THE ORTHOGONAL POLYNOMIALS ASSOCIATED WITH THE ITERATION OF A SECOND DEGREE POLYNOMIAL

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ABSTRACT

THE ORTHOGONAL POLYNOMIALS ASSOCIATED WITH
THE ITERATION OF A SECOND DEGREE POLYNOMIAL

By

Daniel Arac Nussbaum

We consider the polynomial $P(z) = z^2 - p$ and its iterates $P_n(z)$ defined by $P_n(z) = P(P_{n-1}(z))$. Define F as the set of points for which $\{P_n(z)\}$ is not a normal family and μ as the equilibrium distribution for F. These definitions are due to Brolin [1].

Kinney and Pitcher have shown in [4] that the family of iterates of $P(z)$ form an orthogonal family over F with respect to the measure μ in the sense that $\int_F Q_n(z)\,\overline{Q_m(z)}\,d\mu(z) = 0$ if $n \neq m$. Since $P_n(z)$ has degree 2^n, we have a lacunary set of orthogonal polynomials. In this paper we attempt to complete the set by finding polynomials $Q_n(z)$ of degree n, with leading coefficient 1, for which $\int_F Q_n(z)\,\overline{Q_m(z)}\,d\mu(z) = 0$ for $n \neq m$ and so that $Q_{2^n}(z) = P_n(z)$. As a matter of convenience we introduce $z^{\langle n\rangle}$ where $z^{\langle n\rangle} = \prod_{i=0}^{\infty} (P_i(z))^{\varepsilon(i,n)}$, where $\sum_{i=0}^{\infty} \varepsilon(i,n)2^i$ is the binary expansion of n.

We achieve a matrix representation for $Q_n(z)$ and obtain results about the matrix entries. Specifically, we obtain a closed form for calculating the matrix entries.

We show that the $Q_n(z)$ satisfy a linear difference equation which depends only on n, $n-1$, $n-2$ and a constant, $K(n)$, which depends only on n and p. In the case that F is a subset of the reals ($p \geq 2$ is sufficient) we obtain non-linear recursion formulas for $K(n)$. We also relate these constants to a function which represents the ratio of successive Q's.

We also show that $Q_n(z)$ can be expanded as a linear combination of $z^{<0>}, z^{<1>}, z^{<2>}, \ldots$, and we obtain results about the coefficients in this expansion.

Finally, we extend the orthogonality of the family $\{Q_n(z)\}$ to certain sets which contain F. These sets are level curves of a Green's function, and the measure we use is the normal derivative of $\int_F \log |z - w| d\mu(w) |dz|$.

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TABLE OF CONTENTS

INTRODUCTION

This paper deals with the set of polynomials which are orthogonal with respect to a certain measure.

Consider a polynomial $P(z)$ of degree N and its iterates $P_n(z)$ given by $P_n(z) = P(P_{n-1}(z))$. Brolin [1] has discussed the set of points F where the family $\{P_n(z)\}$ is not a normal family. Thus, for $z \notin F$ and for every neighborhood, N, of z, there is a subsequence of $\{P_n(z)\}$ which converges uniformly to an analytic function on compact subsets of N. Brolin shows that F is compact, contains no open set, is invariant under P and P_{-1}, and has capacity one. Thus $\exp\{-\inf(\iint_{FF} \log \frac{1}{|z-w|} d\nu(z)d\nu(w))\} = 1$ where the infimum is taken over all measures ν for which $\int_F d\nu = 1$. We denote the measure which minimizes the above energy integral by μ. It turns out that μ is invariant under P_{-1} (the inverse of P). That is, $d\mu(z) = d\mu(P_{-1}(z))$.

In [4], Kinney and Pitcher show that $\{P_n\}$ form an orthogonal family in the sense that $\int_F P_n(z)\, \overline{P_m(z)}\, d\mu(z) = 0$ if n and m are unequal.

In this paper we seek methods to complete this lacunary set of orthogonal functions. That is, we seek polynomials, $Q_n(z)$, with leading coefficient 1 so that $Q_{N^n}(z) = P_n(z)$ and $\int Q_n(z)\, \overline{Q_m(z)}\, d\mu(z) = 0$ for $n \neq m$.

While we could attempt the above problem for an arbitrary polynomial $P(z)$, for the sake of brevity we restrict our attention to the case $N = 2$. For reasons given in Brolin [1] and Kinney-Pitcher [4], we may take $P(z) = z^2 - p$. We know from Brolin [1] that F is a real set if $p \geq 2$.

In the first chapter we derive a representation of our orthogonal polynomials. Using a technique developed by Walsh [9] (see also Fine [2]) in his completion of the Rademacher functions, we introduce $z^{\langle n \rangle} = \prod_{i=0}^{\infty} (P_i(z))^{\epsilon(i,n)}$, where $n = \sum_{i=0}^{\infty} \epsilon(i,n)2^i$ is the binary expansion of n. The powers $z^{\langle n \rangle}$ give rise to an inner product matrix, $\mathcal{C}$, where $c(m,n) = \int_F z^{\langle m \rangle} \overline{z^{\langle n \rangle}} d\mu(z)$ for $m,n = 0,1,\ldots$.

We derive some results about the matrix $\mathcal{C}$. In particular we find a closed form for evaluating $c(m,n)$ (Theorem 1.12), and a result about entries $c(m,n)$ where $m + n$ is constant (Theorem 1.36).

We use $c(m,n)$ to construct a matrix representation of our orthogonal polynomials (Theorem 1.57) essentially as Szego does in [5] and [7]. We do not use true moments in our matrix, and we do not use arc length in our measure until Chapter IV.

In Chapter II we approach the problem of completing the orthogonal family differently. We find that the $Q_n(z)$'s satisfy a linear difference equation (Theorem 2.8), and that the coefficients in the equation satisfy non-linear recursion formulae (Theorem 2.9) in the case that F is a subset of the reals. We also obtain continued fraction representations for the coefficients (Theorem 2.13), and a link between the methods of Chapters I and II (Theorem 2.16).

In Chapter II we also consider $T_n(z)$, the ratio of $Q_n(z)$ to $Q_{n-1}(z)$ and show it is related to the coefficients of the linear difference equation (Lemma 2.20).

In Chapter III we construct $Q_n(z)$ as a linear combination of the $z^{<n>}$'s. We establish recursion formulae among the coefficients and in so doing we link this method with Chapter II.

In Chapter IV we extend the orthogonality of the family $\{Q_n(z)\}$ to sets other than F. Specifically we show in Theorem 4.9 that $Q_n(z)$ and $Q_m(z)$ are orthogonal on the level curves of a Green's function with respect to the measure $\frac{d}{dn}\int_F \log|z - w|d\mu(w)\cdot|dz|$.

Using the results of the previous chapters, we present two examples. The first, $P(z) = z^2$, for which the F set is the unit circle.

The second example we present is $P(z) = z^2 - 2$ for which the real interval [-2,2] is the F set. We show that the Tchebysheff polynomials $T_n(z) = 2 \operatorname{Cos} n \operatorname{Cos}^{-1} z/2$ are orthogonal on [-2,2] with respect to μ, and we derive a 1934 result of Walsh's on the orthogonality of Tchebysheff polynomials on certain confocal ellipses.

CHAPTER I

Let $P(z) = z^2 - p$ and its iterates $\{P_n(z)\}$ be defined as follows: $P_0(z) = z$, $P_1(z) = P(z)$, and the iterates $P_n(z)$ are recursively defined by

$$(1.1) \qquad P_n(z) = P(P_{n-1}(z)).$$

Throughout, F denotes the set of points in the plane at which $\{P_n(z)\}$ is not a normal family. Brolin shows in [1] that when $P(z)$ is as above then F is compact, has capacity one, is invariant under $P(z)$ and $P_{-1}(z)$, and often has 2-dimensional measure zero.

As Brolin does in [1], we introduce the measure μ on F as follows:

(1.2) Let z_0 be any point in the plane with at most 2 exceptions. We define μ_n to be the discrete measure which places weight 2^{-n} at each of the roots $P_n(z) = z_0$; that is, at the 2^n points $\pm\sqrt{p \pm \sqrt{p \pm \cdots \pm \sqrt{p + z_0}}}$. Then μ_n converges weakly to μ, where μ is independent of z_0 [1; Theorem 16.1]. That is, for f continuous and zero outside a compact set containing F, $\lim_{n\to\infty} \int_F f \, d\mu_n = \int_F f \, d\mu$. We know that μ is concentrated on F and $\int_F d\mu = 1$, [1]. Moreover, μ is symmetric with respect to the origin by its construction. Thus for $f(z)$ an odd function of z, (1.3) $$\int_F f(z)d\mu(z) = 0 .$$

It is known [4; pg. 25] that μ is invariant under $P_{-1}(z)$. Together with the invariance of F under $P_{-1}(z)$, this invariance yields the following relation:

(1.4) $$\int_F f(z)d\mu(z) = \int_F f(P_{-1}(z))d\mu(z) .$$

We will use this "shift" extensively.

Walsh [9] completes the Rademacher functions with what have come to be known as Walsh (or Walsh-Rademacher) functions. His technique, described also by Fine in [2], leads us to make the following definition: for $n = \sum_{i=0}^{\infty} \varepsilon(i,n)2^i$, where $\varepsilon(i,n) = 0$ or 1 we take

(1.5) $$z^{<n>} = \prod_{i=0}^{\infty} (P_i(z))^{\varepsilon(i,n)} .$$

We have

(1.6) $$(P(z))^{<n>} = z^{<2n>} .$$

When we multiply $z^{<n>}$ by z we obtain:

<u>Lemma 1.7</u>.

(i) If $\varepsilon(0,n) = 0$, then $z\cdot z^{<n>} = z^{<n+1>}$.

(ii) If $\varepsilon(0,n) = \ldots = \varepsilon(s,n) = 1$ and $\varepsilon(s+1,n) = 0$ where $s \geq 0$, then $z\cdot z^{<n>} = z^{<n+1>} + p \sum_{k=1}^{s+1} z^{<n+1-2^k>}$.

<u>Proof</u>. By definition, $z\cdot z^{<n>} = z \prod_{k=0}^{\infty} (P_k(z))^{\varepsilon(k,n)}$, which by

assumption equals $z \prod_{k=1}^{\infty} (P_k(z))^{\epsilon(k,n)}$. Since $P_0(z) = z$, we have $z \cdot z^{<n>} = P_0(z) \prod_{k=1}^{\infty} (P_k(z))^{\epsilon(k,n)}$, which equals $z^{<n+1>}$. Thus, (i) is true.

To prove (ii), we have $z \cdot z^{<n>} = \lambda z \prod_{k=0}^{s} P_k(z)$, where $\lambda = \prod_{k=s+2}^{\infty} (P_k(z))^{\epsilon(k,n)}$. Since $P_0(z) = z$, $z \cdot z^{<n>} = \lambda \prod_{k=1}^{s} z^2 P_k(z)$. Since $z^2 = P(z) + p$, $z \cdot z^{<n>} = \lambda \prod_{k=1}^{s} (P(z) + p) P_k(z) = \lambda \prod_{k=2}^{\infty} P_1^2(z) P_k(z) + p z^{<n-1>}$. Since $P_1^2(z) = P_2(z) + p$,

$$\begin{aligned} z \cdot z^{<n>} &= \lambda \prod_{k=2}^{\infty} (P_2(z) + p) P_k(z) + p\, z^{<n-1>} \\ &= \lambda \prod_{k=3}^{\infty} P_2^2(z) P_k(z) + p\, z^{<n-3>} + p\, z^{<n-1>} . \end{aligned}$$

Repeating this argument s times, we have

$$\begin{aligned} z \cdot z^{<n>} &= \lambda \;\; P_s^2(z) + p \sum_{k=0}^{s-1} z^{<n-2^k>} \\ &= (P_{s+1}(z) + p)\lambda + p \sum_{k=0}^{s-1} z^{<n-2^k>} \qquad (1.8) \\ &= z^{<n+1>} + p \sum_{k=0}^{s-1} z^{<n-2^k>} . \end{aligned}$$

This proves (ii).

