcal{U} \neq \emptyset$, we have $\mathcal{G} \neq \emptyset$, where every function in $\mathcal{G}$ is a Lipschitz solution to (1.2). Thus $\mathcal{U} = \mathcal{G} = \{\bar{u}\}$, where $\bar{u}$ is the *unique* Lipschitz solution to (1.2). $\square$

We also have the following general property for Lipschitz solutions to (1.2) when the flux $A(p)$ satisfies Fourier's inequality.

Proposition 2.2.6. *Let* $A\colon \mathbb{R}^n \to \mathbb{R}^n$ *satisfy Fourier's inequality:* $A(p) \cdot p \geq 0$ *for all* $p \in \mathbb{R}^n$*. Then any Lipschitz solution* u *to* (1.2) *satisfies*

$$\min_{\bar{\Omega}} u_0 \leq u(x,t) \leq \max_{\bar{\Omega}} u_0 \quad \text{in } \Omega_T. \tag{2.7}$$

Proof. Let $u \in W^{1,\infty}(\Omega_T)$ be any Lipschitz solution to (1.2). By (1.4), for all $\zeta \in C^\infty(\bar{\Omega}_T)$,

$$\int_{\Omega_T} u_t(x,t)\zeta(x,t)dxdt = -\int_{\Omega_T} A(Du) \cdot D\zeta dxdt;$$

hence by approximation, this equality holds for all $\zeta \in W^{1,\infty}(\Omega_T)$. Taking $\zeta(x,t) = \phi(x,t)\psi(t)$ with arbitrary $\phi \in W^{1,\infty}(\Omega_T)$ and $\psi \in W^{1,\infty}(0,T)$, we deduce that

$$\int_\Omega u_t(x,t)\phi(x,t)\,dx = -\int_\Omega A(Du(x,t)) \cdot D\phi(x,t)\,dx$$

for a.e. $t \in (0,T)$ and all $\phi \in W^{1,\infty}(\Omega_T)$. Now taking $\phi = u^{2k-1}$ with $k = 1, 2, \cdots$, it follows from the Fourier inequality of the flux $A(p)$ that for a.e. $t \in (0,T)$,

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega u^{2k}dx\Big) &= 2k\int_\Omega u_t\phi\,dx = -2k\int_\Omega A(Du)\cdot D\phi dx\\ &= -2k(2k-1)\int_\Omega u^{2k-2}A(Du)\cdot Dudx \le 0.\end{aligned}$$

From this we deduce that the $L^{2k}(\Omega)$-norm of $u(\cdot,t)$ is non-increasing on $t \in [0,T]$; in particular,

$$\|u(\cdot,t)\|_{L^{2k}(\Omega)} \le \|u_0\|_{L^{2k}(\Omega)} \quad \forall\, t\in[0,T],\ \ k=1,2,\cdots.$$

Letting $k\to\infty$, we obtain $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}$; hence

$$\|u\|_{L^\infty(\Omega_T)} = \|u_0\|_{L^\infty(\Omega)}. \tag{2.8}$$

Let

$$m_1 = \min_{\bar\Omega} u_0, \quad m_2 = \max_{\bar\Omega} u_0,$$

and define $\tilde A(p) = -A(-p)$. Observe that $\tilde A(p)$ also satisfies the Fourier inequality: $\tilde A(p)\cdot p \ge 0\ \ \forall p\in\mathbb{R}^n$.

We show $m_1 \le u(x,t) \le m_2$ for all $(x,t)\in\Omega_T$ to complete the proof. We consider three cases.

Case 1: $m_2 > 0$ and $|m_1| \le m_2$. In this case, $\|u_0\|_{L^\infty(\Omega)} = m_2$; so by (2.8), for all $(x,t)\in\Omega_T$,

$$u(x,t) \le \|u\|_{L^\infty(\Omega_T)} = \|u_0\|_{L^\infty(\Omega)} = m_2.$$

To obtain the lower bound, let $\tilde u_0 = -u_0 + m_2 + m_1$ and $\tilde u = -u + m_2 + m_1$. Then $\tilde u$ is a Lipschitz solution to (1.2) with A, u_0 replaced by $\tilde A$, $\tilde u_0$, respectively. Since $m_1 \le \tilde u_0(x) \le$

m_2, we have $\tilde{u}(x,t) \le m_2$ as above; hence $u(x,t) \ge m_1$ for all $(x,t) \in \Omega_T$.

Case 2: $m_2 > 0$ and $m_1 < -m_2$. Let $\tilde{u}_0 = -u_0$ and $\tilde{u} = -u$. Then $\tilde{u}$ is a Lipschitz solution to (1.2) with A, u_0 replaced by $\tilde{A}$, $\tilde{u}_0$, respectively. Since $-m_2 \le \tilde{u}_0(x) \le -m_1$, $-m_1 > 0$, and $|-m_2| = m_2 \le -m_1$, it follows from Case 1 that $-m_2 \le \tilde{u}(x,t) \le -m_1$; hence $m_1 \le u(x,t) \le m_2$ for all $(x,t) \in \Omega_T$.

Case 3: $m_2 \le 0$. In this case, $m_1 \le 0$. If $m_1 = 0$ then $m_2 = 0$ and hence $u_0 \equiv 0$; so, by (2.8), $u \equiv 0$. Next assume $m_1 < 0$. Let again as in Case 2 $\tilde{u}_0 = -u_0$ and $\tilde{u} = -u$. Since $-m_2 \le \tilde{u}_0(x) \le -m_1$ and $-m_1 > 0$, $|-m_2| = -m_2 \le -m_1$, it follows again from Case 1 that $-m_2 \le \tilde{u}(x,t) \le -m_1$; hence $m_1 \le u(x,t) \le m_2$ for all $(x,t) \in \Omega_T$. □

2.3 Existence theorems on anisotropic diffusions

In what follows, we study problem (1.2) for non-monotone diffusion fluxes $A(p)$ of the form

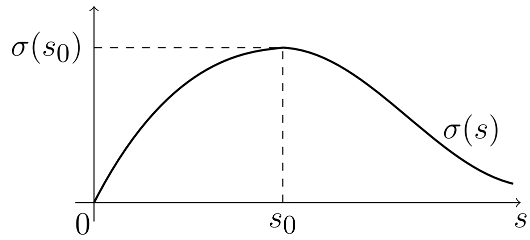
$$A(p) = f(|p|^2)p \quad (p \in \mathbb{R}^n), \tag{2.9}$$

where $f\colon [0,\infty) \to \mathbb{R}$ is a function with profile $\sigma(s) = sf(s^2)$ having one of the graphs in Figure 2.1 below. Precise structural assumptions on $\sigma(s)$ will be given in the following subsections.

Concerning the domain Ω and initial datum u_0, we assume the following hereafter:

$$\begin{cases} \Omega \subset \mathbb{R}^n \text{ is a bounded domain with } \partial\Omega \text{ of } C^{2+\alpha}, \\ u_0 \in C^{2+\alpha}(\bar{\Omega}) \text{ is non-constant with } Du_0 \cdot \mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{2.10}$$

where $\alpha \in (0,1)$ is a given number.



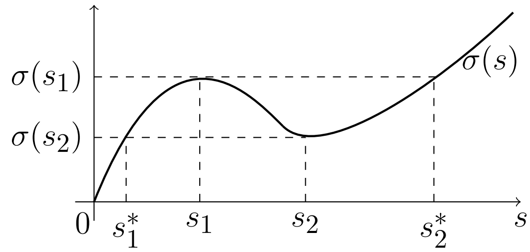


Figure 2.1: Graphs of two profiles $\sigma(s)$. **Case I:** Perona-Malik type. **Case II:** Höllig type.

We aim to apply Theorem 2.2.4 to study the existence of Lipschitz solutions to (1.2). The rest of the dissertation is devoted to constructing some admissible sets $\mathcal{U}$ satisfying the density property (2.5). Of course, such constructions depend on the initial datum u_0 and profile $\sigma(s)$ illustrated in Figure 2.1.

2.3.1 Case I: Perona-Malik type equations

In this case, we assume the following on the profile $\sigma(s) = sf(s^2)$:

Hypothesis (PM) (See Figure 2.1.)

(i) There exists a number $s_0 > 0$ such that

$$f \in C^0([0,\infty)) \cap C^3([0,s_0^2)) \cap C^1(s_0^2,\infty).$$

(ii) $\sigma'(s) > 0 \ \ \forall s \in [0, s_0)$, $\sigma'(s) < 0 \ \ \forall s \in (s_0, \infty)$, and

$$\lim_{s \to \infty} \sigma(s) = 0.$$

We now state the existence result for the Perona-Malik type equations. In this case, for each $r \in (0, \sigma(s_0))$, let $s_-(r) \in (0, s_0)$ and $s_+(r) \in (s_0, \infty)$ denote the unique numbers with $r = \sigma(s_\pm(r))$.

Theorem 2.3.1 (Perona-Malik type)**.** *Let Ω and u_0 satisfy* (2.10) *with Ω convex, and let $\Omega_T = \Omega \times (0, T)$ for a given $T > 0$. Then there exist infinitely Lipschitz solutions u to* (1.2)*.*

Depending on the size of $||Du_0||_{L^\infty(\Omega)}$, our solutions satisfy further properties as described in the theorem below; we prove this detailed version that implies Theorem 2.3.1.

Theorem 2.3.2. *In Theorem 2.3.1, let $M_0 = ||Du_0||_{L^\infty(\Omega)}$. Then for each $r \in (0, \sigma(M_0))$, there exists a number $l = l_r \in (0, r)$ such that for all $\tilde{r} \in (l, r)$ and all but at most countably many $\bar{r} \in (0, \tilde{r})$, there exist two disjoint open sets $\Omega_T^1, \Omega_T^2 \subset \Omega_T$ with $|\Omega_T^1 \cup \Omega_T^2| = |\Omega_T|$ and infinitely many Lipschitz solutions u to* (1.2) *satisfying*

$$u \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega}_T^1), \quad u_t = \operatorname{div}(A(Du)) \ \ \textit{pointwise in} \ \ \Omega_T^1,$$

$$|Du(x,t)| < s_-(\bar{r}) \ \ \forall (x,t) \in \Omega_T^1, \quad \Omega_0^{\bar{r}} \subset \partial\Omega_T^1,$$

$$|S_r| + |L_{r,\tilde{r}}| = |\Omega_T^2|, \ \ |L_{r,\tilde{r}}| > 0,$$

where

$$\Omega_0^{\bar{r}} = \{(x,0) \,|\, x \in \Omega, \ |Du_0(x)| < s_-(\bar{r})\},$$

$$S_r = \{(x,t) \in \Omega_T^2 \,|\, |Du(x,t)| \le s_-(r)\}, \; L_{r,\tilde{r}} = \{(x,t) \in \Omega_T^2 \,|\, s_+(r) \le |Du(x,t)| \le s_+(\tilde{r})\}.$$

Chapter 4 is devoted to the complete proof of this theorem.

Remark 2.3.3. By Hypothesis (PM), $\lim_{r\to 0^+} s_-(r) = 0$ and $\lim_{r\to 0^+} s_+(r) = \infty$. So if $0 < r \ll \sigma(M_0)$ is fixed, then corresponding Lipschitz solutions u have *large* and *small* gradient regimes $L_{r,\tilde{r}}$ and $\Omega_T^1 \cup S_r$ up to measure zero that represent *sharp edge* and *almost constant* parts of u in Ω_T, respectively. These properties together with (2.7) for solutions u are *somehow* reflected in numerical simulations; see Figure 2.2 taken from Perona and Malik [35]. On the other hand, it had been observed in [2, 3] that as the limits of solutions to a class of regularized equations, infinitely many different evolutions may arise under the same initial datum u_0. Our non-uniqueness result seems to reflect this pathological behavior of forward-backward problem (1.2).

2.3.2 Case II: Höllig type equations

Next, we assume the following on the profile $\sigma(s) = sf(s^2)$:

Hypothesis (H) (See Figure 2.1.)

(i) There exist two numbers $s_2 > s_1 > 0$ such that

$$f \in C^0([0,\infty)) \cap C^{1+\alpha}([0, s_1^2) \cup (s_2^2, \infty)).$$

(ii) $\sigma'(s) > 0 \;\; \forall s \in [0, s_1) \cup (s_2, \infty)$, $\sigma(s_1) > \sigma(s_2) > 0$, and $\lambda \le \sigma'(s) \le \Lambda \;\; \forall s \ge 2s_2$ for some constants $\Lambda \ge \lambda > 0$. Let $s_1^* \in (0, s_1)$, $s_2^* \in (s_2, \infty)$ denote the unique numbers with $\sigma(s_1^*) = \sigma(s_2)$, $\sigma(s_2^*) = \sigma(s_1)$, respectively.

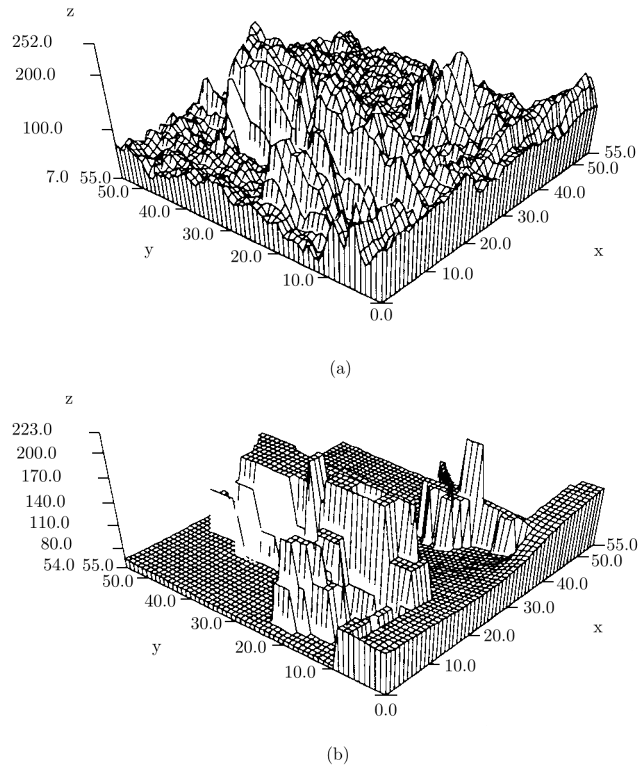


Figure 2.2: Scale-space using anisotropic diffusion with flux $A(p) = \frac{p}{1+|p|^2/s_0^2}$. Three dimensional plot of the brightness of Figure 12 in [35]. (a) Original image, (b) after smoothing with anisotropic diffusion.

We state the existence result for the Höllig type equations. In this case, for each $r \in (\sigma(s_2), \sigma(s_1))$, let $s_-(r) \in (s_1^*, s_1)$ and $s_+(r) \in (s_2, s_2^*)$ denote the unique numbers with $r = \sigma(s_\pm(r))$.

Theorem 2.3.4 (Höllig type)**.** *Let* Ω *and* u_0 *satisfy* (2.10) *with* $|Du_0(x_0)| \in (s_1^*, s_2^*)$ *for some* $x_0 \in \Omega$, *and let* $\Omega_T = \Omega \times (0,T)$ *for a given* $T > 0$. *Then there exist infinitely many Lipschitz solutions* u *to* (1.2).

Chapter 5 deals with the complete proof of this theorem.

2.3.3 Radial and non-radial solutions

We introduce here the coexistence of radial and non-radial Lipschitz solutions to problem (1.2) when the domain Ω is a ball and the initial datum u_0 is radial. For convenience, we focus only on **Case I:** Perona-Malik type equations, although one could equally justify the same for **Case II:** Höllig type equations. So we assume the flux $A(p)$ fulfills **Hypothesis (PM)**.

Let $\Omega = B_R(0)$ be the open ball in $\mathbb{R}^n$ with center 0 and a given radius $R > 0$. Let the initial datum $u_0 \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega})$ satisfy the compatibility condition

$$A(Du_0) \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega.$$

We say that a function u defined in Ω_T [Ω, resp.] is *radial* if $u(x,t) = u(y,t)$ $\forall x, y \in \Omega$, $|x| = |y|$, $\forall t \in (0,T)$ [$u(x) = u(y)$ $\forall x, y \in \Omega$, $|x| = |y|$, resp.].

We now have the following.

Theorem 2.3.5. *Assume u_0 is radial. Then there are infinitely many radial and non-radial Lipschitz solutions to* (1.2)*.*

The proof of this theorem is given at the end of Chapter 4.

Chapter 3

Preliminaries

This chapter prepares some essential ingredients for the proofs of existence theorems, Theorems 2.3.2 and 2.3.4.

3.1 Uniformly parabolic equations

We refer to the standard references [29, 30] for some notations concerning functions and domains of class $C^{k+\alpha}$ with an integer $k \geq 0$ and a number $0 < \alpha < 1$.

Assume $\tilde{f} \in C^{1+\alpha}([0,\infty))$ is a function satisfying

$$\theta \leq \tilde{f}(s) + 2s\tilde{f}'(s) \leq \Theta \quad \forall\, s \geq 0, \tag{3.1}$$

where $\Theta \geq \theta > 0$ are constants. This condition is equivalent to $\theta \leq (s\tilde{f}(s^2))' \leq \Theta$ for all $s \in \mathbb{R}$; hence, $\theta \leq \tilde{f}(s) \leq \Theta$ for all $s \geq 0$. Let

$$\tilde{A}(p) = \tilde{f}(|p|^2)p \quad (p \in \mathbb{R}^n).$$

Then we have

$$\tilde{A}^i_{p_j}(p) = \tilde{f}(|p|^2)\delta_{ij} + 2\tilde{f}'(|p|^2)p_i p_j \quad (i,j = 1,2,\cdots,n;\ p \in \mathbb{R}^n)$$

and hence the *uniform ellipticity condition*:

$$\theta|q|^2 \le \sum_{i,j=1}^{n} \tilde{A}^i_{p_j}(p) q_i q_j \le \Theta|q|^2 \quad \forall\, p,\ q \in \mathbb{R}^n. \tag{3.2}$$

Theorem 3.1.1. *Assume* (2.10) *holds. Then the initial-Neumann boundary value problem*

$$\begin{cases} u_t = \operatorname{div}(\tilde{A}(Du)) & \text{in } \Omega_T, \\ \partial u/\partial \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega \end{cases} \tag{3.3}$$

has a unique solution $u \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega}_T)$*. Moreover, if* $\tilde{f} \in C^3([0,\infty))$ *and* Ω *is convex, then the* gradient maximum principle *holds:*

$$\|Du\|_{L^\infty(\Omega_T)} = \|Du_0\|_{L^\infty(\Omega)}. \tag{3.4}$$

Proof. Let us divide the proof into three steps.

1. As problem (3.3) is uniformly parabolic by (3.2), the existence of a unique classical solution u in $C^{2+\alpha, \frac{2+\alpha}{2}}(\bar{\Omega}_T)$ follows from the standard theory; see [30, Theorem 13.24]. To prove the gradient maximum principle (3.4), we assume $\tilde{f} \in C^3([0,\infty))$ and Ω is convex. Note that, since $\tilde{A} \in C^3(\mathbb{R}^n)$, a standard bootstrap argument based on the regularity theory of linear parabolic equations [29, 30] shows that the solution u has all continuous partial derivatives $u_{x_i x_j x_k}$ and $u_{x_i t}$ within Ω_T for $1 \le i,j,k \le n$; see [24] for details.

2. Let $v = |Du|^2$. Then, within Ω_T, we compute

$$\Delta v = 2Du \cdot D(\Delta u) + 2|D^2 u|^2,$$

$$u_t = \operatorname{div}(\tilde{A}(Du)) = \operatorname{div}(\tilde{f}(v)Du) = \tilde{f}'(v)Dv \cdot Du + \tilde{f}(v)\Delta u,$$

$$\begin{aligned} Du_t =& \tilde{f}''(v)(Dv \cdot Du)Dv + \tilde{f}'(v)(D^2u)Dv \\ & + \tilde{f}'(v)(D^2v)Du + \tilde{f}'(v)(\Delta u)Dv + \tilde{f}(v)D(\Delta u). \end{aligned}$$

Putting these equations into $v_t = 2Du \cdot Du_t$, we obtain

$$v_t - \mathcal{L}(v) - B \cdot Dv = -2\tilde{f}(|Du|^2)|D^2u|^2 \le 0 \quad \text{in } \Omega_T, \tag{3.5}$$

where operator $\mathcal{L}(v)$ and coefficient B are defined by

$$\mathcal{L}(v) = \tilde{f}(|Du|^2)\Delta v + 2\tilde{f}'(|Du|^2)Du \cdot (D^2v)Du,$$

$$B = 2\tilde{f}''(v)(Dv \cdot Du)Du + 2\tilde{f}'(v)(D^2u)Du + 2\tilde{f}'(v)(\Delta u)Du.$$

We write $\mathcal{L}(v) = \sum_{i,j=1}^n a_{ij} v_{x_i x_j}$, with coefficients $a_{ij} = a_{ij}(x,t)$ given by

$$a_{ij} = \tilde{A}^i_{p_j}(Du) = \tilde{f}(|Du|^2)\delta_{ij} + 2\tilde{f}'(|Du|^2)u_{x_i}u_{x_j} \quad (i, j = 1, \cdots, n).$$

Note that on $\bar{\Omega}_T$ all eigenvalues of the matrix (a_{ij}) lie in $[\theta, \Theta]$.

3. We show

$$\max_{(x,t)\in\bar{\Omega}_T} v(x,t) = \max_{x\in\bar{\Omega}} v(x,0),$$

which proves (3.4). We prove this by contradiction. Suppose

$$M := \max_{(x,t)\in\bar{\Omega}_T} v(x,t) > \max_{x\in\bar{\Omega}} v(x,0). \tag{3.6}$$

Let $(x_0,t_0)\in\bar{\Omega}_T$ be such that $v(x_0,t_0)=M$; then $t_0>0$. If $x_0\in\Omega$, then the strong maximum principle applied to (3.5) would imply that v is constant on Ω_{t_0}, which yields $v(x,0)\equiv M$ on $\bar{\Omega}$, a contradiction to (3.6). Consequently $x_0\in\partial\Omega$ and thus $v(x_0,t_0)=M>v(x,t)$ for all $(x,t)\in\Omega_T$. We can then apply Hopf's Lemma for parabolic equations [36] to (3.5) to deduce $\partial v(x_0,t_0)/\partial\mathbf{n}>0$. However, a result of [1, Lemma 2.1] (see aslo [21, Theorem 2]) asserts that $\partial v/\partial\mathbf{n}\le 0$ on $\partial\Omega\times[0,T]$, which gives a desired contradiction. □

3.2 Modification of profile functions

The following elementary results can be proved in a similar way as in [44, 45]; we omit the proofs (see Figures 3.1 and 3.2).

Lemma 3.2.1 (Case I: Perona-Malik type**).** *Assume* Hypothesis (PM)*. For every* $0<r_1<r_2<\sigma(s_0)$*, there exists a function* $\tilde{\sigma}\in C^3([0,\infty))$ *such that*

$$\tilde{\sigma}(s)\begin{cases} =\sigma(s), & 0\le s\le s_-(r_1),\\ <\sigma(s), & s_-(r_1)<s<s_+(r_2),\end{cases}$$
$$\theta\le\tilde{\sigma}'(s)\le\Theta\ \ (0\le s<\infty)$$

for some constants $\Theta\ge\theta>0$*. With such function* $\tilde{\sigma}$*, define* $\tilde{f}(s)=\tilde{\sigma}(\sqrt{s})/\sqrt{s}$ *(*$s>0$*) and* $\tilde{f}(0)=f(0)$*; then* $\tilde{f}\in C^3([0,\infty))$ *fulfills condition* (3.1)*.*

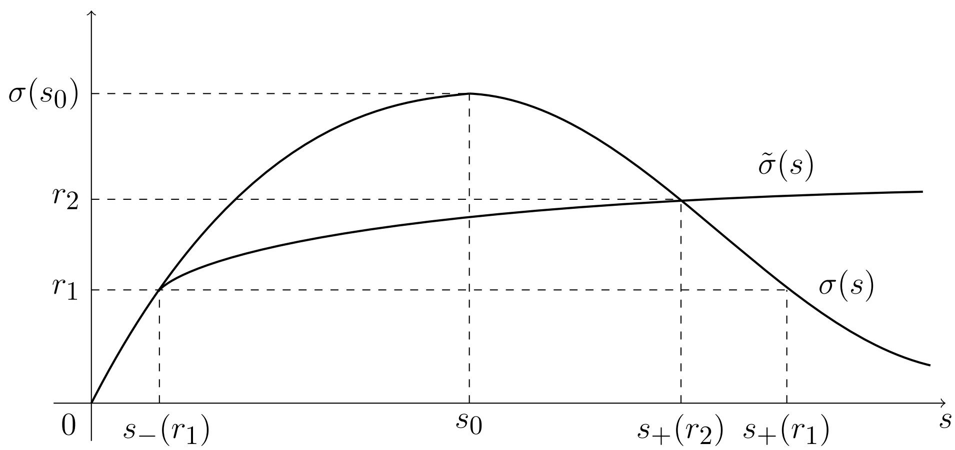


Figure 3.1: **Case I:** Perona-Malik type profile $\sigma(s)$ and modified function $\tilde{\sigma}(s)$.