It will be necessary to form inner products of these "powers". We make the following definition:

<u>Definition</u>. For $m,n = 0,1,\ldots$

$$c(m,n) = \langle z^{<m>}, z^{<n>} \rangle = \int_F z^{<m>} \overline{z^{<n>}} \, d\mu(z). \qquad (1.9)$$

It follows immediately from (1.9) that $c(m,n) = \overline{c(n,m)}$. We would like to have a procedure for computing $c(m,n)$. To that end, we assume that $p \geq 2$. Then F is a subset of the real axis [1], μ is a symmetric real measure, $c(m,n) = c(n,m)$, and we have

Lemma 1.10. For $p \geq 2$,

(i) $c(2n,2m) = c(n,m)$

(ii) $c(2n+1,2m) = 0$

(iii) $c(4n+1,4m+1) = p\cdot c(n,m)$

(iv) $c(4n+1,4m+3) = c(2n+1,2m+1)$

(v) $c(4n+3,4m+3) = p\cdot c(2n+1,2m+1)$.

Proof. The first assertion follows from (1.4). Since $c(2n+1,2m) = \langle z^{\langle 2n\rangle}\cdot z, z^{\langle 2m\rangle}\rangle$, the integrand is odd. Thus, applying (1.3) gives $c(2n+1,2m) = 0$, so the second assertion follows.

Since $c(4n+1,4m+1) = \langle z^{\langle 4n\rangle}\cdot z,\ z^{\langle 4m\rangle}\cdot z\rangle = \langle z^2\cdot z^{\langle 4n\rangle}, z^{\langle 4m\rangle}\rangle$, we use $z^2 = P(z) + p$ to get $c(4n+1,4m+1) = \langle P(z)z^{\langle 4n\rangle}, z^{\langle 4m\rangle}\rangle + p\langle z^{\langle 4n\rangle}, z^{\langle 4m\rangle}\rangle$. Using (1.7) and (1.3) we see that the first inner product equals 0. Applying (1.7) twice we see that the second inner product equals $p\cdot c(n,m)$. Thus, (iii) is proved.

Since $c(4n+1,4m+3) = \langle z^2\cdot z^{\langle 4n\rangle}, z^{\langle 4m+2\rangle}\rangle$, we may use $P(z) + p = z^2$ to obtain $c(4n+1,4m+3) = \langle P(z)z^{\langle 4n\rangle}, z^{\langle 4m+2\rangle}\rangle + p\langle z^{\langle 4n\rangle}, z^{\langle 4m+2\rangle}\rangle$. Using (1.7) and (1.4), we see that the first inner product is $\langle z\, z^{\langle 2n\rangle}, z^{\langle 2m+1\rangle}\rangle$. Applying Lemma 1.8 we see that this is $\langle z^{\langle 2n+1\rangle}, z^{\langle 2m+1\rangle}\rangle$, which is $c(2n+1,2m+1)$. From (1.4) we see that $p\langle z^{\langle 4n\rangle}, z^{\langle 4m+2\rangle}\rangle = p\langle z^{\langle 2n\rangle}, z^{\langle 2m+1\rangle}\rangle$, which is 0 from (1.3). Thus (iv) follows.

To show (v) we use $P(z) + p = z^2$ to get $c(4n+3,4m+3) = \langle P(z)z^{\langle 4n+2\rangle}, z^{\langle 4m+2\rangle}\rangle + p\, c(4n+2,4m+2)$. Using (1.7) and (1.4) we see that the first inner product is $\langle z\cdot z^{\langle 2n+1\rangle}, z^{\langle 2m+1\rangle}\rangle = \langle z^3\, z^{\langle 2n\rangle}, z^{\langle 2m\rangle}\rangle$, which is 0, by using (1.3). Thus (v) follows.

Before proceeding to the main theorem on $c(m,n)$, we need the following definition:

<u>Definition 1.11</u>. For $i,n,m = 0,1,\ldots$

$$A(i;\, n,m) = (\varepsilon(i+1,n),\ \varepsilon(i,n),\ \varepsilon(i+1,m),\ \varepsilon(i,m))$$

and $f(i;\, n,m) = f(A(i;\, n,m)) = 0$ if $A(i;\, n,m)$ $= (0,0,1,0)$ or $(1,0,0,0)$ and equals 1 otherwise.

We may now state

<u>Theorem 1.12</u>. For $p \geq 2$,

$$c(n,m) = \prod_{i=0}^{\infty} f(i;\, n,m)\cdot p^{\sum_{i=0}^{\infty} \varepsilon(i,n)\varepsilon(i,m)}$$

if $m+m$ is even and 0 if $n+m$ is odd.

<u>Proof</u>. We may assume that $m \leq n$ since $c(m,n) = c(n,m)$.

If $n+m$ is odd the theorem follows from Lemma 1.10, part 2.

For $n+m$ even, we use induction on n. For $n = 0$, $c(0,0) = 1$. For $n = 1$, $c(1,1) = \int_F z^2 d\mu = \int (P(z) + p)d\mu =$ $\int z\, d\mu + p\int d\mu = p$, by applying (1.4) to the first integral, which is 0 by (1.3). We now assume that the theorem is true for $n \leq k$. For $n = k+1$ we distinguish four cases. Case (i). For $n = 2s$ and $m = 2t$ we have $\varepsilon(0,n) = \varepsilon(0,m) = 0$, and for $k \geq 0$ we have $\varepsilon(k+1,n) = \varepsilon(k,s)$, $\varepsilon(k+1,m) = \varepsilon(k+1,t)$ and $f(k+1;\, n,m) = f(k;\, s,t)$. If we apply Lemma 1.10, part 1 we obtain

$$\begin{aligned} c(n,m) &= \prod_{i=0}^{\infty} f(i;\, s,t)\cdot p^{\sum_{i=0}^{\infty} \varepsilon(i,s)\cdot\varepsilon(i,t)} \\ &= \prod_{i=1}^{\infty} f(i;\, n,m)\ p^{\sum_{i=1}^{\infty} \varepsilon(i,n)\cdot\varepsilon(i,m)} \end{aligned}$$

.

If $\varepsilon(1,n) \neq \varepsilon(1,m)$, then $f(0;\, n,m) = 0$ and $s + t$ is odd. The theorem holds in this case. On the other hand, if $\varepsilon(1,n) = \varepsilon(1,m)$, then $f(0;\, n,m) = 1$ and $c(n,m) = c(s,t)$

$$= \prod_{i=0}^{\infty} f(i;\, n,m)\, p^{\sum_{i=0}^{\infty} \varepsilon(i,n)\cdot\varepsilon(i,m)} .$$

Case (ii). If $n = 4s + 1$ and $m = 4t + 1$, then $\varepsilon(0,n) = \varepsilon(1,n) = 1 = f(0;\, n,m)$, and for $k \geq 0$ we have $\varepsilon(k,2s) = \varepsilon(k+1,n)$, $\varepsilon(k,2t) = \varepsilon(k+1,m)$ and $f(k;\, 2s,2t) = f(k+1;\, n,m)$. Using Lemma 1.10, part 3, we obtain

$$c(n,m) = p\; c(2s,2t) = p \prod_{i=1}^{\infty} f(i;\, n,m)\, p^{\sum_{i=1}^{\infty} \varepsilon(i,n)\varepsilon(i,m)}$$

$$= p \prod_{i=0}^{\infty} f(i;\, n,m)\, p^{-1} p^{\sum_{i=0}^{\infty} \varepsilon(i,n)\cdot\varepsilon(i,m)} ,$$

and the theorem holds in this case.

Case (iii). If $n = 4s + 1$ and $m = 4t + 3$ then

$$1 = \varepsilon(0,2s+1) = \varepsilon(0,n) = \varepsilon(0,2t+1) = \varepsilon(0,m) = \varepsilon(1,m),$$

while $\varepsilon(1,n) = 0$, $f(0;\, n,m) = f(1;\, n,m) = f(0,2s+1,2t+1) = 1$, and for $k \geq 1$ $\varepsilon(k,2t+1) = \varepsilon(k+1,m)$ and $f(k;\, 2s+1,2t+1) = f(k+1;\, n,m)$. Thus, using Lemma 1.10, part 4, we obtain

$$c(n,m) = c(2s+1,2t+1) = \prod_{i=2}^{\infty} f(i;\, n,m)\, p\; p^{\sum_{i=1}^{\infty} \varepsilon(i,n)\varepsilon(i,m)}$$

$$= \prod_{i=0}^{\infty} f(i;\, n,m)\, p^{\sum_{i=0}^{\infty} \varepsilon(i,n)\varepsilon(i,m)} ,$$

and the theorem is valid for this case.

Case (iv). If $n = 4s + 3$ and $m = 4t + 3$, then $\varepsilon(0,n) = \varepsilon(1,n) = \varepsilon(0,m) = \varepsilon(1,m) = f(0;\, n,m) = 1$, and for $k \geq 0$ we have $\varepsilon(k,2s+1) =$

$\varepsilon(k+1,n)$, $\varepsilon(k,2t+1) = \varepsilon(k+1,m)$, and $f(k;\ 2s+1,2t+1) = f(k+1;\ n,m)$. Thus, from Lemma 1.10, part 5, we obtain

$$c(n,m) = p\ c(2s+1,2t+1) = p \prod_{k=1}^{\infty} f(k;\ n,m) p^{\sum_{k=1}^{\infty} \varepsilon(k,n)\varepsilon(k,m)}$$

$$= p \prod_{k=0}^{\infty} f(k;\ n,m) p^{-1} p^{\sum_{k=0}^{\infty} \varepsilon(k,n)\varepsilon(k,m)} ,$$

and the theorem holds in this case. This completes the proof of the theorem.

The following results follow from the above theorem:

<u>Corollary 1.13</u>. If $p \geq 2$, then

(i) For $k \neq 0$, $c(k,0) = 0$.

(ii) For $0 \leq k \leq 2^n + 2^{n+1}$, $c(2^n,k) = 0$ unless $k = 2^n$, in which case $c(2^n,2^n) = p$.

(iii) For $0 \leq k \leq 2^n-1$, $c(2^n - 1,k) = p^{\sum_{i=0}^{\infty} \varepsilon(i,k)}$.

(iv) For all k, $c(k,k) = p^{\sum_{i=0}^{\infty} \varepsilon(i,k)}$.

(v) For $2 \leq j \leq n < \infty$, $c(2^n - 1,\ 2^j - 1) = p^j$.

(vi) For $s \geq 0$ and for all n, $c(2^n,2^n + 2^{n+1} +\ldots+ 2^{n+s}) = p$.

(vii) For all k, except as noted in (vi) above, $c(2^n,k) = 0$.

(viii) For all n and for $k \leq 2^n$, $c(2^n + 1,k) = p$ if $k = 2^n - 1$ and 0 otherwise.

(ix) For $n < 2^{s+1} \leq m$ and $t > s+1$, $c(n,m+2^t) = c(n,m)$ or 0. Thus for n,m,s as above at $X \geq 2^{s+1}$, $c(n,m+x) = c(n,m)$ or 0.

(x) For $2m+4 < 2^n - 2$ and for $c(2^n + 2,2m) = 0$, we have $c(2^n + 2,\ 2m + 4) = 0$.

(xi) For $2m+4 < 2^n - 2$ and for $c(2^n + 2,2m) \neq 0$, we have $c(2^n + 2,\ 2m + 2) = 0$.

(xii) For $2m+4 < 2^n - 2$ and for $c(2^n - 2, 2n) \neq 0$, we have $c(2^n - 2, 2m + 2) = 0$.