Lemma 3.2.2 (**Case II:** Cubic-like type)**.** *Assume* Hypothesis (C)*. For every* $\sigma(s_2) < r_1 < r_2 < \sigma(s_1)$*, there exists a function* $\tilde{\sigma} \in C^{1+\alpha}([0,\infty))$ *such that*

$$\tilde{\sigma}(s) \begin{cases} = \sigma(s), & s \in [0, s_-(r_1)] \cup [s_+(r_2), \infty), \\ < \sigma(s), & s_-(r_1) < s \leq s_-(r_2), \\ > \sigma(s), & s_+(r_1) \leq s < s_+(r_2), \end{cases}$$
$$\theta \leq \tilde{\sigma}'(s) \leq \Theta \ \ (0 \leq s < \infty)$$

for some constants $\Theta \geq \theta > 0$*. With such function* $\tilde{\sigma}$*, define* $\tilde{f}(s) = \tilde{\sigma}(\sqrt{s})/\sqrt{s}$ *(*$s > 0$*) and* $\tilde{f}(0) = f(0)$*; then* $\tilde{f} \in C^{1+\alpha}([0,\infty))$ *fulfills condition* (3.1)*.*

3.3 Right inverse of the divergence operator

To deal with linear constraint $\operatorname{div} v = u$, we follow an argument of [4, Lemma 4] to construct a right inverse $\mathcal{R}$ of the divergence operator: $\operatorname{div} \mathcal{R} = Id$ (in the sense of distributions in Ω_T). For the purpose of this dissertation, the construction of $\mathcal{R}$ is restricted to *box* domains,

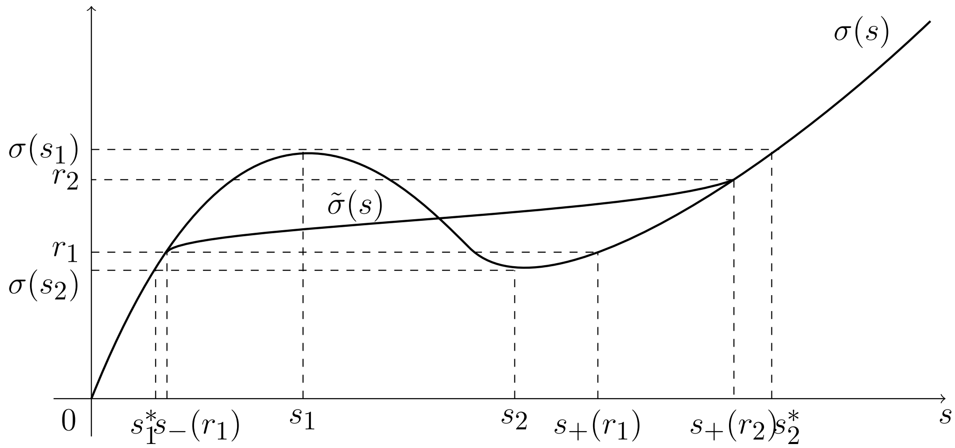


Figure 3.2: **Case II:** Höllig type profile $\sigma(s)$ and modified function $\tilde{\sigma}(s)$.

by which we mean domains given by $Q = J_1 \times J_2 \times \cdots \times J_n$, where $J_i = (a_i, b_i) \subset \mathbb{R}$ is a finite open interval.

Given a box Q, we define a linear operator $\mathcal{R}_n \colon L^\infty(Q) \to L^\infty(Q; \mathbb{R}^n)$ inductively on dimension n as follows. If $n = 1$, for $u \in L^\infty(J_1)$, we define $v = \mathcal{R}_1 u$ by

$$v(x_1) = \int_{a_1}^{x_1} u(s)ds \quad (x_1 \in J_1).$$

Assume $n = 2$. Let $u \in L^\infty(J_1 \times J_2)$. Set $\tilde{u}(x_1) = \int_{a_2}^{b_2} u(x_1, s)\, ds$ for $x_1 \in J_1$. Then $\tilde{u} \in L^\infty(J_1)$. Let $\tilde{v} = \mathcal{R}_1 \tilde{u}$; that is,

$$\tilde{v}(x_1) = \int_{a_1}^{x_1} \tilde{u}(s)ds = \int_{a_1}^{x_1} \int_{a_2}^{b_2} u(s, \tau)\, d\tau ds \quad (x_1 \in J_1).$$

Let $\rho_2 \in C_c^\infty(a_2, b_2)$ be such that $0 \le \rho_2(s) \le \frac{C_0}{b_2 - a_2}$ and $\int_{a_2}^{b_2} \rho_2(s)ds = 1$. Define $v = \mathcal{R}_2 u \in L^\infty(J_1 \times J_2; \mathbb{R}^2)$ by $v = (v^1, v^2)$ with $v^1(x_1, x_2) = \rho_2(x_2)\tilde{v}(x_1)$ and

$$v^2(x_1, x_2) = \int_{a_2}^{x_2} u(x_1, s)ds - \tilde{u}(x_1) \int_{a_2}^{x_2} \rho_2(s)ds.$$

Note that if $u \in W^{1,\infty}(J_1 \times J_2)$ then $\tilde{u} \in W^{1,\infty}(J_1)$; hence $v = \mathcal{R}_2 u \in W^{1,\infty}(J_1 \times J_2; \mathbb{R}^2)$ and $\operatorname{div} v = u$ a.e. in $J_1 \times J_2$. Moreover, if $u \in C^1(\overline{J_1 \times J_2})$ then v is in $C^1(\overline{J_1 \times J_2}; \mathbb{R}^2)$.

Assume that we have defined the operator $\mathcal{R}_{n-1}$. Let $u \in L^\infty(Q)$ with $Q = J_1 \times J_2 \times \cdots \times J_n$ and $x = (x', x_n) \in Q$, where $x' \in Q' = J_1 \times \cdots \times J_{n-1}$ and $x_n \in J_n$. Set $\tilde{u}(x') = \int_{a_n}^{b_n} u(x', s)\, ds$ for $x' \in Q'$. Then $\tilde{u} \in L^\infty(Q')$. By the assumption, $\tilde{v} = \mathcal{R}_{n-1}\tilde{u} \in L^\infty(Q'; \mathbb{R}^{n-1})$ is defined. Write $\tilde{v}(x') = (Z^1(x'), \cdots, Z^{n-1}(x'))$, and let $\rho_n \in C_c^\infty(a_n, b_n)$ be a function satisfying $0 \le \rho_n(s) \le \frac{C_0}{b_n - a_n}$ and $\int_{a_n}^{b_n} \rho_n(s) ds = 1$. Define $v = \mathcal{R}_n u \in L^\infty(Q; \mathbb{R}^n)$ as follows. For $x = (x', x_n) \in Q$, $v(x) = (v^1(x), v^2(x), \cdots, v^n(x))$ is defined by

$$
\begin{aligned}
v^k(x', x_n) &= \rho_n(x_n) Z^k(x') \quad (k = 1, 2, \cdots, n-1), \\
v^n(x', x_n) &= \int_{a_n}^{x_n} u(x', s) ds - \tilde{u}(x') \int_{a_n}^{x_n} \rho_n(s) ds.
\end{aligned}
$$

Then $\mathcal{R}_n \colon L^\infty(Q) \to L^\infty(Q; \mathbb{R}^n)$ is a well-defined linear operator; moreover,

$$
\|\mathcal{R}_n u\|_{L^\infty(Q)} \le C_n \left(|J_1| + \cdots + |J_n|\right) \|u\|_{L^\infty(Q)}, \tag{3.7}
$$

where $C_n > 0$ is a constant depending only on n.

As in the case $n = 2$, we see that if $u \in W^{1,\infty}(Q)$ then $v = \mathcal{R}_n u \in W^{1,\infty}(Q; \mathbb{R}^n)$ and $\operatorname{div} v = u$ a.e. in Q. Also, if $u \in C^1(\bar{Q})$ then $v = \mathcal{R}_n u$ is in $C^1(\bar{Q}; \mathbb{R}^n)$. Moreover, if $u \in W_0^{1,\infty}(Q)$ satisfies $\int_Q u(x) dx = 0$, then one can easily show that $v = \mathcal{R}_n u \in W_0^{1,\infty}(Q; \mathbb{R}^n)$.

Let I be a finite open interval in $\mathbb{R}$. We now extend the operator $\mathcal{R}_n$ to an operator $\mathcal{R}$ on $L^\infty(Q \times I)$ by defining, for a.e. $(x, t) \in Q \times I$,

$$
(\mathcal{R}u)(x, t) = (\mathcal{R}_n u(\cdot, t))(x) \quad \forall\, u \in L^\infty(Q \times I).
$$

Then $\mathcal{R}\colon L^\infty(Q\times I)\to L^\infty(Q\times I;\mathbb{R}^n)$ is a bounded linear operator.

We have the following result.

Theorem 3.3.1. *Let $u\in W_0^{1,\infty}(Q\times I)$ satisfy $\int_Q u(x,t)\,dx=0$ for all $t\in I$. Then $v=\mathcal{R}u\in W_0^{1,\infty}(Q\times I;\mathbb{R}^n)$, $\operatorname{div} v=u$ a.e. in $Q\times I$, and*

$$\|v_t\|_{L^\infty(Q\times I)}\le C_n\,(|J_1|+\cdots+|J_n|)\|u_t\|_{L^\infty(Q\times I)}, \tag{3.8}$$

where $Q=J_1\times\cdots\times J_n$ and C_n is the same constant as in (3.7)*. Moreover, if $u\in C^1(\overline{Q\times I})$ then $v=\mathcal{R}u\in C^1(\overline{Q\times I};\mathbb{R}^n)$.*

Proof. Given $u\in W_0^{1,\infty}(Q\times I)$, let $v=\mathcal{R}u$. We easily verify that v is Lipschitz continuous in t and hence v_t exists. It also follows that $v_t=\mathcal{R}(u_t)$. Clearly, if $\int_Q u(x,t)dx=0$ then $v(x,t)=0$ whenever $t\in\partial I$ or $x\in\partial Q$. This proves $v\in W_0^{1,\infty}(Q\times I;\mathbb{R}^n)$ and the estimate (3.8) follows from (3.7). Finally, from the definition of $\mathcal{R}u$, we see that if $u\in C^1(\overline{Q\times I})$ then $v=\mathcal{R}u\in C^1(\overline{Q\times I};\mathbb{R}^n)$. $\square$

Chapter 4

Perona-Malik type equations

In this chapter, we completely prove the existence result on **Case I:** Perona-Malik type equations, that is, Theorem 2.3.2. In order to so, we assume **Hypothesis (PM)** throughout this chapter.

4.1 Geometry of relevant matrix set

We begin this section by introducing an approach for solving problem (1.2) that turns out to be unsuccessful; however, it provides us with the main idea of solving (1.2) in the context of our method, Theorem 2.2.4. Then we embark on an extensive analysis of *partial* rank-one structure of some relevant matrix set that eventually yields Theorem 4.1.6.

4.1.1 Non-homogeneous differential inclusion and its limitation

Let $u_0 \in W^{1,\infty}(\Omega)$. Assume $\Phi = (u^*, v^*) \in W^{1,\infty}(\Omega_T; \mathbb{R}^{1+n})$ is a boundary function, that is, it satisfies

$$\begin{cases} u^*(x,0) = u_0(x), & x \in \Omega, \\ \operatorname{div} v^*(x,t) = u^*(x,t), & \text{a.e. } (x,t) \in \Omega_T, \\ v^*(\cdot,t) \cdot \mathbf{n}|_{\partial\Omega} = 0, & t \in [0,T]. \end{cases}$$

If a function $w = (u, v) \in W^{1,\infty}(\Omega_T; \mathbb{R}^{1+n})$ solves the Dirichlet problem of *non-homogeneous* differential inclusion

$$\begin{cases} \nabla w(x,t) \in K(u(x,t)), & \text{a.e. } (x,t) \in \Omega_T, \\ w(x,t) = \Phi(x,t), & (x,t) \in \partial\Omega_T, \end{cases} \tag{4.1}$$

then it can be easily seen that u is a Lipschitz solution to (1.2). Here, ∇w denotes the space-time Jacobian matrix of w that lies in $\mathbb{M}^{(1+n)\times(n+1)}$, the space of $(1+n)\times(n+1)$ real matrices, and for each $l \in \mathbb{R}$, $K(l)$ is the subset of $\mathbb{M}^{(1+n)\times(n+1)}$ defined by

$$K(l) = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \,\Big|\, p \in \mathbb{R}^n,\, c \in \mathbb{R},\, B \in \mathbb{M}^{n\times n},\, \operatorname{tr} B = l \right\}. \tag{4.2}$$

The Dirichlet problem (4.1) falls into the framework of general non-homogeneous partial differential inclusions that have been studied by Dacorogna and Marcellini [10] using Baire's category method and by Müller and Sychev [34] using the convex integration method following the works [18, 32, 33]; see also [28]. Recently, the methods of differential inclusion have been successfully applied to other important problems in partial differential equations [8, 11, 31, 38, 41].

We point out that the existence result of [34] is not applicable to problem (4.1) even in dimension $n = 1$, as has already been noticed in [44, 45]. A key condition in the main existence theorem of [34], when applied to (4.1), would require that the boundary function Φ satisfy

$$\nabla\Phi(x,t) \in U(u^*(x,t)) \cup K(u^*(x,t)) \quad a.e.\ (x,t) \in \Omega_T,$$

where $U(s) \subset \mathbf{M}^{(1+n)\times(n+1)}$ $(s \in \mathbb{R})$ are bounded sets that are *reducible to* $K(s)$ in the sense that, for every $s_0 \in \mathbb{R}$, $\xi_0 \in U(s_0)$, $\epsilon > 0$, and bounded Lipschitz domain $G \subset \mathbb{R}^{n+1}$, there exist a piecewise affine function $w \in W_0^{1,\infty}(G;\mathbb{R}^{1+n})$ and a $\delta > 0$ satisfying, for a.e. $z = (x,t) \in G$,

$$\xi_0 + \nabla w(z) \in \bigcap_{|s-s_0|<\delta} U(s), \quad \int_G \operatorname{dist}(\xi_0 + \nabla w(z), K(s_0))\, dz < \epsilon |G|.$$

The second condition would imply $\operatorname{tr} B_0 = s_0$ for each $\xi_0 = \begin{pmatrix} p_0 & c_0 \\ B_0 & \beta_0 \end{pmatrix} \in U(s_0)$ and $s_0 \in \mathbb{R}$; but then $\cap_{|s-s_0|<\delta} U(s) = \emptyset$, which makes the first condition impossible.

However, certain structures of set $K(0)$ turn out to be still quite useful, especially when it comes to the relaxation of *homogeneous* differential inclusion $\nabla \omega(z) \in K(0)$ with $z = (x,t)$ and $\omega = (\varphi, \psi)$. We investigate these structures and establish such relaxation result through the rest of this section and Section 4.2.

4.1.2 Geometry of the matrix set F_0

Fix any two numbers $0 < r_1 < r_2 < \sigma(s_0)$, and let $F_0 = F_{r_1,r_2}(0)$ be the subset of $K(0)$ defined by

$$F_0 = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \;\middle|\; \begin{array}{l} p \in \mathbb{R}^n,\ |p| \in (s_-(r_1), s_-(r_2)) \cup (s_+(r_2), s_+(r_1)), \\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0 \end{array} \right\}.$$

We decompose the set F_0 into two disjoint subsets as follows:

$$F_- = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \;\middle|\; \begin{array}{l} p \in \mathbb{R}^n,\ |p| \in (s_-(r_1), s_-(r_2)), \\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0 \end{array} \right\},$$

$$F_+ = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \;\middle|\; \begin{array}{l} p \in \mathbb{R}^n,\ |p| \in (s_+(r_2), s_+(r_1)), \\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \mathrm{tr}\, B = 0 \end{array} \right\}.$$

In order to extract more detailed information on solutions as stated in Theorem 2.3.2, we focus on the homogeneous differential inclusion $\nabla\omega(z) \in F_0$; thus we first scrutinize the rank-one structure of the set F_0. We introduce the following:

Definition 4.1.1. For a given set $E \subset \mathbb{M}^{(1+n)\times(n+1)}$, $L(E)$ is defined to be the set of all matrices $\xi \in \mathbb{M}^{(1+n)\times(n+1)}$ that are not in E but are representable by $\xi = \lambda\xi_1 + (1-\lambda)\xi_2$ for some $\lambda \in (0,1)$ and $\xi_1, \xi_2 \in E$ with $\mathrm{rank}(\xi_1 - \xi_2) = 1$, or equivalently,

$$L(E) = \{\xi \notin E \mid \xi + t_\pm\eta \in E \text{ for some } t_- < 0 < t_+ \text{ and } \mathrm{rank}\,\eta = 1\}.$$

For the matrix set F_0, we define

$$R(F_0) = \bigcup_{\xi_\pm \in F_\pm,\, \mathrm{rank}(\xi_+ - \xi_-) = 1} (\xi_-, \xi_+),$$

where (ξ_-, ξ_+) is the open line segment in $\mathbb{M}^{(1+n)\times(n+1)}$ joining $\xi_\pm$.

From a careful analysis, one can actually deduce

$$L(F_0) = R(F_0) \cup L(F_+). \tag{4.3}$$

Here, due to the backward nature of profile $\sigma(s)$ on $(s_+(r_2), s_+(r_1))$, the set $L(F_+)$ is non-empty, and its structural analysis seems quite difficult to be accomplished by the presence of some *degeneracy*. Fortunately, it turns to be harmless and even better to only stick to the analysis of the set $R(F_0)$ towards the existence result, Theorem 2.3.2.

We perform the step-by-step analysis of the set $R(F_0)$.

1. **Alternate expression for $R(F_0)$.** We derive an equivalent condition for the membership of a matrix in $R(F_0)$.

Lemma 4.1.2. *Let $\xi \in \mathbb{M}^{(1+n)\times(n+1)}$. Then $\xi \in R(F_0)$ if and only if there exist numbers $t_- < 0 < t_+$ and vectors $q,\ \gamma \in \mathbb{R}^n$ with $|q| = 1$, $\gamma \cdot q = 0$ such that for each $b \in \mathbb{R}\setminus\{0\}$, if $\eta = \begin{pmatrix} q & b \\ \frac{1}{b} q\otimes\gamma & \gamma \end{pmatrix}$, then $\xi + t_\pm \eta \in F_\pm$.*

Proof. Assume $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0)$. By definition, $\xi + t_\pm \tilde\eta \in F_\pm$, where $t_- < 0 < t_+$ and $\tilde\eta$ is a rank-one matrix given by

$$\tilde\eta = \begin{pmatrix} a \\ \alpha \end{pmatrix} \otimes (q, \tilde b) = \begin{pmatrix} aq & a\tilde b \\ \alpha\otimes q & \tilde b\alpha \end{pmatrix}, \quad a^2 + |\alpha|^2 \neq 0, \quad \tilde b^2 + |q|^2 \neq 0,$$

for some $a,\ \tilde b \in \mathbb{R}$ and $\alpha,\ q \in \mathbb{R}^n$; here $\alpha\otimes q$ denotes the rank-one or zero matrix $(\alpha_i q_j)$ in $\mathbb{M}^{n\times n}$. Condition $\xi + t_\pm\tilde\eta \in F_\pm$ with $t_- < 0 < t_+$ is equivalent to the following:

$$\begin{gathered} \operatorname{tr} B = 0, \quad \alpha\cdot q = 0, \quad A(p + t_\pm a q) = \beta + t_\pm \tilde b \alpha, \\ |p + t_+ a q| \in (s_+(r_2), s_+(r_1)), \quad |p + t_- a q| \in (s_-(r_1), s_-(r_2)). \end{gathered} \tag{4.4}$$

Therefore, $aq \neq 0$. Upon rescaling $\tilde\eta$ and $t_\pm$, we can assume $a = 1$ and $|q| = 1$; namely,

$$\tilde\eta = \begin{pmatrix} q & \tilde b \\ \alpha\otimes q & \tilde b\alpha \end{pmatrix}, \quad |q| = 1, \quad \alpha\cdot q = 0.$$

We now let $\gamma = \tilde{b}\alpha$. Let $b \in \mathbb{R} \setminus \{0\}$ and

$$\eta = \begin{pmatrix} q & b \\ \frac{1}{b}\gamma \otimes q & \gamma \end{pmatrix}.$$

From (4.4), it follows that $\xi + t_\pm \eta \in F_\pm$.

The converse directly follows from the definition of $R(F_0)$. $\square$

2. **Diagonal components of matrices in $R(F_0)$.** The following gives a description for the diagonal components of matrices in $R(F_0)$.

Lemma 4.1.3.

$$R(F_0) = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \,\middle|\, c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0,\ (p,\beta) \in \mathcal{S} \right\} \tag{4.5}$$

for some set $\mathcal{S} = \mathcal{S}_{r_1,r_2} \subset \mathbb{R}^{n+n}$.

Proof. Let (c, B), $(c', B') \in \mathbb{R} \times \mathbb{M}^{n\times n}$ be such that $\operatorname{tr} B = \operatorname{tr} B' = 0$, and define

$$\mathcal{S}_{(c,B)} = \left\{ (p,\beta) \in \mathbb{R}^{n+n} \,\middle|\, \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0) \right\},$$

$$\mathcal{S}_{(c',B')} = \left\{ (p,\beta) \in \mathbb{R}^{n+n} \,\middle|\, \begin{pmatrix} p & c' \\ B' & \beta \end{pmatrix} \in R(F_0) \right\}.$$

It is sufficient to show that $\mathcal{S}_{(c,B)} = \mathcal{S}_{(c',B')} =: \mathcal{S}$. Let $(p,\beta) \in \mathcal{S}_{(c,B)}$, that is, $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0)$. Then $\xi_\pm := \xi + t_\pm \eta \in F_\pm$ for some $t_- < 0 < t_+$ and $\operatorname{rank} \eta = 1$. Observe that

$\xi = \lambda\xi_+ + (1-\lambda)\xi_-$ with $\lambda = \frac{-t_-}{t_+ - t_-} \in (0,1)$ and that

$$\xi' := \begin{pmatrix} p & c' \\ B' & \beta \end{pmatrix} = \xi + \begin{pmatrix} 0 & \tilde{c} \\ \tilde{B} & 0 \end{pmatrix} = \lambda\tilde{\xi}_+ + (1-\lambda)\tilde{\xi}_-$$

where $\tilde{c} = c' - c$, $\tilde{B} = B' - B$, and $\tilde{\xi}_\pm = \xi_\pm + \begin{pmatrix} 0 & \tilde{c} \\ \tilde{B} & 0 \end{pmatrix}$. Since $\xi_\pm \in F_\pm$ and $\operatorname{tr}\tilde{B} = 0$, we have $\tilde{\xi}_\pm \in F_\pm$, and so $\xi' \in R(F_0)$. This implies $(p,\beta) \in \mathcal{S}_{(c',B')}$; hence $\mathcal{S}_{(c,B)} \subset \mathcal{S}_{(c',B')}$. Likewise, $\mathcal{S}_{(c',B')} \subset \mathcal{S}_{(c,B)}$; that is, $\mathcal{S}_{(c,B)} = \mathcal{S}_{(c',B')}$. $\square$

3. **Selection of approximate collinear rank-one connections for $R(F_0)$.** We begin with a 2-dimensional description for the rank-one connections of diagonal components of matrices in $R(F_0)$ in a general form.

Lemma 4.1.4. *For all positive numbers a, b, c with $b > a$, there exists a continuous function*

$$h(a,b,c,\cdot,\cdot,\cdot) : I_{a,c} = [0,a) \times [0,\infty) \times [0,c) \to [0,\infty)$$

with $h(a,b,c,0,0,0) = 0$ satisfying the following:

Let δ_1, δ_2 and η be any positive numbers with

$$0 < a - \delta_1 < a < b < b + \delta_2, \quad 0 < c - \eta < c,$$

and let $R_1 \in [a-\delta_1, a]$, $R_2 \in [b, b+\delta_2]$, and $\tilde{R}_1, \tilde{R}_2 \in [c-\eta, c]$. Suppose $\theta \in [-\pi/2, \pi/2]$ and

$$\Big(\tilde{R}_1\big(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta)\big) - \tilde{R}_2\big(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta)\big)\Big)$$

$$\cdot\Big(R_1\big(\cos(\frac{\pi}{2}+\theta),\sin(\frac{\pi}{2}+\theta)\big)-R_2\big(\cos(\frac{\pi}{2}-\theta),\sin(\frac{\pi}{2}-\theta)\big)\Big)=0.$$

Then $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, $\tilde{R}_1\geq\tilde{R}_2$, *and*

$$\max\Big\{\big|(0,a)-R_1\big(\cos(\frac{\pi}{2}+\theta),\sin(\frac{\pi}{2}+\theta)\big)\big|,\ \big|(0,b)-R_2\big(\cos(\frac{\pi}{2}-\theta),\sin(\frac{\pi}{2}-\theta)\big)\big|$$

$$\big|(0,c)-\tilde{R}_1\big(\cos(\frac{\pi}{2}+\theta),\sin(\frac{\pi}{2}+\theta)\big)\big|,\ \big|(0,c)-\tilde{R}_2\big(\cos(\frac{\pi}{2}-\theta),\sin(\frac{\pi}{2}-\theta)\big)\big|\Big\}$$

$$\leq h(a,b,c,\delta_1,\delta_2,\eta).$$

Proof. By assumption,

$$0=(\tilde{R}_1(-\sin\theta,\cos\theta)-\tilde{R}_2(\sin\theta,\cos\theta))\cdot(R_1(-\sin\theta,\cos\theta)-R_2(\sin\theta,\cos\theta))$$

$$=(-(\tilde{R}_1+\tilde{R}_2)\sin\theta,(\tilde{R}_1-\tilde{R}_2)\cos\theta)\cdot(-(R_1+R_2)\sin\theta,(R_1-R_2)\cos\theta)$$

$$=(\tilde{R}_1+\tilde{R}_2)(R_1+R_2)\sin^2\theta+(\tilde{R}_1-\tilde{R}_2)(R_1-R_2)\cos^2\theta,$$

that is,

$$(R_2-R_1)(\tilde{R}_1-\tilde{R}_2)\cos^2\theta=(R_1+R_2)(\tilde{R}_1+\tilde{R}_2)\sin^2\theta;$$

hence, $\theta\neq\pm\frac{\pi}{2}$, $\tilde{R}_1\geq\tilde{R}_2$, and

$$\theta=\pm\tan^{-1}\left(\sqrt{\frac{(R_2-R_1)(\tilde{R}_1-\tilde{R}_2)}{(R_1+R_2)(\tilde{R}_1+\tilde{R}_2)}}\right).$$

So

$$|\theta|\leq\tan^{-1}\left(\sqrt{\frac{(b-a+\delta_1+\delta_2)\eta}{2(a+b-\delta_1)(c-\eta)}}\right)=:g(a,b,c,\delta_1,\delta_2,\eta).$$

Note that the function $g(a,b,c,\cdot,\cdot,\cdot):I_{a,c}\to[0,\pi/2)$ is well-defined and continuous and that $g(a,b,c,\delta_1,\delta_2,\eta) = 0$ for all $(\delta_1,\delta_2,\eta) \in I_{a,c}$ with $\eta = 0$.

Observe now that

$$|(0,a) - R_1(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta))|$$

$$\leq \max\{|(0,a) - a(-\sin\theta, \cos\theta)|, |(0,a) - (a-\delta_1)(-\sin\theta, \cos\theta)|\}$$

$$= \max\left\{\sqrt{a^2\sin^2\theta + a^2(1-\cos\theta)^2}, \sqrt{(a-\delta_1)^2\sin^2\theta + (a-(a-\delta_1)\cos\theta)^2}\right\}$$

$$= \max\left\{\sqrt{2}a\sqrt{1-\cos\theta}, \sqrt{(a-\delta_1)^2 + a^2 - 2a(a-\delta_1)\cos\theta}\right\}$$

$$\leq \max\left\{\sqrt{2}a\sqrt{1-\cos(g(a,b,c,\delta_1,\delta_2,\eta))},\right.$$

$$\left.\sqrt{(a-\delta_1)^2 + a^2 - 2a(a-\delta_1)\cos(g(a,b,c,\delta_1,\delta_2,\eta))}\right\}$$

$$=: h_1(a,b,c,\delta_1,\delta_2,\eta),$$

$$|(0,b) - R_2(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta))|$$

$$\leq \max\{|(0,b) - b(\sin\theta, \cos\theta)|, |(0,b) - (b+\delta_2)(\sin\theta, \cos\theta)|\}$$

$$= \max\left\{\sqrt{b^2\sin^2\theta + b^2(1-\cos\theta)^2}, \sqrt{(b+\delta_2)^2\sin^2\theta + (b-(b+\delta_2)\cos\theta)^2}\right\}$$

$$= \max\left\{\sqrt{2}b\sqrt{1-\cos\theta}, \sqrt{(b+\delta_2)^2 + b^2 - 2b(b+\delta_2)\cos\theta}\right\}$$

$$\leq \max\left\{\sqrt{2}b\sqrt{1-\cos(g(a,b,c,\delta_1,\delta_2,\eta))},\right.$$

$$\left.\sqrt{(b+\delta_2)^2 + b^2 - 2b(b+\delta_2)\cos(g(a,b,c,\delta_1,\delta_2,\eta))}\right\}$$

$$=: h_2(a,b,c,\delta_1,\delta_2,\eta),$$

$$|(0,c) - \tilde{R}_1(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta))|$$

$$\leq \max\Big\{\sqrt{2}c\sqrt{1-\cos(g(a,b,c,\delta_1,\delta_2,\eta))},$$

$$\sqrt{(c-\eta)^2+c^2-2c(c-\eta)\cos(g(a,b,c,\delta_1,\delta_2,\eta))}\Big\}$$

$$=: h_3(a,b,c,\delta_1,\delta_2,\eta),$$

$$|(0,c)-\tilde{R}_2(\cos(\frac{\pi}{2}-\theta),\sin(\frac{\pi}{2}-\theta))| \leq h_{i,3}(a,b,c,\delta_1,\delta_2,\eta).$$

Define $h(a,b,c,\delta_1,\delta_2,\eta) = \max_{1\leq j\leq 3} h_j(a,b,c,\delta_1,\delta_2,\eta)$; then it is trivial to see that the function $h(a,b,c,\cdot,\cdot,\cdot) : I_{a,c} \to [0,\infty)$ is well-defined and satisfies the desired properties. $\square$

We now apply the previous lemma to choose *approximate* collinear rank-one connections for the diagonal components of matrices in $R(F_0)$.

Theorem 4.1.5. *Let $p_\pm \in \mathbb{R}^n$ satisfy*

$$s_-(r_1) < |p_-| < s_-(r_2) < s_+(r_2) < |p_+| < s_+(r_1)$$

and $(A(p_+)-A(p_-))\cdot(p_+-p_-)=0$. Then there exists a vector $\zeta^0 \in \mathbb{S}^{n-1}$ such that, with $p^0_\pm = s_\pm(r_2)\zeta^0$, $A(p^0_\pm) = r_2\zeta^0$, we have

$$\max\{|p^0_- - p_-|, |p^0_+ - p_+|, |A(p^0_-) - A(p_-)|, |A(p^0_+) - A(p_+)|\}$$

$$\leq h(s_-(r_2), s_+(r_2), r_2, s_-(r_2)-s_-(r_1), s_+(r_1)-s_+(r_2), r_2-r_1)$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ and h is the function in Lemma 4.1.4.

Proof. Let Σ_2 denote the 2-dimensional linear subspace of $\mathbb{R}^n$ spanned by the two vectors $p_\pm$. (In the case that $p_\pm$ are collinear, we choose Σ_2 to be any 2-dimensional space containing

$p_\pm$.) Set

$$\zeta^0 = \frac{\frac{p_+}{|p_+|} + \frac{p_-}{|p_-|}}{\left|\frac{p_+}{|p_+|} + \frac{p_-}{|p_-|}\right|} \in \mathbb{S}^{n-1} \cap \Sigma_2.$$

Since vectors $p_\pm$, $A(p_\pm)$ and ζ^0 all lie in Σ_2, we can recast the problem into the setting of the previous lemma via one of the two linear isomorphisms of Σ_2 onto $\mathbb{R}^2$ with correspondence $\zeta^0 \leftrightarrow (0,1) \in \mathbb{R}^2$. Then the result follows, where $a = s_-(r_2)$, $b = s_+(r_2)$, $c = r_2$, $\delta_1 = s_-(r_2) - s_-(r_1)$, $\delta_2 = s_+(r_1) - s_+(r_2)$, $\eta = r_2 - r_1$, $R_1 = |p_-|$, $R_2 = |p_+|$, $\tilde{R}_1 = \sigma(|p_-|)$, $\tilde{R}_2 = \sigma(|p_+|)$, and $\theta \in [0, \pi/2]$ is the half of the angle between p_+ and p_- in applying Lemma 4.1.4. $\square$

4. **Final characterization of $R(F_0)$.** We are now ready to establish the result concerning essential structures of $R(F_0)$.

Theorem 4.1.6. *Let $0 < r_2 < \sigma(s_0)$. Then there exists a number $l_2 = l_{r_2} \in (0, r_2)$ such that for any $l_2 < r_1 < r_2$, the set $\mathcal{S} = \mathcal{S}_{r_1,r_2} \subset \mathbb{R}^{n+n}$ in (4.5) satisfies the following:*

(i) $\sup_{(p,\beta)\in\mathcal{S}} |p| \le s_+(r_1)$ and $\sup_{(p,\beta)\in\mathcal{S}} |\beta| \le r_2$; hence $\mathcal{S}$ is bounded.

(ii) $\mathcal{S}$ is open.

(iii) For each $(p_0, \beta_0) \in \mathcal{S}$, there exist an open set $\mathcal{V} \subset\subset \mathcal{S}$ containing (p_0, β_0) and C^1 functions $q : \bar{\mathcal{V}} \to \mathbf{S}^{n-1}$, $\gamma : \bar{\mathcal{V}} \to \mathbb{R}^n$, $t_\pm : \bar{\mathcal{V}} \to \mathbb{R}$ with $\gamma \cdot q = 0$ and $t_- < 0 < t_+$ on $\bar{\mathcal{V}}$ such that for every $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0) = R(F_{r_1,r_2}(0))$ with $(p, \beta) \in \bar{\mathcal{V}}$, we have

$$\xi + t_\pm \eta \in F_\pm,$$

where $t_\pm = t_\pm(p, \beta)$, $\eta = \begin{pmatrix} q(p,\beta) & b \\ \frac{1}{b}\gamma(p,\beta) \otimes q(p,\beta) & \gamma(p,\beta) \end{pmatrix}$, and $b \neq 0$ is arbitrary.

Proof. Fix any $0 < r_2 < \sigma(s_0)$. For the moment, we let r_1 be any number in $(0, r_2)$ and prove (i). Then we choose later a lower bound $l_2 = l_{r_2} \in (0, r_2)$ of r_1 for the validity of (ii) and (iii) above.

We now divide the proof into several steps.

1. To show that (i) holds, choose any $(p, \beta) \in \mathcal{S}$. By Lemma 4.1.3, $\xi := \begin{pmatrix} p & 0 \\ O & \beta \end{pmatrix} \in R(F_0) = R(F_{r_1,r_2}(0))$, where O is the $n \times n$ zero matrix. By the definition of $R(F_0)$, there exist two matrices $\xi_\pm = \begin{pmatrix} p_\pm & c_\pm \\ B_\pm & \sigma(|p_\pm|)\frac{p_\pm}{|p_\pm|} \end{pmatrix} \in F_\pm$ and a number $0 < \lambda < 1$ such that $\xi = \lambda\xi_+ + (1-\lambda)\xi_-$. So

$$|p| = |\lambda p_+ + (1-\lambda)p_-| \le s_+(r_1),$$

$$|\beta| = \left|\lambda\sigma(|p_+|)\frac{p_+}{|p_+|} + (1-\lambda)\sigma(|p_-|)\frac{p_-}{|p_-|}\right| \le r_2\,;$$

hence, $\sup_{(p,\beta)\in\mathcal{S}} |p| \le s_+(r_1)$, $\sup_{(p,\beta)\in\mathcal{S}} |\beta| \le r_2$, and $\mathcal{S}$ is bounded. So (i) is proved.

2. We now turn to the remaining assertions that for all $r_1 < r_2$ sufficiently close to r_2, $\mathcal{S} = \mathcal{S}_{r_1,r_2}$ satisfies (ii) and (iii). But in this step, we still assume r_1 is any fixed number in $(0, r_2)$.

Let $(p_0, \beta_0) \in \mathcal{S}$. Since $\xi_0 := \begin{pmatrix} p_0 & 0 \\ O & \beta_0 \end{pmatrix} \in R(F_0)$, it follows from Lemma 4.1.2 that there exist numbers $s_0 < 0 < t_0$ and vectors q_0, $\gamma_0 \in \mathbb{R}^n$ with $|q_0| = 1$, $\gamma_0 \cdot q_0 = 0$ such that $\xi_0 + s_0\eta_0 \in F_-$ and $\xi_0 + t_0\eta_0 \in F_+$, where $\eta_0 = \begin{pmatrix} q_0 & b \\ \frac{1}{b}q_0 \otimes \gamma_0 & \gamma_0 \end{pmatrix}$ and $b \neq 0$ is any fixed number. Let $q_0' = t_0 q_0 \neq 0$, $\gamma_0' = t_0\gamma_0$, and $s_0' = s_0/t_0 < 0$; then

$$\gamma_0' \cdot q_0' = 0, \quad s_-(r_1) < |p_0 + s_0' q_0'| < s_-(r_2),$$

$$s_+(r_2) < |p_0 + q_0'| < s_+(r_1),$$

$$\sigma(|p_0 + s_0'q_0'|)\frac{p_0 + s_0'q_0'}{|p_0 + s_0'q_0'|} = \beta_0 + s_0'\gamma_0', \quad \sigma(|p_0 + q_0'|)\frac{p_0 + q_0'}{|p_0 + q_0'|} = \beta_0 + \gamma_0'. \tag{4.6}$$

Observe also

$$t_0 - s_0 \geq |(p_0 + t_0 q_0)| - |(p_0 + s_0 q_0)| > s_+(r_2) - s_-(r_2). \tag{4.7}$$

Next, consider the function F defined by

$$\begin{aligned} F(\gamma', q', s'; p, \beta) \quad = \quad & (\sigma(|p + s'q'|)\frac{p + s'q'}{|p + s'q'|} - \beta - s'\gamma', \\ & \sigma(|p + q'|)\frac{p + q'}{|p + q'|} - \beta - \gamma', \gamma' \cdot q') \in \mathbb{R}^{n+n+1} \end{aligned}$$

for all γ', q', p, $\beta \in \mathbb{R}^n$ and $s' \in \mathbb{R}$ with $s_-(r_1) < |p + s'q'| < s_-(r_2)$, $s_+(r_2) < |p + q'| < s_+(r_1)$. Then F is C^1 in the described open subset of $\mathbb{R}^{n+n+1+n+n}$, and the above observation gives

$$F(\gamma_0', q_0', s_0'; p_0, \beta_0) = 0.$$

Suppose for the moment that the Jacobian matrix $D_{(\gamma', q', s')}F$ is invertible at the point $(\gamma_0', q_0', s_0'; p_0, \beta_0)$. Then the Implicit Function Theorem implies the following: There exist a bounded domain $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_{(p_0, \beta_0)} \subset \mathbb{R}^{n+n}$ containing (p_0, β_0) and C^1 functions $\tilde{q}$, $\tilde{\gamma} \in \mathbb{R}^n$, $\tilde{s} \in \mathbb{R}$ of $(p, \beta) \in \tilde{\mathcal{V}}$ such that

$$\tilde{\gamma}(p_0, \beta_0) = \gamma_0', \quad \tilde{q}(p_0, \beta_0) = q_0', \quad \tilde{s}(p_0, \beta_0) = s_0'$$

and that

$$\tilde{s}(p, \beta) < 0, \quad s_-(r_1) < |p + \tilde{s}(p, \beta)\tilde{q}(p, \beta)| < s_-(r_2),$$

$$s_+(r_2) < |p + \tilde{q}(p,\beta)| < s_+(r_1),$$

$$F(\tilde{\gamma}(p,\beta), \tilde{q}(p,\beta), \tilde{s}(p,\beta); p,\beta) = 0 \quad \forall (p,\beta) \in \tilde{\mathcal{V}}.$$

Define

$$\gamma = \frac{\tilde{\gamma}}{|\tilde{q}|}, \quad q = \frac{\tilde{q}}{|\tilde{q}|}, \quad t_- = \tilde{s}|\tilde{q}|, \quad t_+ = |\tilde{q}| \quad \text{in } \tilde{\mathcal{V}};$$

then

$$s_-(r_1) < |p + t_- q| < s_-(r_2),$$

$$s_+(r_2) < |p + t_+ q| < s_+(r_1),$$

$$\sigma(|p + t_\pm q|)\frac{p + t_\pm q}{|p + t_\pm q|} = \beta + t_\pm \gamma, \quad |q| = 1, \quad \gamma \cdot q = 0, \quad t_- < 0 < t_+,$$

where $(p,\beta) \in \tilde{\mathcal{V}}$, $\gamma = \gamma(p,\beta)$, $q = q(p,\beta)$, and $t_\pm = t_\pm(p,\beta)$.

Let $(p,\beta) \in \tilde{\mathcal{V}}$, $B \in \mathbb{M}^{n\times n}$, $\operatorname{tr} B = 0$, $b, c \in \mathbb{R}$, $b \neq 0$, $q = q(p,\beta)$, $\gamma = \gamma(p,\beta)$, $t_\pm = t_\pm(p,\beta)$, $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix}$, and $\eta = \begin{pmatrix} q & b \\ \frac{1}{b}\gamma \otimes q & \gamma \end{pmatrix}$. Then $\xi_\pm := \xi + t_\pm \eta \in F_\pm$. By the definition of $R(F_0)$, $\xi \in (\xi_-, \xi_+) \subset R(F_0)$. By Lemma 4.1.3, we thus have $(p,\beta) \in \mathcal{S}$; hence $\tilde{\mathcal{V}} \subset \mathcal{S}$. Choosing any open set $\mathcal{V} \subset\subset \tilde{\mathcal{V}}$ with $(p_0,\beta_0) \in \mathcal{V}$, the assertions (ii) and (iii) hold with $\mathcal{S} = \cup_{(p_0,\beta_0)\in\mathcal{S}} \tilde{\mathcal{V}}_{(p_0,\beta_0)}$ open.

3. In this step, we continue Step 2 to deduce an equivalent condition for the invertibility of the Jacobian matrix $D_{(\gamma',q',s')}F$ at $(\gamma_0', q_0', s_0'; p_0, \beta_0)$. By direct computation,

$$D_{(\gamma',q',s')}F = \begin{pmatrix} -s' I_n & M_{s'} & \omega_{s'}^- \\ -I_n & M_1 & 0 \\ q' & \gamma' & 0 \end{pmatrix} \in \mathbb{M}^{(n+n+1)\times(n+n+1)},$$

where I_n is the $n \times n$ identity matrix,

$$M_{s'} = s'(\sigma'(|p+s'q'|) - \frac{\sigma(|p+s'q'|)}{|p+s'q'|})\frac{p+s'q'}{|p+s'q'|} \otimes \frac{p+s'q'}{|p+s'q'|} + s'\frac{\sigma(|p+s'q'|)}{|p+s'q'|}I_n,$$

$$\omega^{\pm}_{s'} = (\sigma'(|p+s'q'|) - \frac{\sigma(|p+s'q'|)}{|p+s'q'|})(\frac{p+s'q'}{|p+s'q'|} \cdot q')\frac{p+s'q'}{|p+s'q'|} + \frac{\sigma(|p+s'q'|)}{|p+s'q'|}q' \pm \gamma'.$$

For notational simplicity, we write $(\gamma', q', s'; p, \beta) = (\gamma'_0, q'_0, s'_0; p_0, \beta_0)$. Applying suitable elementary row operations, where $s' < 0$,

$$D_{(\gamma',q',s')}F \rightarrow \begin{pmatrix} -s'I_n & M_{s'} & \omega^-_{s'} \\ O & M_1 - \frac{1}{s'}M_{s'} & -\frac{1}{s'}\omega^-_{s'} \\ 0 & \gamma' + \frac{q'_1}{s'}(M_{s'})^1 + \cdots + \frac{q'_n}{s'}(M_{s'})^n & \frac{1}{s'}q' \cdot \omega^-_{s'} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -s'I_n & M_{s'} & \omega^-_{s'} \\ O & s'M_1 - M_{s'} & -\omega^-_{s'} \\ 0 & s'\gamma' + q'_1(M_{s'})^1 + \cdots + q'_n(M_{s'})^n & q' \cdot \omega^-_{s'} \end{pmatrix},$$

where O is the $n \times n$ zero matrix, and $(M_{s'})^i$ is the ith row of $M_{s'}$. Since $|q'| = t_0$, $\gamma' \cdot q' = 0$, and $s_-(r_1) < |p+s'q'| < s_-(r_2)$, we have

$$q' \cdot \omega^-_{s'} = (\sigma'(|p+s'q'|) - \frac{\sigma(|p+s'q'|)}{|p+s'q'|})(\frac{p+s'q'}{|p+s'q'|} \cdot q')^2 + \frac{\sigma(|p+s'q'|)}{|p+s'q'|}t_0^2$$

$$= t_0^2(\cos^2\theta'\sigma'(|p+s'q'|) + (1-\cos^2\theta')\frac{\sigma(|p+s'q'|)}{|p+s'q'|}) > 0,$$

where $\theta' \in [0, \pi]$ is the angle between $p + s'q'$ and q'. Observe here that the forward part of σ in the definition of F_- becomes essential to guarantee that $\sigma'(|p+s'q'|) > 0$. After some elementary column operations to the last matrix from the above row operations, we obtain

$$D_{(\gamma',q',s')}F \to \begin{pmatrix} -s'I_n & M_{s'} - N_{s'} & \omega^-_{s'} \\ O & s'M_1 - M_{s'} + N_{s'} & -\omega^-_{s'} \\ 0 & 0 & q' \cdot \omega^-_{s'} \end{pmatrix},$$

where the jth column of $N_{s'} \in \mathbb{M}^{n\times n}$ is $\frac{s'\gamma'_j + q'\cdot(M_{s'})_j}{q'\cdot\omega^-_{s'}}\omega^-_{s'}$. So $D_{(\gamma',q',s')}F$ is invertible if and only if the $n \times n$ matrix $M_1 - \frac{1}{s'}M_{s'} + \frac{1}{s'}N_{s'}$ is invertible. We compute

$$M_1 - \frac{1}{s'}M_{s'} + \frac{1}{s'}N_{s'} = (\sigma'(|p+q'|) - \frac{\sigma(|p+q'|)}{|p+q'|})\frac{p+q'}{|p+q'|} \otimes \frac{p+q'}{|p+q'|}$$

$$+\frac{\sigma(|p+q'|)}{|p+q'|}I_n - (\sigma'(|p+s'q'|) - \frac{\sigma(|p+s'q'|)}{|p+s'q'|})\frac{p+s'q'}{|p+s'q'|} \otimes \frac{p+s'q'}{|p+s'q'|}$$

$$-\frac{\sigma(|p+s'q'|)}{|p+s'q'|}I_n + \frac{1}{q'\cdot\omega^-_{s'}}\omega^-_{s'} \otimes (\gamma'$$

$$+(\sigma'(|p+s'q'|) - \frac{\sigma(|p+s'q'|)}{|p+s'q'|})(\frac{p+s'q'}{|p+s'q'|}\cdot q')\frac{p+s'q'}{|p+s'q'|} + \frac{\sigma(|p+s'q'|)}{|p+s'q'|}q')$$

$$= (a_1 - a_{s'})I_n + (b_1 - a_1)\frac{p+q'}{|p+q'|} \otimes \frac{p+q'}{|p+q'|}$$

$$-(b_{s'} - a_{s'})\frac{p+s'q'}{|p+s'q'|} \otimes \frac{p+s'q'}{|p+s'q'|} + \frac{1}{q'\cdot\omega^-_{s'}}\omega^-_{s'} \otimes \omega^+_{s'},$$

and set (with an assumption $a_1 \neq a_{s'}$)

$$B = \frac{1}{a_1 - a_{s'}}(M_1 - \frac{1}{s'}M_{s'} + \frac{1}{s'}N_{s'}) = I_n + \frac{b_1 - a_1}{a_1 - a_{s'}}\frac{p+q'}{|p+q'|} \otimes \frac{p+q'}{|p+q'|}$$

$$-\frac{b_{s'} - a_{s'}}{a_1 - a_{s'}}\frac{p+s'q'}{|p+s'q'|} \otimes \frac{p+s'q'}{|p+s'q'|} + \frac{1}{(a_1 - a_{s'})q'\cdot\omega^-_{s'}}\omega^-_{s'} \otimes \omega^+_{s'},$$

where $a_{s'} = \frac{\sigma(|p+s'q'|)}{|p+s'q'|}$, $b_{s'} = \sigma'(|p+s'q'|)$; then $D_{(\gamma',q',s')}F$ is invertible if and only if the matrix $B \in \mathbb{M}^{n\times n}$ is invertible.

4. To close the argument in Step 2 and thus to finish the proof, we choose a suitable $l_2 = l_{r_2} \in (0, r_2)$, depending on r_2, in such a way that for any $r_1 \in (l_2, r_2)$, the matrix B, determined through Steps 2 and 3 for any given $(p_0, \beta_0) \in \mathcal{S} = \mathcal{S}_{r_1,r_2}$, is invertible.