We now construct the matrix of these inner products. We will use this matrix to construct the orthogonal polynomials.

Definition 1.14. $\mathcal{C}$ is the matrix whose entries are $c(m,n)$ for $m,n = 0,1,\ldots$.

In what follows, we will assume that $p \geq 2$. Thus F is real, $F \subseteq [-\frac{1}{2} - \sqrt{\frac{1}{4} + p}, +\frac{1}{2} + \sqrt{\frac{1}{4} + p}]$ [1; Theorem 12.1], and $\mathcal{C}$ is a symmetric matrix.

We note: For $0 \leq n,m \leq 2^s - 1$, we have

(1.15) $\epsilon(k,n) = \epsilon(k,n + 2^s)$ for $0 \leq k \leq s-1$

(1.16) $\epsilon(k,m) = \epsilon(k,m + 2^s)$ for $0 \leq k \leq s-1$

(1.17) $\epsilon(s,n) = \epsilon(2,m) = 0$

(1.18) $\epsilon(s,n + 2^s) = \epsilon(s,m + 2^s) = 1$.

For $2^s \leq n \leq 2^{s+1} - 1$ and $0 \leq m \leq 2^s - 1$ we have

(1.19) $\epsilon(k,n) = \epsilon(k,n + 2^s) = \epsilon(k,n + 2^{s+1})$ for $0 \leq k \leq s-1$

(1.20) $\epsilon(s,n) = \epsilon(s,n + 2^s) = \epsilon(s,n + 2^{s+1}) = \epsilon(s+1,n + 2^{s+1}) = 1$

(1.21) $\epsilon(s+1,n) = \epsilon(s,n + 2^s) = \epsilon(s,m) = \epsilon(s+1,m) = 0$.

For $2^s \leq n \leq 2^{s+1} - 1$ and $m + n \leq 2^{s+1} - 1$, we have

(1.22) $\epsilon(k,n) = \epsilon(k,n + 2^s) = \epsilon(k,n + 2^{s+1})$ for $0 \leq k \leq s-1$

(1.23) $\epsilon(s,n) = \epsilon(s,n + 2^{s+1}) = \epsilon(s+1,n + 2^s) = \epsilon(s+1,n + 2^{s+1}) = 1$

(1.24) $\epsilon(s+1,n) = \epsilon(s,n + 2^s) = \epsilon(s,m) = \epsilon(s+1,m) = 0$.

For $0 \le n \le 2^s - 1,\ 2^s \le m \le 2^{s+1} - 1$

(1.25) $\quad \varepsilon(k,n) = \varepsilon(k,n + 2^s) = \varepsilon(k,n + 2^{s+1}) = \varepsilon(k,n + 2^s + 2^{s+1})$

for $k = 0,1,\ldots,s-1$.

(1.26) $\quad \varepsilon(s,n) = \varepsilon(s+1,n) = \varepsilon(s+1,n + 2^s) = \varepsilon(s,n + 2^{s+1}) = 0$.

(1.27) $\quad \varepsilon(s,n + 2^s) = \varepsilon(s+1,n + 2^{s+1}) = \varepsilon(s,n + 2^s + 2^{s+1}) =$

$\varepsilon(s+1,n + 2^s + 2^{s+1}) = 1.$

Thus, the following formulae are valid for $s \ge 0$:

Theorem 1.28. (i) For $0 \le n,m \le 2^s - 1$, $c(n + 2^s,\ m + 2^s) = p\ c(n,m)$.

(ii-iii). For $2^s \le n \le 2^{s+1} - 1$ and $0 \le m \le 2^s - 1$,

$$c(n + 2^{s+1},m) = c(n,m) \quad \text{and} \quad c(n + 2^s,m) = 0 .$$

(iv-v). For $2^s \le n \le 2^{s+1} - 1$ and $m + n \le 2^{s+1} - 1$,

$$c(n + 2^{s+1},m) = c(n,m) \quad \text{and} \quad c(n + 2^s,m) = 0 .$$

(vi-vii). For $0 \le n \le 2^s - 1$ and $2^s \le m \le 2^{s+1} - 1$,

$$c(n + 2^{s+1},m) = c(n,m) \quad \text{and} \quad c(n + 2^s + 2^{s+1},m) = c(n + 2^s,m).$$

Proof. To show (i) we apply (1.15-1.18) to (1.12). To show (ii) and (iii) we apply (1.19-1.21) to (1.12). To show (iv) and (v) we apply (1.22-1.24) to (1.12). To show (vi) and (vii) we need some additional remarks: Using (1.25-1.27) we obtain $\sum_{k=0}^{\infty} \varepsilon(k,n)\cdot\varepsilon(k,m) = \sum_{k=0}^{\infty} \varepsilon(k,n + 2^{s+1})\varepsilon(k,m)$ and $A(k;\ n,m) = A(k;\ n + 2^{s+1},m)$, for $0 \le k \le s$. Therefore $f(k;\ n,m) = f(k;\ n + 2^{s+1},m)$ for $0 \le k \le s$. So, using (1.12), $c(n,m) = c(n + 2^{s+1},m)$, which proves (vi). Again, using (1.25-1.27), $\sum_{k=0}^{\infty} \varepsilon(k,n + 2^s)\varepsilon(k,m) = \sum_{k=0}^{\infty} \varepsilon(k,n + 2^s + 2^{s+1})\varepsilon(k,m)$ and $A(k;\ n + 2^s,m) = A(k;\ n + 2^s + 2^{s+1})$ for $0 \le k \le s$. Therefore, $f(k;\ n + 2^s,m) = f(k;\ n + 2^s + 2^{s+1},m)$ for $0 \le k \le s$. So, by applying (1.12) we have $c(n + 2^s,m) = c(n + 2^s + 2^{s+1},m)$, which proves (vii).

<u>Corollary 1.29</u>. For $m,n \le 2^s - 1$ and $m + n \ge 2^s$, $c(n,m) = c(n + 2^s,m)$.

<u>Proof</u>. We distinguish 3 cases. In the first case we have $n,m \le 2^{s-1} - 1$ and $n + m \ge 2^{s-1}$. Under the assumption of the corollary $m \ge 2^s - n$; under the assumption of the first case $-n \ge -2^{s-1} + 1$. Thus, $m \ge 2^s - 2^{s-1} + 1 = 2^{s-1} + 1 \ge 2^{s-1}$. We have

$$(1.30) \qquad 2^{s-1} \le m \le 2^{s-1} - 1 \le 2^s \quad \text{and} \quad 0 \le n \le 2^{s-1} - 1 .$$

We apply (1.28 (vi)) to get $c(n,m) = c(n + 2^s,m)$.

In the second case we take $n \ge 2^{s-1}$, $m \le 2^s - 1$. Let $N = n - 2^{s-1}$. Then

$$(1.31) \qquad 0 \le N \le 2^{s-1} \qquad \text{and}$$

$$(1.32) \qquad 2^{s-1} \le m \le 2^s - 1 .$$

We may apply (1.28 (vii)) to get $c(N + 2^{s-1},m) = c(N + 2^{s-1} + 2^s,m)$. That is, $c(n,m) = c(n + 2^s,m)$.

In the final case, $n \le 2^s - 1$, $m \le 2^{s-1} - 1$, and $n + m \ge 2^s$. By assumption in this case, $n \ge 2^s - m \ge 2^s - 2^{s-1} + 1 = 2^{s-1} + 1 \ge 2^{s-1}$. Thus

$$(1.33) \qquad 0 \le m \le 2^{s-1} - 1 \quad \text{and} \quad 2^{s-1} \le n \le 2^s - 1 .$$

We apply (1.28 (ii)) to get $c(n,m) = c(n + 2^s,m)$. This completes the proof of the corollary.

<u>Theorem 1.34</u>.

$$c(n,m) = 0 \quad \text{for} \quad 2^s \le n \le 2^{s+1} - 1 \quad \text{and} \quad n + m \le 2^{s+1} - 1.$$

Proof. We prove the theorem by induction on s. For $s = 0$ and $s = 1$, we have, using (1.12), $c(0,1) = c(0,2) = c(0,3) = c(2,1) = 0$. We assume the theorem is true for $s \le k-1$. That is,

$$(1.35) \qquad c(n,m) = 0 \quad \text{for} \quad 2^{k-1} \le n \le 2^k - 1 \quad \text{and} \quad n+m \le 2^k - 1.$$

For $s = k$ we have 3 cases.

In the first case, we take m,n in the rectangle defined by $2^k \le n \le 2^k + 2^{k-1} - 1$ and $0 \le m \le 2^{k-1} - 1$. We take $s = k-1$ and use (1.28 (iii)) and (1.35) to get our result.

In the second case we take m,n in the triangle defined by $2^k + 2^{k-1} \le n \le 2^{k+1} - 1$ and $m + n \le 2^{k+1} - 1$. We take $s = k-1$ and use (1.28 (iv)) and (1.35) to get our result.

In the third case, we take m,n in the triangle defined by $2^k \le n \le 2^k + 2^{k-1}$ and $m + n \le 2^{k+1} - 1$. We use (1.28 (vi)), symmetry of the matrix $\mathcal{C}$ and (1.35) to get our result. Thus the theorem is proved.

The next theorem tells us that certain diagonals in $\mathcal{C}$ can be classified by a single parameter.

Theorem 1.36 (Diagonal Theorem). If $n + m =$ constant, then

$$(1.36) \qquad c(n,m) = 0 \quad \text{or} \quad p^{\chi(n + m)},$$

where $\chi(n + m) = \sum_{t=0}^{\infty} \varepsilon(t,n) \cdot \varepsilon(t,m)$.

Proof. If $n + m$ is odd the theorem follows easily since $c(n,m) = 0$ by using (1.12).

If $n + m$ is even we let s and λ be defined by $n + m = 2s$ and $2^\lambda \le 2s \le 2^{\lambda+1} - 2$. We proceed by induction on λ. Using (1.12), $c(0,0) = 1$, $c(2,0) = c(0,2) = c(1,1) = p$.

Thus we may assume $\lambda \geq 2$. Since $c(m,n) = c(n,m)$, we may assume $m \leq n$. Using (1.34), $c(n,m) = 0$ for $n > 2^{\lambda}$.

We partition $\mathcal{C}$ and show that the theorem is true in each member of the partition. We will then show that the members of the partition give the same answers. We partition $\mathcal{C}$ as follows: For $2^{\lambda-1} \leq m \leq n \leq 2^{\lambda} - 1$, we have $0 \leq m - 2^{\lambda-1} \leq n - 2^{\lambda-1} \leq 2^{\lambda-1} - 1$. Therefore, using (1.28 (i)) we obtain $c(n,m) = c(n - 2^{\lambda-1} + 2^{\lambda-1}, m - 2^{\lambda-1} + 2^{\lambda-1}) = p\, c(n - 2^{\lambda-1}, m - 2^{\lambda-1})$. Thus, if we restrict our attention to (the triangle) $2^{\lambda-1} \leq m \leq n \leq 2^{\lambda} - 1$, the relation (1.36) holds by induction. On the other hand, for $m \leq 2^{\lambda-1} -1 \leq n \leq 2^{\lambda} - 1$, we have, using (1.29), $c(n,m) = c(n - 2^{\lambda-1},m)$. Thus, if we restrict our view to (the triangle) $m \leq 2^{\lambda-1} - 1 \leq n \leq 2^{\lambda} - 1$, the relation (1.36) is true by induction.

It is now sufficient to show that one entry from $2^{\lambda-1} \leq m \leq n \leq 2^{\lambda} - 1$ equals one entry from $m \leq 2^{\lambda-1} - 1 \leq n \leq 2^{\lambda} - 1$. Moreover, we may assume

(1.37) $$s \leq 2^{\lambda-1} + 2^{\lambda-2} - 1 ,$$

for if $s > 2^{\lambda-1} + 2^{\lambda-2} - 1$, then we would have $m + n = 2s \geq 2^{\lambda} - 2^{\lambda-1}$. But for $m \leq 2^{\lambda-1} - 1 \leq n \leq 2^{\lambda} - 1$ we have $m < 2^{\lambda-1}$ and $n \leq 2^{\lambda} - 1$, whereby $m + n = 2s < 2^{\lambda} + 2^{\lambda-1} - 1$. Thus (1.37) holds.

Let $k = s - 2^{\lambda-1}$. Using $2^{\lambda} \leq 2s \leq 2^{\lambda+1} + 2$ (as stated at the beginning of this proof), we have

(1.38) $$0 \leq k \leq 2^{\lambda-1} - 1 .$$

Using (1.37) we have

$$(1.39) \qquad 0 \le k \le 2^{\lambda-2} - 1 .$$

We choose $c(2^{\lambda-1} + k,\ 2^{\lambda-1} + k)$ as an entry for which $2^{\lambda-1} \le m \le n \le 2^{\lambda} - 1$. We choose $c(2^{\lambda-1} + 2^{\lambda-2} + k,\ 2^{\lambda-2} + k)$ as an entry for which $m \le 2^{\lambda-1} - 1 \le n \le 2^{\lambda-1}$. It is sufficient to show $c(2^{\lambda-1} + 2^{\lambda-2} + k,\ 2^{\lambda-2} + k) = c(2^{\lambda-1} + k,\ 2^{\lambda-1} + k)$.

We may use (1.38) and (1.28 (i)) to get

$$(1.40) \qquad c(2^{\lambda-1} + k,\ 2^{\lambda-1} + k) = p\ c(k,k) .$$

Adding $2^{\lambda-2}$ to (1.39) we get

$$(1.41) \qquad 2^{\lambda-2} \le k + 2^{\lambda-2} \le 2^{\lambda-1} - 1 .$$

Thus, applying (1.28 (vii)),

$$(1.42) \qquad c(k + 2^{\lambda-2},\ k + 2^{\lambda-2}) = c(k + 2^{\lambda-1} + 2^{\lambda-2},\ k + 2^{\lambda-2}).$$

Using (1.39), $0 \le k \le 2^{\lambda-2} - 1$. Applying (1.28 (i)),

$$(1.43) \qquad p\ c(k,k) = c(k + 2^{\lambda-2},\ k + 2^{\lambda-2}) .$$

Combining (1.40), (1.42), and (1.43), we have

$$c(2^{\lambda-1} + k,\ 2^{\lambda-1} + k) = c(k + 2^{\lambda-1} + 2^{\lambda-2},\ k + 2^{\lambda-2}),$$

and this completes the proof.

We are now prepared to construct the monic polynomials $Q_n(z)$ of degree n which are orthogonal with respect to μ. That is, let $Q_0(z) = 1$ and $Q_n(z)$ be a polynomial of degree n with leading coefficient 1 such that

$$(1.44) \qquad \int_F Q_n(z)\ \overline{Q_m(z)}\ d\mu(z) = 0 \quad \text{if} \quad n \ne m .$$

For example, if $Q_1(z) = z + A$, then

$$0 = \int_F Q_0(z)\, \overline{Q_1(z)}\, d\mu = \int_F 1\cdot\overline{z}\, d\mu + \overline{A} \int_F 1\cdot d\mu \;.$$

The first integral is 0 since its integrand is odd, while (1.2) tells us that the second integral is 1. Therefore $A = 0$ and

$$Q_1(z) = z \;. \tag{1.45}$$

Kinney and Pitcher [4; pg. 27] find

$$Q_{2^n}(z) = P_n(z) \;. \tag{1.46}$$

We complete this lacunary set of orthogonal polynomials to the complete set.

Szegö uses Gram-Schmidt Orthogonalization on linearly independent functions to obtain orthogonal polynomials in [5; pp. 227-228]. We use a slight variation, suggested by the technique of Walsh, to obtain $Q_n(z)$.

Since the degree of $z^{<n>}$ is n, $\{z^{<n>}\}$ form a linearly independent set of functions. For $s \le n = 0,1,2,\ldots$, let $\alpha(n,s)$ be defined by

$$Q_n(z) = \sum_{s=0}^{n} \alpha(n,s) z^{<s>} \;. \tag{1.47}$$

Then for $0 \le k < n$,

$$\begin{aligned} 0 &= \langle Q_n(z), z^{<k>} \rangle = \int_F Q_n(z)\, \overline{z^{<k>}}\, d\mu(z) \\ &= \int_F \Big[\sum_{s=0}^{n} \alpha(n,s) z^{<s>} \Big] \overline{z^{<k>}}\, d\mu(z). \end{aligned}$$

Interchanging integration and summation, we get

$$0 = \sum_{s=0}^{n} \alpha(n,s) \int_F z^{<s>} \overline{z^{<k>}} \, d\mu(z)$$

$$= \sum_{s=0}^{n} \alpha(n,s)c(s,k) \ .$$

Also, since $z^{<n>}$ can be expressed as a linear combination of the Q_n's,

$$<Q_n(z), z^{<n>}> = \int_F Q_n(z) \sum_{j=0}^{n} \beta_j Q_j(z) d\mu(z)$$

$$= \int |Q_n(z)|^2 d\mu \ .$$

We denote this by $\|Q_n(z)\|^2$. Thus we have

$$(1.48) \qquad 0 = \sum_{s=0}^{n} \alpha(n,s)c(s,k) \quad \text{for} \quad 0 \le k \le n-1$$

and

$$(1.49) \qquad \|Q_n(z)\|^2 = \sum_{s=0}^{n} \alpha(n,s)c(s,n) \ .$$

(1.48) and (1.49) represent $n+1$ linear, non-homogeneous equations in the $n+1$ unknowns $\alpha(n,0), \alpha(n,1), \ldots, \alpha(n,n)$. Thus, as in Szegö's paper [5; pp. 227-228],

$$(1.50) \qquad \alpha(n,s) = \Delta_n^{-1} \begin{vmatrix} c(0,0) & \ldots & c(0,s-1) & 0 & c(0,s+1) & \ldots & c(0,n) \\ c(1,0) & \ldots & c(1,s-1) & 0 & c(1,s+1) & \ldots & c(1,n) \\ \vdots & & & \vdots & & & \\ c(n,0) & \ldots & c(n,s-1) & \|Q_n\|^2 & c(n,s+1) & \ldots & c(n,n) \end{vmatrix}$$

where

$$(1.51) \qquad \Delta_0 = 1 \quad \text{and} \quad \Delta_n = \begin{vmatrix} c(0,0) & \ldots & c(0,n) \\ \vdots & & \vdots \\ c(n,0) & \ldots & c(n,n) \end{vmatrix}$$

According to Theorem 1.12, $c(0,s) = c(s,0)$ and both are $\delta_{s,0}$; that is, 0 if $s \neq 0$ and 1 if $s = 0$. Thus,

$$(1.52)\quad \alpha(n,s) = \Delta_n^{-1}\delta_{s,0}\|Q_n(z)\|^2 \begin{vmatrix} c(1,1) & \ldots & c(1,s-1) & c(1,s+1) & \ldots & c(1,n) \\ \vdots & & & & & \\ c(n,1) & \ldots & c(n,s-1) & c(n,s+1) & \ldots & c(n,n) \end{vmatrix}$$

In particular,

$$(1.53)\qquad \alpha(n,0) = 0 \quad \text{for} \quad n = 1,2,\ldots\ .$$

Combining (1.47) with (1.52), we have shown that the following representation holds for $Q_n(z)$:

$$(1.54)\quad Q_n(z) = \|Q_n(z)\|^2\Delta_n^{-1} \begin{vmatrix} c(1,1) & \ldots. & c(1,n) \\ \vdots & & \\ c(n-1,1) & \ldots. & c(n-1,n) \\ z^{<1>} & \ldots. & z^{<n>} \end{vmatrix}$$

Since the leading term of $Q_n(z)$ is z^n, we have $Q_n(z) = z^n$ + lower order terms. But (1.54) indicates that $Q_n(z) = \|Q_n(z)\|^2\ \Delta_n^{-1}\Delta_{n-1}z^{<n>}$ + lower order terms. Thus,

$$(1.55)\quad \|Q_n(z)\|^2\ \Delta_{n-1}\Delta_n^{-1} = 1 \quad \text{or} \quad \Delta_n = \Delta_{n-1}\ \|Q_n(z)\|^2\ .$$

Thus, by induction we have

$$(1.56)\qquad \Delta_n = \prod_{k=0}^{n} \|Q_k(z)\|^2\ .$$

We have thus established the following:

<u>Theorem 1.57</u>. For $n \geq 1$,

$$Q_n(z) = D_n \begin{vmatrix} c(1,1) & \cdots & c(1,n) \\ \vdots & & \vdots \\ c(n-1,1) & \cdots & c(n-1,n) \\ z^{<1>} & \cdots & z^{<n>} \end{vmatrix} ,$$

where $D_n = \|Q_n(z)\|^2\Delta_n^{-1} = \Delta_{n-1}^{-1} = (\prod_{k=0}^{n} \|Q_k(z)\|^2)^{-1}$.

In special cases we can be more specific about Δ_n, as the next lemma shows.

Lemma 1.58. (i) $\Delta_{2^n} = p\,\Delta_{2^n-1}$

(ii) $\|Q_{2^n}(z)\|^2 = p$.

Proof. To show (i) we note that using (1.13 (2)) we obtain $c(2^n,k) = p\,\delta_{k,2^n}$ (where $\delta_{i,j} = 0$ for $i \neq j$ and 1 for $i = j$). When we expand Δ_{2^n} by the row containing $c(2^n,k)$, the result follows.

To show (ii) we recall that (1.46) implies $Q_{2^n}(z) = P_n(z)$. Therefore, $\|Q_{2^n}(z)\|^2 = \|P_n(z)\|^2 = c(2^n,2^n) = p$ when we use (1.13 (2)) or (1.13 (4)). This completes the proof of the lemma.

In [5], Szegö uses $\int z^n \overline{z^m}\,|dz|$ as the matrix entries. We have used $\int z^{<n>}\,\overline{z^{<m>}}\,d\mu$. In the case that F is a subset of the real line (this occurs for $p \geq 2$), we might have considered $\int z^N d\mu$. If N is odd, we apply (1.3) to find this integral equals 0, but if N is even, then the integral is a polynomial in p of degree N/2. Moreover, the coefficients in this polynomial have a complex pattern. As examples, we note that

$$\int z^2 d\mu = \int (P(z) + p) d\mu = \int z \, d\mu + \int p \, d\mu = p$$

$$\int z^4 d\mu = \int (P(z) + p)^2 d\mu = \int z^2 d\mu + 2p \int z \, d\mu + p^2 \int d\mu = p^2 + p$$

$$\int z^6 d\mu = 2p^2 + p^3$$

$$\int z^{2s} d\mu = \int (P(z) + p)^s d\mu = \sum_{\lambda=0}^{s} \binom{s}{\lambda} p^{s-\lambda} \int z^{\lambda} d\mu$$

$$= \sum_{\lambda=0}^{[s/2]} \binom{s}{2\lambda} p^{s-2\lambda} \int z^{2\lambda} d\mu \; .$$

The use of $c(n,m)$ for our matrix entries seems to be simpler since $c(n,m)$ is 0 or a power of p (Theorem 1.12) and $c(n,m) = c(s,t)$ or 0 if $n + m = s + t$ (Theorem 1.36).

CHAPTER II

We develop a difference equation for our orthogonal polynomials as Szegö does in [1].