First, by Hypothesis (PM), $\tilde{r}_2 \in (0, r_2)$ can be chosen close enough to r_2 so that

$$\frac{\sigma(k)}{k} < \frac{\sigma(l)}{l} \quad \forall l \in [s_-(\tilde{r}_2), s_-(r_2)], \ \ \forall k \in [s_+(r_2), s_+(\tilde{r}_2)].$$

Then define a real-valued continuous function (to express the determinant of the matrix B from Step 3)

$$\mathrm{DET}(u,v,q,\gamma) = \det\Big(I_n + \frac{\sigma'(|u|) - \frac{\sigma(|u|)}{|u|}}{\frac{\sigma(|u|)}{|u|} - \frac{\sigma(|v|)}{|v|}}\frac{u}{|u|} \otimes \frac{u}{|u|} - \frac{\sigma'(|v|) - \frac{\sigma(|v|)}{|v|}}{\frac{\sigma(|u|)}{|u|} - \frac{\sigma(|v|)}{|v|}}\frac{v}{|v|} \otimes \frac{v}{|v|}$$

$$+\frac{1}{(\frac{\sigma(|u|)}{|u|} - \frac{\sigma(|v|)}{|v|})((\sigma'(|v|) - \frac{\sigma(|v|)}{|v|})(\frac{v}{|v|}\cdot q)^2 + \frac{\sigma(|v|)}{|v|})}\big((\sigma'(|v|) - \frac{\sigma(|v|)}{|v|})(\frac{v}{|v|}\cdot q)\frac{v}{|v|}$$
$$+\frac{\sigma(|v|)}{|v|}q - \gamma\big) \otimes \big((\sigma'(|v|) - \frac{\sigma(|v|)}{|v|})(\frac{v}{|v|}\cdot q)\frac{v}{|v|} + \frac{\sigma(|v|)}{|v|}q + \gamma\big)\Big)$$

on the compact set $\mathcal{M}$ of points $(u,v,q,\gamma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n$ with

$$|u| \in [s_+(r_2), s_+(\tilde{r}_2)], \ \ |v| \in [s_-(\tilde{r}_2), s_-(r_2)], \ \ |\gamma| \leq 1.$$

Set $\bar{k} = s_+(r_2)$ and $\bar{l} = s_-(r_2)$; then for each $q \in \mathbb{S}^{n-1}$,

$$\mathrm{DET}(\bar{k}q, \bar{l}q, q, 0) = \det\left(I_n + \frac{\sigma'(\bar{k}) - \frac{\sigma(\bar{k})}{\bar{k}} + \frac{\sigma(\bar{l})}{\bar{l}}}{\frac{\sigma(\bar{k})}{\bar{k}} - \frac{\sigma(\bar{l})}{\bar{l}}} q \otimes q\right) \neq 0,$$

since $\sigma'(\bar{k}) \neq 0$ and hence the fraction in front of $q \otimes q$ is different from -1. So

$$d := \min_{q \in \mathbb{S}^{n-1}} |\mathrm{DET}(\bar{k}q, \bar{l}q, q, 0)| > 0.$$

Next, choose a number $\delta > 0$ such that for all $(u, v, q, \gamma), (\tilde{u}, \tilde{v}, \tilde{q}, \tilde{\gamma}) \in \mathcal{M}$ with $|u - \tilde{u}|$, $|v - \tilde{v}|$, $|q - \tilde{q}|$, $|\gamma - \tilde{\gamma}| < \delta$, we have

$$|\mathrm{DET}(u, v, q, \gamma) - \mathrm{DET}(\tilde{u}, \tilde{v}, \tilde{q}, \tilde{\gamma})| < d/2. \tag{4.8}$$

Let $l_2 \in (\tilde{r}_2, r_2)$ be sufficiently close to r_2 so that for all $r_1 \in (l_2, r_2)$,

$$h(s_-(r_2), s_+(r_2), r_2, s_-(r_2) - s_-(r_1), s_+(r_1) - s_+(r_2), r_2 - r_1) < \tau,$$

where h is the function in Theorem 4.1.5, and let

$$\tau := \min\{\delta, \delta(s_+(r_2) - s_-(r_2))/4\}.$$

Now, fix any $r_1 \in (l_2, r_2)$, and let B be the $n \times n$ matrix determined through Steps 2 and 3 in terms of any given $(p_0, \beta_0) \in \mathcal{S} = \mathcal{S}_{r_1, r_2}$. Let $p_+ = p_0 + t_0 q_0$ and $p_- = p_0 + s_0 q_0$ from Step 2; then $p_\pm$ and $A(p_\pm)$ fulfill the conditions in Theorem 4.1.5. So this theorem implies that there exists a vector $\zeta^0 \in \mathbb{S}^{n-1}$ such that

$$\max\{|p^0_- - p_-|, |p^0_+ - p_+|, |A(p^0_-) - A(p_-)|, |A(p^0_+) - A(p_+)|\} < \tau,$$

where $p^0_+ = \bar{k}\zeta^0$, $p^0_- = \bar{l}\zeta^0$, and $A(p^0_\pm) = r_2\zeta^0$. Using (4.6) and (4.7),

$$|p_+ - \bar{k}\zeta^0| < \delta, \quad |p_- - \bar{l}\zeta^0| < \delta,$$

$$\begin{aligned} |q_0 - \zeta^0| = |\frac{p_+ - p_-}{t_0 - s_0} - \zeta^0| &\le \frac{|(p_+ - p_-) - (\bar{k} - \bar{l})\zeta^0| + |(\bar{k} - \bar{l}) - (t_0 - s_0)|}{t_0 - s_0} \\ &\le \frac{2\tau + ||p^0_+ - p^0_-| - |p_+ - p_-||}{t_0 - s_0} < \frac{4\tau}{t_0 - s_0} < \delta, \\ |\gamma_0| = |\frac{A(p_+) - A(p_-)}{t_0 - s_0}| &\le \frac{|A(p_+) - A(p^0_+)| + |A(p^0_-) - A(p_-)|}{t_0 - s_0} < \delta. \end{aligned}$$

Since $\det(B) = \mathrm{DET}(p_+, p_-, q_0, \gamma_0)$ and $|\mathrm{DET}(\bar{k}\zeta^0, \bar{l}\zeta^0, \zeta^0, 0)| \ge d$, it follows from (4.8) that

$$|\det(B)| > d/2 > 0.$$

The proof is now complete. $\square$

4.2 Relaxation of $\nabla\omega(z) \in F_0$

The following result is important for the convex integration with linear constraint; the function φ determined here plays a similar role as the tile function g used in [44, 45]. For a more general case, see [37, Lemma 2.1].

Lemma 4.2.1. *Let $\lambda_1, \lambda_2 > 0$ and $\eta_1 = -\lambda_1\eta,\ \eta_2 = \lambda_2\eta$ with*

$$\eta = \begin{pmatrix} q & b \\ \frac{1}{b}\gamma \otimes q & \gamma \end{pmatrix}, \quad |q| = 1,\ \gamma \cdot q = 0,\ b \neq 0.$$

Let $G \subset \mathbb{R}^{n+1}$ be a bounded domain. Then for each $\epsilon > 0$, there exists a function $\omega = (\varphi, \psi) \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{1+n})$ with $\operatorname{supp}(\omega) \subset\subset G$ that satisfies the following properties:

(a) $\operatorname{div}\psi = 0$ *in G,*

(b) $|\{z \in G \mid \nabla\omega(z) \notin \{\eta_1,\ \eta_2\}\}| < \epsilon$,

(c) $\operatorname{dist}(\nabla\omega(z), [\eta_1, \eta_2]) < \epsilon$ *for all $z \in G$,*

(d) $\|\omega\|_{L^\infty(G)} < \epsilon$,

(e) $\int_{\mathbb{R}^n} \varphi(x,t)\,dx = 0$ *for all $t \in \mathbb{R}$.*

Proof. The proof follows a simplified version of [37, Lemma 2.1].

1. We define a map $\mathcal{P}\colon C^1(\mathbb{R}^{n+1}) \to C^0(\mathbb{R}^{n+1}; \mathbb{R}^{1+n})$ by setting $\mathcal{P}(h) = (u, v)$, where, for $h(x,t) \in C^1(\mathbb{R}^{n+1})$,

$$u(x,t) = q \cdot Dh(x,t), \quad v(x,t) = \frac{1}{b}(\gamma \otimes q - q \otimes \gamma)Dh(x,t).$$

We easily see that $\mathcal{P}(h) = (u,v) \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{1+n})$, $\operatorname{supp}(\mathcal{P}(h)) \subset \operatorname{supp}(h)$, $\operatorname{div} v \equiv 0$, and $\int_{\mathbb{R}^n} u(x,t)\,dx = 0$ for all $t \in \mathbb{R}$, for all $h \in C_c^\infty(\mathbb{R}^{n+1})$. For $h(x,t) = f(q \cdot x + bt)$ with $f \in C^\infty(\mathbb{R})$, $w = (u,v) = \mathcal{P}(h)$ is given by $u(x,t) = f'(q \cdot x + bt)$ and $v(x,t) = f'(q \cdot x + bt)\frac{\gamma}{b}$, and hence $\nabla w(x,t) = f''(q \cdot x + bt)\eta$. We also note that $\mathcal{P}(gh) = g\mathcal{P}(h) + h\mathcal{P}(g)$ and hence

$$\nabla\mathcal{P}(gh) = g\nabla\mathcal{P}(h) + h\nabla\mathcal{P}(g) + \mathcal{B}(\nabla g, \nabla h) \quad \forall\ g,\ h \in C^\infty(\mathbb{R}^{n+1}), \tag{4.9}$$

where $\mathcal{B}(\nabla g, \nabla h)$ is a bilinear map of ∇g and ∇h; so $|\mathcal{B}(\nabla h, \nabla g)| \le C|\nabla h||\nabla g|$ for some constant $C > 0$.

2. Let $G_\epsilon \subset\subset G$ be a smooth sub-domain such that $|G \setminus G_\epsilon| < \epsilon/2$, and let $\rho_\epsilon \in C_c^\infty(G)$ be a cut-off function satisfying $0 \le \rho_\epsilon \le 1$ in G, $\rho_\epsilon = 1$ on G_ϵ. As G is bounded, $G \subset \{(x,t) \mid k < q \cdot x + bt < l\}$ for some numbers $k < l$. For each $\tau > 0$, we can find a function $f_\tau \in C_c^\infty(k,l)$ satisfying

$$-\lambda_1 \le f_\tau'' \le \lambda_2,\ |\{s \in (k,l) \mid f_\tau''(s) \notin \{-\lambda_1,\ \lambda_2\}\}| < \tau,\ \|f_\tau\|_{L^\infty} + \|f_\tau'\|_{L^\infty} < \tau.$$

3. Define $\omega = (\varphi, \psi) = \mathcal{P}(\rho_\epsilon(x,t)h_\tau(x,t))$, where $h_\tau(x,t) = f_\tau(q\cdot x + bt)$. Then $\|h_\tau\|_{C^1} \le C\|f_\tau\|_{C^1} \le C\tau$, $\omega \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{1+n})$, $\operatorname{supp}(\omega) \subset \operatorname{supp}(\rho_\epsilon) \subset\subset G$, and (a) and (e) are satisfied. Note that

$$|\omega| \le |\rho_\epsilon||\mathcal{P}(h_\tau)| + |h_\tau||\mathcal{P}(\rho_\epsilon)| \le C_\epsilon \tau,$$

where $C_\epsilon > 0$ is a constant depending on $\|\rho_\epsilon\|_{C^1(G)}$. So we can choose a $\tau_1 > 0$ so small that (d) is satisfied for all $0 < \tau < \tau_1$. Note also that

$$\{z \in G \mid \nabla\omega(z) \notin \{\eta_1, \eta_2\}\} \subseteq (G \setminus G_\epsilon) \cup \{z \in G_\epsilon \mid f_\tau''(q \cdot x + bt) \notin \{-\lambda_1,\ \lambda_2\}\}.$$

Since $|\{z \in G_\epsilon \mid f_\tau''(q \cdot x + bt) \notin \{-\lambda_1,\ \lambda_2\}| \le N|\{s \in (k,l) \mid f_\tau''(s) \notin \{-\lambda_1,\ \lambda_2\}\}|$ for some constant $N > 0$ depending only on set G, there exists a $\tau_2 > 0$ such that

$$|\{z \in G \mid \nabla\omega(z) \notin \{\eta_1, \eta_2\}\}| \le \frac{\epsilon}{2} + N\tau < \epsilon$$

for all $0 < \tau < \tau_2$. Therefore, (b) is satisfied. Finally, note that

$$\rho_\epsilon \nabla \mathcal{P}(h_\tau(x,t)) = \rho_\epsilon f_\tau''(q \cdot x + bt)\eta \in [\eta_1, \eta_2] \quad \text{in } G$$

and, by (4.9), for all $z = (x,t) \in G$,

$$|\nabla \omega(z) - \rho_\epsilon \nabla \mathcal{P}(h_\tau(x,t))| \leq |h_\tau||\nabla \mathcal{P}(\rho_\epsilon)| + |\mathcal{B}(\nabla h_\tau, \nabla \rho_\epsilon)| \leq C_\epsilon' \tau < \epsilon$$

for all $0 < \tau < \tau_3$, where $C_\epsilon' > 0$ is a constant depending on $\|\rho_\epsilon\|_{C^2(G)}$, and $\tau_3 > 0$ is another constant. Hence (c) is satisfied. Taking $0 < \tau < \min\{\tau_1, \tau_2, \tau_3\}$, the proof is complete. $\square$

We now state the relaxation theorem for homogeneous differential inclusion $\nabla \omega(z) \in F_0$ in a form that is more suitable for later use; we restrict the inclusion to only (p, β) components.

Theorem 4.2.2. *Let $0 < r_2 < \sigma(s_0)$, and let $l_2 = l_{r_2} \in (0, r_2)$ be some number determined by Theorem 4.1.6. Let $l_2 < r_1 < r_2$, and let $\mathcal{K}$ be a compact subset of $\mathcal{S} = \mathcal{S}_{r_1,r_2}$. Let $\tilde{Q} \times \tilde{I}$ be a box in $\mathbb{R}^{n+1}$. Then given any $\epsilon > 0$, there exists a $\delta > 0$ such that for each box $Q \times I \subset \tilde{Q} \times \tilde{I}$, point $(p, \beta) \in \mathcal{K}$, and number $\rho > 0$ sufficiently small, there exists a function $\omega = (\varphi, \psi) \in C_c^\infty(Q \times I; \mathbb{R}^{1+n})$ satisfying the following properties:*

(a) $\operatorname{div} \psi = 0$ *in* $Q \times I$,

(b) $(p' + D\varphi(z), \beta' + \psi_t(z)) \in \mathcal{S}$ *for all* $z \in Q \times I$ *and* $|(p', \beta') - (p, \beta)| \leq \delta$,

(c) $\|\omega\|_{L^\infty(Q \times I)} < \rho$,

(d) $\int_{Q \times I} |\beta + \psi_t(z) - A(p + D\varphi(z))| dz < \epsilon |Q \times I| / |\tilde{Q} \times \tilde{I}|$,

(e) $\int_Q \varphi(x,t) dx = 0$ *for all* $t \in I$,

(f) $\|\varphi_t\|_{L^\infty(Q \times I)} < \rho$.

Proof. By Theorem 4.1.6, there exist finitely many open balls $\mathcal{B}_1, \cdots, \mathcal{B}_N \subset\subset \mathcal{S}$ covering $\mathcal{K}$ and C^1 functions $q_i : \bar{\mathcal{B}}_i \to \mathbb{S}^{n-1}$, $\gamma_i : \bar{\mathcal{B}}_i \to \mathbb{R}^n$, $t_{i,\pm} : \bar{\mathcal{B}}_i \to \mathbb{R}$ $(1 \le i \le N)$ with $\gamma_i \cdot q_i = 0$ and $t_{i,-} < 0 < t_{i,+}$ on $\bar{\mathcal{B}}_i$ such that for each $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0)$ with $(p, \beta) \in \bar{\mathcal{B}}_i$, we have

$$\xi + t_{i,\pm}\eta_i \in F_\pm,$$

where $t_{i,\pm} = t_{i,\pm}(p, \beta)$, $\eta_i = \begin{pmatrix} q_i(p,\beta) & b \\ \frac{1}{b}\gamma_i(p,\beta) \otimes q_i(p,\beta) & \gamma_i(p,\beta) \end{pmatrix}$, and $b \neq 0$ is arbitrary.

Let $1 \le i \le N$. We write $\xi_i = \xi_i(p, \beta) = \begin{pmatrix} p & 0 \\ O & \beta \end{pmatrix} \in R(F_0)$ for $(p, \beta) \in \bar{\mathcal{B}}_i \subset \mathcal{S}$, where O is the $n \times n$ zero matrix. We omit the dependence on $(p, \beta) \in \bar{\mathcal{B}}_i$ in the following whenever it is clear from the context. Given any $\rho > 0$, we choose a constant b_i with

$$0 < b_i < \min_{\bar{\mathcal{B}}_i} \frac{\rho}{t_{i,+} - t_{i,-}}.$$

With this choice of $b = b_i$, let η_i be defined on $\bar{\mathcal{B}}_i$ as above. Then

$$\xi_{i,\pm} = \begin{pmatrix} p_{i,\pm} & c_{i,\pm} \\ B_{i,\pm} & \beta_{i,\pm} \end{pmatrix} := \xi_i + t_{i,\pm}\eta_i \in F_\pm,$$

$$\xi_i = \lambda_i \xi_{i,+} + (1 - \lambda_i)\xi_{i,-}, \quad \lambda_i = \frac{-t_{i,-}}{t_{i,+} - t_{i,-}} \in (0,1) \quad \text{on } \bar{\mathcal{B}}_i.$$

By the definition of $R(F_0)$, on $\bar{\mathcal{B}}_i$, both $\xi^\tau_{i,-} = \tau\xi_{i,+} + (1-\tau)\xi_{i,-}$ and $\xi^\tau_{i,+} = (1-\tau)\xi_{i,+} + \tau\xi_{i,-}$ belong to $R(F_0)$ for all $\tau \in (0,1)$. Let $0 < \tau < \min_{1 \le j \le N} \min_{\bar{\mathcal{B}}_j} \min\{\lambda_j, 1 - \lambda_j\} \le \frac{1}{2}$ be a small number to be selected later. Let $\lambda'_i = \frac{\lambda_i - \tau}{1 - 2\tau}$ on $\bar{\mathcal{B}}_i$. Then $\lambda'_i \in (0,1)$, $\xi_i = \lambda'_i \xi^\tau_{i,+} + (1 - \lambda'_i)\xi^\tau_{i,-}$ on $\bar{\mathcal{B}}_i$. Moreover, on $\bar{\mathcal{B}}_i$, $\xi^\tau_{i,+} - \xi^\tau_{i,-} = (1 - 2\tau)(\xi_{i,+} - \xi_{i,-})$ is rank-one,

$[\xi^\tau_{i,-}, \xi^\tau_{i,+}] \subset (\xi_{i,-}, \xi_{i,+}) \subset R(F_0)$, and

$$c\tau \le |\xi^\tau_{i,+} - \xi_{i,+}| = |\xi^\tau_{i,-} - \xi_{i,-}| = \tau|\xi_{i,+} - \xi_{i,-}| = \tau(t_{i,+} - t_{i,-})|\eta_i| \le C\tau,$$

where $C = \max_{1\le j\le N} \max_{\bar{\mathcal{B}}_j}(t_{j,+} - t_{j,-})|\eta_j| \ge \min_{1\le j\le N} \min_{\bar{\mathcal{B}}_j}(t_{j,+} - t_{j,-})|\eta_j| = c > 0$. By continuity, $H_\tau = \bigcup_{(p,\beta)\in\bar{\mathcal{B}}_j, 1\le j\le N}[\xi^\tau_{j,-}(p,\beta), \xi^\tau_{j,+}(p,\beta)]$ is a compact subset of $R(F_0)$, where $R(F_0)$ is open in the space

$$\Sigma_0 = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \;\Big|\; \mathrm{tr}B = 0 \right\},$$

by Lemma 4.1.3 and Theorem 4.1.6. So $d_\tau = \mathrm{dist}(H_\tau, \partial|_{\Sigma_0} R(F_0)) > 0$, where $\partial|_{\Sigma_0}$ is the relative boundary in Σ_0.

Let $\eta_{i,1} = -\lambda_{i,1}\eta_i = -\lambda'_i(1-2\tau)(t_{i,+}-t_{i,-})\eta_i$, $\eta_{i,2} = \lambda_{i,2}\eta_i = (1-\lambda'_i)(1-2\tau)(t_{i,+}-t_{i,-})\eta_i$ on $\bar{\mathcal{B}}_i$, where $\lambda_{i,1} = \tau(-t_{i,+}) + (1-\tau)(-t_{i,-}) > 0$, $\lambda_{i,2} = (1-\tau)t_{i,+} + \tau t_{i,-} > 0$ on $\bar{\mathcal{B}}_i$, and $\tau > 0$ is so small that

$$\min_{1\le j\le N} \min_{\bar{\mathcal{B}}_j} \lambda_{j,k} > 0 \quad (k = 1,\, 2).$$

Applying Lemma 4.2.1 to matrices $\eta_{i,1} = \eta_{i,1}(p,\beta)$, $\eta_{i,2} = \eta_{i,2}(p,\beta)$ for a fixed $(p,\beta) \in \bar{\mathcal{B}}_i$ and a given box $G = Q \times I$, we obtain that for each $\rho > 0$, there exist a function $\omega =$

$(\varphi, \psi) \in C_c^\infty(Q \times I; \mathbb{R}^{1+n})$ and an open set $G_\rho \subset\subset Q \times I$ satisfying the following conditions:

$$
\begin{cases}
(1) \quad \operatorname{div} \psi = 0 \text{ in } Q \times I, \\
(2) \quad |(Q \times I) \setminus G_\rho| < \rho; \;\; \xi_i + \nabla\omega(z) \in \{\xi^\tau_{i,-},\ \xi^\tau_{i,+}\} \text{ for all } z \in G_\rho, \\
(3) \quad \xi_i + \nabla\omega(z) \in [\xi^\tau_{i,-}, \xi^\tau_{i,+}]_\rho \text{ for all } z \in Q \times I, \\
(4) \quad \|\omega\|_{L^\infty(Q\times I)} < \rho, \\
(5) \quad \int_Q \varphi(x,t)\,dx = 0 \text{ for all } t \in I, \\
(6) \quad \|\varphi_t\|_{L^\infty(Q\times I)} < 2\rho,
\end{cases}
\tag{4.10}
$$

where $[\xi^\tau_{i,-}, \xi^\tau_{i,+}]_\rho$ denotes the ρ-neighborhood of closed line segment $[\xi^\tau_{i,-}, \xi^\tau_{i,+}]$. Here, from (4.10.3), (4.10.6) follows as

$$
|\varphi_t| < |c_{i,+} - c_{i,-}| + \rho = (t_{i,+} - t_{i,-})|b_i| + \rho < 2\rho \quad \text{in } Q \times I.
$$

Note (a), (c), (e), and (f) follow from (4.10), where 2ρ in (4.10.6) can be adjusted to ρ as in (f). By the uniform continuity of A on $J = \{p' \in \mathbb{R}^n \mid |p'| \leq s_+(\bar{r}_2)\}$, we can find a $\delta' > 0$ such that $|A(p') - A(p'')| < \frac{\epsilon}{3|\tilde{Q}\times\tilde{I}|}$ whenever p', $p'' \in J$ and $|p' - p''| < \delta'$. We then choose a $\tau > 0$ so small that

$$
C\tau < \delta', \;\; C|\tilde{Q} \times \tilde{I}|\tau < \frac{\epsilon}{3}.
$$

Next, we choose a $\delta > 0$ such that $\delta < \frac{d_\tau}{2}$. If $0 < \rho < \delta$, then by (4.10.1) and (4.10.3), for all $z \in Q \times I$ and $|(p', \beta') - (p, \beta)| \le \delta$,

$$\xi_i(p', \beta') + \nabla\omega(z) \in \Sigma_0, \quad \operatorname{dist}(\xi_i(p', \beta') + \nabla\omega(z), H_\tau) < d_\tau,$$

and so $\xi_i(p', \beta') + \nabla\omega(z) \in R(F_0)$, that is, $(p' + D\varphi(z), \beta' + \psi_t(z)) \in \mathcal{S}$. Thus (b) holds for all $0 < \rho < \delta$. In particular, $(p + D\varphi(z), \beta + \psi_t(z)) \in \mathcal{S}$ and so $|p + D\varphi(z)| \le s_+(r_1) < s_+(\bar{r}_2)$ and $|\beta + \psi_t(z)| \le r_2$ for all $z \in Q \times I$, by (i) of Theorem 4.1.6. Thus

$$\begin{aligned}
&\int_{Q\times I} |\beta + \psi_t - A(p + D\varphi)|dz \\
&\qquad \le \int_{G_\rho} |\beta + \psi_t - A(p + D\varphi)|dz + (r_2 + M_\sigma)\rho \\
&\qquad \le |Q \times I| \max\{|\beta^\tau_{i,\pm} - A(p^\tau_{i,\pm})|\} + (r_2 + M_\sigma)\rho \\
&\qquad \le C|Q \times I|\tau + |Q \times I| \max\{|A(p_{i,\pm}) - A(p^\tau_{i,\pm})|\} + (r_2 + M_\sigma)\rho \\
&\qquad \le \frac{2\epsilon|Q \times I|}{3|\tilde{Q} \times \tilde{I}|} + (r_2 + M_\sigma)\rho,
\end{aligned}$$

where $\xi^\tau_{i,\pm} = \begin{pmatrix} p^\tau_{i,\pm} & c^\tau_{i,\pm} \\ B^\tau_{i,\pm} & \beta^\tau_{i,\pm} \end{pmatrix}$ and $M_\sigma = \sigma(s_0)$. Thus, (d) holds for all $\rho > 0$ satisfying $(r_2 + M_\sigma)\rho < \frac{\epsilon|Q\times I|}{3|\tilde{Q}\times\tilde{I}|}$.