To insure that divisions occurring later are legitimate we need the following lemma:

<u>Lemma 2.1.</u>

$$\|Q_n(z)\|^2 \neq 0 \quad \text{for} \quad n = 0,1,2,\ldots$$

<u>Proof</u>: $\int_F |Q_n(z)|^2 d\mu(z) = 0$ if and only if μ is concentrated on the roots of $Q_n(z)$, $n = 0,1,2,\ldots$. Since μ is concentrated on F [1; Lemma 15.2], we would have F consisting of the roots of $Q_n(z)$, $n = 0,1,2,\ldots$. Then F would be a denumerable set and have capacity 0 [8; Theorem III.8]. But F has capacity 1 [1; Lemma 15.1]. This contradiction completes the proof.

Since $Q_n(z)$ are orthogonal polynomials, the difference equation

$$(2.2) \qquad Q_n(z) = z\,Q_{n-1}(z) + A_{n-2}Q_{n-2}(z) + A_{n-1}Q_{n-1}(z)$$

holds for some A_{n-2}, A_{n-1} [7; Theorem 3.2.1]. We can improve the above difference equation; after a few lemmas, we will be able to delete the middle term.

<u>Lemma 2.3.</u> $Q_{2n}(z) = Q_n(P(z))$ for $n = 0,1,\ldots$

<u>Proof.</u> It is sufficient to show that $\int_F Q_n(P(z))\overline{z^s}\, d\mu(z) = 0$ for $0 \leq s < 2n$.

If s is even we let $s = 2m$. Then $0 \le m < n$, and

$$\int Q_n(P(z))\overline{z^s}\, d\mu = \int Q_n(P(z))\overline{z^{2m}}\, d\mu = \int Q_n(P(z))\overline{(P(z) + p)^m} d\mu$$

Shifting, the last integral is $\int Q_n(z)\overline{(z + p)^m}\, d\mu$.

Using the binomial theorem on $(z + p)^m$ and interchanging integration and summation we get $\int Q_n(P(z))\overline{z^s}\, d\mu = \sum_{j=0}^{m} \binom{m}{j} p^{m-j} \int Q_n(z)\overline{z^j} d\mu$. But $j \le m < n$ and thus the integral is 0 by induction. Thus we have our result if s is even.

If s is odd, then we claim that $Q_n(P(z))\overline{z^s}$ is an odd function of z. This is true because $P(z)$ is even, hence $Q_n(P(z))$ is even and $\overline{z^s}$ is odd. Thus $\int Q_n(P(z))\overline{z^s}\, d\mu = 0$ by (1.3). This concludes the proof of the lemma.

We have, immediately, the following corollary:

Corollary 2.4. $\int_F |Q_{2n}(z)|^2 d\mu = \int_F |Q_n(z)|^2 d\mu$.

Proof. Use the previous lemma and shift (1.4).

Our next result tells us about the evenness or oddness of the $Q_n(z)$.

Lemma 2.5.

(i) $Q_{2n}(z)$ is an even function of z for $n = 0,1,\ldots$

(ii) $Q_{2n+1}(z)$ is an odd function of z for $n = 0,1,\ldots$

Proof. Since $Q_{2n}(z) = Q_n(P(z))$ by lemma 2.3, we obtain (i).

To prove (ii), we note that by orthogonality, $\int Q_{2n}(z)\, \overline{Q_{2n+1}(z)}\, d\mu = 0$. By (2.2),

$$(2.5) \quad \int Q_{2n}(z)\, \overline{Q_{2n+1}(z)}\, d\mu(z) = \int Q_{2n}(z)\, \overline{z\, Q_{2n}(z)}\, d\mu$$
$$+ \bar{A}_{2n-1} \int Q_{2n}(z)\, \overline{Q_{2n-1}(z)}\, d\mu + \overline{A_{2n}} \int Q_{2n}(z)\, \overline{Q_{2n}(z)}\, d\mu \quad .$$

Thus,

$$(2.6)\quad 0 = \int Q_{2n}(z)\,\overline{zQ_{2n}(z)}\,d\mu + \bar{A}_{2n-1}\int Q_{2n}(z)\,\overline{Q_{2n-1}(z)}\,d\mu + A_{2n}\int |Q_{2n}(z)|^2 d\mu .$$

The first integral on the right hand side of (2.6) is zero because its integrand is an odd function of z. The second integral is zero by orthogonality of the $Q_n(z)$. Thus, since $\int |Q_n(z)|^2 d\mu \neq 0$ by lemma 3.1, we must have $\bar{A}_{2n} = 0$. Thus $A_{2n} = 0$. We have shown that

$$(2.7)\qquad Q_{2n+1}(z) = zQ_{2n}(z) + A_{2n-1}\,Q_{2n-1}(z).$$

Therefore, by induction on n and the facts that $zQ_2(z) = z(z^2 - P) = z^3 - pz$ and $Q_1(z) = z$, we have our result.

We combine (2.2) with lemma (2.5), and we write $-K(n)$ instead of A_{n-2}. Then we have shown that the following theorem holds:

Theorem 2.8.

$$Q_0(z) = 1\ ,\ Q_1(z) = z\ ,\quad \text{and}$$

$$(2.8)\qquad Q_n(z) = zQ_{n-1}(z) - K(n)\,Q_{n-2}(z) \quad \text{for} \quad n \geq 2.$$

We have reduced the problem to finding relationships among the $K(n)$. This problem admits a solution in some cases. As the next theorem shows, when our F set is a real set, the $K(n)$ satisfy a pair of non-linear recursion relations.

Theorem 2.9. If $p \geq 2$, then

(i) $K(2) = p$

(ii) $K(2n+1) = K(n+1)/K(2n)$ for $n = 1,2,\ldots$

(iii) $K(2n) = p - K(2n-1)$ for $n = 2,3,\ldots$

Proof. Since $Q_o(z) = 1$, $Q_1(z) = z$, $Q_2(z) = z^2 - p$, we may take $n = 2$ in Theorem 2.8. Thus $z^2 - p = z(z) - K(2)$. We then have $K(2) = p$, giving us (i). To show (ii), we multiply (2.8) by $Q_{n-2}(z)$ and integrate:

$$(2.10) \qquad 0 = \int zQ_{n-1}(z)Q_{n-2}(z)d\mu - K_n \|Q_{n-2}(z)\|^2 ,$$

but $zQ_{n-2}(z) = Q_{n-1}(z) + K(n-1)Q_{n-3}(z)$ from (2.8), so

$$\int zQ_{n-1}(z)Q_{n-2}(z)d\mu = \int |Q_{n-1}(z)|^2 d\mu + K(n-1)\int Q_{n-1}(z)Q_{n-3}(z)d\mu$$

which equals $\|Q_{n-1}(z)\|^2$.

Thus, (2.10) becomes

$$0 = \|Q_{n-1}(z)\|^2 - K(n) \|Q_{n-2}(z)\|^2$$

or

$$(2.11) \qquad K(n) = \|Q_{n-1}(z)\|^2/\|Q_{n-2}(z)\|^2 \qquad \text{for} \quad n = 2,3,\ldots$$

Thus using Corollary 2.4 and taking $n \geq 1$, we have

$$K(2n+1) = \frac{\|Q_{2n}\|^2}{\|Q_{2n-1}\|^2} = \frac{\|Q_n\|^2}{\|Q_{2n-1}\|^2} .$$

Dividing numerator and denominator by $\|Q_{n-1}\|^2$, we have

$$K(2n+1) = \frac{\|Q_n\|^2/\|Q_{n-1}\|^2}{\|Q_{2n-1}\|^2/\|Q_{n-1}\|^2} = \frac{\|Q_n\|^2/\|Q_{n-1}\|^2}{\|Q_{2n-1}\|^2/\|Q_{2n-2}\|^2} = \frac{K(n+1)}{K(2n)} ,$$

which proves (ii).

To show (iii), we write (2.8) as $Q_{2n+1}(z) = zQ_{2n}(z) - K(2n+1)Q_{2n-1}(z)$ and note that

(2.12) $$zQ_{2n-1}(z) = Q_{2n}(z) + K(2n)Q_{2n-2}(z) .$$

Thus,

$$\begin{aligned}\|Q_{2n+1}(z)\|^2 &= \int Q_{2n+1}(z)[zQ_{2n}(z) - K(2n+1)Q_{2n-1}(z)]d\mu \\ &= \int zQ_{2n}(z)Q_{2n+1}(z)\,d\mu - K(2n+1)\cdot 0 \\ &= \int zQ_{2n}(z)[zQ_{2n}(z) - K(2n+1)Q_{2n-1}(z)]d\mu \\ &= \int z^2Q_{2n}^2(z)d\mu - K(2n+1)\int zQ_{2n-1}(z)Q_{2n}(z)d\mu .\end{aligned}$$

Using (1.6) in the first integral and (2.12) in the second integral, we get

$$\|Q_{2n+1}(z)\|^2 = \int (P(z)+p)Q_{2n}^2(z)d\mu - K(2n+1)\int[Q_{2n}^2(z) + \\ + K(2n)Q_{2n-2}(z)Q_{2n}(z)]d\mu .$$

Using (1.3) and (1.4), we have

$$\|Q_{2n+1}(z)\|^2 = \int zQ_n^2(z)d\mu + p\int Q_{2n}^2(z)d\mu - K(2n+1)\int Q_{2n}^2(z)d\mu - K(2n+1)\cdot K(2n)\cdot 0 .$$

The first integral above is zero because its integrand is an odd function of z. Thus

$$\|Q_{2n+1}(z)\|^2 = p\|Q_n(z)\|^2 - K(2n+1)\ \|Q_n(z)\|^2 .$$

That is,

$$p - K(2n+1) = \|Q_{2n+1}(z)\|^2/\|Q_n(z)\|^2 .$$

But by (2.4), $\|Q_n(z)\|^2 = \|Q_{2n}(z)\|^2$ and by (2.11)

$$K(2n+2) = \|Q_{2n+1}\|^2/\|Q_{2n}\|^2 .$$

Therefore,

$$K(2n+2) = p - K(2n+1)$$

or

$$K(2n) = p - K(2n-1) ,$$

which completes the proof of (iii).

Since $K(2n+1) = \frac{K(n+1)}{K(2n)}$ and $K(2n) = p - K(2n-1)$, we have $K(2n+1) = \frac{K(n+1)}{p-K(2n-1)}$. Continuing this way we can develop a continued fraction expansion for $K(2n+1)$. Similarly, since $K(2n) = p - K(2n-1)$ and $K(2n-1) = \frac{K(n)}{K(2n-2)}$, we have $K(2n) = p - \frac{K(n)}{K(2n-2)}$, and we can develop a continued fraction expansion for $K(2n)$. We exhibit these continued fractions in our next theorem.

Theorem 2.13.

(2.14)
$$K(2n+1) = \cfrac{K(n+1)}{p - \cfrac{K(n)}{p - K(n-1) \cdots \overline{p - K(3)}}}$$
for $n \geq 3$ and $p \geq 2$

(2.15)
$$K(2n) = p - \cfrac{K(n)}{p - \cfrac{K(n-1)}{p - \cfrac{K(n-2)}{\ddots \overline{p - K(3)}}}}$$
for $n \geq 2$

We have so far developed two approaches to the problem of finding the orthogonal polynomials $Q_n(z)$. One method uses a matrix representation (Theorem 1.57) in which the determinant Δ_n appears. The other approach is by difference equation techniques (Theorem 2.8) in which we use the coefficients $K(n)$. The next theorem relates these two methods.