We have verified (a) – (f) for any $(p, \beta) \in \bar{\mathcal{B}}_i$ and $1 \le i \le N$, where $\delta > 0$ is independent of the index i. Since $\mathcal{B}_1, \cdots, \mathcal{B}_N$ cover $\mathcal{K}$, the proof is now complete. □

4.3 Construction of admissible set $\mathcal{U}$

We first construct a suitable boundary function $\Phi = (u^*, v^*) \in W^{1,\infty}(\Omega_T; \mathbb{R}^{1+n})$. Assume Ω and u_0 satisfy (2.10) with Ω convex. Let $\Omega_T = \Omega \times (0, T)$ for a given $T > 0$ and

$M_0 = \|Du_0\|_{L^\infty(\Omega)}$. Recall that we assume (2.1); that is,

$$\int_\Omega u_0(x)\,dx = 0. \tag{4.11}$$

To tailor the detailed result of Theorem 2.3.2 into the general existence theorem, Theorem 2.2.4, we assume the following: Let $0 < r = r_2 < \sigma(M_0)$, and let $l = l_r \in (0, r)$ be some number determined by Theorem 4.1.6. Choose any $\tilde{r} = r_1 \in (l, r)$.

With these numbers $r_1 = \tilde{r}$, $r_2 = r$, we apply Lemma 3.2.1 to obtain functions $\tilde{\sigma}$, $\tilde{f} \in C^3([0,\infty))$ satisfying its conclusion. Also, let $\tilde{A}(p) = \tilde{f}(|p|^2)p$ ($p \in \mathbb{R}^n$). Then:

Lemma 4.3.1. *We have*

$$(p, \tilde{A}(p)) \in \mathcal{S} \quad \forall\, s_-(r_1) < |p| < s_+(r_2),$$

where $\mathcal{S} = \mathcal{S}_{r_1,r_2}$ is the set in Lemma 4.1.3.

Proof. Let $s = |p|$, $r = \tilde{\sigma}(s)$ and $\zeta = p/|p|$, so that $s_-(r_1) < s < s_+(r_2)$, $\zeta \in \mathbf{S}^{n-1}$ and $\tilde{A}(p) = r\zeta$. By Lemma 3.2.1, $s_-(r) < s < s_+(r)$ and $r_1 < r < r_2$. Set $p_\pm = s_\pm(r)\zeta$ and $\beta_\pm = r\zeta$. Then $A(p_\pm) = r\zeta = \beta_\pm$. Define $\xi = \begin{pmatrix} p & 0 \\ O & \tilde{A}(p) \end{pmatrix}$ and $\xi_\pm = \begin{pmatrix} p_\pm & 0 \\ O & \beta_\pm \end{pmatrix}$. Then $\xi = \lambda\xi_+ + (1-\lambda)\xi_-$ for some $0 < \lambda < 1$. Since $\xi_\pm \in F_\pm$ and $\operatorname{rank}(\xi_+ - \xi_-) = 1$, it follows from the definition of $R(F_0) = R(F_{r_1,r_2}(0))$ that $\xi \in (\xi_-, \xi_+) \subset R(F_0)$. Thus, by Lemma 4.1.3, $(p, \tilde{A}(p)) \in \mathcal{S}$. $\square$

By Lemma 3.2.1, equation $u_t = \operatorname{div}(\tilde{A}(Du))$ is uniformly parabolic. So by Theorem 3.1.1

together with the convexity of Ω, the initial-Neumann boundary value problem

$$\begin{cases} u_t^* = \operatorname{div}(\tilde{A}(Du^*)) & \text{in } \Omega_T \\ \partial u^*/\partial \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T) \\ u^*(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{4.12}$$

admits a unique classical solution $u^* \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T)$ satisfying

$$|Du^*(x,t)| \le M_0 \quad \forall (x,t) \in \Omega_T.$$

From conditions (2.10) and (4.11), we can find a function $h \in C^{2+\alpha}(\bar{\Omega})$ satisfying

$$\Delta h = u_0 \ \text{ in } \ \Omega, \quad \partial h/\partial \mathbf{n} = 0 \ \text{ on } \ \partial\Omega.$$

Let $v_0 = Dh \in C^{1+\alpha}(\bar{\Omega}; \mathbb{R}^n)$ and define, for $(x,t) \in \Omega_T$,

$$v^*(x,t) = v_0(x) + \int_0^t \tilde{A}(Du^*(x,s))\, ds. \tag{4.13}$$

Then it is easily seen that $\Phi := (u^*, v^*) \in C^1(\bar{\Omega}_T; \mathbb{R}^{1+n})$ satisfies (2.4); that is,

$$\begin{cases} u^*(x,0) = u_0(x) \ (x \in \Omega), \\ \operatorname{div} v^* = u^* \ \text{ a.e. in } \Omega_T, \\ v^*(\cdot,t) \cdot \mathbf{n}|_{\partial\Omega} = 0 \ \ \forall\, t \in [0,T]. \end{cases} \tag{4.14}$$

Hence Φ is a boundary function in the sense of Definition 2.2.2.

Next, let

$$\mathcal{F} = \{(p, A(p)) \mid |p| \in [0, s_-(r_1)]\}.$$

Then we have the following:

Lemma 4.3.2.

$$(Du^*(x,t), v_t^*(x,t)) \in \mathcal{S} \cup \mathcal{F} \quad \forall\, (x,t) \in \Omega_T.$$

Proof. Let $(x,t) \in \Omega_T$ and $p = Du^*(x,t)$; then $|p| \leq M_0$.

If $|p| \leq s_-(r_1)$, then $\tilde{A}(p) = A(p)$ and hence by (4.13)

$$(Du^*(x,t), v_t^*(x,t)) = (p, \tilde{A}(p)) = (p, A(p)) \in \mathcal{F}.$$

If $s_-(r_1) < |p| \leq M_0$, then by Lemma 4.3.1 and (4.13)

$$(Du^*(x,t), v_t^*(x,t)) = (p, \tilde{A}(p)) \in \mathcal{S}.$$

Therefore $(Du^*, v_t^*) \in \mathcal{S} \cup \mathcal{F}$ in Ω_T. □

Definition 4.3.3. We say a function u is *piecewise* C^1 in Ω_T and write $u \in C^1_{piece}(\Omega_T)$ if there exists a sequence of disjoint open sets $\{G_j\}_{j=1}^{\infty}$ in Ω_T such that

$$u \in C^1(\bar{G}_j) \ \ \forall j \in \mathbb{N}, \quad |\Omega_T \setminus \cup_{j=1}^{\infty} G_j| = 0.$$

Note that in this definition, we necessarily have $|\partial G_j| = 0$ for all $j \in \mathbb{N}$.

(Selection of interface of measure zero for classical and Lipschitz parts of

solutions) Observe that

$$|\{(x,t) \in \Omega_T \,|\, |Du^*(x,t)| = s_-(\bar{r})\}| > 0$$

for *at most countably many* $\bar{r} \in (0, \tilde{r})$. We fix any $\bar{r} \in (0, \tilde{r})$ with

$$|\{(x,t) \in \Omega_T \,|\, |Du^*(x,t)| = s_-(\bar{r})\}| = 0,$$

and let

$$\Omega_T^1 = \{(x,t) \in \Omega_T \,|\, |Du^*(x,t)| < s_-(\bar{r})\},$$

$$\Omega_T^2 = \{(x,t) \in \Omega_T \,|\, |Du^*(x,t)| > s_-(\bar{r})\},$$

so that Ω_T^1 and Ω_T^2 are disjoint subsets of Ω_T whose union has measure $|\Omega_T|$. Clearly, $\Omega_0^{\bar{r}} \subset \partial\Omega_T^1$, where $\Omega_0^{\bar{r}}$ is as in Theorem 2.3.2.

Let $m = \|u_t^*\|_{L^\infty(\Omega_T)} + 1$. We finally define the admissible set $\mathcal{U}$ as follows:

$$\begin{aligned}\mathcal{U} = \Big\{ & u \in C^1_{piece} \cap W^{1,\infty}_{u^*}(\Omega_T) \;\Big|\; u = u^* \text{ in } \Omega_T^1,\ \|u_t\|_{L^\infty(\Omega_T)} < m, \\ & \exists\, v \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T;\mathbb{R}^n) \text{ such that} \\ & \operatorname{div} v = u \text{ and } (Du, v_t) \in \mathcal{S} \cup \mathcal{F} \text{ a.e. in } \Omega_T \Big\}.\end{aligned} \tag{4.15}$$

For each $\epsilon > 0$, let $\mathcal{U}_\epsilon$ be the subset of $\mathcal{U}$ given by

$$\begin{aligned}\mathcal{U}_\epsilon = \Big\{ & u \in \mathcal{U} \mid \exists\, v \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T;\mathbb{R}^n) \text{ such that } \operatorname{div} v = u \text{ and} \\ & (Du, v_t) \in \mathcal{S} \cup \mathcal{F} \text{ a.e. in } \Omega_T, \text{ and } \textstyle\int_{\Omega_T} |v_t - A(Du)| dxdt \le \epsilon |\Omega_T| \Big\}.\end{aligned}$$

Remark 4.3.4. From (4.14), Lemma 4.3.2, and the definition of $\mathcal{U}$, it follows that $u^* \in \mathcal{U}$ with its corresponding vector function v^*; so $\mathcal{U}$ is non-empty. Also $\mathcal{U}$ is a bounded subset of $W^{1,\infty}_{u^*}(\Omega_T)$ as $\mathcal{S} \cup \mathcal{F}$ is bounded. Moreover, by (i) of Theorem 4.1.6 and the definition of $\mathcal{F}$, for each $u \in \mathcal{U}$, its corresponding vector function v satisfies $\|v_t\|_{L^\infty(\Omega_T)} \leq r_2 = r$. Thus $\mathcal{U}$ is indeed an admissible set in the sense of Definition 2.2.3 with respect to the boundary function $\Phi = (u^*, v^*)$. Finally, note that $s_-(r_1) < |Du^*| < s_+(r_2)$ on some non-empty open subset of Ω_T, and so $\tilde{A}(Du^*) \neq A(Du^*)$ on this set; so u^* itself is not a Lipschitz solution to (1.2). In terms of Theorem 2.2.4, it only remains to verify the density property (2.5) to obtain multiple Lipschitz solutions to problem (1.2). We accomplish this in the next section.

4.4 Completion of proof of Theorem 2.3.2

Following Section 4.3, we complete the proof of Theorem 2.3.2. The density theorem below is the last preparation for the proof.

Theorem 4.4.1. *For each $\epsilon > 0$, $\mathcal{U}_\epsilon$ is dense in $\mathcal{U}$ under the L^∞-norm.*

Proof. Let $u \in \mathcal{U}$, $\eta > 0$. The goal is to construct a function $\tilde{u} \in \mathcal{U}_\epsilon$ such that $\|\tilde{u} - u\|_{L^\infty(\Omega_T)} < \eta$. For clarity, we divide the proof into several steps.

1. Note $\|u_t\|_{L^\infty(\Omega_T)} < m - \tau_0$ for some $\tau_0 > 0$ and there exists a vector function $v \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T; \mathbb{R}^n)$ such that $\operatorname{div} v = u$ and $(Du, v_t) \in \mathcal{S} \cup \mathcal{F}$ a.e. in Ω_T. Since both u and v are piecewise C^1 in Ω_T, there exists a sequence of disjoint open sets $\{G_j\}_{j=1}^\infty$ in Ω_T with $|\partial G_j| = 0$ such that

$$u \in C^1(\bar{G}_j),\ v \in C^1(\bar{G}_j; \mathbb{R}^n)\ \ \forall j \geq 1, \quad |\Omega_T \setminus \cup_{j=1}^\infty G_j| = 0.$$

2. Let $j \in \mathbb{N}$ be fixed, where $\mathbb{N}$ is the set of positive integers. Note that $(Du(z), v_t(z)) \in \bar{\mathcal{S}} \cup \mathcal{F}$ for all $z = (x,t) \in G_j$ and that $H_j = \{z \in G_j \mid (Du(z), v_t(z)) \in \partial\mathcal{S}\}$ is a (relatively) closed set in G_j with measure zero. So $\tilde{G}_j = G_j \backslash H_j$ is an open subset of G_j with $|\tilde{G}_j| = |G_j|$, and $(Du(z), v_t(z)) \in \mathcal{S} \cup \mathcal{F}$ for all $z \in \tilde{G}_j$.

3. For each $\tau > 0$, let $\mathcal{G}_\tau = \{(p,\beta) \in \mathcal{S} \mid |\beta - A(p)| > \tau, \operatorname{dist}((p,\beta), \partial\mathcal{S}) > \tau)\}$; then $\mathcal{S} = (\cup_{\tau>0}\mathcal{G}_\tau) \cup \{(p,\beta) \in \mathcal{S} \mid A(p) = \beta\}$ as $\mathcal{S}$ is open. Since $A(p) = \beta \ \ \forall (p,\beta) \in \mathcal{F}$, we have

$$\int_{\tilde{G}_j} |v_t(z) - A(Du(z))|\, dz = \lim_{\tau\to 0^+} \int_{\{z\in\tilde{G}_j \mid (Du(z),v_t(z))\in\mathcal{G}_\tau\}} |v_t(z) - A(Du(z))|\, dz;$$

so we can find a $\tau_j > 0$ such that

$$\int_{F_j} |v_t(z) - A(Du(z))|\, dz < \frac{\epsilon}{3 \cdot 2^j}|\Omega_T| \ \text{ and } \ |\partial O_j| = 0, \tag{4.16}$$

where $F_j = \{z \in \tilde{G}_j \mid (Du(z), v_t(z)) \notin \mathcal{G}_{\tau_j}\}$ and $O_j = \tilde{G}_j \setminus F_j$ is open. Let J be the set of all indices $j \in \mathbb{N}$ with $O_j \neq \emptyset$. Then for $j \notin J$, $F_j = \tilde{G}_j$.

4. We now fix a $j \in J$. Note that $O_j = \{z \in \tilde{G}_j \mid (Du(z), v_t(z)) \in \mathcal{G}_{\tau_j}\}$ and that $\mathcal{K}_j := \bar{\mathcal{G}}_{\tau_j}$ is a compact subset of $\mathcal{S}$. Let $\tilde{Q} \subset \mathbb{R}^n$ be a box with $\Omega \subset \tilde{Q}$ and $\tilde{I} = (0,T)$. Applying Theorem 4.2.2 to box $\tilde{Q} \times \tilde{I}$, $\mathcal{K}_j \subset\subset \mathcal{S} = \mathcal{S}_{r_1,r_2}$ (recall $r_2 = r$, $r_1 = \tilde{r}$), and $\epsilon' = \frac{\epsilon|\Omega_T|}{12}$, we obtain a constant $\delta_j > 0$ that satisfies the conclusion of the theorem. By the uniform continuity of A on compact subsets of $\mathbb{R}^n$, we can find a $\theta = \theta_{\epsilon,r_1} > 0$ such that

$$|A(p) - A(p')| < \frac{\epsilon}{12} \tag{4.17}$$

whenever $|p|$, $|p'| \leq 2s_+(r_1)$ and $|p - p'| \leq \theta$. Also by the uniform continuity of u, v, and their gradients on $\bar{G}_j$, there exists a $\nu_j > 0$ such that

$$\begin{aligned}
&|u(z) - u(z')| + |\nabla u(z) - \nabla u(z')| + |v(z) - v(z')| \\
&+|\nabla v(z) - \nabla v(z')| < \min\{\frac{\delta_j}{2}, \frac{\epsilon}{12}, \theta, s_+(r_1)\}
\end{aligned} \tag{4.18}$$

whenever $z, z' \in \bar{G}_j$ and $|z - z'| \leq \nu_j$. We now cover O_j (up to measure zero) by a sequence of disjoint open cubes $\{Q_j^i \times I_j^i\}_{i=1}^{\infty}$ in O_j whose sides are parallel to the axes with center z_j^i and diameter $l_j^i < \nu_j$.

5. Fix an $i \in \mathbb{N}$ and write $w = (u, v)$, $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} = \nabla w(z_j^i) = \begin{pmatrix} Du(z_j^i) & u_t(z_j^i) \\ Dv(z_j^i) & v_t(z_j^i) \end{pmatrix}$.
By the choice of $\delta_j > 0$ in Step 4 via Theorem 4.2.2, since $Q_j^i \times I_j^i \subset \tilde{Q} \times \tilde{I}$ and $(p, \beta) \in \mathcal{K}_j$, for all sufficiently small $\rho > 0$, there exists a function $\omega_j^i = (\varphi_j^i, \psi_j^i) \in C_c^{\infty}(Q_j^i \times I_j^i; \mathbb{R}^{1+n})$ satisfying

(a) $\operatorname{div} \psi_j^i = 0$ in $Q_j^i \times I_j^i$,

(b) $(p' + D\varphi_j^i(z), \beta' + (\psi_j^i)_t(z)) \in \mathcal{S}$ for all $z \in Q_j^i \times I_j^i$ and all $|(p', \beta') - (p, \beta)| \leq \delta_j$,

(c) $\|\omega_j^i\|_{L^\infty(Q_j^i \times I_j^i)} < \rho$,

(d) $\int_{Q_j^i \times I_j^i} |\beta + (\psi_j^i)_t(z) - A(p + D\varphi_j^i(z))| dz < \epsilon' |Q_j^i \times I_j^i| / |\tilde{Q} \times \tilde{I}|$,

(e) $\int_{Q_j^i} \varphi_j^i(x, t) dx = 0$ for all $t \in I_j^i$,

(f) $\|(\varphi_j^i)_t\|_{L^\infty(Q_j^i \times I_j^i)} < \rho$.

Here, we let $0 < \rho \leq \min\{\tau_0, \frac{\delta_j}{2C}, \frac{\epsilon}{12C}, \eta\}$, where C_n is the constant in Theorem 3.3.1 and C is the product of C_n and the sum of the lengths of all sides of $\tilde{Q}$. By (e), we can apply Theorem 3.3.1 to φ_j^i on $Q_j^i \times I_j^i$ to obtain a function $g_j^i = \mathcal{R}\varphi_j^i \in C^1(\overline{Q_j^i \times I_j^i}; \mathbb{R}^n) \cap W_0^{1,\infty}(Q_j^i \times I_j^i; \mathbb{R}^n)$

such that $\operatorname{div} g^i_j = \varphi^i_j$ in $Q^i_j \times I^i_j$ and

$$\|(g^i_j)_t\|_{L^\infty(Q^i_j \times I^i_j)} \leq C\|(\varphi^i_j)_t\|_{L^\infty(Q^i_j \times I^i_j)} \leq \frac{\delta_j}{2}. \text{ (by (f))} \tag{4.19}$$

6. As v_t and $A(Du)$ are essentially bounded in Ω_T, we can select a finite index set $\mathcal{I} \subset J \times \mathbb{N}$ so that

$$\int_{\bigcup_{(j,i)\in(J\times\mathbb{N})\setminus\mathcal{I}} Q^i_j \times I^i_j} |v_t(z) - A(Du(z))|dz \leq \frac{\epsilon}{3}|\Omega_T|. \tag{4.20}$$

We finally define

$$(\tilde{u}, \tilde{v}) = (u, v) + \sum_{(j,i)\in\mathcal{I}} \chi_{Q^i_j \times I^i_j}(\varphi^i_j, \psi^i_j + g^i_j) \quad \text{in } \Omega_T.$$

As a side remark, note here that only *finitely* many functions $(\varphi^i_j, \psi^i_j + g^i_j)$ are disjointly patched to the original (u, v) to obtain a new function $(\tilde{u}, \tilde{v})$ towards the goal of the proof. The reason for using only finitely many pieces of gluing is due to the lack of control over the spatial gradients $D(\psi^i_j + g^i_j)$, and overcoming this difficulty is at the heart of our method.

7. Let us finally check that $\tilde{u}$ together with $\tilde{v}$ indeed gives the desired result. By construction, it is clear that $\tilde{u} \in C^1_{piece} \cap W^{1,\infty}_{u^*}(\Omega_T)$, $\tilde{v} \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T; \mathbb{R}^n)$. By the choice of ρ in (f) as $\rho \leq \tau_0$, we have $\|\tilde{u}_t\|_{L^\infty(\Omega_T)} < m$. Next, let $(j, i) \in \mathcal{I}$, and observe that for $z \in Q^i_j \times I^i_j$, with $(p, \beta) = (Du(z^i_j), v_t(z^i_j)) \in \mathcal{G}_{\tau_j}$, since $|z - z^i_j| < l^i_j < \nu_j$, it follows from (4.18) and (4.19) that

$$|(Du(z), v_t(z) + (g^i_j)_t(z)) - (p, \beta)| \leq \delta_j,$$

and so $(D\tilde{u}(z), \tilde{v}_t(z)) \in \mathcal{S}$ from (b) above. From (a) and $\operatorname{div} g^i_j = \varphi^i_j$, for $z \in Q^i_j \times I^i_j$,

$$\operatorname{div} \tilde{v}(z) = \operatorname{div}(v + \psi^i_j + g^i_j)(z) = u(z) + 0 + \varphi^i_j(z) = \tilde{u}(z).$$

Therefore, $\tilde{u} \in \mathcal{U}$. Next, observe

$$\int_{\Omega_T} |\tilde{v}_t - A(D\tilde{u})| dz = \int_{\cup_{j\in\mathbb{N}} F_j} |v_t - A(Du)| dz$$

$$+ \int_{\cup_{(j,i)\in(J\times\mathbb{N})\setminus\mathcal{I}} Q^i_j \times I^i_j} |v_t - A(Du)| dz + \int_{\cup_{(j,i)\in\mathcal{I}} Q^i_j \times I^i_j} |\tilde{v}_t - A(D\tilde{u})| dz$$

$$=: I_1 + I_2 + I_3.$$

From (4.16) and (4.20), we have $I_1 + I_2 \leq \frac{2\epsilon}{3}|\Omega_T|$. Note also that for $(j,i) \in \mathcal{I}$ and $z \in Q^i_j \times I^i_j$, from (4.18), (4.19), and (f),

$$|\tilde{v}_t(z) - A(D\tilde{u}(z))| = |v_t(z) + (\psi^i_j)_t(z) + (g^i_j)_t(z) - A(Du(z) + D\varphi^i_j(z))|$$

$$\leq |v_t(z) - v_t(z^i_j)| + |v_t(z^i_j) + (\psi^i_j)_t(z) - A(Du(z^i_j) + D\varphi^i_j(z))|$$

$$+ |(g^i_j)_t(z)| + |A(Du(z^i_j) + D\varphi^i_j(z)) - A(Du(z) + D\varphi^i_j(z))|$$

$$\leq \frac{\epsilon}{6} + |v_t(z^i_j) + (\psi^i_j)_t(z) - A(Du(z^i_j) + D\varphi^i_j(z))|$$

$$+ |A(Du(z^i_j) + D\varphi^i_j(z)) - A(Du(z) + D\varphi^i_j(z))|.$$

From (i) of Theorem 4.1.6 and (4.18), we have $|Du(z^i_j) + D\varphi^i_j(z)| \leq 2s_+(r_1)$. As $(D\tilde{u}(z), \tilde{v}_t(z)) \in \mathcal{S}$, we also have $|Du(z) + D\varphi^i_j(z)| = |D\tilde{u}(z)| \leq s_+(r_1)$, and by (4.18), $|Du(z^i_j) - Du(z)| < \theta$.

From (4.17) we thus have

$$|A(Du(z_j^i) + D\varphi_j^i(z)) - A(Du(z) + D\varphi_j^i(z))| < \frac{\epsilon}{12}.$$

Integrating the above inequality over $Q_j^i \times I_j^i$, we now obtain from (d) that

$$\int_{Q_j^i \times I_j^i} |\tilde{v}_t(z) - A(D\tilde{u}(z))|dz \leq \frac{\epsilon}{4}|Q_j^i \times I_j^i| + \frac{\epsilon|\Omega_T|}{12}\frac{|Q_j^i \times I_j^i|}{|\tilde{Q} \times \tilde{I}|} \leq \frac{\epsilon}{3}|Q_j^i \times I_j^i|,$$

which yields that $I_3 \leq \frac{\epsilon}{3}|\Omega_T|$. Hence $I_1 + I_2 + I_3 \leq \epsilon|\Omega_T|$, and so $\tilde{u} \in \mathcal{U}_\epsilon$. Lastly, from (c) with $\rho \leq \eta$ and the definition of $\tilde{u}$, we have $\|\tilde{u} - u\|_{L^\infty(\Omega_T)} < \eta$.

The proof is now complete.

□

Completion of Proof of Theorem 2.3.2

Proof of Theorem 2.3.2. Combining Remark 4.3.4 and Theorem 4.4.1, we can see that there are infinitely many Lipschitz solutions u to problem (1.2).

We now follow the proof of Theorem 2.2.4 for detailed information on any fixed Lipschitz solution $u \in \mathcal{G}$ to (1.2). Here Du is the a.e.-pointwise limit of the gradient sequence Du_j of some sequence $u_j \in \mathcal{U}_{1/j}$ converging to u in $L^\infty(\Omega_T)$. Since $u_j \equiv u^*$ in Ω_T^1, we also have $u \equiv u^* \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T^1)$, so that

$$u_t = \operatorname{div}(A(Du)) \text{ and } |Du| < s_-(\bar{r}) \text{ in } \Omega_T^1.$$

As $\|Du_j\|_{L^\infty(\Omega_T)} \leq s_+(\tilde{r}) = s_+(r_1)$, we have $\|Du\|_{L^\infty(\Omega_T)} \leq s_+(\tilde{r})$. Also $(v_j)_t \rightharpoonup v_t$ in $L^2(\Omega_T;\mathbb{R}^n)$, where v_j is the corresponding vector function of u_j and $v \in W^{1,2}((0,T);L^2(\Omega;\mathbb{R}^n))$.

From (2.6), we can even deduce that $(v_j)_t \to v_t$ pointwise a.e. in Ω_T, where $\|(v_j)_t\|_{L^\infty(\Omega)} \leq r = r_2$ by the definition of $\mathcal{U}$ and (i) of Theorem 4.1.6; hence $\|v_t\|_{L^\infty(\Omega)} \leq r$. Note that

$$v_t = A(Du) \quad \text{a.e. in } \Omega_T,$$

and so $r \geq |v_t| = \sigma(|Du|)$ a.e. in Ω_T. On the other hand, $|Du| \leq s_+(\tilde{r})$ a.e. in Ω_T. So the graph of σ forces to give

$$|S_r| + |L_{r,\tilde{r}}| = |\Omega_T^2|.$$

If $|L_{r,\tilde{r}}| = 0$, then $|Du| \leq s_-(r)$ a.e. in Ω_T, which implies that $u = u^*$ in Ω_T by uniqueness. This contradicts the fact that $\|Du^*(\cdot, 0)\|_{L^\infty(\Omega)} = \|Du_0\|_{L^\infty(\Omega)} = M_0 > s_-(r)$. Thus, $|L_{r,\tilde{r}}| > 0$.

□

4.5 Proof of Theorem 2.3.5

In this final section, we complete the proof of Theorem 2.3.5 on the coexistence of radial and non-radial Lipschitz solutions to problem (1.2) when Ω is a ball and u_0 is radial.

Proof of Theorem 2.3.5. Using Theorem 3.1.1, the existence of infinitely many *radial* Lipschitz solutions to (1.2) follows from [25]. We remark that these radial solutions are not obtained through the existence result of this dissertation, Theorem 2.3.2.

The existence of infinitely many *non-radial* Lipschitz solutions to (1.2) can be shown by modifying the proof of Theorem 2.3.2. We proceed the proof as below.

It is easy to check that the function $u^* \in \mathcal{U}$ constructed in Section 4.3 is radial in Ω_T. Our strategy is to imitate the procedure of the density proof in Section 4.4 to the function

(u^*, v^*). We choose a space-time box in Ω_T with positive distance from the central axis of Ω_T where v_t^* is sufficiently away from $A(Du^*)$ in L^1-sense. Then as in the density proof, we perform the surgery on (u^*, v^*) only in this box to obtain a function (u_{nr}^*, v_{nr}^*) with membership $u_{nr}^* \in \mathcal{U}$ maintained. Such surgery breaks down the radial symmetry of u^*; hence, u_{nr}^* is non-radial. Note also that this u_{nr}^* cannot be a Lipschitz solution to (1.2).

Suppose there is no non-radial Lipschitz solution to (1.2). In the context of the proof of Theorem 2.2.4, this means that every $u \in \mathcal{G}$ is a radial solution. The L^∞-density of $\mathcal{G}$ in $\mathcal{X}$ then implies that every function in $\mathcal{X}$ is radial. This contradicts the existence of a non-radial function u_{nr}^* in $\mathcal{U} \subset \mathcal{X}$ above. Thus there exists a non-radial Lipschitz solution to (1.2).

Suppose there are only finitely many non-radial Lipschitz solutions to (1.2). This forces that the non-radial function u_{nr}^* should be the L^∞-limit of some sequence of radial functions in $\mathcal{G}$, a contradiction. Therefore, there are infinitely many non-radial Lipschitz solutions to (1.2). $\square$

Chapter 5

Höllig type equations

This chapter deals with the existence result on **Case II:** Höllig type equations, that is, Theorem 2.3.4. We thus assume **Hypothesis (H)** throughout this chapter.

5.1 Geometry of relevant matrix set

This section proceeds almost in the same way as in Section 4.1, and so we skip many details unless there should some change to be made.

For each $l \in \mathbb{R}$, let $K(l)$ be the subset of $\mathbb{M}^{(1+n)\times(n+1)}$ defined by (4.2) with flux $A(p)$ with profile $\sigma(s)$ satisfying Hypothesis (H).

Fix any two numbers $\sigma(s_2) < r_1 < r_2 < \sigma(s_1)$, and let $F_0 = F_{r_1,r_2}(0)$ be the subset of $K(0)$ given by

$$F_0 = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \;\middle|\; \begin{array}{l} p \in \mathbb{R}^n,\ |p| \in (s_-(r_1), s_-(r_2)) \cup (s_+(r_1), s_+(r_2)), \\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0 \end{array} \right\}.$$

The set F_0 is decomposed into two disjoint subsets as follows:

$$F_- = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \;\middle|\; \begin{array}{l} p \in \mathbb{R}^n,\ |p| \in (s_-(r_1), s_-(r_2)), \\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0 \end{array} \right\},$$

$$F_+ = \left\{ \begin{pmatrix} p & c \\ B & A(p) \end{pmatrix} \;\middle|\; \begin{array}{l} p \in \mathbb{R}^n,\ |p| \in (s_+(r_1), s_+(r_2)), \\ c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0 \end{array} \right\}.$$

As in **Case I:** Perona-Malik type equations, we focus on the homogeneous differential inclusion $\nabla\omega(z) \in F_0$; thus we first study the rank-one structure of the set F_0.

For the matrix set F_0, we define

$$R(F_0) = \bigcup_{\xi_\pm \in F_\pm,\, \operatorname{rank}(\xi_+ - \xi_-) = 1} (\xi_-, \xi_+).$$

From a careful analysis, one can actually deduce

$$L(F_0) = R(F_0). \tag{5.1}$$

This is a drastic difference to (4.3) where $L(F_+) \neq \emptyset$. However, in the current case, as only forward parts of σ are involved in F_0, no such set appears in (5.1); so it is even more natural to only stick to the analysis of the set $R(F_0)$ towards the existence result, Theorem 2.3.4.

We perform the step-by-step analysis of the set $R(F_0)$.

1. **Alternate expression for** $R(F_0)$**.** The proof of the following lemma just follows the lines of that of Theorem 4.1.2 with minor changes. So we skip the proof.

Lemma 5.1.1. *Let $\xi \in \mathbb{M}^{(1+n)\times(n+1)}$. Then $\xi \in R(F_0)$ if and only if there exist numbers $t_- < 0 < t_+$ and vectors $q, \gamma \in \mathbb{R}^n$ with $|q| = 1$, $\gamma \cdot q = 0$ such that for each $b \in \mathbb{R} \setminus \{0\}$, if $\eta = \begin{pmatrix} q & b \\ \frac{1}{b} q \otimes \gamma & \gamma \end{pmatrix}$, then $\xi + t_\pm \eta \in F_\pm$.*

2. **Diagonal components of matrices in** $R(F_0)$**.** The proof of the lemma below is precisely the same as that of Lemma 4.1.3.

Lemma 5.1.2.

$$R(F_0) = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \,\Big|\, c \in \mathbb{R},\ B \in \mathbb{M}^{n\times n},\ \operatorname{tr} B = 0,\ (p,\beta) \in \mathcal{S} \right\} \tag{5.2}$$

for some set $\mathcal{S} = \mathcal{S}_{r_1,r_2} \subset \mathbb{R}^{n+n}$.

3. **Selection of approximate collinear rank-one connections for** $R(F_0)$**.** We first equip with a 2-dimensional description for the rank-one connections of diagonal components of matrices in $R(F_0)$ in a general form whose proof is similar to that of Lemma 4.1.4 but has several minor changes.

Lemma 5.1.3. *For all positive numbers* a, b, c *with* $b > a$*, there exists a continuous function*

$$h(a,b,c,\cdot,\cdot,\cdot) : I_{a,b,c} = [0,a) \times [0,b-a) \times [0,c) \to [0,\infty)$$

with $h(a,b,c,0,0,0) = 0$ *satisfying the following:*

Let δ_1, δ_2 *and* η *be any positive numbers with*

$$0 < a - \delta_1 < a < b - \delta_2 < b, \quad 0 < c - \eta < c,$$

and let $R_1 \in [a-\delta_1, a]$*,* $R_2 \in [b-\delta_2, b]$*, and* $\tilde{R}_1, \tilde{R}_2 \in [c-\eta, c]$*. Suppose* $\theta \in [-\pi/2, \pi/2]$ *and*

$$\Big(\tilde{R}_1\big(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta)\big) - \tilde{R}_2\big(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta)\big)\Big)$$
$$\cdot\Big(R_1\big(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta)\big) - R_2\big(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta)\big)\Big) = 0.$$

Then $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$*,* $\tilde{R}_1 \geq \tilde{R}_2$*, and*

$$\max\Big\{\big|(0,a) - R_1\big(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta)\big)\big|,\ \big|(0,b) - R_2\big(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta)\big)\big|$$

$$\big|(0,c) - \tilde{R}_1\big(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta)\big)\big|,\ \big|(0,c) - \tilde{R}_2\big(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta)\big)\big|\Big\}$$

$$\leq h(a,b,c,\delta_1,\delta_2,\eta).$$

Proof. By assumption,

$$0 = (\tilde{R}_1(-\sin\theta, \cos\theta) - \tilde{R}_2(\sin\theta,\cos\theta)) \cdot (R_1(-\sin\theta,\cos\theta) - R_2(\sin\theta,\cos\theta))$$

$$= (-(\tilde{R}_1 + \tilde{R}_2)\sin\theta, (\tilde{R}_1 - \tilde{R}_2)\cos\theta) \cdot (-(R_1+R_2)\sin\theta, (R_1 - R_2)\cos\theta)$$

$$= (\tilde{R}_1 + \tilde{R}_2)(R_1+R_2)\sin^2\theta + (\tilde{R}_1 - \tilde{R}_2)(R_1 - R_2)\cos^2\theta,$$

that is,

$$(R_2 - R_1)(\tilde{R}_1 - \tilde{R}_2)\cos^2\theta = (R_1+R_2)(\tilde{R}_1 + \tilde{R}_2)\sin^2\theta;$$

hence, $\theta \neq \pm\frac{\pi}{2}$, $\tilde{R}_1 \geq \tilde{R}_2$, and

$$\theta = \pm\tan^{-1}\left(\sqrt{\frac{(R_2-R_1)(\tilde{R}_1 - \tilde{R}_2)}{(R_1+R_2)(\tilde{R}_1+\tilde{R}_2)}}\right).$$

So

$$|\theta| \leq \tan^{-1}\left(\sqrt{\frac{(b-a+\delta_1)\eta}{2(a+b-\delta_1-\delta_2)(c-\eta)}}\right) =: g(a,b,c,\delta_1,\delta_2,\eta).$$

Note that the function $g(a,b,c,\cdot,\cdot,\cdot) : I_{a,b,c} \to [0,\pi/2)$ is well-defined and continuous and that $g(a,b,c,\delta_1,\delta_2,\eta) = 0$ for all $(\delta_1,\delta_2,\eta) \in I_{a,b,c}$ with $\eta = 0$.

Observe now that

$$|(0,a) - R_1(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta))|$$

$$\leq \max\{|(0,a) - a(-\sin\theta, \cos\theta)|, |(0,a) - (a-\delta_1)(-\sin\theta, \cos\theta)|\}$$

$$= \max\left\{\sqrt{a^2\sin^2\theta + a^2(1-\cos\theta)^2}, \sqrt{(a-\delta_1)^2\sin^2\theta + (a-(a-\delta_1)\cos\theta)^2}\right\}$$

$$= \max\left\{\sqrt{2}a\sqrt{1-\cos\theta}, \sqrt{(a-\delta_1)^2 + a^2 - 2a(a-\delta_1)\cos\theta}\right\}$$

$$\leq \max\left\{\sqrt{2}a\sqrt{1-\cos(g(a,b,c,\delta_1,\delta_2,\eta))},\right.$$

$$\left.\sqrt{(a-\delta_1)^2 + a^2 - 2a(a-\delta_1)\cos(g(a,b,c,\delta_1,\delta_2,\eta))}\right\}$$

$$=: h_1(a,b,c,\delta_1,\delta_2,\eta),$$

$$|(0,b) - R_2(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta))|$$

$$\leq \max\left\{\sqrt{2}b\sqrt{1-\cos(g(a,b,c,\delta_1,\delta_2,\eta))},\right.$$

$$\left.\sqrt{(b-\delta_2)^2 + b^2 - 2b(b-\delta_2)\cos(g(a,b,c,\delta_1,\delta_2,\eta))}\right\}$$

$$=: h_2(a,b,c,\delta_1,\delta_2,\eta),$$

$$|(0,c) - \tilde{R}_1(\cos(\frac{\pi}{2}+\theta), \sin(\frac{\pi}{2}+\theta))|$$

$$\leq \max\left\{\sqrt{2}c\sqrt{1-\cos(g(a,b,c,\delta_1,\delta_2,\eta))},\right.$$

$$\left.\sqrt{(c-\eta)^2 + c^2 - 2c(c-\eta)\cos(g(a,b,c,\delta_1,\delta_2,\eta))}\right\}$$

$$=: h_3(a,b,c,\delta_1,\delta_2,\eta),$$

$$|(0,c) - \tilde{R}_2(\cos(\frac{\pi}{2}-\theta), \sin(\frac{\pi}{2}-\theta))| \leq h_3(a,b,c,\delta_1,\delta_2,\eta).$$

Define $h(a,b,c,\delta_1,\delta_2,\eta) = \max_{1\le j\le 3} h_j(a,b,c,\delta_1,\delta_2,\eta)$. Then it is trivial to see that the function $h(a,b,c,\cdot,\cdot,\cdot) : I_{a,b,c} \to [0,\infty)$ is well-defined and satisfies the desired properties.

$\square$

Next, we apply the previous lemma to choose *approximate* collinear rank-one connections for the diagonal components of matrices in $R(F_0)$.

Theorem 5.1.4. *Let $p_\pm \in \mathbb{R}^n$ satisfy*

$$s_-(r_1) < |p_-| < s_-(r_2) < s_+(r_1) < |p_+| < s_+(r_2)$$

and $(A(p_+) - A(p_-)) \cdot (p_+ - p_-) = 0$. Then there exists a vector $\zeta^0 \in \mathbb{S}^{n-1}$ such that, with $p^0_\pm = s_\pm(r_2)\zeta^0$, $A(p^0_\pm) = r_2\zeta^0$, we have

$$\max\{|p^0_- - p_-|, |p^0_+ - p_+|, |A(p^0_-) - A(p_-)|, |A(p^0_+) - A(p_+)|\}$$

$$\le h(s_-(r_2), s_+(r_2), r_2, s_-(r_2) - s_-(r_1), s_+(r_2) - s_+(r_1), r_2 - r_1),$$

where h is the function in Lemma 5.1.3.

Proof. The proof is the same as that of Lemma 4.1.5 except that we let $\delta_2 = s_+(r_2) - s_+(r_1)$ in applying Lemma 5.1.3. $\square$

4. **Final characterization of $R(F_0)$.** Now we can deduce the result on essential structures of $R(F_0)$. Although the proof is very similar to that of Theorem 4.1.6, we include it here for completeness.

Theorem 5.1.5. *Let $\sigma(s_2) < r_2 < \sigma(s_1)$. Then there exists a number $l_2 = l_{r_2} \in (\sigma(s_2), r_2)$ such that for any $l_2 < r_1 < r_2$, the set $\mathcal{S} = \mathcal{S}_{r_1,r_2} \subset \mathbb{R}^{n+n}$ in (5.2) satisfies the following:*

(i) $\sup_{(p,\beta)\in\mathcal{S}}|p| \leq s_+(r_2)$ *and* $\sup_{(p,\beta)\in\mathcal{S}}|\beta| \leq r_2$*; hence* $\mathcal{S}$ *is bounded.*

(ii) $\mathcal{S}$ *is open.*

(iii) For each $(p_0,\beta_0)\in\mathcal{S}$*, there exist an open set* $\mathcal{V}\subset\subset\mathcal{S}$ *containing* (p_0,β_0) *and* C^1 *functions* $q:\bar{\mathcal{V}}\to\mathbb{S}^{n-1}$*,* $\gamma:\bar{\mathcal{V}}\to\mathbb{R}^n$*,* $t_\pm:\bar{\mathcal{V}}\to\mathbb{R}$ *with* $\gamma\cdot q=0$ *and* $t_-<0<t_+$ *on* $\bar{\mathcal{V}}$ *such that for every* $\xi=\begin{pmatrix} p & c \\ B & \beta\end{pmatrix}\in R(F_0)=R(F_{r_1,r_2}(0))$ *with* $(p,\beta)\in\bar{\mathcal{V}}$*, we have*

$$\xi+t_\pm\eta\in F_\pm,$$

where $t_\pm=t_\pm(p,\beta)$*,* $\eta=\begin{pmatrix} q(p,\beta) & b \\ \frac{1}{b}\gamma(p,\beta)\otimes q(p,\beta) & \gamma(p,\beta)\end{pmatrix}$*, and* $b\neq 0$ *is arbitrary.*

Proof. Fix any $\sigma(s_2)<r_2<\sigma(s_1)$. For the moment, we let r_1 be any number in $(\sigma(s_2),r_2)$ and prove (i). Then we choose later a lower bound $l_2=l_{r_2}\in(\sigma(s_2),r_2)$ of r_1 for the validity of (ii) and (iii) above.

We divide the proof into several steps.

1. To show that (i) holds, choose any $(p,\beta)\in\mathcal{S}$. By Lemma 5.1.2, $\xi:=\begin{pmatrix} p & 0 \\ O & \beta\end{pmatrix}\in R(F_0)=R(F_{r_1,r_2}(0))$, where O is the $n\times n$ zero matrix. By the definition of $R(F_0)$, there exist two matrices $\xi_\pm=\begin{pmatrix} p_\pm & c_\pm \\ B_\pm & \sigma(|p_\pm|)\frac{p_\pm}{|p_\pm|}\end{pmatrix}\in F_\pm$ and a number $0<\lambda<1$ such that $\xi=\lambda\xi_+ +(1-\lambda)\xi_-$. So

$$|p|=|\lambda p_+ +(1-\lambda)p_-|\leq s_+(r_2),$$

$$|\beta|=\left|\lambda\sigma(|p_+|)\frac{p_+}{|p_+|}+(1-\lambda)\sigma(|p_-|)\frac{p_-}{|p_-|}\right|\leq r_2\,;$$

hence, $\sup_{(p,\beta)\in\mathcal{S}}|p| \le s_+(r_2)$, $\sup_{(p,\beta)\in\mathcal{S}}|\beta| \le r_2$, and $\mathcal{S}$ is bounded. So (i) is proved.

2. We now turn to the remaining assertions that for all $r_1 < r_2$ sufficiently close to r_2, $\mathcal{S} = \mathcal{S}_{r_1,r_2}$ fulfills (ii) and (iii). In this step, we still assume r_1 is any fixed number in $(\sigma(s_2), r_2)$.

Let $(p_0,\beta_0) \in \mathcal{S}$. Since $\xi_0 := \begin{pmatrix} p_0 & 0 \\ O & \beta_0 \end{pmatrix} \in R(F_0)$, it follows from Lemma 5.1.1 that there exist numbers $s_0 < 0 < t_0$ and vectors q_0, $\gamma_0 \in \mathbb{R}^n$ with $|q_0| = 1$, $\gamma_0 \cdot q_0 = 0$ such that $\xi_0 + s_0\eta_0 \in F_-$ and $\xi_0 + t_0\eta_0 \in F_+$, where $\eta_0 = \begin{pmatrix} q_0 & b \\ \frac{1}{b} q_0 \otimes \gamma_0 & \gamma_0 \end{pmatrix}$ and $b \neq 0$ is any fixed number. Let $q_0' = t_0 q_0 \neq 0$, $\gamma_0' = t_0\gamma_0$, and $s_0' = s_0/t_0 < 0$; then

$$\gamma_0' \cdot q_0' = 0, \quad s_-(r_1) < |p_0 + s_0' q_0'| < s_-(r_2),$$

$$s_+(r_1) < |p_0 + q_0'| < s_+(r_2),$$

$$\sigma(|p_0 + s_0'q_0'|)\frac{p_0 + s_0'q_0'}{|p_0 + s_0'q_0'|} = \beta_0 + s_0'\gamma_0', \;\; \sigma(|p_0 + q_0'|)\frac{p_0 + q_0'}{|p_0 + q_0'|} = \beta_0 + \gamma_0'. \tag{5.3}$$

Observe also

$$t_0 - s_0 \ge |(p_0 + t_0 q_0)| - |(p_0 + s_0 q_0)| > s_+(r_1) - s_-(r_2). \tag{5.4}$$

Next, consider the function F defined by

$$\begin{aligned} F(\gamma', q', s'; p, \beta) \;\; &= \;\; (\sigma(|p + s'q'|)\frac{p + s'q'}{|p + s'q'|} - \beta - s'\gamma', \\ &\quad\;\; \sigma(|p + q'|)\frac{p + q'}{|p + q'|} - \beta - \gamma', \gamma' \cdot q') \in \mathbb{R}^{n+n+1} \end{aligned}$$

for all γ', q', p, $\beta \in \mathbb{R}^n$ and $s' \in \mathbb{R}$ with $s_-(r_1) < |p + s'q'| < s_-(r_2)$, $s_+(r_1) < |p + q'| < s_+(r_2)$. Then F is C^1 in the described open subset of $\mathbb{R}^{n+n+1+n+n}$, and the above observation gives

$$F(\gamma_0', q_0', s_0'; p_0, \beta_0) = 0.$$

Suppose for the moment that the Jacobian matrix $D_{(\gamma', q', s')}F$ is invertible at the point $(\gamma_0', q_0', s_0'; p_0, \beta_0)$; then the Implicit Function Theorem implies the following: There exist a bounded domain $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_{(p_0,\beta_0)} \subset \mathbb{R}^{n+n}$ containing (p_0, β_0) and C^1 functions $\tilde{q}$, $\tilde{\gamma} \in \mathbb{R}^n$, $\tilde{s} \in \mathbb{R}$ of $(p, \beta) \in \tilde{\mathcal{V}}$ such that

$$\tilde{\gamma}(p_0, \beta_0) = \gamma_0', \;\; \tilde{q}(p_0, \beta_0) = q_0', \;\; \tilde{s}(p_0, \beta_0) = s_0'$$

and that

$$\tilde{s}(p, \beta) < 0, \;\; s_-(r_1) < |p + \tilde{s}(p, \beta)\tilde{q}(p, \beta)| < s_-(r_2),$$

$$s_+(r_1) < |p + \tilde{q}(p, \beta)| < s_+(r_2),$$

$$F(\tilde{\gamma}(p, \beta), \tilde{q}(p, \beta), \tilde{s}(p, \beta); p, \beta) = 0 \quad \forall (p, \beta) \in \tilde{\mathcal{V}}.$$

Define

$$\gamma = \frac{\tilde{\gamma}}{|\tilde{q}|}, \;\; q = \frac{\tilde{q}}{|\tilde{q}|}, \;\; t_- = \tilde{s}|\tilde{q}|, \;\; t_+ = |\tilde{q}| \quad \text{in } \tilde{\mathcal{V}};$$

then

$$s_-(r_1) < |p + t_- q| < s_-(r_2),$$

$$s_+(r_1) < |p + t_+ q| < s_+(r_2),$$

$$\sigma(|p + t_\pm q|)\frac{p + t_\pm q}{|p + t_\pm q|} = \beta + t_\pm \gamma, \;\; |q| = 1, \;\; \gamma \cdot q = 0, \;\; t_- < 0 < t_+,$$

where $(p, \beta) \in \tilde{\mathcal{V}}$, $\gamma = \gamma(p, \beta)$, $q = q(p, \beta)$, and $t_\pm = t_\pm(p, \beta)$.

Let $(p,\beta)\in\tilde{\mathcal{V}}$, $B\in\mathbb{M}^{n\times n}$, $\operatorname{tr}B=0$, $b,c\in\mathbb{R}$, $b\neq 0$, $q=q(p,\beta)$, $\gamma=\gamma(p,\beta)$, $t_\pm=t_\pm(p,\beta)$, $\xi=\begin{pmatrix} p & c \\ B & \beta\end{pmatrix}$, and $\eta=\begin{pmatrix} q & b \\ \frac{1}{b}\gamma\otimes q & \gamma\end{pmatrix}$. Then $\xi_\pm:=\xi+t_\pm\eta\in F_\pm$. By the definition of $R(F_0)$, $\xi\in(\xi_-,\xi_+)\subset R(F_0)$. By Lemma 5.1.2, we thus have $(p,\beta)\in\mathcal{S}$; hence $\tilde{\mathcal{V}}\subset\mathcal{S}$. Choosing any open set $\mathcal{V}\subset\subset\tilde{\mathcal{V}}$ with $(p_0,\beta_0)\in\mathcal{V}$, the assertions (ii) and (iii) hold with $\mathcal{S}=\cup_{(p_0,\beta_0)\in\mathcal{S}}\tilde{\mathcal{V}}_{(p_0,\beta_0)}$ open.