Theorem 2.16.

$$K(n) = \Delta_{n-1} \Delta_{n-2}^{-2} \Delta_{n-3} \quad \text{for} \quad n \geq 3 .$$

Proof. By (2.11) we have

$$K(n) = \|Q_{n-1}(z)\|^2 / \|Q_{n-2}(z)\|^2 \quad \text{for} \quad n \geq 2 .$$

We also have, by (1.55)

$$\|Q_n(z)\|^2 = \Delta_n \Delta_{n-1}^{-1} \quad \text{for} \quad n \geq 1 .$$

Therefore $K(n) = \dfrac{\Delta_{n-1} \Delta_{n-2}^{-1}}{\Delta_{n-2} \Delta_{n-3}^{-1}} = \Delta_{n-1} \Delta_{n-2}^{-2} \Delta_{n-3}$, which is what we wish to show.

Another approach to computing $Q_n(z)$ for $p \geq 2$, using the coefficients $K(n)$ of the linear difference equation is to consider ratios of the $Q_n(z)$.

Definition.

$$(2.17) \qquad T_n(z) = Q_n(z)/Q_{n-1}(z) \quad \text{for } n \geq 1 \text{ and } Q_{n-1}(z) \neq 0 .$$

If we divide (2.8) by $Q_{n-1}(z)$ we have

$$(2.18) \qquad T_n(z) = z - \frac{K(n)}{T_{n-1}(z)} .$$

Since, by definition, $T_1(z) = z$, $T_2(z) = z - \frac{p}{z}$, we have the following continued fraction representation for $T_n(z)$:

(2.19)
$$T_n(z) = z - \cfrac{K(n)}{z - \cfrac{K(n-1)}{z - \cfrac{K(n-2)}{\ddots \; z - \frac{p}{z}}}} \quad .$$

This continued fraction reminds us of the continued fraction for $K(2n)$ in (2.15). In fact, if we replace z in (2.19) by p, the two continued fractions (2.15) and (2.19) are identical. Thus we have shown the following lemma.

<u>Lemma 2.20</u>. For $p \geq 2$ and $n \geq 1$

$$T_n(p) = K(2n) \quad .$$

We are in the following situation:

If we know $K(2),\ldots,K(n)$ then we can compute, by (2.19), $T_1(z)$, $T_2(z),\ldots,T_n(z)$. Then, by lemma 2.20, we get $K(2n)$ by evaluating $T_n(p)$. Moreover, since $K(2n-1) = p - K(2n)$, by (2.9 (iii)), we have $K(3)$, $K(5),\ldots$, $K(2n-1)$. Thus, starting with the first $n-1$ K's, we readily get the first $2n-1$ K's. Repeating this argument gets us as many coefficients $K(n)$ as we want.

<u>Example</u>. We know (2.9 (i)) that $K(2) = p$. Thus $T_2(z) = z - \frac{p}{z}$. Therefore $K(4) = T_2(p) = p - 1$ and $K(3) = p - K(4) = 1$. Continuing,

$$T_3(z) = z - \frac{1|}{|z} - \frac{p|}{|z} \quad \text{and}$$

$$T_4(z) = z - \frac{p-1|}{|z} - \frac{1|}{|z} - \frac{p|}{|z} ,$$

from which $K(6) = T_3(p) = \dfrac{p(p-1)-1}{p-1}$ and $K(5) = p - K(6) = \dfrac{1}{p-1}$.

Therefore

$$T_5(z) = z - \frac{\frac{1}{p-1}|}{|z} - \frac{p-1|}{|z} - \frac{1|}{|z} - \frac{p|}{|z} \quad \text{and}$$

$$T_6(z) = z - \frac{\frac{p(p-1)-1}{p-1}|}{|z} - \frac{\frac{1}{p-1}|}{|z} - \frac{p-1|}{z} - \frac{1|}{|z} - \frac{p|}{|z} .$$

A simple observation allows us another approach to computing $Q_n(z)$. Namely

$$T_1(z) \cdot T_2(z) \ldots T_n(z) = \frac{Q_n(z)}{Q_o(z)} .$$

Since $Q_o(z) \equiv 1$ we have observed that

(2.21)
$$Q_n(z) = \prod_{j=1}^{n} T_j(z) .$$

Using the $T_n(z)$ we computed above we have, as examples:

$$Q_1(z) = T_1(z) = z$$

$$Q_2(z) = T_1(z)T_2(z) = z(z - \frac{p}{z}) = z^2 - p$$

$$Q_3(z) = (T_1(z)T_2(z))T_3(z) = Q_2(z)T_3(z) = (z^2-p)(z - \frac{1}{z-\frac{p}{z}}) = z^3 - (p+1)z.$$

We already have one continued fraction representation for $T_n(z)$ in (2.19). Another may be obtained by solving (2.18) for $T_{n-1}(z)$. We have

(2.22)
$$T_{n-1}(z) = \frac{K(n)}{z-T_n(z)} .$$

Since $T_n(z)$ in the denominator of (2.22) can be expressed as $\frac{K(n+1)}{z-T_{n+1}(z)}$, we obtain the following continued fraction representation for $T_n(z)$:

$$T_n(z) = \cfrac{K(n+1)}{z - \cfrac{K(n+2)}{z - \cfrac{K(n+3)}{z - \ddots}}} \tag{2.23}$$

As an example, we choose $T_1(z) = z$ and obtain

$$z = \cfrac{K(2)}{z - \cfrac{K(3)}{z - \cfrac{K(4)}{z - \ddots}}} \tag{2.24}$$

CHAPTER III

In this chapter we use the linear independence of the set $\{z^{<n>}\}$ to find representations for $Q_n(z)$.

As in (1.47) let

$$Q_n(z) = \sum_{k=0}^{n} \alpha(n,k)\, z^{<k>} . \tag{3.1}$$

We remark that since $z^{<n>}$ is an n-th degree polynomial, the $z^{<n>}$ are linearly independent; thus the coefficients $\alpha(n,k)$ are unique.

We obtain some relationships among the $\alpha(n,k)$ in the following lemmas.

<u>Lemma 3.2</u>.

3.2 (i) $\qquad \alpha(2n,2m) = \alpha(n,m) \quad$ for $\quad 0 \le m \le n < \infty$

3.2 (ii) $\qquad \alpha(2n,2m+1) = 0 \quad$ for $\quad 0 \le m \le n < \infty$.

<u>Proof</u>. Since $Q_{2n}(z) = Q_n(P(z))$ (Lemma 2.3) we can write $Q_{2n}(z)$ as $\sum_{k=0}^{2n} \alpha(2n,k)z^{<k>}$ or as $\sum_{k=0}^{n} \alpha(n,k)(P(z))^{<k>}$. But $(P(z))^{<k>} = z^{<2k>}$ by (1.7). Thus

$$\sum_{k=0}^{2n} \alpha(2n,k)z^{<k>} = \sum_{k=0}^{n} \alpha(n,k)z^{<k>} . \tag{3.3}$$

Comparing coefficients in (3.3) we obtain (3.2).

We next prove:

Lemma 3.4.

(3.4) $\alpha(2n+1,2m) = 0 \quad \text{for} \quad 0 \le m \le n < \infty$

(3.5) $\alpha(2k+3,2k+3) = \alpha(2k+2,2k+2)$

(3.6) $\alpha(2k+3,2s+1) = \alpha(2k+2,2s) - K(2k+3)\alpha(2k+1,2s+1)$

for $0 \le s \le k$.

Proof. To show (3.4) we use induction on n. For $n = 0$ and 1 we use $Q_o(z) = 1$ to obtain $\alpha(0,0) = 1$; we use $Q_1(z) = P_o(z) = z$ to obtain $\alpha(1,1) = 1$ and $\alpha(1,0) = 0$; we use $Q_2(z) = P_1(z) = z^2 - p$ to obtain $\alpha(2,2) = 1$, $\alpha(2,1) = 0$, and $\alpha(2,0) = -p$.

We assume (3.4) is true for $n \le k$. That is $\alpha(2n+1,2m) = 0$ for $0 \le m \le n < k$. For $n = k+1$, we use $n = k+1$ in (2.8), and we get

$$Q_{2(k+1)+1}(z) = zQ_{2(k+1)}(z) - K(2(k+1)+1)Q_{2(k+1)-1}(z) .$$

So,

$$\sum_{s=0}^{2k+3} \alpha(2k+3,s)z^{<s>} = z\sum_{s=0}^{2k+2} \alpha(2k+2,s)z^{<s>} - K(2k+3)\sum_{s=0}^{2k+1} \alpha(2k+1,s)z^{<s>} .$$

Applying (3.2 (ii)) to the first expression on the right hand side and the inductive assumption to the second expression, we may write

$$\sum_{s=0}^{2k+3} \alpha(2k+3,s)z^{<s>} = z\sum_{\substack{s=0 \\ s \text{ even}}}^{2k+2} \alpha(2k+2,s)z^{<s>} - K(2k+3)\sum_{\substack{s=0 \\ s \text{ odd}}}^{2k+1} \alpha(2k+1,s)z^{<s>}.$$

So, by (1.8 (i)),

$$(3.7) \quad \sum_{s=0}^{2k+3} \alpha(2k+3,s)z^{<s>} = \sum_{s=0}^{2k+2} \alpha(2k+2,s)z^{<s+1>} - K(2k+3)\sum_{\substack{s=0 \\ s \text{ odd}}}^{2k+1} \alpha(2k+1,s)z^{<s>}.$$

Comparing coefficients in (3.7), we obtain $\alpha(2k+3,2s) = 0$ for $s = 0,1,\ldots,k+1$. Thus (3.4) is true for $n = k+1$ and thus true by induction.

Comparing coefficients in (3.7) also establishes (3.5) and (3.6).

We are now able to show that when we expand $Q_n(z)$ in powers of $z^{<n>}$, the coefficient of $z^{<o>}$ is always zero.

Lemma 3.8.

$$\alpha(n,0) = 0 \quad \text{for} \quad n = 1,2,\ldots$$

Proof. If n is odd, this follows from Lemma 3.4. If n is even, we use (3.2 (i)) to write $\alpha(n,0) = \alpha(n/2,0)$. Our result follows by induction.

Having shown that there is no "constant coefficient", we now show that $\sum_{k=0}^{n} \alpha(n,k)z^{<k>}$ is a "monic" expansion. That is

Lemma 3.9.

$$\alpha(n,n) = 1 \quad \text{for} \quad n = 0,1,\ldots$$

Proof. We prove the lemma by induction on n. We have already shown, in the proof of Lemma 3.4, that the result is true for $n = 0,1,2$.

We assume that the theorem is true for $n \le N$. That is,

$$\alpha(n,n) = 1 \quad \text{for} \quad n \le N \; . \tag{3.10}$$

If $n = N+1$, we have to distinguish between $N+1$ being even or odd. If $N+1$ is even we take $N+1 = 2s$. Then $\alpha(N+1,N+1) = \alpha(2s,2s)$, which by (3.2 (i)) equals $\alpha(s,s)$, which by (3.10) equals 1.

If $N+1$ is odd, we let $N+1 = 2s+1$.

Then writing the difference equation for $Q_n(z)$ as expansions in $z^{<n>}$ and utilizing (1.8 (i)) we obtain:

$$(3.11) \quad \sum_{j=0}^{s} \alpha(2s+1,2j+1)z^{<2j+1>} = \sum_{j=0}^{s} \alpha(2s,2j)z^{<2j+1>} - K(2s+1) \sum_{j=0}^{s-1} \alpha(2s-1,2j+1)z^{<2j+1>} .$$

The second sum on the right hand side of (3.11) has no terms involving $z^{<2s+1>}$. Thus, comparing coefficients in (3.11) we obtain

$$\alpha(N+1,N+1) = \alpha(2s+1,2s+1) = \alpha(2s,2s) = \alpha(s,s) = 1 .$$

This completes the proof of the lemma.