3. In this step, we continue Step 2 to deduce an equivalent condition for the invertibility of the Jacobian matrix $D_{(\gamma',q',s')}F$ at $(\gamma_0',q_0',s_0';p_0,\beta_0)$. By direct computation,

$$D_{(\gamma',q',s')}F=\begin{pmatrix} -s'I_n & M_{s'} & \omega_{s'}^- \\ -I_n & M_1 & 0 \\ q' & \gamma' & 0\end{pmatrix}\in\mathbb{M}^{(n+n+1)\times(n+n+1)},$$

where I_n is the $n\times n$ identity matrix,

$$M_{s'}=s'(\sigma'(|p+s'q'|)-\frac{\sigma(|p+s'q'|)}{|p+s'q'|})\frac{p+s'q'}{|p+s'q'|}\otimes\frac{p+s'q'}{|p+s'q'|}+s'\frac{\sigma(|p+s'q'|)}{|p+s'q'|}I_n,$$

$$\omega_{s'}^\pm=(\sigma'(|p+s'q'|)-\frac{\sigma(|p+s'q'|)}{|p+s'q'|})(\frac{p+s'q'}{|p+s'q'|}\cdot q')\frac{p+s'q'}{|p+s'q'|}+\frac{\sigma(|p+s'q'|)}{|p+s'q'|}q'\pm\gamma'.$$

For notational simplicity, we write $(\gamma',q',s';p,\beta)=(\gamma_0',q_0',s_0';p_0,\beta_0)$. Applying suitable elementary row operations, where $s'<0$,

$$D_{(\gamma',q',s')}F\to\begin{pmatrix} -s'I_n & M_{s'} & \omega_{s'}^- \\ O & M_1-\frac{1}{s'}M_{s'} & -\frac{1}{s'}\omega_{s'}^- \\ 0 & \gamma'+\frac{q_1'}{s'}(M_{s'})^1+\cdots+\frac{q_n'}{s'}(M_{s'})^n & \frac{1}{s'}q'\cdot\omega_{s'}^-\end{pmatrix}$$

$$\to \begin{pmatrix} -s'I_n & M_{s'} & \omega^-_{s'} \\ O & s'M_1 - M_{s'} & -\omega^-_{s'} \\ 0 & s'\gamma' + q'_1(M_{s'})^1 + \cdots + q'_n(M_{s'})^n & q'\cdot\omega^-_{s'} \end{pmatrix},$$

where O is the $n\times n$ zero matrix, and $(M_{s'})^i$ is the ith row of $M_{s'}$. Since $|q'| = t_0$, $\gamma'\cdot q' = 0$, and $s_-(r_1) < |p+s'q'| < s_-(r_2)$, we have

$$q'\cdot\omega^-_{s'} = (\sigma'(|p+s'q'|) - \frac{\sigma(|p+s'q'|)}{|p+s'q'|})(\frac{p+s'q'}{|p+s'q'|}\cdot q')^2 + \frac{\sigma(|p+s'q'|)}{|p+s'q'|}t_0^2$$

$$= t_0^2(\cos^2\theta'\sigma'(|p+s'q'|) + (1-\cos^2\theta')\frac{\sigma(|p+s'q'|)}{|p+s'q'|}) > 0,$$

where $\theta' \in [0,\pi]$ is the angle between $p+s'q'$ and q'. Observe here that the forward part of σ in the definition of F_- becomes essential to guarantee that $\sigma'(|p+s'q'|) > 0$. After some elementary column operations to the last matrix from the above row operations, we obtain

$$D_{(\gamma',q',s')}F \to \begin{pmatrix} -s'I_n & M_{s'} - N_{s'} & \omega^-_{s'} \\ O & s'M_1 - M_{s'} + N_{s'} & -\omega^-_{s'} \\ 0 & 0 & q'\cdot\omega^-_{s'} \end{pmatrix},$$

where the jth column of $N_{s'} \in \mathbb{M}^{n\times n}$ is $\frac{s'\gamma'_j + q'\cdot(M_{s'})_j}{q'\cdot\omega^-_{s'}}\omega^-_{s'}$. So $D_{(\gamma',q',s')}F$ is invertible if and only if the $n\times n$ matrix $M_1 - \frac{1}{s'}M_{s'} + \frac{1}{s'}N_{s'}$ is invertible. We compute

$$M_1 - \frac{1}{s'}M_{s'} + \frac{1}{s'}N_{s'} = (\sigma'(|p+q'|) - \frac{\sigma(|p+q'|)}{|p+q'|})\frac{p+q'}{|p+q'|}\otimes\frac{p+q'}{|p+q'|}$$

$$+\frac{\sigma(|p+q'|)}{|p+q'|}I_n - (\sigma'(|p+s'q'|) - \frac{\sigma(|p+s'q'|)}{|p+s'q'|})\frac{p+s'q'}{|p+s'q'|}\otimes\frac{p+s'q'}{|p+s'q'|}$$

$$-\frac{\sigma(|p+s'q'|)}{|p+s'q'|}I_n+\frac{1}{q'\cdot\omega^-_{s'}}\omega^-_{s'}\otimes(\gamma'$$

$$+(\sigma'(|p+s'q'|)-\frac{\sigma(|p+s'q'|)}{|p+s'q'|})(\frac{p+s'q'}{|p+s'q'|}\cdot q')\frac{p+s'q'}{|p+s'q'|}+\frac{\sigma(|p+s'q'|)}{|p+s'q'|}q')$$

$$=(a_1-a_{s'})I_n+(b_1-a_1)\frac{p+q'}{|p+q'|}\otimes\frac{p+q'}{|p+q'|}$$

$$-(b_{s'}-a_{s'})\frac{p+s'q'}{|p+s'q'|}\otimes\frac{p+s'q'}{|p+s'q'|}+\frac{1}{q'\cdot\omega^-_{s'}}\omega^-_{s'}\otimes\omega^+_{s'},$$

and set (with an assumption $a_1\neq a_{s'}$)

$$B=\frac{1}{a_1-a_{s'}}(M_1-\frac{1}{s'}M_{s'}+\frac{1}{s'}N_{s'})=I_n+\frac{b_1-a_1}{a_1-a_{s'}}\frac{p+q'}{|p+q'|}\otimes\frac{p+q'}{|p+q'|}$$

$$-\frac{b_{s'}-a_{s'}}{a_1-a_{s'}}\frac{p+s'q'}{|p+s'q'|}\otimes\frac{p+s'q'}{|p+s'q'|}+\frac{1}{(a_1-a_{s'})q'\cdot\omega^-_{s'}}\omega^-_{s'}\otimes\omega^+_{s'},$$

where $a_{s'}=\frac{\sigma(|p+s'q'|)}{|p+s'q'|}$, $b_{s'}=\sigma'(|p+s'q'|)$; then $D_{(\gamma',q',s')}F$ is invertible if and only if the matrix $B\in\mathbb{M}^{n\times n}$ is invertible.

4. To close the arguments in Step 2 and thus to finish the proof, we choose a suitable $l_2=l_{r_2}\in(\sigma(s_2),r_2)$ in such a way that for any $r_1\in(l_2,r_2)$, the matrix B, determined through Steps 2 and 3 for any given $(p_0,\beta_0)\in\mathcal{S}=\mathcal{S}_{r_1,r_2}$, is invertible.

First, by Hypothesis (C), $\tilde r_2\in(\sigma(s_2),r_2)$ can be chosen close enough to r_2 so that

$$\frac{\sigma(k)}{k}<\frac{\sigma(l)}{l}\quad\forall l\in[s_-(\tilde r_2),s_-(r_2)],\ \forall k\in[s_+(\tilde r_2),s_+(r_2)].$$

Then define a real-valued continuous function (to express the determinant of the matrix B

from Step 3)

$$\mathrm{DET}(u,v,q,\gamma)=\det\Big(I_n+\frac{\sigma'(|u|)-\frac{\sigma(|u|)}{|u|}}{\frac{\sigma(|u|)}{|u|}-\frac{\sigma(|v|)}{|v|}}\frac{u}{|u|}\otimes\frac{u}{|u|}-\frac{\sigma'(|v|)-\frac{\sigma(|v|)}{|v|}}{\frac{\sigma(|u|)}{|u|}-\frac{\sigma(|v|)}{|v|}}\frac{v}{|v|}\otimes\frac{v}{|v|}$$

$$+\frac{1}{(\frac{\sigma(|u|)}{|u|}-\frac{\sigma(|v|)}{|v|})((\sigma'(|v|)-\frac{\sigma(|v|)}{|v|})(\frac{v}{|v|}\cdot q)^2+\frac{\sigma(|v|)}{|v|})}\big((\sigma'(|v|)-\frac{\sigma(|v|)}{|v|})(\frac{v}{|v|}\cdot q)\frac{v}{|v|}$$

$$+\frac{\sigma(|v|)}{|v|}q-\gamma\big)\otimes\big((\sigma'(|v|)-\frac{\sigma(|v|)}{|v|})(\frac{v}{|v|}\cdot q)\frac{v}{|v|}+\frac{\sigma(|v|)}{|v|}q+\gamma\big)\Big)$$

on the compact set $\mathcal{M}$ of points $(u,v,q,\gamma)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}^{n-1}\times\mathbb{R}^n$ with

$$|u|\in[s_+(\tilde r_2),s_+(r_2)],\quad |v|\in[s_-(\tilde r_2),s_-(r_2)],\quad |\gamma|\le 1.$$

With $\bar k=s_+(r_2)$ and $\bar l=s_-(r_2)$, for each $q\in\mathbb{S}^{n-1}$,

$$\mathrm{DET}(\bar kq,\bar lq,q,0)=\det\Big(I_n+\frac{\sigma'(\bar k)-\frac{\sigma(\bar k)}{\bar k}+\frac{\sigma(\bar l)}{\bar l}}{\frac{\sigma(\bar k)}{\bar k}-\frac{\sigma(\bar l)}{\bar l}}q\otimes q\Big)\neq 0,$$

since $\sigma'(\bar k)\neq 0$ and hence the fraction in front of $q\otimes q$ is different from -1. So

$$d:=\min_{q\in\mathbb{S}^{n-1}}|\mathrm{DET}(\bar kq,\bar lq,q,0)|>0.$$

Next, choose a number $\delta>0$ such that for all $(u,v,q,\gamma),(\tilde u,\tilde v,\tilde q,\tilde\gamma)\in\mathcal{M}$ with $|u-\tilde u|$, $|v-\tilde v|$, $|q-\tilde q|$, $|\gamma-\tilde\gamma|<\delta$, we have

$$|\mathrm{DET}(u,v,q,\gamma)-\mathrm{DET}(\tilde u,\tilde v,\tilde q,\tilde\gamma)|<d/2. \tag{5.5}$$

Let $l_2 \in (\tilde{r}_2, r_2)$ be sufficiently close to r_2 so that for all $r_1 \in (l_2, r_2)$,

$$h(s_-(r_2), s_+(r_2), r_2, s_-(r_2) - s_-(r_1), s_+(r_2) - s_+(r_1), r_2 - r_1) < \tau,$$

where h is the function in Theorem 5.1.4 and

$$\tau := \min\{\delta, \delta(s_2 - s_1)/4\}.$$

Now, fix any $r_1 \in (l_2, r_2)$, and let B be the $n \times n$ matrix determined through Steps 2 and 3 in terms of any given $(p_0, \beta_0) \in \mathcal{S} = \mathcal{S}_{r_1, r_2}$. Let $p_+ = p_0 + t_0 q_0$ and $p_- = p_0 + s_0 q_0$ from Step 2; then $p_\pm$ and $A(p_\pm)$ fulfill the conditions in Theorem 5.1.4. So this theorem implies that there exists a vector $\zeta^0 \in \mathbb{S}^{n-1}$ such that

$$\max\{|p_-^0 - p_-|, |p_+^0 - p_+|, |A(p_-^0) - A(p_-)|, |A(p_+^0) - A(p_+)|\} < \tau,$$

where $p_+^0 = \bar{k}\zeta^0$, $p_-^0 = \bar{l}\zeta^0$, and $A(p_\pm^0) = r_2\zeta^0$. Using (5.3) and (5.4),

$$|p_+ - \bar{k}\zeta^0| < \delta, \quad |p_- - \bar{l}\zeta^0| < \delta,$$

$$|q_0 - \zeta^0| = |\frac{p_+ - p_-}{t_0 - s_0} - \zeta^0| \leq \frac{|(p_+ - p_-) - (\bar{k} - \bar{l})\zeta^0| + |(\bar{k} - \bar{l}) - (t_0 - s_0)|}{t_0 - s_0}$$

$$\leq \frac{2\tau + ||p_+^0 - p_-^0| - |p_+ - p_-||}{t_0 - s_0} < \frac{4\tau}{t_0 - s_0} < \delta,$$

$$|\gamma_0| = |\frac{A(p_+) - A(p_-)}{t_0 - s_0}| \leq \frac{|A(p_+) - A(p_+^0)| + |A(p_-^0) - A(p_-)|}{t_0 - s_0} < \delta.$$

Since $\det(B) = \mathrm{DET}(p_+, p_-, q_0, \gamma_0)$ and $|\mathrm{DET}(\bar{k}\zeta^0, \bar{l}\zeta^0, \zeta^0, 0)| \geq d$, it follows from (5.5) that

$$|\det(B)| > d/2 > 0.$$

The proof is now complete. $\square$

5.2 Relaxation of $\nabla\omega(z) \in F_0$

Lemma 4.2.1 is in common use for both Theorems 4.2.2 and 5.2.1, and so we do not restate it here.

We state the relaxation theorem for homogeneous differential inclusion $\nabla\omega(z) \in F_0$ in a form that is more suitable for later use; we restrict the inclusion to only (p, β) components. Although the proof of this theorem is quite similar to that of its companion version, Theorem 4.2.2, we include it here for the sake of completeness.

Theorem 5.2.1. *Let $\sigma(s_2) < r_2 < \sigma(s_1)$, and let $l_2 = l_{r_2} \in (\sigma(s_2), r_2)$ be some number determined by Theorem 5.1.5. Let $l_2 < r_1 < r_2$, and let $\mathcal{K}$ be a compact subset of $\mathcal{S} = \mathcal{S}_{r_1,r_2}$. Let $\tilde{Q} \times \tilde{I}$ be a box in $\mathbb{R}^{n+1}$. Then, given any $\epsilon > 0$, there exists a $\delta > 0$ such that for each box $Q \times I \subset \tilde{Q} \times \tilde{I}$, point $(p, \beta) \in \mathcal{K}$, and number $\rho > 0$ sufficiently small, there exists a function $\omega = (\varphi, \psi) \in C_c^\infty(Q \times I; \mathbb{R}^{1+n})$ satisfying the following properties:*

(a) $\operatorname{div}\psi = 0$ *in* $Q \times I$,

(b) $(p' + D\varphi(z), \beta' + \psi_t(z)) \in \mathcal{S}$ *for all* $z \in Q \times I$ *and* $|(p', \beta') - (p, \beta)| \leq \delta$,

(c) $\|\omega\|_{L^\infty(Q\times I)} < \rho$,

(d) $\int_{Q\times I} |\beta + \psi_t(z) - A(p + D\varphi(z))| dz < \epsilon |Q \times I| / |\tilde{Q} \times \tilde{I}|$,

(e) $\int_Q \varphi(x, t) dx = 0$ *for all* $t \in I$,

(f) $\|\varphi_t\|_{L^\infty(Q\times I)} < \rho$.

Proof. By Theorem 5.1.5, there exist finitely many open balls $\mathcal{B}_1, \cdots, \mathcal{B}_N \subset\subset \mathcal{S}$ covering $\mathcal{K}$ and C^1 functions $q_i : \bar{\mathcal{B}}_i \to \mathbb{S}^{n-1}$, $\gamma_i : \bar{\mathcal{B}}_i \to \mathbb{R}^n$, $t_{i,\pm} : \bar{\mathcal{B}}_i \to \mathbb{R}$ $(1 \le i \le N)$ with $\gamma_i \cdot q_i = 0$ and $t_{i,-} < 0 < t_{i,+}$ on $\bar{\mathcal{B}}_i$ such that for each $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \in R(F_0)$ with $(p, \beta) \in \bar{\mathcal{B}}_i$, we have

$$\xi + t_{i,\pm}\eta_i \in F_\pm,$$

where $t_{i,\pm} = t_{i,\pm}(p, \beta)$, $\eta_i = \begin{pmatrix} q_i(p,\beta) & b \\ \frac{1}{b}\gamma_i(p,\beta) \otimes q_i(p,\beta) & \gamma_i(p,\beta) \end{pmatrix}$, and $b \neq 0$ is arbitrary.

Let $1 \le i \le N$. We write $\xi_i = \xi_i(p, \beta) = \begin{pmatrix} p & 0 \\ O & \beta \end{pmatrix} \in R(F_0)$ for $(p, \beta) \in \bar{\mathcal{B}}_i \subset \mathcal{S}$, where O is the $n \times n$ zero matrix. We omit the dependence on $(p, \beta) \in \bar{\mathcal{B}}_i$ in the following whenever it is clear from the context. Given any $\rho > 0$, we choose a constant b_i with

$$0 < b_i < \min_{\bar{\mathcal{B}}_i} \frac{\rho}{t_{i,+} - t_{i,-}}.$$

With this choice of $b = b_i$, let η_i be defined on $\bar{\mathcal{B}}_i$ as above. Then

$$\xi_{i,\pm} = \begin{pmatrix} p_{i,\pm} & c_{i,\pm} \\ B_{i,\pm} & \beta_{i,\pm} \end{pmatrix} := \xi_i + t_{i,\pm}\eta_i \in F_\pm,$$

$$\xi_i = \lambda_i \xi_{i,+} + (1 - \lambda_i)\xi_{i,-}, \quad \lambda_i = \frac{-t_{i,-}}{t_{i,+} - t_{i,-}} \in (0, 1) \quad \text{on } \bar{\mathcal{B}}_i.$$

By the definition of $R(F_0)$, on $\bar{\mathcal{B}}_i$, both $\xi^\tau_{i,-} = \tau\xi_{i,+} + (1-\tau)\xi_{i,-}$ and $\xi^\tau_{i,+} = (1-\tau)\xi_{i,+} + \tau\xi_{i,-}$ belong to $R(F_0)$ for all $\tau \in (0, 1)$. Let $0 < \tau < \min_{1 \le j \le N} \min_{\bar{\mathcal{B}}_j} \min\{\lambda_j, 1 - \lambda_j\} \le \frac{1}{2}$

be a small number to be selected later. Let $\lambda_i' = \frac{\lambda_i - \tau}{1-2\tau}$ on $\bar{\mathcal{B}}_i$. Then $\lambda_i' \in (0,1)$, $\xi_i = \lambda_i'\xi_{i,+}^\tau + (1-\lambda_i')\xi_{i,-}^\tau$ on $\bar{\mathcal{B}}_i$. Moreover, on $\bar{\mathcal{B}}_i$, $\xi_{i,+}^\tau - \xi_{i,-}^\tau = (1-2\tau)(\xi_{i,+} - \xi_{i,-})$ is rank-one, $[\xi_{i,-}^\tau, \xi_{i,+}^\tau] \subset (\xi_{i,-}, \xi_{i,+}) \subset R(F_0)$, and

$$c\tau \le |\xi_{i,+}^\tau - \xi_{i,+}| = |\xi_{i,-}^\tau - \xi_{i,-}| = \tau|\xi_{i,+} - \xi_{i,-}| = \tau(t_{i,+} - t_{i,-})|\eta_i| \le C\tau,$$

where $C = \max_{1\le j\le N}\max_{\bar{\mathcal{B}}_j}(t_{j,+} - t_{j,-})|\eta_j| \ge \min_{1\le j\le N}\min_{\bar{\mathcal{B}}_j}(t_{j,+} - t_{j,-})|\eta_j| = c > 0$. By continuity, $H_\tau = \bigcup_{(p,\beta)\in\bar{\mathcal{B}}_j, 1\le j\le N}[\xi_{j,-}^\tau(p,\beta), \xi_{j,+}^\tau(p,\beta)]$ is a compact subset of $R(F_0)$, where $R(F_0)$ is open in the space

$$\Sigma_0 = \left\{ \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} \;\Big|\; \mathrm{tr}B = 0 \right\},$$

by Lemma 5.1.2 and Theorem 5.1.5. So $d_\tau = \mathrm{dist}(H_\tau, \partial|_{\Sigma_0}R(F_0)) > 0$, where $\partial|_{\Sigma_0}$ is the relative boundary in Σ_0.

Let $\eta_{i,1} = -\lambda_{i,1}\eta_i = -\lambda_i'(1-2\tau)(t_{i,+}-t_{i,-})\eta_i$, $\eta_{i,2} = \lambda_{i,2}\eta_i = (1-\lambda_i')(1-2\tau)(t_{i,+}-t_{i,-})\eta_i$ on $\bar{\mathcal{B}}_i$, where $\lambda_{i,1} = \tau(-t_{i,+}) + (1-\tau)(-t_{i,-}) > 0$, $\lambda_{i,2} = (1-\tau)t_{i,+} + \tau t_{i,-} > 0$ on $\bar{\mathcal{B}}_i$, and $\tau > 0$ is so small that

$$\min_{1\le j\le N}\min_{\bar{\mathcal{B}}_j}\lambda_{j,k} > 0 \;\; (k = 1,\, 2).$$

Applying Lemma 4.2.1 to matrices $\eta_{i,1} = \eta_{i,1}(p,\beta)$, $\eta_{i,2} = \eta_{i,2}(p,\beta)$ with a fixed $(p,\beta) \in \bar{\mathcal{B}}_i$ and a given box $G = Q \times I$, we obtain that for each $\rho > 0$, there exist a function $\omega =$

$(\varphi, \psi) \in C_c^\infty(Q \times I; \mathbb{R}^{1+n})$ and an open set $G_\rho \subset\subset Q \times I$ satisfying the following conditions:

$$\begin{cases} (1) \quad \operatorname{div} \psi = 0 \text{ in } Q \times I, \\ (2) \quad |(Q \times I) \setminus G_\rho| < \rho; \ \ \xi_i + \nabla\omega(z) \in \{\xi_{i,-}^\tau, \ \xi_{i,+}^\tau\} \text{ for all } z \in G_\rho, \\ (3) \quad \xi_i + \nabla\omega(z) \in [\xi_{i,-}^\tau, \xi_{i,+}^\tau]_\rho \text{ for all } z \in Q \times I, \\ (4) \quad \|\omega\|_{L^\infty(Q \times I)} < \rho, \\ (5) \quad \int_Q \varphi(x,t)\, dx = 0 \text{ for all } t \in I, \\ (6) \quad \|\varphi_t\|_{L^\infty(Q \times I)} < 2\rho, \end{cases} \tag{5.6}$$

where $[\xi_{i,-}^\tau, \xi_{i,+}^\tau]_\rho$ denotes the ρ-neighborhood of closed line segment $[\xi_{i,-}^\tau, \xi_{i,+}^\tau]$. Here, from (5.6.3), (5.6.6) follows as

$$|\varphi_t| < |c_{i,+} - c_{i,-}| + \rho = (t_{i,+} - t_{i,-})|b_i| + \rho < 2\rho \quad \text{in } Q \times I.$$

Note (a), (c), (e), and (f) follow from (5.6), where 2ρ in (5.6.6) can be adjusted to ρ as in (f). By the uniform continuity of A on $J = \{p' \in \mathbb{R}^n \mid |p'| \leq s_2^*\}$, we can find a $\delta' > 0$ such that $|A(p') - A(p'')| < \frac{\epsilon}{3|\tilde{Q} \times \tilde{I}|}$ whenever p', $p'' \in J$ and $|p' - p''| < \delta'$. We then choose a $\tau > 0$ so small that

$$C\tau < \delta', \ \ C|\tilde{Q} \times \tilde{I}|\tau < \frac{\epsilon}{3}.$$

Next, we choose a $\delta > 0$ such that $\delta < \frac{d_\tau}{2}$. If $0 < \rho < \delta$, then by (5.6.1) and (5.6.3), for all $z \in Q \times I$ and $|(p', \beta') - (p, \beta)| \leq \delta$,

$$\xi_i(p', \beta') + \nabla\omega(z) \in \Sigma_0, \quad \operatorname{dist}(\xi_i(p', \beta') + \nabla\omega(z), H_\tau) < d_\tau,$$

and so $\xi_i(p', \beta') + \nabla\omega(z) \in R(F_0)$, that is, $(p' + D\varphi(z), \beta' + \psi_t(z)) \in \mathcal{S}$. Thus (b) holds for all $0 < \rho < \delta$. In particular, $(p + D\varphi(z), \beta + \psi_t(z)) \in \mathcal{S}$ and so $|p + D\varphi(z)| \leq s_+(r_2) < s_2^*$ and $|\beta + \psi_t(z)| \leq r_2$ for all $z \in Q \times I$, by (i) of Theorem 5.1.5. Thus

$$
\begin{aligned}
&\int_{Q\times I} |\beta + \psi_t - A(p + D\varphi)|dz \\
&\qquad \leq \int_{G_\rho} |\beta + \psi_t - A(p + D\varphi)|dz + (r_2 + M_\sigma)\rho \\
&\qquad \leq |Q \times I| \max\{|\beta^\tau_{i,\pm} - A(p^\tau_{i,\pm})|\} + (r_2 + M_\sigma)\rho \\
&\qquad \leq C|Q \times I|\tau + |Q \times I| \max\{|A(p_{i,\pm}) - A(p^\tau_{i,\pm})|\} + (r_2 + M_\sigma)\rho \\
&\qquad \leq \frac{2\epsilon|Q \times I|}{3|\tilde{Q} \times \tilde{I}|} + (r_2 + M_\sigma)\rho,
\end{aligned}
$$

where $\xi^\tau_{i,\pm} = \begin{pmatrix} p^\tau_{i,\pm} & c^\tau_{i,\pm} \\ B^\tau_{i,\pm} & \beta^\tau_{i,\pm} \end{pmatrix}$ and $M_\sigma = \sigma(s_2^*)$. Thus, (d) holds for all $\rho > 0$ satisfying $(r_2 + M_\sigma)\rho < \frac{\epsilon|Q\times I|}{3|\tilde{Q}\times\tilde{I}|}$.

We have verified (a) – (f) for any $(p, \beta) \in \bar{\mathcal{B}}_i$ and $1 \leq i \leq N$, where $\delta > 0$ is independent of the index i. Since $\mathcal{B}_1, \cdots, \mathcal{B}_N$ cover $\mathcal{K}$, the proof is now complete. $\square$

5.3 Construction of admissible set $\mathcal{U}$

We first construct a suitable boundary function $\Phi = (u^*, v^*) \in W^{1,\infty}(\Omega_T; \mathbb{R}^{1+n})$. Assume Ω and u_0 satisfy (2.10) and $|Du_0(x_0)| \in (s_1^*, s_2^*)$ for some $x_0 \in \Omega$. Let $\Omega_T = \Omega \times (0, T)$ for a given $T > 0$ and $M_0 = \|Du_0\|_{L^\infty(\Omega)}$. Recall that we assume (2.1); hence,

$$\int_\Omega u_0(x)\, dx = 0. \tag{5.7}$$

Note $M_0 \geq |Du_0(x_0)| > s_1^*$. We now assume the following: If $M_0 < s_1$, we fix any $\sigma(s_2) < r_2 < \sigma(M_0) < \sigma(s_1)$. If $M_0 \geq s_1$, we fix any $\sigma(s_2) < r_2 < \sigma(s_1)$. Then let $l_2 = l_{r_2} \in (\sigma(s_2), r_2)$ be some number determined by Theorem 5.1.5. Now, fix any $r_1 \in (l_2, r_2)$.

With these numbers r_1, r_2, we apply Lemma 3.2.2 to determine functions $\tilde{\sigma}$, $\tilde{f} \in C^3([0,\infty))$ satisfying its conclusion. Also, let $\tilde{A}(p) = \tilde{f}(|p|^2)p$ ($p \in \mathbb{R}^n$). Then:

Lemma 5.3.1. *We have*

$$(p, \tilde{A}(p)) \in \mathcal{S} \quad \forall\, s_-(r_1) < |p| < s_+(r_2),$$

where $\mathcal{S} = \mathcal{S}_{r_1,r_2}$ is the set in Lemma 5.1.2.

Proof. Let $s = |p|$, $r = \tilde{\sigma}(s)$ and $\zeta = p/|p|$, so that $s_-(r_1) < s < s_+(r_2)$, $\zeta \in \mathbb{S}^{n-1}$ and $\tilde{A}(p) = r\zeta$. By Lemma 3.2.2, $s_-(r) < s < s_+(r)$ and $r_1 < r < r_2$. Set $p_\pm = s_\pm(r)\zeta$ and $\beta_\pm = r\zeta$. Then $A(p_\pm) = r\zeta = \beta_\pm$. Define $\xi = \begin{pmatrix} p & 0 \\ O & \tilde{A}(p) \end{pmatrix}$ and $\xi_\pm = \begin{pmatrix} p_\pm & 0 \\ O & \beta_\pm \end{pmatrix}$. Then $\xi = \lambda\xi_+ + (1-\lambda)\xi_-$ for some $0 < \lambda < 1$. Since $\xi_\pm \in F_\pm$ and $\operatorname{rank}(\xi_+ - \xi_-) = 1$, it follows from the definition of $R(F_0) = R(F_{r_1,r_2}(0))$ that $\xi \in (\xi_-, \xi_+) \subset R(F_0)$. Thus, by Lemma 5.1.2, we have $(p, \tilde{A}(p)) \in \mathcal{S}$. $\square$

By Lemma 3.2.2, equation $u_t = \operatorname{div}(\tilde{A}(Du))$ is uniformly parabolic. So by Theorem 3.1.1, the initial-Neumann boundary value problem

$$\begin{cases} u_t^* = \operatorname{div}(\tilde{A}(Du^*)) & \text{in } \Omega_T \\ \partial u^*/\partial \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T) \\ u^*(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{5.8}$$

admits a unique classical solution $u^* \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_T)$.

Note here that we may not have the gradient maximum principle (3.4) for the solution u^* since we do not assume the convexity of Ω in Case II: Höllig type equations. However, in the case that Ω is convex, such gradient maximum principle holds, and it invokes advantageous effects on the existence result, Theorem 2.3.4, in two folds:

1. The profile $\sigma(s)$ in Hypothesis (H) can be allowed to have unbounded derivative for large values of $s > 0$.

2. Lipschitz solutions to problem (1.2) can be chosen to satisfy certain gradient estimates in terms of the initial gradient Du_0.

Despite of these advantages coming from the convexity assumption on the domain Ω, we plan not to pursue those here. Instead, we are including a larger class of domains for the existence result.

From conditions (2.10) and (5.7), we can find a function $h \in C^{2+\alpha}(\bar{\Omega})$ satisfying

$$\Delta h = u_0 \text{ in } \Omega, \quad \partial h/\partial \mathbf{n} = 0 \text{ on } \partial\Omega.$$

Let $v_0 = Dh \in C^{1+\alpha}(\bar{\Omega}; \mathbb{R}^n)$ and define, for $(x,t) \in \Omega_T$,

$$v^*(x,t) = v_0(x) + \int_0^t \tilde{A}(Du^*(x,s))\, ds. \tag{5.9}$$

Then it is easily seen that $\Phi := (u^*, v^*) \in C^1(\bar{\Omega}_T; \mathbb{R}^{1+n})$ satisfies (2.4); that is,

$$\begin{cases} u^*(x,0) = u_0(x) \ (x \in \Omega), \\ \operatorname{div} v^* = u^* \ \text{ a.e. in } \Omega_T, \\ v^*(\cdot, t) \cdot \mathbf{n}|_{\partial\Omega} = 0 \ \ \forall\, t \in [0,T]. \end{cases} \tag{5.10}$$

Hence Φ is a boundary function in the sense of Definition 2.2.2.

Next, set $M = \max\{s_2^* + 1, \|Du^*\|_{L^\infty(\Omega_T)}\}$, $r = \sigma(M)$ and define

$$\mathcal{F} = \{(p, A(p)) \mid |p| \in [0, s_-(r_1)] \cup [s_+(r_2), M]\}.$$

Then we have the following:

Lemma 5.3.2.

$$(Du^*(x,t), v_t^*(x,t)) \in \mathcal{S} \cup \mathcal{F} \quad \forall\, (x,t) \in \Omega_T.$$

Proof. Let $(x,t) \in \Omega_T$ and $p = Du^*(x,t)$; then $|p| \le M$.

If $|p| \le s_-(r_1)$ or $s_+(r_2) \le |p| \le M$, then $\tilde{A}(p) = A(p)$ and hence by (5.9)

$$(Du^*(x,t), v_t^*(x,t)) = (p, \tilde{A}(p)) = (p, A(p)) \in \mathcal{F}.$$

If $s_-(r_1) < |p| < s_+(r_2)$, then by Lemma 5.3.1 and (5.9)

$$(Du^*(x,t), v_t^*(x,t)) = (p, \tilde{A}(p)) \in \mathcal{S}.$$

Therefore $(Du^*, v_t^*) \in \mathcal{S} \cup \mathcal{F}$ in Ω_T. $\square$

Let $m = \|u^*_t\|_{L^\infty(\Omega_T)} + 1$. We finally define the admissible set $\mathcal{U}$ as follows:

$$\begin{aligned}\mathcal{U} = \Big\{ u \in C^1_{piece} \cap W^{1,\infty}_{u^*}(\Omega_T) \;\big|\; & \|u_t\|_{L^\infty(\Omega_T)} < m, \\ & \exists\, v \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T; \mathbb{R}^n) \text{ such that} \\ & \operatorname{div} v = u \text{ and } (Du, v_t) \in \mathcal{S} \cup \mathcal{F} \text{ a.e. in } \Omega_T \Big\}.\end{aligned} \tag{5.11}$$

For each $\epsilon > 0$, let $\mathcal{U}_\epsilon$ be given by

$$\begin{aligned}\mathcal{U}_\epsilon = \Big\{ u \in \mathcal{U} \mid \exists\, v \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T; \mathbb{R}^n) \text{ such that } \operatorname{div} v = u \text{ and} \\ (Du, v_t) \in \mathcal{S} \cup \mathcal{F} \text{ a.e. in } \Omega_T, \text{ and } \textstyle\int_{\Omega_T} |v_t - A(Du)| dxdt \le \epsilon |\Omega_T| \Big\}.\end{aligned}$$

Remark 5.3.3. From (5.10), Lemma 5.3.2, and the definition of $\mathcal{U}$, it follows that $u^* \in \mathcal{U}$ with its corresponding vector function v^*; so $\mathcal{U}$ is non-empty. Also $\mathcal{U}$ is a bounded subset of $W^{1,\infty}_{u^*}(\Omega_T)$ as $\mathcal{S} \cup \mathcal{F}$ is bounded. Moreover, by (i) of Theorem 5.1.5 and the definition of $\mathcal{F}$, for each $u \in \mathcal{U}$, its corresponding vector function v satisfies $\|v_t\|_{L^\infty(\Omega_T)} \le r$. Thus $\mathcal{U}$ is indeed an admissible set in the sense of Definition 2.2.3 with respect to the boundary function $\Phi = (u^*, v^*)$. Finally, note that $s_-(r_1) < |Du^*| < s_+(r_2)$ on some non-empty open subset of Ω_T, and so $\tilde{A}(Du^*) \ne A(Du^*)$ on a non-empty open subset of this set; so u^* itself is not a Lipschitz solution to (1.2). In terms of Theorem 2.2.4, it only remains to verify the density property (2.5) to obtain multiple Lipschitz solutions to problem (1.2). We complete this task in the next section.

5.4 Completion of proof of Theorem 2.3.4

Following Section 5.3, we complete the proof of Theorem 2.3.4. As mentioned in Remark 5.3.3, it only remains to prove the following density theorem. Although the proof of this density theorem is very similar to that of Theorem 4.4.1, since the ingredients towards it are coming from the current chapter, we sacrifice conciseness for the sake of completeness.

Theorem 5.4.1. *For each $\epsilon > 0$, $\mathcal{U}_\epsilon$ is dense in $\mathcal{U}$ under the L^∞-norm.*

Proof. Let $u \in \mathcal{U}$, $\eta > 0$. The goal is to construct a function $\tilde{u} \in \mathcal{U}_\epsilon$ such that $\|\tilde{u} - u\|_{L^\infty(\Omega_T)} < \eta$. For clarity, we divide the proof into several steps.

1. Note $\|u_t\|_{L^\infty(\Omega_T)} < m - \tau_0$ for some $\tau_0 > 0$ and there exists a vector function $v \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T; \mathbb{R}^n)$ such that $\operatorname{div} v = u$ and $(Du, v_t) \in \mathcal{S} \cup \mathcal{F}$ a.e. in Ω_T. Since both u and v are piecewise C^1 in Ω_T, there exists a sequence of disjoint open sets $\{G_j\}_{j=1}^\infty$ in Ω_T with $|\partial G_j| = 0$ such that

$$u \in C^1(\bar{G}_j),\ v \in C^1(\bar{G}_j; \mathbb{R}^n)\ \ \forall j \geq 1, \quad |\Omega_T \setminus \cup_{j=1}^\infty G_j| = 0.$$

2. Let $j \in \mathbb{N}$ be fixed. Note that $(Du(z), v_t(z)) \in \bar{\mathcal{S}} \cup \mathcal{F}$ for all $z = (x,t) \in G_j$ and that $H_j = \{z \in G_j \mid (Du(z), v_t(z)) \in \partial\mathcal{S}\}$ is a (relatively) closed set in G_j with measure zero. So $\tilde{G}_j = G_j \setminus H_j$ is an open subset of G_j with $|\tilde{G}_j| = |G_j|$, and $(Du(z), v_t(z)) \in \mathcal{S} \cup \mathcal{F}$ for all $z \in \tilde{G}_j$.

3. For each $\tau > 0$, let $\mathcal{G}_\tau = \{(p,\beta) \in \mathcal{S} \mid |\beta - A(p)| > \tau, \operatorname{dist}((p,\beta), \partial\mathcal{S}) > \tau)\}$; then $\mathcal{S} = (\cup_{\tau>0}\mathcal{G}_\tau) \cup \{(p,\beta) \in \mathcal{S} \mid A(p) = \beta\}$ as $\mathcal{S}$ is open. Since $A(p) = \beta\ \ \forall (p,\beta) \in \mathcal{F}$, we have

$$\int_{\tilde{G}_j} |v_t(z) - A(Du(z))|\, dz = \lim_{\tau \to 0^+} \int_{\{z \in \tilde{G}_j \mid (Du(z), v_t(z)) \in \mathcal{G}_\tau\}} |v_t(z) - A(Du(z))|\, dz;$$

thus we can find a $\tau_j > 0$ such that

$$\int_{F_j} |v_t(z) - A(Du(z))|\, dz < \frac{\epsilon}{3 \cdot 2^j} |\Omega_T| \ \text{ and } \ |\partial O_j| = 0, \tag{5.12}$$

where $F_j = \{z \in \tilde{G}_j \mid (Du(z), v_t(z)) \notin \mathcal{G}_{\tau_j}\}$ and $O_j = \tilde{G}_j \setminus F_j$ is open. Let J be the set of all indices $j \in \mathbb{N}$ with $O_j \neq \emptyset$. Then for $j \notin J$, $F_j = \tilde{G}_j$.

4. We now fix a $j \in J$. Note that $O_j = \{z \in \tilde{G}_j \mid (Du(z), v_t(z)) \in \mathcal{G}_{\tau_j}\}$ and that $\mathcal{K}_j := \bar{\mathcal{G}}_{\tau_j}$ is a compact subset of $\mathcal{S}$. Let $\tilde{Q} \subset \mathbb{R}^n$ be a box with $\Omega \subset \tilde{Q}$ and $\tilde{I} = (0, T)$. Applying Theorem 5.2.1 to box $\tilde{Q} \times \tilde{I}$, $\mathcal{K}_j \subset\subset \mathcal{S} = \mathcal{S}_{r_1, r_2}$, and $\epsilon' = \frac{\epsilon |\Omega_T|}{12}$, we obtain a constant $\delta_j > 0$ that satisfies the conclusion of the theorem. By the uniform continuity of A on compact subsets of $\mathbb{R}^n$, we can find a $\theta = \theta_{\epsilon, s_2^*} > 0$ such that

$$|A(p) - A(p')| < \frac{\epsilon}{12} \tag{5.13}$$

whenever $|p|$, $|p'| \leq 2s_2^*$ and $|p - p'| \leq \theta$. Also by the uniform continuity of u, v, and their gradients on $\bar{G}_j$, there exists a $\nu_j > 0$ such that

$$\begin{gathered} |u(z) - u(z')| + |\nabla u(z) - \nabla u(z')| + |v(z) - v(z')| \\ + |\nabla v(z) - \nabla v(z')| < \min\{\tfrac{\delta_j}{2}, \tfrac{\epsilon}{12}, \theta, s_2^*\} \end{gathered} \tag{5.14}$$

whenever $z, z' \in \bar{G}_j$ and $|z - z'| \leq \nu_j$. We now cover O_j (up to measure zero) by a sequence of disjoint open cubes $\{Q_j^i \times I_j^i\}_{i=1}^{\infty}$ in O_j whose sides are parallel to the axes with center z_j^i and diameter $l_j^i < \nu_j$.

5. Fix an $i \in \mathbb{N}$ and write $w = (u, v)$, $\xi = \begin{pmatrix} p & c \\ B & \beta \end{pmatrix} = \nabla w(z_j^i) = \begin{pmatrix} Du(z_j^i) & u_t(z_j^i) \\ Dv(z_j^i) & v_t(z_j^i) \end{pmatrix}$.

By the choice of $\delta_j > 0$ in Step 4 via Theorem 5.2.1, since $Q^i_j \times I^i_j \subset \tilde{Q} \times \tilde{I}$ and $(p, \beta) \in \mathcal{K}_j$, for all sufficiently small $\rho > 0$, there exists a function $\omega^i_j = (\varphi^i_j, \psi^i_j) \in C^\infty_c(Q^i_j \times I^i_j; \mathbb{R}^{1+n})$ satisfying

(a) $\operatorname{div} \psi^i_j = 0$ in $Q^i_j \times I^i_j$,

(b) $(p' + D\varphi^i_j(z), \beta' + (\psi^i_j)_t(z)) \in \mathcal{S}$ for all $z \in Q^i_j \times I^i_j$ and all $|(p', \beta') - (p, \beta)| \le \delta_j$,

(c) $\|\omega^i_j\|_{L^\infty(Q^i_j \times I^i_j)} < \rho$,

(d) $\int_{Q^i_j \times I^i_j} |\beta + (\psi^i_j)_t(z) - A(p + D\varphi^i_j(z))| dz < \epsilon' |Q^i_j \times I^i_j| / |\tilde{Q} \times \tilde{I}|$,

(e) $\int_{Q^i_j} \varphi^i_j(x,t) dx = 0$ for all $t \in I^i_j$,

(f) $\|(\varphi^i_j)_t\|_{L^\infty(Q^i_j \times I^i_j)} < \rho$.

Here, we let $0 < \rho \le \min\{\tau_0, \frac{\delta_j}{2C}, \frac{\epsilon}{12C}, \eta\}$, where C_n is the constant in Theorem 3.3.1 and C is the product of C_n and the sum of the lengths of all sides of $\tilde{Q}$. By (e), we can apply Theorem 3.3.1 to φ^i_j on $Q^i_j \times I^i_j$ to obtain a function $g^i_j = \mathcal{R}\varphi^i_j \in C^1(\overline{Q^i_j \times I^i_j}; \mathbb{R}^n) \cap W^{1,\infty}_0(Q^i_j \times I^i_j; \mathbb{R}^n)$ such that $\operatorname{div} g^i_j = \varphi^i_j$ in $Q^i_j \times I^i_j$ and

$$\|(g^i_j)_t\|_{L^\infty(Q^i_j \times I^i_j)} \le C \|(\varphi^i_j)_t\|_{L^\infty(Q^i_j \times I^i_j)} \le \frac{\delta_j}{2}. \text{ (by (f))} \tag{5.15}$$

6. As v_t and $A(Du)$ are essentially bounded in Ω_T, we can select a finite index set $\mathcal{I} \subset J \times \mathbb{N}$ so that

$$\int_{\bigcup_{(j,i) \in (J \times \mathbb{N}) \setminus \mathcal{I}} Q^i_j \times I^i_j} |v_t(z) - A(Du(z))| dz \le \frac{\epsilon}{3} |\Omega_T|. \tag{5.16}$$

We finally define

$$(\tilde{u}, \tilde{v}) = (u, v) + \sum_{(j,i)\in\mathcal{I}} \chi_{Q_j^i\times I_j^i}(\varphi_j^i, \psi_j^i + g_j^i) \quad \text{in } \Omega_T.$$

7. Let us finally check that $\tilde{u}$ together with $\tilde{v}$ indeed gives the desired result. By construction, it is clear that $\tilde{u} \in C^1_{piece} \cap W^{1,\infty}_{u^*}(\Omega_T)$, $\tilde{v} \in C^1_{piece} \cap W^{1,\infty}_{v^*}(\Omega_T; \mathbb{R}^n)$. By the choice of ρ in (f) as $\rho \le \tau_0$, we have $\|\tilde{u}_t\|_{L^\infty(\Omega_T)} < m$. Next, let $(j, i) \in \mathcal{I}$, and observe that for $z \in Q_j^i \times I_j^i$, with $(p, \beta) = (Du(z_j^i), v_t(z_j^i)) \in \mathcal{G}_{\tau_j}$, since $|z - z_j^i| < l_j^i < \nu_j$, it follows from (5.14) and (5.15) that

$$|(Du(z), v_t(z) + (g_j^i)_t(z)) - (p, \beta)| \le \delta_j,$$

and so $(D\tilde{u}(z), \tilde{v}_t(z)) \in \mathcal{S}$ from (b) above. From (a) and $\operatorname{div} g_j^i = \varphi_j^i$, for $z \in Q_j^i \times I_j^i$,

$$\operatorname{div} \tilde{v}(z) = \operatorname{div}(v + \psi_j^i + g_j^i)(z) = u(z) + 0 + \varphi_j^i(z) = \tilde{u}(z).$$

Therefore, $\tilde{u} \in \mathcal{U}$. Next, observe

$$\int_{\Omega_T} |\tilde{v}_t - A(D\tilde{u})|dz = \int_{\cup_{j\in\mathbb{N}} F_j} |v_t - A(Du)|dz$$

$$+ \int_{\cup_{(j,i)\in(J\times\mathbb{N})\setminus\mathcal{I}} Q_j^i\times I_j^i} |v_t - A(Du)|dz + \int_{\cup_{(j,i)\in\mathcal{I}} Q_j^i\times I_j^i} |\tilde{v}_t - A(D\tilde{u})|dz$$

$$=: I_1 + I_2 + I_3.$$

From (5.12) and (5.16), we have $I_1 + I_2 \le \frac{2\epsilon}{3}|\Omega_T|$. Note also that for $(j, i) \in \mathcal{I}$ and $z \in Q_j^i \times I_j^i$, from (5.14), (5.15), and (f),

$$|\tilde{v}_t(z) - A(D\tilde{u}(z))| = |v_t(z) + (\psi_j^i)_t(z) + (g_j^i)_t(z) - A(Du(z) + D\varphi_j^i(z))|$$

$$\leq |v_t(z) - v_t(z_j^i)| + |v_t(z_j^i) + (\psi_j^i)_t(z) - A(Du(z_j^i) + D\varphi_j^i(z))|$$

$$+|(g_j^i)_t(z)| + |A(Du(z_j^i) + D\varphi_j^i(z)) - A(Du(z) + D\varphi_j^i(z))|$$

$$\leq \frac{\epsilon}{6} + |v_t(z_j^i) + (\psi_j^i)_t(z) - A(Du(z_j^i) + D\varphi_j^i(z))|$$

$$+|A(Du(z_j^i) + D\varphi_j^i(z)) - A(Du(z) + D\varphi_j^i(z))|.$$

From (i) of Theorem 5.1.5 and (5.14), we have $|Du(z_j^i)+D\varphi_j^i(z)| \leq 2s_2^*$. As $(D\tilde{u}(z), \tilde{v}_t(z)) \in \mathcal{S}$, we also have $|Du(z) + D\varphi_j^i(z)| = |D\tilde{u}(z)| \leq s_2^*$, and by (5.14), $|Du(z_j^i) - Du(z)| < \theta$. From (5.13) we thus have

$$|A(Du(z_j^i) + D\varphi_j^i(z)) - A(Du(z) + D\varphi_j^i(z))| < \frac{\epsilon}{12}.$$

Integrating the above inequality over $Q_j^i \times I_j^i$, we now obtain from (d) that

$$\int_{Q_j^i \times I_j^i} |\tilde{v}_t(z) - A(D\tilde{u}(z))|dz \leq \frac{\epsilon}{4}|Q_j^i \times I_j^i| + \frac{\epsilon|\Omega_T|}{12}\frac{|Q_j^i \times I_j^i|}{|\tilde{Q} \times \tilde{I}|} \leq \frac{\epsilon}{3}|Q_j^i \times I_j^i|,$$

which yields that $I_3 \leq \frac{\epsilon}{3}|\Omega_T|$. Hence $I_1 + I_2 + I_3 \leq \epsilon|\Omega_T|$, and so $\tilde{u} \in \mathcal{U}_\epsilon$. Lastly, from (c) with $\rho \leq \eta$ and the definition of $\tilde{u}$, we have $\|\tilde{u} - u\|_{L^\infty(\Omega_T)} < \eta$.

The proof is now complete.

$\square$

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