We now show:

<u>Lemma 3.12</u>.

$$\alpha(2n+1,2m+1) = \alpha(n,m) - K(2n+1)\alpha(2n-1,2m+1) .$$

<u>Proof</u>. We use, again, the difference equation for the Q_n (2.8) written as expansions in $z^{<n>}$.

$$\sum_{s=0}^{2n+1} \alpha(2n+1,s)z^{<s>} = z \sum_{s=0}^{2n} \alpha(2n,s)z^{<s>} - K(2n+1) \sum_{s=0}^{2n-1} \alpha(2n-1,s)z^{<s>} .$$

By (3.2 (ii)) and (3.4), $\alpha(2n,2s+1)$, $\alpha(2n-1,2s)$ and $\alpha(2n+1,2s)$ are zero for $0 \le s < n$.

Thus

$$(3.13) \quad \sum_{s=0}^{n} \alpha(2n+1,2s+1)z^{<2s+1>} = z \sum_{s=0}^{n} \alpha(2n,2s)z^{<2s>} - K(2n+1) \sum_{s=0}^{n-1} \alpha(2n-1,2s+1)z^{<2s+1>} .$$

Recalling that $z \cdot z^{<2s>} = z^{<2s+1>}$, and combining like terms we have

$$\sum_{s=0}^{n} \alpha(2n+1,2s+1)z^{<2s+1>} = \sum_{s=0}^{n-1} [\alpha(2n,2s)-K(2n+1)\alpha(2n-1,2s+1)]z^{<2s+1>} + \alpha(2n,2n)z^{<2n+1>} .$$

By Lemma 3.9, $\alpha(2n,2n) = 1$. Thus,

$$(3.14) \quad \sum_{s=0}^{n} \alpha(2n+1,2s+1)z^{<2s+1>} = \sum_{s=0}^{n-1} [\alpha(2n,2s)-K(2n+1)\alpha(2n-1,2s+1)]z^{<2s+1>} + z^{<2n+1>} .$$

Comparing coefficients in (3.14),

$$\alpha(2n+1,2s+1) = \alpha(2n,2s) - K(2n+1)\alpha(2n-1,2s+1) \quad \text{for} \quad n \geq 0 .$$

Since $\alpha(2n,2s) = \alpha(n,s)$, we obtain our result:

$$\alpha(2n+1,2m+1) = \alpha(n,m) - K(2n+1)\alpha(2n-1,2m+1) .$$

Combining our results from (3.2), (3.4), and (3.12) we have:

<u>Theorem 3.15</u>.

$\alpha(n,m) = 0$ if $n+m$ is odd

$\alpha(n,m) = \alpha(\frac{n}{2}, \frac{m}{2})$ if n and m are even

$\alpha(n,m) = \alpha(\frac{n-1}{2}, \frac{m-1}{2}) - K(n)\alpha(n-2,m)$ if n and m are odd

$\alpha(n,m) = 1$ if $n = m = 1$.

As corollaries to this theorem we have

<u>Corollary 3.16</u>.

1. $\alpha(2^n+1,2m+1) = -K(2^n+1)\alpha(2^n-1,2m+1)$ for $0 \leq m \leq 2^{n-2}$

2. $\alpha(2^n-1,2m+1) = \alpha(2^{n-1}-1,m) - K(2^n-1)\alpha(2^n-3,2m+1)$

for $0 \le m \le 2^{n-1}-1$

3. $\alpha(2n+1,2n-1) = -K(2n+1)$

4. $\alpha(2n+1,1) = (-1)^n \prod_{j=1}^{n} K(2j+1)$

5. $\alpha(2n+1,2n-3) = \alpha(n,n-2) + K(2n+1)K(2n-1)$

6. $\alpha(2n+1,2n-5) = -K(2n+1)K(2n-1)K(2n-3)$ if n is odd and

$K(2n+1)[K(n-1) - K(2n-3)K(2n-1)]$ if n is even.

CHAPTER IV

We have developed representations for $Q_n(x)$, the monic orthogonal polynomials over F with respect to μ. We now extend the orthogonality to sets other than F by using a measure μ^* induced by use of the Green's function.

We let $G(z,z_o,\mathcal{C})$ be the Green's function for the region bounded by the closed analytic curve $\mathcal{C}$ with pole at z_o. We recall that $G(z,z_o,\mathcal{C}) = 0$ on $\mathcal{C}$, $G(z,z_o,\mathcal{C}) > 0$ in the region bounded by $\mathcal{C}$, and $G(z,z_o,\mathcal{C}) + \log|z - z_o|$ is harmonic in the region bounded by $\mathcal{C}$.

<u>Theorem 4.1</u>. Let $\mathcal{C}$ be any closed, analytic curve containing F in its interior. Let $w \in F$. For $z \in \mathcal{C}$ we take

$$\varphi(z) = \int_F G(z,w,\mathcal{C})\,d\mu(w)$$

and $d\mu^*(z) = \frac{d}{dn}\varphi(z)|dz|$ where n is normal to $\mathcal{C}$ at z. Then

$$\int_C Q_n(z)\,\overline{Q_m(z)}\,d\mu^*(z) = 0 \quad \text{if} \quad n \neq m .$$

<u>Proof</u>. By definition we have

$$\int_C Q_n(z)\,\overline{Q_m(z)}\,d\mu^*(z) = \int_C Q_n(z)\,\overline{Q_m(z)}\,\left(\frac{d}{dn}\int_F G(z,w,\mathcal{C})d\mu(w)\right)|dz|$$

$$= \int_C\left(\int_F Q_n(z)\,\overline{Q_m(z)}\,G(z,w,\mathcal{C})d\mu(w)\right)|dz| .$$

By the Fubini Theorem, the last expression equals

$$\int_F \left(\int_C Q_n(z) \overline{Q_m(z)} \frac{\partial}{\partial n} G(z,w,\mathcal{C}) |dz| \right) d\mu(w) \ .$$

In [8] we find the following result:

<u>Theorem</u> [8; Theorem I.21]. Let $f(\zeta)$ be a bounded measurable function defined on the closed analytic curve Γ. For $\zeta \in \Gamma$, we take n to be the inner normal to Γ at ζ. We put

$$u(z) = \int_\Gamma f(\zeta) \frac{\partial}{\partial n} G(z,\zeta,\Gamma) \ |d\zeta| \ .$$

Then $u(z)$ is a bounded harmonic function in the region bounded by Γ, and if $f(\zeta)$ is continuous at $\zeta_o \in \Gamma$, then

$$\lim_{z \to \zeta_o} u(z) = f(\zeta_o) \ ,$$

for z belonging to the region bounded by Γ. Thus

$$\int_F \left(\int_C Q_n(z) \overline{Q_m(z)} \frac{\partial}{\partial n} G(z,w,\mathcal{C}) |dz| \right) d\mu(w)$$

$$= \int_F Q_n(w) \overline{Q_m(w)} \ d\mu(w) \ ,$$

which equals zero by orthogonality. This completes the proof of the theorem.

We examine the equilibrium measures concentrated on the equipotential curves associated with the measure μ concentrated on F. We take

$$\mathcal{C}_r = \{z : \int_F \log |z - w| d\mu(w) = r\} \ . \tag{4.2}$$

We always assume that r is large enough so that $\mathcal{C}_r$ is simply connected and contains F in its interior.

We have the following relationship between Green's functions and $\int_F \log |z - w| d\mu(w)$:

Lemma 4.3.

$$G(z,\infty,C_r) = \int_F \log |z - w| d\mu(w) - r .$$

Proof. In order to prove the lemma we must show two things. First we must prove that as z tends to infinity, $\int_F \log |z-w| d\mu(w) - r$ tends to $\log |z|$. This is so because as $z \to \infty$, $\log |z-w| \to \log |z|$; thus,

$$\int_F \log |z - w| d\mu(w) \sim \int_F \log |z| d\mu(w) = \log |z| .$$

Therefore as $z \to \infty$,

$$\int_F \log |z - w| d\mu(w) - r \sim \log |z| - r \sim \log |z| .$$

Secondly, we must show that for $z \in \mathcal{C}_r$, $\int_F \log |z-w| d\mu(w) - r = 0$. This is true by definition of $\mathcal{C}_r$. Thus, the lemma is proved.

There is another representation for $G(z,\infty,C_r)$, as follows:

Lemma 4.4.

$$G(z,\infty,\mathcal{C}_r) = -\int_F G(z,w,\mathcal{C}_r) d\mu(w) .$$

Proof. By definition of $G(z,w,\mathcal{C}_r)$, we know that

(4.5) $$G(z,w,\mathcal{C}_r) + \log |z - w| = h(z,w,\mathcal{C}_r)$$

where h is a harmonic function of z throughout the region bounded by $\mathcal{C}_r$. Thus,

(4.6) $$\int_F G(z,w,\mathcal{C}_r) d\mu(w) = \int_F -\log |z-w| d\mu(w) + \int_F h(z,w,\mathcal{C}_r) d\mu(w) .$$

The first integral on the right hand side of (4.6) equals $-G(z,\infty,\mathcal{C}_r) - r$ by Lemma 4.3. The second integral on the right hand side of (4.6) is a harmonic function of z inside $\mathcal{C}_r$ since the integrand is a harmonic function of z inside $\mathcal{C}_r$. By definition of $G(z,z_o,\mathcal{C})$, $h(z,w,\mathcal{C}_r) = +\log|z - w|$ for $z \in \mathcal{C}_r$. Thus, for $z \in \mathcal{C}_r$,

$$\int_F h(z,w,\mathcal{C}_r)d\mu(w) = +\int_F \log|z - w|d\mu(w) \equiv +r.$$

Since $\int_F h(z,w,\mathcal{C}_r)d\mu(w)$ is harmonic throughout the region bounded by $\mathcal{C}_r$ and constant on $\mathcal{C}_r$, $\int h(z,w,\mathcal{C}_r)d\mu(w) \equiv +r$. Thus,

$$\int_F G(z,w,\mathcal{C}_r)d\mu(w) = -G(z,\infty,C_r) + r - r = -G(z,\infty,C_r)$$

Since $G(z,\infty,\mathcal{C}_r)$ is unique, we combine the results of Lemma 5.3 and Lemma 5.4 to get

$$(4.7) \qquad \int_F \log|z - w|d\mu(w) - r = -\int_F G(z,w,C_r)d\mu(w) .$$

Therefore

$$(4.8) \qquad \frac{d}{dn}\int_F G(z,w,\mathcal{C}_r)d\mu(w) = -\frac{d}{dn}\int_F \log|z - w|d\mu(w)$$

and $d\mu^*$, which we have previously defined as $\frac{d}{dn}\int_F G(z,w,\mathcal{C}_r)d\mu(w)\cdot|dz|$ may now be taken as $\frac{d}{dn}\int_F \log|z-w|d\mu(w)\cdot|dz|$.

We have proved the following theorem:

<u>Theorem 4.9</u>.

$$\int_{C_r} Q_n(z)\,\overline{Q_m(z)}\left[\frac{d}{dn}\int_F \log|z-w|d\mu(w)\right]|dz| = 0 \quad \text{if} \quad m \neq n.$$

CHAPTER V

We conclude with two examples; namely, $P(z) = z^2$ in which case the F set is the unit circle, and $P(z) = z^2 - 2$ in which case the F set is the interval $[-2,2]$ on the real axis.

<u>Example 1</u>. $P(z) = z^2$.

We will show that the F set is $\{z: |z| = 1\}$, that $d\mu$ is arc length, and that the orthogonal polynomials are powers of z. We observe that $P_2(z) = z^4$ and $P_n(z) = z^{2^n}$.

If $|z_0| > 1$, then for all z sufficiently close to z_0, $\{P_n(z)\}$ converges uniformly to ∞. Thus $\{P_n(z)\}$ is normal at z_0, so that no point of $|z| > 1$ belongs to F.

If $|z_0| < 1$, then $P_n(z)$ converges uniformly to 0 for all z sufficiently close to z_0. Thus $\{P_n(z)\}$ is normal at z_0, so that no point of $|z| < 1$ belongs to F.

On the other hand $P_n(e^{i\theta})$ is not normal for each point $e^{i\theta}$ of the unit circle. This is clear since each open neighborhood of $e^{i\theta}$ contains points whose absolute value is greater than one and points whose absolute value is less than one. Under iteration the former points converge to ∞ while the latter points converge to 0.

Thus $F = \{z: |z| = 1\}$.

To construct μ we take $z_0 = 1$ and we let μ_n be the discrete measure placing weight 2^{-n} at each root of $P_n(z) = 1$.

Since these roots are the roots of unity, and since they are distributed symmetrically around the unit circle, we have $\lim_{n\to\infty} \mu_n$ = Lebesgue measure ($= \frac{1}{2\pi}$ · arc length). That is, $d\mu_n \to \frac{1}{2\pi} \cdot |dz|$ as $n \to \infty$.

We now use the method of Chapter I to construct $Q_n(z)$.

<u>Lemma 5.1</u>. $z^{<n>} = z^n = Q_n(z)$ for $n = 0,1,\ldots$.

<u>Proof</u>. For $n = \sum_{s=0}^{\infty} \epsilon(s,n)2^s$ we have

$$z^{<n>} = \prod_{s=0}^{\infty} (P_s(z))^{\epsilon(s,n)} = \prod_{s=0}^{\infty} (z^{2^s})^{\epsilon(s,n)} = z^{\sum_{s=0}^{\infty} \epsilon(s,n)2^s} = z^n .$$

Thus we have the first equality.

We have $c(m,n) = \int_F z^{<m>} \overline{z^{<n>}} d\mu(z) = \frac{1}{2\pi} \int_{|z|=1} z^m \overline{z^n} |dz| = 0$ if $m \neq n$ and 1 if $m = n$. Hence $\Delta_n = 1$ for all n. Hence

$$Q_n(z) = \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & 1 & \vdots \\ 1 & z & & z^{n-1} & z^n \end{vmatrix} = z^n$$

and this illustrates the second equality.

Now that we have F (= unit circle), μ ($= \frac{1}{2\pi}$ · arc length), and $Q_n(z)$ ($= z^n$) we can use the results of Chapter IV to extend orthogonality. For $\mathcal{C}_r = \{z: |z| = r\}$ we have $G(z,\infty,\mathcal{C}_r) = \log \frac{1}{r} \cdot |z|$. In this case $\frac{d}{dn}$ is taken in the radial direction, so that $\frac{d}{dn} \log \frac{1}{r} \cdot |z| = \frac{d}{dr} \log |z| = \frac{d}{dr} \log r = \frac{1}{r}$. Therefore $d\mu^*(z) = \frac{1}{r} \cdot |dz|$. We have obtained the well known result that for $z = r\,e^{i\theta}$, $\int_0^{2\pi} z^n \overline{z^m} \frac{1}{r} |dz| = 0$ if $m \neq n$.

Example 2. $P(z) = z^2 - 2$.

We will use the results of Chapter II to show that $Q_n(z)$ is the Tchebycheff polynomial $2 \operatorname{Cos} n \operatorname{arc} \operatorname{Cos} (x/2)$. F will be the real segment $[-2,2]$ and $d\mu(z) = \frac{1}{2} \cdot (4 - z^2)^{-\frac{1}{2}} dz$. We conclude with remarks about the orthogonality of $Q_n(z)$ on certain confocal ellipses of which F is a limiting case.

Brolin [1; Theorem 12.1] shows that $F = [-2,2]$. Kinney and Pitcher [4; pg. 27] remark, without proof, that $d\mu(x) = \frac{1}{2} (4 - x^2)^{-\frac{1}{2}} dx$. We show this in the following lemma.

Lemma 5.2. $d\mu(x) = \frac{1}{2} \cdot (4 - x^2)^{-\frac{1}{2}} dx$.

Proof. We must show that μ is invariant under $P_{-1}(x)$. This follows from $d\mu(P_{-1}(x)) = \frac{1}{2} \cdot [4 - ((x + 2)^{\frac{1}{2}})^2]^{-\frac{1}{2}} d(x + 2)^{\frac{1}{2}} = \frac{1}{2} d\mu(x)$. But when we consider that there are two branches of $P_{-1}(x)$, each contributing equally to our measure, and we have considered only the positive branch, we get our result: $d\mu(P_{-1}(x)) = d\mu(x)$.

We use our non-linear recursion relations to find $k(n)$ and thus to find $Q_n(x)$.

From the recurrence formulae of Theorem 2.9 we have $k(2) = 2$, $k(3) = k(4) = \ldots = 1$. Thus, $Q_0(x) = 1$, $Q_1(x) = x$, $Q_2(x) = x^2 - 2$ and

$$(5.3) \qquad \text{for } n \geq 3,\ Q_n(x) = xQ_{n-1}(x) - Q_{n-2}(x) .$$

That $Q_n(x) = 2 \operatorname{Cos} n \operatorname{arc} \operatorname{Cos} (x/2)$ satisfies (5.3) may be seen by setting $A = n \operatorname{arc} \operatorname{Cos} (x/2)$, $B = \operatorname{arc} \operatorname{Cos} (x/2)$ in the identity $\operatorname{Cos} (A + B) + \operatorname{Cos} (A - B) = 2 \operatorname{Cos} A \cdot \operatorname{Cos} B$.

Now we use the results of Chapter IV to extend the orthogonality of the Tchebycheff polynomials to other curves.

We define $\mathscr{E}_R$ as the ellipse

$$[x(R + R^{-1})^{-1}]^2 + [y(R - R^{-1})^{-1}]^2 = 1 ,$$

where $z = x + iy$, and $R > 1$. We note that $\mathscr{E}_R$ has $x = \pm 2$ as foci , and that as $R \to 1$, $\mathscr{E}_R$ approaches the segment $[-2,2]$.

Since $z = w + w^{-1}$ maps $|w| > R$ conformally onto the exterior of $\mathscr{E}_R$ with the point at ∞ going into the point at ∞, $\log \frac{1}{R} \cdot |w| = G(z,\infty,\mathscr{E}_R)$. That is,

$$(5.4) \qquad \log \frac{1}{R} \cdot |(z + \sqrt{z^2 - 4})| = G(z,\infty,\mathscr{E}_R) .$$

According to our work at the beginning of Chapter IV,

$$(5.5) \qquad \int_{\mathscr{E}_R} 2 \operatorname{Cos} n \operatorname{arc} \operatorname{Cos} (z/2) \cdot \overline{2 \operatorname{Cos} m \operatorname{arc} \operatorname{Cos} (z/2)}\, d\mu^*(z) = 0$$

if $m \neq n$ where $d\mu^*(z) = \frac{d}{dn} G(z,\infty,\mathcal{C}_r) \cdot |dz|$. In our next lemma we compute $d\mu^*$.

<u>Lemma 5.6</u>. $\frac{d}{dn} G(z,\infty,\mathscr{E}_R) = \left|\frac{1}{\sqrt{4-z^2}}\right|$.

<u>Proof</u>. By (5.4), we must show

$$\frac{d}{dn} \log |z + \sqrt{z^2 - 4}| = \frac{1}{|\sqrt{4-z^2}|} .$$

If $w = r e^{i\theta}$ $(r > R)$ corresponds to z, then $G(z,\infty,\mathscr{E}_R) = \log r - \log R$ and $\frac{d}{dn} G(z,\infty,\mathscr{E}_R) = \frac{1}{r} \cdot \frac{dr}{dn}$. It is thus sufficient to show that for $z \in \mathscr{E}_R$,

$$(5.7) \qquad R^{-1} \cdot \frac{\partial R}{\partial n} = \frac{1}{|\sqrt{4-z^2}|} .$$

For $z = x + iy \in \mathscr{E}_R$, we have

$$(5.8) \qquad x = A \operatorname{Cos} \theta , \quad y = B \operatorname{Sin} \theta$$

where

(5.9) $$A = R + R^{-1}\ , \quad B = R - R^{-1}\ .$$

We note that

(5.10) $$\frac{\partial A}{\partial R} = 1 - R^{-2} = R^{-1}\cdot B \quad \text{and} \quad \frac{\partial B}{\partial R} = 1 + R^{-2} = R^{-1}A\ .$$

Taking the partial derivative with respect to x in both equations of (5.8) and solving for $\frac{\partial R}{\partial x}$, we get

(5.11) $$\frac{\partial R}{\partial x} = R\cdot B\cdot D^{-2}\ \mathrm{Cos}\ \theta\ ,$$

where $D^2 = B^2\mathrm{Cos}^2\theta + A^2\mathrm{Sin}^2\theta$.

Similarly, taking the partial derivative with respect to y and solving for $\frac{\partial R}{\partial y}$, we obtain

(5.12) $$\frac{\partial R}{\partial y} = R\cdot A\cdot D^{-2}\mathrm{Sin}\ \theta\ .$$

Therefore,

(5.13) $$\mathrm{grad}\ R = R\cdot D^{-2}(B\ \mathrm{Cos}\ \theta,\ A\ \mathrm{Sin}\ \theta)\ .$$

We still need to find n, the unit normal to $\mathcal{E}_R$. Since $(2A^{-2}x,\ 2B^{-2}y)$ is normal to $A^{-2}x^2 + b^{-2}y^2 = 1$, we have $n = (4x^2A^{-4} + 4y^2B^{-4})^{-\frac{1}{2}}\ (2A^{-2}x,\ 2B^{-2}y)$. Using (5.8) we have

$$\begin{aligned} n &= (A^{-2}\mathrm{Cos}^2\theta + B^{-2}\mathrm{Sin}^2\theta)^{-\frac{1}{2}}(A^{-1}\mathrm{Cos}\ \theta,\ B^{-1}\mathrm{Sin}\ \theta) \\ &= (B^2\mathrm{Cos}^2\theta + A^2\mathrm{Sin}^2\theta)^{-\frac{1}{2}}(B\ \mathrm{Cos}\ \theta,\ A\ \mathrm{Sin}\ \theta) \\ &= D^{-1}(B\ \mathrm{Cos}\ \theta,\ A\ \mathrm{Sin}\ \theta)\ . \end{aligned}$$

Thus $\frac{dR}{dn} = \frac{1}{R} \cdot R \cdot D^{-2}(B \text{ Cos } \theta, A \text{ Sin } \theta) \cdot D^{-1}(B \text{ Cos } \theta, A \text{ Sin } \theta)$

$$= D^{-3}(B^2 \text{Cos}^2\theta + A^2 \text{Sin}^2\theta) = D^{-3} \cdot D^2 = D^{-1} .$$

Now we show $\frac{1}{|\sqrt{4-z^2}|} = D^{-1}$.

Since $z = w + w^{-1}$, we have

$$|4-z^2|^{-\frac{1}{2}} = |4-(w+w^{-1})^2|^{-\frac{1}{2}} = |(w-w^{-1})^2|^{-\frac{1}{2}} = |w-w^{-1}|^{-1} =$$

$$= |Re^{i\theta} - R^{-1}e^{-i\theta}|^{-1} = |(R-R^{-1})\text{Cos } \theta + i(R+R^{-1})\text{Sin } \theta|^{-1},$$

which is $|A \text{ Cos } \theta + i\, B \text{ Sin } \theta|^{-1}$, which equals D^{-1} when $w = R\, e^{i\theta}$. This proves the lemma.

Hence, (5.5) reduces to the following relation, first obtained by Walsh [10]:

(5.14) $$\int_{\gamma_R} T_n(z)\, \overline{T_m(z)}\, \frac{|dz|}{|\sqrt{4-z^2}|} = 0 \quad \text{if} \quad m \neq n ,$$

where $T_n(z)$ is the n-th Tchebycheff polynomial $2 \text{ Cox } n \text{ arc Cos } (z/2)$.

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